Mathematisches Forschungsinstitut Oberwolfach

Report No. 18/2015

DOI: 10.4171/OWR/2015/18

Representations of Finite Groups

Organised by
Joseph Chuang, London
Meinolf Geck, Stuttgart
Markus Linckelmann, London
Gabriel Navarro, Valencia

5 April – 11 April 2015

Abstract. The workshop *Representations of Finite Groups* was organised by Joseph Chuang (London), Meinolf Geck (Stuttgart), Markus Linckelmann (London), and Gabriel Navarro (Valencia). It covered a wide variety of aspects of the representation theory of finite groups and related objects, such as algebraic groups.

Mathematics Subject Classification (2010): 20Cxx, 20Gxx, 20Jxx.

Introduction by the Organisers

The workshop *Representations of Finite Groups* was organised by Joseph Chuang (London), Meinolf Geck (Stuttgart), Markus Linckelmann (London), and Gabriel Navarro (Valencia). It was attended by 52 participants with broad geographic representation. It covered a wide variety of aspects of the representation theory of finite groups and related objects, notably algebraic groups.

In 25 lectures, either 30 or 50 minutes long, recent progress in the representation theory of finite groups was presented and interesting new research directions were proposed. Besides the lectures, there was plenty of time for informal discussion between the participants, either continuing ongoing research cooperation or starting new projects.

The representation theory of finite groups seems to have entered an exciting stage where some parts of the fundamental conjectures which have been open for decades and which drive this area in a major way are being proved. In the previous meeting at the MFO (2012), the proof of one implication of Brauer's height zero conjecture from 1955 was announced by Kessar and Malle. In the present meeting, a proof of the case $p = 2$ of McKay's conjecture from the early 1970s was announced by Malle and Späth. Both cases rely on the classification of finite simple groups and the Clifford theory developed in recent years by Malle, Navarro, Späth, Tiep, amongst others. These results point on the one hand to the need to develop general methods in block theory, and on the other hand to the interaction with the representation theory of algebraic groups, which has a prominent place at this meeting, as much for its own sake as for the study of finite groups of Lie type, part of any approach using the classification of finite simple groups.

A highlight of the conference was Malle's talk, announcing the proof for $p = 2$ of the McKay conjecture, stating that if S is a Sylow 2-subgroup of a finite group G, then G and $N_G(S)$ have the same number of irreducible characters of odd degree. Tiep described a number of results with a focus on the MacKay conjecture at odd primes, and Schaeffer Fry considered consequences of refinements of the McKay conjecture. Lower bounds on the numbers of characters of degree prime to a given prime number p appeared as well in Maroti's talk.

The conference opened with a talk by Srinivasan on a conjecture of Dudas and Malle, describing the unipotent part of a projective character of the general linear group in nondefining characteristic in terms of Kazhdan-Lusztig polynomials associated with a Hecke algebra of type A. Another side of the same representation theory was explored by Dipper, describing a relationship with supercharacters of the subgroup of lower unitriangular matrices. Keeping to the theme of algebraic groups, and more specifically representations in nondefining characteristic, progress for finite unitary groups was reported in talks by Hiss and by Dudas, including new results on Harish-Chandra series, branching graphs and Broué's conjecture. Taylor spoke on connections between the representation theory of a finite group of Lie type and the geometry of unipotent classes of the ambient algebraic group, with applications to the construction of projective characters. Finally, Williamson presented a conjectural combinatorial model for the category of tilting modules over a connected reductive algebraic group.

The talks of Park and of Puig had as main focus cohomological aspects of abstract fusion systems, with implications for block theory via the fusion systems associated with block algebras. Closely related cohomological methods appeared in Stancu's talk on the description of simple biset functors.

Some of the most fundamental invariants of finite symmetric groups, alternating groups, and their double covers are unknown to this date - and thus, not surprisingly, these groups were the main players in a number of talks. Livesey showed that Broué's perfect isometry conjecture holds for the aforementioned double covers. Evseev considered generalised Cartan invariants of symmetric groups, and Giannelli investigated the vertices of Specht modules and simple modules for symmetric groups. Bessenrodt extended results of Tong-Viet on symmetric and alternating groups to their double covers, showing that these are determined by their character degree sets. The relation of character degrees and group orders was further developed in the talk by Tong-Viet.

Several talks had a general block theoretic focus. This includes Kessar's talk on conjugacy classes of p-subgroups canonically associated with irreducible characters, Boltje's talk on p-permutation equivalences, and talks by Mazza and Koshitani on endotrivial modules. The relevance of the latter stems from the fact that endotrivial modules determine a significant part of the Picard group of the stable category of a finite group. The stable category - one of the difficult, and at present insufficiently understood triangulated categories associated with blocks - was also the main player in Benson's talk, investigating analogous concepts for finite group schemes. Murray developed the theory of symmetric vertices of symmetric modules, and, well off the beaten tracks, Bouc and Thévenaz explored uncharted territory in the representation theory of finite sets.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Pham Huu Tiep in the "Simons Visiting Professors" program at the MFO.

Workshop: Representations of Finite Groups

Table of Contents

Abstracts

On a conjecture of Dudas and Malle

Bhama Srinivasan

We study the ℓ-modular characters of $G_n = GL(n,q)$ where ℓ is a prime not dividing q. The unipotent characters and the Brauer characters of G_n are indexed by $\mathcal{P}_n$, the set of all partitions of n. Let χ_μ and ϕ_μ be the unipotent character and the Brauer character of G_n indexed by $\mu \in \mathcal{P}_n$. Let $\mathcal{P}$ be the set of all partitions of $n, n \geq 0$.

A conjecture of Dudas and Malle (C.R. Acad. Sci. Paris (2014)) is stated as follows.

Let $Q_w = \sum_\psi \psi(C_w) R_\psi$. Then Q_w is the unipotent part of a projective character.

Here the ψ are characters of a Hecke algebra corresponding to W of type A, the $C_w, w \in W$ are the basis of this algebra defined by Kazhdan-Lusztig, and the R_ψ are "almost characters" of G_n, which are in fact unipotent characters in this case.

We refer to the paper [2] for the theorem below.

A Fock space $\mathcal{F}$ is a vector space over $\mathbb{C}$ with a standard basis $\{\lambda >\}$ indexed by $\mathcal{P}$. It also has two canonical bases $G^+(\lambda)$ and $G^-(\lambda)$, $\lambda \in \mathcal{P}$. The ℓ-decomposition numbers of G_n are described in terms of the bases as follows.

Theorem. Let $\lambda \in \mathcal{P}_n$. Then, for large ℓ, $(\chi_\mu, \phi_\lambda) = (G^-(\lambda), |\mu >)$, where $(G^-(\lambda), |\mu >)$ is the coefficient of $|\mu >)$ in the expansion of the canonical basis of $\mathcal{F}$ in terms of the standard basis.

Leclerc and Thibon have shown that these coefficients can be expressed as Kazhdan-Lusztig polynomials. More precisely, we get the following Kazhdan-Lusztig polynomials. In [1, 5.12] it is stated that $(G^-(\lambda), |\mu) = \Sigma_\mu \ell_{\lambda\mu}(-q^{=1})|\mu >$, and $\ell_{\lambda\mu} = P^-_{\mu,\lambda}$, a parabolic Kazhdan-Lusztig polynomial. From the theorem we have expressions for the Brauer characters in terms of unipotent characters, which lead to expressions for the (unipotent parts of) projective characters in terms of unipotent characters. From the above result of Leclerc-Thibon, we see below that the coefficients are Kazhdan-Lusztig polynomials.

Let χ_λ, ϕ_λ and Q_λ be unipotent characters, Brauer characters, and projective characters of G_n, where $\lambda \in \mathcal{P}_n$. We then have $(G^-(\lambda), |\mu >) = P^-_{\mu,\lambda}(-v^{-1})$, by [1, p. 186]. Hence: $(Q_\mu, \chi_\lambda) = P^-_{\mu,\lambda}(-1)$, expressing projective characters in terms of unipotent characters with coefficients which are Kazhdan-Lusztig polynomials as expected by Dudas and Malle.

References

[1] B.Leclerc, J-Y. Thibon, *Littlewood-Richardson coefficients and Kazhdan-Lusztig polynomials*, Advanced Studies in Pure Math. 28 (2000), 155-220.

[2] B. Srinivasan, *On CRDAHA and finite general linear and unitary groups*, arXiv.1411.1714

Stratifying the stable module category for a finite group scheme

David Benson

The purpose of this talk is to describe work in progress with Srikanth Iyengar, Henning Krause and Julia Pevtsova, the goal of which is to classify the localising subcategories of the stable module category of a finite group scheme. The framework for this classification was developed in joint work with Iyengar and Krause [1, 2, 3, 4].

Let k be a field of characteristic $p > 0$ (characteristic zero is for our purposes uninteresting). Recall that giving a finite group scheme G is equivalent to giving a finite dimensional cocommutative Hopf algebra kG, the group algebra of G, or dually a finite dimensional commutative Hopf algebra $k[G]$, the coordinate ring of G. Giving a representation of G over k is equivalent to giving a kG-module. The kG-modules form an abelian category in which the projectives and injectives coincide, and quotienting out maps which factor through a projective gives the stable module category $\mathsf{StMod}\,(kG)$. In the language of the above cited papers, our goal is to show that $\mathsf{StMod}\,(kG)$ is stratified by the action of the cohomology ring $H^*(kG) = \mathsf{Ext}^*_{kG}(k, k)$. Namely, the theory of support induces a one to one correspondence between the tensor ideal localising subcategories of $\mathsf{StMod}\,(kG)$ and sets of non-maximal homogeneous prime ideals in $H^*(kG)$.

A finite group scheme G is said to be connected if $k[G]$ is a local ring, and unipotent if kG is a local ring. Following Friedlander and Pevtsova, a π-point of a finite group scheme G consists of a field extension $K \supseteq k$ and a flat map of algebras $\alpha\colon K[t]/(t^p) \to KG = K \otimes_k kG$ that factors through an abelian unipotent subgroup scheme. An equivalence relation $\alpha \sim \beta$ on π-points of G is imposed, whereby $\alpha\colon K[t]/(t^p) \to KG$ and $\beta\colon K'[t]/(t^p) \to K'G$ are equivalent if for all finitely generated kG-modules M, $\alpha^*(K \otimes_k M)$ is projective if and only if $\beta^*(K' \otimes_k M)$ is projective. Note that this only needs to be tested on Carlson's L_ζ modules, and it implies the same property for all kG-modules. A theorem of Friedlander and Pevtsova [5, Theorem 3.6] says that the equivalence class of α is determined by the kernel of the composite

$$H^*(kG) \hookrightarrow H^*(KG) \xrightarrow{\alpha^*} H^*(K[t]/(t^p)) \to H^*(K[t]/(t^p))/\text{Nil Rad},$$

and that this gives a one to one correspondence between equivalence classes of π-points in G and homogeneous prime ideals in $H^*(kG)$ other than the maximal ideal of positive degree elements. We write $\mathsf{Proj}\,H^*(kG)$ for this set of prime ideals, and for $\mathfrak{p} \in \mathsf{Proj}\,H^*(kG)$ we write $\alpha_{\mathfrak{p}}$ for a representative of the corresponding equivalence class of π-points. The π-support of a kG-module M is defined to be

$$\pi\text{-}\mathsf{supp}(M) = \{\mathfrak{p} \in \mathsf{Proj}\,H^*(kG) \mid \alpha^*_{\mathfrak{p}}(K \otimes_k M) \text{ is not projective}\}.$$

Another theorem of Friedlander and Pevtsova [5, Theorem 5.3] states that M is projective if and only if $\pi\text{-}\mathsf{supp}(M) = \varnothing$. There is a gap in the proof of this theorem, which has now been fixed as a consequence of our work.

One of the key ideas we use to make progress on the problem is the notion dual to π-support, namely the π-cosupport. Write M_K for $K \otimes_k M$ and M^K for $\mathsf{Hom}_k(K, M)$ as KG-modules. The π-cosupport of M is then defined to be

$$\pi\text{-}\mathsf{cosupp}(M) = \{\mathfrak{p} \in \mathsf{Proj}\, H^*(kG) \mid \alpha_{\mathfrak{p}}^*(M^K) \text{ is not projective}\}.$$

The main properties of π-support and π-cosupport are:

(i) For finite dimensional modules M, we have

$$\pi\text{-}\mathsf{cosupp}(M) = \pi\text{-}\mathsf{supp}(M).$$

(ii) For tensor products with diagonal action defined through the comultiplication on kG, we have

$$\pi\text{-}\mathsf{supp}(M \otimes_k N) = \pi\text{-}\mathsf{supp}(M) \cap \pi\text{-}\mathsf{supp}(N).$$

(iii) For internal homs, we have

$$\pi\text{-}\mathsf{cosupp}(\mathsf{Hom}_k(M, N)) = \pi\text{-}\mathsf{supp}(M) \cap \pi\text{-}\mathsf{cosupp}(N).$$

In particular, for duals we have

$$\pi\text{-}\mathsf{cosupp}(M^*) = \pi\text{-}\mathsf{supp}(M).$$

(iv) The analogue of Dade's lemma says that for an elementary abelian p-group E, a kE-module M is projective if and only if $\pi\text{-}\mathsf{supp}(M) = \varnothing$, and also M is projective if and only if $\pi\text{-}\mathsf{cosupp}(M) = \varnothing$.

The key step in classifying localising subcategories of $\mathsf{StMod}\,(kG)$ is the following statement:

($*$) If M is a kG-module with $\pi\text{-}\mathsf{cosupp}(M) = \varnothing$ then M is projective.

Given this statement, we prove first that π-support and π-cosupport agree with support $\mathsf{supp}(M)$ and cosupport $\mathsf{cosupp}(M)$ as defined in [1, 2, 3, 4]. We then show that this implies stratification, namely that there is a one to one correspondence between subsets of the set of homogeneous non-maximal prime ideals in $H^*(kG)$ and the set of localising subcategories of $\mathsf{StMod}\,(kG)$. In this bijection, a set S of primes is sent to the full subcategory consisting of those modules M such that $\pi\text{-}\mathsf{supp}(M) \subseteq S$, or equivalently, such that $\mathsf{supp}(M) \subseteq S$.

At the time of giving the talk, we could only prove ($*$) for rather restricted types of finite group schemes. However, immediately following the Oberwolfach meeting, the four authors met in Barcelona for two weeks, and we can now prove ($*$) for all finite group schemes G. This therefore completes the classification.

References

[1] D. J. Benson, S. B. Iyengar, and H. Krause, *Local cohomology and support for triangulated categories*, Ann. Scient. Éc. Norm. Sup. (4) **41** (2008), 575–621.
[2] ———, *Stratifying triangulated categories*, J. Topology **4** (2011), 641–666.
[3] ———, *Stratifying modular representations of finite groups*, Ann. of Math. **174** (2011), 1643–1684.
[4] ———, *Colocalising subcategories and cosupport*, J. Reine & Angew. Math. **673** (2012), 161–207.
[5] E. M. Friedlander and J. Pevtsova, *Π-supports for modules for finite groups schemes*, Duke Math. J. **139** (2007), 317–368.

The McKay conjecture and beyond

Pham Huu Tiep

In this talk, we discuss recent progress on some global-local conjectures in representation theory of finite groups, obtained in joint work of the speaker with Gabriel Navarro and other collaborators.

We begin with the classical *Ito-Michler theorem*, stating that "*For a given prime p and a finite group G, $p \nmid \chi(1)$ for every complex irreducible character χ of G if and only if G has a normal, abelian Sylow p-subgroup P*". Several generalizations of this theorem have been obtained (by work of Dolfi-Navarro-Tiep, Navarro-Tiep, Marinelli-Tiep, etc.), by imposing the coprimeness condition not on the entire set $\mathrm{Irr}(G)$ of irreducible characters of G, but only on some subset of it, say the set of real-valued, or rational-valued irreducible characters of G. We will formulate one recent result in this direction:

Theorem 1. [5], [10] Let p be a prime, G a finite group, $p > 2$ a prime, and suppose $p \nmid \chi(1)$ for every $\chi = \bar{\chi} \in \mathrm{Irr}(G)$. Then $\mathbf{O}^{p'}(G)$ is solvable. Moreover, if $K := \mathbf{O}^{2'}(G)$ and $Q \in \mathrm{Syl}_2(K)$, then $Q \lhd K$ and $Q' \leq \mathbf{Z}(K)$.

There is also a quantitative way of generalizing the Ito-Michler theorem. To do this, we let $\mathsf{acd}_p(G)$ denote the average degree of those $\chi \in \mathrm{Irr}(G)$ that have degree either equal to 1 or divisible by p.

Theorem 2. [3] Let G be a finite group. If $\mathsf{acd}_2(G) < 5/2$ then G is solvable. If moreover $\mathsf{acd}_2(G) < 4/3$, then G has a normal Sylow 2-subgroup.

The odd-p analogue of this result is being obtained in our forthcoming work.

Next we focus our attention on the *McKay conjecture*. This was originally formulated by John McKay for the prime $p = 2$ and finite groups G with self-normalizing Sylow 2-subgroups $P \in \mathrm{Syl}_2(G)$ (and concerned with the set $\mathrm{Irr}_{p'}(G)$ of irreducible characters of G of degree coprime to p). In 2007, I. M. Isaacs, G. Malle, and G. Navarro [4] proved a reduction theorem for the McKay conjecture, which reduces the conjecture to the verification of a collection of (much strongger) conditions for all finite non-abelian simple groups. It is impossible to overestimate the importance of this reduction theorem. On the other hand, in the particular case of self-normalizing Sylow 2-subgroups, can one obtain a *direct* reduction of the conjecture to the almost simple groups? This is indeed the case, as shown recently in [8]. A key ingredient of our proof of this reduction is the following extension theorem, which is also of independent interest:

Theorem 3. Let G be a finite group and $P \in \mathrm{Syl}_2(G)$. Suppose that $P = \mathbf{N}_G(P)$. Let $N \lhd G$, $\chi \in \mathrm{Irr}_{2'}(G)$, and $\theta \in \mathrm{Irr}(N)$ be under χ. Then θ extends to $I_G(\theta)$.

What about self-normalizing Sylow p-subgroups for odd p? The structure of such groups is quite limited, as shown in [1], and so the McKay conjecture holds for G by [4]. But can one expect to have a *natural* bijection between $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}(P/P')$ in this case? An affirmative answer to this question is given by

Theorem 4. [9] Let G be a finite group, let $p > 2$ be an odd prime, and let $P \in \mathrm{Syl}_p(G)$. Suppose that $\mathbf{N}_G(P) = P$. Then for any $\chi \in \mathrm{Irr}_{p'}(G)$, χ_P contains a unique linear constituent χ^*, and the map $\chi \mapsto \chi^*$ yields a bijection between $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}(P/P')$.

In fact, a version of Theorem 4 also holds the p'-degree characters that belong to the principal p-blocks of G and of $\mathbf{N}_G(P)$ if $\mathbf{N}_G(P) = P\mathbf{C}_G(P)$, see [9].

More recently, in [2] we have been able to classify all possible non-abelian composition factors of finite groups G with $|\mathbf{N}_G(P)|$ being *odd*. Together with [4] and some recent work of Späth and Koshitani-Späth, our result yields

Corollary 5. [2] Suppose that $|\mathbf{N}_G(P)|$ is odd for $P \in \mathrm{Syl}_p(G)$. Then G satisfies the (Alperin-)McKay conjecture and (the blockwise version of) the Alperin weight conjecture, for the prime p.

In another direction, we have been able to extend some classical results on *exceptional characters* to the finite groups G for which $\mathbf{N}_G(P)/P'$ is a Frobenius group. This allows us to prove the following theorem:

Theorem 6. [6] Let G be a finite group, $p > 2$ a prime, and $P \in \mathrm{Syl}_p(G)$. Suppose that $\mathbf{N}_G(P)/P'$ is a Frobenius group with kernel P/P'. Then the McKay conjecture, as well as the Isaacs-Navarro refinenement, and the Navarro refinement of it, hold for G.

