Mathematisches Forschungsinstitut Oberwolfach

Report No. 19/2015

DOI: 10.4171/OWR/2015/19

Mathematical Theory of Water Waves

Organised by
Jerry Bona, Chicago
Mark Groves, Saarbrücken
Mariana Haragus, Besancon
Erik Wahlén, Lund

12 April – 18 April 2015

Abstract. The *water-wave problem* is the study of the two- and three-dimensional flow of a perfect fluid bounded above by a free surface subject to the forces of gravity and surface tension. From a mathematical viewpoint, the water-wave equations pose surprisingly deep and subtle challenges for mathematical analysis. The governing equations are widely accepted and there has been substantial research into their validity and limitations. However, a rigorous theory of their solutions is extremely complex due not only to the fact that the water-wave problem is a classical free-boundary problem (where the problem domain, specifically the water surface, is one of the unknowns), but also because the boundary conditions (and, in some cases, the equations) are strongly nonlinear. In contrast to other meetings on water waves, which usually focus upon modelling and numerical issues, this workshop was devoted to the rigorous mathematical theory for the exact hydrodynamic equations.

Mathematics Subject Classification (2010): 76B15.

Introduction by the Organisers

Although attempts at a theory for water waves were made in the later middle ages, it was not until the middle of the nineteenth century that the modern theory (incorporating nonlinear effects) appeared in the work of Stokes, who first wrote down the governing equations in their modern form. The nineteenth century also saw Scott Russell's discovery of solitary waves, the emergence of model equations for water waves (Korteweg-deVries, Boussinesq), Gerstner's explicit solutions for rotational waves and the Kelvin-Helmholtz instability. With the emergence of modern mathematics in the early twentieth century, significant progress was made in the theory of steady-wave solutions to the exact equations. The early results in the mathematical theory of two-dimensional steady gravity waves (local and global bifurcation theory, proof of Stokes's conjecture on the existence of a wave with a corner of angle $2\pi/3$) now form the classical mathematical theory of the subject. The last quarter of the twentieth century saw a new surge of interest in the mathematical theory of water waves, mainly in the areas of steady water waves with surface tension (local bifurcation theory for solitary waves and three-dimensional doubly periodic waves) and time-dependent water waves (Hamiltonian formulations, local well-posedness theory, justification of the Boussinesq and Korteweg-deVries scaling limits, Nash-Moser theory for standing waves).

The vigorous activity in water waves in the last decade of the twentieth century prompted the organisation of a mini-workshop in 2001 and a half-workshop in 2006 at the Mathematical Research Institute at Oberwolfach. The meetings were very successul, bringing together many leading figures in the analysis of water waves and leading to great progress on a range of outstanding problems in water waves. Several other significant programmes on the analysis of water waves have also taken place in the last ten years (*Surface Water Waves* (Isaac Newton Institute, 2001), *Workshop on Free Surface Water Waves* (Fields Institute, 2004), *Wave Motion* (Mittag-Leffler Institute, 2005), *Nonlinear Water Waves* (Schrödinger Institute, 2011), *Theory of Water Waves* (Isaac Newton Institute, 2015)). Taken together, these events have inspired new and increasing interest in the mathematical theory of water waves and in particular the arrival of a whole new generation of young researchers in this field. Significant progress has been made in particular in the theory of waves with vorticity, local and global well-posedness, the mathematical justification of various model equations and stability results.

In view of the dramatic new developments of the last ten years it appeared timely to hold another half-workshop at Oberwolfach. Its aims were to review the state of the art and stimulate research in major open problems in the following themes.

- The initial-value problem and time-dependent water waves;
- Mathematical justification of model equations;
- New existence theories for steady water waves;
- Stability of steady water waves;
- Waves with vorticity;
- Qualitative properties of waves.

Significant new results in these areas were reported at the conference and are summarised in the extended abstracts below. The workshop was attended by twenty-four participants from ten countries; there was a good mix of senior and junior researchers. Nineteen lectures were held in a friendly and informal atmosphere and many collaborative discussions took place.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

Workshop: Mathematical Theory of Water Waves

Table of Contents

Abstracts

Ill-posedness of some water wave models

Jerry L. Bona

1. Précis of the Lecture

This lecture dealt with model equations for surface waves in both shallow and deep water. The evolution of disturbances on the surface of a large body of water, such as an ocean or a lake, is reasonably well approximated by the Euler system for ideal, free–surface fluid mechanics under the influence of gravity. The origins of this theory lie in the 18^{th} century. Well–posedness results for initial-value problems for these equations has a distinguished, but more modern pedigree. We know now that solutions exist, are unique, and depend continuously upon initial data in various function–space contexts. This theory is subtle and the design of stable, accurate, numerical schemes is likewise challenging.

Starting already in the 19^{th} century, when concrete issues have arisen concerning wave propagation, attention has been turned to model equations which formally approximate the full, Euler system. This latter thrust, which also has a long history, has been a mainstay of developments in oceanography and theoretical fluid mechanics in the 19^{th} and 20^{th} centuries.

Depending upon the wave regime in question, there are many different approximate models that can be formally derived from the Euler equations. As the Euler system is known to be well posed, it seems appropriate that associated approximate models should also have this property. Indeed, without well-posedness, the model is probably useless for practical application. Certain approximations of the Euler equations are known to be well posed. However, we will indicate here that there are naturally occuring classes of weakly nonlinear models for which well posedness does not in fact obtain.

One widely used, modern method of deriving model systems was introduced by Craig and Sulem [7]. They start by writing the water wave evolution equations in the Zakharov formulation [10], which involves the Dirichlet-to-Neumann operator for the fluid domain. Appealing to a result of Coifman and Meyer, one is able to expand the Dirichlet-to-Neumann operator as a series. Craig and Sulem make such an expansion, truncate the relevant series, and use the resulting equations for simulations and study of the water wave problem.

In the deep water situation, if such an expansion is truncated at quadratic order, the resulting model is seen to be

$$\eta_t = \Lambda\xi - \partial_x \{[\mathcal{H}, \eta]\Lambda\xi\}, \tag{1a}$$

$$\xi_t = -g\eta + \frac{1}{2}(\Lambda\xi)^2 - \frac{1}{2}(\xi_x)^2, \tag{1b}$$

where $\mathcal{H}$ and Λ are Fourier multiplier operators; indeed, $\mathcal{H}$ is the Hilbert transform, $\Lambda = \mathcal{H}\partial_x$, $[A, B]$ is the commutator of A and B while g is the gravity constant.

The dependent variable $\eta(x,t)$ is the deviation of the free surface from its rest position at the spatial point represented by x at time t while ξ is the velocity potential at the free surface at x at time t.

In [2], the authors considered this weakly nonlinear, quadratic approximation of the Euler equations supplemented with an artificial viscosity. These models combine two primary ingredients: The method of operator expansions of Craig and Sulem and the artificial viscosity ideas put forward by Dias, Dyachenko, and Zakharov [8]. In [2], it was established that if artificial viscosity effects are included, then the resulting model system is indeed globally well–posed in a reasonable range of function classes. However, the constants in the relevant *a priori* energy-type estimates depend in an unfortunate way upon the 'viscosity' parameter. This is not surprising. However, it was troubling that more detailed analysis did not yield bounds which can be controlled as the viscosity vanishes. We then considered the possibility that inviscid models, constructed in the spirit of Craig and Sulem, may not actually be well–posed.

It turns out that indeed, this system, and even the system obtained by going to third order, are both ill–posed in the standard L_2–based Sobolev spaces. The lecture sketched theory, joint with Ambrose and Nicholls [3], and later Dai [4], showing that this is in fact the case.

Attention was then turned to shallow-water models. The long-wavelength Boussinesq regime was considered. For long-crested waves in this regime, there is a class of models derived in [5] that have the form

$$\left\{\begin{aligned} &\partial_t\eta + \partial_x w + \partial_x(w\eta) + a\partial_x^3 w - b\partial_x^2\partial_t\eta = 0,\\ &\partial_t w + \partial_x\eta + w\partial_x w + c\partial_x^3\eta - d\partial_x^2\partial_t w = 0.\end{aligned}\right. \tag{2}$$

Here, η is as above, the deviation of the free surface from its rest postion and w is the horizontal velocity at a particular height above the bottom. (Since the flow is assumed irrotational and incompressible, the velocity potential is harmonic and hence knowledge of u and η suffices to infer the velocity field everywhere in the flow domain.) While the *abcd*-systems appear to depend upon four parameters, these are not in fact independent. In particular, in the standard scaling for this problem, it must be the case that $a+b+c+d=\frac{1}{3}$ (for more details see [5]).

The question raised in this shallow-water regime has to do with a well known member of the *abcd*-systems that was shown already in [5] to be linearly ill-posed. Choosing the constants as $a=\frac{1}{3}$ and $b=c=d=0$, which is admissible within the detailed formulas for the values of these constants, yields the Kaup system,

$$\left\{\begin{aligned} &\eta_t + w_x + (w\eta)_x + \tfrac{1}{3}w_{xxx} = 0,\\ &w_t + \eta_x + ww_x = 0.\end{aligned}\right. \tag{3}$$

This system was derived by Kaup in [9] as an early example of a coupled system of equations that admits an inverse-scattering formalism. The system has been the object of a number of studies connected with inverse scattering theory.

Despite its attractiveness due to its inverse scattering formalism, it transpires that this model, too, is ill-posed in L_2–based Sobolev spaces. The remainder of the lecture was spent providing an indication of how this is established. The details will appear in [1].

References

[1] D.M. Ambrose, J.L. Bona and T. Milgrom. Global solutions and ill-posedness for the Kaup system and related Boussinesq systems. To appear.

[2] D.M. Ambrose, J.L. Bona and D.P. Nicholls. Well-posedness of a model for water waves with viscosity. *Discrete Contin. Dyn. Syst. Ser. B* **17**, 1113–1137 (2012).

[3] D.M. Ambrose, J.L. Bona and D.P. Nicholls. On ill-posedness of truncated series models for water waves. *Proc. Royal Soc. London, Series A* **470**, 1–16 (1975).

[4] J.L. Bona and M. Dai. Manuscript in preparation.

[5] J.L. Bona, M. Chen and J.-C. Saut. Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media: Part I. Derivation and linear theory. *J. Nonlinear Sci.* **12**, 283–318 (2002).

[6] J.L. Bona, M. Chen and J.-C. Saut. Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media: Part II. Nonlinear theory. *Nonlinearity* **17**, 925–952 (2004).

[7] W. Craig and C. Sulem. Numerical simulation of gravity waves. *J. Comput. Phys.* **108**, 73–83 (1993).

[8] F. Dias, A.I. Dyachenko, and V.E. Zakharov. Theory of weakly damped free-surface flows: A new formulation based on potential flow solutions. *Phys. Lett. A* **372**, 1297–1302 (2008).

[9] D.J. Kaup. Finding eigenvalue problems for solving nonlinear evolution equations. *Progress of Theoretical Physics*, **54**, 72–78 (1975).

[10] V. Zakharov. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Journal of Applied Mechanics and Technical Physics* **9**, 190–194 (1968).

Phase dynamics and water waves

Thomas J. Bridges

Phase dynamics had its earliest appearance in wave equations with geometrical optics for high frequency solutions of inhomogeneous linear wave equations (e.g. §7.7 of [8]). Geometrical optics later played a prominent role in water wave theory, for example, in ray theory for shoaling and refraction of waves approaching a beach. However, it was the introduction of Whitham modulation theory (WMT), exactly 50 years ago this June [7], that showed the importance of "phase dynamics" in the theory of water waves.

The talk started by discussing WMT in its simplest context, starting with a Lagrangian formulation of field equations, introducing a basic state, averaging, and deriving the Whitham modulation equations. By introducing slow time and space scales: $T = \varepsilon t$ and $X = \varepsilon x$, and taking (ω, k) in a neighborhood of some fixed TW parameters,

$$\omega \mapsto \omega + \varepsilon\Omega(X,T,\varepsilon) \quad \text{and} \quad k \mapsto k + \varepsilon q(X,T,\varepsilon)\,, \tag{1}$$

it was shown that, to leading order, q and Ω satisfy

$$q_T = \Omega_X \quad \text{and} \quad \mathscr{A}_\omega \Omega_T + \mathscr{A}_k q_T + \mathscr{B}_\omega \Omega_X + \mathscr{B}_k q_X = 0\,. \tag{2}$$

With (ω, k) fixed, this equation is linear and, with the assumption $\mathscr{A}_\omega \neq 0$, the linear WMEs can be written in the standard form

$$\begin{pmatrix} q \\ \Omega \end{pmatrix}_T + \mathbf{A}(\omega, k) \begin{pmatrix} q \\ \Omega \end{pmatrix}_X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3}$$

with

$$\mathbf{A}(\omega, k) = \frac{1}{\mathscr{A}_\omega} \begin{bmatrix} 0 & -\mathscr{A}_\omega \\ \mathscr{B}_k & \mathscr{A}_k + \mathscr{B}_\omega \end{bmatrix}. \tag{4}$$

The WMEs (3) are *hyperbolic* if the eigenvalues of $\mathbf{A}(\omega, k)$ are real and *elliptic* if the eigenvalues of $\mathbf{A}(\omega, k)$ are complex. Ellipticity is an indication that the basic state is unstable to long wave modulational instability.

This form of the WMEs is linear, in contrast to most applications of WMT. The linearity arises due to the scaling of (ω, k) in (1).

The talk then discussed the role of singularities in producing new natural scaling and different modulation equations. The primary example was the morphing of conservation of wave action into the KdV equation near a singularity. Suppose that the linear system (3) has a codimension one singularity: simple zero characteristic speed: $\mathscr{A}_\omega \neq 0$ but $\mathscr{B}_k = 0$.

With $\mathscr{B}_k = 0$, expand $\mathscr{B}(\omega + \varepsilon\Omega, k + \varepsilon q)_X$ to the next order

$$\mathscr{A}_\omega \Omega_T + (\mathscr{A}_k + \mathscr{B}_\omega) q_T + \varepsilon \mathscr{B}_{kk} q q_X + \cdots = 0.$$

Change scales $X = \varepsilon x$ and $T = \varepsilon^3 t$ and $q \sim \varepsilon^2$ (KdV scaling). Then conservation of waves requires $\Omega \sim \varepsilon^4$. With this new scaling conservation of wave action morphs into

$$\mathscr{A}_\omega \underbrace{\Omega_T}_{\varepsilon^7} + \underbrace{(\mathscr{A}_\omega + \mathscr{B}_k) \underbrace{q_T}_{\varepsilon^5} + \mathscr{B}_{kk} \underbrace{q q_X}_{\varepsilon^5} + \mathscr{K} \underbrace{q_{XXX}}_{\varepsilon^5}}_{\text{leading order terms}} + \cdots = 0.$$

To leading order the KdV equation emerges. It remains to show why the q_{XXX} term should appear, and the details of the argument are given in [1]. An example of the theory is the reduction of defocussing NLS to KdV and that example is treated in [1].

Although the classic emergence of the KdV equation in shallow water would seem to arise through a different mechanism, it in fact emerges via a similar modulation argument, and that argument extends to the KP equation [2]. The connection between modulation in the above sense and WMT is discussed in [3]. The modulation strategy leads to new equations. Validity is generally done independently. For example, [4] give a validity proof for the reduction from defocussing NLS to KdV.

The talk also discussed connections with dissipative phase modulation (e.g. Burgers' equation, Cross-Newell, Kuramoto, Kopell-Howard, Haragus-Scheel, Kuramoto-Sivashinsky, etc: see [5] and references therein), and the emerging field of dislocations of water waves (e.g. [6] and references therein).

References

[1] T.J. Bridges. A universal form for the emergence of the Korteweg-de Vries equation. *Proc. Roy. Soc. Lond. A* **469**, 20120707 (2013).
[2] T.J. Bridges. Emergence of dispersion in shallow water hydrodynamics via modulation of uniform flow. *J. Fluid Mech.* **761**, R1–R9 (2014).
[3] T.J. Bridges. Breakdown of the Whitham modulation theory and the emergence of dispersion. *Stud. Appl. Math.*, in press (2015).
[4] M. Chirlius-Bruckner, W.-P. Düll and G. Schneider. Validity of the KdV equation for the modulation of periodic traveling waves in the NLS equation. J. Math. Anal. Appl. **414**, 166–175 (2014).
[5] A. Doelman, A. Scheel, B. Sandstede and G. Schneider. The dynamics of modulated wave-trains. *Memoirs Amer. Math. Soc.* **199**, (2009).
[6] N. Karjanto and E. van Groesen. Note on wavefront dislocation in surface water waves. *Phys. Lett. A* **371**, 173–179 (2007).
[7] G.B. Whitham. A general approach to linear and non-linear dispersive waves using a Lagrangian, *J. Fluid Mech.* **22**, 273–283 (1965).
[8] G.B. Whitham. *Linear and Nonlinear Waves.* Wiley: New York (1974).

Existence of fully localised water waves with weak surface tension

Boris Buffoni

Let the free upper surface of a three-dimensional layer of fluid be parametrised by $y = \eta(x, z, t)$, where the variables x and z are for the two horizontal directions. At time t, the fluid fills the domain $\Omega_t = \{(x, y, z) \in \mathbb{R}^3 : 0 < y < 1 + \eta(x, z, t)\}$, its asymptotic depth being equal to 1 and $\eta(\cdot, t) \in H^3(\mathbb{R}^2, \mathbb{R})$ (totally localised waves). The velocity field is of the form $\vec{v}(x, y, z, t) = \nabla\varphi(x, y, z, t)$, where the gradient is with respect to (x, y, z), $\varphi(\cdot, t) \in H^1_{loc}(\Omega_t)$ and $\nabla\varphi(\cdot, t) \in L^2(\Omega_t)$. The flow is therefore irrotational. Assuming in addition that the density is constant and the flow divergence free, we get the classical water-wave equation:

$$\begin{cases} \Delta\varphi = 0 & \text{on } \ \Omega_t, \\ \varphi_y = 0 & \text{if } \ y = 0, \\ \eta_t = \varphi_y - \eta_x\varphi_x - \eta_z\varphi_z & \text{if } \ y = 1 + \eta, \\ \varphi_t = -\frac{1}{2}|\nabla\varphi|^2 - g\eta + \beta\left[\frac{\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_x + \beta\left[\frac{\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_z & \text{if } \ y = 1 + \eta, \end{cases}$$

with $g = 1$ (gravity) and where $\beta > 0$ measures the strength of the surface tension. A travelling wave of speed $c > 0$ is of the form

$$\varphi(x, y, z, t) = \widetilde{\varphi}(x + ct, y, z) \ \text{ and } \ \eta(x, z, t) = \widetilde{\eta}(x + ct, z),$$

the moving domain $\widetilde{\Omega}$ being defined by $\widetilde{\Omega} = \{(x, y, z) \in \mathbb{R}^3 : 0 < y < 1 + \widetilde{\eta}(x, z)\}$.

In [4], M. D. Groves and S.-M. Sun proved the existence of a fully localised travelling water wave when the surface tension is strong, that is, $\beta > 1/3$. See also the minimisation approach in [2]. The aim of the talk is to report on a joint work in progress with M. D. Groves and E. Wahlén extending that existence result to the case $0 < \beta < 1/3$ as follows:

Theorem 1. *There exists $\Lambda \in (0,1)$ such that, for all small $\varepsilon > 0$, there exists $(\widetilde{\eta}_\varepsilon, \widetilde{\eta}_\varepsilon)$ such that*

- *it gives rise to a non-trivial solution with $c = \sqrt{\Lambda(1-\varepsilon^2)}$,*
- *$\widetilde{\eta}_\varepsilon \in H^3(\mathbb{R}^2, \mathbb{R})$, $\widetilde{\varphi}_\varepsilon \in H^1_{loc}(\widetilde{\Omega})$ and $\nabla\widetilde{\varphi}_\varepsilon \in L^2(\widetilde{\Omega})$,*
- *$\lim_{\varepsilon\to 0} ||\widetilde{\eta}_\varepsilon||_{W^{1,\infty}(\mathbb{R}^2)} = 0$.*

We refer for example to the works [5, 6, 7] for numerical simulations and to the model equation by Davey and Stewartson (see e.g. [3]).