Recently, we have also been investigating possible global-local statements that involve more than one prime. In particular, we have proved

Theorem 7. [7] Let G be a finite group, $p \neq q$ be primes, and $\{p, q\} \neq \{5, 7\}$. Then $\mathrm{Irr}_{p'}(G) = \mathrm{Irr}_{q'}(G)$ if and only if G admits an abelian $P \in \mathrm{Syl}_p(G)$ and an abelian $Q \in \mathrm{Syl}_q(G)$ such that $\mathbf{N}_G(P) = \mathbf{N}_G(Q)$.

Acknowledgements. The author gratefully acknowledges the support of the NSF (grant DMS-1201374), the Simons Foundation Fellowship 305247, and the Mathematisches Forschungsinstitut Oberwolfach.

References

[1] R. M. Guralnick, G. Malle, and G. Navarro, *Self-normalizing Sylow subgroups*, Proc. Amer. Math. Soc. **132** (2003), 973–979.
[2] R. M. Guralnick, G. Navarro, and Pham Huu Tiep, *Finite groups with odd Sylow normalizers*, (in preparation).
[3] Nguyen Ngoc Hung and Pham Huu Tiep,*Irreducible characters of even degree and normal Sylow 2-subgroups*, (submitted).
[4] I. M. Isaacs, G. Malle, and G. Navarro, *A reduction theorem for the McKay conjecture*, Invent. Math. **170** (2007), 33–101.
[5] I. M. Isaacs and G. Navarro, Groups whose real irreducible characters have degree coprime to p, *J. Algebra* **356** (2012), 195–206.
[6] G. Navarro and Pham Huu Tiep, *Natural character correspondences for finite groups with Frobenius local subgroups*, (in preparation).
[7] G. Navarro and Pham Huu Tiep, *Characters, degrees, and sets of primes*, (in preparation).
[8] G. Navarro and Pham Huu Tiep, *Irreducible representations of odd degree*, (submitted).
[9] G. Navarro, Pham Huu Tiep, and C. Vallejo, *McKay natural correspondences on characters*, *Algebra Number Theory* **8** (2014), 1839–1856.
[10] Pham Huu Tiep, *Real ordinary characters and real Brauer characters*, *Trans. Amer. Math. Soc.* **367** (2015), 1273–1312.

Generalised Gelfand–Graev Characters in Small Characteristics

Jay Taylor

Let $\mathbf{G}$ be a connected reductive algebraic group defined over an algebraic closure $\overline{\mathbb{F}_p}$ of the finite field $\mathbb{F}_p$ of characteristic $p > 0$. We will assume that p is a *good prime* for $\mathbf{G}$ (the assumption $p > 5$ is sufficient for all cases). Furthermore, let us assume that $F : \mathbf{G} \to \mathbf{G}$ is a Frobenius endomorphism of $\mathbf{G}$ defining an $\mathbb{F}_q$-rational structure so that the fixed point group $G = \mathbf{G}^F$ is a finite reductive group. We will denote by $\mathrm{Irr}(G)$ the set of ordinary irreducible characters of G (for example over $\mathbb{C}$).

In [4, 5] Kawanaka has constructed for every unipotent element $u \in G$ a character Γ_u of G called a *generalised Gelfand–Graev character* (GGGC). The definition of Γ_u is somewhat delicate and depends upon the study of the unipotent conjugacy classes of $\mathbf{G}$. However, here are some of the important properties:

- $\Gamma_u = \Gamma_v$ if $u = gvg^{-1}$ for some $g \in G$,
- $\Gamma_u = \mathrm{Ind}_V^G(\varphi_u)$ where $V \leqslant G$ is a p-subgroup (depending upon the $\mathbf{G}$-conjugacy class containing u) and φ_u is a linear character of V,
- if $u = 1$ then $V = \{1\}$ is the trivial subgroup and Γ_1 is the character of the regular representation of G,
- if u is a regular unipotent element of $\mathbf{G}$ (i.e., $\dim C_{\mathbf{G}}(u)$ is minimal) then V is a Sylow p-subgroup of G and Γ_u is a Gelfand–Graev character (see [2, 14.29]).

In his study of GGGCs Kawanaka conjectured the following surprising relationship between the unipotent conjugacy classes of $\mathbf{G}$ and the irreducible characters of G (see [4, 3.3.3]).

Conjecture. For any irreducible character $\chi \in \mathrm{Irr}(G)$ there exists a unique F-stable unipotent class $\mathcal{O}_\chi^* \subseteq \mathbf{G}$, called the *wave front set* of χ, such that:

(WF1) $\langle \Gamma_u, \chi\rangle \neq 0$ for some $u \in \mathcal{O}_\chi^{*F}$,
(WF2) $\langle \Gamma_u, \chi\rangle \neq 0$ implies $\dim \mathcal{O}_u < \dim \mathcal{O}_\chi^*$ or $u \in \mathcal{O}_\chi^*$,

where $\mathcal{O}_u$ is the $\mathbf{G}$-conjugacy class containing $u \in G$.

We point out here that Kawanaka's conjecture is related to, and inspired by, an earlier conjecture of Lusztig on the existence of a *unipotent support* for $\chi \in \mathrm{Irr}(G)$. See [8] or [9] for more details. Let us also note that as Γ_1 is the regular character of G there always exists a class satisfying (WF1). Hence, another way to state the conjecture is that there is a unique class of maximal dimension satisfying (WF1).

Recall that we have a partial ordering $\leqslant$ on the set of unipotent conjugacy classes of $\mathbf{G}$ defined by $\mathcal{O}' \leqslant \mathcal{O}$ if and only if $\mathcal{O}' \subseteq \overline{\mathcal{O}}$ (the Zariski closure). It is well known that if $\mathcal{O}' \leqslant \mathcal{O}$ then we have $\dim \mathcal{O}' \leqslant \dim \mathcal{O}$ with equality if and only if $\mathcal{O}' = \mathcal{O}$. Hence one could ask whether the following stronger geometric condition holds

(WF2') $\langle \Gamma_u, \chi\rangle \neq 0$ implies $\mathcal{O}_u \leqslant \mathcal{O}_\chi^*$.

In [8] Lusztig showed that Kawanaka's conjecture is true under the assumption that p and q are sufficiently large. Furthermore, using Lusztig's work Achar and Aubert showed that the geometric refinement (WF2') is satisfied in [1] (again under the assumption that p and q are sufficiently large). The following removes these restrictions and is proved in [9].

Theorem. Assume only that p is a good prime for $\mathbf{G}$ then the wave front set exists for every irreducible character $\chi \in \mathrm{Irr}(G)$ and the geometric condition (WF2') is satisfied.

Lusztig's proof of Kawanaka's conjecture is based on a formula which decomposes the GGGCs in terms of characteristic functions of intersection cohomology complexes on the closures of unipotent conjugacy classes with coefficients in various local systems. Our approach is to show that this formula holds whenever p is an *acceptable prime* for $\mathbf{G}$; this is the main result of [9]. The assumption that p is an acceptable prime comes from the work of Letellier [6], on which our result relies heavily. Although this is a stronger assumption than p good it gives us enough information to prove the theorem by using various reduction arguments.

We finish by noting the following application of our result. The GGGCs are projective characters in characteristic $\ell \neq p$ a prime different from p. By Brauer reciprocity the multiplicities of irreducible characters in GGGCs provides information on the ℓ-decomposition numbers. In particular, the existence of the wave-front set can be used to ensure that certain decomposition numbers will be zero. This information was recently used by Dudas and Malle in [3] as part of their determination of new decomposition matrices for unipotent blocks of exceptional-type groups.

References

[1] P. N. Achar and A.-M. Aubert, *Supports unipotents de faisceaux caractères*, J. Inst. Math. Jussieu **6** (2007), no. 2, 173–207.
[2] F. Digne and J. Michel, *Representations of finite groups of Lie type*, vol. 21, London Mathematical Society Student Texts, Cambridge: Cambridge University Press, 1991.
[3] O. Dudas and G. Malle, *Decomposition matrices for exceptional groups at $d = 4$*, preprint (Oct. 2014), arXiv:1410.3754 [math.RT].
[4] N. Kawanaka, *Generalized Gelfand–Graev representations and Ennola duality*, in: *Algebraic groups and related topics (Kyoto/Nagoya, 1983)*, vol. 6, Adv. Stud. Pure Math. Amsterdam: North-Holland, 1985, 175–206.
[5] N. Kawanaka, *Generalized Gelfand-Graev representations of exceptional simple algebraic groups over a finite field. I*, Invent. Math. **84** (1986), no. 3, 575–616.
[6] E. Letellier, *Fourier transforms of invariant functions on finite reductive Lie algebras*, vol. 1859, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2005.
[7] G. Lusztig, *Characters of reductive groups over a finite field*, vol. 107, Annals of Mathematics Studies, Princeton, NJ: Princeton University Press, 1984.
[8] G. Lusztig, *A unipotent support for irreducible representations*, Adv. Math. **94** (1992), no. 2, 139–179.
[9] J. Taylor, *Generalised Gelfand–Graev representations in small characteristics*, preprint (Aug. 2014), arXiv:1408.1643 [math.RT].

Broué's perfect isometry conjecture holds for the double covers of the symmetric and alternating groups

Michael Livesey

Let p be a prime and $(K, \mathcal{O}, k)$ a p-modular system big enough for all groups considered. It has long been the goal of modular representation theory to relate the representation theory of a block to that of its Brauer correspondent. In particular we have Broué's abelian defect group conjecture:

Conjecture 1.[1] If B is a block of $\mathcal{O}G$ with an abelian defect group P then B and its Brauer correspondent in $N_G(P)$ are derived equivalent.

We call the conjecture with derived equivalent replaced by perfectly isometric Broué's perfect isometries conjecture. The conjecture has been proved for symmetric groups. Recall that to every block of an S_n is associated an integer w called its weight and its defect group is abelian if and only if $w < p$. The proof consists of two main steps. The first step involves proving the conjecture for a particular block.

Theorem 2.[2, §3, Theorem 2] For each w with $(0 \leq w < p)$ there exists an $n \geq 1$ and a block B of $\mathcal{O}S_n$, with weight w and defect group P, which is Morita equivalent to the principal block of $\mathcal{O}(S_p \wr S_w)$.

Theorem 3.[3, Theorem 4.3(b)] The principal block of $\mathcal{O}(S_p \wr S_w)$ is derived equivalent to the Brauer correspondent of B in $N_{S_n}(P)$.

The second step involves proving that having the conjecture for this particular block is enough.

Observation 4. Any two blocks B, B' of $\mathcal{O}S_n$, $\mathcal{O}S_{n'}$ respectively, of the same weight, have Morita equivalent Brauer correspondents.

Theorem 5.[4, Theorem 7.2] Any two blocks B, B' of $\mathcal{O}S_n$, $\mathcal{O}S_{n'}$ respectively, of the same weight, are derived equivalent.

We would like to generalise all of the above to the double covers of symmetric and alternating groups. This is a hard problem but we do have all the analogous theorems at the level of characters (in other words with derived equivalence replaced by perfect isometry). Recall that to each faithful block of $\tilde{S}_n$ is associated a weight w and a sign $\{\pm\}$ and its defect group is abelian if and only if $w < p$.

Theorem 6.[5, Theorem 4.15] Let p be an odd prime and let B and B' be blocks with weight $w > 0$ of $\tilde{S}_m$ and $\tilde{S}_n$ respectively and $\overline{B}$ and $\overline{B}'$ the corresponding blocks of $\tilde{A}_m$ and $\tilde{A}_n$.

(1) If B and B' have the same sign then they are perfectly isometric and so are $\overline{B}$ and $\overline{B}'$.
(2) If B and B' have opposite signs then B and $\overline{B}'$ are perfectly isometric and so are $\overline{B}$ and B'.

The proof of this involves using the analogue of the Murnaghan-Nakayama rule for $\tilde{S}_n$ as well as proving and using a Murnaghan-Nakayama rule for $\tilde{A}_n$. We apply the methods and results of this paper to prove the following:

Theorem 7.[6, Theorem 1.0.1] Let p be an odd prime and B a block of $\tilde{S}_n$ or $\tilde{A}_n$ with abelian defect group. Then there exists a perfect isometry between B and its Brauer correspondent.

In other words Broué's perfect isometries conjecture holds for $\tilde{S}_n$ and $\tilde{A}_n$. The proof of this involves constructing a Murnaghan-Nakayama rule for $N_{\tilde{S}_n}(P)$, where P is an abelian defect group of a block of $\tilde{S}_n$ and heavily relies on the results of Brunat and Gramain. Using this theorem one can then show the following:

Corollary 8. For all weight $w < p$ and sign ϵ there exists a block B of an $\tilde{S}_n$ of weight w and sign ϵ that is perfectly isometric to its Brauer correspondent in $\theta_n^{-1}(S_{n-pw} \times (S_p \wr S_w))$ with all signs postive, where θ_n is the canonical group homomorphism $\theta_n : \tilde{S}_n \to S_n$.

There is an analogous statement for $\tilde{A}_n$. This corollary suggests the following conjecture which is the analogue of Theorem for the double covers.

Conjecture 9. The blocks in the above corollary are Morita equivalent.

This is work in progress but the methods being used are analogous to those used by Evseev in [7] where he proves a strengthening of Theorem 2 using KLR-algebras. The hope is that similar techniques can be applied to the double covers using quiver Hecke-Clifford superalgebras.

References

[1] M. Broué, *Isométries parfaites, types de blocs, catégories dérivées*, Astérisque **181–182** (1990), 61–92.

[2] J. Chuang and R. Kessar, *Symmetric groups, wreath products, Morita equivalences, and Broué's abelian defect group conjecture*, Bull. London Math. Soc. **34**(2) (2002), 174–185.

[3] A. Marcus, *On the equivalences between blocks of group algebras: Reduction to the simple components*, J. Algebra **184**(2) (1996), 372–396.

[4] J. Chuang and R. Rouquier, *Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification*, Ann. of Math. **167**(1) (2008), 245–298.

[5] O. Brunat and J. B. Gramain, *Perfect isometries and Murnaghan-Nakayama rules*, preprint: arXiv:1305.7449 (2015).

[6] M. Livesey, *Broué's perfect isometry conjecture holds for the double covers of the symmetric and alternating groups*, Algebras and Representation Theory (submitted August 2014, available at arXiv:1408.4709) (2015).

[7] A. Evseev, *Wreath products and graded rock blocks of symmetric groups*, Preprint not yet available (2015).

Endotrivial Modules for Very Important Groups

Nadia Mazza

(joint work with Jon Carlson and Dan Nakano, and Caroline Lassueur)

Endotrivial modules have been studied since the late 70s. These are modular representations of finite groups whose stable isomorphism classes form a part of the Picard group of selfequivalences of the stable module category of a finite group.

Given an algebraically closed field k of positive characteristic p and a finite group G of order divisible by p, an *endotrivial kG-module* is a finitely generated kG-module M such that $\mathrm{End}_k M \cong k \oplus (\mathrm{proj})$ as kG-modules ([9]). The latter k denotes the trivial kG-module.

The stable isomorphism classes of endotrivial modules form a finitely generated abelian group $T(G)$ for the composition induced by $\otimes_k$. The torsion-free rank is known to only depend on the p-local structure of G, while Green correspondence allows us to bound the size of the torsion subgroup of $T(G)$ by that of $T(N)$ where N is the normaliser of a Sylow p-subgroup of G. In case $N = G$, i.e. G has a normal Sylow p-subgroup, and other special situations, the group $T(G)$ can be given by generators and relations, while in many other cases its isomorphism type is determined. But the unsufficient knowledge on the behaviour of modules under induction-restriction prevents the generalisation of the results to arbitrary finite groups. In particular, the main difficulty is to find all the indecomposable trivial source endotrivial modules. The objective of both research projects is to determine the structure of $T(G)$, and in doing so classifying the endotrivial modules, for certain families of "important" finite groups, based on the results obtained so far, in particular [2, 3, 4, 7, 8, 13].

With J. Carlson and D. Nakano: We find $T(G)$ for all finite groups of Lie type A in nondefining characteristic. We exploit two methods: *Young modules for general linear groups* ([10]), and results from [8], which build on Balmer's method of *weak homomorphisms* ([1]).

Our main results can be summarised as follows:

Theorem. *Let G be a finite group of order divisible by p such that* $\mathrm{SL}(n,q) \subseteq G \subseteq \mathrm{GL}(n,q)$*, and let Z be a central subgroup of G. Write e for the multiplicative order of q modulo p (i.e. $0 < e < p$ and p divides $q^e - 1$).*

(1) *If a Sylow p-subgroup S of G/Z is cyclic, then $T(G/Z) \cong T(N_G(S))$ and an explicit description can be given* (cf. [14] too).

(2) *Assume that the following conditions hold:* (a) *In all cases, $n \geq 2e$,* (b) *If $e = 1$, then $n > 2$ if p is odd, and $n > 3$ if $p = 2$,* (c) *If $n = p = 3$ and $e = 1$, then Z does not contain a Sylow 3-subgroup of $Z(G)$.*
Then $T(G/Z) \cong \mathbb{Z} \oplus X(G/Z)$, where $X(G/Z)$ is the group under tensor product of the group of $k(G/Z)$-modules of dimension one and the torsion free part is generated by the class of $\Omega(k)$.

Details, including the results for the excluded cases, are given in [5, 6].

With C. Lassueur: Using mainly computational algebra (MAGMA and GAP) and character theory as in [13] we handle sporadic groups, their Schur covers and the Schur covers of symmetric and alternating groups. We determine $T(G)$ for these quasi-simple groups in all characteristics, except for some "sporadic" cases still unsolved. Notably, we obtain several instances of trivial source endotrivial modules of dimension greater than one. Tables and detailed analysis in [11, 12]. In particular for:

- $G = 3.\mathfrak{A}_6$ with $p = 2$, for which $K(G) \cong \mathbb{Z}/3$ is generated by a 9-dimensional faithful trivial source endotrivial module (whose Green correspondent for $3.\mathfrak{A}_7$ is not endotrivial).
- $G = \mathrm{M}_{22}$ and its covers with $p = 3$, for which $K(G) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $K(4.G) \cong K(2.G) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$ (whereas $K(d.G)$ is trivial whenever 3 divides d).
- $G = 3.\mathrm{McL}$ with $p = 5$, for which $K(G) \cong \mathbb{Z}/24$.
- $G = \mathrm{Ru}$ and $2.\mathrm{Ru}$ with $p = 3$, for which $K(\mathrm{Ru}) = \mathbb{Z}/2$ and $K(2.\mathrm{Ru}) \cong \mathbb{Z}/4$.
- $G = \mathrm{Fi}_{22}$ and its covers with $p = 5$, for which $K(G) \cong \mathbb{Z}/2$, $K(2.G) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, $K(3.G) \cong \mathbb{Z}/6$ and $K(6.G) \cong \mathbb{Z}/6 \oplus \mathbb{Z}/2$.

Work in progress: (1) with Carlson and Thévenaz, we investigate the trivial source endotrivial kG-modules, (2) with Carlson and Nakano, we aim to find $T(G)$ for all finite groups of Lie type in nondefining characteristic.

References

[1] P. Balmer, *Modular representations of finite groups with trivial restriction to Sylow subgroups*, J. European Math. Soc., **15**, (2013), 2061–2079.

[2] J. Carlson, D. Hemmer and N. Mazza, *The group of endotrivial modules for the symmetric and alternating groups*, Proc. Edinburgh Math. Soc., **53**, (2010), 83–95.

[3] J. Carlson, N. Mazza, D. Nakano, *Endotrivial modules for finite groups of Lie type*, J. Reine Angew. Math., **595** (2006), 93–120.

[4] J. Carlson, N. Mazza and D. Nakano, *Endotrivial modules for the symmetric and alternating groups*, Proc. Edinburgh Math. Soc., **52**, (2009), 45–66.

[5] J. Carlson, N. Mazza and D. Nakano, *Endotrivial modules for the general linear group in a nondefining characteristic*, Math. Zeit., **278**, (2014), 901–925.