The energy and the so called total horizontal momentum are preserved:

$$\mathcal{E}(\eta,\varphi) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}\int_0^{1+\eta(x,z,t)} |\nabla\varphi|^2 dy + \frac{1}{2}\eta^2 + \beta[\sqrt{1+\eta_x^2+\eta_z^2} - 1] \right\} dxdz,$$

$$\mathcal{I}(\eta,\varphi) = \int_{\mathbb{R}^2} \eta_x \varphi|_{y=1+\eta} dxdz.$$

At a formal level, it is well known that travelling waves can be obtained as critical points of a "Lagrangian" functional (see e.g. [1]):

$$\delta\Big(\mathcal{E}(\widetilde{\eta},\widetilde{\varphi}) - c\mathcal{I}(\widetilde{\eta},\widetilde{\varphi})\Big) = 0.$$

From now on, we only deal with travelling waves and thus we omit the tildes.

For given $\eta \in H^3(\mathbb{R}^2,\mathbb{R})$, we minimise with respect to φ, which gives a minimizer φ_η, and we set $c = \sqrt{\Lambda(1-\varepsilon^2)}$. This leads to the reduced Lagrangian

$$\mathcal{E}(\eta,\varphi_\eta) - c\mathcal{I}(\eta,\varphi_\eta) = \mathcal{K}(\eta) - (1-\varepsilon^2)\mathcal{L}(\eta) := \mathcal{J}(\eta)$$

with

$$\mathcal{K}(\eta) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}\eta^2 + \beta[\sqrt{1+\eta_x^2+\eta_z^2} - 1] \right\} dxdz$$

and the quadratic part of $\mathcal{L}$ given by

$$\mathcal{L}_2(\eta) = \frac{\Lambda}{2}\int_{\mathbb{R}^2} \frac{k_1^2}{|k|^2}\frac{|k|\cosh|k|}{\sinh|k|}|\widehat{\eta}(k)|^2 dk_1 dk_2.$$

Clearly the quadratic part of $\mathcal{K}$ is given by

$$\mathcal{K}_2(\eta) = \frac{1}{2}\int_{\mathbb{R}^2} \left(\eta^2 + \beta\eta_x^2 + \beta\eta_z^2\right) dxdz.$$

The parameter $\Lambda \in (0,1)$ is chosen such that

$$1 + \beta s^2 - \Lambda\frac{s\cosh s}{\sinh s} \geq 0 \ \text{ with equality at } \ \pm\omega \neq 0.$$

Such Λ and $\omega > 0$ exist and are unique because $\beta \in (0, 1/3)$. This implies that $\mathcal{K}_2 - (1-\varepsilon^2)\mathcal{L}_2$ is positive definite for all $\varepsilon > 0$ small enough.

Further reduction. Let $0 < \delta < \omega/3$ and

$$\chi(k) = \begin{cases} 1 & \text{if } (|k_1| - \omega)^2 + k_2^2 \leq \delta^2, \\ 0 & \text{if not.} \end{cases}$$

We set

$$\eta = \eta_1 + \eta_2, \ \widehat{\eta}_1 = \chi\widehat{\eta}, \ \widehat{\eta}_2 = (1-\chi)\widehat{\eta}$$

and $H^3(\mathbb{R}^2) = \mathcal{X}_1 + \mathcal{X}_2$ accordingly.

For fixed η_1, we solve

$$\forall v_2 \in \mathcal{X}_2 \ \mathcal{J}'(\eta_1 + \eta_2)v_2 = 0,$$

which gives $\eta_2 = \eta_2(\eta_1)$. We then introduce a new function $\phi : \mathbb{R}^2 \to \mathbb{C}$ defined by

$$\widehat{\eta}_1(k) = b(k)\widehat{\zeta}(k), \ \zeta(x,z) = \mathrm{Re}\Big(\varepsilon\phi(\varepsilon x, \varepsilon z)e^{i\omega x}\Big),$$

where $b, \frac{1}{b} : \mathbb{R}^2 \to \mathbb{R}$ are bounded and smooth. The new reduced functional reads as

$$\widetilde{J}_\varepsilon(\phi) = \varepsilon^{-2}\mathcal{J}(\eta_1 + \eta_2(\eta_1)) = \int_{\mathbb{R}^2}\{a_1|\phi_x|^2 + a_2|\phi_z|^2 + a_3|\phi|^2\}dxdz$$
$$-C_1\int_{\mathbb{R}^2} \frac{k_1^2}{(1-\Lambda)k_1^2+k_2^2}\left|\widehat{|\phi|^2}\right|^2 dk_1dk_2 - C_2\int_{\mathbb{R}^2}|\phi|^4 dk_1dk_2 + \varepsilon^{1/2}\mathcal{R}_\varepsilon(\phi),$$

where $a_1, a_2, a_3, C_1, C_2 > 0$ and $\mathcal{R}_\varepsilon$ is an error term that can be estimated accurately together with its two first derivatives. Hence we are lead to a perturbed Davey-Stewartson functional and a non-trivial critical point is then obtained by minimisation on a Nehari constraint (any vector space of complex-valued functions being regarded as a real vector space).

References

[1] T.B. Benjamin and P.J. Olver. Hamiltonian structure, symmetries and conservation laws in surface-tension-dominated flows. *SIAM J. Appl. Math.* **61**, 731-750 (1982).

[2] B. Buffoni, M.D. Groves, S.-M. Sun and E. Wahlén. Existence and conditional energetic stability of three-dimensional fully localised solitary-capillary waves. *J. Diff. Eqns* **254**, 1006-1096 (2013).

[3] R. Cipolatti. On the existence of standing waves for a Davey-Stewartson system. *Comm. PDE* **17**, 967-988 (1992).

[4] M.D. Groves and S.-M. Sun. Fully localised solitary-wave solutions of the three-dimensional gravity-capillary water-wave problem. *Arch. Rat. Mech. Anal.* **188**, 1-91 (2008).

[5] B. Kim and T.R. Akylas, On gravity-capillary lumps. *J. Fluid Mech.* **540**, 337-351 (2005)

[6] P.A. Milewski, Three-dimensional localized solitary gravity-capillary waves. *Comm. Math. Sci.* **3**, 89-99 (2005)

[7] E.I. Părău, J.-M. Vanden-Broeck and M.J. Cooker. Three-dimensional gravity-capillary solitary waves in water of finite depth and related problems. *Phy. Fluids* **17**, 122101 (2005).

Splash singularities for the free boundary Navier-Stokes equations

Angel Castro

The free boundary Navier-Stokes equations model the motion of and incompressible and viscous fluid in vacuum. These equations can be written in the following way

$$\begin{aligned} \partial_t u + u\cdot\nabla u &= -\nabla p + \nu\Delta u \quad \text{in } \Omega(t)\\ \nabla\cdot u &= 0 \quad \text{in } \Omega(t)\\ u(x,0) &= u_0(x), \end{aligned}$$

where $u:\mathbb{R}^2\to\mathbb{R}^2$ is the velocity of the fluid (we will work in dimension 2), $p:\mathbb{R}\to\mathbb{R}$ is the pressure and ν is the viscosity.

The domain $\Omega(t)$, where we solve the system, is transported by the flow. That means that if we define the trajectories $X(\alpha,t)$ by solving the equation

$$\begin{aligned} \frac{d}{dt}X(\alpha,t) &= u(X(\alpha,t),t) \quad \text{in } \Omega_0\\ X(\alpha,0) &= \alpha \quad \alpha\in\Omega_0 \end{aligned} \tag{1}$$

then the domain $\Omega(t)$ is given by

$$\Omega(t) = X(\Omega_0,t).$$

Also the following boundary conditions are imposed for the velocity and the pressure

$$(p\mathbb{I} - \nu(\nabla u + \nabla u^*)\,n = p_{\text{atm}} n \quad \text{on } \Omega(t).$$

Here n is the normal unit vector to $\partial\Omega(t)$, pointing out $\Omega(t)$.

These system has been extensively studied. For example is well know the local in time existence of solutions and there are also several works dealing with the long time behaviour for small initial data. However there are no previous result concerning the existence of singularities.

The initial domain Ω_0 is assumed to be open, bounded, simply connected and with an smooth boundary that satisfies the chord-arc condition, i.e., there is no self-intersections. We will say that the Navier-Stokes system develops a splash singularity if at some finite time $T>0$ the boundary of the domain $\Omega(t)$ touch itself in a point. In our resent work we show that this kind of singularities can be formed for free boundary N-S.

The existence of splash singularities has been already proven in [1] for the water waves problem, i.e.,

$$\begin{aligned} \partial_t u + u\cdot\nabla u &= -\nabla p - (0,g) \quad \text{in } \Omega(t)\\ \nabla\cdot u &= 0 \quad \text{in } \Omega(t)\\ \nabla^{\perp}\cdot u &= 0 \quad \text{in } \Omega(t)\\ u(x,0) &= u_0(x). \end{aligned}$$

The domain $\Omega(t)$ is again given by (1).

Two of the main ingredients in the proof of the splash singularities for the water waves problem are the following:

- One can take a transformation of the equations in such a way that the new equations do not see the singularity.
- The water waves system has a symmetry under time reversal and then it can be solved backwards in time.

The Navier-Stokes system has not this symmetry anymore and a different argument has to be carried out. Instead to solve the equations backwards in time, the existence of a splash singularity is shown by solving the transformed equations forward and then proving stability under small perturbation.

References

[1] A. Castro, D. Córdoba, C. Fefferman, F. Gancedo and J. Gómez-Serrano. Finite time singularities for the free boundary incompressible Euler equations. *Ann. Math.* **178**, 1061–1134 (2013).

Validity of the KdV and the NLS approximation of the water wave problem

Wolf-Patrick Düll

We consider the two-dimensional water wave problem for waves over an incompressible and inviscid fluid in an infinitely long canal of finite depth both with and without surface tension. We additionally assume that the flow is irrotational. Then the velocity field is the gradient field of a harmonic potential and the vertical component v_2 of the velocity is uniquely determined by the horizontal component v_1, more precisely, there exists an operator $\mathcal{K}$ depending on the height η of the free top surface Γ such that $v_2 = \mathcal{K}v_1$. Consequently, the water wave problem can be completely described by the evolution equations for η and for v_1 restricted on Γ. Using Eulerian coordinates, we have

$$\eta_t = \mathcal{K}v_1 - v_1\eta_x \qquad \text{at } \Gamma(t), \tag{1}$$

$$(v_1)_t = -g\eta_x - \tfrac{1}{2}(v_1^2 + (\mathcal{K}v_1)^2)_x + bgh^2\left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)_{xx} \qquad \text{at } \Gamma(t), \tag{2}$$

where g is the constant of gravity, h is the depth of the canal, b is the Bond number, which is proportional to the strength of the surface tension, and κ is the curvature of Γ.

Despite this simplification the water wave problem still remains so complicated that a qualitative understanding of the solutions to the above equations being usable for practical applications does not seem within reach for the near future, neither analytically nor numerically. Therefore, it is reasonable to approximate the water wave problem in various parameter regimes by appropriate reduced models whose qualitative properties are more easily accessible. To understand to which extent these reduced models yield correct predictions of the behavior of the original problem it is important to justify the validity of these approximations by estimates of the approximation errors in the typical length and time scales.

In this talk, we discuss mathematically rigorous justifications of the approximations of the water wave problem by the Korteweg-de Vries equation and the Nonlinear Schrödinger equation.

The Korteweg-de Vries (KdV) equation can be derived as an approximation equation for small and slow modulations of the trivial solution $\eta = v_1 = 0$. Inserting the long-wave ansatz

$$\begin{pmatrix} \eta \\ v_1 \end{pmatrix}(x,t) = \varepsilon^2 A\left(\varepsilon(x \pm t), \varepsilon^3 t\right)\begin{pmatrix} 1 \\ \mp 1 \end{pmatrix} + \mathcal{O}(\varepsilon^3)$$

with $A : \mathbb{R}^2 \to \mathbb{R}$ and a small perturbation parameter $\varepsilon > 0$ into (1)–(2) and equating the terms with the lowest power of ε one can derive that A has to satisfy in lowest order with respect to ε the KdV equation

$$A_\tau = \pm(\frac{1}{6} - \frac{b}{2})A_{\xi\xi\xi} \pm \frac{3}{2}AA_\xi \tag{3}$$

with $\tau = \varepsilon^3 t$ and $\xi = \varepsilon(x \pm t)$. For $b = \frac{1}{3} + 2\nu\varepsilon^2$ one obtains, by making the ansatz

$$\begin{pmatrix} \eta \\ v_1 \end{pmatrix}(x,t) = \varepsilon^4 A\left(\varepsilon(x \pm t), \varepsilon^5 t\right)\begin{pmatrix} 1 \\ \mp 1 \end{pmatrix} + \mathcal{O}(\varepsilon^5)$$

with $A : \mathbb{R}^2 \to \mathbb{R}$ the Kawahara equation

$$\partial_\tau A = \mp\nu\partial_\xi^3 A \pm \frac{1}{90}\partial_\xi^5 A \pm \frac{3}{2}A\partial_\xi A \tag{4}$$

with $\tau = \varepsilon^5 t$ and $\xi = \varepsilon(x \pm t)$ as an approximation equation.

The Nonlinear Schrödinger (NLS) equation can be derived as an approximation equation for small and slow modulations of an oscillating wave packet. The solution of the NLS equation describes the evolution of the envelope of the wave packet. Inserting the ansatz

$$\begin{pmatrix} \eta \\ v_1 \end{pmatrix}(x,t) = \varepsilon A\left(\varepsilon(x - c_g t), \varepsilon^2 t\right) e^{i(k_0 x - \omega(k_0)t)}\varphi(k_0) + c.c. + \mathcal{O}(\varepsilon^2),$$

where $\omega(k) = \text{sign}(k)\sqrt{k\,\tanh(k)}$, $c_g = \partial_k\omega(k_0)$ and $\varphi(k_0) \in \mathbb{C}^2$ in (1)–(2) yields in the case $b = 0$ at leading order in ε the NLS equation

$$A_\tau \quad = \quad i\nu_1 A_{\xi\xi} + i\nu_2 A|A|^2 \tag{5}$$

with $\tau = \varepsilon^2 t, \xi = \varepsilon(x - c_g t)$ and $\nu_j = \nu_j(k_0) \in \mathbb{R}$.

We justify the validity of the KdV approximation by the following approximation theorem.

Theorem 1. *For all $b_0, C_0, \tau_0 > 0$ there exist an $\varepsilon_0 > 0$ such that for all $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon \le \varepsilon_0$ and all $b \in \mathbb{R} \setminus \{\frac{1}{3}\}$ with $0 \le b \le b_0$ the following is true. Let*

$$\eta|_{t=0}(x) = \varepsilon^2\Phi_1(\varepsilon x), \qquad v_1|_{t=0}(x) = \varepsilon^2\Phi_2(\varepsilon x)$$

with $\max\{\|(\Phi_1(\cdot),\Phi_2(\cdot))\|_{H^{s+8}_\xi}, \|(\rho^k\Phi_1(\cdot),\rho^k\Phi_2(\cdot))\|_{H^{s+3}_\xi}\} \leq C_0\varepsilon^l$, *where* $\xi = \varepsilon x$, $s \geq 7$, $k > 1$, $l \geq 0$ *and* $\rho(\xi) = (1+\xi^2)^{1/2}$. *Split the initial conditions into*

$$A_1|_{\tau=0} = \frac{1}{2}(\Phi_1 + \Phi_2), \qquad A_2|_{\tau=0} = \frac{1}{2}(\Phi_1 - \Phi_2)$$

and let the amplitudes $A_1 = A_1(\xi,\tau)$ *and* $A_2 = A_2(\xi,\tau)$ *satisfy*

$$(A_1)_\tau = (\frac{b}{2} - \frac{1}{6})(A_1)_{\xi\xi\xi} - \frac{3}{2}A_1(A_1)_\xi, \qquad (A_2)_\tau = (\frac{1}{6} - \frac{b}{2})(A_2)_{\xi\xi\xi} + \frac{3}{2}A_2(A_2)_\xi.$$

Then there exists a unique solution of the 2-D water wave problem (1)–(2) with the above initial conditions satisfying

$$\sup_{t\in[0,\tau_0/\varepsilon^3]} \left\| \begin{pmatrix} \eta \\ v_1 \end{pmatrix} (\cdot,t) - \psi(\cdot,t) \right\|_{H^s_\xi \times H^{s-1/2}_\xi} \lesssim \varepsilon^{4+l}$$

where

$$\psi(x,t) = \varepsilon^2 A_1\left(\varepsilon(x-t),\varepsilon^3 t\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \varepsilon^2 A_2\left(\varepsilon(x+t),\varepsilon^3 t\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We prove the theorem by using the arc length formulation of the water wave problem, where the top surface is parametrized by arc length and the velocity on the top surface is decomposed into its tangential and its normal component. In this formulation, the term $b\,(\eta_x(1+\eta_x^2)^{-1/2})_{xx}$ becomes linear, which allows to prove error estimates being uniform with respect to b as b and ε go to 0. Consequently, the cases with and without surface tension can be handled together in one proof. The error estimates are proven with the help of a suitably constructed energy.

For the Kawahara approximation there is a similar approximation theorem which can be proven analogously to Theorem 1.

The validity of the NLS approximation is justified by the following approximation theorem.

Theorem 2. *Let* $b = 0$ *and* $s \geq 7$. *Then for all* $k_0 > 0$ *and for all* $C_1, \tau_0 > 0$ *there exist* $\tau_1 > 0$ *and* $\varepsilon_0 > 0$ *such that for all solutions* $A \in C([0,\tau_0], H^s(\mathbb{R},\mathbb{C}))$ *of the NLS equation (5) with*

$$\sup_{\tau\in[0,\tau_0]} \|A(\cdot,\tau)\|_{H^s(\mathbb{R},\mathbb{C})} \leq C_1$$

the following holds. For all $\varepsilon \in (0,\varepsilon_0)$ *there exists a solution*

$$(\eta, v_1) \in C([0,\tau_1/\varepsilon^2], (H^s(\mathbb{R},\mathbb{R}))^2)$$

of the 2-D water wave problem (1)–(2) which satisfies

$$\sup_{t\in[0,\tau_1/\varepsilon^2]} \left\| \begin{pmatrix} \eta \\ v_1 \end{pmatrix} (\cdot,t) - \psi(\cdot,t) \right\|_{(H^s(\mathbb{R},\mathbb{R}))^2} \lesssim \varepsilon^{3/2}$$

where

$$\psi(x,t) = \varepsilon A\left(\varepsilon(x-c_g t),\varepsilon^2 t\right) e^{i(k_0x-\omega(k_0)t)}\varphi(k_0) + c.c.\,.$$

We prove the theorem by using Lagrangian coordinates. We eliminate all terms of order $\mathcal{O}(\varepsilon)$ in the error equations by a suitable normal-form transform which can be constructed although the error equations possess non-trivial resonances. Even though the normal-form transform loses regularity the structure of the error equations allow to invert the normal-form transform. Having performed the normal-form transform the error can be estimated by using an appropriate time dependent analytic norm to overcome the loss of regularity in the error equations. The use of an analytic norm is possible because the approximation function is compactly supported in Fourier space up to an error of order $\mathcal{O}(\varepsilon^{3/2})$ in $(H^s(\mathbb{R},\mathbb{R}))^2$.