[6] J. Carlson, N. Mazza and D. Nakano, *Endotrivial modules for finite groups of Lie type A in nondefining characteristic*, submitted.
[7] J. Carlson, N. Mazza and J. Thévenaz, *Endotrivial modules over groups with quaternion or semi-dihedral Sylow* 2*-subgroup*, J. European Math. Soc, **15**, (2013), 157–177.
[8] J. Carlson and J. Thévenaz, *A local method for computing the group of endotrivial modules*, Algebra and Number Theory, to appear, 2015.
[9] E. C. Dade, *Endo-permutation modules over p-groups, I and II*, Ann. Math., **107** and **108**, (1978), 459–494 and 317–346.
[10] K. Erdmann, S. Schroll, *On Young modules of general linear groups*, J. Algebra, **310**, (2007), no. 1, 434–451.
[11] C. Lassueur, N. Mazza, *Endotrivial modules for the sporadic simple groups and their covers*, submitted.
[12] C. Lassueur, N. Mazza, *Endotrivial modules for the Schur covers of the symmetric and alternating groups*, submitted.
[13] C. Lassueur, G. Malle, E. Schulte, *Simple endotrivial modules for quasi-simple groups*, J. reine angew. Math. (2014), DOI: 10.1515/crelle-2013-0100.
[14] N. Mazza and J. Thévenaz, *Endotrivial modules in the cyclic case*, Arch. Math., **89**, (2007), 497–503.

Harish-Chandra series and characters of odd degree

Gunter Malle
(joint work with Britta Späth)

In our talk we announced the proof of the following result:

Theorem. *Let G be a finite group. Then the numbers of odd degree irreducible characters of G and of the normaliser of a Sylow 2-subgroup of G agree.*

This is an instance of the so-called McKay conjecture from 1972 which claims the same identity for any prime in place of the prime 2.

The starting point of our proof is the reduction theorem of Isaacs-Malle-Navarro from 2007 which shows that the McKay conjecture for a given prime ℓ is true if a stronger so-called inductive McKay condition holds for all finite simple groups at ℓ. This inductive condition has been verified for several of the families of simple groups, but remains open for seven out of the 16 families of groups of Lie type for non-defining primes ℓ.

As an intermediate result on the way to the main theorem and as an important contribution towards the general McKay conjecture we are able to show that the inductive McKay condition holds for five out of those seven families of groups G for all primes ℓ such that $\ell|(q-1)$, respectively $4|(q-1)$ when $\ell = 2$, where q is the size of the underlying field of G. This latter result is established using Späth's reformulation of the inductive McKay conditions and has two main ingredients. On the global level it is based on the earlier classification of ℓ'-characters of groups of Lie type in terms of Harish-Chandra series and on the new explicit description of the action of outer automorphisms on characters in arbitrary Harish-Chandra series once the underlying cuspidal character satisfies suitable properties.

The heart of the argument concerns the local level. There we construct equivariant extension maps to obtain a labelling of the irreducible characters of the normaliser of a maximally split torus together with the action of outer automorphisms on these labels. Putting together the local and the global parametrisations yields the required equivariant bijection of ℓ'-characters.

The proof of our main theorem in the case $4|(q+1)$ requires several additional ad hoc considerations tailored to the $\ell = 2$ situation, for example the investigation of an additional non-principal Harish-Chandra series.

Anchors of irreducible characters

Radha Kessar

(joint work with Burkhard Kuelshammer, Markus Linckelmann)

Let $\mathcal{O}$ be a complete discrete valuation ring with residue field k and field of fractions K. We assume that k has characteristic p, that K has characteristic 0 and that K is a splitting field for all finite groups and their subgroups considered below. For χ an irreducible K-valued character of a finite group G we denote by e_χ the unique primitive idempotent in the center $Z(KG)$ of the group algebra KG with $\chi(e_\chi) \neq 0$. Then the $\mathcal{O}$-order $\mathcal{O}Ge_\chi$ is a G-interior $\mathcal{O}$-algebra, via the map $G \to (\mathcal{O}Ge_\chi)^\times$ sending $g \in G$ to ge_χ. Since $(\mathcal{O}Ge_\chi)^G = Z(\mathcal{O}Ge_\chi) = \mathcal{O}e_\chi \cong \mathcal{O}$ is a local $\mathcal{O}$-order, $\mathcal{O}Ge_\chi$ is a primitive G-interior $\mathcal{O}$-algebra and thus has a defect group, uniquely defined up to conjugacy in G. These have been considered earlier by L. Barker [1].

Definition Let χ be an irreducible K-valued character of G. We call each defect group of the primitive interior G-algebra $\mathcal{O}Ge_\chi$ an *anchor* of χ.

We prove various properties of anchors which demonstrate that they may be considered as analogues of vertices of indecomposable $\mathcal{O}G$-modules in the sense of J. A. Green. Here we note that there may be infinitely many non-isomorphic indecomposable $\mathcal{O}$-modules affording the character χ and these may have different vertices [3].

Theorem. Let G be a finite group and let χ be an irreducible K-valued character of G. Let B be the block of $\mathcal{O}G$ containing χ and let L be an $\mathcal{O}G$-lattice affording χ. Let P be an anchor of χ. Then,

(1) P is contained in a defect group of B.
(2) P contains a vertex of L.
(3) If χ restricts to an irreducible p-Brauer character of G, then L is unique up to isomorphism, and P is a vertex of L.
(4) $O_p(G) \leq P$.
(5) Let τ be a local point of P on $\mathcal{O}Ge_\chi$. Then the multiplicity module of τ is simple. In particular, $O_p(N_G(P_\tau)) = P$ and P is centric in a fusion system of B.
(6) If B has an abelian defect group D, then D is an anchor of χ.
(7) If χ has height zero, then P is a defect group of B.

(8) Suppose that G' is a finite group and B' is a block of $\mathcal{O}G'$ which is source algebra equivalent to B. Let χ' be an irreducible K-valued character of G corresponding to χ under a source algebra equaivelence between B and B'. Then the anchors of χ' are isomorphic to P.
(9) The suborder $\mathcal{O}Pe_\chi$ of $\mathcal{O}Ge_\chi$ is local and $\mathcal{O}Ge_\chi$ is a separable extension of $\mathcal{O}Pe_\chi$.

We also investigate the relationship between anchors and vertices of irreducible characters for p-solvable groups in the sense of Navarro [2].

References

[1] L. Barker, *Defects of irreducible characters of p-soluble groups* J. Algebra **202** (1998), 178-184.
[2] G. Navarro, *Vertices for characters of p-solvable groups*, Trans. Amer. Math. Soc. **354** (2002), 2759-2773.
[3] W. Plesken, *Vertices of irreducible lattices over p-groups*, Comm. Algebra **10** (1982), 227-236.

Generalized Cartan invariants of symmetric groups

Anton Evseev
(joint work with Shunsuke Tsuchioka)

Let A be a finite-dimensional algebra over an algebraically closed field F. Let $P^1, \dots, P^N$ be the representatives of isomorphism classes of projective indecomposable A-modules, and denote by S^i the (simple) head of P^i. The *Cartan matrix* of A is the matrix $C_A = ([P^i : S^j])_{1\le i,j\le N}$, where $[P^i : S^j]$ is the multiplicity of S^j as a composition factor of P^i. If X and Y are two $N \times N$-matrices over a commutative ring R, we write $X \equiv_R Y$ and say that X and Y are R-*equivalent* if there exist $U, V \in \mathrm{GL}_N(R)$ such that $UXV = Y$.

Külshammer, Olsson and Robinson [7] generalized many aspects of the ℓ-modular character theory of the symmetric group $\mathfrak{S}_n$ to the case when ℓ is an arbitrary positive integer, not necessarily prime. In particular, they defined a Cartan matrix $C_{\mathfrak{S}_n,\ell}$ in this context. By a result of Donkin [3], $C_{\mathfrak{S}_n,\ell} \equiv_{\mathbb{Z}} C_{\mathcal{H}_n(\xi)}$ where $\mathcal{H}_n(\xi)$ is the Iwahori–Hecke algebra of type A_{n-1} defined over a field containing a primitive ℓ-th root of unity ξ.

For $k > 0$, let $\ell_k = \ell/(\ell, k)$. If π is a set of primes and m is an integer, let m_π be the largest divisor of m such that all prime divisors of m_π belong to π. Further, $\pi(m)$ denotes the set of all prime divisors of m. If $\lambda = (\lambda_1, \dots, \lambda_s)$ is a partition of n, we write $\lambda \vdash n$ and $m_k(\lambda) = \#\{j \mid \lambda_j = k\}$ for all $k \ge 1$. Let $\mathrm{diag}(\{a_1, \dots, a_m\})$ denote the diagonal matrix with diagonal entries $a_1, \dots, a_m$. The following result was conjectured in [7] and proved in [4]. Hill [6] previously proved the result in certain special cases.

Theorem 1. We have

$$C_{\mathfrak{S}_n,\ell} \equiv_{\mathbb{Z}} \operatorname{diag}(\{r_\ell(\lambda) \mid \lambda \vdash n \text{ and } \ell \nmid \lambda_i \ \forall i\}),$$

where

$$r_\ell(\lambda) = \prod_{k\geq 1,\, \ell\nmid k} \ell_k^{\lfloor m_k(\lambda)/\ell \rfloor} \lfloor \frac{m_k(\lambda)}{\ell} \rfloor !_{\pi(\ell_k)}.$$

It appears that greater understanding of structural properties behind this result may be obtained by considering *graded Cartan matrices* $C^v_{\mathcal{H}_n(\xi)}$, which are defined as follows. The algebra $\mathcal{H}_n(\xi)$ is isomorphic to certain cyclotomic Khovanov–Lauda–Rouquier algebras and therefore possesses a remarkable $\mathbb{Z}$-grading (see [2]). The projective indecomposable $\mathcal{H}_n(\xi)$-modules $P^1, \dots, P^N$ are graded uniquely up to graded shift. If S^j is the graded head of P^j, define

$$\langle P^i, P^j \rangle^v = \sum_{s\in\mathbb{Z}} [P^i : S^j\langle s \rangle] v^s \in \mathbb{Z}[v, v^{-1}] =: \mathscr{A},$$

where $S^j\langle s\rangle$ is the same module as S^j with the grading shifted by s, i.e. $S^j\langle s\rangle_m = (S^j)_{m-s}$ for $m \in \mathbb{Z}$. Further, $C^v_{\mathcal{H}_n(\xi)} := (\langle P^i, P^j\rangle)_{1\leq i,j\leq N}$. We have $C^v_{\mathcal{H}_n(\xi)}|_{v=1} = C_{\mathcal{H}_n(\xi)}$.

For $s \in \mathbb{Z}$ and $t \in \mathbb{Z}_{>0}$, define $[s]_t = (v^{st} - v^{-st})/(v^t - v^{-t}) \in \mathscr{A}$. If π is a set of primes, denote by π' its complement in the set of all primes. The following conjecture is stated in [5]; an equivalent conjecture in the special case when ℓ is a prime power was proposed in [8].

Conjecture A. We have

$$C^v_{\mathcal{H}_n(\xi)} \equiv_{\mathscr{A}} \operatorname{diag}(\{r^v_\ell(\lambda) \mid \lambda \vdash n \text{ and } \ell \nmid \lambda_i \ \forall i\}),$$

where

$$r_\ell(\lambda) = \prod_{k\geq 1} \prod_{t=1}^{\lfloor m_k(\lambda)/\ell \rfloor} [\ell_k t_{\pi(\ell_k)}]_{(\ell,k)t_{\pi(\ell_k)'}}.$$

Theorem 2 ([5]). Let D be the diagonal matrix on the right-hand side of the equivalence in Conjecture A. Then

(a) $C^v_{\mathcal{H}_n(\xi)} \equiv_{\mathbb{Q}[v,v^{-1}]} D$;
(b) If $0 \neq \theta \in \mathbb{Q}$, then $C^v_{\mathcal{H}_n(\xi)}|_{v=\theta} \equiv_{\mathbb{Z}[\theta,\theta^{-1}]} D|_{v=\theta}$.

The case $\theta = 1$ of part (b) is Theorem 1. Both Theorems 1 and 2 are proved using a result that reduces Conjecture A to a problem stated purely in terms of combinatorics and linear algebra. This reduction result was established in [6, 1] in the ungraded case and in [5] in the graded case.

Conjecture A is closely related to problems in modular representation theory of quantum groups. Consider the quantum group U_v of affine type $A^{(1)}_{\ell-1}$, defined over $\mathbb{Q}(v)$, and its basic representation $V(\Lambda_0)$, with highest weight vector v_{Λ_0}. Let $V(\Lambda_0)^{\mathscr{A}}$ be the $\mathscr{A}$-lattice spanned by $f_i^{(m)} v_{\Lambda_0}$ for $m \geq 0$, $i = 0, \dots, \ell - 1$, where $f_i^{(m)}$ are the usual divided powers. By a result of Brundan and Kleshchev, the Gram matrix of the Shapovalov form on each weight space of $V(\Lambda_0)^{\mathscr{A}}$ is $\mathscr{A}$-equivalent to the graded Cartan matrix of a certain block of $\mathcal{H}_n(\xi)$ for some n (see [5] for further details). As a consequence, Conjecture A and Theorems 1 and 2 may provide information on modular reductions of $V(\Lambda_0)$ and, in particular, on Jantzen filtrations in those reductions.

References

[1] C. Bessenrodt and D. Hill, *Cartan invariants of symmetric groups and Iwahori-Hecke algebras*, J. London Math. Soc. **81** (2010), 113–128.

[2] J. Brundan and A. Kleshchev, *Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras*, Invent. Math. **178** (2009), 451–484.

[3] S. Donkin, *Representations of Hecke algebras and characters of symmetric groups*, Studies in memory of Issai Schur, 49–67, Progr. Math. **210**, Birkhaüser Boston, Boston, MA, 2003.

[4] A. Evseev, *Generalised Cartan invariants of symmetric groups*, Trans. Amer. Math. Soc. **367** (2015), 2823–2851.

[5] A. Evseev and S. Tsuchioka, *On graded Cartan invariants of symmetric groups and Hecke algebras*, arXiv:1503.00921 (2015).

[6] D. Hill, *Elementary divisors of the Shapovalov form on the basic representation of Kac-Moody Lie algebras*, J. Algebra **319** (2008), 5208–5246.

[7] B. Külshammer, J.B. Olsson and G.R. Robinson, *Generalized blocks for symmetric groups*, Invent. Math. **151** (2003), 513–552.

[8] S. Tsuchioka, *Graded Cartan determinants of the symmetric groups*, Trans. Amer. Math. Soc. **366** (2014), 2019–2040.

Lower bounds for the number of conjugacy classes of a finite group

Attila Maróti

Let $k(G)$ denote the number of conjugacy classes of a finite group G.

1. Bounding $k(G)$ in terms of $|G|$

Answering a question of Frobenius, Landau [9] proved in 1903 that for a given k there are only finitely many groups having k conjugacy classes. Making this result explicit, we have $\log_2 \log_2 |G| < k(G)$ for any non-trivial finite group G. Problem 3 of Brauer's list of problems [2] is to give a substantially better lower bound for $k(G)$ than this. Pyber [13] proved that there exists a constant $\epsilon > 0$ so that for every finite group G of order at least 3 we have $\epsilon \log_2 |G|/(\log_2 \log_2 |G|)^8 < k(G)$. Almost 20 years later Keller [8] replaced the 8 in the previous bound by 7. The recent paper [1] gives a further improvement to Pyber's theorem.

Theorem (Baumeister, M, Tong-Viet, 2015). For every $\epsilon > 0$ there exists a $\delta > 0$ so that for every finite group G of order at least 3 we have

$$\delta \log_2 |G|/(\log_2 \log_2 |G|)^{3+\epsilon} < k(G).$$

The conjecture whether there exists a universal constant $c > 0$ so that $c \log_2 |G| < k(G)$ for any finite group G has been intensively studied by many mathematicians including Bertram. He speculates whether $\log_3 |G| < k(G)$ is true for every finite group G.

Theorem (Baumeister, M, Tong-Viet, 2015). Let G be a finite group with a trivial solvable radical. Then $\log_3 |G| < k(G)$.

2. Bounding $k(G)$ in terms of a prime divisor of $|G|$

Pyber asked various questions concerning lower bounds for $k(G)$ in terms of the prime divisors of $|G|$. In response to these questions (and motivated by trying to find explicit lower bounds for the number of complex irreducible characters in a block) Héthelyi and Külshammer obtained various results [3], [4] for solvable groups. For example they proved in [3] that every solvable finite group G whose order is divisible by p has at least $2\sqrt{p-1}$ conjugacy classes. Later Malle [10, Section 2] showed that if G is a minimal counterexample to the inequality $k(G) \geq 2\sqrt{p-1}$ with p dividing $|G|$ then G has the form HV where V is an irreducible faithful H-module for a finite group H with $(|H|, |V|) = 1$ where p is the prime dividing $|V|$. He also showed that H cannot be an almost quasisimple group. Using these results, Keller [7] showed that there exists a universal constant C so that whenever $p > C$ then $k(G) \geq 2\sqrt{p-1}$. In a later paper Héthelyi, Horváth, Keller and M [5] proved that by disregarding at most finitely many non-solvable p-solvable groups G, we have $k(G) \geq 2\sqrt{p-1}$ with equality if and only if $\sqrt{p-1}$ is an integer, $G = C_p \rtimes C_{\sqrt{p-1}}$ and $C_G(C_p) = C_p$. However since the constant C in Keller's theorem was unspecified, there had been no quantitative information on what was meant by at most finitely many in the afore-mentioned theorem. In [12] the following is proved.

Theorem (M, 2014). Every finite group G whose order is divisible by a prime p has at least $2\sqrt{p-1}$ conjugacy classes. Equality occurs if and only if $\sqrt{p-1}$ is an integer, $G = C_p \rtimes C_{\sqrt{p-1}}$ and $C_G(C_p) = C_p$.

This section is also related to Landau's theorem via Problem 21 of [2].

3. Bounding the number of p' degree characters

Let p be a prime and G a finite group. Denote the set of complex irreducible characters of G whose degrees are prime to p by $\mathrm{Irr}_{p'}(G)$. The McKay Conjecture states that $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(N_G(P))|$ where $N_G(P)$ is the normalizer of a Sylow p-subgroup P in G. Some known cases (easy consequence of a result of Dade and a special case of [6]) of this problem together with [12] allows us in [11] to generalize the main result of [12].

Theorem (Malle, M, 2015). Let G be a finite group and p a prime divisor of the order of G. Then $|\mathrm{Irr}_{p'}(G)| \geq 2\sqrt{p-1}$.

The second result in [11] gives a full description of group G for which equality occurs in the previous theorem for some prime divisor p of $|G|$. The proof of both theorems reduces to a question on finite simple groups of Lie type. The hardest case of this question is when the prime p is different from the natural characteristic.

References

[1] B. Baumeister, A. Maróti, H. P. Tong-Viet, *Finite groups have more conjugacy classes*, submitted for publication.
[2] R. Brauer, *Representations of finite groups*, 1963 Lectures on Modern Mathematics, Vol. I pp. 133–175 Wiley, New York.
[3] L. Héthelyi, B. Külshammer, *On the number of conjugacy classes of a finite solvable group*, Bull. London Math. Soc. **32** (2000), no. 6, 668–672.
[4] L. Héthelyi, B. Külshammer, *On the number of conjugacy classes of a finite solvable group. II,* J. Algebra **270** (2003), no. 2, 660–669.
[5] L. Héthelyi, E. Horváth, T. M. Keller, A. Maróti, *Groups with few conjugacy classes*, Proc. Edinb. Math. Soc. (2) **54** (2011), no. 2, 423–430.
[6] I. M. Isaacs, G. Malle, G. Navarro, *A reduction theorem for the McKay conjecture*, Invent. Math. **170** (2007), 33–101.
[7] T. M. Keller, *Lower bounds for the number of conjugacy classes of finite groups*, Math. Proc. Cambridge Philos. Soc. **147** (2009), no. 3, 567–577.
[8] T. M. Keller, *Finite groups have even more conjugacy classes*, Israel J. Math. **181** (2011), 433–444.
[9] E. Landau, *Über die Klassenzahl der binären quadratischen Formen von negativer Discriminante*, Math. Ann. **56** (1903), no. 4, 671–676.
[10] G. Malle, *Fast-einfache Gruppen mit langen Bahnen in absolut irreduzibler Operation*, J. Algebra **300** (2006), no. 2, 655–672.
[11] G. Malle, A. Maróti, *On the number of p'-degree characters in a finite group*, submitted for publication.
[12] A. Maróti, *A lower bound for the number of conjugacy classes of a finite group*, submitted for publication.
[13] L. Pyber, *Finite groups have many conjugacy classes*, J. London Math. Soc. (2) **46** (1992), no. 2, 239–249.

Symmetric Vertices for Symmetric Modules in Characteristic 2

John Murray

Let k be an algebraically closed field of characteristic 2. Then each finite dimensional k-vector space affords at most two isometry classes of non-degenerate k-valued symmetric bilinear forms: the symplectic forms (in even dimension) and the non-symplectic forms (which are diagonalizable). Let G be a finite group of even order. We say that a kG-module M has symmetric type if M affords a non-degenerate G-invariant symmetric bilinear form B. Now the endomorphism ring $\mathrm{End}_k(M)$ is an interior G-algebra [11]. The adjoint of B is a G-involution σ on $\mathrm{End}_k(M)$; so σ is an involutary k-algebra anti-automorphism of $\mathrm{End}_k(M)$ which commutes with the G-algebra action.

The purpose of this talk is to outline the results of our investigations into orthogonal decompositions, induction from subgroups and relative projectivity of symmetric G-forms using the theory of involutary G-algebras. Given $(\mathrm{End}_k(M), \sigma)$ and $H \leq G$, the involution σ acts on $\mathrm{End}_{kH}(M)$ and on its maximal ideals and points. Given a σ-fixed point, the associated multiplicity algebra is an involutary $N_G(H)$-algebra. Frequently this endows the associated multiplicity module with an equivariant symmetric form. If I is a σ-fixed two sided ideal of $\mathrm{End}_{kH}(M)$, then

σ-invariant idempotents of $\mathrm{End}_{kH}(M)/I$ can be lifted to σ-invariant idempotents of $\mathrm{End}_{kH}(M)$ [8].

Recall that if M is a kG-module, and $H \leq G$, then M is said to be H-projective if it is a direct summand of a module induced from H to G. Now if (L, B_L) is a symmetric kH-module, there is a standard construction of the induced symmetric kG-module $\mathrm{Ind}_H^G(L, B_L)$. See [9] or [10]. So the notion of H-projectivity makes sense for symmetric kG-modules: a symmetric G-form B is H-projective if (M, B) is an orthogonal direct summand of a symmetric module induced from H to G. Note in particular that if M has a H-projective form, then M is itself H-projective.

Now suppose that M is indecomposable. A vertex of M is a 2-subgroup V of G which is minimal subject to M being V-projective [6]. J. A. Green showed that V is determined up to G-conjugacy. For forms, we need to be a bit careful. If M has symmetric type, a symmetric vertex of M is a 2-subgroup T of G which is minimal subject to M having a symmetric G-form which is T-projective. We do not know whether the symmetric vertices of M are determined up to G-conjugacy. However, we can prove:

Theorem 1. Let M be an indecomposable kG-module which has symmetric type. Then each symmetric vertex of M contains a vertex of M with index at most 2.

If we fix a symmetric G-form B on M, we can consider $T \leq G$ minimal subject to B being T-projective. Again, we do not know whether T is determined up to G-conjugacy, with one important exception:

Theorem 2. If B is T-projective, where T is a vertex of M, then B is H-projective, for $H \leq G$, if and only if some G-conjugate of T is contained in H.

The trivial module is of symmetric type, and P. Fong noted [2] that each non-trivial self-dual irreducible kG-module has a symplectic G-form, determined up to a non-zero scalar. So Theorem 2 implies:

Theorem 3. Each self-dual irreducible kG-module is of symmetric type and its symmetric vertices are determined up to G-conjugacy.

The conclusion holds for any indecomposable kG-module which has one isometry class of symmetric G-forms.

Our motivation for looking at the projectivity of symmetric kG-modules arose from the theory of real 2-blocks. Let e be a primitive idempotent in $Z(kG)$ which is invariant under the contragredient map $x \to x^o$ on kG. Then kGe is called a real 2-block of kG. Note that kGe is a self-dual indecomposable $kG \times G$-module.

According to [7] each vertex of this module has the form $\Delta(D)$ where D is a defect group of the block, in the sense of R. Brauer [1]. In addition to defect groups, kGe has extended defect groups, in the sense of R. Gow [3]. Each extended defect group of kGe has the form $\Delta(E)$ where E is a 2-subgroup of G which can be chosen so that $[E : D] \leq 2$. Our main result on 2-blocks is:

Theorem 4. Let E be an extended defect group of a real 2-block kGe. Then ΔE is a symmetric vertex of kGe, as $G \times G$-module.

In fact, let B_1 be the $G \times G$-invariant bilinear form on kG for which the elements of G form an orthonormal basis. Then B_1 restricts to a non-degenerate ΔE-projective form on kGe. In the light of Theorem 2, this proves that the extended defect groups of a real 2-block are determined up to G-conjugacy, reproving a result of R. Gow.

We note that [4], [5] and [12] were key inspirations for this work. Results were submitted for publication in January 2015.

References

[1] R. Brauer, 'Zur Darstellungstheorie der Gruppen endlicher Ordnung', *Math. Zeit.* 63 (1956) 406–444.

[2] P. Fong, 'On decomposition numbers of J_1 and $R(q)$', *Symposia Mathematica* Vol. XIII (Convegno di Gruppi e loro Rappresentazioni, INDAM, Rome, 1972), pp. 415–422. (Academic Press, London, 1974).

[3] R. Gow, 'Real 2-blocks of characters of finite groups', *Osaka J. Math.* 25 (1) (1988) 135–147.

[4] R. Gow and W. Willems, 'Quadratic geometries, projective modules and idempotents', *J. Algebra* 160 (1993) 257–272.

[5] R. Gow and W. Willems, 'A note on Green correspondence and forms', *Comm. Algebra* 23 (4) (1995) 1239–1248.

[6] J. A. Green, 'On the indecomposable representations of a finite group', *Math. Zeit.* 70 (1959) 430–445.

[7] J. A. Green, 'Blocks of modular representations', *Math. Zeit.* 79 (1962) 100–115.

[8] P. Landrock and O. Manz, 'Symmetric forms, idempotents and involutory anti-isomorphisms', *Nagoya Math. J.* 125 (1992) 33–51.

[9] G. Nebe, 'Orthogonal Frobenius reciprocity', *J. Algebra* 225 (2000) 250–260.

[10] E. Pacifici, 'On tensor induction for representations of finite groups', *J. Algebra* 288 (2) (2005) 287–320.

[11] J. Thévenaz, *G-algebras and Modular Representation Theory,* Oxford Science Publications, (Clarendon Press, Oxford, 1995).

[12] W. Willems, 'Metrische moduln Über gruppenringen', Ph.D. Thesis, Johannes Gutenberg-Universität, Mainz, 1976.

p-Parts of character degrees

Hung P. Tong-Viet

(joint work with Mark L. Lewis, Gabriel Navarro and Pham Huu Tiep)

Let G be a finite group and p be a prime. We denote by $\mathrm{Irr}(G)$ the set of complex irreducible characters of G and by $\mathrm{cd}(G)$ the character degrees of G. The celebrated Itô-Michler theorem states that p does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$ if and only if G has a normal abelian Sylow p-subgroup. Many variations of this theorem have been proposed and studied in the literature. Recently, Lewis, Navarro and Wolf [2] proved that if G is a finite solvable group and for every $\chi \in \mathrm{Irr}(G)$, $\chi(1)_p \leq p$, then $|G : \mathbf{F}(G)|_p \leq p^2$, where $\mathbf{F}(G)$ is the Fitting subgroup of G and m_p is the p-part of $m \in \mathbb{N}$. Furthermore, if P is a Sylow p-subgroup of G, then P' is subnormal in G. For arbitrary groups, they proved that if every character $\chi \in \mathrm{Irr}(G)$ satisfies $\chi(1)_2 \leq 2$, then $|G : \mathbf{F}(G)|_2 \leq 2^3$ and P'' is subnormal in G where P is a Sylow 2-subgroup of G. The simple group A_7 shows that this bound is best possible.

For odd primes, we can prove the following.

Theorem 1. *Let G be a finite group, and let p be an odd prime. If $\chi(1)_p \leq p$ for all $\chi \in \mathrm{Irr}(G)$, then $|G : \mathbf{O}_p(G)|_p \leq p^4$. Moreover, if P is a Sylow p-subgroup of G, then P'' is subnormal in G.*

Notice that $|G : \mathbf{F}(G)|_p = |G : \mathbf{O}_p(G)|_p$ for a finite group G and a prime p. For p-solvable groups, we obtained a stronger result.

Theorem 2. *Let p be an odd prime and let G be a finite p-solvable group. If $\chi(1)_p \leq p$ for all $\chi \in \mathrm{Irr}(G)$, then $|G/\mathbf{R}(G)|_p \leq p$ and $|G/\mathbf{O}_p(G)|_p \leq p^3$.*

Recall that $\mathbf{R}(G)$ is the solvable radical of G. We suspect that the correct bound in both Theorems 1 and 2 is $|G : \mathbf{O}_p(G)|_p \leq p^2$. Our study also suggests the following.

Conjecture 3. *If p^a is the largest power of p dividing the irreducible characters of G, then $|G : \mathbf{O}_p(G)|_p$ is bounded by $p^{f(a)}$ where $f(a)$ is a function in a and $P^{(a+1)}$ is subnormal in G.*

We now turn to p-Brauer characters. Let $\mathrm{IBr}_p(G)$ be the set of irreducible p-Brauer characters of G. There are some significant differences between ordinary character degrees and p-Brauer character degrees; for example, the Brauer degrees need not divide the order of the group, and a Brauer character version of the Itô-Michler theorem only holds for the given prime p.

Fong-Swan theorem implies that for p-solvable groups G, if $\chi(1)_p \leq p$ for all $\chi \in \mathrm{Irr}(G)$ then $\varphi(1)_p \leq p$ for all $\varphi \in \mathrm{IBr}_p(G)$. Moving away from p-solvable groups, this condition does not hold. For example, if one takes $G = \mathrm{M}_{22}$ and $p = 2$, then $\mathbf{O}_2(G) = 1$ and $\beta(1)_2 \leq 2$ for all $\beta \in \mathrm{IBr}_p(G)$ but $|G|_2 = 2^7$ and there exists a character $\chi \in \mathrm{Irr}(G)$ with $\chi(1)_2 = 2^3$. However, if the group has an abelian Sylow p-subgroup, then a recent result of Kessar and Malle [1] on Brauer's height zero conjecture implies the following.

Theorem 4. *Let p be a prime and let G be a finite group with $\mathbf{O}_p(G) = 1$. If G has an abelian Sylow p-subgroup and $\varphi(1)_p \leq p$ for every $\varphi \in \mathrm{IBr}_p(G)$, then $\chi(1)_p \leq p$ for every $\chi \in \mathrm{Irr}(G)$.*

As a corollary, we deduce that for a p-solvable group G, $\chi(1)_p \leq p$ for all $\chi \in \mathrm{Irr}(G)$ if and only if $\varphi(1)_p \leq p$ for all $\varphi \in \mathrm{IBr}(G)$. Obviously, for arbitrary finite groups, we do not have such an equivalence. Nevertheless, we obtain the following.

Theorem 5. *Let p be a prime and G be a finite group with $\mathbf{O}_p(G) = 1$. If $\beta(1)_p \leq p$ for all $\beta \in \mathrm{IBr}_p(G)$, then the following hold.*

(*i*) *If $p = 2$, then $|G|_2 \leq 2^9$.*
(*ii*) *if $p \geq 5$ or if $p = 3$ and A_7 is not involved in G, then $|G|_p \leq p^4$.*
(*iii*) *If $p = 3$ and A_7 is involved in G, then $|G|_3 \leq 3^5$.*

It seems that the bounds in Theorem 5 are probably not best possible. We conjecture that the correct bounds should be 2^7 in (*i*), p^2 in (*ii*), and 3^3 in (*iii*).

References

[1] R. Kessar, G. Malle, *Quasi-isolated blocks and Brauer's height zero conjecture*, Ann. of Math. (2) **178** (2013), 321–384.

[2] M. L. Lewis, G. Navarro, T.R. Wolf, *p-Parts of character degrees and the index of the Fitting subgroup*, J. Algebra **411** (2014), 182–190.

[3] M. Lewis, G. Navarro, P.H. Tiep, H.P. Tong-Viet, *p-Parts of character degrees*, submitted.

Harish-Chandra branching graphs of classical groups and branching graphs of Iwahori-Hecke algebras

Gerhard Hiss

(joint work with Thomas Gerber and Nicolas Jacon)

In the recent paper [3], Thomas Gerber, Nicolas Jacon and the author introduced a Harish-Chandra branching graph for unipotent ℓ-modular representations of certain classical groups. Here ℓ is a prime different from the defining characteristic of the group. The Harish-Chandra branching graph of [3] is defined in a way analogous to Kleshchev's branching graph for irreducible representations of symmetric groups in characteristic ℓ, where we replace restriction from S_n to S_{n-1} by Harish-Chandra restriction from $\mathrm{GU}_n(q)$ to $\mathrm{GU}_{n-2}(q)$, and so on for other classical groups. We have also formulated a series of conjectures relating the Harish-Chandra branching graph for the unitary groups with the crystal graph of a certain integrable module for an affine quantum algebra of type A. Furthermore, we have introduced the concept of weak Harish-Chandra series partitioning the usual Harish-Chandra series.

Current joint work with Thomas Gerber [2] takes some steps towards proving parts of the conjectures of [3]. The first new result asserts that every connected components of the Harish-Chandra branching graph has a unique source vertex, i.e., one with only outgoing edges. As such a source vertex corresponds to a weakly cuspidal module, the connected components of the Harish-Chandra branching graph correspond to the weak Harish-Chandra series introduced in [3]. In our second result we prove that a connected component arising from a weakly cuspidal pair $(\mathrm{GU}_n(q), X)$ is isomorphic to the usual branching graph of the corresponding series of Iwahori-Hecke algebras

$$\mathcal{H}_0 \hookrightarrow \mathcal{H}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{H}_m \hookrightarrow \cdots ,$$

where

$$\mathcal{H}_m := \mathrm{End}_{kG_m}(R_{G_0}^{G_m}(X))$$

with $G_m := \mathrm{GU}_{n+2m}(q)$ and k an algebraically closed field of characteristic ℓ. From [3] we know that the algebras $\mathcal{H}_m$ are Iwahori-Hecke algebras of type B_m, and we have a precise conjecture on the parameters of these algebras.

A result of Ariki [1] states that the branching graph of a series of Iwahori-Hecke algebras as above, provided their parameters are as conjectured, is isomorphic to the connected component of the crystal graph containing the trivial weight.

References

[1] S. Ariki, *Proof of the modular branching rule for cyclotomic Hecke algebras*, J. Algebra, **306** (2007), 290–300.
[2] T. Gerber and G. Hiss, *Branching graphs for finite unitary groups in non-defining characteristic*, preprint, 2015, arXiv:1502.01868.
[3] T. Gerber, G. Hiss and N. Jacon, *Harish-Chandra series in finite unitary groups and crystal graphs*, Int. Math. Res. Not., 2015, doi: 10.1093/imrn/rnv058, Advance Access published March 9, 2015.

Cohomology of fusion systems

Sejong Park
(joint work with Antonio Díaz)

The classical Cartan–Eilenberg stable elements theorem says that the mod-p cohomology of a finite group G is determined by its p-fusion system $\mathcal{F}_P(G)$ with P a Sylow p-subgroup of G, as the limit of the cohomology functor $H^*(-;\mathbb{F}_p)$, viewed as a contravariant functor over $\mathcal{F}_P(G)$. Thus it is natural to define the cohomology of a fusion system $\mathcal{F}$ as

$$H^*(\mathcal{F}) := \varprojlim_{\mathcal{F}} H^*(-;\mathbb{F}_p).$$

With this definition, Mislin's theorem [4] on isomorphism of cohomology and control of fusion for finite groups generalizes to saturated fusion systems.

Theorem.([5]) Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P and let $\mathcal{E}$ be a saturated subsystem of $\mathcal{F}$ on the same p-group P. Suppose $H^*(\mathcal{F}) = H^*(\mathcal{E})$. Then $\mathcal{F} = \mathcal{E}$.

It is conjectured that Dwyer's subgroup sharpness [3] for the classifying space BG of a finite group G generalizes to arbitrary saturated fusion systems $\mathcal{F}$ and arbitrary Mackey functors $M = (M*, M_*)$ for $\mathcal{F}$.

Conjecture.([2]) Let $\mathcal{F}$ be a saturated fusion system on a finite p-group and let $M = (M^*, M_*)\colon \mathcal{O}(\mathcal{F}) \to \mathbf{mod}(\mathbb{F}_p)$ be a Mackey functor over the full orbit category of $\mathcal{F}$ with values finite dimensional $\mathbb{F}_p$-vector spaces. Then the higher limits of the contravariant part M^* over the centric orbit category vanish:

$$\varprojlim{}^i_{\mathcal{O}(\mathcal{F}^c)} M^*|_{\mathcal{O}(\mathcal{F}^c)} = 0 \qquad \forall i > 0.$$

This conjecture is confirmed for the group case and for some small exotic fusion systems.

Proposition.([2]) The conjecture holds for any fusion system $\mathcal{F} = \mathcal{F}_P(G)$ of a finite group G with P a Sylow p-subgroup of G.

Proposition.([2]) Let P be a nonabelian finite p-group with an abelian subgroup of index p. The conjecture holds for any saturated fusion system $\mathcal{F}$ on P.

In the above results, what really matters is that the cohomology functor is a Mackey functor, or a biset functor. As saturated fusion systems are induced by certain (P, P)-bisets, biset functors mix well with fusion systems. Indeed this was the key ingredient of all the above results. The sharpness conjecture is analogous to the existence and uniqueness of a centric linking system [1], which is about vanishing higher limits of the center functor.

References

[1] A. Chermak, Fusion systems and localities, *Acta Math.* **211** (2013), no. 1, 47–139.

[2] A. Díaz and S. Park, *Mackey functors and sharpness for fusion systems*, to appear in Homology, Homotopy Appl.

[3] W.G. Dwyer, Sharp homology decompositions for classifying spaces of finite groups, Group representations: cohomology, group actions and topology (Seatle, WA, 1996), *Amer. Math. Soc.*, Providence, RI, (1998), 197–220

[4] Guido Mislin, *On group homomorphisms inducing mod-p cohomology isomorphisms*, Comment. Math. Helv. **65** (1990), no. 3, 454–461.

[5] S. Park, *Mislin's theorem for fusion systems via Mackey functors*, preprint.

Existence, uniqueness and functoriality of the perfect locality

Lluís Puig

If P is a finite p-group and G a finite group containing P as a Sylow p-subgroup, the Frobenius P-category $\mathcal{F}_G$ of G is the category where the subgroups of P are the objects and the set of $\mathcal{F}_G$-morphisms from R to Q is the quotient $T_G(R,Q)/C_G(R)$; but, we also have a category $\mathcal{P}_G$ over the same objects where the set of $\mathcal{P}_G$-morphisms are the quotients $T_G(R,Q)/\mathbb{O}^p\big(C_G(R)\big)$. In 1994 Dave Benson, dealing with any abstract Frobenius P-category $\mathcal{F}$ that I had already defined, asked for the existence of a suitable *regular extension category* $\mathcal{P}$ of $\mathcal{F}$ analogous to $\mathcal{P}_G$.

Indeed, I had already proved that the group $\mathcal{F}(Q)$ of $\mathcal{F}$-automorphisms of $Q \subset P$ admited a suitable extension $\mathcal{P}(Q)$ by the *hyperfocal quotient* of $C_{\mathcal{F}}(Q)$ — called the $\mathcal{F}$-*localizer* of Q . Thus, the category we were looking for was an extension of $\mathcal{F}$ admitting the $\mathcal{F}$-*localizers* as groups of automorphisms of the objects, endowed with a functor from the category $\mathcal{T}_P$ of *transporters* in P . Actually, in that time the effort was restricted to the $\mathcal{F}$-*selfcentralizing* subgroups Q of P [3, Chap. 4].