References

[1] W.-P. Düll. Validity of the Korteweg-de Vries approximation for the two-dimensional water wave problem in the arc length formulation. *Comm. Pure Appl. Math.* **65**, 381-429 (2012).

[2] W.-P. Düll, G. Schneider, and C. E. Wayne. Justification of the nonlinear Schrödinger equation for the evolution of gravity driven 2D surface water waves in a canal of finite depth. Submitted to *Arch. Rat. Mech. Anal.*.

On Whitham's conjecture of a highest cusped wave for a nonlocal shallow water wave equation

Mats Ehrnström

The main aim of this report is to present the existence of a highest, cusped, periodic travelling-wave solution for the Whitham equation, thereby positively resolving the Whitham conjecture. For $m(\xi) = (\frac{\tanh(\xi)}{\xi})^{1/2}$, let

$$K(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} m(\xi) \exp(-ix\xi)\, \mathrm{d}\xi$$

denote the Fourier transform $\mathcal{F}m$ of the function m at a point $x \in \mathbb{R}$. The function m is smooth with decay rate $\mathcal{O}(|\xi|^{-1/2})$ as $|x| \to \infty$, so that $(\mathcal{F}m)(x) = \mathcal{O}(|x|^{-k})$ as $|x| \to \infty$, for all $k \in \mathbb{N}$. Our normalisation of $\mathcal{F}$ is

$$(\mathcal{F}f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) \exp(-ix\xi)\, dx, \qquad \text{for} \qquad f \in \mathbb{S},$$

which makes $\mathcal{F}$ a unitary operator on $L_2(\mathbb{R})$. Since m is even and real-valued, so is the kernel K. It is furthermore smooth for all $x \neq 0$ and all its derivates have rapid decay. As made precise below, it is singular at the origin. If we denote by $L\colon f \mapsto K * f$ the action by convolution with the kernel K, the Whitham equation is the nonlinear, nonlocal evolution equation $u_t + (u^2 + Lu)_x = 0$. In steady variables $\varphi(x) = u(x - \mu t)$ it takes the form

$$-\mu\varphi + L\varphi + \varphi^2 = 0, \tag{1}$$

where we have integrated the equation once and used Galilean invariance to normalise the constant of integration to zero. The equation (1) models waves travelling right-ward with a permanent form and a normalised wave speed μ. With a solution of the steady Whitham equation we denote a real-valued, bounded and continuous function φ that satisfies (1) pointwise. Our first result is the following.

The singularity of K. *The Whitham kernel K satisfies $K(x) \sim |x|^{-1/2}$ and $-K'(x) \sim \frac{1}{2}\operatorname{sgn}(x)|x|^{-3/2}$ as $|x| \to 0$.*

To see the difficulty involved in handling derivatives of K away from the origin, consider first the formal equality

$$D_x K(x) = -\frac{2}{\pi}\int_0^\infty (\xi \tanh(\xi))^{1/2} \sin(x\xi)\,\mathrm{d}\xi,$$

obtained by differentiating under the integral sign. Since $\xi \mapsto (\xi \tanh(\xi))^{1/2}$ is monotone increasing on $(0,\infty)$, the integral in the right-hand side is not well defined. To circumvent this difficulty, we consider instead of $D_x K(x)$ the product $-xD_xK(x)$. This is an element of $\mathbb{S}'$, and we show that its Fourier transform is positive definite. Hence, $xD_xK(x)$ is (almost) everywhere negative:

Global form of K. *The Fourier transform of $(m(\xi))^{2\alpha}$ is positive and one-sided monotone for all $\alpha \in (0,1]$. In addition, the Whitham kernel K is strictly convex for all $x \neq 0$.*

We now introduce the periodised Whitham kernel,

$$K_P(x) := \sum_{n\in\mathbb{N}} K(x+nP),$$

for $P \in (0,\infty)$. It follows from the above that for any $P > 0$ the periodised Whitham kernel K_P is strictly decreasing in the interval $(0, P/2)$. The operator L is furthermore strictly monotone, parity-preserving, and $Lf(x) > 0$ on $(-P/2, 0)$ for f odd and continuous with $f \gneq 0$ on $(-P/2, 0)$. One of the essential ingredients of the proof of the highest wave is preservation of the nodal pattern along the main bifurcation branch.

Nodal pattern. *Any non-constant and even solution $\varphi \in C^1(\mathbb{R})$ of the steady Whitham equation (1) which is non-decreasing on $(-P/2, 0)$ satisfies*

$$\varphi' > 0,\ \varphi < \frac{\mu}{2} \qquad \text{on} \quad (-P/2, 0).$$

If $\varphi \in C^2(\mathbb{R})$, then $\varphi < \frac{\mu}{2}$ everywhere, with $\varphi''(0) < 0$. For $P < \infty$ one furthermore has $\varphi''(\pm P/2) > 0$.

To formulate the main regularity result, define $C^{1/2+}(\mathbb{R}) = \cup_{\alpha\in(\frac{1}{2},1)} C^\alpha(\mathbb{R})$ and $C^{1/2-}(\mathbb{R}) = \cap_{\alpha\in(0,\frac{1}{2})} C^\alpha(\mathbb{R})$. We shall say that a solution φ is *Hölder continuous of regularity* $\alpha \in (0,1)$ at a point $x \in \mathbb{R}$ if $\sup_{h>0} |\varphi(x+h) - \varphi(x)|/h^\alpha < \infty$. If φ is Hölder continuous of regularity β at a point x for all $\beta < \alpha \leq 1$, we similarly say that φ is of Hölder regularity α^- at that point. Part two of the subsequent result proves that a limiting wave is cusped.

Regularity of solutions. *Let $\varphi \leq \frac{\mu}{2}$ be a solution of the steady Whitham equation.*

(i) *If* $\varphi < \frac{\mu}{2}$, *uniformly on* $\mathbb{R}$, *then* $\varphi \in C^\infty(\mathbb{R})$.

(ii) *If* φ *is even, non-constant, and non-decreasing on* $(-P/2, 0)$ *with* $\varphi(0) = \frac{\mu}{2}$, *then* $\varphi \in C^{1/2-}(\mathbb{R}) \setminus C^{1/2+}(\mathbb{R})$ *with Hölder regularity* 1^- *wherever* $\varphi(x) \neq \frac{\mu}{2}$, *and Hölder regularity* $(\frac{1}{2})^-$ *at* $x = 0$.

We now fix $\alpha \in (\frac{1}{2}, 1)$, and consider $C^\alpha_{\text{even}}(\mathbb{S})$, the space of even and α-Hölder continuous real-valued functions on the unit circle $\mathbb{S}$. Let furthermore $F\colon C^\alpha_{\text{even}}(\mathbb{S}) \times \mathbb{R} \to C^\alpha_{\text{even}}(\mathbb{S})$ be the operator defined by

$$F(\varphi, \mu) := \mu\varphi - L\varphi - \varphi^2,$$

With $U := \{(\varphi, \mu) \in C^\alpha_{\text{even}}(\mathbb{S}) \times \mathbb{R}\colon \varphi < \mu/2\}$, we let

$$S := \{(\varphi, \mu) \in U\colon F(\varphi, \mu) = 0\}$$

be our set of solutions.

Global bifurcation. *For each integer* $k \geq 1$, *there exist* $\mu_k := (\tanh(k)/k)^{1/2}$ *and a local, analytic curve*

$$\varepsilon \mapsto (\varphi(\varepsilon), \mu(\varepsilon)) \in C^\alpha_{\text{even}}(\mathbb{S}) \times (0, 1)$$

of nontrivial $2\pi/k$*-periodic Whitham solutions with* $D_\varepsilon\varphi(0) = \cos(kx)$ *that bifurcates from the trivial solution curve* $\mu \mapsto (0, \mu)$ *at* $(\varphi(0), \mu(0)) = (0, \mu_k)$. *The local curves extend to global continuous curves of solutions* $\mathbb{R}_{\geq 0} \to S$. *One of the following alternatives holds:*

(i) $\|(\varphi(\varepsilon), \mu(\varepsilon))\|_{C^\alpha(\mathbb{S})\times\mathbb{R}} \to \infty$ *as* $\varepsilon \to \infty$.

(ii) *The pair* $(\varphi(\varepsilon), \mu(\varepsilon))$ *approaches the boundary of* S *as* $\varepsilon \to \infty$.

(iii) *The function* $\varepsilon \mapsto (\varphi(\varepsilon), \mu(\varepsilon))$ *is* T*-periodic, for some* $T \in (0, \infty)$.

One can prove that any sequence of Whitham solutions $(\varphi_n, \mu_n) \in S$ has a subsequence which converges uniformly to a solution $\varphi \in C^\alpha(\mathbb{S})$, with $\alpha \in (0, \frac{1}{2})$ arbitrary. Since furthermore alternative (i) in the above bifurcation result can happen only if

$$\liminf_{\varepsilon\to\infty} \inf_{x\in\mathbb{R}} \left(\frac{\mu(\varepsilon)}{2} - \varphi(x; \varepsilon) \right) = 0,$$

alternative (i) implies alternative (ii). The alternative (iii) is ruled out using preservation of the nodal pattern along the main bifurcation branch. One can show the the only remaining alternative, (i), implies the existence of a limiting wave peaking at $\varphi(0) = \frac{\mu}{2}$ with $\mu \in [0, 1]$. Finally, the following result excludes the limiting wave from being trivial.

Uniform lower bound of the size of solutions. *Let* φ *be an even, non-constant* C^1*-solution of the steady Whitham equation* (1) *with* $\varphi < \frac{\mu}{2}$ *and* $\varphi' \geq 0$ *in* $(-P/2, 0)$. *Then there exists a universal constant* $\lambda_{K,P} > 0$, *depending only on the kernel* K *and the period* P, *such that*

$$\tfrac{\mu}{2} - \varphi(\tfrac{P}{2}) \geq \lambda_{K,P}.$$

More generally, $\frac{\mu}{2} - \varphi(x) \gtrsim_{K,P} |x_0|^{1/2}$, *uniformly for all* $x \in [-P/2, x_0]$, *with* $x_0 < 0$.

Spatial dynamics methods for axisymmetric solitary waves on a ferrofluid jet

Mark D. Groves

We consider the irrotational flow of an incompressible, inviscid ferrofluid of constant density ρ surrounding a metal wire in a vaccum. In terms of cylindrical polar coordinates (r, θ, z), where z is aligned with the wire, the fluid domain is $\{(r, \theta, z) : 0 < r < R + \zeta(\theta, z)\}$ (see Figure 1). We examine whether the magnetic force in the ferrofluid induced by a current J flowing in the wire can, together with surface tension, support axisymmetric solitary waves on its free surface.

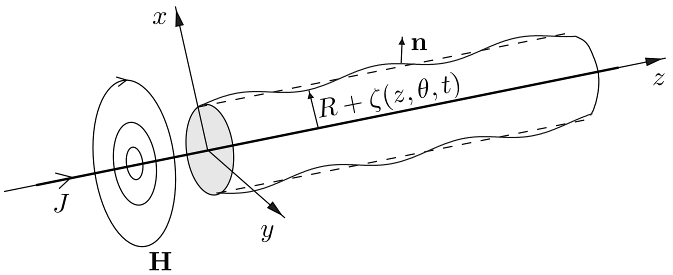


Figure 1. A ferrofluid jet surrounding a metal wire in a vacuum. The current J in the wire generates a magnetic field $\mathbf{H} = J\mathbf{e}_\theta/(2\pi r)$ which is unchanged by axisymmetric flows.

An axisymmetric solitary-wave flow is described by a scalar velocity potential $\phi(r, z - ct)$ which satisfies the equations

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0, \qquad r < 1 + \zeta, \tag{1}$$

$$\zeta_z + \phi_r - \phi_z\zeta_z = 0, \qquad r = 1 + \zeta, \tag{2}$$

$$-\phi_z + \frac{1}{2}(\phi_r^2 + \phi_z^2) - \alpha\left(\nu\left(\tfrac{1}{1+\zeta}\right) - \nu(1)\right) + \beta\left(\frac{(1+\zeta_z^2)^{-\frac{1}{2}}}{(1+\zeta)} - \frac{\zeta_{zz}}{(1+\zeta_z^2)^{-\frac{1}{2}}} - 1\right) = 0, \qquad r = 1 + \zeta \tag{3}$$

and asymptotic conditions ζ, ϕ_r, $\phi_z \to 0$ as $z \to \infty$ (see Blyth & Parau [1]). Here dimensionless variables have been introduced, the dimensionless parameters α and β are given by the formulae

$$\alpha = \frac{\mu_0 J^2}{4\pi^2 R^2 c^2}, \qquad \beta = \frac{\sigma}{Rc^2},$$

in which μ_0 is the magnetic permeability in a vacuum and σ is the coefficient of surface tension, and

$$\nu(s) = \int_0^s |\mathbf{M}(t)|\, \mathrm{d}t,$$

where $\mathbf{M}(|\mathbf{H}|)$ is the magnetic force in the fluid generated by a magnetic field $\mathbf{H}$. Note that (1)-(3) follow from the formal variational principle

$$\delta \int \Bigg\{ \int_0^{1+\zeta} \Big(-r\phi_z + \frac{1}{2}(r\phi_r^2 + r\phi_z^2) \Big)\, \mathrm{d}r$$

$$- \alpha T(\zeta) + \beta(1+\zeta)(1+\zeta_z^2)^{\frac{1}{2}} - \tfrac{1}{2}\beta(1+\zeta)^2 \Bigg\}\, \mathrm{d}z = 0, \tag{4}$$

where

$$T(\zeta) = \int_0^{\zeta} \Big(\nu\left(\tfrac{1}{1+s}\right) - \nu(1) \Big) (1+s)\, \mathrm{d}s.$$

Solitary-wave solutions to (1)–(3) can be found by adapting the results available in the literature on the classical water-wave problem. A complete Hamiltonian spatial dynamics theory for small-amplitude solitary gravity-capillary water waves with a general distribution of vorticity was given by Groves & Wahlén [4], and applying their theory to the present problem yields analogous results. The phrase 'Hamiltonian spatial dynamics' refers to an approach where a system of partial differential equations arising from a variational principle is formulated (by means of an appropriately constructed Legendre transform) as a (typically ill-posed) Hamiltonian evolutionary equation

$$u_\xi = L(u) + N(u), \tag{5}$$

in which an unbounded spatial coordinate ξ plays the role of the time-like variable.

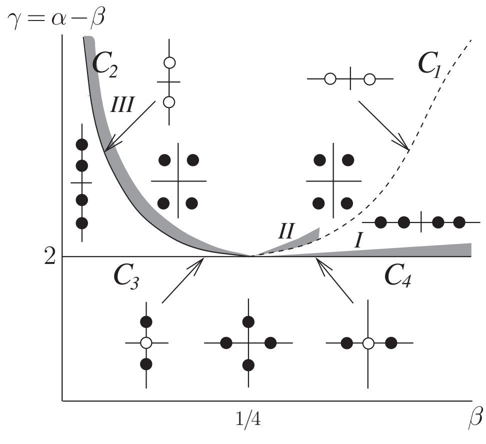


FIGURE 2. Bifurcation curves in the (γ, α)-plane; homoclinic bifurcation is detected in the shaded regions.

We study (1)–(3) using spatial dynamics by formulating this problem as an evolutionary equation of the form (5) with $\xi = z$ in an infinite-dimensional phase space consisting of functions of the radial coordinate r. Solutions of (5) are found using *centre-manifold reduction* (Mielke [7]). The equation admits a finite-dimensional invariant manifold called the *centre manifold* which contains all its small, bounded solutions; the dimension of the centre manifold is given by the number of purely imaginary eigenvalues of L, and the reduced flow on the centre manifold is determined by a reversible Hamiltonian system with finitely many degrees of freedom.

The reduction procedure is especially helpful in detecting bifurcations which are associated with a change in the number of purely imaginary eigenvalues. In the present problem there are three critical curves C_2, C_3, C_4 in the (β, γ) parameter plane at which the number of purely imaginary eigenvalues of L changes (see Figure 2), together with a fourth curve C_1 at which the number of real eigenvalues changes; an explicit parametrisation of each of these curves is available. We introduce a bifurcation parameter ε by perturbing β and γ around fixed reference values, and the reduction procedure delivers an ε-dependent manifold which captures the small-amplitude dynamics for small values of this parameter. Homoclinic solutions to the reduced system correspond to solitary ferrofluid waves.

Region I: A *Hamiltonian* 0^2*-resonance* takes place at C_4, that is two imaginary eigenvalues collide at the origin and become real as one crosses the curve from below. The flow on the two-dimensional centre manifold is controlled by the reversible, Hamiltonian equation

$$u_{zz} = u + \frac{3}{2}u^2 + O(\delta^{1/2}),$$

where $0 < \delta \ll 1$ is the bifurcation parameter $\gamma - 2$. This equation admits a homoclinic solution which corresponds to a solitary wave of depression whose tail decays exponentially and monotonically (cf. Kirchgässner [6]) (Figure 3(a) and 4).

Region II lies on the 'complex side' of the curve C_1, at points of which two pairs of small-magnitude real eigenvalues collide and become complex. The centre manifold is four-dimensional, and it the flow on the centre manifold is controlled by the reversible, Hamiltonian equation

$$u_{zzzz} - 2(1+\delta)u_{zz} + u - u^2 = 0(\mu), \tag{6}$$

where $0 < \mu \ll 1$ measures the distance from the point $(\frac{1}{4}, 2)$ and $0 < \delta \ll 1$ is the bifurcation parameter (measuring the distance from C_1). This equation has an infinite family of *multipulse* homoclinic solutions which make several large excursions away from the origin (cf. Buffoni, Groves & Toland [3]). The corresponding ferrofluid waves are solitary waves of depression with $2, 3, 4, \ldots$ large troughs separated by $2, 3, \ldots$ small oscillations; their tails are oscillatory and decay exponentially (Figures 3(b) and 5).

Region III: A *Hamiltonian-Hopf bifurcation* takes place at C_2 (two pairs of purely imaginary eigenvalues collide at non-zero points $\pm \mathrm{i}q$ and become complex). The two-degree-of-freedom reduced Hamiltonian system is conveniently studied using complex coordinates (A, B) and a normal-form transformation. Introducing a bifurcation parameter δ so that positive values of δ correspond to points on the 'complex' side C_2, one obtains the reduced Hamiltonian system

$$A_z = \frac{\partial H}{\partial \bar{B}}, \qquad B_z = -\frac{\partial H}{\partial \bar{A}},$$

$$H = \mathrm{i}q(A\bar{B} - \bar{A}B) + |B|^2 + H_{\mathrm{NF}}(|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B), \delta) + O(|(A,B)|^2 |(\delta, A, B)|^{n_0}),$$

where H_{NF} is a real polynomial which satisfies $H_{\mathrm{NF}}(0,0,\delta) = 0$; it contains the terms of order 3, ..., $n_0 + 1$ in the Taylor expansion of H. Supposing that the coefficients of certain terms in H_{NF} have the correct sign (a condition which can be explicitly verified in the present context), one finds two 'basic' symmetric homoclinic solutions (Iooss & Pérouème [5]) and a family of geometrically distinct homoclinic solutions which generically resemble multiple copies of one of the basic homoclinic solutions (Buffoni & Groves [2]). The corresponding ferrofluid waves are symmetric solitary waves which take the form of periodic wave trains modulated by exponentially decaying envelopes (Figure 3(c)).