After many tentatives, in 2012 Andy Chermak has found a proof of the existence and the uniqueness of $\mathcal{P}$ over the $\mathcal{F}$-*selfcentralizing* subgroups—noted $\mathcal{P}^{\mathrm{sc}}$—based on the Classification of Finite Simple Groups (CFSG).

Of course, I was very disappointed on the use of CFSG in such a subject and came back towards my own tentatives. I had already proved that the existence and the uniqueness of $\mathcal{P}^{\mathrm{sc}}$ imply the existence and the uniqueness of $\mathcal{P}$ [3, Chap. 20], opening the possibility of discussing on *functoriality*. More generally, I considered the $\mathcal{F}$-*localities*, namely the category extensions $\mathcal{L}$ of $\mathcal{F}$ endowed with a functor from the category $\mathcal{T}_P$ of *transporters* in P.

Proposition 1. *For any p-coherent $\mathcal{F}$-locality $\mathcal{L}$, any $\mathcal{F}^{\rm sc}$-locality functor $\mathfrak{h}^{\rm sc}$ from $\mathcal{P}^{\rm sc}$ to $\mathcal{L}^{\rm sc}$ can be extended to a unique $\mathcal{F}$-locality functor $\mathfrak{h} : \mathcal{P} \to \mathcal{L}$.*

Actually, from any $\mathcal{F}$-*basic* $P \times P$-*set* Ω [3, Chap. 21] we can construct an $\mathcal{F}$-*locality* $\mathcal{L}^{\Omega}$. Conversely, in [3, Chap. 24] we already proved that the existence of $\mathcal{P}^{\rm sc}$ allows to construct a suitable $\mathcal{F}$-*basic* $P \times P$-*set* $\Omega^{\rm n}$ in such a way that then $\mathcal{P}^{\rm sc}$ is "contained" in a suitable quotient $(\overline{\mathcal{L}^{\Omega^{\rm n}}})^{\rm sc}$ of $(\mathcal{L}^{\Omega^{\rm n}})^{\rm sc}$. The starting point in my proof of the existence and the uniqueness of $\mathcal{P}^{\rm sc}$ is that I can prove the existence of such a $\Omega^{\rm n}$ — called *natural*.

Proposition 2. *Ω^n contains the union $\bigsqcup_Q \bigsqcup_{\tilde\varphi} (P \times P)/\Delta_\varphi(Q)$, where Q runs over a set of representatives for the set of P-conjugacy classes of $\mathcal{F}$-selfcentralizing subgroups of P and $\tilde\varphi$ over a set of representatives for the set of $\tilde{\mathcal{F}}_P(Q)$-orbits in $\tilde{\mathcal{F}}(P,Q)_{\tilde\iota_Q^P}$. Moreover, we have $|(\Omega^{\rm n})^{\Delta(Q)}| = |Z(Q)|$.*

As a matter of fact, all this is true restricted to a nonempty set $\mathfrak{X}$ of $\mathcal{F}$-*selfcentralizing* subgroups of P, closed by $\mathcal{F}$-isomorphisms and by up-inclusions. Thus, arguing by induction on $|\mathfrak{X}|$, considering a minimal element U in $\mathfrak{X}$ and denoting by $\mathfrak{Y}$ the complement in $\mathfrak{X}$ of the $\mathcal{F}$-isomorphism class of U, we may assume that $(\overline{\mathcal{L}^{\Omega^{\rm n}}})^{\mathfrak{Y}}$ contains the unique *perfect* $\mathcal{F}^{\mathfrak{Y}}$- *locality* $\mathcal{P}^{\mathfrak{Y}}$. At this point, the main problem is of lifting $\mathcal{P}^{\mathfrak{Y}}$ to $(\overline{\mathcal{L}^{\Omega^{\rm n}}})^{\mathfrak{X}}$ since the *perfect* $\mathcal{F}^{\mathfrak{X}}$-*locality* $\mathcal{P}^{\mathfrak{X}}$ then appears in a suitable quotient of $(\overline{\mathcal{L}^{\Omega^{\rm n}}})^{\mathfrak{X}}$. Of course, the lifting question leads to a $\mathfrak{Ker}(\rho^{\mathfrak{X}})$-*valued* 2-*cocycle* k over the *exterior quotint* $\tilde{\mathcal{F}}^{\mathfrak{X}}$ where $\rho^{\mathfrak{X}} : \bar{\mathcal{M}}^{\mathfrak{X}} \to \mathcal{F}^{\mathfrak{X}}$ is the structural functor of a suitable $\mathcal{F}^{\mathfrak{X}}$-*locality* obtained from the converse image of $\mathcal{P}^{\mathfrak{Y}}$ in $(\overline{\mathcal{L}^{\Omega^{\rm n}}})^{\mathfrak{X}}$.

Theorem 3. *The* 2-*cocycle k is a* 2-*coboundary.*

The Proposition 3.2 in [1] and a result in [5] allow to reduce the question to the restriction of k to the *normalizer* $N_{\tilde{\mathcal{F}}^{\mathfrak{X}}}(U)$ of U.

Once we have the whole *perfect $\mathcal{F}$-locality* $\mathcal{P}$, it makes sense to formulate the *functoriaity question*: with obvious notation, if $f : P' \to P$ is a group homomorphism inducing a functor $\mathfrak{f} : \mathcal{F}' \to \mathcal{F}$, is there a functor $\mathfrak{g} : \mathcal{P}' \to \mathcal{P}$ lifting $\mathfrak{f}$? If f is surjective, in [3, Chap. 17] I already gave a positive answer. If f is injective, I only am able to give a positive answer for the quotient $\bar{\mathcal{P}}$ of $\mathcal{P}$ by the commutator group of any $C_G(Q)$.

The problem is that, since the functor $\mathfrak{f}$ needs not preserve *selfcentrality*, the fact that $\mathcal{P}^{\rm sc}$ is "contained" in $(\overline{\mathcal{L}^{\Omega^{\rm n}}})^{\rm sc}$ of $(\mathcal{L}^{\Omega^{\rm n}})^{\rm sc}$ does not help very much. The solution comes from the *basic $\mathcal{F}$-locality* $\mathcal{L}^{\rm b}$, namely the $\mathcal{F}$-*locality* $\mathcal{L}^{\Omega}$ for a $\mathcal{F}$-*basic* $P \times P$-*set* Ω which is "big enough".

Proposition 4. *For any $\mathcal{F}$-basic $P \times P$-set Ω, $\mathcal{L}^{\rm b}$ contains $\mathcal{L}^{\Omega}$.*

The proof of this result needs my classification of $\mathcal{F}$-*basic* $P \times P$-*sets* in [4, Theorem 6.6]. Then, the strategy goes through the following result.

Theorem 5. *The basic $\mathcal{F}$-locality $\mathcal{L}^{\mathrm{b}}$ contains $\bar{\mathcal{P}}$.*

According to Proposition 1, in order to define a functor from the *perfect* to the *basic* $\mathcal{F}$-localities, it suffices get a functor $\mathfrak{h}^{\mathrm{sc}} : \mathcal{P}^{\mathrm{sc}} \to (\mathcal{L}^{\mathrm{b}})^{\mathrm{sc}}$; actually, from Proposition 4 it is not difficult to prove that $(\overline{\mathcal{L}^{\Omega^{\mathrm{n}}}})^{\mathrm{sc}}$ is "contained" in a suitable quotient $(\overline{\mathcal{L}^{\mathrm{b}}})^{\mathrm{sc}}$ of $(\mathcal{L}^{\mathrm{b}})^{\mathrm{sc}}$ and therefore $\mathcal{P}^{\mathrm{sc}}$ is also "contained" in $(\overline{\mathcal{L}^{\mathrm{b}}})^{\mathrm{sc}}$. Thus, it remains to lift this inclusion to a functor to $(\mathcal{L}^{\mathrm{b}})^{\mathrm{sc}}$; the kernel of the canonical functor $(\mathcal{L}^{\mathrm{b}})^{\mathrm{sc}} \to (\overline{\mathcal{L}^{\mathrm{b}}})^{\mathrm{sc}}$ admits a reasonable filtration which, once again, allows to apply a result in [5].

References

[1] Carles Broto, Ran Levi and Bob Oliver, *The homotopy theory of fusion systems*, Journal of Amer. Math. Soc. 16(2003), 779-856.
[2] Stefan Jackowski and James McClure, *Homotopy decomposition of classifying spaces via elementary abelian subgroups*, Topology, 31(1992), 113-132.
[3] Lluís Puig, *"Frobenius categories versus Brauer blocks"*, Progress in Math. 274(2009)
[4] Lluís Puig, *The Hecke algebra of a Frobenius Pcategory*, Algebra Colloquium 21(2014).
[5] Lluís Puig, *A criterion on vanishing cohomology*, submitted to J. of Algebra.

Categorical actions on unipotent representations of finite unitary groups

Olivier Dudas
(joint work with Michela Varagnolo and Éric Vasserot)

We are interested in the representation theory of finite classical groups, and in particular finite unitary groups, over a field of positive characteristic ℓ (different from the defining characteristic of the algebraic group). Finite unitary groups are defined by

$$\mathrm{GU}_n(q) = \{M \in \mathrm{GL}_n(q^2) \,|\, M^t\mathrm{Fr}(M) = I_n\}$$

where $\mathrm{Fr} : (m_{i,j}) \longmapsto (m_{i,j}^q)$ is the standard Frobenius endomorphism of $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$. Representations of $\mathrm{GU}_n(q)$ over an algebraic closed field $k = \overline{k}$ of characteristic ℓ form the abelian category $k\mathrm{GU}_n(q)$-mod. We will focus on the Serre subcategory formed by the so-called *unipotent representations*, which we will denote by $k\mathrm{GU}_n(q)$-umod. When $\ell \neq q$ (non-defining characteristic case), there is an adjoint pair of exact functors

$$F_{n,n+2} : \, k\mathrm{GU}_n(q)\text{-umod} \rightleftharpoons k\mathrm{GU}_{n+2}(q)\text{-umod} : E_{n,n+2}$$

given by Harish-Chandra (or parabolic) induction and restriction. These functors yield endofunctors $F = \bigoplus F_{n,n+2}$ and $E = \bigoplus E_{n,n+2}$ of the category

$$\mathcal{C} = \bigoplus_{n\geq 0} k\mathrm{GU}_n(q)\text{-umod}.$$

Motived by the pioneering work of Chuang and Rouquier on the case of finite general linear groups [1], we show that F and E endow $\mathcal{C}$ with a structure of *higher representation* for a well-identified Kac-Moody algebra $\mathfrak{g}$.

A categorical action of $\mathfrak{g}$ should consist of an abelian category acted on by endofunctors inducing an action of the Lie algebra $\mathfrak{g}$ on the Grothendieck group. This notion is actually too weak, and Chuang-Rouquier [1] and Rouquier [4] have axiomatized the good notion of such a higher action. For our category $\mathcal{C}$, it will be given by natural transformations of F and F^2 satisfying the relations of an affine Hecke algebra of type A.

Proposition. There exist natural transformations $X \in \mathrm{End}(F)^\times$ and $T \in \mathrm{End}(F^2)$ such that

(i) $(T^2 - q^2 1_{F^2}) \circ (T^2 + 1_{F^2}) = 0$
(ii) $(T1_F) \circ (1_F T) \circ (T1_F) = (1_F T) \circ (T1_F) \circ (1_F T)$
(iii) $T \circ (1_F X) T = q^2 X 1_F$

From this datum we can construct a proper categorical action. First, we define the functors F_a and E_a by the generalized a-eigenspace of X on F and E. Then we consider the Kac-Moody algebra $\mathfrak{g}$ associated with the quiver whose vertices are the eigenvalues of X (here $(-q)^{\mathbb{Z}}$) and with arrows $a \longrightarrow aq^2$. Let e be the order of $-q$ modulo ℓ. If $e = \infty$ (*i.e.* $\mathrm{char}\, k = 0$) then $\mathfrak{g} = \mathfrak{sl}_\infty \oplus \mathfrak{sl}_\infty$; otherwise if e is finite we can distinguish two cases:

- if e is odd then $\mathfrak{g} = \widehat{\mathfrak{sl}}_e$ (*unitary prime* case)
- if e is even then $\mathfrak{g} = \widehat{\mathfrak{sl}}_{e/2} \oplus \widehat{\mathfrak{sl}}_{e/2}$ (*linear prime* case)

Then the Lie algebra $\mathfrak{g}$ acts on the category $\mathcal{C}$ in the following sense:

Theorem. Let $V = \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{C})$ and $f_a = [F_a]$, $e_a = [E_a]$ be the linear endomorphisms of V induced by the functors E_a and F_a. Then

(i) the action of $\langle e_a, f_a \rangle_{a \in (-q)^{\mathbb{Z}}}$ induces an integrable action of $\mathfrak{g}$ on V
(ii) there is a decomposition $\mathcal{C} = \bigoplus \mathcal{C}_\omega$ with $V_\omega = \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{C}_\omega)$ and such that $F_a \mathcal{C}_\omega \subset \mathcal{C}_{\omega - \alpha_a}$ and $E_a \mathcal{C}_\omega \subset \mathcal{C}_{\omega + \alpha_a}$

It turns out that this action encodes a large part of the representation theory of finite unitary groups, as it does in the case of linear groups. For example, cuspidal representations will correspond to highest weight vectors, and when e is odd (resp. when e is even), blocks correspond to weight spaces (resp. weight spaces in a fixed ordinary Harish-Chandra series).

It is important to note that in our case the structure of $\mathfrak{g}$-module of V can be made explicit. It is isomorphic to direct sum of level 2 Fock space representations and can be related to the level 1 (corresponding to $\mathrm{GL}(q)$) by Uglov's construction. This gives, at least on the Grothendieck group, an explicit relation between Harish-Chandra induction and 2-Harish-Chandra induction (defined by Lusztig induction from 2-split Levi subgroups). It will be interesting to show the same relation between the functors.

To finish, let us give some applications of the existence of categorical actions on $\mathcal{C}$. By [1], the action of the affine Weyl group lifts to derived self-equivalences of $\mathcal{C}$. Using the work of Livesey [3] on the existence of good blocks for linear primes, we deduce the following:

Theorem. Assume the order of $-q$ modulo ℓ is even. Then Broué's abelian defect group conjecture holds for $\mathrm{GU}_n(q)$.

Also, from the explicit structure of $\mathfrak{g}$-module of $K_0(\mathcal{C})$ we can deduce the modular branching rule for Harish-Chandra induction, as well as the various Hecke algebras occuring as endomorphism algebras of induced representations. This gives a partial proof of the recent conjectures of Gerber-Hiss-Jacon [2].

Theorem. The modular branching graph for Harish-Chandra induction is isomorphic to the crystal graph of the level 2 Fock space representation V.

Note that many of these results have already been generalized to other classical groups, but this is still a work in progress.

References

[1] J. Chuang and R. Rouquier, *Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification*, Annals of Math. 167 (2008), 245-298.

[2] T. Gerber, G. Hiss and N. Jacon, *Harish-Chandra series in finite unitary groups and crystal graphs*, 2014, to appear in Int. Math. Res. Not.

[3] Livesey, M. *On Rouquier Blocks for Finite Classical Groups at Linear Primes*, J. Algebra 432, 2015.

[4] Rouquier, R., *2-Kac-Moody algebras*, arXiv:0812.5023v1.

Representations of finite sets

Serge Bouc and Jacques Thévenaz

We develop the representation theory of finite sets and correspondences. Given a commutative ring k, let $k\mathcal{C}$ be the category of finite sets, in which morphisms are k-linear combinations of correspondences. We define a *correspondence functor* (over k) to be a k-linear functor from $k\mathcal{C}$ to the category k-Mod of all k-modules. Let $\mathcal{F}_k$ be the category of correspondence functors, an abelian category.

We first obtain a series of results on the category $\mathcal{F}_k$ when k is a field.

Theorem 1. *There is a parametrization of all simple correspondence functors by isomorphism classes of triples (E, R, V) consisting of a finite set E, a partial order relation R on E, and a simple $k\,\mathrm{Aut}(E, R)$-module V, where $\mathrm{Aut}(E, R)$ is the automorphism group of the poset (E, R).*

This is based on our previous work [1] on the algebra of essential relations on a finite set E, which is a quotient of the algebra $k\mathcal{C}(E, E)$ of all relations on E. The simple modules for this algebra of essential relations are parametrized by pairs (R, V), where R and V are as above, and this implies the classification of Theorem 1.

Theorem 2. *If k is a field, a correspondence functor F is finitely generated if and only if $\dim_k \big(F(X)\big)$ is bounded above by an exponential function of $|X|$.*

Actually there exists a positive integer c and a positive rational number r such that the dimension of $F(X)$ is equivalent to $r \cdot c^{|X|}$ as $|X| \to +\infty$.

Our next result is specific to correspondence functors, because it does not hold for other kinds of categories of functors (e.g. biset functors [2]).

Theorem 3. *If k is a field, any finitely generated correspondence functor has finite length.*

When k is a field, the finitely generated correspondence functors form an abelian subcategory $\mathcal{F}_k^f$ of $\mathcal{F}_k$, in which the Krull-Remak-Schmidt theorem holds. Moreover, an object of $\mathcal{F}_k^f$ is projective if and only if it is injective, so the global dimension of $\mathcal{F}_k^f$ is infinite. We also show (for an arbitrary commutative ring k) that the algebra $k\mathcal{C}(E,E)$ of all relations on a finite set E is symmetric, by constructing an explicit trace map, and it follows (when k is a field) that the head of a finitely generated projective correspondence functor is isomorphic to its socle. In particular, the (infinite) Cartan matrix of $\mathcal{F}_k^f$ is symmetric.

Questions. The previous results open a number of questions about the representation theory of finite sets and correspondences. Decomposition theory? Non degeneracy of the Cartan matrix? Extensions? Stable category? Growth of projective resolutions? Blocks? etc.

For an arbitrary commutative ring k, we also show that there is a natural tensor structure on $\mathcal{F}_k$, and an adjoint internal hom-functor. Another special feature of our work is the construction which associates a correspondence functor F_T to any finite lattice T. For a finite set X, the evaluation $F_T(X)$ is the free k-module with basis the set T^X of all maps from X to T. We introduce a suitable k-linear category $\mathcal{L}_k$ of lattices for which the assignment $T \to F_T$ becomes a fully faithful k-linear functor $\mathcal{L}_k \to \mathcal{F}_k$. We show that the functor F_T is projective in $\mathcal{F}_k$ if and only if the lattice T is distributive.

The case where T is a total order is of particular interest : the endomorphism algebra of F_T turns out to be naturally isomorphic to a direct product of matrix algebras over k. Consequently, when k is a field and R is a total order on E, the simple functor $S_{E,R,k}$ parametrized by (E,R,k) is projective (and injective).

More generally, for an arbitrary finite poset (E,R), we obtain an explicit description of the simple functor $S_{E,R,V}$, indexed by the triple (E,R,V), as a quotient of a functor F_T, for a suitable lattice T containing E as a full subposet. Using this description and some arguments which are rather involved, we obtain a formula for the dimension of evaluations of simple functors.

Theorem 4. *Let k be a field and let (E,R,V) be as above. There is an (explicit) subset G with $E \subseteq G \subseteq T$ such that*

$$\dim_k\big(S_{E,R,V}(X)\big) = \frac{\dim_k(V)}{|\operatorname{Aut}(E,R)|}\sum_{i=0}^{|E|}(-1)^i\binom{|E|}{i}(|G|-i)^{|X|}\,.$$

This formula gives an explicit version, for simple functors, of the exponential behaviour of dimensions mentioned in Theorem 2.

References

[1] S. Bouc, J. Thévenaz, *The algebra of essential relations on a finite set*, J. reine angew. Math. (2015), to appear.

[2] S. Bouc, *Biset functors for finite groups*, Springer Lecture Notes in Mathematics no. 1990 (2010).

Simple biset functors and double Burnside ring

Radu Stancu

(joint work with Serge Bouc, Jacques Thévenaz)

Let G be a finite group and let k be a field. The double Burnside ring $B(G,G)$ of all (G,G)-bisets appears in the theory of biset functors developped by Serge Bouc [2, 4], which was successfully used for the classification of endo-permutation modules for a p-group [3], and also in homotopy theory, where the subring of bi-free bisets of $B(G,G)$ appears in the theory of p-completed classifying spaces [8, 1], fusion systems and p-local finite groups [7, 9, 10]. An important problem concerning the ring structure is to understand the simple $kB(G,G)$-modules. This is a difficult task in general, as, for instance, $kB(G,G)$ is semi-simple only for cyclic groups in suitable characteristic, e.g. in characteristic zero. Moreover, it was shown recently by Rognerud [11] that in general this algebra is not quasi-hereditary.

The main purpose of this work [5, 6] is to analyse the simple $kB(G,G)$-modules, using their connection with simple biset functors. This connection is quite deep since any simple $kB(G,G)$-module determines uniquely a simple biset functor, and conversely any (k-linear) simple biset functor has an evaluation at G which is a simple $kB(G,G)$-module (provided that this evaluation is non-zero).

Consider the finite-dimensional $kB(G,G)$-module $kB(G,H) = k \otimes_{\mathbb{Z}} B(G,H)$, where G and H are finite groups and $B(G,H)$ is the Grothendieck group of (G,H)-bisets, and also the standard quotient $k\overline{B}(G,H) = kB(G,H)/kI(G,H)$, where $kI(G,H)$ is the k-subspace generated by all bisets which factor through a proper subquotient of H. We define a canonical bilinear form on $k\overline{B}(G,H)$, depending on a semisimple quotient E of $k\overline{B}(H,H) = k\mathrm{Out}(H)$ and pass to the quotient by the right kernel $R(G,H)$ of this form. Allowing G to vary, we obtain a biset functor $k\overline{B}(-,H)$ and a subfunctor $R(-,H)$. We prove that $k\overline{B}(-,H)/R(-,H)$ is semi-simple. Moreover, when $E = k\mathrm{Out}(H)/J(k\mathrm{Out}(H))$, the biset functor $k\overline{B}(-,H)/R(-,H)$ the largest semi-simple quotient of $k\overline{B}(-,H)$.

Evaluation of the functor $k\overline{B}(-,H)/R(-,H)$ at G gives rise to a semi-simple $kB(G,G)$-module $k\overline{B}(G,H)/R(G,H)$ with an explicit decomposition into simple summands. This provides the main tool for analysing simple $kB(G,G)$-modules, or equivalently, evaluation of simple biset functors. In particular, if V is a simple $k\mathrm{Out}(H)$-module and $E = \mathrm{End}_k(V)$, we obtain a formula for the dimension of the evaluation $S_{H,V}(G)$ of a simple biset functor $S_{H,V}$, in terms of the rank of the bilinear form we introduced. The following theorem summarize the above results.

Theorem. Let H and G be finite groups and let $\pi : kB(H,H) \to E$ be the projection on a semi-simple quotient E and $\tau : E \to k$ be a symmetrizing form. Define

$$<-,->_G \,: k\overline{B}(G,H) \times k\overline{B}(G,H) \longrightarrow k\,, \qquad <\overline{\phi},\overline{\psi}>_G = \tau\pi(\phi^{\mathrm{op}}\psi)\,,$$

where $\phi, \psi \in kB(G,H)$ are representatives of $\overline{\phi}$ and $\overline{\psi}$ respectively. Let $R(G,H)$ be the right kernel of the form $<-,->_G$, that is, the set of all elements $\overline{\psi} \in k\overline{B}(G,H)$ such that $<\overline{\phi},\overline{\psi}>_G = 0$ for all $\overline{\phi} \in k\overline{B}(G,H)$. We have the following results.

(1) The map $<-,->_G$ is well-defined and k-bilinear.
(2) $R(-,H)$ is a subfunctor of $k\overline{B}(-,H)$ and $R(G,H)$ is a right $k\mathrm{Out}(H)$-submodule of $k\overline{B}(G,H)$.
(3) The quotient functor $k\overline{B}(-,H)/R(-,H)$ is isomorphic to $S_{H,E}$.
(4) Let V be a simple $k\mathrm{Out}(H)$-module, $E = \mathrm{End}_k(V)$, and $\tau = \tau_V$ be the trace map on $\mathrm{End}_k(V)$. Then

$$\dim(S_{H,V}(G)) = \frac{\mathrm{rank}<-,->_G}{\dim(V)}\,.$$

(5) Let $E = k\mathrm{Out}(H)/J(k\mathrm{Out}(H)) \cong \prod_{i=1}^{r} \mathrm{End}_k(V_i)$, where $V_1, \ldots, V_r$ are the simple $k\mathrm{Out}(H)$-modules, and $\tau = \sum_{i=1}^{r} \tau_{V_i}$. Then $k\overline{B}(-,H)/R(-,H)$ is the largest semi-simple quotient of the biset functor $k\overline{B}(-,H)$. In other words, $R(-,H)$ is the Jacobson radical of $k\overline{B}(-,H)$.

Using the theorem above for $E = k\mathrm{Out}(H)/J(k\mathrm{Out}(H))$, we found that $R(-,H)$ is the Jacobson radical of $k\overline{B}(-,H)$, as biset functors. A natural question is whether, on evaluation at G, the $kB(G,G)$-module $R(G,H)$ is the Jacobson radical of $k\overline{B}(G,H)$. The answer to this question is positive if we can find $\alpha \in kB(H,G)$ and $\beta \in kB(G,H)$ such that $\alpha\beta \equiv \mathrm{Id} \mod kI(H,H)$.

Proposition. Suppose that there exists $\alpha \in kB(H,G)$ and $\beta \in kB(G,H)$ such that $\alpha\beta \equiv \mathrm{Id} \mod kI(H,H)$. Then

$$J\big(k\overline{B}(G,H)\big) = R(G,H)$$

and the only simple quotients of $k\overline{B}(G,H)$ (as $kB(G,G)$-module) are the simple modules $S_{H,V}(G)$, where V is a simple $k\mathrm{Out}(H)$-module.

Moreover, the above assumption is satisfied in each of the following cases.

(1) H is isomorphic to a quotient of G.
(2) G is abelian and H is a subquotient og G.
(3) H is isomorphic to a subgroup Z of G such that $N_G(Z) = ZC_G(Z)$, and $|N_G(Z) : Z|$ is non-zero in k.
(4) H is isomorphic to a central subgroup Z of G, and $|G : Z|$ is non-zero in k.

The answer in general is negative, and one of the smallest example that we can give is $G = \mathrm{Alt}(5)$, $H = C_3$ and k a field of characteristic different from 2. In this particular case we have $\dim_k k\overline{B}(G,H) = 3$, $\dim_k k\overline{B}(G,H)/R(G,H) = 1$ and $J(k\overline{B}(G,H)) = 1$.

References

[1] D. Benson, M. Feshbach. *Stable splittings of classifying spaces of finite groups*, Topology **31** (1992), 157–176.
[2] S. Bouc. *Foncteurs d'ensembles munis d'une double action*, J. Algebra **183** (1996), 664–736.
[3] S. Bouc. *The Dade group of a p-group*, Invent. Math. **164** (2006), 189–231.
[4] S. Bouc. Biset functors for finite groups, *Springer Lecture Notes in Mathematics* no. **1990** (2010).
[5] S. Bouc, J. Thévenaz, R. Stancu *Simple biset functors and double Burnside ring*, J. Pure Appl. Algebra **217** (2013), no. 3, 546–566.
[6] S. Bouc, J. Thévenaz, R. Stancu *Vanishing evaluations of simple functors*, J. Pure Appl. Algebra **218** (2014), no. 2, 218–227.
[7] C. Broto, R. Levi, B. Oliver. *The theory of p-local groups: a survey*, In: Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory, 51–84, Contemp. Math. **346**, Amer. Math. Soc., Providence, 2004.
[8] J. Martino, S. Priddy. *Stable homotopy classification of* $B\hat{G_p}$, Topology **34** (1995), 633–649.
[9] K. Ragnarsson. *Classifying spectra of saturated fusion systems*, Algebr. Geom. Topol. **6** (2006), 195–252.
[10] K. Ragnarsson, R. Stancu. *Saturated fusion systems as idempotents in the double Burnside ring*, Geom. Topol. **17** (2013), no. 2, 839–904.
[11] B. Rognerud, *Quasi-hereditary property of double Burnside algebras*, preprint 2015, to appear in Comptes rendus Mathematique, 7 pages.

Relative projectivity of Specht and simple modules of the symmetric group

Eugenio Giannelli

Given any finite group G and a field F of prime characteristic p, Green's classical theory of vertices, sources and Green correspondents of indecomposable FG-modules provides an important link between the global and the local representation theory of FG, see [6]. Determining these local invariants explicitly is in general an extremely hard task, even when looking at concrete examples. In this talk I will present a series of new results concerning the determination of Green vertices of Specht and simple modules for the symmetric group $\mathfrak{S}_n$.

The problem of determining the vertices of all the simple $F\mathfrak{S}_n$-modules is very difficult, and seems for the moment out of reach. Since the beginning of the century, mathematicians focused on the study of the subfamily of simple modules labelled by hook partitions.

We denote by D^λ the simple $F\mathfrak{S}_n$-module labelled by the partition λ on n. The vertices of $D^{(n-r,1^r)}$ have been studied before by Wildon in [8], by Müller and Zimmermann in [7], and by Danz in [2]. In consequence of these results, the vertices of $D^{(n-r,1^r)}$ have been known, except in the case where $p > 2$, $r = p-1$ and $n \equiv p \pmod{p^2}$. In [3] Danz and myself prove the following theorem, which together with [2, Corollary 5.5] proves [7, Conjecture 1.6(a)].

Theorem 1. Let $p > 2$, let F be a field of characteristic p, and let $n \in \mathbb{N}$ be such that $n \equiv p \pmod{p^2}$. Then the vertices of the simple $F\mathfrak{S}_n$-module $D^{(n-p+1,1^{p-1})}$ are precisely the Sylow p-subgroups of $\mathfrak{S}_n$.

To summarize, the above mentioned results in [2, 7, 8] and [3] lead to the following exhaustive description of the vertices of the modules $D^{(n-r,1^r)}$:

Theorem 2. Let F be a field of characteristic $p > 0$, and let $n \in \mathbb{N}$. Let further $r \in \{0, 1 \ldots, p-1\}$, and let Q be a vertex of the simple $F\mathfrak{S}_n$-module $D^{(n-r,1^r)}$.

(a) If $p \nmid n$ then Q is $\mathfrak{S}_n$-conjugate to a Sylow p-subgroup of $\mathfrak{S}_{n-r-1} \times \mathfrak{S}_r$.

(b) If $p \mid n$ then Q is a Sylow p-subgroup of $\mathfrak{S}_n$, unless $p = 2$, $n = 4$ and $r = 1$. In this last particular case Q is the unique Sylow 2-subgroup of $\mathfrak{A}_4$.

In [4], we are able to determine new lower bounds on the vertices of the entire family of Specht modules. In particular we prove the following theorem.

Theorem 3. Let λ be a partition of n such that the corresponding Specht module S^λ is indecomposable. Let t be a λ-tableau. Denote by $H(t)$ the subgroup of $\mathfrak{S}_n$ consisting of all the permutations $\sigma \in \mathfrak{S}_n$ such that σ permutes rows and columns of t as bocks for its action. Let Q_λ be a Sylow p-subgroup of $H(t)$. Then there exists a vertex Q of S^λ containing Q_λ.

The lower bounds obtained in Theorem 3 are attained in some remarkable cases. For example, if $p \nmid n$ and $\lambda = (n-k, 1^k)$ is a hook-partition then Q_λ is a vertex of S^λ.

A fundamental tool that we used to prove all the results presented above is the extensive application of the Brauer construction, as developed by Broué in [1]. This technique was extremely powerful when combined to the following new description of the Sylow p-subgroups of symmetric (and alternating) groups, obtained in [5] by Lim, Wildon and myself.

Theorem 4. Let G be either $\mathfrak{S}_n$ or the alternating group $\mathfrak{A}_n$. Let P be a p-subgroup of G. Then the following are equivalent.

(a) P is a Sylow p-subgroup of G.

(b) P contains a G-conjugate of every elementary abelian p-subgroup of G.

In this case we say that $\mathfrak{S}_n$ and $\mathfrak{A}_n$ are *p-elementarily large*. I am currently involved in an ongoing joint project with Livesey, aimed to determine whether classical groups satisfy the property described in Theorem 4. So far we showed that for $q = p^k$, both $GL_n(q)$ and $SL_n(q)$ are p-elementarily large.

References

[1] Broué, M. On Scott modules and p-permutation modules: an approach through the Brauer homomorphism. *Proc. Amer. Math. Soc. 93, no. 3* (1985), 401–408.

[2] Danz, S. On vertices of exterior powers of the natural simple module for the symmetric group in odd characteristic. *Arch. Math.* 89 (2007), 485–496.

[3] Danz, S. Giannelli, E. Vertices of simple modules of symmetric groups labelled by hook partitions. *J. Group Theory.* Volume 18, Issue 2, Pages 313–334, 2014.

[4] Giannelli, E. A lower bound on the vertices of Specht modules of the symmetric group. *Arch. Math. (Basel)*, issue 1, volume 103 (2014), pp. 1-9.

[5] Giannelli, E. Lim, K.J. and Wildon, M. Sylow subgroups of symmetric and alternating groups and the vertex of $S^{(kp-p,1^p)}$. *Submitted.* arXiv:1407.1845.

[6] Green, J. A. On the indecomposable representations of a finite group. *Math. Zeitschrift 70* (1958/59), 430–445.
[7] Müller, J., and Zimmermann, R. Green vertices and sources of simple modules of the symmetric group. *Arch. Math. 89* (2007), 97–108.
[8] Wildon, M. Two theorems on the vertices of Specht modules. *Arch. Math. (Basel) 81*, 5 (2003), 505–511.

Self-Normalizing Sylow Subgroups and Galois Automorphisms: Type A in Characteristic 2

Amanda A. Schaeffer Fry

The problem at hand is motivated by the McKay conjecture, which asserts that given a finite group G, the number $|\mathrm{Irr}_{\ell'}(G)|$ of irreducible complex characters with degree relatively prime to a prime ℓ should be the same as that for $N_G(P)$, where P is a Sylow ℓ-subgroup. Although perhaps relatively easy to state, this statement is a bit mysterious - although there is much evidence for this conjecture (and now even a proof due to G. Malle and B. Späth for McKay's original conjecture when $\ell = 2$), the question of *why* it should be true remains unclear.

For this reason, many refinements to the McKay conjecture have been proposed with the hope of finding the "right" version, providing more insight into a more structural reason for the phenomenon. The refinement of interest in this talk is due to G. Navarro and incorporates the role of the action of the Galois group into the conjectured bijection between $\mathrm{Irr}_{\ell'}(G)$ and $\mathrm{Irr}_{\ell'}(N_G(P))$.

Let $\mathbb{Q}_{|G|}$ denote the extension field of $\mathbb{Q}$ obtained by adjoining a primitive $|G|$th root of unity. Then the Galois group $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$ acts on the irreducible characters of G in a natural way, and Navarro's Galois automorphism refinement of the McKay conjecture says that for the proper choices of Galois automorphisms σ in $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$, the number of members of $\mathrm{Irr}_{\ell'}(G)$ fixed by σ should be the same as that for $\mathrm{Irr}_{\ell'}(N_G(P))$ (see [2]). Specifically, these Galois automorphisms $\sigma := \sigma_e$ are chosen to send every ℓ'-root of unity ξ to ξ^{ℓ^e}, where e is a nonnegative integer.

In his paper, Navarro shows that, among other interesting consequences, the validity of his conjecture would lead to a necessary and sufficient condition for a Sylow ℓ-subgroup to be self-normalizing. (In particular, this consequence implies that we can determine from the character table whether a Sylow subgroup is self-normalizing.) For odd primes, Navarro - Tiep - Turull [3] have shown that this particular consequence is true without assuming the conjecture. Namely, they show that for ℓ an odd prime, $P \in \mathrm{Syl}_\ell(G)$ is self-normalizing if and only if there is no nontrivial irreducible ℓ-rational character of G of ℓ'-degree.

The statement takes a different form when $\ell = 2$, and I have been working toward proving this statement without assuming Navarro's Galois automorphism refinement. Namely, the statement for $\ell = 2$ says that a finite group has a self-normalizing Sylow 2-subgroup if and only if every irreducible character of odd degree is fixed by the Galois automorphism fixing 2-power roots of unity and squaring any $2'$-root of unity. In [4], I prove a reduction to simple groups for this statement and show that the alternating, sporadic, and most groups of Lie type in characteristic 2 satisfy this reduction.

What remains is to prove that most simple groups of Lie type in odd characteristic, as well as the groups $E_6^{\pm}(2^a)$ with nontrivial center, satisfy these statements. One of the main issues that arises in these cases is determining how the automorphism group and Galois group act on the Lusztig series. In my talk, I discuss my success in working around this issue in specific cases. Namely, in [4, Section 4], I was able to use the ideas of Cabanes and Späth [1], who make use of generalized Gelfand-Graev representations in their work with the inductive McKay conditions for type A, to show that $PSL_n(2^a)$ and $PSU_n(2^a)$ satisfy my reduction by using multiplicity arguments. It now seems reasonable that an extension of this argument can also be used to complete the case of type A in odd characteristic, and that there is a simplification to my original argument using (usual) Gelfand-Graev representations which will extend to complete the case of $E_6^{\pm}(2^a)$.

References

[1] Marc Cabanes and Britta Späth. *Equivariant character correspondences and inductive Mckay condition for type A*. ArXiv e-prints, to appear in J. reine angew. Math., (1305.6407), 2013.

[2] Gabriel Navarro, *The McKay conjecture and Galois automorphisms*, Ann. of Math. (2), 160(3) (2004), 1129-1140.

[3] Navarro, Gabriel and Tiep, Pham Huu and Turull, Alexandre, *p-rational characters and self- normalizing Sylow p-subgroups*, Represent. Theory 11 (2007), 84-94

[4] Amanda A. Schaeffer Fry, *Odd-degree characters and self-normalizing Sylow subgroups: A reduction to simple groups*, to appear in Comm. Algebra, 2015.

Endo-trivial modules revisited

Shigeo Koshitani

(joint work with Caroline Lassueur)

For a finite group G and an algebraically closed field k of characteristic $p > 0$ a finitely generated kG-module M is called *endo-trivial* if $M \otimes_k M^* \cong k_G \oplus (\text{proj})$ as kG-modules, where k_G is the trivial kG-module and M^* is the k-dual of M. It is interesting and important to know the structure of the abelian group $T(G)$ of endo-trivial kG-modules. The addition in $T(G)$ is defined as follows. First the elements $[M]$ in $T(G)$ are elements in the stable module category St-mod(kG) of all finitely generated kG-modules, and hence $[M \oplus (\text{proj})] = [M]$ for a kG-module M, and $[M] + [N] := [M \otimes_k N]$. Then, $T(G)$ becomes an abelian group, which is defined by E.C. Dade in [8]. It is known that $T(G)$ is finitely generated, essentially due to L. Puig in [14]. For the case that $G = P$ is a finite p-group, a complete classification has been done by J.F. Carlson and J. Thévenaz in [5, 6, 7].

Let P be a Sylow p-subgroup of G. For the case where P is cyclic (i.e. kG is of finite-representation-type), the structure of $T(G)$ is determined by N. Mazza and J. Thévenaz in [12]. Then it is quite natural to move on to the case where kG is of tame-representation-type. It is well-known that there are only three cases happening whenever kG is of tame-representation-type. Namely, p has to be 2 and exactly three cases: (1) P is generalized quaternion, (2) P is semi-dihedral, and (3) P is dihedral including the case $P = C_2 \times C_2$ (the Klein-four group). The cases (1) and (2) have been treated and the structure of $T(G)$ is almost completed in a paper by J.F. Carlson, N. Mazza and J. Thévenaz in [4]. However, so far the case (3) is missing. In fact the torsion free part $TF(G)$ has been known, see [3]. So the problem which has been left is determining the structure of the torsion part $TT(G)$ of $T(G)$ for the case (3). Actually, our result here is the following:

Theorem. The structure $TT(G)$ for the case (3) is determined.