References

[1] M. Blyth and E. Parau. Solitary waves on a ferrofluid jet. *J. Fluid Mech.* **750**, 401–420 (2014)

[2] B. Buffoni and M. D. Groves. A multiplicity result for solitary gravity-capillary waves in deep water via critical-point theory. *Arch. Rat. Mech. Anal.* **146**, 183–220 (1999).

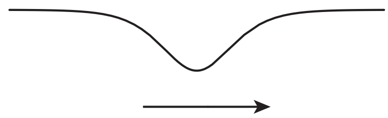
[3] B. Buffoni, M. D. Groves and J. F. Toland. A plethora of solitary gravity-capillary water waves with nearly critical Bond and Froude numbers. *Phil. Trans. Roy. Soc. Lond. A* **354**, 575–607 (1996).

[4] M. D Groves and E. Wahlén. Spatial dynamics methods for solitary gravity-capillary water waves with an arbitrary distribution of vorticity. *SIAM J. Math. Anal.* **750**, 932–964 (2007).

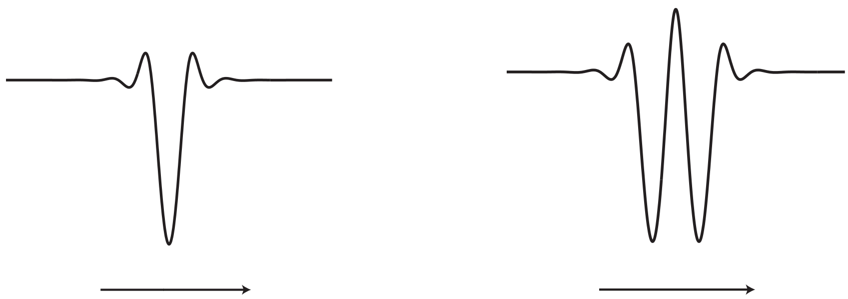
[5] G. Iooss and M.-C. Pérouème Perturbed homoclinic solutions in reversible 1:1 resonance vector fields. *J. Diff. Eqns.* **102**, 62–88 (1993).

[6] K. Kirchgässner Nonlinearly resonant surface waves and homoclinic bifurcation. *Adv. Appl. Mech.* **26**, 135–181 (1988).

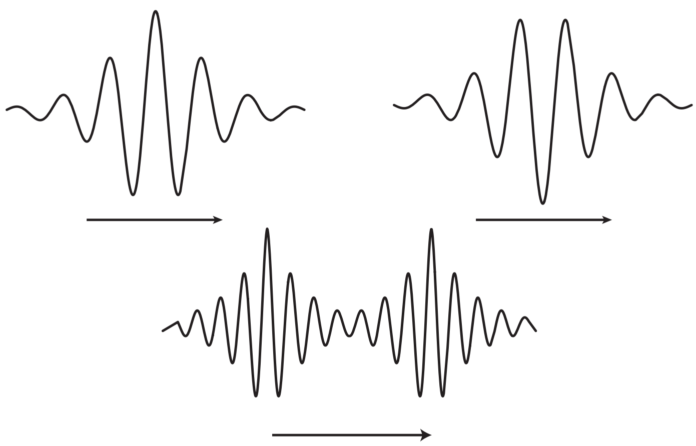
[7] A. Mielke. *Hamiltonian and Lagrangian Flows on Center Manifolds*. Berlin: Springer-Verlag (1991).



(a) A symmetric solitary wave of depression is found in region I.



(b) Region II contains an infinite family of multi-troughed solitary waves which decay in an oscillatory fashion.



(c) Symmetric unipulse modulated solitary waves (left and centre) co-exist with an infinite family of multipulse modulated solitary waves (right) in region III.

Figure 3. Sketches of the function ζ for the solitary waves found by centre-manifold reduction and homoclinic bifurcation theory.

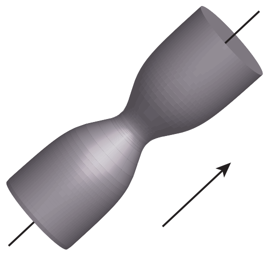


Figure 4. The ferrofluid solitary wave in region I.

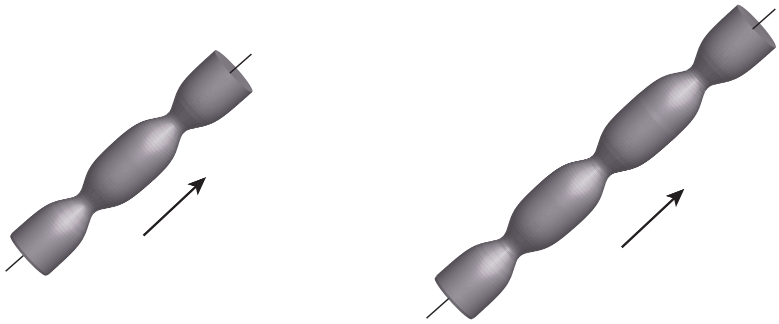


Figure 5. Multipulse ferrofluid solitary waves in region II.

Instabilities in the Whitham equation for shallow water waves

Vera Hur

As Whitham said in his celebrated treatise on linear and nonlinear waves, "the breaking phenomenon is one of the most intriguing long-standing problems of water wave theory." The *shallow water equations*

$$\begin{aligned} &\partial_t h + d\partial_x u + \partial_x(uh) = 0, \\ &\partial_t u + g\partial_x h + u\partial_x u = 0 \end{aligned} \tag{1}$$

approximate the water wave problem when waves are long compared to the fluid depth, and furthermore, they explain *wave breaking* – the solution remains bounded but its slope becomes unbounded in finite time. Here, $t \in \mathbb{R}$ denotes the temporal variable, $x \in \mathbb{R}$ is the spatial variable; $h = h(x,t)$ represents the surface displacement from the undisturbed depth d, and $u = u(x,t)$ is the horizontal velocity of a fluid particle at the free surface. Moreover g denotes the constant of gravitational acceleration.

But the shallow water theory goes too far. It predicts that *all* waves carrying an increase of elevation break. Observations have been long since established that some waves do not break. Clearly, the neglected dispersion effects inhibit breaking. Note that the phase speed of a plane wave with the wave number k near the quintessential state of water is

$$c(k) = \sqrt{\frac{g\tanh(kd)}{k}} = \sqrt{gd}\Big(1 - \frac{1}{6}k^2d^2\Big) + O(k^4d^4)$$

as $kd \to 0$. Note moreover that solutions of the linear part of (1) translate at the speed $\sqrt{gd}$, regardless of their wave number.

But a simple theory including some of dispersion effects, namely the *Korteweg-de Vries (KdV) equation*,

$$\partial_t h + \sqrt{gd}\Big(1 + \frac{1}{6}d^2\partial_x^2\Big)\partial_x h + \frac{3}{2}\sqrt{\frac{g}{d}}h\partial_x h = 0, \tag{2}$$

in turn, goes too far and predicts that *n*o waves break.

One concludes that some dispersion is necessary to explain wave breaking[1] but the dispersion of the KdV equation is too strong (for short wavelengths). It is intriguing to find a simple mathematical equation that could capture breaking.

Whitham therefore put forward

$$\partial_t h + \mathcal{M}\partial_x h + \frac{3}{2}\sqrt{\frac{g}{d}}h\partial_x h = 0, \tag{3}$$

where

$$\widehat{\mathcal{M}f}(k) = c(k)\widehat{f}(k) = \sqrt{\frac{g\tanh(kd)}{k}}\widehat{f}(k),$$

as an alternative to the KdV equation, combining the dispersion relation of surface water waves and the nonlinearity of the shallow water equations, and he advocated that (3) would explain breaking. The kernel associated with the integral representation of $\mathcal{M}$ is difficult to handle. Nevertheless, recently in [1], wave breaking for (3) was analytically confirmed, provided that the initial datum is sufficiently steep. The proof is based upon ordinary differential equations along characteristics with nonlocal forcing terms and their asymptotic behavior near zero.

Furthermore in [2], a $2\pi/k$-periodic traveling wave of (3) with a sufficiently small amplitude was shown to be spectrally unstable with respect to long wavelengths perturbations if $kd > 1.145\ldots$. The proof involves a spectral perturbation of the associated linearized operator with respect to the Floquet exponent and the small amplitude parameter. Incidentally Benjamin and Feir and, independently, Whitham in the mid 1960's formally argued that a $2\pi/k$-periodic traveling wave in water would be unstable, leading to sidebands growth, namely the Benjamin-Feir instability, provided that $kd > 1.363\ldots$.

[1] When gradients are no longer small, the long wavelengths assumption under which (1) and (2) are derived is no longer valid. Yet breaking does occur and in some circumstances does not seem to be too far away from what (1) describes.

References

[1] V.M. Hur. Breaking in the Whitham equation for shallow water waves. Preprint (2015).
[2] V.M. Hur and M.A. Johnson. Modulational instability in the Whitham equation for water waves. *Stud. Appl. Math.* **134**, 120–143 (2015).

Solvability of the initial value problem to a model system for water waves

Tatsuo Iguchi

The water wave problem is mathematically formulated as a free boundary problem for an irrotational flow of an inviscid and incompressible fluid under the gravitational field. The basic equations for water waves are complicated due to the nonlinearity of the equations together with the presence of an unknown free surface. Therefore, until now many approximate equations have been proposed and analyzed to understand natural phenomena for water waves. Famous examples of such approximate equations are the shallow water equations, the Green–Naghdi equations, Boussinesq type equations, the Korteweg–de Vries equation, the Kadomtsev–Petviashvili equation, the Benjamin–Bona–Mahony equation, the Camassa–Holm equation, the Benjamin–Ono equations, and so on. All of them are derived from the water wave problem under the shallowness assumption of the water waves, which means that the mean depth of the water is sufficiently small compared to the typical wavelength of the water surface.

On the other hand, it is well-known that the water wave problem has a variational structure. In fact, J. C. Luke [6] gave a Lagrangian in terms of the velocity potential and the surface variation, and showed that the corresponding Euler–Lagrange equations are the basic equations for water waves. M. Isobe [1, 2] and T. Kakinuma [3, 4, 5] derived model equations for water waves without any shallowness assumption. The model equations are the Euler–Lagrange equations to an approximated Lagrangian, which is obtained by approximating the velocity potential in Luke's Lagrangian. I would like to talk on the initial value problem to one of the model equations

$$\begin{cases} \eta_t + \nabla\cdot\Big(H\nabla\phi^0 + \dfrac{1}{3}H^3\nabla\phi^1 - H^2\phi^1\nabla b\Big) = 0, \\ H^2\eta_t + \nabla\cdot\Big(\dfrac{1}{3}H^3\nabla\phi^0 + \dfrac{1}{5}H^5\nabla\phi^1 - \dfrac{1}{2}H^4\phi^1\nabla b\Big) \\ \quad + H^2\nabla b\cdot\nabla\phi^0 + \dfrac{1}{2}H^4\nabla b\cdot\nabla\phi^1 - \dfrac{4}{3}H^3(1+|\nabla b|^2)\phi^1 = 0, \\ \phi^0_t + H^2\phi^1_t + g\eta + \dfrac{1}{2}|\nabla\phi^0|^2 + \dfrac{1}{2}H^4|\nabla\phi^1|^2 \\ \quad + H^2\nabla\phi^0\cdot\nabla\phi^1 - 2H\phi^1\nabla b\cdot\nabla\phi^0 - 2H^3\phi^1\nabla b\cdot\nabla\phi^1 + 2H^2(1+|\nabla b|^2)(\phi^1)^2 = 0 \end{cases}$$

under the initial conditions

$$(\eta, \phi^0, \phi^1) = (\eta_0, \phi^0_0, \phi^1_0) \qquad \text{at} \quad t = 0,$$

where $\eta = \eta(x,t)$ is the surface elevation, $b = b(x)$ represents the bottom topography, $\phi^0 = \phi^0(x,t)$ and $\phi^1 = \phi^1(x,t)$ are related to the velocity potential $\Phi = \Phi(x,z,t)$ of the water by an approximate formula $\Phi(x,z,t) = \phi^0(x,t) + (z - b(x))^2\phi^1(x,t)$ and $H = H(x,t)$ is the depth of water and is given by $H(x,t) = h + \eta(x,t) - b(x)$. Here, g is the gravitational constant and h is the mean depth of the water. t is the time, $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$ is the horizontal spatial coordinates, and z is the vertical spatial coordinate.

The model is a nonlinear dispersive system, in fact, the linear dispersion relation is given by $(6h^2|\xi|^2 + 15)\omega^2 - gh|\xi|^2(h^2|\xi|^2 + 15) = 0$, where $\xi \in \mathbf{R}^n$ is the wave vector and $\omega \in \mathbf{C}$ is the angular frequency. Therefore, the phase speed $c_{IK}(\xi)$ is given by

$$\big(c_{IK}(\xi)\big)^2 = gh\frac{1 + \frac{1}{15}h^2|\xi|^2}{1 + \frac{2}{5}h^2|\xi|^2},$$

which is the [2/2] Padé approximant of $(c_{WW}(\xi))^2$, where $c_{WW}(\xi) = \pm\sqrt{\frac{g\tanh(h|\xi|)}{|\xi|}}$ is the phase speed of the linear water waves. Since the square of the phase speed of the Green–Naghdi equations $(c_{GN}(\xi))^2 = gh\frac{1}{1+\frac{1}{3}h^2|\xi|^2}$ is the [0/2] Padé approximant of $(c_{WW}(\xi))^2$, the model gives a better approximation than the Green–Naghdi equations in the shallow water regime $h|\xi| \ll 1$, at least, in the linear level and in the case of a flat bottom.

As in the case of the full water wave problem, the model has a conserved energy

$$E(t) := \frac{1}{2}\int_{\mathbf{R}^n}\left\{\int_{b(x)}^{h+\eta(x,t)}\big|\nabla_X\big(\phi^0(x,t) + (z-b(x))^2\phi^1(x,t)\big)\big|^2\mathrm{d}z + g\big(\eta(x,t)\big)^2\right\}\mathrm{d}x,$$

where $\nabla_X = (\nabla, \partial_z)$. In fact, the energy function $E(t)$ is a conserved quantity in time for any smooth solution (η, ϕ^0, ϕ^1) of the model system.

A severe drawback of the model is the fact that the hypersurface $t = 0$ in the space-time $\mathbf{R}^n \times \mathbf{R}$ is characteristic for the model, so that the initial value problem to the model is not solvable in general. In fact, if the problem has a solution (η, ϕ^0, ϕ^1), then by eliminating the time derivative η_t from the first two equations in the model, we see that the solution has to satisfy the relation

$$\begin{aligned}
&H^2\nabla\cdot\Big(H\nabla\phi^0 + \frac{1}{3}H^3\nabla\phi^1 - H^2\phi^1\nabla b\Big)\\
&= \nabla\cdot\Big(\frac{1}{3}H^3\nabla\phi^0 + \frac{1}{5}H^5\nabla\phi^1 - \frac{1}{2}H^4\phi^1\nabla b\Big)\\
&\quad + H^2\nabla b\cdot\nabla\phi^0 + \frac{1}{2}H^4\nabla b\cdot\nabla\phi^1 - \frac{4}{3}H^3(1 + |\nabla b|^2)\phi^1.
\end{aligned}$$

Therefore, as a necessary condition the initial date $(\eta_0, \phi_0^0, \phi_0^1)$ and the bottom topography b have to satisfy the above relation for the existence of the solution.

In this talk, we report that if the initial data and the bottom topography satisfy the above condition together with the so-called Rayleigh–Taylor sign condition, then the initial value problem for the model has a unique solution locally in time. We refer to [7] for the details.

References

[1] M. Isobe. A proposal on a nonlinear gentle slope wave equation. *Proceedings of Coastal Engineering, Japan Society of Civil Engineers* **41**, 1–5 (1994) [Japanese].

[2] M. Isobe. Time-dependent mild-slope equations for random waves,. *Proceedings of 24th International Conference on Coastal Engineering, ASCE*, 285–299 (1994).

[3] T. Kakinuma. [Title in Japanese]. *Proceedings of Coastal Engineering, Japan Society of Civil Engineers* **47**, 1–5 (2000) [Japanese].

[4] T. Kakinuma. A set of fully nonlinear equations for surface and internal gravity waves. *Coastal Engineering V: Computer Modelling of Seas and Coastal Regions*, 225–234, WIT Press (2001).

[5] T. Kakinuma. A nonlinear numerical model for surface and internal waves shoaling on a permeable beach,. *Coastal engineering VI: Computer Modelling and Experimental Measurements of Seas and Coastal Regions*, 227–236, WIT Press (2003).

[6] J. C. Luke. A variational principle for a fluid with a free surface. *J. Fluid Mech.* **27**, 395–397 (1967).

[7] Y. Murakami and T. Iguchi. Solvability of the initial value problem to a model system for water waves. *Kodai Math. J.*, to appear.

Mechanical balance laws in long wave models

Henrik Kalisch

Consider wave motion at the surface of an inviscid incompressible fluid of unit density in the absence of capillarity. Suppose the depth of the fluid in the undisturbed state is given by h_0, and gravity is denoted by g. For waves which respect an approximate relationship $\alpha \sim \beta$ between the nondimensional amplitude $\alpha = a/h_0$ and the long-wave parameter $\beta = h_0^2/\lambda^2$, there are a variety of Boussinesq-type equations which may be used to describe the wave motion for waves which have sufficiently long wavelength λ when compared to the undisturbed depth h_0.

The derivation of such systems is well understood [14], and there exist a large number of systems with various requisite properties. For instance, the systems may be optimized with respect to the description of shorter waves, or with respect to smoothing properties, or amenability to numerical study. An overview is given in [11]. Here we focus on a class of models derived and studied in [4, 5]. Denote the limiting long-wave speed by $c_0 = \sqrt{gh_0}$, and define non-dimensional variables by

$$\tilde{x} = \frac{x}{\lambda}, \quad \tilde{z} = \frac{z + h_0}{h_0}, \quad \tilde{\eta} = \frac{\eta}{a}, \quad \tilde{t} = \frac{c_0 t}{\lambda}, \quad \tilde{\phi} = \frac{c_0 \phi}{g a \lambda}.$$

Assuming irrotaional fluid motion, expanding the velocity potential ϕ in an asymptotic series, and substituting into the governing Euler equations and free-surface boundary conditions yields

$$\tilde{\eta}_{\tilde{t}} + \tilde{w}_{\tilde{x}} + \alpha(\tilde{\eta}\tilde{w})_{\tilde{x}} - \tfrac{1}{2}\big(\theta^2 - \tfrac{1}{3}\big)\beta\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{t}} = \mathcal{O}(\alpha\beta, \beta^2), \tag{1}$$

$$\tilde{w}_{\tilde{t}} + \tilde{\eta}_{\tilde{x}} + \alpha\tilde{w}\tilde{w}_{\tilde{x}} - \tfrac{1}{2}\big(1 - \theta^2\big)\beta\tilde{w}_{\tilde{x}\tilde{x}\tilde{t}} = \mathcal{O}(\alpha\beta, \beta^2). \tag{2}$$

From these relations it appears that if α and β are sufficiently small, terms of order $\mathcal{O}(\alpha\beta, \beta^2)$ can be disregarded, and one may use the following system as approximate equations of motion:

$$\begin{aligned} \eta_t + h_0 w_x + (\eta w)_x - b\eta_{xxt} &= 0, \\ w_t + g\eta_x + w w_x - d w_{xxt} &= 0. \end{aligned} \tag{3}$$

Here $\eta(x,t)$ represents the excursion of the free surface at a spatial point x and at time t, while $w(x,t)$ represents the horizontal velocity at a given height $h_0\theta$ in the fluid column. The parameters b and d are given in terms of $0 \le \theta \le 1$ by

$$b = \tfrac{1}{2}\big(\theta^2 - \tfrac{1}{3}\big)h_0^2, \qquad d = \tfrac{1}{2}\big(1-\theta^2\big)h_0^2.$$

This point of view can also be made rigorous by proving that solutions of the free-surface problem based on the Euler equations converge to solutions of (3) in an appropriate sense on a certain time scale [6, 11].