Proof. We use several initiated results such as Gorenstein-Walter [10], Bender-Walter [1, 15], Navarro-Robinson [13], Bender-Suzuki Theorem [2], Fong-Reynolds Theorem [9] and so on. See also [11]. □

Question. In order to know the structure of $TT(G)$, is it true that the problem can be reduced essentially to the case where $O_{p'}(G) = 1$?

References

[1] H. Bender, *On groups with abelian Sylow 2-subgroups*, Math. Z. **117** (1970), 164-176.

[2] H. Bender, *Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festlässt*, J. Algebra **17** (1971), 527–554.

[3] J.F. Carlson, N. Mazza, D. Nakano, *Endotrivial modules for finite groups of Lie type*, J. reine. angew. math. **595** (2006), 284–306.

[4] J.F. Carlson, N. Mazza, J. Thévenaz, *Endotrivial modules over groups with quaternion or semi-dihedral Sylow 2-subgroup*, J. Eur. Math. Soc. **15** (2013), 157–177.

[5] J.F. Carlson, J. Thévenaz, *Torsion endo-trivial modules*, Algebras and Rep. Theory **3** (2000), 303–335.

[6] J.F. Carlson, J. Thévenaz, *The classification of endo-trivial modules*, Invent. math. **158** (2004), 389–411.

[7] J.F. Carlson, J. Thévenaz, *The classification of torsion endo-trivial modules*, Ann. of Math. **165** (2002), 823–883.

[8] E.C. Dade, *Endo-permutation modules over p-groups, I, II*, Ann. of Math. **107** (1978), 459–494, **108** (1978), 317–346.

[9] P. Fong, *On the characters of p-solvable groups*, Trans. Amer. Math. Soc. **98** (1961), 263–284.

[10] D. Gorenstein, J.H. Walter, *The characterization of finite groups with dihedral Sylow 2-subgroups I, II, III*, J. Algebra **2** (1965), 85-151, 218–270, 354–393.

[11] S. Koshitani, C. Lassueur, *Endo-trivial modules for finite groups with Klein-four Sylow 2-subgroups*, to appear in Manuscripta Mathematica DOI: 10.1007/s00229-015-0739-5.

[12] N. Mazza, J. Thévenaz, *Endotrivial modules in the cyclic case*, Arch. Math. **89** (2007), 497–503.

[13] G. Navarro, G.R. Robinson, *On endo-trivial modules for p-solvable groups*, Math. Z. **270** (2012), 983–987.

[14] L. Puig, *Affirmative answer to a question of Feit*, J. Algebra **131** (1990), 513–526.

[15] J.H. Walter, *The characterization of finite groups with abelian Sylow 2-subgroups*, Ann. of Math. **89** (1969), 405–514.

Tilting modules and the anti-spherical module

Geordie Williamson
(joint work with Simon Riche)

Let G denote a connected reductive algebraic group over a field $\Bbbk = \overline{\Bbbk}$ of positive characteristic and Rep its category of rational representations. Let X^+ denote the dominant weights of G (with respect to a fixed $T \subset B$). To any $\lambda \in X^+$ we have a standard ("Weyl") module Δ_λ, a costandard ("induced") ∇_λ and a simple module L_λ all with highest weight λ and canonical maps

$$\Delta_\lambda \twoheadrightarrow L_\lambda \hookrightarrow \nabla_\lambda$$

identifying L_λ as the head (resp. socle) of Δ_λ (resp. ∇_λ). The set $\{L_\lambda\}_{\lambda \in X^+}$ coincides with the set of isomorphism classes of simple rational G-modules.

Denote by Tilt $\subset$ Rep the full additive subcategory of tilting modules. Recall that a module is tilting if it can be written both as a successive extension of Δ and of ∇ modules. By a theorem of Ringel and Donkin for every $\lambda \in X^+$ there exists an indecomposable tilting module T_λ with highest weight λ and the set $\{T_\lambda\}_{\lambda \in X^+}$ coincides with the set of isomorphism classes of indecomposable tilting modules.

Why are we interested in tilting modules? Here are a few pointers:

(1) From the fundamental vanishing $\mathrm{Ext}^i(\Delta, \nabla) = 0$ for $i > 0$ it follows immediately that $\mathrm{Ext}^i(T, T') = 0$ for $T, T' \in$ Tilt. It is then not difficult to see that one has an equivalence
$$K^b(\mathrm{Tilt}) = D^b(\mathrm{Rep})$$
and one can use the order on X^+ to define a t-structure on $K^b(\mathrm{Tilt})$ which recovers Rep. Thus Tilt provides a "minimal homological skeleton" of Rep.

(2) The usual meaning of tilting object is an E which generates the derived category, and satisfies $\mathrm{Ext}^i(E, E) = 0$. Such an E leads to a derived equivalence with $\mathrm{End}(E)$-modules, with one heart "tilted" with respect to the other. Hence one should really call each T_λ a partial tilting object. Rep contains many other tilting objects in this larger sense.

(3) Steinberg's tensor product theorem leads to a recursive "fractal" structure on the simple characters in Rep [5]. In a similar but more complicated manner, the characters of tilting modules display an (only partially understood) fractal behaviour [6].

(4) An extremely important basic theorem on tilting modules: $T \otimes T' \in$ Tilt if $T, T' \in$ Tilt. This is easily deduced from the statement (due to Humphreys, ..., Donkin, Mathieu, Kaneda, ...) that any tensor product $\Delta_\lambda \otimes \Delta_\mu$ of Weyl modules has a Δ-filtration. This theorem has the consequence (important later) that translation functors preserve Tilt.

(5) Now suppose that $G = GL_n$. Then the natural module V is a Weyl, induced, simple and tilting module. Applying the previous point we see that any tensor power $V^{\otimes m}$ decomposes (non-canonically) as a direct sum of tilting modules. It is an now an easy consequence of Schur-Weyl duality

that if we knew how to write $V^{\otimes m}$ as direct sum of tilting modules then we would know dimensions of all simple modules for all symmetric groups indexed by partitions with $\leq n$ parts. Of course we could do this if we knew the characters of indecomposable tilting modules, but this seems very difficult. For example we know all tilting modules for GL_2, but already for GL_3 there are infinitely many unknown cases. (Compare this to the fact that the characters of the simple rational $G = GL_3$-modules were already known to Jantzen prior to Lusztig's conjecture.)

In my talk I outlined a conjecture giving a combinatorial model for the category of tilting modules. Let Rep_0 denote the principal block, and Tilt_0 its full subcategory of tilting modules. Let us assume that G is semi-simple, simply connected and that the characteristic p of $\Bbbk$ is larger than the Coxeter number of G.[1]

Let W denote the affine Weyl group associated to G which acts by affine translations on $\mathfrak{h}^*$, the real vector space containing the root system of G. Once we have fixed a fundamental alcove we may identify W with the set of alcoves and we have a bijection:

$$\{\text{dominant alcoves}\} \xrightarrow{\sim} {}^f W$$

where ${}^f W$ denotes minimal coset representatives for $W_f \setminus W$, where $W_f \subset W$ is the finite Weyl group. Given any dominant alcove A we may associate to it in a unique way a highest weight of a module in the principal block, and hence objects $L_A, \Delta_A, \nabla_A, T_A \in \mathrm{Rep}_0$. (These facts are consequences of the linkage principal, which we will not explain here.)

For every simple reflection $s \in W$ we have an exact biadjoint functor θ_s : $\mathrm{Rep}_0 \to \mathrm{Rep}_0$ ("translation through the s-wall"). These functors preserve Tilt_0. It is an easy exercise in the combinatorics of these functors to see that we have an isomorphism of Grothendieck groups

$$\mathrm{sgn} \otimes_{\mathbb{Z}W_f} \mathbb{Z}W \xrightarrow{\sim} [\mathrm{Tilt}_0] = [\mathrm{Rep}_0]$$

such that the (right) action of a translation functor $[\Theta_s]$ on the right hand side corresponds to multiplication by $(1 + s)$ on the left hand side. The left hand module is sometimes called the "anti-spherical module", which explains the title of this talk.

Now let $\mathcal{D}$ denote the diagrammatic category of Soergel bimodules determined by the affine Cartan matrix corresponding to G over $\Bbbk$, as defined in [2]. This is a graded monoidal category with hom spaces enriched in graded $\Bbbk[\alpha_0, \dots, \alpha_m]$-modules, where $\alpha_0, \dots, \alpha_m$ denote the simple (affine) roots and $\deg \alpha_i = 2$. In [2] it is proved that the isomorphism classes of indecomposable objects are classified up to shift by W, and that the split Grothendieck group of $\mathcal{D}$ is isomorphic to the Hecke algebra H of W. (This theorem is a natural generalization of a theorem of Soergel.) We write B_w for the self-dual indecomposable object indexed by w.

A natural categorification of the anti-spherical module is given by

$$\mathcal{AS} := \Bbbk \otimes_{\mathcal{D}_f} \mathcal{D}$$

[1] These assumptions are for simplicity only, there should be variants for small p.

where the tensor product means that we quotient by the ideal of $\mathcal{D}$ generated by morphisms which factor through objects B_w for $w \notin {}^f W$ as well as morphisms of the form $r \cdot m$ where $r \in R^{>0}$ and m is any morphism in $\mathcal{D}$. The category $\mathcal{AS}$ is a right $\mathcal{D}$-module category enriched in graded finite dimensional vector spaces. Let $\mathcal{AS}_{\text{deg}}$ denote the category obtained from $\mathcal{AS}$ by forgetting the grading on hom spaces (the "degrading").

Main Conjecture: We can equip Tilt_0 with the structure of a right $\mathcal{D}$-module category with $B_s := \theta_s$. Moreover, we have an equivalence of right $\mathcal{D}$-module categories:

$$\mathcal{AS}_{\text{deg}} \xrightarrow{\sim} \text{Tilt}_0$$

Notes and consequences of the conjecture:

(1) Basically it says that two natural ways of categorifying the anti-spherical module are equivalent. Hence it can be seen as a "uniqueness of categorification" statement. Rather surprisingly (for me) the first statement in the conjecture implies the second.
(2) The conjecture implies that Tilt_0 admits a grading (defined by $\mathcal{AS}$) and that this grading is defined over $\mathbb{Z}$ (because $\mathcal{D}$ is defined over $\mathbb{Z}$). Over fields of characteristic zero we understand $\mathcal{D}$ quite well (Kazhdan-Lusztig conjectures . . .). It follows that the conjecture immediately implies Lusztig's conjecture for large p.
(3) There is an analogue of this conjecture over $\mathbb{C}$ where Tilt_0 is replaced by the principal block of representations of the quantum group at an ℓ^{th} root of unity. This conjecture has recently been checked for $U_q(\mathfrak{sl}_2)$ by Andersen and Tubbenhauer [1].
(4) This conjecture should be Koszul dual to a conjecture of Finkelberg and Mirkovic [4]. Perhaps it is more tractable because translation functors are "built in" to the equivalence.
(5) A version of the conjecture for singular weights should imply a conjecture of Rickard [7] and the version for the quantum group should imply a conjecture of Chuang-Miyachi [3].
(6) There is also a part of the conjecture explaining the tilting tensor product theorem in terms of the action of Gaitsgory's central sheaves, but we're running out of space...

References

[1] H. H. Andersen, D. Tubbenhauer, *Diagram categories for U_q-tilting modules at roots of unity*, arXiv:1409.2799.
[2] B. Elias, G. Williamson, *Soergel calculus*, arXiv:1309.0865.
[3] J. , Chuang, H. Miyachi, *Runner removal Morita equivalences*, Representation theory of algebraic groups and quantum groups, 55–79, Progr. Math., 284, Birkhuser/Springer, New York, 2010.
[4] M. Finkelberg, I. Mirkovic, *Semiinfinite Flags. I. Case of global curve P^1*, arXiv:9707010.
[5] G. Lusztig, *On the character of certain modular irreducible representations*, arxiv:1407.5346.
[6] G. Lusztig, G. Williamson, *On the character of certain tilting modules*, arxiv:1502.04904.

[7] J. Rickard, *Translation functors and equivalences of derived categories for blocks of algebraic groups*, Finite-dimensional algebras and related topics (Ottawa, ON, 1992), 255–264,NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 424, Kluwer Acad. Publ., Dordrecht, 1994.

On the character degree sets of the symmetric and alternating groups and their double covers

Christine Bessenrodt

In his famous 1963 list of problems on representations of finite groups, Richard Brauer asked in his Problem 2 [3]:

When do nonisomorphic groups have isomorphic group algebras?

We focus on the situation where the ground field is the field of complex numbers. Stated in a wider form, one wants to know:

What do the degrees of its irreducible complex characters tell about a group?

This report is about contributions to both types of questions, obtained in joint work with Nguyen, Olsson, Tong-Viet [1] and joint work with Tong-Viet and Zhang [2], respectively.

In the following, we only consider finite groups. For a group G, $\mathrm{Irr}(G)$ denotes the set of its complex irreducible characters, and $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ the set of degrees; the corresponding multiset of degrees is denoted by $\mathrm{cd}^*(G)$.

1. Complex group algebras of the symmetric and alternating groups and their double covers

First we recall some recent results on the questions above due to Hung Tong-Viet. We denote by S_n the symmetric group and by A_n the alternating group of degree n, respectively.

Theorem (Tong-Viet, 2011/2012).

(1) Let G be a group with $\mathbb{C}G \cong \mathbb{C}S_n$. Then $G \cong S_n$.
(2) Let S be a nonabelian simple group, G a group with $\mathbb{C}G \cong \mathbb{C}S$. Then $G \cong S$.
(3) Let G be a nontrivial group with $\mathrm{cd}^*(G) \subseteq \mathrm{cd}^*(A_n)$. Then $G \cong A_n$.

The covering groups of the alternating and symmetric groups were studied in recent joint work with Hung N. Nguyen, Jørn B. Olsson, Hung P. Tong-Viet [1]. This led to the following main results.

Theorem A. Let $n \geq 5$ and let $\tilde{A}_n$ be the double cover of A_n. If G is a group with $\mathbb{C}G \cong \mathbb{C}\tilde{A}_n$, then $G \cong \tilde{A}_n$.

A similar result for the triple and 6-fold (perfect central) covers of A_6 and A_7, together with earlier work by Nguyen and Tong-Viet, gives:

Theorem B. Let G be a group and H a quasi-simple group with $\mathbb{C}G \cong \mathbb{C}H$. Then $G \cong H$.

We denote by $\tilde{S}_n^{\pm}$ the double covers of S_n. Using as a crucial ingredient in the proof a non-existence result for special character degrees of $\tilde{S}_n^{\pm}$, we have:

Theorem C. Let $n \geq 5$, and let G be a group. Then $\mathbb{C}G \cong \mathbb{C}\tilde{S}_n^+$ (or equivalently $\mathbb{C}G \cong \mathbb{C}\tilde{S}_n^-$) if and only if $G \cong \tilde{S}_n^+$ or $G \cong \tilde{S}_n^-$.

2. Huppert's conjecture for alternating groups

Around 1999, based on some evidence, Huppert conjectured that the nonabelian simple groups are almost characterized by their character degree sets:

Huppert's Conjecture. Let H be a nonabelian simple group and G a finite group with $\mathrm{cd}(G) = \mathrm{cd}(H)$. Then $G \cong H \times A$, where A is abelian.

Huppert proved the conjecture for many simple groups H, including alternating groups of degree up to $n = 11$; for A_{12} and A_{13}, it was proved by Nguyen, Tong-Viet and Wakefield. In recent joint work with Hung P. Tong-Viet and Jiping Zhang [2] we confirmed Huppert's conjecture for alternating groups:

Theorem. Let $n \geq 5$. Let G be a finite group such that $\mathrm{cd}(G) = \mathrm{cd}(A_n)$. Then $G \cong A_n \times A$, where A is abelian.

Here is an outline of the main steps of the proof. We assume that G is a group such that $\mathrm{cd}(G) = \mathrm{cd}(A_n)$; because of the previous results, we may take $n \geq 14$.

Step 1: Show that G is nonsolvable.

Here, a result of G. R. Robinson (1992) on the minimal degree of nonlinear irreducible characters of finite solvable groups is crucial. Note that it is essential that we compare G with a simple group: Navarro recently constructed a perfect group H and a solvable group G such that $\mathrm{cd}(G) = \mathrm{cd}(H)$. Navarro and Rizo even found a perfect group H and a nilpotent group G with $\mathrm{cd}(G) = \mathrm{cd}(H)$.

Step 2: If L/M is any nonabelian chief factor of G, then $L/M \cong A_n$.

Here, besides the classification of finite simple groups, classification results for prime power degree characters and information on small degrees are used.

Step 3: If L is a finite perfect group and M is a minimal normal elementary abelian subgroup of L such that $L/M \cong A_n$, then some degree of L divides no degree of A_n.

Here, for $n \geq 17$, a result due to Guralnick and Tiep (2005) on the non-coprime $k(GV)$ problem is a crucial ingredient.

Up to this point, we are able to prove that $G \cong A_n \times A$, or $G \cong (A_n \times A) \cdot 2$ and $G/A \cong S_n$ with A abelian. Then the Theorem follows by applying the following recent result that confirms a conjecture by Tong-Viet.

Theorem (Debaene, 2014). For $n \geq 5$, $\mathrm{cd}(S_n) \nsubseteq \mathrm{cd}(A_n)$.

References

[1] C. Bessenrodt, H. N. Nguyen, J.B. Olsson, H.P. Tong-Viet, *Complex group algebras of the double covers of the symmetric and alternating groups*, Algebra and Number Theory **9** (2015), 601–628.

[2] C. Bessenrodt, H.P. Tong-Viet, J. Zhang, *Huppert's conjecture for alternating groups*, arXiv:1502.03425.

[3] R. Brauer, *Representations of finite groups*, in *Lectures on Modern Mathematics* I, 133–175 Wiley, New York, 1963.

[4] B. Huppert, *Some simple groups which are determined by the set of their character degrees. I*, Illinois J. Math. **44** (2000), 828–842.

More on p-permutation equivalences

Robert Boltje

(joint work with Philipp Perepelitsky)

Let F be an algebraically closed field of positive characteristic p, let G and H be finite groups, and let A and B be blocks of FG and FH, respectively. Recall that a p-permutation equivalence between A and B is an element $\gamma \in T^{\Delta}(A,B)$ satisfying

$$(*) \qquad \gamma \otimes_B \gamma^* = [A] \in T^{\Delta}(A,A) \quad \text{and} \quad \gamma^* \otimes_A \gamma = [B] \in T^{\Delta}(B,B)\,.$$

Here, the Grothendieck group $T^{\Delta}(A,B)$ is a free abelian group over the set of isomorphism classes of finitely generated indecomposable (A,B)-bimodules M with the following property: Viewed as $F[G\times H]$-module, M is isomorphic to a direct summand of a permutation $F[G\times H]$ module and M has a twisted diagonal vertex $\Delta(P,\alpha,Q) := \{(\alpha(y),y) \mid y \in Q\}$, where P is a p-subgroup of G, Q is a p-subgroup of H, and $\alpha\colon Q \to P$ is an isomorphism. $\gamma^* \in T^{\Delta}(B,A)$ denotes the F-dual of γ. We denote the set of elements γ satisfying $(*)$ by $T^{\Delta}_o(A,B)$.

If C is a splendid Rickard complexes of (A,B)-bimodules then

$$\gamma := \sum_{n\in\mathbb{Z}} (-1)^n [C_n] \in T^{\Delta}(A,B)$$

is a p-permutation equivalence. Although p-permutation equivalences are defined for arbitrary pairs (G,A) and (H,B), we are particularly interested in the case that A is a block with abelian defect group D, $H = N_G(D)$, and B is the Brauer correspondent of A, which is the situation of Broué's Abelian Defect Group Conjecture. Reverting back to the general situation, the following theorem states that the existence of a p-permutation equivalence between A and B implies that all relevant invariants of the blocks A and B coincide.

Theorem. Assume that $\gamma \in T^{\Delta}_o(A,B)$.

(a) There exists a unique indecomposable constituent M of γ whose vertex is of the form $\Delta(D,\alpha,D')$, where D and D' are respective defect groups of A and B. The multiplicity of M in γ is ± 1, and every other indecomposable constituent of γ has a vertex contained in $\Delta(D,\alpha,D')$.