Since the system (3) was obtained by a procedure which is not based on the conservation of mass and momentum (such as the derivation of the shallow-water system for example), one may ask whether the system (3) allows the conservation of mass, momentum, or indeed conservation of energy. As it happens, if $\theta^2 = \frac{2}{3}$, the system takes the form

$$\begin{aligned} \eta_t + h_0 w_x + (\eta w)_x - \tfrac{h_0^2}{6}\eta_{xxt} &= 0, \\ w_t + g\eta_x + w w_x - \tfrac{h_0^2}{6} w_{xxt} &= 0, \end{aligned} \tag{4}$$

and in this case, the system is Hamiltonian with Hamiltonian function

$$H = \int_{-\infty}^{\infty} \big\{ \tfrac{g}{2}\eta^2 + \tfrac{h_0}{2}w^2 + \tfrac{1}{2}\eta w^2 \big\}\, dx.$$

However, since the derivation of (4) was not based on preserving the Hamiltonian structure, it remains to be shown that this functional represents the total mechanical energy due to the wave motion. Moreover, the question also arises how to express the energy of the wave motion in the more general system (3).

While in the study of system of this type, the prevailing point of view is to consider conservation of functionals usually interpreted as total excess mass, momentum and energy[1] a different way to proceed is to focus on approximate local conservation. As explained in [2], this approach entails substituting the expansion for the velocity potential into the conservation equations based on the Euler description of the flow, and requiring the approximate balance law

$$\frac{\partial}{\partial \tilde t}\tilde E(\tilde\eta, \tilde w) + \frac{\partial}{\partial \tilde x}\tilde q_E(\tilde\eta, \tilde w) = \mathcal{O}(\alpha^2, \alpha\beta, \beta^2),$$

which defines the energy density E and energy flux q_E.

In the case of (3), the dimensional versions of the energy density and energy flux are obtained in the form

$$E_\theta = \tfrac{g}{2}\eta^2 + \tfrac{h_0}{2}w^2 + \tfrac{1}{2}w^2\eta + \tfrac{h_0^3}{2}\big(\theta^2 - \tfrac{1}{3}\big) w w_{xx} + \tfrac{h_0^3}{6} w_x^2 \tag{5}$$

[1]The general system (3) features conservation of total excess mass through the conserved integral $\int_{-\infty}^{\infty} \eta\, dx$. Moreover, for the system (4) the integral $\int_{-\infty}^{\infty} \eta w + b\eta_x w_x\, dx$ is also formally conserved. However, it is not clear if this last integral has any physical significance.

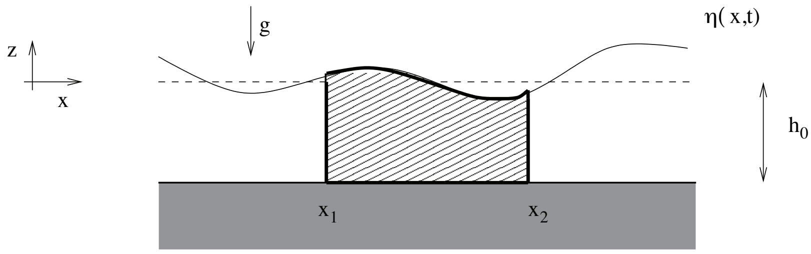


FIGURE 1. Geometric setup of the problem. The undisturbed water depth is h_0, and the x-axis is aligned with the free surface at rest. The free surface is described by a function $\eta(x,t)$. The figure shows a control interval delimited by x_1 and x_2 on the abscissa.

and

$$q_{E_\theta} = \tfrac{h_0}{2} w^3 + c_0^2 \eta w + \tfrac{c_0^2 h_0^2}{2}\big(\theta^2 - \tfrac{1}{3}\big)\eta w_{xx} - \tfrac{h_0^3}{3} w w_{xt} + \tfrac{c_0^2}{h_0} w \eta^2. \tag{6}$$

In particular, $q_{E_\theta}(x,t)$ gives the energy flux and work done by the pressure force due to the wave motion at a point x and a time t. Integrating $E_\theta(x,t)$ over an interval $[x_1, x_2]$ yields the energy due to the wave motion in the control interval shown in Figure 1 at a time t, and to the same order of approximation as the system (3) is valid.

If the surface disturbance is localized, so that η and w decay to zero at infinity, and the integration of E is taken over the entire real line, then the Hamiltonian of (4) is recovered in the case when $\theta^2 = 2/3$: $H = \int_{-\infty}^{\infty} E_\theta \, dx$.

Similar approximate balance laws can be sought for the mass density and flux, and for the momentum density and flux. Since it was already decided that the system (3) is the governing system in the current description, these balance laws will generally not hold exactly, but only up to some order in β and α.

One application of the analysis detailed above has been used to understand the energy budget in an undular bore as approximated by different model equations [1, 9].

Similar considerations can be applied to the KdV equation

$$\eta_t + c_0 \eta_x + \tfrac{3}{2}\tfrac{c_0}{h_0}\eta\eta_x + \tfrac{c_0 h_0^2}{6}\eta_{xxx} = 0, \tag{7}$$

which is a unidirectional model for surface waves. In this case, it was found in [3] that the energy density and flux are given by

$$E = c_0^2 \left(\tfrac{1}{h_0}\eta^2 + \tfrac{1}{4h_0^2}\eta^3 + \tfrac{h_0}{6}\eta\eta_{xx} + \tfrac{h_0}{6}\eta_x^2 \right),$$

and

$$q_E = c_0^3 \left(\tfrac{1}{h_0}\eta^2 + \tfrac{5}{4h_0^2}\eta^3 + \tfrac{h_0}{2}\eta\eta_{xx} \right). \tag{8}$$

One interesting application where these quantities can be put to use concerns the the shoaling of periodic wavetrains and solitary waves. Consider a wave which experiences a decrease in depth over a gentle slope with no variation in the transverse direction. From a practical point of view, the waveheight of the shoaling waves is of particular interest, and one may use the conservation of energy flux in an adiabatic setting to obtain a first approximation for the waveheight. The linear theory is well known [7], and there have also been many studies making use of cnoidal wave solutions of (7) for periodic shoaling.

However, there is a deep-water limit beyond which cnoidal solutions of the the KdV equation cannot be used to describe periodic wave trains. Because of this limitation, it is necessary in the shoaling problem to compute the initial transition from deep water to intermediate depths by linear wave theory [12].

However, one problem which the authors of [12] faced was that at the point where linear and cnoidal theory were to be matched, a discontinuity in waveheight appeared in the shoaling curve. This problem was overcome later in [13] by imposing continuity in waveheight directly, but at the cost of incurring a discontinuity in the energy flux. Using the nonlinearly defined energy flux q_E in the shoaling equation

$$\int_0^T q_{E_A}\, dt = \int_0^T q_E\, dt, \tag{9}$$

eliminates the problem of discontinuities in waveheight or energy flux at the matching point between linear and cnoidal theory [10].

A comparison between the shoaling computations based on (9) and the numerical results for the full water-wave problem [8] is shown in Figure 2 for a wave of initial wavelength L_0 and waveheight H_0. It can be seen that the waveheight increases initially more slowly than predicted by Green's law, but the shoaling curve then turns up, and reaches a slope similar to Boussinesq's law. The curve based on (9) matches the curve obtained in [8] rather well. One aspect in which the comparison is not favorable is the termination of the shoaling curve based on (9) before the breaking point. This issue has not been investigated further so far.

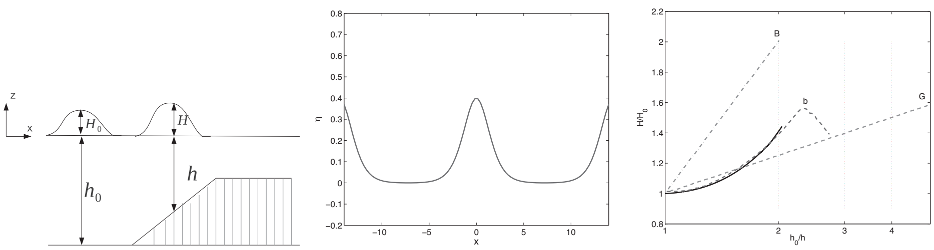


FIGURE 2. Left panel: definition sketch for h_0, h, H_0 and H. Center panel: incident wave profile with $\frac{L_0}{h_0} = 14.5$, $\frac{H_0}{h_0} = 0.4$. Right panel: b: shoaling curve after Grilli et al., black solid curve: shoaling curve based on (8) and (9), G: Green's law and B: Boussinesq's law.

References

[1] A. Ali and H. Kalisch. Energy balance for undular bores. *C. R. Mécanique* **338**, 67–70 (2010).

[2] A. Ali and H. Kalisch. Mechanical balance laws for Boussinesq models of surface water waves. *J. Nonlinear Sci.* **22**, 371–398 (2012).

[3] A. Ali and H. Kalisch. On the formulation of mass, momentum and energy conservation in the KdV equation. *Acta Appl. Math.* **133**, 113–131 (2014).

[4] J.L. Bona, M. Chen and J.-C. Saut. Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I: Derivation and linear theory. *J. Nonlinear Sci.* **12**, 283–318. (2002).

[5] J.L. Bona, M. Chen and J.-C. Saut. Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II: The nonlinear theory. *Nonlinearity* **17**, 925–952 (2004).

[6] J.L. Bona, T. Colin and D. Lannes. Long wave approximations for water waves. *Arch. Ration. Mech. Anal.* **178**, 373–410 (2005).

[7] R.G. Dean and R.A. Dalrymple. *Water Wave Mechanics for Engineers and Scientists.* World Scientific, Singapore (1991).

[8] S.T. Grilli, I.A. Svendsen and R. Subramanya. Breaking criterion and characteristics for solitary waves on slopes. *J. Wtrwy., Port, Coast., and Oc. Engrg.* **123**, 102–112 (1997).

[9] H. Kalisch and M. Bjørkavåg. Energy budget in a dispersive model for undular bores. *Proc. Est. Acad. Sci.* **59**, 172–181 (2010).

[10] Z. Khorsand and H. Kalisch. On the shoaling of solitary waves in the KdV equation, *Proc. 34th Conf. Coastal Engineering, Seoul, Korea*, 34, 10 pp. (2014).

[11] D. Lanne., *The Water Waves Problem* (*Mathematical Surveys and Monographs* **188**). Amer. Math. Soc.: Providence (2013).

[12] I.A. Svendsen and O. Brink-Kjær. Shoaling of cnoidal waves. *Proc. 13th Conf. Coastal Engng, Vancouver*, 365–383 (1972).

[13] I.A. Svendsen and J. Buhr Hansen. The wave height variation for regular waves in shoaling water. *Coastal Engineering* **1**, 261–284 (1977).

[14] G.B. Whitham. *Linear and Nonlinear Waves.* Wiley: New York (1974).

Water waves with vorticity and asymptotic models

David Lannes

Motivated by the study of nonlinear wave-current interactions (such as rip-currents) we study the influence of vorticity on surface water waves. We first derive a generalization of the classical hamiltonian Zakharov-Craig-Sulem formulation of irrotational water waves that takes into account the effects of the vorticity. The canonical variables for this formally Hamiltonian generalization are the surface elevation, the ?gradient component? of the horizontal component of the tangential vector field, and the vorticity. It allows therefore to keep track of the influence of vorticity on the flow. We show that this formulation is formally Hamiltonian and that it is well-posed. We also establish the stability of the lifes pan with respect to shallow water limits and provide some bounds on the solution. These results can be found in [1].

Based on the bounds thus obtained, we turn to derive shallow water asymptotic models. The big difference with irrotational flows is that the dynamics of the vorticity is fully three dimensional, while shallow water models are typically two-dimensional (through vertical averaging). We show however that the vorticity contribution can be reduced to two-dimensional equations; the idea, based on an analogy with turbulence theory, is that the vorticity contributes to the averaged momentum equation through a Reynolds-like tensor. A cascade of equations is then derived for this tensor, but contrary to standard turbulence theory, closure of the equations is obtained after a finite number of steps. The models thus obtained generalize the classical Green-Naghdi equation in the sense that there are additional terms in the momentum equation that take into account the presence of vorticity. The evolution of these additional terms is essentially an advection at the mean velocity. These results can be found in [2].

The structure of these extended equations allow for numerical simulations based on a finite volume scheme; we present some preliminary result that will appear in [3], and we also analyze the equations in the 1d case. We prove in particular that there exist solitary waves of maximal amplitude with an angle at the crest that depend on the vorticity.

References

[1] A. Castro and D. Lannes. Well-posedness and shallow-water stability for a new Hamiltonian formulation of the water waves equations with vorticity. *Indiana U. Math. J.*, in press (2015).

[2] A. Castro and D. Lannes. Fully nonlinear long-wave models in the presence of vorticity. *J. Fluid Mech.* **759**, 642–675 (2014).

[3] D. Lannes and F. Marche. Numerical approximation of the Green-Naghdi equations with vorticity. Manuscript in preparation.

Non-existence of solitary waves of depression in the presence of vorticity

Evgeniy Lokharu

Statement of the problem. Let an open channel of uniform rectangular cross-section be bounded below by a horizontal rigid bottom and let water occupying the channel be bounded above by a free surface not touching the bottom. In appropriate Cartesian coordinates (x, y), the bottom coincides with the x-axis and gravity acts in the negative y-direction. We use the non-dimensional variables proposed by Keady and Norbury [2] (see also Appendix A in [3] for details of scaling); namely, lengths and velocities are scaled to $(Q^2/g)^{1/3}$ and $(Qg)^{1/3}$ respectively. Here Q and g are the dimensional quantities for the rate of flow and the gravity acceleration respectively, whereas $(Q^2/g)^{1/3}$ is the depth of the critical uniform stream in the irrotational case.

The steady water motion is supposed to be two-dimensional and rotational; the surface tension is neglected on the free surface of the water, where the pressure is constant. These assumptions and the fact that water is incompressible allow us to seek the velocity field in the form $(\psi_y, -\psi_x)$, where $\psi(x, y)$ is referred to as the *stream function*. The vorticity distribution ω is supposed to be a prescribed continuous function depending on ψ.

We choose the frame of reference so that the velocity field is time-independent as well as the unknown free-surface profile. The latter is assumed to be the graph of $y = \eta(x)$, $x \in \mathbb{R}$, where η is a positive continuous function, and so the longitudinal section of the water domain is $D = \{x \in \mathbb{R},\ 0 < y < \eta(x)\}$. The following free-boundary problem for ψ and η which describes all kinds of waves has long been known (cf. [2]):

$$\psi_{xx} + \psi_{yy} + \omega(\psi) = 0, \quad (x, y) \in D; \tag{1}$$

$$\psi(x, 0) = 0, \quad x \in \mathbb{R}; \tag{2}$$

$$\psi(x, \eta(x)) = 1, \quad x \in \mathbb{R}; \tag{3}$$

$$|\nabla\psi(x, \eta(x))|^2 + 2\eta(x) = 3r, \quad x \in \mathbb{R}. \tag{4}$$

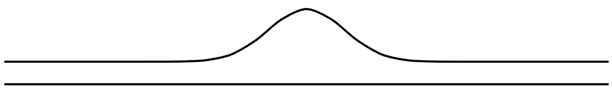
In condition (4) (Bernoulli's equation), r is a constant considered as the problem's parameter and referred to as Bernoulli's constant/the total head. In what follows, we suppose that ψ is a strictly monotonic function of y, say

$$\psi_y(x, y) > 0 \quad \text{for all } (x, y) \in \bar{D}, \tag{5}$$

which means that the flows we are going to study are unidirectional.

By a *stream* (shear-flow) solution we mean a pair $(u(y),\, d)$ determining a solution for the problem (1)-(4), where u stands for the stream function instead of ψ and the constant depth of flow d replaces the wave profile η.

Solitary waves. In 1834 a remarkable phenomena was observed by John Scott Russell. During his experiments with a boat in a rectangular water channel he generated a wave of special form moving with a constant speed and shape. The wave surface had only one crest and was monotone around the crest (see the picture below).



Such waves are now known as solitary waves of elevation. The first rigorous proof of the existence for irrotational solitary waves was given by Lavrentiev in 1954, where a solitary wave was constructed as a limit of periodic waves. Since that time a lot of research has been done, however there are many open questions left, especially for waves with vorticity and surface tension. In 2007 appeared paper [1] by Mark Groves and Erik Wahlén on small-amplitude waves with surface tension. Using methods of spacial dynamics they constructed solitary type waves of more complicated geometry. In particular, solitary waves of depression were obtained (see the picture below).

Note, that in the irrotational case, it is well known that only solitary waves of elevation exist. However, for rotational waves without surface tension it is still unclear, if solitary waves of depression exist or not. Our goal is to answer this question. We prove that in the class of unidirectional waves only waves of elevation exist. A precise statement is given in the from of the following theorem:

Theorem 1 (Kozlov, Kuznetsov, Lokharu, 2015)**.** *Let* (Ψ, η) *be a solution of the problem* (1)-(4) *such that* $\eta(x) \leq d$ *for all* $x \in \mathbb{R}$ *and* $\lim_{x\to+\infty} \eta(x) = d$ *for some* $d > 0$. *Assume, moreover, that* $\Psi_y > \delta$ *in* D *for some* $\delta > 0$. *Then* (Ψ, η) *is a stream solution with depth* d.

With similar methods we can also prove the absence of solitary type waves with profiles that oscillate around a certain level and approach it as $x \to \infty$.

References

[1] M.D. Groves and E. Wahlén. Spatial dynamics methods for solitary gravity-capillary water waves with an arbitrary distribution of vorticity. *SIAM J. Math. Anal.* **39**, 932-964 (2007).

[2] G. Keady and J. Norbury. Waves and conjugate streams with vorticity. *Mathematika* **25**, 129–150 (1978).

[3] V. Kozlov, N. Kuznetsov. Dispersion equation for water waves with vorticity and Stokes waves on flows with counter-currents. *Arch. Rat. Mech. Anal.* **214**, 971–1018 (2014).

Asymptotic stability of solitary waves in a water wave model with indefinite variational structure

Robert L. Pego

We study asymptotic stability properties for solitary waves in the Benney-Luke model equation for water waves. One feature that this model shares with the full water wave problem is that solitary wave profiles are critical points of an energy-momentum functional that is infinitely indefinite and not useful for estimates.

In one space dimension, the Benney-Luke equation takes the form

$$\partial_t^2\varphi - \partial_x^2\varphi + a\partial_x^4\varphi - b\partial_x^2\partial_t^2\varphi + (\partial_t\varphi)(\partial_x^2\varphi) + 2(\partial_x\varphi)(\partial_x\partial_t\varphi) = 0. \tag{1}$$

This is the one-dimensional version of an equation originally derived by Benney and Luke [1] as an isotropic model for three-dimensional water waves, formally valid as an approximation for describing small-amplitude, long water waves in water of finite depth. We take the parameters $a,\ b > 0$ to be such that $a - b = \hat{\tau} - \frac{1}{3} < 0$, where $\hat{\tau}$ is the *inverse* Bond number. This corresponds to the case of small or zero surface tension. We remark that (1) is an approximation formally valid for describing *two-way* water wave propagation, in contrast to one-way equations such as the KdV, BBM, or KP equations.

We write evolution equations for the system in terms of

$$u = \begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} \partial_x\varphi \\ \partial_t\varphi \end{pmatrix}, \quad A = 1 - a\partial_x^2, \quad B = 1 - b\partial_x^2.$$

The system takes the form

$$\partial_t u = Lu + f(u), \quad L = \begin{pmatrix} 0 & \partial_x \\ B^{-1}A\partial_x & 0 \end{pmatrix}, \quad f(u) = \begin{pmatrix} 0 \\ -B^{-1}(r\partial_x q + 2q\partial_x r) \end{pmatrix}.$$

There is a two-parameter family of solitary waves $(q,r) = (q_c, r_c)(x - ct - x_0)$ with

$$q_c(x) = \frac{c^2-1}{c}\operatorname{sech}^2\left(\frac{1}{2}\alpha_c x\right), \quad r_c = -cq_c, \qquad \alpha_c = \sqrt{\frac{c^2-1}{bc^2-a}}.$$

There are about 19 short steps involved in our stability analysis of these waves. These are organized roughly as follows.

- Description of solitary waves and linearization. Hamiltonian structure. Zero modes.
- Spectral stability implies linear stability: Fourier symbol estimate with weights e^{ax}. Reduction of resolvent to scalar form. Verify the Gearhart-Prüss stability criterion using compactness.
- Spectral stability in the KdV limit: KdV scaling of the reduced resolvent. Bundle convergence theorem (Uniform symbol estimates. KdV limit of Fourier multipliers: low-frequency Taylor expansions, high-frequency bounds. Commutator estimates). Null multiplicity (using the Gohberg-Sigal-Rouché theorem).
- Linear implies nonlinear stability: Mizumachi's wave-background decomposition. Speed/phase modulation equations. Linear estimates via recentering. Hamiltonian energy estimate. Virial estimate (on nonlinear transport). Stability proof (A priori estimates. Continuation. Rate estimates.)

We study linear stability in a space L^2_α with an exponential weight $e^{\alpha x}$ designed to discount energy transport that is *slower* than the solitary wave, having group velocity that is negative with respect to the co-moving frame:

$$\|g\|_{L^2_\alpha} = \left(\int_{\mathbb{R}} |e^{\alpha x} g(x)|^2\,dx\right)^{1/2}.$$

The linear analysis goes like our work [2] for water waves. The linearized equations in the co-moving frame take the form

$$\partial_t u = \mathcal{L}_c u, \quad \mathcal{L}_c = \begin{pmatrix} c\partial & \partial \\ -B^{-1}(-A\partial + r_c\partial + 2r'_c) & c\partial - B^{-1}(2q_c\partial + q'_c) \end{pmatrix}.$$

Perhaps the main theorem worth featuring here is the following. (The spectral stability condition on $\mathcal{L}_{c_0}$ is known to hold for sufficiently small waves.)

Theorem 1 (Spectral stability implies nonlinear stability)**.** *Suppose $c_0 > \sigma > 1$ and $0 < \alpha < \frac{1}{2}\alpha_{c_0}$, and assume that in L^2_α, $\mathcal{L}_{c_0}$ has no nonzero eigenvalue λ satisfying $\Re\lambda \geq 0$. Then there exists $\delta > 0$ satisfying the following: If $u_0(x) = u_{c_0}(x - x_0) + v_0(x)$ where $x_0 \in \mathbb{R}$ and $\|v_0\|_{H^1} < \delta$, then there exist $c_\star > 1$ and a*

C^1-function $x(t)$ such that

$$\lim_{t\to\infty} x'(t) = c_\star, \tag{2}$$

$$|c_\star - c_0| + \sup_{t>0} |x'(t) - c_0| = O(\|v_0\|_{H^1}), \tag{3}$$

$$\sup_{t\geq 0} \|u(t,\cdot) - u_{c_0}(\cdot - x(t))\|_{H^1}^2 = O(\|v_0\|_{H^1}), \tag{4}$$

$$\lim_{t\to\infty} \|u(t,\cdot) - u_{c_\star}(\cdot - x(t))\|_{H^1(x\geq\sigma t)} = 0. \tag{5}$$

If one makes stronger assumptions on the decay of the perturbation v_0, so that it is square-integrable against a weight with sufficient polynomial growth, one can prove a polynomial rate of convergence for the phase difference $x(t) - c_\star t$.

References

[1] D.J. Benney and J.C. Luke, Interactions of permanent waves of finite amplitude. *J. Math. Phys.* **43**, 309–313 (1964).

[2] R.L. Pego and S.-M. Sun. Asymptotic linear stability of solitary water waves. Submitted.

Multi-solitons and related constructions for the water-waves system

Frédéric Rousset

We consider the motion of an irrotational, incompressible fluid with constant density in the situation where the fluid domain is a strip with a rigid bottom and a free surface:

$$\Omega_t = \{Y = (X,z) \in \mathbb{R}^{d+1} \,:\, -H < z < \eta(t,X)\},$$

where t is the time, $d = 1,2$ is the horizontal dimension, H is a parameter defining the fixed bottom $z = -H$ and $z = \eta(t,X)$ is the equation of the unknown free surface at time t. We shall say that we are in the one dimensional case when $X = x \in \mathbb{R}$ and in the two-dimensional case when $X = (x,y) \in \mathbb{R}^2$. We denote by u the speed of the fluid, since the motion is irrotational, it is given by $u = \nabla_Y \Phi = (\nabla_X \Phi, \partial_z \Phi)$ for some scalar function Φ and hence we find that inside the fluid domain Ω_t,

$$\nabla_Y \cdot u = \Delta_Y \Phi = (\Delta_X + \partial_z^2)\Phi = 0\,. \tag{1}$$

On the boundaries of Ω_t, we make the usual assumption that no fluid particles cross the boundary. At the bottom of the fluid this reads

$$\partial_z \Phi(t,X,-H) = 0 \tag{2}$$

and on the free surface, this yields the kinematic condition

$$\partial_t \eta(t,X) + \nabla_X \Phi(t,X,\eta(t,X)) \cdot \nabla_X \eta(t,X) - \partial_z \Phi(t,X,\eta(t,X)) = 0\,. \tag{3}$$

On the free surface, we also need to impose the pressure, taking into account the surface tension and using the Bernouilli law to eliminate the pressure, we find that:

(4)

$$\partial_t \Phi(t,X,\eta(t,X)) + \frac{1}{2}|\nabla_Y \Phi(t,X,\eta(t,X))|^2 + g\eta(t,X) = b\nabla_X \cdot \frac{\nabla_X \eta(t,X)}{\sqrt{1+|\nabla_X \eta(t,X)|^2}}\,.$$

The number b is the surface tension coefficient and g is the gravitational constant. The term $g\eta(t,X)$ is the trace of the gravitational force gz on the free surface.

It is classical to rewrite the system (1), (3), (4) as a system involving unknowns defined on the free surface only [20]. For that purpose, let us define the following Dirichlet-Neumann operator: for given $\eta(X)$ $\varphi(X)$, we define $\Phi(X,z)$ as the (well-defined) solution of the elliptic boundary value problem

$$(\Delta_X + \partial_z^2)\Phi = 0, \quad \text{in} \quad \{(X,z) \,:\, -H < z < \eta(X)\},$$
$$\Phi(X,\eta(X)) = \varphi(X), \quad \partial_z\Phi(X,-H) = 0,$$

and we define the Dirichlet-Neumann operator as

$$\begin{aligned}(G[\eta]\varphi)(X) &:= (\partial_z\Phi - \nabla_X\eta\cdot\nabla_X\Phi)|_{z=\eta(X)}\\ &= \sqrt{1+|\nabla_X\eta|^2}(\nabla_{X,z}\Phi\cdot\mathbf{n})|_{z=\eta(X)},\end{aligned}$$

where $\mathbf{n}$ is the unit outward normal vector on the free surface at the point $z = \eta(X)$.

This allows to rewrite the system only in terms of the unknowns

$$(\eta(t,X),\varphi(t,X)) := (\eta(t,X),\Phi(t,X,\eta(t,X)))\,.$$

In the one-dimensional case, the 1D water-wave problem can thus be written as

$$\tag{5} \begin{cases} \partial_t\eta = G[\eta]\varphi \\ \partial_t\varphi = -\dfrac{1}{2}|\partial_x\varphi|^2 + \dfrac{1}{2}\dfrac{(G[\eta]\varphi + \partial_x\varphi\partial_x\eta)^2}{1+|\partial_x\eta|^2} - g\eta + b\partial_x\Big(\dfrac{\partial_x\eta}{\sqrt{1+|\partial_x\eta|^2}}\Big)\end{cases}$$

By introducing the notations $U = (\eta,\varphi)^t$ and

$$\mathcal{F}(U) = \Big(G[\eta]\varphi, \quad -\frac{1}{2}|\partial_x\varphi|^2 + \frac{1}{2}\frac{(G[\eta]\varphi + \partial_x\varphi\partial_x\eta)^2}{1+|\partial_x\eta|^2} - g\eta + b\partial_x\Big(\frac{\partial_x\eta}{\sqrt{1+|\partial_x\eta|^2}}\Big)\Big)^t,$$

we shall write the water-wave system (5) in the abstract form

$$\tag{6} \partial_t U = \mathcal{F}(U).$$

We know from [1] that for suitable parameters g, b and h, there exist solitary wave solutions $Q_c(x-ct) = (\eta_c(x-ct),\varphi_c(x-ct))^t$ at speed $c \sim \sqrt{gH}$.

Theorem 1 (Amick-Kirchgässner [1])**.** *Suppose that*

$$\tag{7} \frac{gH}{c^2} = 1+\varepsilon^2, \quad \frac{b}{Hc^2} > \frac{1}{3}\,.$$

Then there exists ε_0 such that for every $\varepsilon \in (0,\varepsilon_0)$ (which fixes the speed) there is a solution of (5) under the form

$$Q_c(x-ct) = \big(\eta_c(x-ct),\varphi_c(x-ct)\big)^t = \Big(H\eta_\varepsilon(H^{-1}(x-ct)),\, cH\varphi_\varepsilon(H^{-1}(x-ct))\Big)^t$$

with

$$\eta_\varepsilon(x) = \varepsilon^2\Theta_1(\varepsilon x,\varepsilon), \quad \varphi_\varepsilon(x) = \varepsilon\Theta_2(\varepsilon x,\varepsilon),$$

where Θ_1 and Θ_2 satisfy:

$$\exists\, d>0, \quad \forall\,\alpha\geq 0, \quad \exists\, C_\alpha > 0, \quad \forall\,(x,\varepsilon)\in\mathbb{R}\times(0,\varepsilon_0), \quad |(\partial_x^\alpha\Theta_1)(x,\varepsilon)| \leq C_\alpha e^{-d|x|}$$

and

$$\exists\, d > 0, \quad \forall\, \alpha \geq 1, \quad \exists\, C_\alpha > 0, \quad \forall\, (x,\varepsilon) \in \mathbb{R}\times(0,\varepsilon_0), \quad |(\partial_x^\alpha \Theta_2)(x,\varepsilon)| \leq C_\alpha e^{-d|x|}.$$

Moreover Θ_1 *is even and* Θ_2 *is odd.*

We thus consider two solitons $Q_{c_1}(x - c_1 t)$ and $Q_{c_2}(x - h - c_2 t)$ of (5) with $c_1 < c_2$. We suppose that c_1 and c_2 satisfy (7) with suitable choices of the small parameters $\varepsilon_{1,2}$. We also suppose that $h > 0$ is large enough. We define

$$M(t,x) := Q_{c_1}(x - c_1 t) + Q_{c_2}(x - h - c_2 t) \tag{8}$$

as the two-soliton function. We will focus on the case where each solitary wave is stable in the following sense. Under our assumptions (7) on the speed c of a solitary wave, it was proven in [15] that for sufficiently small corresponding parameter ε, the solitary wave Q_c is stable in the sense that the second derivative of the Hamiltonian at the solitary wave restricted to a natural co-dimension 2 subspace is positive. We shall assume that the speeds c_1, c_2 are such that this property is verified. Our main result reads:

Theorem 2. *Let us fix* $s \geq 0$. *Suppose that the speeds* $c_1 < c_2$ *satisfy* (7) *with parameters* ε_1, ε_2. *Define* M *by* (8). *Then there exists* ε^* *such that for* $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon^*]$ *and* h *sufficiently large, we have that there exists a (semi) global solution* $U(t) = (\eta, \varphi)^t$, $t \geq 0$ *to the water-wave system (5) satisfying*

$$U - M \in \mathcal{C}_b([0,\infty); H^s(\mathbb{R}) \times H^s(\mathbb{R}))$$

and

$$\lim_{t\to+\infty} \|U(t) - M(t)\|_{H^s\times H^s} = 0.$$

We have focused on water waves with surface tension, nevertheless, since the existence of solitary waves is also known (see [7], [6], [10] for example) and since some of them are linearly stable, [17] it could be possible to perform a related construction for water waves without surface tension.

Finally, let us point out that the assumption that h is sufficiently large is only used in order to get a solution on $[0, +\infty[$, an equivalent statement would be to take $h = 0$ and to get a multi-soliton solution on the interval $[T_0, +\infty[$ with T_0 sufficiently large.

The main arguments that are used to prove Theorem 2 can also be used in order to sharpen the transverse instability result proven in [18] and construct for the two-dimensional water-waves system that is to say when the fluid domain is

$$\Omega_t = \{(X, z) \in \mathbb{R}^3,\ -H < z < \eta(t, X)\},$$

a solution on $[0, +\infty)$ of the system which is different from the solitary wave (and all its translates) and converge to the solitary wave as time goes to infinity. The result that we shall prove is the following.

Theorem 3. *Let us fix* $s \geq 0$. *Suppose that* c *satisfies* (7). *For* ε *sufficiently small there exists a global solution* U *of the 2-D water waves system with initial data* U_0

satisfying $U - Q_c \in \mathcal{C}_b([0,\infty); H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2))$*. Moreover, one has*

$$\partial_y U_0 \neq 0 \tag{9}$$

and

$$\lim_{t\to+\infty} \|U(t,x,y) - Q_c(x-ct)\|_{H^s\times H^s} = 0\,.$$

Remark 1. *By the remark after* [19, Theorem 1.5] *this result implies the transverse instability of the solitary wave.*

This result can be compared to classical results about the existence of strongly stable manifolds for ordinary differential equations or semilinear partial differential equations. Results as in Theorem 3 were in particular obtained for semilinear partial differential equations in [5, 2] for example and in [19] for the KP-I equation (see also [11]). The proof of Theorem 3 also relies on the construction of a well-chosen approximate solution and of a remainder that solves a nonlinear system.

References

[1] C.J. Amick and K. Kirchgassner. A theory of solitary water-waves in the presence of surface tension. *Arch. Rat. Mech. Anal* **105**, 1–49 (1989).

[2] V. Combet. Construction and characterization of solutions converging to solitons for supercritical gKdV equations. *Differential Integral Equations* **23**, 513–568 (2010).

[3] R. Côte and S. Le Coz. High-speed excited multi-solitons in nonlinear Schrödinger equations. *J. Math. Pures Appl.* **96**, 135-166 (2011).

[4] R. Côte, Y. Martel and F. Merle. Construction of multi-soliton solutions for the L2-supercritical gKdV and NLS equations. *Rev. Mat. Iberoam.* **27**, 273-302 (2011)

[5] T. Duyckaerts and F. Merle. Dynamic of threshold solutions for energy-critical NLS. *Geom. Funct. Anal.* **18**, 1787–1840 (2009).

[6] K. O. Friedrichs and D. H. Hyers. The existence of solitary waves. *Comm. Pure Appl. Math.* **7**, 517-550 (1954).

[7] G. Iooss and K. Kirchgässner. Water waves for small surface tension: an approach via normal form. *Proc. Roy. Soc. Edinb. A* **122**, 267-299 (1992).

[8] E. Grenier. On the nonlinear instability of Euler and Prandtl equations. *Comm. Pure Appl. Math.* **53**, 1067–1091 (2000).

[9] J. Krieger, Y. Martel and P. Raphael. Two-soliton solutions to the three-dimensional gravitational Hartree equation. *Comm. Pure Appl. Math.* **62**, 1501–1550 (2009).

[10] M. A. Lavrentiev. On the theory of long waves. II. A contribution to the theory of long waves. *Amer. Math. Soc. Translation* **102**, 53 pp. (1954).

[11] Z. Lin and Ch. Zeng. Unstable manifolds of Euler equations. Preprint, arXiv:1112.4525.

[12] Y. Martel. Asymptotic N-soliton-like solutions for the subcritical and critical generalized KdV equations. *Amer. J. Math.* **127**, 1103–1140 (2005).

[13] Y. Martel and F. Merle. Multi solitary waves for nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **23**, 849–864 (2006).

[14] F. Merle, Construction of solutions with exactly k blow-up points for the Schrödinger equation with critical nonlinearity. *Comm. Maths. Phys.* **129**, 223–240 (1990).

[15] A. Mielke. On the energetic stability of solitary water waves. *Phil. Trans. R. Soc. Lond. A* **360**, 2337-2358 (2002).

[16] T. Mizumachi. N-soliton states of the Fermi-Pasta-Ulam lattices. *SIAM J. Math. Anal.* **43**, 2170–2210 (2011).

[17] R. L. Pego and S.-M. Sun. Asymptotic linear stability of solitary water-waves. Preprint, arXiv:1009.0494.

[18] F. Rousset and N. Tzvetkov. Transverse instability of the line solitary water-waves. *Invent. Math.* **184**, 257-388 (2011)
[19] F. Rousset and N. Tzvetkov. Stability and instability of the KdV solitary wave under the KP-I flow. *Comm. Math. Phys.* **313**, 155-173 (2012).
[20] V. Zakharov. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Journal of Applied Mechanics and Technical Physics* **9**, 190–194 (1968).

Approximation results for amplitude equations describing water wave problem like dispersive systems

Guido Schneider

We are interested in the valdity of various amplitude equations for dispersive wave systems which have the following properties with the water wave problem in common, namely two curves of eigenvalues vanishing at the wave number $k = 0$ and associated nonlinear terms which do the same. Famous examples which fall into this class are the FPU system and the polyatomic FPU system. Among the amplitude equations which can be derived via a multiple scaling perturbation ansatz are the KdV equation, the NLS equation, the TWI system, the FWI system, and the Whitham system. Except for the TWI approximation the proof of the approximation results in general is non-trivial since solutions of order $\mathcal{O}(\varepsilon^{\alpha})$ have to be approximated on an $\mathcal{O}(\varepsilon^{-(1+\alpha)})$ time scale where $0 < \varepsilon \ll 1$ is the small perturbation parameter. Herein, $\alpha = 2$ for the KdV approximation, $\alpha = 1$ for the NLS and FWI approximation, and $\alpha = 0$ for the Whitham approximation. Here we are especially interested in approximation results for dispersive wave systems with spatially periodic coefficients with non-small contrast.

We refrain from giving a complete overview about the existing literature and restrict ourselves to the ones which we think are most fundamental. Error estimates that these approximations predict the dynamics of solutions of the original system correctly can be found for the KdV approximation in [7, 16, 17, 3, 10] via energy estimates and in [13] with the explanations from [15] with a Cauchy-Kowalevskaya like approach. For the NLS approximation such estimates can be found in [8, 18, 20, 21, 11]. In [19] a counterexample for the failure of the NLS approximation for water wave problem with surface tension in certain situations has been constructed. The proof of the NLS approximation results are based on normal form transformations.

Systems with spatially periodic coefficients can serve as a toy model for the water problem over a periodic bottom or as a toy problem for the modulations of spatially periodic wave trains. KdV justification results for the water wave problem with a long wave small amplitude periodic perturbations of the bottom can be found in [12, 6]. KDV justification results for modulations of periodic wave trains in the NLS equation can be found in [2, 5] using energy estimates in Sobolev spaces and in [9] for the Whitham approximation using the Cauchy-Kowalevskaya like approach. We explain that for the Boussinesq equation

$$\partial_t^2 u = \partial_x(a\partial_x u) - \partial_x^2(b\partial_x^2 u) + \partial_x(c\partial_x(u^2))$$

with spatially periodic coefficient functions a, b, and c, an energy can be found which allows us to prove the required estimates for the KdV and Whitham approximation [1]. We explain that this approach surprisingly allows to get rid of oscillatory quadratic terms without normal form transforms and of a number of resonances. For the derivation of the approximations we use Bloch wave analysis. We explain that for the NLS approximation of standing waves the Klein-Gordon-Zakharov system can serve as a toy model. NLS approximation results for this system can be found for instance in [14]. See also [4].

References

[1] R. Bauer. *The KdV and Whitham limit for a spatially periodic Boussinesq model.* PhD Thesis, Universität Stuttgart 2015. In preparation.

[2] F. Bethuel, P. Gravejat, J.-C. Saut and D. Smets. On the Korteweg-de Vries long-wave approximation of the Gross-Pitaevskii equation. I. *Int. Math. Res. Not.* **14**, 2700–2748 (2009).

[3] J. Bona, T. Colin and D. Lannes. Long wave approximations for water waves. *Arch. Ration. Mech. Anal.*, **178**, (2005), 373–410.

[4] K. Busch, G. Schneider, L. Tkeshelashvili and H. Uecker. Justification of the Nonlinear Schrödinger equation in spatially periodic media. *ZAMP*, **57**, 1–35 (2006).

[5] D. Chiron and F. Rousset. The KdV/KP-I limit of the nonlinear Schrdinger equation. *SIAM J. Math. Anal.* **42**, 64–96 (2010).

[6] F. Chazel. On the Korteweg-de Vries approximation for uneven bottoms. *Eur. J. Mech. B Fluids* **28**, 234–252 (2009).

[7] W. Craig. An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits. *Comm. PDE*, **10**, 787–1003 (1985).

[8] W.-P. Düll and G. Schneider. Justification of the nonlinear Schrdinger equation for a resonant Boussinesq model. *Indiana Univ. Math. J.* **55**, 1813–1834 (2006).

[9] W.-P. Düll and G. Schneider. Validity of Whitham's equations for the modulation of wavetrains in the NLS equation. *J. Nonlinear Science* **19**, 453–466 (2009).

[10] W.-P. Düll. Validity of the Korteweg-de Vries approximation for the two-dimensional water wave problem in the arc length formulation. *Comm. Pure Appl. Math.* **65**, 381–429 (2012).

[11] W.-P. Düll, G. Schneider and C.E. Wayne. Justification of the Nonlinear Schrödinger equation for the evolution of gravity driven 2D surface water waves in a canal of finite depth. Preprint (2014).

[12] T. Iguchi. A long wave approximation for capillary-gravity waves and an effect of the bottom. *Comm. PDE* **32**, 37–85 (2007).

[13] T. Kano and T. Nishida. A mathematical justification for Korteweg-de Vries equation and Boussinesq equation of water surface waves. *Osaka J. Math.* **23**, 389–413 (1986).

[14] N. Masmoudi and K. Nakanishi. From nonlinear Klein-Gordon equation to a system of coupled nonlinear Schrödinger equations. *Math. Ann.* **324**, 359–389 (2002).

[15] G. Schneider. Limits for the Korteweg-de Vries-approximation. *ZAMM* **76**, Suppl. 2, 341–344 (1996).

[16] G. Schneider, C.E. Wayne. The long-wave limit for the water wave problem. I. The case of zero surface tension. *Comm. Pure Appl. Math.* **53**, 1475–1535 (2000).

[17] G. Schneider and C. Eugene Wayne. The rigorous approximation of long-wavelength capillary-gravity waves. *Arch. Rat. Mech. Anal.* **162**, 247–285 (2002).

[18] G. Schneider. Bounds for the Nonlinear Schrödinger approximation of the Fermi-Pasta-Ulam-system. *Applicable Analysis* **89**, 1523–1540 (2010)

[19] G. Schneider, D.A. Sunny and D. Zimmermann. The NLS approximation makes wrong predictions for the water wave problem in case of small surface tension and spatially periodic boundary conditions. *Journal of Dynamics and Differential Equations*, published online 2014.

[20] N. Totz and S. Wu. A rigorous justification of the modulation approximation to the 2D full water wave problem. *Comm. Math. Phys.* **310**, 817–883 (2012).
[21] N. Totz. A justification of the modulation approximation to the 3D full water wave problem. *Comm. Math. Phys.* **335**, 369–443 (2015).

Two-hump surface waves for water-wave problems

Shu-Ming Sun

The problem concerns two-dimensional surface waves on water of finite depth with surface tension. Let the wave propagate with a constant speed c in the horizontal direction (denoted as x-axis) on an inviscid and incompressible fluid of constant density (usually called water) subject to the forces of gravity and surface tension on the free surface. The fluid is bounded above by a free surface and below by a horizontal rigid bottom. The existence of traveling waves is determined by two non-dimensional constants: the Froude number $F = c/\sqrt{gh}$ (non-dimensional wave speed) and the Bond number $\tau = T/\rho hc^2$ (non-dimensional surface tension), where g is the gravity constant, h is the depth of the fluid at infinity, T is the surface tension coefficient on the free surface, and ρ is the density of the fluid.

When F is near its critical value 1 with usual long-wave assumption, the model equation to describe the motion of the surface wave is the KdV equation,

$$2\eta_t + \lambda\eta_x + 3\eta\eta_x + (1/3 - \tau)\eta_{xxx} = 0$$

where the free surface is approximately determined by $h + h\varepsilon\eta(\varepsilon^{1/2}hx, \varepsilon^{3/2}ct/h)$ with $F^{-2} = 1 + \lambda\varepsilon$ and $\varepsilon > 0$ a small constant. The KdV equation has a traveling-wave solution

$$\eta(x,t) = (2a - \lambda)\mathrm{sech}^2\left(\sqrt{(\tau - 1/3)(\lambda - 2a)}(x - at)/2\right),$$

which is called solitary-wave solution, if the expression in the square root is positive. The validation of the solitary-wave solution was proved using the exact governing equations of the fluid flow with zero surface tension by Lavrentiev, Friedrichs and Hyers, and Beale.

The existence of two-hump waves on water with surface tension has not been studied. Notice that the time independent KdV equation has no two-hump solutions and it was proved by Craig and Sternberg that the exact equations for water waves without surface tension have no two-hump solutions, either. Therefore, it is necessary to include the surface tension in the equations when two-hump solutions are considered. Moreover, it is conceivable that the multi-hump waves exist in the real world (here, we are not talking about the solitons) since two-hump waves can be generated in laboratories if the time gap between producing two equal solitary waves is large enough.

To make the problem simper, we study approximate model equations. For water-wave problems, if τ is near $1/3$, the KdV equation is not valid and the following fifth-order KdV equation can be derived,

$$\eta_t + \lambda\eta_x + \tau_0\eta_{xxx} - (3/2)\eta\eta_x + (1/45)\eta_{xxxxx} = 0$$

where $\tau_0 > 0$ corresponds to $\tau > 1/3$ and $\tau_0 < 0$ for $\tau < 1/3$. For the time-independent case, the equation is

$$\lambda\eta + \tau_0\eta_{xx} - (3/4)\eta^2 + (1/45)\eta_{xxxx} = 0\,.$$

By change of variables, the time-independent equation can be transformed to

$$u'''' + Pu'' \pm u - u^2 = 0$$

where P is a real constant. The case with $+$ sign was studied by Buffoni, Champneys, Groves, Toland and many others. It was shown that for $P < -2$, the equation has a unique solitary-wave solution (up to translation). For $P \in (-2, 2)$, there are single- and multi-hump solutions with decaying oscillatory tails at infinity. Each multi-hump solution is formed approximately by patching several copies of the single-hump solution together. Intuitively, existence of such multi-hump solutions depends on the oscillatory behavior of the single-hump solution at infinity.

For the equation with $-$ sign, we only consider the case

$$\varepsilon u'''' + u'' - u - u^2 = 0$$

where $\varepsilon > 0$ is small (or the fifth-order KdV equation, i.e., if $\varepsilon = 0$, it becomes the KdV equation). This equation has been studied extensively in the past. It can be shown that there are no solutions decaying to zero at infinity (proved by Amick and McLeod, and many others). Also, it was proved that for small $\varepsilon > 0$ the equation has solitary-wave solutions with small non-decaying oscillations at infinity (called generalized solitary-wave solutions). The amplitude of the oscillations at infinity can be either algebraically small (i.e., of order ε^n) or exponentially small (i.e., of order $e^{-\delta\varepsilon^{-1/2}}$). Note that since these generalized solitary-wave solutions have oscillations at infinity, it is very reasonable to expect that multi-hump solutions with oscillations at infinity exist for the equation.

When $\tau < 1/3$ and F near 1, similar to the derivation of the KdV equation, it can be obtained that up to the order ε, the approximate time-independent model equation is

$$\lambda\eta - (3/4)\eta^2 - (1/2)\Big((1/3) - \tau + (\varepsilon/2)\tau\lambda\Big)\eta_{xx} + (\varepsilon/24)\Big(3\eta^3 \\ - (10 - 15\tau)\eta\eta_{xx} - ((13 + 3\tau)/2)\eta_x^2 - ((19/15) - 3\tau)\eta_{xxxx}\Big) = 0\,.$$

Note that the coefficients of η_{xx} and η_{xxxx} have negative sign and to have solitary-wave solution for $\varepsilon = 0$, it is required that $\lambda > 0$.

To prove the existence of generalized solitary-wave solutions, consider a more general equation

$$\varepsilon\eta_{xxxx} + \eta_{xx} - \lambda\eta + \beta\eta^k = \varepsilon\mathcal{P}\left[\varepsilon, \eta, \eta_x, \eta_{xx}\right], \tag{1}$$

where $\varepsilon > 0$ is a small parameter, $\mathcal{P}$ is a polynomial with respect to its variables for η and its derivatives satisfying that

$$\mathcal{P}[\varepsilon, \alpha_1, \alpha_2, \alpha_3] = \mathcal{P}[\varepsilon, \alpha_1, -\alpha_2, \alpha_3] \quad \text{with} \quad \mathcal{P}[\varepsilon, 0, 0, 0] = 0\,,$$

and $\mathcal{P}$ is at most linear in α_3 and quadratic in α_2 (an example of such an equation is given above for water-wave problem). For $\varepsilon = 0$, if $k \geq 2, \lambda > 0$ ($\beta > 0$ if k is odd), there is a solitary-wave solution

$$S(x) = \Big((k+1)\lambda/2\beta\Big)^{1/(k-1)} \operatorname{sech}^{2/(k-1)}\Big((k-1)\lambda^{1/2}x/2\Big).$$

Then, for $\varepsilon > 0$, the following theorem holds [1].

Theorem 1. *Let $\sigma(x)$ be even and $C^\infty(R)$ with $\sigma(x) = 0$ for $|x| \leq 1/2$ and $\sigma(x) = 1$ for $|x| \geq 1$ and $\hat{r}_2 = \big((\sqrt{1+4\lambda\varepsilon}+1)/(2\varepsilon)\big)^{1/2}$. For small $\varepsilon > 0$ and $A = a\varepsilon^{(n+2)/2}$ with an integer $n \geq 2$ and a fixed constant $a \neq 0$, there are two constants γ and δ and two functions $R_0(x)$ and $R^+(x)$ such that*

$$\eta(x) = S(x) + \varepsilon R_0(x) + A\sigma(x)\Big(\cos(\hat{r}_2 x\sqrt{1-\gamma\varepsilon} - \delta) + AR^+(\hat{r}_2 x\sqrt{1-\gamma\varepsilon} - \delta)\Big)$$

for $x \geq 0$ (evenly extended to $x \leq 0$) is a solution of equation (1), where $|\gamma| \leq C|A|$, $\delta = c_0\varepsilon^{1/2} + O(\varepsilon)$ for some constant c_0, and $R_0(x)$ and $R^+(x)$ are bounded, even, and differentiable functions in x satisfying $R_0(x) = O(\exp(-d|x|))$ and $|R^+(x)| \leq C$ as $x \to +\infty$ with some small constant $d > 0$. $R^+(x)$ is periodic with period 2π and C is a generic constant independent of ε.

The existence result of two-hump solutions is the following [1].

Theorem 2. *Let $k = 2$ and $\mathcal{P} = 0$ in (1). For small $\varepsilon > 0$ and $A^+ = a\varepsilon^{(n+2)/2}$ with an integer $n \geq 2$ and a fixed constant $a \neq 0$, there exist two constants $x_0 > 0$ and A^- with $|x_0| \sim C\ln(1/\varepsilon)$ and $|A^- - A^+| \leq C\varepsilon^{2n+1}$ such that the equation (1) has an even solution $\eta(x)$ respect to $x = -x_0$ satisfying that for $x \geq 0$, $\eta(x) = w(x)$, where*

$$\begin{aligned} w(x) =& S(x) + \varepsilon R_0(x) + A^+\sigma(x)\Big(\cos(\hat{r}_2 x\sqrt{1-\gamma\varepsilon} - \delta) \\ &+ A^+R^+(\hat{r}_2 x\sqrt{1-\gamma\varepsilon} - \delta)\Big) = w_0(x; A^+, \delta) \end{aligned}$$

for $x \geq 0$ obtained similarly as the solution in Theorem 1 and $\eta(x) = w_0(x; A^-, -\delta)$ for $-x_0 \leq x \leq 0$. Thus, the equation has a two-hump solution with small oscillations at infinity and two peaks at $x = 0, -2x_0$.

A similar existence result of two-hump solutions can be proved for the stationary Swift-Hohenberg equation

$$cw - (\partial_x^2 + k_0^2)^2 w - w^3 = 0,$$

where $c > 0$ and $k_0^2 > 0$ are parameters satisfying the relationship, $k_0^2 = \sqrt{c} - \varepsilon$ with $\varepsilon > 0$ a small parameter [2].

References

[1] J.W. Choi, D.S. Lee, S.H. Oh, S.-M. Sun and S.I. Whang. Multi-hump solutions of some singularly-perturbed equations of KdV type. *Discrete Contin. Dyn. Syst. Ser. A* **34**, 5181–5209 (2014).

[2] S. Deng and S.-M. Sun. Multi-hump solutions with small oscillations at infinity for stationary Swift-Hohenberg equation. Preprint (2015).

Singularities of steady free surface water flows under gravity

Eugen Varvaruca

We present our recent results [4], [5], [6], which provide a characterization, by means of geometric analysis methods, of all possible singularities of the free surface in two related free-boundary problems concerning steady water flows under gravity: that of two-dimensional travelling waves, and that of three-dimensional axisymmetric flows without swirl. In the two-dimensional problem, which had been extensively studied, we have given a new proof of the well known *Stokes conjecture*, that at a stagnation point the free surface necessarily has lateral tangents enclosing a symmetric angle of 120° and the velocity field has an asymptotic profile described by the so-called *Stokes corner flow*. In this talk we concentrate on describing the new results we have obtained for the three-dimensional axisymmetric problem, which had been much less studied previously.

We consider a steady axisymmetric solution of the Euler equations describing the irrotational flow without swirl of an incompressible inviscid fluid acted on by gravity and with a free surface. Using cylindrical coordinates and the Stokes stream function Ψ to describe the flow, we obtain in a subset of the half-plane $\{(x_1, x_2) : x_1 \geq 0\}$ the free-boundary problem

$$\begin{aligned} \operatorname{div}\left(\frac{1}{x_1}\nabla\Psi(x_1,x_2)\right) &= 0 \quad \text{in the water phase } \{\Psi > 0\}, \\ \frac{1}{x_1^2}|\nabla\Psi(x_1,x_2)|^2 &= -x_2 \quad \text{on the free surface } \partial\{\Psi > 0\}, \end{aligned}$$

where the gravitational constant g has been normalized by scaling. For such a flow, the velocity field in three-dimensional space is given by

$$V(X,Y,Z) = \left(-\frac{1}{x_1}\partial_2\Psi\cos\varphi, -\frac{1}{x_1}\partial_2\Psi\sin\varphi, \frac{1}{x_1}\partial_1\Psi\right),$$

where $(X,Y,Z) = (x_1\cos\varphi, x_1\sin\varphi, x_2)$. (Note that the equations above describe, apart from a steady flow, also the case of a traveling wave moving at constant speed along the direction of the axis of symmetry, in which situation the flow is steady with respect to the moving frame.)

In [1] Garabedian gave an example of an explicit solution of the above problem in which Ψ is a homogeneous function of degree 5/2, while the water domain is above the air domain and occupies, in the half-plane $\{(x_1, x_2) : x_1 \geq 0\}$, a cone with vertex at the origin and of opening angle approximately 114.799°; this explicit solution has been reexamined in [2]. In [3] it has been suggested, by means of formal asymptotic expansions supported by numerical computations, that at a point on $\{x_2 = 0\}$ different from the origin, the free boundary has lateral tangents making an angle of 120° while the velocity field is described asymptotically by the Stokes corner flow.

Here we focus on precisely the degenerate sets $\{x_1 = 0\}$ (the axis of symmetry) and $\{x_2 = 0\}$ (containing all stagnation points) and rigorously analyze the possible profiles of the velocity field and of the free boundary close to points in those sets. The degeneracy of the free-boundary condition

$$|\nabla\Psi(x_1, x_2)|^2 = -x_1^2 x_2$$

at points $x^0 = (x_1^0, x_2^0)$ with $x_1^0 x_2^0 = 0$, enables us to identify *four* (almost) invariant scalings of the problem,

$$\frac{\Psi(x^0 + rx)}{r} \quad \text{in the case } x_1^0 \neq 0 \text{ and } x_2^0 \neq 0,$$
$$\frac{\Psi(x^0 + rx)}{r^{3/2}} \quad \text{in the case } x_1^0 \neq 0 \text{ and } x_2^0 = 0,$$
$$\frac{\Psi(x^0 + rx)}{r^2} \quad \text{in the case } x_1^0 = 0 \text{ and } x_2^0 \neq 0,$$
$$\frac{\Psi(x^0 + rx)}{r^{5/2}} \quad \text{in the case } x_1^0 = x_2^0 = 0,$$

which suggest the appropriate form of the *blow-up sequences* to be considered.

We first determine the asymptotic profile of the rescaled solutions as $r \to 0$. In the case $x_1^0 \neq 0$ and $x_2^0 \neq 0$ the only nontrivial asymptotics possible is constant velocity flow parallel to the free surface. In the case $x_1^0 \neq 0$ and $x_2^0 = 0$ the only nontrivial asymptotics possible is the Stokes corner flow. Due to the perturbed nature of the equation, the situation is actually not unlike the two-dimensional steady free-surface water-wave problem in the presence of vorticity, see [5]. In the case $x_1^0 = 0$ and $x_2^0 \neq 0$ the only nontrivial asymptotics possible is constant velocity flow in the gravity direction. This suggests the possibility of *air cusps* pointing in the gravity direction. In the case $x_1^0 = x_2^0 = 0$ the only nontrivial asymptotics possible is the *Garabedian pointed bubble* solution with water above air. This means that there is no nontrivial asymptotic profile at all with air above water and with the invariant scaling. However there remains at this stage the possibility that the solution has a higher growth than that suggested by the invariant scaling. The key ingredient in the above analysis is a new *monotonicity formula*, which implies that the blow-up limits are homogeneous functions.

We then analyze the possible shapes of the free surface close to stagnation points and close to points on the axis of symmetry. Assuming that the free surface is given by an injective curve and assuming also a strong Bernstein inequality (corresponding to a Rayleigh-Taylor condition) we obtain the following result:

In the case $x_1^0 \neq 0$ and $x_2^0 = 0$ the only asymptotics possible are the well-known *Stokes corner* (an angle of opening 120° in the direction of the axis of symmetry), and a *horizontal point*.

In the case $x_1^0 = 0$ and $x_2^0 \neq 0$ the only asymptotics possible are *cusps* in the direction of the axis of symmetry.

In the case $x_1^0 = x_2^0 = 0$ the only asymptotics possible are the *Garabedian pointed bubble* asymptotics, and a *horizontal point*.

A finer analysis of the velocity profile in the last case ($x_1^0 = x_2^0 = 0$ and a horizontal point) is then given in the case of air above water. We prove that the velocity field scales almost like $\sqrt{X^2+Y^2+Z^2}$ and is asymptotically given by

$$V(X,Y,Z) = c_0(X,Y,-2Z),$$

where c_0 is a nonzero constant. The main tool is a new nonlinear *frequency formula* (which is not merely a perturbation of Almgren's frequency formula), valid at each point of *highest density*. In combination with a concentration compactness result for the axially symmetric Euler equations by Delort, this leads to the already mentioned profile for the velocity field. Note that while the concentration compactness result alone does *not* lead to strong convergence in general, we prove the convergence to the limiting velocity field to be strong in our application.

References

[1] P. R. Garabedian. A remark about pointed bubbles. *Comm. Pure Appl. Math.* **38**, 609–612 (1985).

[2] P. Milewski, J.-M. Vanden-Broeck and J.B. Keller. Singularities on free surfaces of fluid flows. *Stud. Appl. Math.* **100**, 245–267 (1998)

[3] J.-M. Vanden-Broeck and J.B. Keller. An axisymmetric free surface with a 120 degree angle along a circle. *J. Fluid Mech.* **342**, 403–409 (1997).

[4] E. Varvaruca and G.S. Weiss. A geometric approach to generalized Stokes conjectures. *Acta Math.* **206**, 363–403 (2011).

[5] E. Varvaruca and G.S. Weiss. The Stokes conjecture for waves with vorticity. *Ann. Inst. H. Poincar Anal. Non Linaire* **29**, 861–885 (2012).

[6] E. Varvaruca and G.S. Weiss. Singularities of steady axisymmetric free surface flows with gravity. *Comm. Pure Appl. Math.* **67**, 1263–1306 (2014).

Continuous dependence on the density for stratified steady water waves

Samuel Walsh

An important property of ocean waves is *stratification* — a heterogeneity in the density distribution caused by salinity in the water or heating from the sun. Even small amounts of stratification may have profound effects on the dynamics, e.g., giving rise to so-called internal waves that can propagate for hundreds of kilometers beneath the surface.

We are interested specifically in traveling waves, which means that by shifting to a moving reference, we can eliminate time dependence from the system. Say then that the water occupies a domain

$$\Omega = \{(x,y) \in \mathbb{R}^2 : -d < y < \eta(x)\}.$$

Here η is (an a priori unknown) free surface profile. We are considering the two-dimensional case and coordinates have been fixed so that the wave propagates in the positive x-direction with speed $c > 0$ and the ocean depth is $d > 0$. The flow is described mathematically by a velocity field $(u,v) : \Omega \to \mathbb{R}^2$, a pressure $P : \Omega \to \mathbb{R}$, and a density $\varrho : \Omega \to \mathbb{R}_+$. Assume that the wave is periodic, i.e., u,

v, P, ϱ, and η are periodic in x. For (u, v, ϱ, P, η) to represent a water wave, they must satisfy the free boundary steady Euler equations.

In the actual ocean, the waves one observes are usually structured as large regions of nearly constant density separated by thin transition layers where the density may vary sharply. A very common practice among applied scientists is to simply collapse these transition layers to material lines, i.e., to imagine these waves as consisting of two or more immiscible layers. The density in each layer is assumed to be smooth (say, constant) and a jump discontinuity is permitted over the interfaces.

This distinguishes two separate regimes: a wave is said to be *continuously stratified* provided that ϱ is continuous throughout the entire fluid domain, and *layer-wise smooth* if Ω can be partitioned into finitely many immiscible fluid regions

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega_i},$$

where each $\Omega_i \subset \Omega$ is an open set with smooth boundary, and the restriction $\varrho|_{\Omega_i}$ is smooth. Note that in both regimes, the governing equations are the steady free surface Euler problem. For layer-wise smooth waves, however, the solutions are weak and the internal interfaces $\partial\Omega_i \cap \partial\Omega_{i+1}$ are additional free boundaries that must be determined.

The goal of this work is to quantify the degree to which a continuously stratified water wave can be approximated by a layer-wise smooth wave. Our main result states that, in a certain small-amplitude regime, the wave depends *continuously* on the stratification. That is, if one fixes a continuously stratified wave of this type, there exists nearby many-layered traveling waves that converge to the smooth wave as the number of layers is taken to infinity. In fact, these layer-wise smooth waves are parameterized by the density in a Lipschitz continuous fashion.

To make sense of this, recall that the steady Euler system can be recast in terms of the (pseudo) relative stream function, which is a scalar function $\psi : \Omega \to \mathbb{R}$ satisfying

$$\partial_x \psi = -\sqrt{\varrho} v, \qquad \partial_y \psi = \sqrt{\varrho}(u - c) \qquad \text{in } \Omega. \tag{1}$$

The level sets of ψ, called the *streamlines*, are the integral curves of the relative velocity field. Note that these include the ocean bed and all of the free surfaces in the system. For the unidirectional waves that we consider, the streamlines must be simple curves extending from $x = -\infty$ to $x = +\infty$.

By the conservation of mass, the density ϱ is in fact constant along streamlines. That is, there exists some *streamline density function* ρ such that

$$\varrho(x, y) = \rho(-\psi(x, y)), \qquad \text{in } \Omega.$$

We view ρ as an infinite-dimensonal parameter for the problem; the continuity result we obtain is a continuous dependence of the solutions on ρ.

Informal statement of results. Fix a Hölder exponent $\alpha \in (0, 1)$, and put $r := 2/(1 - \alpha)$. Let $\rho_* \in C^{1,\alpha}$ be a stably stratified streamline density function, and suppose that $(u_*, v_*, \varrho_*, P_*, \eta_*)$ is a traveling wave with streamline density function ρ_*. Assume further that (i) its Bernoulli function is the same as for a solitary wave with uniform velocity in the far field, (ii) its wave speed c_* is supercritical, (iii) it is a wave of strict elevation, and (iv) it is sufficiently small-amplitude. Then the following statements hold.

(A1) *Existence of nearby many-layered solutions.* There is a neighborhood $\mathcal{U}$ of ρ_* in $L^\infty([-p_0, 0])$ such that, for any $\rho \in \mathcal{U}$ that is non-increasing and piecewise smooth, there exists a solution (u, v, ϱ, P, η) to the steady Euler equations with streamline density function ρ, period L, and wave speed c. Moreover, u and η are even in x, while v is odd in x.

Observe that since $\mathcal{U}$ is an open set in L^∞, it contains streamline densities functions with arbitrarily many jump discontinuities.

(A2) *Convergence of the height function and wave speed.* For each $(q,p) \in \mathcal{R} := \mathbb{R} \times [p_0, 0]$, let $h_*(q,p)$ denote the height above the bed $\{y = -d\}$ of the point with x-coordinate q that lies on the streamline $\{\psi_* = -p\}$ for the wave with velocity field (u_*, v_*); let h designate the corresponding height for the wave furnished by statement (A1). Then

$$h_* = h + \mathcal{O}(\|\rho - \rho_*\|_{L^\infty}) \qquad \text{in } W^{1,r}_{\text{per}}(\mathcal{R}) \subset C^{0,\alpha}_{\text{per}}(\overline{\mathcal{R}}).$$

Likewise, the wave speed c satisfies

$$c_* = c + \mathcal{O}(\|\rho - \rho_*\|_{L^\infty}).$$

In fact, (A1) and (A2) are a form of continuity result. Let $\mathscr{D}$ denote the set of bounded, layer-wise smooth, stable streamline density functions; $\mathscr{D}$ can be viewed as a convex subset of $L^\infty([p_0, 0])$. Then (A1) proves that there exists a mapping $\rho \in \mathscr{D} \cap \mathcal{U} \mapsto h \in W^{1,r}_{\text{per}}(\mathcal{R})$, and (A2) follows from the fact that this mapping is Lipschitz continuous.

Away from the internal interfaces, the solutions enjoy improved regularity:

(B) *Improved regularity.* Let $I \subset\subset [p_0, 0] \setminus \{p_1, \ldots, p_{N-1}\}$ be a connected set for which $\rho \in C^{1,\alpha}(\overline{I})$. Then

$$\|h - h_*\|_{C^{1,\alpha}_{\text{per}}(\mathbb{R}\times\overline{I})} \leq C_1 \left(\|\rho - \rho_*\|_{L^\infty([p_0,0])} + \|\rho - \rho_*\|_{C^{1,\alpha}(\overline{I})} \right),$$

where $C_1 > 0$ depends on the length of I, ρ_*, and h_*.

In general, C_1 will increase as the length of I decreases. One consequence of (B) is that, if ρ_* is constant in some region, then the approximation by a layer-wise constant density stratification converges in a higher regularity norm there.

Lastly, we prove a result on the convergence of the pressure. This is specifically aimed at the surface reconstruction problem (cf. [2]).

(C) *Convergence of the pressure.* Let a connected set $I \subset\subset [p_0, 0] \setminus \{p_1, \ldots, p_{N-1}\}$ be given with $p_0 \in I$, and assume that $\rho \in C^{1,\alpha}(\overline{I})$. Denote by P_{b} the trace of the pressure on the ocean bed for the traveling wave with density ρ, and let $P_{\text{b}*}$ be the trace of P_* on the bed. Then

$$\|P_{\text{b}} - P_{\text{b}*}\|_{C^{0,\alpha}_{\text{per}}(\mathbb{R})} \leq C_2 \left(\|\rho - \rho_*\|_{L^\infty([p_0,0])} + \|\rho - \rho_*\|_{C^{1,\alpha}(\overline{I})} \right),$$

where $C_2 > 0$ depends on the length of I, ρ_*, and h_*.

The proof is based on an implicit function argument, supplemented with a (fairly intricate) penalization scheme in the spirit of Turner [3]. For more details, see [1].

References

[1] R.M. Chen and S. Walsh. Continuous dependence on the density for stratified steady water waves. Preprint, arXiv:1408.5030.

[2] R.M. Chen and S. Walsh. Reconstruction of stratified steady water waves from pressure readings. Preprint, arXiv:1502.07775.

[3] R.E.L. Turner. A variational approach to surface solitary waves. *J. Diff. Eqns* **55**, 401–438 (1984).

Bounds on the slope of steady water waves with vorticity

Miles H. Wheeler

We consider the angle θ between the free surface of a steady two-dimensional water wave and the horizontal. The famous "extreme Stokes wave" has a corner at its crest with $\theta = 30°$, but McLeod [4] proved that nearly extreme waves in fact have larger angles $\theta > 30°$ at points near their (rounded) crests. Earlier numerical work suggests that the largest possible angle is 30.37° [5, 3, 2], while Amick [1] proved the remarkable upper bound $\theta < 31.15°$ for all steady irrotational waves.

In the presence of *adverse* vorticity (positive vorticity in our formulation), Amick's bound fails dramatically: there are explicit Gerstner waves with 90° cusps at their crests, and there is also numerical evidence for the existence of steady waves with overturning free surfaces [7]. In this talk, we instead focus on a large class of waves with *favorable* vorticity (for instance constant negative vorticity), for which we prove the upper bound $\theta < 45°$ [6].

Working in a frame moving with the wave, we denote the velocity by (u, v), the vorticity by ω, and the stream function by ψ. We assume that $u < 0$. This rules out stagnation points where $u = v = 0$, and also guarantees that the vorticity ω is a function of the stream function ψ alone. An informal version of our main result is as follows, where by a streamline we mean a level curve of ψ and by a trivial wave we mean a shear flow with a flat free surface.

Theorem 1. *Let $\mathscr{C}$ be a connected set (containing a trivial wave) of symmetric periodic finite-depth water waves with a single crest and trough per period for which $u < 0$ (non-stagnation) and for which the streamlines are strictly decreasing from crest to trough. We assume that the vorticity ω satisfies*

$$\omega \leq 0, \quad \frac{d\omega}{d\psi} \leq 0, \quad \frac{d^2\omega}{d\psi^2} \leq 0.$$

At least until $u\omega = g$ at the troughs, the waves in $\mathscr{C}$ that bifurcate from a trivial wave have angle strictly less than 45°. (The trivial wave has $u\omega < g$ everywhere.)

The same statement is true for symmetric solitary waves (instead of periodic waves) at least until $\lim_{x\to\infty}(u\omega)(x, \eta(x)) = g$.

Our argument is somewhat related to Amick's proof in [1] of the weaker bound $\theta < 38.2^\circ$. This proof hinges on the function

$$f_\alpha = \mathrm{Re}[(-u+iv)^\alpha] = [u^2+v^2]^{\alpha/2}\cos(\alpha\theta),$$

together with its derivative along streamlines,

$$W_\alpha = \frac{u\partial_x + v\partial_y}{u^2+v^2} f_\alpha.$$

For a given $\alpha \geq 1$, if W_α has the appropriate sign in each half-period, then an easy argument shows that $f_\alpha > 0$ and hence $|\theta| < \pi/\alpha$. Because the flow is irrotational, both f_α and W_α are the real parts of analytic functions of $z = x+iy$ and hence harmonic. By combining maximum principle arguments for W_α with a continuation in α, Amick eventually shows that $|\theta| < \pi/\alpha$ for $1 \leq \alpha \leq \beta$ where $\beta = (9+\sqrt{97})/8 > 38.2^\circ$.

There are many difficulties with generalizing even this first part of Amick's argument to include vorticity. The complex analysis methods and Nekrasov formulation that he relies on are no longer available, and, more seriously, the function W_α is no longer harmonic. Indeed, for general α, it does not seem possible to apply maximum principle arguments to W_α. Nevertheless, we are able to make some progress in the special case $\alpha = 2$ where formulas are simpler. Under our assumptions on ω, we obtain several inequalities by applying the Hopf lemma to the quotients u_x/u and v/u. We then replace Amick's continuation in α with a continuation along the connected set of solutions $\mathscr{C}$, and in this final step the quantity $g - u\omega$ plays an important role.

Finally, we return to the overhanging waves mentioned at the start of the talk, and show that, in the absence of stagnation points on the free surface, every overturning wave (periodic or solitary) must have a pressure sink.

References

[1] C.J. Amick. Bounds for water waves. *Arch. Rational Mech. Anal.* **99**, 91–114 (1987).

[2] B. Chen and P.G. Saffman. Numerical evidence for the existence of new types of gravity waves of permanent form on deep water. *Stud. Appl. Math.* **62**,1–21 (1980).

[3] M.S. Longuet-Higgins and M.J.H. Fox. Theory of the almost-highest wave: the inner solution. *J. Fluid Mech.* **80**, 721–741 (1977).

[4] J.B. McLeod. The Stokes and Krasovskii conjectures for the wave of greatest height. *Stud. Appl. Math.* **98**, 311–333 (1997). (In preprint form: Univ. of Wisconsin MRC Report no. 2041, 1979.).

[5] K. Sasaki and T. Murakami. Irrotational, progressive surface gravity waves near the limiting height. *Journal of the Oceanographical Society of Japan* **29**, 94–105 (1973).

[6] W.A. Strauss and M.H. Wheeler. Bound on the slope of steady water waves with favorable vorticity. Submitted.

[7] A.F. Teles da Silva and D.H. Peregrine. Steep, steady surface waves on water of finite depth with constant vorticity. *J. Fluid Mech.* **195**, 281–302 (1988).

Reporter: Mark D. Groves

Participants

Prof. Dr. Jerry L. Bona
Dept. of Mathematics, Statistics
and Computer Science, M/C 249
University of Illinois at Chicago
851 S. Morgan Street
Chicago, IL 60607-7045
UNITED STATES

Prof. Dr. Thomas J. Bridges
Department of Mathematics
University of Surrey
Guildford
Surrey GU2 7XH
UNITED KINGDOM

Dr. Boris Buffoni
Section de Mathématiques
Station 8
École Polytechnique Fédérale de
Lausanne
1015 Lausanne
SWITZERLAND

Dr. Angel Castro
Departamento de Matematicas de la
UAM
Instituto de Ciencias Matematicas CSIC
Campus de Cantoblanco
28049 Madrid
SPAIN

Wolf-Patrick Düll
Institut für Analysis, Dynamik und
Modellierung, Fak. für Math. & Physik
Universität Stuttgart
70569 Stuttgart
GERMANY

Dr. Mats Ehrnstrom
Department of Mathematical Sciences
NTNU Trondheim
Alfred Getz vei 1
7491 Trondheim
NORWAY

Cyril Godey
Faculté des Sciences et Techniques
Laboratoire Mathématiques de Besancon
Université de Franche-Comte
16, route de Gray
25030 Besancon Cedex
FRANCE

Prof. Dr. Mark D. Groves
Fachrichtung Mathematik
Universität des Saarlandes
Postfach 151150
66041 Saarbrücken
GERMANY

Prof. Dr. Mariana Haragus
Laboratoire de Mathématiques
Université de Franche-Comté
16, route de Gray
25030 Besancon Cedex
FRANCE

Prof. Dr. Vera Mikyoung Hur
Dept. of Mathematics, University of
Illinois at Urbana Champaign
273 Altgeld Hall
1409 West Green Street
Urbana, IL 61801
UNITED STATES

Prof. Dr. Tatsuo Iguchi
Department of Mathematics
Keio University
14-1 Hiyoshi, 3 Chome
Yokohama 223
JAPAN

Prof. Dr. Henrik Kalisch
Institute of Mathematics
University of Bergen
Allegt 53 - 55
P.O. Box 7800
5020 Bergen
NORWAY

Florian Kogelbauer
Fakultät für Mathematik
Universität Wien
Oskar-Morgenstern-Platz 1
1090 Wien
AUSTRIA

Dr. David Lannes
Dept. de Mathématiques et Applications
École Normale Superieure
45, rue d'Ulm
75005 Paris Cedex
FRANCE

Evgeniy Lokharu
Department of Mathematics
Linköping University
581 83 Linköping
SWEDEN

Dag Nilsson
Centre for Mathematical Sciences
Lund University
Box 118
221 00 Lund
SWEDEN

Prof. Dr. Robert L. Pego
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213-3890
UNITED STATES

Dr. Frédéric Rousset
Laboratoire de Mathématiques
Université Paris Sud (Paris XI)
Batiment 425
91405 Orsay Cedex
FRANCE

Prof. Dr. Guido Schneider
Institut f. Analysis, Dynamik & Modellierung
Fakultät für Mathematik u. Physik
Universität Stuttgart
Pfaffenwaldring 57
70569 Stuttgart
GERMANY

Prof. Dr. Shu-Ming Sun
Department of Mathematics
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061-0123
UNITED STATES

Dr. Eugen Varvaruca
Department of Mathematics
University of Reading
Whiteknights
P.O.Box 220
Reading RG6 6AX
UNITED KINGDOM

Prof. Dr. Erik Wahlén
Centre for Mathematical Sciences
University of Lund
Box 118
221 00 Lund
SWEDEN

Dr. Samuel Walsh
Department of Mathematics
University of Missouri
307 Math. Sciences Bldg.
Columbia, MO 65203
UNITED STATES

Dr. Miles Wheeler
Courant Institute of
Mathematical Sciences
New York University
251, Mercer Street
New York, NY 10012-1110
UNITED STATES