(b) The isomorphism φ is an isomorphism between the fusion systems associated to A and B.

(c) Külshammer-Puig 2-cohomology classes associated to centric subgroups are preserved.

(d) Taking the Brauer construction of γ with respect to subgroups of $\Delta(D,\varphi,D')$ yields p-permutation equivalences between blocks on the local levels and also an isotypy between A and B.

(e) $T_o^\Delta(A,B)$ is finite.

We call the module M in Part (a) of the Theorem the *maximal module* of γ. We want to study the finite group $T_o^\Delta(A,A)$ of p-permutation self equivalences of a block A of FG. By Part (e) of the Theorem, this group is finite and $[A]$ is its identity element. It decomposes into the direct product of $\{\pm[A]\}$ and the subgroup $T_{o,+}^\Delta(A,A)$ of equivalencies whose maximal module occurs with multiplicity $+1$. Let D be a defect group of A, let (D,e) be a maximal A-Brauer pair, let E denote the inertial quotient $N_G(D,e)/DC_G(D)$, and let $\mathcal{F}$ be the fusion system of A with respect to (D,e). The group $T_{o,+}^\Delta(A,A)$ has a sequence of normal subgroups

$$(**)\qquad 1\le \Pi(A)\le \Lambda(A)\le \Sigma(A)\le T_{o,+}^\Delta(A,A)\,,$$

where $\Sigma(A)$ (resp. $\Lambda(A)$) consists of those equivalences whose maximal module has vertex (D,id_D,D) (resp. has maximal module A) and $\Pi(A)$ consists of those of the form $[A]+\pi$, where π is a virtual projective (A,A)-bimodule. These subgroups and subsequent factor groups are invariants of the block A.

Theorem. In general, the factor group $T_{o,+}^\Delta(A,A)/\Sigma(A)$ is isomorphic to a subgroup of $\mathrm{Out}(\mathcal{F})(\le \mathrm{Out}(D\rtimes E))$, the factor group $\Sigma(A)/\Lambda(A)$ is isomorphic to a subgroup of $\hat{E}:=\mathrm{Hom}(E,F^\times)$, and $\Pi(A)$ is isomorphic to a subgroup of $\{\pm1\}\wr \mathrm{Sym}(\mathrm{Irr}(A))$, where $\mathrm{Irr}(A)$ denotes the set of ordinary irreducible characters of A.

In general we don't know what the factor group $\Lambda(A)/\Pi(A)$ looks like. In the case that D is cyclic we have the following more precise statement:

Theorem. Let D be cyclic of order p^n and let $m:=(p^n-1)/|E|$, the number of exceptional characters of A. Then the top factor in $(**)$ is isomorphic to $\mathrm{Out}(D\rtimes E)$, the next factor is isomorphic to $\hat{E}$ and $\Pi(A)$ is isomorphic to the symmetric group of the non-exceptional characters of A if $m>1$ and to $\mathrm{Sym}(\mathrm{Irr}(A))$ if $m=1$. Finally, the factor $\Lambda(A)/\Pi(A)$ is cyclic of order 2, when $p=2$ and $n\ge 2$, and trivial in all other cases.

In the situation of Broué's Abelian Defect Group Conjecture, one has a canonical candidate for the maximal module. Thus, the last theorem implies that in the cyclic defect group case there is a unique normalized stable p-permutation equivalence except in the cases when $|D|$ has order 2^n with $n\ge 2$. In the latter case, there are two such normalized stable equivalences and we suspect that the extra element comes via tensor induction and diagonal embedding from a non-trivial unit in the Burnside ring of D.

$U_n(q)$ acting on flags and supercharacters

Richard Dipper
(joint work with Qiong Guo)

Let q be a power of some prime p, $G = GL_n(q)$, $n \in \mathbb{N}$ and let U be the subgroup $U_n^-(q)$ of lower unitriangular matrices. Let K be a field with $char(K) \neq p$ containing a primitive p-th root of unity. The field with q many elements is denoted by $\mathbb{F}_q$.

To each partition λ of n (denoted by $\lambda \vdash n$), there is exists an unipotent Specht module $S(\lambda)$ for G. For $K = \mathbb{C}$ the set $\{S(\lambda) | \lambda \vdash n\}$ is the set of irreducible modules in the principal series of G. Let P_λ be the standard parabolic subgroup of G containing the Borel subgroup B^+ of upper triangular matrices in G with Levi decomposition $P_\lambda = U_\lambda \rtimes L_\lambda$. We define $M(\lambda) = Ind_{P_\lambda}^G(K)$, where K denotes the trivial module. Note that this is the KG-module obtained from the action of G on λ-flags in $V = \mathbb{F}_q^n$ by right multiplication. The kernel intersection theorem of G.James [5] states that

$$S(\lambda) = \bigcap_{\mu \rhd \lambda} \{ker\phi | \phi \in Hom_{KG}(M(\lambda, M(\mu)\} \leq M(\lambda).$$

Restricting $M(\lambda)$ to U and applying Mackey decompositionto yields: $Res_U^G M(\lambda) = \bigoplus_{\mathfrak{s} \in Rst(\lambda)} M_\mathfrak{s}$, where $M_\mathfrak{s} = Ind_{P_\lambda^d \cap U}^U K$. Here $d = d(\mathfrak{s}) \in \mathfrak{S}_n$ takes the initial λ-tableau $\mathfrak{t}^\lambda$ to $\mathfrak{s}$ and $Rst(\lambda)$ denotes the set of row standard λ-tableaux. As a consequence each $v \in M(\lambda)$ may be written uniquely as sum $v = \sum_{\mathfrak{s} \in Rst(\lambda)} v_\mathfrak{s}$, with $v_\mathfrak{s} \in M_\mathfrak{s}$.

Ordering $Rst(\lambda)$ in reversed lexicographical order we define $last(v)$ to be the last $\mathfrak{s} \in Rst(\lambda)$ such that $v_\mathfrak{s} \neq 0$. S. Lyle showed in [6], that $last(v)$ is always a standard λ-tableau for $0 \neq v \in S(\lambda)$. We denote the set of standard λ-tableaux by $St(\lambda)$. As a consequence, $S(\lambda)$ has a basis $\mathcal{B}$ such that $v_\mathfrak{s} \neq 0$, $v \in \mathcal{B}$, for $\mathfrak{s} \in St(\lambda)$ implies $\mathfrak{s} = last(v)$. For such a basis $\mathcal{B}$ we define $\mathcal{B}_\mathfrak{s} = \{v \in \mathcal{B} | last(v) = \mathfrak{s}\}$. Then $\{v_\mathfrak{s} | v \in \mathcal{B}_\mathfrak{s}\}$ is linearly independent and $|\mathcal{B}_\mathfrak{s}|$ is independent of the choice of $\mathcal{B}$.

Conjecture (James, D.): $\mathcal{B}$ can be chosen independent of K and for each $\mathfrak{s} \in St(\lambda)$ there exists $\mathfrak{p}_\mathfrak{s} \in \mathbb{Z}[x]$ with $\mathfrak{p}_\mathfrak{s}(1) = 1$ such that $|\mathcal{B}_\mathfrak{s}| = \mathfrak{p}_\mathfrak{s}(q)$.

Refinement: $\mathfrak{p}_\mathfrak{s} \in \mathbb{Z}_{\geq 0}[x-1]$.

Both, the conjecture and its refinement hold in the special case of λ being a partition of n into at most two parts ([2], [4]). Indeed the proof in [4] reveals a suprising connection to supercharacter theory for U and its pattern subgroups as defined in [1] and [7].How this connection can be established for general $\lambda \vdash n$ together with some further consequences in the case of two part partitions is discussed below.

For $\lambda \vdash n$, $\mathfrak{s} \in Rst(\lambda)$ and $d = d(\mathfrak{s}) \in \mathfrak{S}_n$ as above the pattern subgroup $U_R = U^d \cap U$ splits into a semidirect product $U_J \rtimes U_L$ of pattern subgroups U_J, U_L where $U_L^{d^{-1}} = U \cap L_\lambda$. Moreover, again by Mackey decomposition, $Res_{U_R}^U M_\mathfrak{s} \tilde{=} Ind_{U_L}^{U_R} K$. We consider the $\mathbb{F}_q$-vector space $V_J = Lie(U_J) = \{u - 1 | u \in U_J\}$ as abelian group with respect to addition. Then combining the action on V_J of U_J by right multiplication and of U_L on V_J by conjugation induces a monomial action of U_R on the set $\hat{V}_J$ of linear characters $\chi : (V_J, +) \mapsto K^* = K \smallsetminus \{0\}$ yielding as well $Ind_{U_L}^{U_R} K$. This turns this transitive permutation module into a monomial representation which splits in general into a direct sum corresponding to many orbits of the action of U_R on $\hat{V}_J$. There is a natural one by one correspondence between V_J and $\hat{V}_J$ and the action of U_R on the linear character $\chi_A \in \hat{V}_J$ corresponding to $A \in V_J$ can be described explicitely.

In the special case $\lambda = (1^n)$ and $d = 1$ we have $\mathfrak{s} = \mathfrak{t}^\lambda$, $U = U_R = U_J$ and $U_L = (1)$. We call linear characters χ_A of $V = Lie(U) = \{u - 1 | u \in U\}$ labelled by $A \in V$ verge, if A has in each row and coulumn at most one nonzero entry. The isomorphism classes of orbit modules of U acting on $\hat{V}$ are in bijection with verges and their characters are precisely the supercharacters of [1], [7].

Similarly, for two part partitions λ and $\mathfrak{s} \in Rst(\lambda)$. the orbits of U_R acting on $\hat{V}_J$ are in bijection with the set of verges χ_A where A runs through V_J. Moreover, each orbit module is an irreducible KU_R-module and the U_R-action extends to U, yielding an irreducible KU-module. If χ_A is such a verge and $\mathcal{O}_A$ is the orbit containing it, then A may be considered as matrix in $V \geq V_J$ and hence corresponds to the verge $\psi_A \in \hat{V}$. Thus $N = \psi_A KU$ affords a supercharacter. It turns out that in this special case $End_{KU}(N)$ is isomorphic to a group algebra and that $K\mathcal{O}_A$ is the unique irreducible constituent of $\psi_A KU$ corresponding to the trivial representation of this group algebra ([3]).

As a consequence applying James' kernel intersection theorem [5] in [4] a standard bases of unipotent Specht modules $S(\lambda)$ fortwo part partitions λ has been exhibited. In addition we prove in [3] that for each $\mathfrak{s} \in Rst(\lambda)$ the $\mathfrak{s}$-component $M_\mathfrak{s}$ of $Res_U^G M(\lambda)$ is the direct sum of orbit module $K\mathcal{O}_A$ where χ_A runs through the set of verges in $\hat{V}_J$. In particular $M_\mathfrak{s}$ is multiplicity free. Moreover the irreducible constituentes of $Res_U^G M(\lambda)$ are of this form as well and the multiplicity of $K\mathcal{O}_A$ as constituent hereof is given as number of $\mathfrak{s} \in Rst(\lambda)$ such that $A \in V_J$, $J = J(\mathfrak{s})$. In particular it is independent of q. Thus the restriction to U of G acting on the cosets of maximal parabolic subgroups can be completely decomposed into irreducibles.

References

[1] C.A.M. Andrr, *Basic characters of the unitriangular group*, J. Algebra **175** (1995), 287–319.

[2] M. Brandt, R. Dipper, G. James, S. Lyle, *Rank polynomials*, Proc. London Math. Soc. **(3), 98** (2009), 1–18.

[3] R. Dipper, Q. Guo, *$U_n(q)$ acting on flags and supercharacters*, preprint (2014), arXiv:1412.3376v1.

[4] Q. Guo, *On the U-module structure of the unipotent Specht modules of finite general linear groups*, preprint (2013), arXiv:1304.4370v2.

[5] G. James, *Representations of general linear groups*, LMS Lecture Notes **94** (1984).

[6] S. Lyle, *On Specht modules of general linear groups*, J. Algebra **269** (2003), 726–734.
[7] N. Yan, *Representations of finite unipotent linear groups by the method of clusters*, preprint (2010), arXiv:1004.2674v1.

Reporter: Ulrich Thiel

Participants

Prof. Dr. David J. Benson
Dept. of Mathematical Sciences
University of Aberdeen
Fraser Noble Bldg.
Aberdeen AB24 3UE
UNITED KINGDOM

Prof. Dr. Christine Bessenrodt
Institut für Algebra, Zahlentheorie
und Diskrete Mathematik
Leibniz Universität Hannover
30167 Hannover
GERMANY

Prof. Dr. Robert Boltje
Department of Mathematics
University of California
Santa Cruz, CA 95064
UNITED STATES

Prof. Dr. Serge Bouc
UFR de Sciences
Université de Picardie Jules Verne
33 rue Saint Leu
80039 Amiens Cedex
FRANCE

Dr. Olivier Brunat
U. F. R. de Mathématiques
Case 7012
Université Paris 7
2, Place Jussieu
75251 Paris Cedex 05
FRANCE

Prof. Dr. Joseph Chuang
Department of Mathematics
The City University
Northampton Square
London EC1V OHP
UNITED KINGDOM

Dr. David A. Craven
School of Mathematics
The University of Birmingham
Edgbaston
Birmingham B15 2TT
UNITED KINGDOM

Prof. Dr. Susanne Danz
Mathematisch-Geographische Fakultät
Kath. Universität Eichstätt
Ostenstr. 26-28
85072 Eichstätt
GERMANY

Prof. Dr. Richard Dipper
Institut für Algebra u. Zahlentheorie
Universität Stuttgart
70569 Stuttgart
GERMANY

Dr. Olivier Dudas
U. F. R. de Mathématiques
Case 7012
Université Paris 7
2, Place Jussieu
75251 Paris Cedex 05
FRANCE

Dr. Charles W. Eaton
School of Mathematics
The University of Manchester
Oxford Road
Manchester M13 9PL
UNITED KINGDOM

Dr. Benjamin Elias
Department of Mathematics
University of Oregon
Eugene, OR 97403-1222
UNITED STATES

Prof. Dr. Karin Erdmann
Mathematical Institute
Oxford University
ROQ
Oxford OX2 6GG
UNITED KINGDOM

Dr. Anton Evseev
School of Mathematics
The University of Birmingham
Edgbaston
Birmingham B15 2TT
UNITED KINGDOM

Dr. Matt Fayers
School of Mathematical Sciences
Queen Mary
University of London
Mile End Road
London E1 4NS
UNITED KINGDOM

Prof. Dr. Meinolf Geck
Fachbereich Mathematik
IAZ - Lehrstuhl für Algebra
Universität Stuttgart
Pfaffenwaldring 57
70569 Stuttgart
GERMANY

Prof. Dr. Michael Geline
Department of Mathematical Sciences
Northern Illinois University
Dekalb, IL 60115
UNITED STATES

Dr. Eugenio Giannelli
Fachbereich Mathematik
T.U. Kaiserslautern
Postfach 3049
67653 Kaiserslautern
GERMANY

Dr. Roderick B. S. Gow
School of Mathematical Sciences
University College Dublin
Belfield
Dublin 4
IRELAND

Dr. Jean-Baptiste Gramain
Department of Mathematics
University of Aberdeen
King's College
Aberdeen AB24 3UE
UNITED KINGDOM

Prof. Dr. Gerhard Hiß
Lehrstuhl D für Mathematik
RWTH Aachen
Pontdriesch 14-16
52062 Aachen
GERMANY

Dr. Lacri Iancu
Fachbereich Mathematik
Institut für Algebra und Zahlentheorie
Universität Stuttgart
Pfaffenwaldring 57
70569 Stuttgart
GERMANY

Dr. Daniel Juteau
Laboratoire LMNO
CNRS UMR 6139
Université de Caen
BP 5186
14032 Caen Cedex
FRANCE

Prof. Dr. Radha Kessar
Department of Mathematics
The City University
Northampton Square
London EC1V OHP
UNITED KINGDOM

Prof. Dr. Shigeo Koshitani
Department of Mathematics and
Informatics, Faculty of Science
Chiba University
1-33 Yayoi-cho, Inage
Chiba-Shi 263-8522
JAPAN

Prof. Dr. Burkhard Külshammer
Mathematisches Institut
Friedrich-Schiller-Universität Jena
07737 Jena
GERMANY

Dr. Caroline Lassueur
Fachbereich Mathematik
T.U. Kaiserslautern
Postfach 3049
67653 Kaiserslautern
GERMANY

Prof. Dr. Martin W. Liebeck
Department of Mathematics
Imperial College of Science,
Technology and Medicine
180 Queen's Gate, Huxley Bldg.
London SW7 2BZ
UNITED KINGDOM

Prof. Dr. Markus Linckelmann
Department of Mathematics
City University
Northampton Square
London EC1V OHB
UNITED KINGDOM

Dr. Michael Livesey
Fachbereich Mathematik
T.U. Kaiserslautern
Postfach 3049
67653 Kaiserslautern
GERMANY

Prof. Dr. Gunter Malle
Fachbereich Mathematik
T.U. Kaiserslautern
67653 Kaiserslautern
GERMANY

Attila Maroti
Fachbereich Mathematik
T.U. Kaiserslautern
Postfach 3049
67653 Kaiserslautern
GERMANY

Dr. Nadia Mazza
Dept. of Mathematics & Statistics
University of Lancaster
Fylde College
Bailrigg
Lancaster LA1 4YF
UNITED KINGDOM

Dr. John Murray
Department of Mathematics
National University of Ireland
Maynooth, Co. Kildare
IRELAND

Prof. Dr. Gabriel Navarro
Department of Mathematics
University of Valencia
Doctor Moliner 50
46100 Burjassot (Valencia)
SPAIN

Dr. Sejong Park
Mathematics Department
National University of Ireland, Galway
University Road
Galway
IRELAND

Prof. Dr. Julia Pevtsova
Department of Mathematics
University of Washington
Padelford Hall
Box 354350
Seattle, WA 98195-4350
UNITED STATES

Prof. Dr. Lluis Puig
6, Avenue Bizet
94340 Joinville-le-Pont
FRANCE

Prof. Dr. Jeremy Rickard
School of Mathematics
University of Bristol
University Walk
Bristol BS8 1TW
UNITED KINGDOM

Prof. Dr. Geoffrey R. Robinson
Department of Mathematics
University of Aberdeen
King's College
Aberdeen AB24 3UE
UNITED KINGDOM

Dr. Amanda Schaeffer Fry
MSU Denver Math and Computer
Sciences
PO Box 173362, Campus Box 38
Denver, CO 80217-3362
UNITED STATES

Dr. Britta Späth
Fachbereich Mathematik
T.U. Kaiserslautern
Postfach 3049
67653 Kaiserslautern
GERMANY

Prof. Dr. Bhama Srinivasan
Dept. of Mathematics, Statistics
and Computer Science, M/C 249
University of Illinois at Chicago
851 S. Morgan Street
Chicago, IL 60607-7045
UNITED STATES

Prof. Dr. Radu Stancu
UFR de Sciences
Université de Picardie Jules Verne
33 rue Saint Leu
80039 Amiens Cedex
FRANCE

Prof. Dr. Peter Symonds
Department of Mathematics
The University of Manchester
Oxford Road
Manchester M13 9PL
UNITED KINGDOM

Dr. Jay Taylor
Dipartimento di Matematica
Universita di Padova
Via Trieste, 63
35121 Padova
ITALY

Prof. Dr. Jacques Thevenaz
Section de Mathématiques
Station 8
École Polytechnique Fédérale de
Lausanne
1015 Lausanne
SWITZERLAND

Dr. Ulrich Thiel
Fachbereich Mathematik
Universität Stuttgart
Pfaffenwaldring 57
70569 Stuttgart
GERMANY

Prof. Dr. Pham Huu Tiep
Department of Mathematics
University of Arizona
P.O.Box 210089
Tucson AZ 85721-0089
UNITED STATES

Carolina Vallejo
Fachbereich Mathematik
T.U. Kaiserslautern
Postfach 3049
67653 Kaiserslautern
GERMANY

Dr. Hung P. Tong-Viet
Department of Mathematics and
Applied Mathematics
University of Pretoria
Private Bag X20 Hatfield
0002 Pretoria
SOUTH AFRICA

Dr. Geordie Williamson
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY