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Tropical Aspects in Geometry, Topology and Physics

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Abstract. The workshop *Tropical Aspects in Geometry, Topology and Physics* was devoted to a wide discussion and exchange of ideas between the leading experts representing various points of view on the subject. The development of tropical geometry is based on deep links between problems in real and complex enumerative geometry, symplectic geometry, quantum fields theory, mirror symmetry, dynamical systems and other research areas. On the other hand, new interesting phenomena discovered in the framework of tropical geometry (like refined tropical enumerative invariants) pose the problem of a conceptual understanding of these phenomena in the "classical" geometry and mathematical physics.

Mathematics Subject Classification (2010): Primary 14-xx (algebraic geometry), Secondary 81Txx (quantum field theory, related classical field theories), 53Dxx (symplectic geometry, contact geometry), 37-xx (dynamical systems and ergodic theory).

Introduction by the Organisers

The workshop *Tropical Aspects in Geometry, Topology and Physics*, organized by Tobias Ekholm (Uppsala), Hannah Markwig (Saarbrücken), Grigory Mikahlkin (Genève), and Eugenii Shustin (Tel Aviv), was held April 26th–May 2nd, 2015. The workshop was well attended by 55 participants from around the world. The program of the workshop consisted of 17 one-hour talks given by leading experts in the subject as well as 4 half hour talks delivered by perspective young researchers. In addition, four informal discussions on open problems and on questions related to the main topics of the workshop were ran during this week. Extended abstracts of the talks and reports on the discussions follow these introductory notes.

The current state of tropical geometry can be characterized by a wide spread of tropical objects, techniques and ideas in various fields of mathematics and physics. For the further development of the topic it becomes vital to understand these interactions deeper and to collect ideas, approaches and problems coming from different sides which are linked to tropical geometry. Thus, the idea of the workshop was to focus on interactions of tropical geometry with algebraic, symplectic, and combinatorial geometry, low-dimensional topology, mathematical physics, on the exchange of ideas coming from different areas as well as the search for new perspective problems and research directions. Two trends come together nowadays: on one side, the development of tropical geometry have been largely motivated and stimulated by its deep links to geometry, topology and physics; on the other hand, new phenomena and techniques discovered within the tropical geometry framework suggest challenging problems of conceptual understanding and elaboration of these results in the "classical" geometry, topology and physics. We shortly comment on these two trends and on how they were reflected in the talks and discussions during the workshop.

Enumeration of real and complex curves by means of tropical tools has been one of the leading directions in tropical geometry from the very beginning. Surprisingly enough the tropical enumerative geometry has brought purely tropical enumerative invariants like real tropical relative and higher genus invariants, broccoli invariants, and, perhaps, the most interesting – refined (quantum) Block-Goettsche invariants. These refined invariants and their "classical" explanation and construction was one of the main topics discussed by the workshop participants, and it was reflected in talks by Y. Soibelman, M. Abouzaid and in informal discussions led by Y. Soibelman and G. Mikhalkin. So, Y. Soibelman has suggested an interpretation of the refined curve count based on the mirror symmetry for the Fukaya category of the total space of a complex integrable system, where non-trivial B-models give rise to quantization of the structure sheaf of the mirror dual. Closely related to this quantization of the family Floer homology, again in the context of mirror symmetry, was the subject of the talk by M. Abouzaid. Another idea to understand the refined curve count came from G. Mikhalkin's observation that, for the rational curves on toric surfaces having fixed real or purely imaginary intersection points with the toric divisors, properly defined areas of their amoebas attain a discrete set of values, and the refined count of these curves coincides with the corresponding refined Block-Goettsche tropical count. In turn, in the talk by B. Kol arose an idea to regard the real (and potentially refined) curve count as a supersymmetric Witten index.

Real and tropical enumerative geometry and their interaction were addressed in talks by I. Itenberg and J. Rau, and in the discussion led by P. Georgieva. In particular, I. Itenberg and P. Georgieva introduced new real enumerative invariants, relative (Itenberg) and positive genus (Georgieva), which so far have no a reasonable tropical interpretation. In turn, the main message of J. Rau's talk was a tropical way to compute real double Hurwitz numbers.

Toric geometry and combinatorics of convex (lattice) polyhedra are an immanent part of constructions and techniques in tropical geometry, and several talks addressed these topics: V. Batyrev presented stringy Chern classes of singular toric varieties, S. Galkin linked quantum cohomology of Fano three-folds with toric degenerations of toric three-folds, N. Kalinin estimated the irrationality degree of hypersurfaces in toric varieties via the combinatorics of their Newton polytopes, a nice application of tropical geometry to the generalized Robinson-Shensted-Knuth correspondence was shown by G. Koshevoy, E. Lupercio has exhibited a theory of non-commutative toric varieties, which, in particular, provides a conventional proof of McMullen inequalities for irrational convex polytopes. T. Nishinou jointly with T. Y. Yu developed a version of the patchworking construction, based, in fact, on specific toric degenerations, in order to prove that immersed trivalent tropical curves in tropical abelian surfaces (equivalently, periodic tropical curves in the plane) are tropicalizations of algebraic curves in abelian surfaces.

The non-Archimedean geometry, Berkovich spaces and tropical varieties came together in the talks by W. Gubler and M. Temkin. The non-Archimedean and Berkovich geometry enrich tropical varieties with new structures, while the tropical picture helps to clarify complicated constructions.

An interaction of tropical geometry with symplectic geometry and mirror symmetry is one of the most perspective and promising research directions. These topics have been presented in the talks by leading experts M. Abouzaid, Y.-G. Oh, Y. Soibelman and in the discussion led by L. Katzarkov. Among the young researchers, H. Argüz demonstrated tropical techniques in development of (symplectic) Floer theory for the Tate curve.

A surprisingly deep and essential manifestation of tropical geometry in the study of integrable systems was shown in the talk of the prominent expert in the field R. Inoue, who worked out an example of box-ball system. Another nice example, a sandpile system, was presented by young researchers M. Shkolnikov and N. Kalinin, who showed that solutions to a certain sandpile model are perfectly described by tropical varieties. The talk by V. Fock brought together integrable systems, geometry of flag varieties and tropical in nature combinatorics, which resulted in very promising and interesting structures.

Problems of real algebraic and symplectic geometry linked to tropical geometry and amoebas appeared in talks by E. Brugall'e, T. de Wolff and A. Renaudineau. So, E. Brugallé extended the notion of a (simple) Harnack curve to the case of pseudo-holomorphic curves and showed that their isotopy classification coincides with that for algebraic Harnack curves. T. de Wolff exhibited an efficient criterion of positivity of real polynomials via the geometry of their amoebas. A. Renaudineau applied the patchworking construction to the study of topology of real surfaces in toric three-folds.

We hope that the very intensive and substantial exchange of a broad spectrum of ideas during the workshop will stimulate the further research in the main discussed problems, which still are far from being completely settled.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Oleg Viro in the "Simons Visiting Professors" program at the MFO.

Workshop: Tropical Aspects in Geometry, Topology and Physics

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Abstracts

Complex integrable systems, Mirror Symmetry, wall-crossing formulas and refined tropical count

Yan Soibelman

This is a summary of my talk as well as the discussion session on the related topic of wall-crossing formulas. The latter were introduced in my joint paper with Maxim Kontsevich in 2008 (see [2]).

Main motivation for the talk was the recent activity on the so-called refined tropical count (Block, Goetsche, Itenberg, Mikhalkin, Filippini, Stoppa). Exposition of the talk was based on several joint papers and discussions with Maxim Kontsevich.

The main idea is to interpret the "quantized" Mikhalkin weights (i.e. Block-Goetsche weights) for tropical curves in terms of the Mirror Symmetry for the Fukaya category of the total space of a complex integrable system (see my joint paper with Kontsevich [3] on this topic). More precisely, one has to consider the Fukaya category with a non-trivial B-field which has rational periods.

It is known since 90's that a non-trivial B-field gives rise to a quantization of the structure sheaf of the mirror dual space. In particular, if we assume that the base of a (real) Liouville integrable system is embedded into the total space as a Lagrangian brane, and the B-field vanishes on it, then the endomorphism algebra of this brane gives (under certain conditions) the quantum torus as a local model for the structure sheaf of the mirror dual.

How this can be related to the refined tropical count? Let us consider a toy-model in the case when the B-field is trivial. Take the mirror dual to an SYZ torus (fiber of the integrable system). It is useful (and even necessary) to consider the mirror dual space as an anlytic space over a non-archimedean field (Novikov field). Then an analytic function in the tube domain which contains the dual torus can be restricted to it and then expanded into a Fourier series. Fourier coefficients c_γ are parametrized by first homology classes γ (equivalently, the fundamental group) of the SYZ torus.

Let us ask the following question: what happens with a given Fourier coefficient when we move the torus from a generic point u of the base to a generic point u_1? It is easy to see that if we do not cross the wall (real codimension one subvariety where SYZ tori contain boundaries of holomorphic discs of Maslov index zero) then the Fourier coefficient changes by the following formula:

$$c_\gamma \mapsto c_\gamma q^{A(V)},$$

where $A(V)$ is the symplectic area of the cylinder V with the boudary of the homology class γ on the tori projected to u and u_1. Here $q^{A(V)}$ is the corresponding element of the Novikov field.

This naive answer should be modified by the wall-crossing formulas when we cross the wall. This agrees with the approach to Mirror Symmetry proposed in our paper with Kontsevich in 2004 [1], which was further developed by Gross and Siebert. The wall-crossing formulas can be written in terms of the virtual number of psedo-holomorphic discs of Maslov index zero with the boundary on the corresponding SYZ tori. Nevertheless explicit formulas are not available in general.

An appropriate foundational framework has been developing in the works of Abouzaid, Fukaya and others under the name of Floer family homology. One of the important issues needed for applications to the refined tropical count is the question on how to relate the wall-crossing formulas arising in "family Floer homology" theory with those of Kontsevich-Soibelman. The latter can be written explicitly in terms of dilogarithm functions and Donaldson-Thomas invariants, hence they are developed in a different framework.

During the discussion session Abouzaid explained that he could prove the coincidence of two wall-crossing formulas in the case when there is only one pseudo-holomorphic disc with the boundary in the class γ. At the same time there is a little doubt that the fact is true in general.

Assuming the agreement of two types of the wall-crossing formulas, let us consider the above story in the case of a complex integrable system $\pi : (M, \omega^{2,0}) \to B$. The corresponding family of real symplectic forms is $Re(\omega^{2,0}/t), t \in \mathbf{C}^*$. In this case there are natural cluster coordinates on the mirror dual manifold. Those coordinates were proposed by Gaiotto-Moore-Netizke in terms of corrections to the "naive" semiflat coordinates (the latter ignore the input from the discs).

It was shown by Filippini and Stoppa that in the tropical limit (i.e. when the SYZ fibers collapse) the GMN explicit formulas for the corrected semiflat coordinates become sums over tropical trees on the base, with Mikhalkin weights assigned to the trees. Although GMN use as enumerative data Donaldson-Thomas invariants (in the form introduced by Kontsevich and myself), one can see that the virtual number of pseudo-holomorphic discs can be used instead (again, assuming that the family Floer homology wall-crossing formulas coinside with our wall-crossing formulas).

As a result, one can obtain a corrected formula for the transported Fourier coefficient c_γ in terms of the sum over tropical trees endowed with Mikhalkin weights.

Then it is natural to guess that when the B-field is non-trivial, we obtain a quantization of the above story. I expect that the family Floer homology wall-crossing automorphisms can be quantized and in the simplest possible case the wall-crossing automorphism coincides with the adjoint action of the quantum dilogarithm of a generator of the quantum torus. The transport of the Fourier coefficient gets quantized, and the analog of the tropical limit of the GMN formula coincides with the sum over the tropical trees on the base, but this time endowed with Block-Goetsche "quantum" weights. Those sums should also coinside with the quantization of GMN formulas obtained by Filippini and Stoppa in a pure formal way. In our approach those formulas have a clear symplectic meaning.

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Relative enumerative invariants of real rational surfaces

Ilia Itenberg

The purpose of the talk is to present real analogs of relative Gromov-Witten invariants in several situations. The Welschinger invariants [8, 9] can be seen as real analogs of absolute genus zero Gromov-Witten invariants and are designed to bound from below the number of real rational curves passing through a given generic real collection of points on a real rational surface. Certain Welschinger invariants can be calculated using the tropical approach (Mikhalkin's correspondence theorem [5] or its modifications).

In some cases, Welschinger type invariants can be defined in a relative situation (meaning that, in addition to point constraints, we fix tangency conditions with respect to some divisor). As examples of such cases, we briefly discuss tropical relative Welschinger invariants (these invariants were defined in [2], and they participate in a real version [2] of the Caporaso-Harris formula [1]), Rasdeaconu-Solomon invariants [7], and the refined invariants introduced very recently by G. Mikhalkin [6]. We present in more details the Hurwitz numbers for real polynomials (joint work with D. Zvonkine [4]) and certain real relative invariants of nodal del Pezzo surfaces; the latter invariants were introduced in the joint work with V. Kharlamov and E. Shustin [3].

In [4] we introduce a signed count of real polynomials which gives rise to a real analog of Hurwitz numbers in the case of polynomials. The invariants obtained allow one to show the abundance of real solutions in the corresponding enumerative problems: in many cases, the number of real solutions is asymptotically equivalent (in the logarithmic scale) to the number of complex solutions.

In [3], for real del Pezzo surfaces with a real (-2)-curve, we suggest (under some assumptions) an invariant signed count of real rational curves that belong to a given divisor class and are tangent to the (-2)-curve at each intersection point; the resulting number does not depend neither on the point constrains, nor on deformation of the surface preserving the real structure and the (-2)-curve.

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Double affine symmetric group action on the toric network, and integrable cellular automata

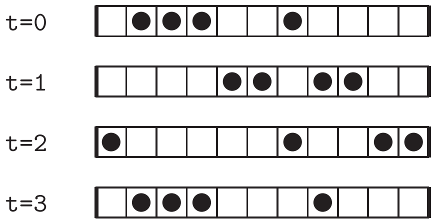
Rei Inoue

In this talk we introduce two types of integrable systems: a cellular automaton called the box-ball system, and a family of commuting rational maps on a toric network. It turns out to be that the former is obtained from a specific case of the latter.

The box-ball system is a cellular automaton introduced by Takahashi and Satsuma in 1990 [TS]. It is described by an algorithm to move finitely many balls in an infinite number of boxes aligned on a line, where a consecutive array of occupied boxes is regarded as a *soliton*. This system has very rich mathematical structure. The time evolution rule of the system is related to known integrable rational maps via a limiting procedure called *ultradiscretization* (which is similar to tropicalization, but we reduce the obtained piecewise-linear map on $\mathbb{R}$ to that on $\mathbb{Z}$); the global dynamics of solitons is related to the discrete Toda lattice, and the local evolution rule is related to the discrete KdV equation. Further, the symmetry of the system is completely explained by the theory of crystal base; the dynamics of balls is induced by the action of the combinatorial R-matrix, the integrals of motion is given by the energy function, and so on. The following diagram is an overview of the related mathematics to the box-ball system.

$$\begin{array}{ccccc}
\text{geom. crystal} & \rightarrow & \text{discrete KdV, Toda} & \leftarrow & \text{algebraic geom.} \\
\Downarrow & & \text{1996-} \Downarrow \text{ultradiscretization} & & \Downarrow \text{tropicalization} \\
\text{crystal base} & \overset{\text{1999-}}{\rightarrow} & \boxed{\text{Box-ball system}} & \overset{\text{2007-}}{\leftarrow} & \text{tropical geom.} \\
\Uparrow & & \text{1999} \Uparrow \text{crystallization} & & \\
\text{quantum group} & \rightarrow & \text{vertex model} & &
\end{array}$$

The periodic box-ball system was defined in 2002 [YT], where a finite number of boxes is aligned on a circle. The following is an example of the time evolution, where we identify the left and right ends.



The initial value problem of the periodic box-ball system was solved in several ways: by using crystal [KTT], combinatorics [MIT], and tropical geometry [IT]. It is remarkable that the tropical Jacobian introduced by Mikhalkin and Zharkov [MZ] works beautifully: the time evolution of the system induces a linear motion on the tropical Jacobian, and the general solution is written in terms of tropical theta function. See [IKT] for a comprehensive review of the box-ball system.

Secondly, we introduce a generalization of the discrete Toda lattice parametrized by a triple of integers (n, m, k), which is defined by using a network on a torus with n horizontal wires, m vertical wires and k *shifts* at the horizontal boundary [ITP]. The phase space $\mathcal{M} \simeq \mathbb{C}^{mn}$ is parametrized by $q_{ij} \in \mathbb{C}$ assigned to each of the crossings of the wires. We construct $m + \gcd(n, k)$ commuting birational maps on $\mathcal{M}$, which come from the *affine symmetric group action* on the parameters on two adjacent parallel wires, either horizontal or vertical. This action originates from the *affine geometric R-matrix* in the theory of affine geometric crystals (see [BK], for example), as a birational lift of the *combinatorial R-matrix* in $A_n^{(1)}$-crystals.

In [ITP] we study the algebro-geometrical structure of the birational maps and solve the corresponding initial value problem. We compute the spectral map from $\mathcal{M}$ to a family of spectral curves with the Picard group of the curve and some additional data, by applying the method of van Moerbeke and Mumford [vMM]. It is shown that each map is linearized on the Jacobian variety of the spectral curve, and the solution is written by using Riemann theta function.

The previously mentioned discrete Toda lattice corresponds to the network of $(n, m, k) = (n, 2, n-1)$. When we consider a generalization of the above diagram, our toric network seems to be a right piece which fits into the first row. There are many interesting problems around the diagram; for instance, "what mathematical object we have in a empty space at the bottom right?"

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Bulk deformations, tropicalizations and non-displaceable Lagrangian toric fibers

Young-Geun Oh

We first explain how one can deform the Lagrangian Floer cohomology using the cycles of the ambient symplectic manifoldsk which Fukaya-Ohta-Oh-Ono [1] called 'bulk deformations'. Consideration of such deformations is prompted on one hand by our attempt to detect as many nondisplaceable Lagrangian submanifolds in a given symplectic manifold and on the other hand by a natural way of getting a conceptual proof of the isomorphism between the quantum cohomology and the Jacobian ring in the big phase of quantum cohomology ring. Fukaya-Ohta-Oh-Ono [1] applied bulk deformations to the case of Lagrangian torus fibers in the toric (or semi-toric) manifolds and called a 'bulk-balanced fiber' when the fiber is detected by a bulk deformation.

In this talk, we explain how one can use tropicalization, or rather the tropicalization relative to the facets of the moment polytope, of the Fukaya-Ohta-Oh-Ono potential function to locate all of the bulk-balanced fibers purely in terms of the moment polytope of the given toric manifold. The first part of the talk is based on the joint work with Fuikaya, Ohta and Ono, and the second part is based on the work by Yoosik Kim and Jaeho Lee, two students of mine in University of Wisconsin-Madison.

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Tropical geometry and high energy physics

Barak Kol

The talk included four parts: introduction to the concept of (p, q) webs; the way supersymmetry implies the (rational) slope condition; a brief review of the speaker's expectation that the Welschinger invariant might be described by a supersymmetric index and finally an introduction to a mathematical audience of the initial parts of the paper "Tropical curves, wall crossing and states of supersymmetric field theory" by Ashoke Sen (2012). In the first part it was described how the speaker and collaborators defined in 1997 the notion of a (p, q) web, describing a collection of strings which satisfy the slope and vertex condition. This definition is essentially equivalent to that of a tropical curve in the plane, and it implies the balancing condition. It was explained that the web is a mechanical model, that a tension can be associated with each string $T_{p,q} := \sqrt{p^2 + q^2}$ and that in this way each vertex is automatically in mechanical equilibrium. In the second part supersymmetry generators were motivated as an odd square root of translation generators. Spinors were introduced as the parameters of a supersymmetry transformation and an equation was presented for the spinors (half dimensional subspace) conserved by a (p,q) string with a prescribed slope in the plane. The spinor (quarter dimension subspace) preserved by all parts of the web was presented. In reply to a question about rotations in the plane it was explained that a (p, q) web is supersymmetric also after a rotation, only it preserves a different spinor. Next the talk briefly discussed the surprising existence of the Welschinger invariant and speaker's expectation that it could be explained by realizing it as a supersymmetric (Witten) index (an idea presented already at the Ein-Gedi meeting in April 2013). Finally the basic set-up of the paper by Sen was explained including the nature of 1/4 BPS states in 4D ($\mathcal{N} = 4$) supersymmetric gauge theory with gauge group $SU(3)$.

Realization of tropical curves in abelian surfaces

Takeo Nishinou

(joint work with Tony Yue Yu)

Algebraic curves in the complex projective plane $\mathbb{CP}^2$ give rise to tropical curves in $\mathbb{R}^2$. Tropical curves in $\mathbb{R}^2$ are balanced piecewise linear graphs. One idea of tropical geometry is to use such combinatoric gadgets to study algebraic curves (see [4, 1, 2]). However, not all tropical curves arise from algebraic curves. The problem of determining whether a tropical curve arises from algebraic curves is called the *realization problem*.

For toric surfaces, it is a consequence of Mikhalkin's correspondence theorem [3] that all trivalent tropical curves in $\mathbb{R}^2$ are realizable. Nishinou-Siebert [6], Nishinou [5] and Tyomkin [7] generalized the correspondence theorem to general toric varieties. As a corollary, all trivalent regular tropical curves in $\mathbb{R}^n$ are realizable.

Nevertheless, the ambient spaces in the results above are always toric varieties. In this talk, we aim to discuss the realization problem for the first non-toric case, i.e. the case of abelian surfaces.

We show that for abelian surfaces of *power-type*, all trivalent tropical curves are realizable. Here is the precise statement.

Let $k := \mathbb{C}((t))$ be the field of formal Laurent series. Let $\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$ be a 2×2 integer matrix with non-zero determinant. Let ϖ_1, ϖ_2 be 2 uniformizing parameters of k. Consider the action of $\mathbb{Z}^2$ on the algebraic torus $\mathbb{G}^2_{\mathrm{m,k}}$ over k via coordinate-wise multiplication by $(\varpi_1^{a_{11}}, \varpi_2^{a_{12}})$ and $(\varpi_1^{a_{21}}, \varpi_2^{a_{22}})$.

Definition. An abelian surface $\mathcal{X}$ over k is said to be of *power-type* if it is isomorphic to the quotient of $\mathbb{G}^2_{\mathrm{m,k}}$ by the action of $\mathbb{Z}^2$ defined above.

Let $\mathcal{X}$ be an abelian surface of power-type as above. Let S denote the quotient of $\mathbb{R}^2$ by the lattice generated by the vectors (a_{11}, a_{12}) and (a_{21}, a_{22}). We consider S to be the tropicalization of the abelian surface $\mathcal{X}$.

Theorem. *Let (Γ, h) be a trivalent immersed tropical curve in S. Then there exists a finite field extension $k \subset k'$, a proper smooth curve $\mathcal{C}$ over k' and a morphism $f\colon \mathcal{C} \to \mathcal{X} \times_k k'$ whose tropicalization is (Γ, h).*

Remark. If the abelian surface $\mathcal{X}$ over $\mathbb{C}((t))$ comes from an actual family of complex tori over a punctured disc, then we can require that the curve $\mathcal{C}$ and the morphism f in Theorem also come from an actual family over a (possibly smaller) punctured disc.

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Family Floer cohomology and Mirror Symmetry

Mohammed Abouzaid

To an integral affine polygon P in $\mathbb{R}^n$, one can associate two spaces: (i) a hypersurface $H_P \subset (\mathbb{C}^*)^n$ and (ii) a toric variety $\overline{Y}_P$ whose fan is the cone on P. In joint work with Auroux and Katzarkov [1], we studied mirror symmetry for these spaces by constructing dual torus fibrations, refining physics predictions. The starting point is to construct two auxiliary spaces: X_P, which is a conic bundle over $(\mathbb{C}^*)^n$ with discriminant locus H_P, and Y_P which is the complement of a regular value of the toric map $\overline{Y}_P \to \mathbb{A}^1$. The pairs (X_P, Y_P) are mirror in the sense that they admit dual torus fibrations. From this duality arise other mirror pairs by adding a divisor at infinity to one side and equipping the other with a potential function. In particular, letting $\bar{X}_P$ denote the blowup of $(\mathbb{C}^*)^n \times \mathbb{C}$ along $H_P \times \{0\}$, the projection to $\mathbb{C}$ is mirror to $\overline{Y}_P$ equipped with the toric map to $\mathbb{A}^1$. To recover mirror symmetry for H_P, we observe that the projection $\bar{X}_P \to \mathbb{C}$ is transversely non-degenerate along H_P, which implies that the theories assigned by symplectic topology and algebraic geometry to H_P are equivalent to the corresponding theories for $\bar{X}_P \to \mathbb{C}$ (Knörrer periodicity). We conclude that H_P is mirror to the toric potential on $\overline{Y}_P$.

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Degenerations, transitions and quantum cohomology

Sergey Galkin

Given a singular variety I discuss the relations between quantum cohomology of its resolution and smoothing. In particular, I explain how toric degenerations help with computing Gromov–Witten invariants, and the role of this story in "Fanosearch" programme [7]. The challenge is to formulate enumerative symplectic geometry of complex 3-folds in a way suitable for extracting invariants under blowups, contractions, and transitions.

Topologically *a conifold transition* is a surgery on a real 6-dimensional manifold switching a 3-sphere with a 2-sphere. The boundary of the tubular neighbourhood of a 2-sphere or a 3-sphere is isomorphic to $S^2 \times S^3$, so one can replace a tubular neighbourhood of one of those spheres with a tubular neighbourghood of another one. Smith–Thomas–Yau [17] show that this procedure makes sense in the context of symplectic manifolds, replacing a Lagrangian S^3 with a symplectic S^2. There are two natural ways of choosing a S^2 related by a *flop*. Note that similar procedure in real dimension 4, namely replacing a Lagrangian S^2 with a symplectic S^2, does not change the topology of underlying manifold (as was observed already by Brieskorn), and can be compensated merely by changing the symplectic form on the same underlying manifold. For the conifold transition of 6-dimensional manifolds topology obviously changes, since the Euler number of two manifolds differs by $2 = e(S^2) - e(S^3)$.

A local algebro-geometric picture for the conifold transition looks as follows. Consider a function $f : \mathbb{C}^4 \to \mathbb{C}$ given in some coordinates x, y, z, w by a non-degenerate quadratic form

$$f = (xy - zw)$$

A 3-fold conifold singularity (or an ordinary double point in dimension 3) is locally given by equation $(f = 0)$, the fiber of map f over 0

$$X_0 := \{u \in \mathbb{C}^4 : f(u) = 0\} = \{(x, y, z, w) \in \mathbb{C}^4 : xy = zw\}.$$

It has *a smoothing* X_t for $t \in \mathbb{C}$ given as a fiber of f over t

$$X_t := \{u \in \mathbb{C}^4 : f(u) = t\} = \{(x, y, z, w) \in \mathbb{C}^4 : xy = zw + t\}$$

and two *small resolutions* $\hat{X}_0 = \overline{\Gamma_\phi}, \hat{X}_0{}' = \overline{\Gamma_{\phi'}}$, given as Zariski closures of the graphs of rational maps $\phi, \phi' : X_0 \to \mathbb{CP}^1$

$$\phi(x, y, z, w) := (x : z) = (w : y)$$

$$\phi'(x, y, z, w) := (x : w) = (z : y)$$

Here all X_t are isomorphic as complex manifolds [1], and as a real 6-manifold X_1 is isomorphic to T^*S^3, the total space of the (co)tangent bundle on S^3 [2]. The small resolutions $\hat{X}_0$ and $\hat{X}_0{}'$ are isomorphic as abstract complex manifolds to the so-called *local* $\mathbb{CP}^1$, that is the total space of the bundle $\mathcal{O}_{\mathbb{CP}^1}(-1)^{\oplus 2}$.

The global picture is similar: we look for a singular complex threefold Y_{sing}, such that $\mathrm{Sing Y_{sing}}$ equals to N ordinary double points. It always has 2^N small resolutions which we denote by Y_{res}, however some of them may fail to be quasi-projective. The question of existence of a smoothing [3] is more subtle: versal deformation space for the conifold singularity is smooth and described above, but in some situations there are local-to-global obstructions. It was observed that for projective manifolds the existence of a projective resolution and of a smoothing are *mirror dual* to each other. Friedman [8] shows that a smoothing always exists and unique (that is the versal deformation space is smooth) for Fano threefolds Y_{sing}. For Calabi–Yau threefolds Y_{sing} a smoothing exists iff there is a linear relation between exceptional symplectic 2-spheres in Y_{res} [8], and a projective resolution exists iff there is a linear relation between vanishing Lagrangian 3-spheres in Y_{sm}. For proper toric varieties Y_{sing} Gelfand–Kapranov–Zelevinsky demonstrate the existence of a projective small resolution.

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[1] Thanks to a $\mathbb{C}^*$-action on the total space $(x, y, z, w) \to (\lambda x, \lambda y, \lambda z, \lambda w)$, compatible with the natural action on the base $t \to \lambda^2 t$.

[2] It is easier to see in coordinates $z_1, \ldots, z_4$, where function $f = xy - zw$ has a form $f = \sum_{k=1}^4 z_k^2$ and for real positive t. Then if $z_k = x_k + iy_k$, two vectors in $\mathbb{R}^4$ $x = (x_1, \ldots, x_4)$ and $y = (y_1, \ldots, y_4)$ are pairwise-orthogonal and square of norms differ by $|x|^2 - |y|^2 = t$

[3] A smoothing of Y_{sing} is a flat projective morphism $f : \mathcal{Y} \to \Delta$ to a disc Δ, such that $f^{-1}(0) = Y_{sing}$ and for some $t \in \Delta$ the fiber $f^{-1}(t)$ is a smooth complex threefold Y_{sm}. In this situation we say that Y_{sm} degenerates to Y_{sing}.

Since in complex dimension 2 conifold transitions do not affect the topology, they can be used to compute Gromov–Witten invariants and find mirror potentials for non-toric del Pezzo surfaces of degrees 5 and 4. These surfaces have toric degenerations with A_1 singularities, and the crepant resolutions of these degenerations are smooth toric weak del Pezzo surfaces. There are variety of methods to compute holomorphic curves and discs on toric surfaces (equivariant method of Givental, tropical method of Mikhalkin, Cho–Oh, Fukaya–Oh–Ohta–Ono, Chan–Lau–Leung, etc), and the results of these computations can be transferred back to non-toric del Pezzo. Also in dimension 2 it is quite easy to construct symplectic birational invariants, for example a surface S is rational iff its quantum cohomology $QH(S)$ is generically semi-simple [3] [4].

For the threefolds situation is quite different. There are theorems relating quantitative aspects of Gromov–Witten theories of a manifold and its blowup [10, 3] [5] or conifold transition [13, 14, 4, 12]. However none of them is suitable yet to demonstrate numerous qualitative relations we expect to hold: birational/transition invariance of symplectic rational connectedness, non-triviality of GW invariants, sharp birational invariants. In the remainder I discuss what we know and what we would like to know about the blowups and transitions of threefolds in the context of "birational symplectic topology".

In my thesis [9] I described all conifold transitions from Fano threefolds to toric threefolds, thus answering a question posed by Batyrev in [2]. The case when Y_{sing} is a toric Fano is arguably the simplest: by Friedman a smoothing Y_{sm} exists and its topology is unique, so one just have to identify it, by computing some invariants and using Fano–Iskovskikh–Mori–Mukai's classification. There are 100 conifold transitions from smooth Fano to smooth toric weak Fano, and approximately one half (44) of all non-toric Fanos have at least one such transition, on average two. To a toric Y_{sing} (or Y_{res}) one can associate a Laurent polynomial $W = \sum_v z^v$ where the summation runs over all vertices v of the fan polytope of Y_{sing} (equiv. Y_{res}, since the resolution is small). Batyrev put a conjecture that constant terms of powers W^d are equal to Gromov–Witten invariants $\langle \psi^{d-2}[pt] \rangle_{0,1,d}$, that count rational curves on Y_{sm} passing through a generic point with "tangency conditions" prescribed by a power of psi-class. Nishinou–Nohara–Ueda shown that in fact W above is the Floer potential function, that counts pseudo-holomorphic discs of Maslov index two on Y_{sm} with a boundary on a Lagrangian torus, obtained as a symplectic transport of a fiber of a moment map $Y_{sing} \to \Delta$. With Bondal in [4] we proved a conjecture of Batyrev by expressing the described GW invariant as a polynomial of the numbers of holomorphic discs, using the relation between holomorphic curves, passing through a point, and holomorphic discs bounded on a Lagrangian torus, in the tropical limit. Same invariants were also computed in [6]

[4] However I do not know any a priori proof that semi-simplicity persists under the birational contractions, even in dimension 2. As an illustration of the problem: non-algebraic $K3$ surface does not have any non-trivial Gromov–Witten invariant, but its blowup in a point does — an invariant counting curves in an exceptional class!

[5] On practice it is often preferable to use less direct ways of computation as in [6], thanks to their more compact collection of the enumerative data.

by alternative (and less uniform) methods for all Fano threefolds. Also Batyrev and Kreuzer described degenerations of Fano threefolds to nodal half-anticanonical hypersurfaces in toric fourfolds, and it turns out that almost every Fano has such a degeneration.

In [13] Li and Ruan give a relation between Gromov–Witten theories of two threefolds linked by a conifold transition, and very recently Iritani and Xiao in [12] reformulated it as a relation between two quantum connections [6]: the quantum connection of Y_{sm} (on cohomology of even degree) is obtained from the quantum connection of Y_{res} as a residue of a sub-quotient. Similar descriptions could (and should) be also obtained for the quantum connections of the blowups.

I expect that some sharp birational invariants could be extracted from Gromov–Witten theory of threefolds [7], and that they should be related to the monodromy group of the quantum connection. Much work still has to be done to understand better the relation between these monodromies, and to extract an invariant out of it.

Finally, the relation between conifold transitions and its effect on classical topology should be clarified further. A hypothesis usually referred to as "Reid's dream" [15] says that *all* Calabi–Yau threefolds may be connected by a network of conifold transitions. There is an extensive experimental evidence towards this — most of known Calabi–Yau threefolds were shown to be connected by a network. But there are some classical invariants of the transition, such as the fundamental group [8]. There are a couple of families of non-simply-connected threefolds without holomorphic 1-forms and $K = 0$ (e.g. 16 families were found by Batyrev and Kreuzer as crepant resolutions of hypersurfaces in toric fourfolds), so these threefolds are clearly disconnected from the conjectural "simply-connected network". Smith–Thomas–Yau [17] using conifold degenerations observed that possibly there are lots of non-Kähler symplectic Calabi–Yau threefolds, and Fine–Panov–Petrunin shown that the fundamental group of symplectic Calabi–Yau threefolds can take infinitely many different values. Non-simply-connected manifold could not be rationally connected, non-rationally-connected manifold cannot be symplectically rationally connected, non-symplectically-rationally-connected manifolds cannot have semi-simple quantum cohomology, which means at least some element in $QH(Y)$ of Y with $\pi_1(Y) \neq 0$ should be nilpotent. This suggests an interesting question: *given a non-contractible loop γ on Y, associate to it a nilpotent in $QH(Y)$*. One possible approach to this could be via Givental's interpretation [11] of the quantum D-module of Y with S^1-equivariant Floer theory on a free loop space $\mathcal{L}Y$. We find amusing the contrast with Seidel's representation $Sei : \pi_1(\mathrm{SympY}) \to \mathrm{QH(Y)}^*$ that associates an invertible element to a loop in symplectomorphisms [16].

[6] This work prompted me to speak on this subject here.

[7] For more details I refer to my recent talk in Stony Brook on "New techniques in birational geometry" conference (available at `http://www.math.stonybrook.edu/Videos/Birational/video.php?f=20150411-1-Galkin`).

[8] Both S^3 (resp. S^2) has codimension at least 3 in Y_{sm} (resp. Y_{res}), hence $\pi_1(Y_{sm}) = \pi_1(Y_{sm} \setminus \bigcup S^3_k) = \pi_1(Y_{res} \setminus \bigcup S^2_k) = \pi_1(Y_{res})$.

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Informal discussion “Quantum indices of real curves”

Grigory Mikhalkin

The discussion was devoted to quantum indices of real curves. This notion was introduced by Mikhalkin very recently before the meeting, see [1]. The quantum index is a half-integer number that is prescribed to any projective real algebraic curve of type I intersecting all coordinate axes in the maximal number of real points. The curve should be enhanced with a choice of a *complex orientation*, see [2] for the definition. It turns our that the signed logarithmic area enclosed by such curves cannot vary continuously, and has a discrete spectrum of values formed by π^2 times the quantum index. This motivates the name *quantum index* as the area must change by jump with the Planck constant equal to $\frac{\pi^2}{2}$. The quantum index also can be expressed through the logarithmic rotation number of the curve. Furthermore, in the case of real rational curves, and more generally, for the case of curves of toric type I, it is possible to determine the quantum number (along with many additional quantities, such as the toric complex orientation formulas) by means of the diagram introduced in [1] and encoding the order of appearance of the coordinate axes on the real curve. This diagram can be viewed as a non-commutative version of the Newton polygon of the curve. Furthermore, the quantum indices allow one to refine 2D real enumerative geometry of rational curves of degree d [3] to enumeration of curves in each quantum index separately. To do this one has to place d points on each coordinate axis subject to the Menelaus condition and then look for the rational curves of degree d such that their *squares* pass through the configuration. The resulting refined enumeration is shown to coincide with the so-called *Block-Göttsche* invariant that is well-defined in tropical geometry.

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Integrable systems, flag configurations and matroids

Vladimir Fock

In the talk we will discuss a special class of matroids, namely the ones defined by bipartite planar graphs. These matroids have a cluster variety as the variety of realizations, that in particular are birationally isomorphic to the configuration spaces of flags in a vector space. On the other hand they are defined over any semi-field, in particular over a tropical one. If moreover we consider infinite matroids but invariant under a discrete group action, we get the space of local systems on surfaces, of pairs (planar algebraic curve of genus g, a line bundle of degree $g-1$ on it) and some other varieties which are kind of mixture of the two.

Consider a planar bipartite graph with the sets W and B of white and black vertices, respectively. A discrete line bundle on such graph is an association of one-dimensional vector space V_v over a field or a semi-field $\mathbb{F}$ to every vertex v. A discrete connection on such a graph is an association to every edge e an isomorphism $A_e : V_w \to V_b$ where w and b are the white and the black end of the edge e, respectively. Two connections are called gauge equivalent if they are conjugated by an automorphism of the line bundle. The space of gauge classes of discrete connections is an algebraic torus isomorphic to the group $H^1(\Gamma, \mathbb{F}^\times) = (\mathbb{F}^\times)^F$, where F is the space of faces of the graph Γ. In other words for every face $i \in F$ the composition of isomorphisms A_e and their inverses around it counterclockwise gives an isomorphism from a one dimensional vector space to itself, that is a number. Denoted by x_i this very number if the number of sides of the face is equal to 2 modulo 4 and minus this number otherwise. The collection of numbers x_i are coordinates on the space $H^1(\Gamma, \mathbb{F}^\times)$.

This space has a canonical Poisson structure defined by the rule $\{x_i, x_j\} = \varepsilon_{ij} x_i x_j$, where ε_{ij} is the number of common edges of the faces i and j counted with signs.

The space of connections can be imagined as a matrix $\mathfrak{D} : \mathbb{F}^B \to \mathbb{F}^W$ with columns (resp. rows) enumerated by white (resp. black) vertices and with the sums of A_e over all edges e connecting given black and white vertex as matrix elements. The gauge equivalence amounts to multiplication by diagonal matrices. We call the graph *minimal* if this matrix has generically the maximal rank. Recoloring the vertices if necessary we may assume that the number of white vertices is not less than the number of black ones.

The projectivised cokernel of the matrix $\mathfrak{D}$ contains distinguished collection of points, namely the images of the coordinate axes corresponding to white vertices of the graph. The points corresponding to all neighbors of a given black vertex belong to a subspace of dimension less than their number by two. Therefore a minimal graph Γ defines a matroid structure on the set W with black vertices corresponding to black vertices. Different gauge classes of connections corresponds to different realizations of the matroid.

Call two matroids equivalent if their realization spaces are birationally isomorphic, i.e., one can construct canonically a realization of one from a generic realization of another. For example one can easily see that if a graph Γ has a two-valent vertex than the corresponding matroid is isomorphic to the one with the two adjacent edges contracted. Less trivial equivalence corresponds to a so-called diamond move. Namely if we have two black vertices b_1 and b_2 having two white ones w_1 and w_2 as common neighbors, one can add two more white and two more black three-valent vertices. Indeed, let the black vertices correspond to a collection b_1 of k_1 and b_2 of k_2 points, respectively belonging to projective subspaces L_1 and L_2 of dimension $k_1 - 2$ and $k_2 - 2$, respectively. The intersection of these projective planes contains at least two points w_1 and w_2, and therefore a line L passing through them. Denote by w_3 and w_4 the intersection of the subspace generated by $b_1 \backslash \{w_1, w_2\}$ and L (respectively, the intersection of $b_2 \backslash \{w_1, w_2\}$) and L and denote $b_3 = \{w_1, w_3, w_4\}$ and $b_4 = \{w_2, w_3, w_4\}$.

The equivalence of matroid is a subtraction-free (more precisely - a cluster map) map in coordinates, (here the sign rule for the coordinates is essential) and therefore it is defined over a semifield.

A path on a graph Γ is called a *zig-zag* if it turns maximally left (resp. maximally right) at every black (resp. right) vertex. If such path separates the graph into two parts Γ^r (resp. Γ^l) to the right (resp. left) of the path (the path itself is included in both parts) than the matroid, corresponding to Γ^r (resp. Γ^l) is a sub-matroid (resp. factor-matroid) of the one, defined by Γ.

Graphs on a disk with only white vertices on the boundary are minimal if any two zig-zags have no more than one common edge. Such graphs up to equivalence are determined by the pattern how the zig-zags connect points on the boundary. The spaces defined by the corresponding matroids are configurations of flags (not necessarily complete).

This construction admits a generalization when the graph Γ on a disk is infinite, but invariant under an action of a subgroup of homeomorphisms of the disk with finite quotient when acting on the graph. The corresponding realization space turn out to be the space of $SL(N)$ local systems on the surface, that is the quotient of the disk by the group action.

Another generalization concerns infinite graphs on a plane invariant under the action of a lattice $\mathbb{Z}^2$. In this case the corresponding realization space is birational to the space of pairs (planar algebraic curve, line bundle on it). In this case point configurations are in an infinite dimensional space, namely in the projectivization of the space of meromorphic sections of degree $g-1$ bundle on the curves with poles at infinity only and the action of the group $\mathbb{Z}^2$ corresponds to multiplication of sections by monomial of coordinates.

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Stringy Chern classes of singular toric varieties and their applications

Victor Batyrev

(joint work with Karin Schaller)

1. Introduction

Let X be a d-dimensional normal projective variety over $\mathbb{C}$ and let $\rho : Y \to X$ be a desingularization of X such that the exceptional locus of ρ is a union of smooth irreducible divisors $D_1, \dots, D_s$ with normal crossings. Assume that X is a $\mathbb{Q}$-Gorenstein, i.e. the canononical class K_X is a $\mathbb{Q}$-Cartier divisor, and we can write

$$K_Y = \rho^* K_X + \sum_{i=1}^{s} a_i D_i$$

for some rational numbers a_i. The variety X is said to have at worst log-terminal singularities if $a_i > -1$ for all $1 \leq i \leq s$. We set $I := \{1, \dots, s\}$ and define $D_\emptyset := Y$, $D_J := \cap_{j \in J} D_j$ for all $\emptyset \neq J \subseteq I$. Then for any subset $J \subseteq I$ the complete intersection D_J is either empty, or a smooth projective subvariety of codimension $|J|$.

For any smooth projective variety V, one considers the E-polynomial

$$E(V;u,v) := \sum_{p,q}(-1)^{p+q}h^{p,q}(V)u^p v^q,$$

where $h^{p,q}(V)$ denotes the Hodge number of V.

It was shown in [6] that if X is a normal projective variety with at worst log-terminal singularities then the function

$$E_{str}(X;u,v) := \sum_{\emptyset\subseteq J\subseteq I} E(D_J;u,v)\prod_{j\in J}\left(\frac{uv-1}{(uv)^{a_j+1}-1}-1\right)$$

is independent on the choice of the resolution $\rho : Y \to X$. This function is called the *stringy E-function* of X. The stringy top Chern class (or the stringy Euler number) of X can be obtained via limit by the formula:

$$c_d^{str}(X) := \lim_{u,v\to 1} E_{str}(X;u,v) = \sum_{\emptyset\subseteq J\subseteq I} c_{n-|J|}(D_J)\prod_{j\in J}\left(\frac{-a_j}{a_j+1}\right),$$

where $c_{d-|J|}(D_J)$ denotes the top Chern class of the smooth subvariety D_J. In particular, if $\rho : Y \to X$ is a crepant resolution (i.e. all a_i are zeros) then $E_{str}(X;u,v) = E(Y;u,v)$ and the top Chern class of Y does not depend on the choice of a crepant resolution.

It was observed by Aluffi [1] that the push forward of any Chern class $\rho_* c_i(Y) \in A_{d-i}(X)$ also does not depend on the choice of a crepant resolution. This observation led to the notion of stringy Chern classes of singular varieties considered in [2, 8, 9, 10, 12]

It was proved in [7] that the Libgober-Wood identity (see [11]) can be generalized for the stringy E-function in the following form:

$$\frac{d^2}{du^2}E_{str}\left(X;u,1\right)\Big|_{u=1} = \frac{3d^2-5d}{12}c_d^{str}(X) + \frac{1}{6}c_1(X).c_{d-1}^{str}(X), \tag{1}$$

where the number $c_1(X).c_{d-1}^{str}(X)$ can be defined by the formula

$$c_1(X).c_{d-1}^{str}(X) := \sum_{\emptyset\subseteq J\subseteq I} \rho^* c_1(X) c_{d-|J|-1}(D_J)\prod_{j\in J}\left(\frac{-a_j}{a_j+1}\right).$$

This is an example of a formula that uses two different stringy Chern classes: $c_{d-1}^{str}(X)$ and the top stringy Chern class $c_d^{str}(X)$. In general, the stringy Chern classes $c_{d-k}^{str}(X)$ can be defined by the following formula:

$$c_{d-k}^{str}(X) := \rho_*\left(\sum_{\emptyset\subseteq J\subseteq I} e_{J*}(c_{d-|J|-k}(D_J))\prod_{j\in J}\left(\frac{-a_j}{a_j+1}\right)\right) \in A_k(X)_{\mathbb{Q}}, \tag{2}$$

where $\rho_* : A_k(Y) \to A_k(X)$ and $e_{J*} : A_k(D_J) \to A_k(Y)$ are homomorphisms of Chow groups corresponding to the proper birational morphism $\rho : Y \to X$ and to the closed embedding $e_J : D_J \to Y$.

2. Stringy Chern classes of toric varieties

Our interest in stringy Chern classes of $\mathbb{Q}$-Gorenstein toric varieties is motivated by the mirror symmetry constructions using Calabi-Yau hypersurfaces and complete intersections in Gorenstein toric Fano varieties [3, 4].

We prove that the total stringy Chern class $c_\bullet^{str}$ of hypersurfaces and complete intersections can be computed via the total stringy Chern class of the embient variety by the same formula as in the smooth case:

Theorem 1. *Let X be a projective algebraic variety of dimension d with at worst log-terminal singularities and let $Z_1, \ldots, Z_r \subseteq X$ be a generic semiample divisors on X and $j : Z := Z_1 \cap \ldots \cap Z_r \hookrightarrow X$ be the closed embedding. Then*

$$j_* c_\bullet^{str}(Z) = c_\bullet^{str}(X) \prod_{i=1}^{r} [Z_i] (1 + [Z_i])^{-1}.$$

Let M be a free abelian group (lattice) of rank d and let $N := \mathrm{Hom}(M, \mathbb{Z})$ be the dual lattice. We denote by $M_\mathbb{R} := M \otimes \mathbb{R}$ and by $N_\mathbb{R} := N \otimes \mathbb{R}$ the corresponding d-dimensional real vector spaces. A projective toric variety X defined by a rational polyhedral fan $\Sigma \subset N_\mathbb{R}$ is $\mathbb{Q}$-Gorenstein if and only if there exists a piecewise linear function $\kappa : N_\mathbb{R} \to \mathbb{R}$ which is linear on every cone $\sigma \in \Sigma$ and has value -1 on every primitive lattice generator in N of a 1-dimensional cone in Σ. We denote by $\mathrm{vol}(*)$ the usual l-dimensional volume on a l-dimensional real vector space with respect to a lattice in it. We set $v(*) := l!\mathrm{vol}(*)$. In particular, for any l-dimensional cone $\sigma \in \Sigma$ we define

$$v(\sigma) := l! \cdot \mathrm{vol}(\theta_\sigma)$$

where $\mathrm{vol}(\theta_\sigma)$ is the l-dimensional volume of the lattice polytope θ_σ obtained as convex hull of 0 and the primitive lattice generators of all 1-dimensional faces of σ.

Theorem 2. *Let X be a $\mathbb{Q}$-Gorenstein n-dimensional projective toric variety X_Σ associated with a fan $\Sigma \subseteq N_\mathbb{R}$. Denote by $\Sigma(i)$ the set of all i-dimensional cones in Σ. Then the total stringy Chern class of X equals*

$$c_\bullet^{str}(X) = \sum_{\sigma \in \Sigma} v(\sigma) \cdot [X_\sigma],$$

where $[X_\sigma]$ is the class of the closed subvariety X_σ corresponding to $\sigma \in \Sigma$.

Using Theorems 1 and 2, one obtains:

Corollary ([5] 7.10). *Let Δ be a d-dimensional reflexive polyhedron* [3]. *Then stringy Euler number of a generic Calabi-Yau hypersurface Z in the Gorenstein toric Fano variety X_Δ can be computed by the formula:*

$$e_{str}(Z) = \sum_{i=1}^{d-2} \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta) = i}} (-1)^{i-1} v(\theta) \cdot v(\theta^*).$$

Similar combinatorial formulas can be obtained for Calabi-Yau complete intersection in toric varieties.

3. Stringy Libgober-Wood identity for toric varieties

Using stringy Chern classes of toric varieties we can find a combinatorial interpretation of the stringy Libgober-Wood indentity for $\mathbb{Q}$-Gorenstein toric varieties.

Theorem 3. *Let X be a toric log del Pezzo surface defined by a rational polyhedral fan $\Sigma \subseteq N_{\mathbb{R}}$. We denote by $\Theta \subseteq N_{\mathbb{R}}$ the convex hull of all primitive lattice points generating 1-dimensional cones in Σ. Let $\Theta^* \subset M_{\mathbb{R}}$ be the dual rational polygon. Then the stringy Libgober-Wood identity for X is equivalent to the equation:*

$$v(\Theta) + v(\Theta^*) = 12 \sum_{n \in \Theta \cap N} (\kappa(n) + 1)^2 .$$

In particular, one always has $v(\Theta) + v(\Theta^*) \geq 12$, and the equality holds if and only if Θ is a reflexive polygon.

Let $\Delta^* \subset N_{\mathbb{Q}}$ be a d-dimensional reflexive polytope [3]. Using the power series

$$P(\Delta^*, t) := \sum_{k \geq 0} |k\Delta^* \cap N| t^k$$

one defines the polynomial $\Psi(\Delta^*, t) := (1-t)^{d+1} P(\Delta^*, t) = \sum_{i=0}^{d} \psi_i(\Delta^*) t^i$ where the coefficients $\psi_i(\Delta^*)$ are nonnegative integers such that $\psi_0(\Delta^*) = \psi_d(\Delta^*) = 1$, $\psi_i(\Delta^*) = \psi_{d-i}(\Delta^*)\ \forall 0 \leq i \leq d$ and $\sum_{i=0}^{d} \psi_i(\Delta^*) = v(\Delta^*)$.

Theorem 4. *The stringy Libgober-Wood identity for the Gorenstein toric Fano variety X defined by the fan of cones in $N_{\mathbb{R}}$ over faces of Δ^* is equivalent to the equality:*

$$\sum_{i=0}^{d} \psi_i(\Delta^*) \left(i - \frac{d}{2} \right)^2 = \frac{d}{12} v(\Delta^*) + \frac{1}{6} \sum_{\substack{\theta^* \preceq \Delta^* \\ \dim(\theta^*) = d-2}} v(\theta) \cdot v(\theta^*) , \tag{3}$$

where θ^* denotes the dual to $\theta \subseteq \Delta$ face of the dual reflexive polytope $\Delta \subset M_{\mathbb{Q}}$.

One can show that for reflexive polytopes Δ of dimension 2 and 3 the equation (3) for Δ and for the dual reflexive polytope Δ^* are equivalent to each other and to the well-known equations

$$v(\Delta^*) + v(\Delta) = 12 \ \ (d = 2)$$

and

$$\sum_{\substack{\theta \preceq \Delta \\ \dim(\theta) = 1}} v(\theta) \cdot v(\theta^*) = 24 \ \ (d = 3)$$

For reflexive polytopes Δ of dimension $d \geq 4$ the equation (3) for Δ is not equivalent to the one for Δ^*. If $d = 4$ then the equation (3) for an arbitraty

4-dimensional reflexive polytope Δ is equivalent to the following one:

$$12 \cdot |\partial\Delta \cap M| = 2 \cdot v(\Delta) + \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=2}} v(\theta) \cdot v(\theta^*) \,. \tag{4}$$

A more "mirror symmetric" identity for arbitrary 4-dimensional reflexive polytopes one obtains by summing the equations (4) for Δ and Δ^*:

$$12\left(|\partial\Delta \cap M| + |\partial\Delta^* \cap N|\right) =$$
$$= 2\left(v(\Delta) + v(\Delta^*)\right) + \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=2}} v(\theta) \cdot v(\theta^*) + \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*) \,.$$

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Informal Discussion "Real Gromov-Witten Theory in All Genera"

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(joint work with Aleksey Zinger)

The study of curves in projective varieties has been central to algebraic geometry since the nineteenth century. It was reinvigorated through its introduction into symplectic topology in Gromov's seminal work [4] and now plays prominent roles in symplectic topology and string theory as well. The foundations of (complex) Gromov-Witten invariants, i.e. counts of J-holomorphic curves in symplectic manifolds, were established in the 1990s and have been spectacularly applied ever since.

However, there has been much less progress in establishing the foundations of and applying real Gromov-Witten invariants, i.e. counts of J-holomorphic curves in symplectic manifolds preserved by anti-symplectic involutions.

A real symplectic manifold is a triple (X,ω,ϕ) consisting of a symplectic manifold (X,ω) and an anti-symplectic involution ϕ. For such a triple, we denote by $\mathcal{J}_\omega^\phi$ the space of ω-compatible almost complex structures J on X such that $\phi^*J=-J$. The fixed locus X^ϕ of ϕ is then a Lagrangian submanifold of (X,ω) which is totally real with respect to any $J\in\mathcal{J}_\omega^\phi$. The basic example of a real Kahler manifold (X,ω,ϕ,J) is the complex projective space $\mathbb{P}^{n-1}$ with the Fubini-Study symplectic form, the coordinate conjugation

$$\tau_n\colon \mathbb{P}^{n-1}\longrightarrow\mathbb{P}^{n-1}, \qquad \tau_n\big([z_1,\ldots,z_n]\big)=\big[\overline{z}_1,\ldots,\overline{z}_n\big],$$

and the standard complex structure. Another example is a real quintic threefold X_5, i.e. a smooth hypersurface in $\mathbb{P}^4$ cut out by a real equation; it plays a prominent role in the interactions with string theory and algebraic geometry. A symmetric Riemann surface $(\Sigma,\sigma,\mathfrak{j})$ is a connected nodal Riemann surface $(\Sigma,\mathfrak{j})$ with an anti-holomorphic involution σ.

Let (X,ω,ϕ) be a real symplectic manifold, $g,l\in\mathbb{Z}^{\ge0}$, $B\in H_2(X;\mathbb{Z})$, and $J\in\mathcal{J}_\omega^\phi$. For a symmetric surface (Σ,σ), we denote by

$$\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}\subset\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$$

the uncompactified moduli space of degree B real J-holomorphic maps from (Σ,σ) to (X,ϕ) with l conjugate pairs of marked points and its stable map compactification. Each codimension-one stratum of $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$ is either a hypersurface in $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$ or a boundary of the spaces $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$ for precisely two topological types of orientation-reversing involutions σ on Σ. Thus, the union of real moduli spaces

$$\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}=\bigcup_\sigma\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$$

over all topological types of orientation-reversing involutions σ on Σ forms a space without boundary. There is a natural forgetful morphism

$$\mathfrak{f}\colon\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}\longrightarrow\overline{\mathbb{R}\mathcal{M}}_{g,l}$$

to the Deligne-Mumford moduli space of marked real curves. An orientation on $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$ determined by some topological data on (X,ω,ϕ) gives rise to invariants of (X,ω,ϕ) that enumerate real J-holomorphic curves in X, just as happens in the complex Gromov-Witten theory.

The two main obstacles to defining real Gromov-Witten invariants (or any other count of real curves) is the potential non-orientability of the moduli space $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ and the fact that its boundary strata have real codimension one. In contrast, the complex analogues of these spaces are canonically oriented and have boundary of real codimension two. These obstacles were overcome in many genus 0 situations in [5, 6], providing lower bounds for counts of real rational curves in the corresponding settings. In [2], we overcome these obstacles in *all* genera for many real symplectic manifolds.

A real bundle pair (V,φ) over a topological space X with an involution ϕ consists of a complex vector bundle V over X and a conjugation φ on V lifting ϕ. If X is a smooth manifold, then $(TX,\mathrm{d}\phi)$ is a real bundle pair over (X,ϕ). The inspiration for our approach comes in part from the topological classification of real bundle pairs over smooth symmetric surfaces in [1].

Definition ([2, Part I]). A real orientation on a real symplectic manifold (X,ω,ϕ) consists of

(1) a rank 1 real bundle pair $(L,\tilde{\phi})$ over (X,ϕ) such that

$$w_2(TX^{\phi})=w_1(L^{\tilde{\phi}})^2 \qquad \text{and} \qquad \Lambda^{\mathrm{top}}_{\mathbb{C}}(TX,\mathrm{d}\phi)\approx(L,\tilde{\phi})^{\otimes 2},$$

(2) a homotopy class of above isomorphisms of real bundle pairs, and
(3) a spin structure on the real vector bundle $TX^{\phi}\oplus 2(L^*)^{\tilde{\phi}^*}$ over X^{ϕ} compatible with the orientation induced by the above homotopy class.

We call a real symplectic manifold (X,ω,ϕ) real-orientable if it admits a real orientation. The examples include $\mathbb{P}^{2n-1}$, X_5, many other projective complete intersections, and simply-connected real symplectic Calabi-Yau and real Kahler Calabi-Yau manifolds with spin fixed locus; see [2, Part III].

Theorem ([2, Part I]). *Let (X,ω,ϕ) be a real-orientable $2n$-manifold, $g,l\in\mathbb{Z}^{\geq 0}$, $B\in H_2(X;\mathbb{Z})$, and $J\in\mathcal{J}^{\phi}_{\omega}$.*

(1) *If n is odd, a real orientation on (X,ω,ϕ) orients $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$.*
(2) *If n is even, a real orientation on (X,ω,ϕ) orients the real line bundle*

$$\Lambda^{\mathrm{top}}_{\mathbb{R}}(\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi})\otimes\mathfrak{f}^*\Lambda^{\mathrm{top}}_{\mathbb{R}}(\overline{\mathbb{R}\mathcal{M}}_{g,l})\longrightarrow\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}.$$

This theorem is fundamentally about orienting tensor products of determinants of Cauchy-Riemann (or CR) operators on real bundle pairs over symmetric surfaces. The most basic such operator is the standard $\bar{\partial}$-operator on the trivial rank 1 real bundle pair over (Σ,σ), denoted by $\bar{\partial}_{\mathbb{C}}|_{(\Sigma,\sigma)}$. The linearization D_u of the $\bar{\partial}_{J,\mathfrak{j}}$-operator at a real $(J,\mathfrak{j})$-holomorphic map u from (Σ,σ) to (X,ϕ) is a CR-operator on the real bundle pair $u^*(TX,\mathrm{d}\phi)$ over (Σ,σ). A key step in our proof is a classification of automorphisms of real bundle pairs over smooth and one-nodal symmetric surfaces; we extend it to arbitrary nodal symmetric surfaces in [3]. It implies that a real orientation on a real bundle pair (V,φ) over (Σ,σ) *determines* a homotopy class of trivializations of $(V\oplus 2L^*,\varphi\oplus 2\tilde{\phi}^*)$. Thus, a real orientation on (X,ω,ϕ) *orients* the tensor product of $(\det D_u)$ and $(\det\bar{\partial}_{\mathbb{C}}|_{(\Sigma,\sigma)})^{\otimes n}$ for every element $[u]$ of $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$ in a continuous fashion. Furthermore, the Kodaira-Spencer map *orients* the tensor product of $T_{(\Sigma,\sigma)}\overline{\mathbb{R}\mathcal{M}}_{g,l}$ and $\det\bar{\partial}_{\mathbb{C}}|_{(\Sigma,\sigma)}$ whenever Σ is smooth. The last orientation extends across $\overline{\mathbb{R}\mathcal{M}}_{g,l}$ after being reversed over smooth symmetric surfaces (Σ,σ) with the parity of the number of the components of the fixed locus Σ^{σ} equal to the parity of the genus g. The tensor product of the resulting orientations on the above two tensor products orients $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$ if n is odd and the line bundle in (2) if n is even.

Our notion of real orientation on (X,ω,ϕ) can be viewed as the real arbitrary-genus analogue of the now standard notion of relative spin structure in the open genus 0 GW-theory. The latter induces orientations on the moduli spaces of J-holomorphic disks and can in some cases be used to orient the moduli spaces of real J-holomorphic maps from $\mathbb{P}^1$ with the standard involution τ_2. In [2, Part II], we show that in these special cases our orientations on these moduli spaces reduce to the orientations induced by the associated relative spin structure up to a topological sign.

As in the complex case, the curve-counting invariants arising from the above theorem are generally rational numbers. For specific real almost Kahler manifolds (X,ω,ϕ,J), they can be converted into signed counts of genus g degree B real J-holomorphic curves passing through specified conjugate pairs of constraints and thus provide lower bounds in real enumerative geometry. If $n=3$ and (X,ω,ϕ,J) is sufficiently positive, e.g. $X=\mathbb{P}^3$, then the $g=1$ real GW-invariants themselves are such signed counts. This is also the case of the real GW-invariants with real points constraints that arise from the above theorem if $g=1$ and $n=3$.

The equivariant localization data needed to compute the real GW-invariants of $\mathbb{P}^{2n-1}$ is described in [2, Part III]. We use it to show that the real genus g degree d GW-invariants with conjugate pairs of constraints vanish whenever $d-g$ is even. We also find that the absolute value of the signed count of real genus 1 degree d curves through d pairs of conjugate points in $\mathbb{P}^3$ is 0 for $d=2$, 1 for $d=4$, and 4 for $d=6$; the details of the last computation appear in [2, Appendix]. These values are consistent with the corresponding complex counts: 0, 1, and 2860.

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A tropical approach to non-Archimedean Arakelov geometry

Walter Gubler
(joint work with Klaus Künnemann)

This is a report on a joint paper with Klaus Künnemann in [5].

1. Arakelov theory

Let X be a regular projective variety over number field K. Algebraic intersection theory gives geometric information about X, e.g. the degree of X. Arakelov theory is an arithmetic intersection theory which transmits arithmetic information about X, e.g. the *height* of X. In dimension 1, it was introduced by Arakelov in 1971. It was used by Faltings in 1983 to prove the Mordell conjecture.

In higher dimension, arithmetic intersection theory was developed by Gillet and Soulé in [3]. The basic idea is to use algebraic intersection theory on a projective regular model over the algebraic integers O_K combined with a $*$-product of Green currents on the complex manifold $X(\mathbb{C})$. This approach has two disadvantages. First, the existence of regular projective models over O_K is often not known. Moreover, canonical heights have sometimes no description as an arithmetic intersection number. This leads to the dream that we can use also differential geometry at non-archimedean places instead of models. The goal of the paper is to pave the way to such an approach.

2. Lagerberg's superforms

Lagerberg introduced in his thesis a *superform* on $\mathbb{R}^r$ of type (p,q) as an element of $A^{p,q}(\mathbb{R}^r) := C^\infty(\mathbb{R}^r) \otimes_{\mathbb{Z}} \bigwedge^p(\mathbb{Z}^r)^* \otimes_{\mathbb{Z}} \bigwedge^q(\mathbb{Z}^r)^*$. This leads to bigraded differential alternating $\mathbb{R}$ algebra with respect to the differential operators d', d''. In coordinates and using multiindex notation, a superform is given by

$$\alpha = \sum_{|I|=p,|J|=q} f_{IJ} d'x_I \wedge d''x_J.$$

The differential operators are defined by

$$d'\alpha = \sum_{i=1}^{r} \sum_{I,J} \frac{\partial f_{IJ}}{\partial x_i} d'x_i \wedge d'x_I \wedge d''x_J$$

and by

$$d''\alpha = \sum_{j=1}^{r} \sum_{I,J} \frac{\partial f_{IJ}}{\partial x_j} d''x_j \wedge d'x_I \wedge d''x_J.$$

In the following, all polyhedra have edges with rational slopes. Then integration of $\alpha \in A_c^{n,n}(\mathbb{R}^r)$ over an n-dimensional polyhedron Δ is well-defined and leads to a *supercurrent* δ_Δ on $\mathbb{R}^r$, i.e. a continuous linear functional on $A_c^{\cdot,\cdot}(\mathbb{R}^r)$. By linearity, we extend this definition to weighted polyhedral complexes.

Proposition 1 ([4], 3.8). *The weighted polyhedral complex (Σ, m) is a tropical cycle if and only if $d'\delta_{(\Sigma,m)} = 0$.*

3. delta-forms and delta-currents

A current in $D^{p,q}(\mathbb{R}^r)$ is a δ-*preform* of type (p,q) if it is of the form

$$\sum_{i=1}^{N} \alpha_i \wedge \delta_{C_i} \tag{1}$$

with $\alpha_i \in A^{p_i,q_i}(\mathbb{R}^r)$ and where C_i is a tropical cycle of codimension k_i with $(p,q) = (p_i + k_i, q_i + k_i)$. This leads to a bigraded differential algebra $P^{\bullet,\bullet}(\mathbb{R}^r)$ with respect to d', d'', where $\delta_{C_i} \wedge \delta_{C'_j} := \delta_{C_i \cdot C'_j}$ uses the tropical intersection product.

In the following, K is always an algebraically closed field which is complete with respect to a given non-trivial non-archimedean absolute value. Let X be an algebraic variety over K with associated Berkovich analytification X^{an}.

A *tropical chart* (V, φ_U) consists of a very affine open subset U of X with associated tropicalization map $\mathrm{trop}_U : U^{\mathrm{an}} \to N_U := \mathbb{R}^r$ such that $V := \mathrm{trop}_U^{-1}(\Omega)$ for an open subset Ω of $\mathrm{Trop}(U)$. Tropical charts form a basis for X^{an} [4, Prop. 4.16].

A δ-*form* α on X^{an} is given by tropical charts $(V_i, \varphi_{U_i})_{i\in I}$ covering X^{an} and $\alpha_i \in P^{\bullet,\bullet}(N_{U_i})$ which agree on overlappings, i.e. on $\mathrm{Trop}(U_i \cap U'_j)$. This leads to a bigraded differential alternating algebra $B^{\bullet,\bullet}(X^{\mathrm{an}})$ with respect to d', d''. The topological dual is by definition the space of δ-*currents*.

Remark. Using only Lagerberg's superforms instead of δ-preforms (i.e. skipping tropical cycles in 1), the same procedure gives the smooth (p,q)-forms of Chambert-Loir and Ducros in [2] (see also [4]) leading to a subalgebra $A^{\bullet,\bullet}(X^{\mathrm{an}})$ of $B^{\bullet,\bullet}(X^{\mathrm{an}})$. We may view δ-forms as analogues of complex differential forms with logarithmic singularities.

4. First Chern forms and first Chern currents

Let L be a line bundle in X endowed with a continuous metric on L^{an}. The *first Chern* δ-*current* is the δ-current on X^{an} given locally by

$$[c_1(L, \|\ \ \|)] := d'd''\,[-\log\|t\|]$$

for a local frame t of L. Similarly as in complex analysis, we have the *Poincaré–Lelong formula*:

Proposition 2. *Let s be a non-trivial meromorphic section of L, then*

$$d'd''\,[-\log\|s\|] = [c_1(L, \|\ \ \|)] - \delta_{\mathrm{div}(s)}$$

as an identity of δ-currents on X^{an}.

A metric $\|\ \|$ on L is called *smooth* if $-\log\|s\| \in A^{0,0}(U^{\mathrm{an}})$ for all frames $t : U \to L$ and any open subset U of X.

A δ-*metric* on L is a continuous metric $\|\ \ \|$ such that the first Chern δ-current $[c_1(L, \|\ \ \|)]$ is represented by a δ-form $c_1(L, \|\ \|)$.

A smooth metric is always a δ-metric, but the converse does not always hold.

Example. Let $\mathfrak{X}$ be a proper model of X over the valuation ring K° and assume that the line bundle $\mathfrak{L}$ on $\mathfrak{X}$ is a model of L, i.e. $\mathfrak{L}|_X = L$. Then we get an associated metric $\| \ \|_{\mathfrak{L}}$ on L^{an} which we call a *model metric* and which encodes the arithmetic information of $\mathfrak{L}$. A model metric is a δ-metric, but not always smooth (see [GK], §8 for more details).

Assume for simplicity, that the special fibre $\mathfrak{X}_s = \mathfrak{X} \otimes_{K^\circ} \tilde{K}$ is reduced over the residue field $\tilde{K}$. Then Chambert–Loir [1] introduced the discrete measure $\mu_{\mathfrak{L}} := \sum_Y \deg_{\mathfrak{L}}(Y)\delta_{\xi_Y}$ on X^{an}, where Y ranges over all irreducible components of $\mathfrak{X}_s$, ξ_Y is the unique point of X^{an} whose reduction to the special fibre is the generic point of Y and where δ_{ξ_Y} is the Dirac measure. These measures are important for equidistribution theorems in arithmetics, see [6].

Theorem ([5], 0.3). *If X is proper variety K of dimension n, then $c_1(L, \| \ \|_{\mathcal{L}})^{\wedge n}$ is a Radon measure on X^{an} equal to $\mu_{\mathfrak{L}}$.*

Remark. In other words, the Chambert–Loir measure $\mu_{\mathfrak{L}}$ can be defined as a Monge–Ampère measure. In [2], there is a similar theorem. Note however, that they need an approximation process by smooth metrics to make sense of the wedge products of the first Chern currents.

5. Non-archimedean Arakelov theory

Let X be a proper n-dimensional variety over the algebraically closed non-archimedean field K. A δ-current g_Z is called a *Green current* for the cycle Z on X if we have the following identity of δ-currents on X^{an}:

$$d'd''g_Z = [\omega_Z] - \delta_Z.$$

Example. Let s be a non-trivial meromorphic section of the line bundle L on X and let $\| \ \|$ be a δ-metric on L. Then the Poincaré–Lelong equation shows that $g_D := -[\log \|s\|]$ is a Green current for $D := \mathrm{div}(s)$.

Proposition ([5], 11.4). *If D intersects Z property, then*

$$g_D * g_Z := g_D \wedge \delta_Z - c_1(L, \| \ \|) \wedge g_Z$$

is a Green current for $D.Z$.

Definition. Let $D_0, \ldots, D_n$ be Cartier divisors intersecting properly on X and let us choose δ-metrics on $\mathcal{O}(D_0), \ldots, \mathcal{O}(D_n)$, then

$$\lambda(X) := \langle g_{D_0} * \cdots * g_{D_n}, 1 \rangle$$

is called the *local height* of X.

Theorem ([5], 0.4). *If we use model-metrics on $\mathcal{O}(D_0), \ldots, \mathcal{O}(D_n)$, then $\lambda(X)$ is the usual local height of X in arithmetic geometry given as the intersection number of the Cartier divisors on a common model.*

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Metrization of differential pluriforms on Berkovich analytic spaces

Michael Temkin

1. Introduction

Kontsevich and Soibelman introduced in [1] a metric on the canonical sheaf of a $\mathbf{C}((t))$-analytic Calabi-Yau manifold, and their definition was extended by Mustaţă and Nicaise in [2] to pluricanonical sheaves $\omega_X^{\otimes m}$ on a smooth algebraizable Berkovich space X over a discretely valued field k. Their definition associates to a pluricanonical form $\phi \in \Gamma(\omega_X^{\otimes m})$ a real-valued weight function wt_ϕ defined on a subset of X. First, one defines wt_ϕ at divisorial points of X and then extends it by continuity to PL subspaces of X.

This lecture discusses a recent progress made in [3]. The main goal of that work was to metrize the sheaves $\omega_X^{\otimes m}$ in a purely analytic way, thereby eliminating unnecessary technical assumptions. In particular, this method applies to all rig-smooth spaces over an arbitrary non-archimedean field, including the trivially valued ones, and deals with all points on an equal footing, thereby providing a norm function $\|\phi\|\ X \to \mathbf{R}_+$. Moreover, for any morphism of k-analytic spaces $X \to S$ we naturally metrize $\Omega_{X/S}$ and all related sheaves, such as $S^n\Omega_{X/S}$, $\bigwedge^m \Omega_{X/S}$, etc. In particular, this applies to families of spaces parameterized by S.

2. Kähler seminorm on $\Omega_{X/S}$

Let $f\ X \to S$ be a morphism of Berkovich spaces. The following theorem constructs a seminorm on $\Omega_{X/S}$, that we call *Kähler seminorm*, and describes its basic properties.

Theorem. *(i) There exists a maximal seminorm* $\|\ \|_{\Omega,X/S}$ *on* $\Omega_{X/S}$ *such that the map* $d\ \mathcal{O}_X \to \Omega_{X/S}$ *is non-expansive.*

(ii) If we provide each stalk $\Omega_{X/S,x}$ *with the induced stalk seminorm* $\|\ \|_{\Omega,x}$ *then the completion is naturally isomorphic as a seminormed module to* $\hat{\Omega}_{\mathcal{H}(x)/\mathcal{H}(s)}$.

(iii) For any section $t \in \Omega_{X/S}(U)$ *the function* $\|t\|\ X \to \mathbf{R}$ *is upper semicontinuous and* $\|t\|_U = \max_{x\in X} \|t\|_x$.

Note that (i) is an abstract existence result by a universal property. This is claim (ii) that provides a real possibility to work with $\| \ \|_{\Omega,X/S}$ and make computations. The crucial point in proving (ii) is that if the spaces are good then $\hat{\Omega}_{\kappa(x)/\kappa(s)} = \hat{\Omega}_{\mathcal{H}(x)/\mathcal{H}(s)}$.

3. Main results

3.1. **Pullbacks.** We say that a seminormed k-algebra $\mathcal{A}$ is *universally spectral* if for any extension of real-valued fields l/k the product seminorm on $\mathcal{A} \otimes_k l$ is power-multiplicative. For example, any tame finite extension K/k is universally spectral.

Theorem. *Let $f\ X \to S$ and $g\ S' \to S$ be morphisms of Berkovich spaces and $S' = S \times_X X'$. Then $\| \ \|_{\Omega,X'/S'}$ is dominated by the pullback of $\| \ \|_{\Omega,X/S}$ and the two seminorms are equal if for any $s' \in S'$ with $s = g(s')$ the extension $\mathcal{H}(s')/\mathcal{H}(s)$ is universally spectral. In particular, this happens when g is a monomorphism (e.g. embedding of a point) or $S' = S \otimes_k l$ for a tame finite extension l/k.*

Remark. The Kähler seminorm can drop under wildly ramified ground field extensions. Therefore, it makes sense to also introduce the *geometric Kähler seminorm* $\| \ \|_{\overline{\Omega},X/S}$ obtained by computing the Kähler seminorm after the ground field extension $\widehat{K}/k$. By the above theorem the two norms coincides when k has no wildly ramified extensions.

3.2. **Monomiality of the geometric Kähler seminorm.** Assume that $k = k^a$ and X is good. Then for any monomial point $x \in X$ there exists a family of tame parameters at x, i.e. elements $t_1, ..., t_n \in \mathcal{O}_{X,x}$ such that $n = \dim_x(X)$ and $\mathcal{H}(x)$ is tame over its subfield $\widehat{k(t_1, ..., t_n)}$.

Theorem. *Keep the above notation, then $\frac{dt_1}{t_1}, ..., \frac{dt_n}{t_n}$ form an orthonormal basis of $\hat{\Omega}_{\mathcal{H}(x)/k}$.*

The theorem allows to compute the Kähler seminorm on the sheaf $\mathcal{F} = S^n(\Omega^l_{X/k})$ in terms of a tame parameter family: one represents $\phi_x \in \mathcal{F}_x$ as $\sum_e \phi_e e$ with e's of the form $\left(\frac{dt_{i_1}}{t_{i_1}} \wedge \cdots \wedge \frac{dt_{i_l}}{t_{i_l}}\right) \otimes \cdots \otimes \left(\frac{dt_{j_1}}{t_{j_1}} \wedge \cdots \wedge \frac{dt_{j_l}}{t_{j_l}}\right)$ and computes $\|\phi\|_x = \max_e |\phi_e|_x$. Using (non-trivial) results on existence of tame parameters on skeletons one also obtains the following important result.

Corollary. *If $\phi \in S^n(\Omega^l_{X/k})$ is a differential pluriform on X then its geometric Kähler seminorm $\|\phi\|$ restricts to a PL function on any PL subspace of X.*

Remark. Probably, this result also holds for Kähler seminorms, but this requires a new argument.

3.3. **The maximality locus. Theorem.** *If X possesses a semistable model $\mathcal{X}$ then for any pluricanonical form $\phi \in \Gamma(\omega_{\mathcal{X}}^{\otimes m})$ the maximality locus of the geometric Kähler seminorm of ϕ is a union of faces of the skeleton associated with $\mathcal{X}$.*

Corollary. *Assume that $\chi(\tilde{k}) = 0$, X is strictly analytic and rig-smooth and $\phi \in \Gamma(\omega_{\mathcal{X}}^{\otimes m})$. Then the maximality locus of the Kähler seminorm of ϕ is a PL subspace of X.*

Proof. We can replace k with $\widehat{K}$ since this does not affect the Kähler seminorm thanks to the $\chi(\tilde{k}) = 0$ hypothesis. By local uniformization of Berkovich spaces of equal characteristic zero, there exists an admissible covering of X by affinoid domains that possess semistable reduction. It remains to use the above theorem.

3.4. **Comparison with the weight function.** It remains to answer the natural question whether our definition of Kähler seminorms on pluricanonical sheaves coincides (up to a constant factor) with the definitions of [1] and [2]. Surprisingly, this is so only in the case of residue characteristic zero. In general, the discrepancy is described by the log different. For any (not necessarily algebraic) extension l/k of real-valued fields we define the log different $\delta_{l/k}^{\log}$ to be the torsion content of the module $\Omega_{l^\circ/k^\circ}^{\log}$ (in the discrete valued case this is the absolute value of the zeroth Fitting ideal of the torsion part of $\Omega_{l^\circ/k^\circ}^{\log}$). The log different is an important invariant of the extension l/k that measures its "wildness".

Theorem. *Assume that k is discretely valued with a uniformizer π_k, X is a smooth algebraizable Berkovich space over k, $\phi \in \omega_X^{\otimes m}$, and $x \in X$ is a monomial point; in particular, the weight function wt_ϕ is defined at x. Then $\mathrm{wt}_\phi(x) = |\pi_k|^m (\delta_{\mathcal{H}(x)/k}^{\log})^m \|\phi\|_x$, where the right-hand side involves the Kähler seminorm on $\omega_X^{\otimes m}$.*

In particular, all reasonable seminorms coincide when k has no wildly ramified extensions. (For a discrete-valued k this happens if and only if $\chi(\tilde{k}) = 0$.) Note also that the weight function drops under wildly ramified extensions, and one can define geometric weight function analogously to geometric Kähler seminorm. Then it follows from the theorem that the geometric Kähler norm coincides with the geometric weight function in the case when the latter is defined.

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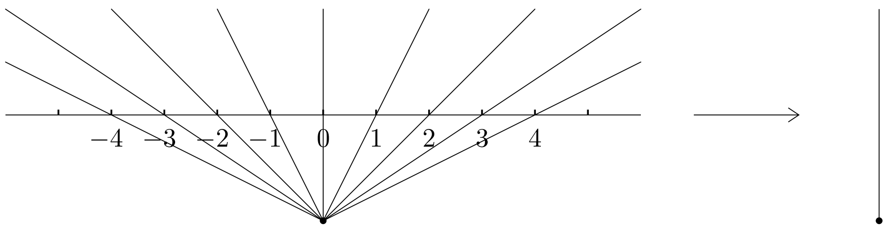
Floer theory for the Tate curve via tropical geometry

Hülya Argüz
(joint work with Bernd Siebert)

We would like to give an algebraic geometric approach to Lagrangian Floer Theory. Homological mirror symmetry suggests a correspondence between the Fukaya category of a variety and the bounded derived category of the mirror, hence a correspondence between symplectic geometry and complex algebraic geometry. Because the symplectic side involves Lagrangians and holomorphic disks it does not appear to be amenable to the refined and computationally effective methods of algebraic geometry. Contrary to such expectations we suggest that a version of log Gromov-Witten theory applied on a certain algebraic-geometric degeneration of the symplectic manifold sometimes does this job. Our object of study is maybe the easiest non-trivial example, the elliptic curve, for which both sides of mirror symmetry have been understood deeply and with explicit methods. The method in principle can however be applied to any variety with a toric degeneration in the sense of the Gross-Siebert program [2].

A toric degeneration of the elliptic curve is known as the Tate curve, constructed torically as follows (see e.g. [1], p.617, for details). Let $\mathcal{P}$ be a $\mathbb{Z}$-periodic polyhedral decomposition of $\mathbb{R}$. Taking the cones over the cells of $\mathcal{P}$, as indicated in the following figure, leads to a fan $\Sigma_{\mathcal{P}}$ in $\mathbb{R}^2$ with support $(\mathbb{R} \times \mathbb{R}_{>0}) \cup \{(0,0)\}$.



By projection onto the second factor we get a non-constant map of fans from $\Sigma_{\mathcal{P}}$ to the fan $\{0, \mathbb{R}_{\geq 0}\}$ of $\mathbb{A}^1_t$. Thus there is a non-constant, hence flat, toric morphism

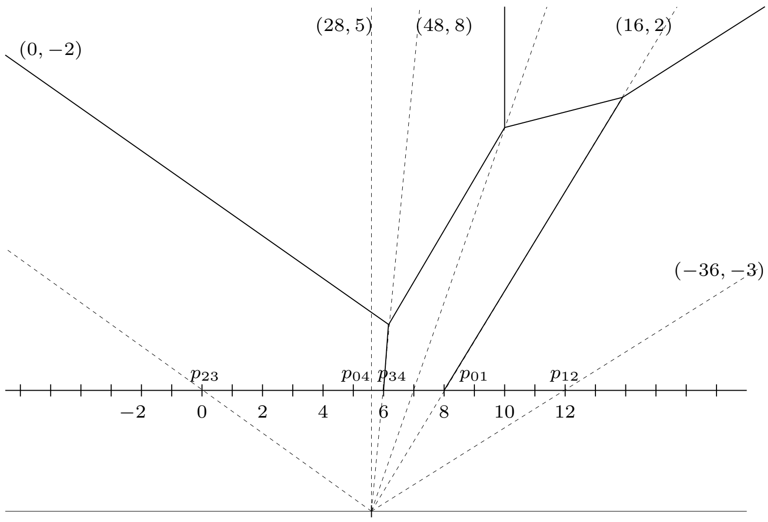
$$\pi : X(\Sigma_{\mathcal{P}}) \longrightarrow \mathbb{A}^1_t.$$

We view π as a degeneration with general fibers $\pi^{-1}(t) = \mathbb{G}_m$, where $\mathbb{G}_m := \operatorname{Spec} \mathbb{C}[x, x^{-1}]$ is the algebraic torus, and central fiber $\pi^{-1}(0)$ an infinite chain of $\mathbb{P}^1$'s.

For technical reasons we take the formal completion around the central fiber of π to obtain $\bar{\pi} : \bar{X}(\Sigma_{\mathcal{P}}) \longrightarrow \operatorname{Spec} \mathbb{C}[[t]]$. The $\mathbb{Z}$ action on $\Sigma_{\mathcal{P}}$ given by translation induces an action on $X(\Sigma_{\mathcal{P}})$ as well as on $\bar{X}(\Sigma_{\mathcal{P}})$. There exists a quotient $\bar{\Pi} : \mathfrak{T} \to \operatorname{Spec} \mathbb{C}[[t]]$ for $\bar{X}(\Sigma_{\mathcal{P}})$ in the category of schemes, while for $X(\Sigma_{\mathcal{P}})$ a quotient $\Pi : T \to D$ exists only in the category of complex analytic spaces after restricting to the unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$. The generic fibre of the algebraic degenerations $\bar{\Pi}$ is an elliptic curve over $\mathbb{C}((t))$, while the fibre over $t \in D \setminus \{0\}$ is the elliptic curve $\mathbb{C}^*/(z \sim t \cdot z)$ viewed as a complex manifold. Both T and $\mathfrak{T}$ are referred as *Tate curve.*

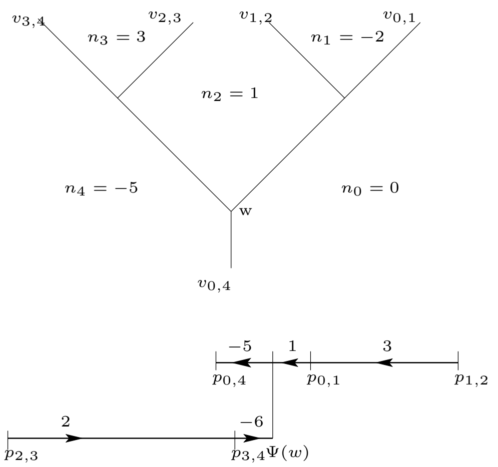
The idea to make holomorphic discs on the general fibre of Π to be approachable by algebraic-geometric techniques is to degenerate the total space of the degeneration itself, by applying a base-change $t \mapsto st$. Note that the central fiber $\tilde{\mathfrak{T}}_0$ of this degeneration defined by $s = 0$ is simply a product of the central fiber $\mathfrak{T}_0$ of the Tate curve with $\mathbb{A}^1$. Torically, $\tilde{\mathfrak{T}}$ is obtained in a similar manner as the Tate curve, this time applied to the polyhedral decomposition of the *truncated cone* $\overline{\mathbf{C}}S^1 := S^1 \times [1, \infty)$ over the circle S^1, or its lifting to $\mathbb{R} \times [1, \infty)$ by the $\mathbb{Z}$-action. This construction yields a $\mathbb{Z}$-periodic fan in $\mathbb{R}^3$ defining $\tilde{\mathfrak{T}}$. We then introduce a version of tropical curves in $\overline{\mathbf{C}}S^1$ that we call *tropical corals*. We show that these are in correspondence with certain stable log morphisms to the central fiber $\tilde{\mathfrak{T}}_0 := \mathfrak{T}_0 \times \mathbb{A}^1$.

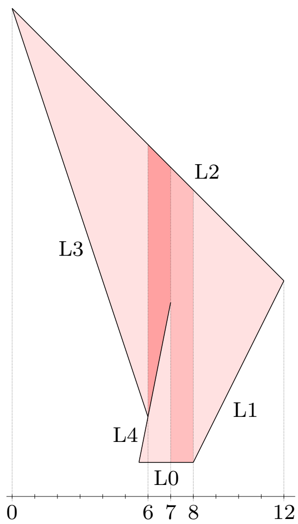
The tropical corals are similar to tropical curves except that the unbounded edges meeting the truncated cone at its lower boundary get cut off. These edges furthermore should lie on affine line segments passing through the origin. We skip the technical definition here and give an example of a tropical coral in the following picture. The undashed lines form the tropical coral, while the dashed lines are drawn to illustrate the asymptotic directions of the unbounded lines for comparison with the associated tropical Morse tree.



Tropical corals are up to some scaling in bijection with *tropical Morse trees* introduced by Abouzaid, Gross and Siebert. These are combinatorial objects corresponding to holomorphic discs between Lagrangians in the framework of the Fukaya category. The correspondence for the elliptic curve has been worked out in [1]. Roughly speaking a tropical Morse tree in S^1 is the image of a map Ψ from a Ribbon graph equipped with some weights. To each region separated by the edges of the Ribbon graph we associate integers n_i and define a weight on the edges by taking differences of the regions the edge lies on. The map Ψ contracts leaves of the Ribbon graph if they have negative weights and it contracts the root if it has positive weight. Furthermore, each external vertex $v_{i,j}$ is mapped to a point $p_{i,j}$ with coordinates in $(\frac{1}{n_j-n_i})\mathbb{Z}$.

Below is an example of a tropical Morse tree. For convenience we draw the images of noncontracted edges on the universal cover of S^1. On the right hand-side we also picture the polygon associated to the tropical Morse tree that enters in the Fukaya category. For details see [1]. The corresponding tropical coral is the one we drew in the picture above. We obtain it by adding rays emanating from the origin passing through the points $p_{i,j}$'s and patching them together with additional segments paying attention to the cyclic order and the balancing condition at each vertex.





Followinging the method of [3] we show that there is a correspondence between a counting problem for tropical corals on $\overline{\mathbf{C}}S^1$ and a counting problem for stable log maps to $\mathfrak{T}_0$. A deformation theory argument is supposed to relate this count of stable log maps to certain relative Gromov-Witten invariant of the total space of the analytic Tate curve. To make contact with symplectic geometry this last step has to be performed with the analytic Tate curve and holomorphic discs.

The relation to the Fukaya category purely in terms of an algebraic-geometric counting problem involves some further steps on the symplectic side as indicated in the following diagram.

wrapped Fukaya category of the Tate curve T	$\longleftrightarrow$	Lagrangian Floer Theory for the elliptic curve	$\longleftrightarrow$	tropical Morse trees in S^1
$\updownarrow$				$\updownarrow$
relative Gromov-Witten invts. of the Tate curve T	$\longleftrightarrow$	stable log maps to $\tilde{\mathfrak{T}}_0$	$\longleftrightarrow$	tropical corals in $\overline{\mathbf{C}}S^1$

In future work we hope to be able to establish all correspondences in this diagram explicitly.

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Topological overlapping property and degree of irrationality of hypersurfaces with given Newton polyhedron

Nikita Kalinin

Given a polyhedron $\Delta \subset \mathbb{Z}^n$ we can take a generic hypersurface H with the Newton polyhedron Δ and ask the following question: what is the minimal degree of a dominant map $H \to \mathbb{P}^{n-1}$ in terms of Δ? What about the genus of $f^{-1}(x)$ where $f : H \dashrightarrow \mathbb{P}^{n-2}$ and $x \in \mathbb{P}^{n-2}$? We address these questions by means of tropical geometry and Gromov's topological overlapping property.

1. The degree of irrationality

For a variety X of dimension k we define its *degree of irrationality* as the minimal d such that there exists a dominant map $X \dashrightarrow \mathbb{P}^k$ of degree k. For example, see [5] for the study of the degree of irrationality for surfaces. The degree of irrationality is called *gonality* in the case of curves. If a hypersurface $H \subset \mathbb{P}^n$ is given by an equation

$$\sum_{i\in I} a_i x^i = 0,$$

where i is multi-index $i = (i_1, i_2, \dots, i_n) \in \mathbb{Z}^n, x^i = x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}, a_i \in \mathbb{C}^*$, then the intersection of the convex hull for I with $\mathbb{Z}^n$ is called *the Newton polyhedron* of X. Recently, it was proven that, for almost all polygons Δ, a generic curve $C \in (\mathbb{C}^*)^2$ with the Newton polygon Δ has gonality equal the *minimal lattice length* of Δ(for details, see [2]). In the case of higher dimensions, for smooth hypersurfaces (in $\mathbb{C}P^n$) and curves (in $\mathbb{C}P^3$) of degree d the degree of irrationality is equal to $d-1$ for most cases([1]).

We take the next case, which is not covered by the current research — a generic hypersurface $X \subset (\mathbb{C}^*)^n$ with a given Newton polyhedron Δ, and address it from the new perspective: tropical geometry. Indeed, instead of studying the maps $X \dashrightarrow (\mathbb{C}^*)^{n-1}$ we can have a look on its tropical counterpart and any estimation there implies the same estimation over $\mathbb{C}$. In the tropical setup we take a generic tropical hypersurface X' with the Newton polyhedron Δ and consider a piecewise linear map $X' \to \mathbb{R}^{n-1}$. Now we can exploit so-called *topological overlapping property*, which allows estimate the maximal number of preimages for a map from an $(n-1)$-dimensional simplicial complex (hypergraph) to $\mathbb{R}^{n-1}$ in terms of homological features of this complex.

So, the plan is as follows. We subdivide Δ by the primitive cubes from $\mathbb{Z}^n$ and then by primitive simplices. Then we take a generic tropical hypersurface X' dual to this subdivision and apply the theory of topological overlapping property.

2. Gromov's topological overlapping property

The basic example is the following: consider a graph $G = (V, E)$ where V is the set of vertices and E is the set of edges. Suppose that there exists $\lambda > 0$ such that for each subset $S \subset V$ of vertices the number of edges between S and $V \setminus S$ is at least $\lambda|S|\frac{\min(|S|,|V\setminus S|)}{|V|}$. Then, for each continuous map $f : G \to$ $mathbbR^1$ there exists a point $p \in \mathbb{R}^1$ such that $|f^{-1}(p)| > \frac{\lambda}{4}|V|(|V|-\frac{1}{|V|})$. Indeed, we can take a point p which divides the set $f(V)$ on two equal parts. In this setup, λ is called *the edge-expansion constant* of G.

Gromov's overlapping property concerns the same problem in higher dimensions, see recent papers [4, 3].

Let X^n be an n-dimensional simplicial complex, X_i the set of its i-dimensional cells, also we consider $C^i(X, \mathbb{Z}_2)$. By $|a|$ we denote the Hamming norm of a chain $a \in C^k(X, \mathbb{Z}_2)$.

Definition. Define *cofiling profiles* $\phi_i(x) = \min(\frac{\delta a}{|X_i|} : a \in X_{i-1}, |[a]| \geq x|X_{i-1}|)$ where $|[a]|$ is the minimal norm of chains homotopic to a.

That means that if $\frac{\delta a}{|X_i|} < \phi_i(\varepsilon)$ then $|[a]| < \varepsilon|X_{i-1}|$.

Theorem (Gromov). Let X be a connected finite d-dimensional simplicial complex with cofiling profiles $\phi_1, \phi_2, \ldots, \phi_d$. Assume $H^i(X) = 0$ for $i = 1, \ldots, d-1$. Then the topological overlapping constant c satisfies:

$$c_d(X) \geq \phi_d(\frac{1}{2}\phi_{d-1}(\ldots \phi_2(\frac{1}{d}\phi_1(\frac{1}{d+1})))).$$

That means that for each continuous map $f : X \to \mathbb{R}^d$ there exists a point $p \in \mathbb{R}^d$ such that $|f^{-1}(p)| \geq c_d(X) \cdot |X_d|$.

Direct applications of these methods gives the answer of right asymptotic: the degree of irrationality is at least the minimal lattice width of Δ multiplied by a constant which depends only on n. Probably, the constant (very small now) can be improved since we stick to some concrete polyhedral complexes.

Furthermore, we can estimate the genus of preimages of points for a map $X' \to \mathbb{R}^{n-1}$. Indeed, X' is homotopically equivalent to the bouquet of spheres. If a preimage intersects a sphere, it has to have a circle in this sphere. Therefore, again, we need to know how many preimages of a point are in **different** cells in X'. Other direction is to look at the *stable* degree of irrationality, this means that we should observe how the cofiling profiles change when we multiply X by an interval.

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Real algebraic surfaces in $(\mathbb{C}P^1)^3$

Arthur Renaudineau

We discuss some topological properties of nonsingular real algebraic surfaces in $(\mathbb{CP}^1)^3$. We consider $(\mathbb{CP}^1)^3$ equipped with the standard real structure c defined by

$$c([u_1 : v_1], [u_2 : v_2], [u_3 : v_3]) = ([\overline{u_1} : \overline{v_1}], [\overline{u_2} : \overline{v_2}], [\overline{u_3} : \overline{v_3}]),$$

where $([u_1 : v_1], [u_2 : v_2], [u_3 : v_3])$ denotes homogeneous coordinates on $(\mathbb{CP}^1)^3$. A nonsingular real algebraic surface in $(\mathbb{CP}^1)^3$ is a nonsingular algebraic surface $X \subset (\mathbb{CP}^1)^3$ such that $c(X) = X$. The fixed point set of $c|_X$ is called the real part of X and is denoted by $\mathbb{R}X$. We first discuss orientability of real parts of nonsingular real algebraic surfaces in $(\mathbb{CP}^1)^3$. Fixing $(d_1, d_2, d_3) \in \mathbb{Z}^3$ with d_1, d_2, d_3 not all even, we show the existence of nonsingular real algebraic surfaces X and Y of degree (d_1, d_2, d_3) in $(\mathbb{CP}^1)^3$ such that $\mathbb{R}X$ is orientable and $\mathbb{R}Y$ is not orientable.

For a topological space A, we put $b_i(A) = \dim_{\mathbb{Z}/2\mathbb{Z}} H_i(A, \mathbb{Z}/2\mathbb{Z})$. The numbers $b_i(A)$ are called *Betti numbers* (*with* $\mathbb{Z}/2\mathbb{Z}$ *coefficients*) *of* A. We focus on the following question.

Question. *What is the maximal possible value of* $b_1(\mathbb{R}X)$ *for a nonsingular real algebraic surface* X *in* $(\mathbb{CP}^1)^3$ *of a given degree?*

Given X a nonsingular algebraic surface, denotes by $h^{p,q}(X)$ the (p, q)-Hodge number of X. In 1980, O. Viro formulated the following conjecture:

Conjecture. (O. Viro) *Let* X *be a compact connected simply-connected projective real surface. Then*

$$b_1(\mathbb{R}X) \leq h^{1,1}(X).$$

The first counterexample to Viro's conjecture was constructed by Itenberg for a double covering of $(\mathbb{CP}^2)$ ramified along a curve of degree 10 (see [2]). Itenberg's construction used Viro's combinatorial patchworking. We construct a nonsingular real algebraic surface X of degree $(4, 4, 2)$ in $(\mathbb{CP}^1)^3$ satisfying $b_1(\mathbb{R}X) = h^{1,1}(X)+$ 4. Classical inequalities in topology of real algebraic varieties implies that if X is

a nonsingular real algebraic surface of degree $(4,4,2)$ in $(\mathbb{CP}^1)^3$, then $b_1(\mathbb{R}X) \leq h^{1,1}(X) + 8$. The existence of a nonsingular real algebraic surface X of degree $(4,4,2)$ in $(\mathbb{CP}^1)^3$ satisfying $b_1(\mathbb{R}X) \geq h^{1,1}(X) + 6$ is still open.
We focus also on the asymptotic behaviour of the first Betti number. Denote by $\mathcal{S}(d_1, d_2)$ the set of nonsingular real algebraic surfaces of degree $(d_1, d_2, 2)$ in $(\mathbb{CP}^1)^3$. Define the following sequence:

$$u_{d_1,d_2} = \frac{\max_{X \in \mathcal{S}(d_1,d_2)} b_1(\mathbb{R}X_{d_1,d_2})}{d_1 d_2}.$$

It is a priori not clear if u_{d_1,d_2} has a limit when d_1 and d_2 go to infinity, but it follows from classical inequalities in Topology of real algebraic varieties that

$$\max b_1(\mathbb{R}X_{d_1,d_2}) \leq 7d_1d_2 - 3d_1 - 3d_2 + 5.$$

In fact, the coefficient of the quadratic term is sharp.

Theorem. *There exist a family $(X_{k,l})$ of non-singular real algebraic surfaces of degree $(2k, 2l, 2)$ in $(\mathbb{CP}^1)^3$, and $A, B, c, d, e \in \mathbb{Z}$ such that*

$$\forall k \geq A, \forall l \geq B, b_1(\mathbb{R}X_{k,l}) \geq 7(2k)(2l) - c(2k) - d(2l) + e.$$

Our constructions use a patchworking theorem, following the work of Shustin (see [3]) and the family of curves constructed by Brugallé in [1].

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Tropical series, sandpiles and symplectic area

Mikhail Shkolnikov

(joint work with Nikita Kalinin)

We will consider a certain class of abelian sandpile models on a convex domain Δ. I will explain how planar tropical curves arise as limiting shapes if Δ is a lattice polygon. We will see that such limiting curves are characterised by the condition of minimizing their symplectic area. In the case of an arbitrary Δ, the analogus scaling limit also exists. Moreover, the limiting picture appears to be a natural infinite generalization of a planar tropical curve described by a tropical series. This is a joint project with Nikita Kalinin [3, 4].

Let Γ be a finite graph with the set of vertices $V(\Gamma)$ and $\partial\Gamma$ be a distinguished subset of $V(\Gamma)$, called *the boundary* of Γ. Vertices in $V(\Gamma) \setminus \partial\Gamma$ are called *internal*.

We denote by $n(v) \subset V(\Gamma)$ the set of vertices adjacent to v. A *state* is a non-negative integer function $\psi : V(\Gamma) \to \mathbb{N}_0$, in the following $\psi(v)$ stands for the number of sand grains in a vertex v.

A *toppling* T_v at $v \in V(\Gamma) \setminus \partial\Gamma$ is a partially defined operation on the space of states. Namely, for a state ϕ such that $\phi(v) > |n(v)| - 1$ we define a new state $\phi' = T_v(\phi)$ by

$$\phi'(v') = \begin{cases} \phi(v') - |n(v)| & \text{if } \ v = v', \\ \phi(v') + 1 & \text{if } v' \in n(v), \\ \phi(v') & \text{otherwise.} \end{cases}$$

We say that ϕ' is the result of *the toppling* at v for a state ϕ. A toppling can be considered as a realization of privatization law: if a vertex v has at least $n(v)$ grains of sand, then all the neighbors of v can take of v by one grain of sand each. Note that we prohibit to apply toppling at the vertices in the boundary of Γ.

A state ϕ is called *stable* if $\phi(v) < |n(v)|$ for all $v \in V(\Gamma) \setminus \partial\Gamma$. Thus, by definition, topplings can be applied only to *non-stable* states. It is easy to see that if each connected component of Γ contains a vertex from $\partial\Gamma$, then after a finite number of topplings the state becomes stable. Indeed, the total amount of sand in $V(\Gamma) \setminus \partial\Gamma$ is finite and topplings at vertices adjacent to the boundary decrease this amount. The process of applying topplings while it is possible is called *the relaxation*. This version of the abelian sandpile model was defined in [1]. The result of relaxation doesn't depend on the order of topplings [2], that is why this model is called *abelian*.

Let Δ be a lattice polygon, and p_1,...,p_n are distinct points in Δ°, the interior of Δ. Now, let N be a positive integer. Consider a graph Γ_N, a full subgraph of $\mathbb{Z}^2$, with a set of vertices equal to $(N \cdot \Delta) \cap \mathbb{Z}^2$. Let $\partial\Gamma$ be the set of vertices of Γ of degree less than 4. Consider a collection of vertices P_1^N,...,$P_n^N \in V(\Gamma_N)$ defined by taking the coordinate-wise integer part of the points $N \cdot p_1$,...,$N \cdot p_n$ in $N \cdot \Delta$. We define a non-stable state φ_N^0 on Γ_N to be equal 4 at the points P_1^N,...,P_n^N and equal 3 at all other internal points. Consider the result of the relaxation process for ϕ_N^0, a new stable state ϕ_N^{end}. We define an exceptional (colored) set of vertices C_N in the relaxation as a set of all such vertices where ψ_N^{end} is not equal to 3, i.e. $C_N = (\phi_N^{\text{end}})^{-1}\{0, 1, 2\}$.

Theorem 1. *The sequence of sets $\frac{1}{N}C_N$ has a limit $\tilde{C}$ in Δ in the Hausdorff sense. Let C be the closure of $\tilde{C}\backslash\partial\Delta$. Then, C is a finite part of a tropical curve. Moreover, C passes through the points $p_1, \dots, p_n$ and the endpoints of C, i.e. $C \cap \partial\Delta$ are exactly the vertices of Δ.*

We call a planar graph *a finite part of a tropical curve* if it can be represented as an intersection of a planar tropical curve and some polygon P, such that P contains all bounded edges of this curve. For the introduction to tropical geometry see [5], [6] or [7]

Definition 1. The curve $C = C(\Delta; p_1, \dots, p_n)$ is called the limiting curve for a configuration of points $p_1, \dots, p_n \in \Delta$.

For the details, proofs and pictures see [3].

Instead of looking at the result of the relaxation one can count a number of topplings at each point. For each state ϕ on Γ there exist a sequence of states $\phi^0, \dots, \phi^{\mathrm{end}}$ such that $\phi^0 = \phi$, the state ϕ^{end} is stable and ϕ^i is a result of a toppling at a vertex q_i of the state ϕ^{i-1}.

Definition 2. Let $Toppl_{\phi^0}(v)$, *the toppling function*, be the number of topplings at v during the relaxation process $\phi^0 \to \cdots \to \phi^{\mathrm{end}}$.

Theorem 2. *A sequence of functions $F_N \colon \Delta \to \mathbb{R}$ given by*

$$F_N(x,y) = \frac{1}{N} Toppl_{\phi_N}([Nx],[Ny])$$

uniformly converges to a continuous function $F(\Delta; p_1, \dots, p_n) = F \colon \Delta \to \mathbb{R}$. This function is concave, piecewise linear with integer slopes and vanishes at the boundary of the polygon. Then, C is the locus where $F|_\Delta$ is not smooth, and thus C coincides with a finite part of a tropical curve as a set.

Definition 3. We call the function $F(\Delta; p_1, \dots, p_n)$ *the limiting toppling function* for the configuration $p_1, \dots, p_n$ of points in Δ.

Theorem 3. *Let V be a set of all tropical polynomials $\tilde{F}$ that vanish at $\partial\Delta$ and the curve defined by $\tilde{F}$ passes through the points $\{p_i\}$. Then the curve $C(\Delta; p_1, \dots, p_n)$ has the minimal tropical symplectic area in the class of all curves given by polynomials in V. Additionally, $F(\Delta; p_1, \dots, p_n)$ is a unique minimum for the functional on V given by*

$$\tilde{F} \longmapsto \int_\Delta \tilde{F} dxdy.$$

Definition 4. (See [8]) The tropical symplectic area of a finite segment l with a rational slope is given by $Area(l) = Length(l) \cdot Length(v)$, where $Length(-)$ denotes a Euclidean length and v is a primitive integer vector parallel to l. If C' is a finite part of a tropical curve, then its symplectic area is the weighted sum of areas for its edges, i.e.

$$Area(C') = \sum_{e \in E(C')} Area(e) \cdot Weight(e).$$

Finally, we managed to generalize the above theorems to the case of an arbitrary boundary (see [4]). Namely, if we replace Δ by essentially any convex domain Ω then most of the results remain true. The essential difference is that some infinite tropical objects will arise in the limit and thus the symplectic area diverges.

Definition 5. A function $F : \Omega \to \mathbb{R}$ is called *a tropical series* if for each $x \in \Omega^\circ$ there exists a neighbourhood $U \subset \Omega$ of x such that $F|_U$ is a tropical polynomial.

We define a tropical analytic curve as a corner locus of a tropical series.

Keeping the old notations the results can be summarized in the following

Theorem 4. *Let $\Omega \subset \mathbb{R}^2$ be a convex closed set with a non-empty interior such that it doesn't contain a line with irrational slope. Then the sequence of sets $\frac{1}{N}C_N$ has a limit $\tilde{C}$ in Ω and $\tilde{C}$ passes through the points $p_1, \dots, p_n$. A sequence of rescaled toppling functions $F_N : \Omega \to \mathbb{R}$ given by*

$$F_N(x, y) = \frac{1}{N} Toppl_{\phi_N}([Nx], [Ny])$$

uniformly converges to a tropical series $F = F : \Omega \to \mathbb{R}_{\geq 0}$. Let C be the closure of $\tilde{C} \backslash \partial\Omega$. Then, C is a tropical analytic curve defined by F. Moreover, F is the minimal tropical series with such properties.

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Informal discussion "Sheaves of categories"

Ludmil Katzarkov

Mirror symmetry originated in physics as a duality between $N = 2$ superconformal quantum field theories. In 1990, Maxim Kontsevich, interpreted this duality in a consistent, powerful mathematical framework called Homological Mirror Symmetry (HMS). The ideas put forth by Kontsevich have led to dramatic developments in how the mathematical community approaches ideas from theoretical physics, and indeed our conception of space itself. These developments created a frenzy of activity in the mathematical community which has led to a remarkable synergy of diverse mathematical disciplines, notably symplectic geometry, algebraic geometry, and category theory. HMS is now the foundation of a wide range of contemporary mathematical research dedicated to the ideas.

In this talk we take a totally new direction. We develop a wide range of new and subtle mathematical structures and study their applications to classical problems in geometry.

Our talk has three parts:

- The theory of categorical linear systems.
- The theory of Kaehler metrics on categories
- Applying these new cutting edge structures to classical problems in geometry.

Non-commutative toric varieties

Ernesto Lupercio
(joint work with Ludmil Katzarkov, Laurent Meerssemann, and Alberto Verjovsky)

In this talk I introduced our non-commutative toric varieties (joint work with Katzarkov, Meersseman and Verjovsky). The image of the moment map (tropicalizations) for such non-commutative Kähler manifolds are irrational polytopes.

I introduced the definition by means of examples stressing the analogy with commutative ordinary toric varieties. For example, the inverse image of the moment map of a point is a non-commutative torus (familiar in non-commutative geometry) rather than a torus.

I also introduced the deformation theory: the moduli space of non-commutative toric varieties is a complex orbifold whose rational points are the classical commutative toric varieties.

Pseudo-holomorphic simple Harnack Curves

Erwan Brugallé

This is an extended abstract of the note [1].

Let L_0, L_1, and L_2 be three distinct real lines in $\mathbb{C}P^2$ with $L_0 \cap L_1 \cap L_2 = \emptyset$. A *simple Harnack curve* is a real algebraic map $\phi : C \to \mathbb{C}P^2$ satisfying the two following conditions:

- C is a non-singular maximal real algebraic curve;
- there exist a connected component $\mathcal{O}$ of $\mathbb{R}C$, and three disjoint arcs l_0, l_1, l_2 contained in $\mathcal{O}$ such that $\phi^{-1}(L_i) \subset l_i$.

We depict in Figure 1 examples of simple Harnack curves with a non-singular image in $\mathbb{C}P^2$ and intersecting transversely all lines L_i. Theorems 1 and 2 below say that these are essentially the only simple Harnack curves.

Let $\phi : C \to \mathbb{C}P^2$ be a simple Harnack curve, and choose an orientation of $\mathcal{O}$. This induces an ordering of the intersection points of $\mathcal{O}$ (or C) with L_i, and we denote by s_i the corresponding sequence of intersection multiplicities. Let s be the sequence (s_0, s_1, s_2) considered up to the equivalence relation generated by

$$(s_0, s_1, s_2) \sim (\overline{s}_0, \overline{s}_1, \overline{s}_2), \quad (s_0, s_1, s_2) \sim (s_2, s_0, s_1), \quad \text{and} \quad (s_0, s_1, s_2) \sim (s_0, s_2, s_1),$$

where $\overline{(u_i)_{1 \le i \le n}} = (u_{n-i})_{1 \le i \le n}$. This equivalence relation is so that s does not depend on the chosen orientation on $\mathcal{O}$, nor on the labeling of the three lines L_i.

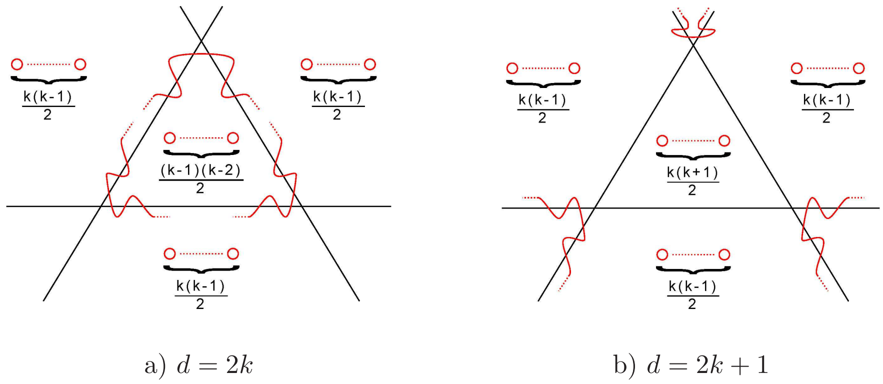

FIGURE 1. Simple Harnack curves of degree d and genus $\frac{(d-1)(d-2)}{2}$.

Theorem 1 (Mikhalkin [5], Mikhalkin-Rullgård [6]). *Let $\phi : C \to \mathbb{C}P^2$ be a simple Harnack curve of degree d, and suppose that $\phi(C)$ is the limit of images of a sequence of simple Harnack curves of degree d and genus $g(C) = \frac{(d-1)(d-2)}{2}$. Then the curve $\phi(C)$ has solitary nodes as only singularities (if any). Moreover if either $g(C) = 0$ or $g(C) = \frac{(d-1)(d-2)}{2}$, then the topological type of the pair $\left(\mathbb{R}P^2, \mathbb{R}\phi(C) \bigcup_{i=0}^{2} \mathbb{R}L_i\right)$ only depends on d, $g(C)$, and s.*

Mikhalkin actually proved Theorem 1 for simple Harnack curves in any toric surface, nevertheless this a priory more general statement can be deduced from the particular case of $\mathbb{C}P^2$. Existence of simple Harnack curves of maximal genus with any Newton polygon, and intersecting transversely all toric divisors, was first established by Itenberg (see [2]). Simple Harnack curves of any degree, genus, and sequence s were first constructed by Kenyon and Okounkov in [3]. In addition, when $g = 0$ they could remove the hypothesis that $\phi(C)$ has to be the limit of images of a sequence of simple Harnack curves of degree d and genus $g(C) = \frac{(d-1)(d-2)}{2}$. In Theorem 1 below, we remove this hypothesis for any g.

Because they are extremal objects, simple Harnack curves play an important role in real algebraic geometry, and Theorem 1 had a deep impact on subsequent developments in this field. However their importance goes beyond real geometry, as showed their connection to dimers discovered by Kenyon, Okounkov, and Sheffield in [4].

The goal of this talk is to give an alternative proof of Theorem 1. Moreover, our proof is also valid for *real pseudoholomorphic curves*, which are also very important objects in real algebraic and symplectic geometry. Note that a real algebraic map $\phi : C \to \mathbb{C}P^2$ is pseudoholomorphic, however the converse is not true in general. Mikhalkin's original proof of Theorem 1 uses amoebas of algebraic curves, and does not a priory apply to real pseudoholomorphic maps which are not algebraic.

We consider $\mathbb{C}P^2$ equipped with the standard Fubini-Study symplectic form ω_{FS}. Recall that an almost complex structure J on $\mathbb{C}P^2$ is said to be *tamed* by ω_{FS} if $\omega_{FS}(v, Jv) > 0$ for any non-null vector $v \in T\mathbb{C}P^2$. Such an almost complex structure is called *real* if the standard complex conjugation $conj$ on $\mathbb{C}P^2$ is J-antiholomorphic (i.e. $conj \circ J = J^{-1} \circ conj$). For example, the standard complex structure on $\mathbb{C}P^2$ is a real almost complex structure.

Let (C, ω) be a compact symplectic surface equipped with a complex structure J_C tamed by ω, and a J_C-antiholomorphic involution $conj_C$, and let J be a real almost complex structure on $\mathbb{C}P^2$. A symplectomorphism $\phi : C \to \mathbb{C}P^2$ is a *real J-holomorphic map* if

$$d\phi \circ J_C = J \circ d\phi \quad \text{and} \quad \phi \circ conj_C = conj \circ \phi.$$

It is of degree d if $\phi_*([C]) = d[\mathbb{C}P^1]$ in $H_2(\mathbb{C}P^2; \mathbb{Z})$. Recall that any intersection of two J-holomorphic curves is positive.

The definition of simple Harnack curves extends immediately to the real J-holomorphic case. Given three distinct real J-holomorphic lines L_0, L_1, and L_2 in $\mathbb{C}P^2$ such that $\bigcap_{i=0}^2 L_i = \emptyset$, a real J-holomorphic curve $\phi : C \to \mathbb{C}P^2$ is a simple Harnack curve if C is maximal, and if there exists a connected component $\mathcal{O}$ of $\mathbb{R}C$, and three disjoint arcs l_0, l_1, l_2 contained in $\mathcal{O}$ such that $\phi^{-1}(L_i) \subset l_i$.

Theorem 2. *Let $\phi : C \to \mathbb{C}P^2$ be a J-holomorphic simple Harnack curve of degree d. Then the curve $\phi(C)$ has solitary nodes as only singularities (if any). Moreover if either $g(C) = 0$ or $g(C) = \frac{(d-1)(d-2)}{2}$, then the topological type of the pair $\left(\mathbb{R}P^2, \mathbb{R}\phi(C) \bigcup_{i=0}^2 \mathbb{R}L_i\right)$ does not depend on J, once d and s are fixed.*

It follows from Theorem 2 that Figure 1 is enough to recover all topological types of pairs $\left(\mathbb{R}P^2, \mathbb{R}\phi(C) \bigcup \cup_{i=0}^2 \mathbb{R}L_i\right)$ where $\phi : C \to \mathbb{C}P^2$ is a simple Harnack curve. As in the case of algebraic curves, one may generalize Theorem 2 to J-holomorphic simple Harnack curves in any toric surface.

The proof of Theorem 2 goes along the following lines: the three projections from $\mathbb{C}P^2 \setminus (L_j \cap L_k)$ to L_i induce three ramified coverings $\pi_i : C \to L_i$; by considering the arrangement of the real Dessins d'enfants $\pi_i^{-1}(\mathbb{R}L_i)$ on $C/conj_C$, we deduce the number of connected components of $\mathbb{R}\phi(C)$ in each quadrant of $\mathbb{R}P^2 \setminus \left(\cup_{i=0}^2 \mathbb{R}L_i\right)$, as well as its complex orientation; the mutual position of all these connected components is then deduced from Rokhlin's complex orientation formula.

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Generalized RSK correspondences

Gleb Koshevoy
(joint work with Arkady Berenstein and Anatol N. Kirillov)

The classical Robinson-Shensted-Knuth correspondence (RSK) can be viewed as a bijection between monomial basis (PBW-basis) in the polynomial algebra $\mathbb{C}[Mat_{p\times q}]$ labeled by $Mat_{p\times q}(\mathbb{Z}_{\geq 0})$, and the tableaux basis labeled by pairs of semistandard Young tableaux of equal shape (on the alphabets $\{1,\ldots,p\}$ and $\{1,\ldots,q\}$ respectively).

There are other bases in $\mathbb{C}[Mat_{p\times q}]$, and one of aims of the talk is to reveal 'good' bases in the polynomial ring $\mathbb{C}[z_1,\ldots,z_m]$, and combinatorics of transitions between them.

First, we say that a map $\nu : \mathbb{C}[z_1,\ldots,z_m]\setminus\{0\} \to \mathbb{Z}^m$ is a *valuation* if $\nu(fg)=\nu(f)+\nu(g)$ and if $\nu(f)\neq\nu(g)$, then $\nu(f+g)=\max(\nu(f),\nu(g))$, where maximum is taken in the lexicographic order on $\mathbb{Z}^m$. We denote by C_ν the image $\nu(\mathbb{C}[z_1,\ldots,z_m]\setminus\{0\})$. By definition, C_ν is a an additive sub-monoid of $\mathbb{Z}^m$.

We also say that ν is *injective* if there exists a basis $\mathbf{B}$ of $\mathbb{C}[z_1,\ldots,z_m]$ such that the restriction of ν to $\mathbf{B}$ is injective. We refer to any such a basis a $\mathbf{B}$ as *adapted* to ν.

We say that a basis $\mathbf{B}$ adapted to ν is *optimal* if $1\in\mathbf{B}$ and each $b\in\mathbf{B}$ such that $\nu(b)$ is decomposable in C_ν factors as $b=b'b''$ for some non-unit elements $b',b''\in\mathbf{B}$.

For each basis $\mathbf{B}$ of $\mathbb{C}[z_1,\ldots,z_m]$ we define a map $K=K_{\nu,\mathbf{B}}:\mathbb{Z}^m_{\geq 0}\to\mathbb{Z}^m$ by

$$K(d)=\min\{\nu(b)\in\mathbb{Z}^m : c_{d,b}\neq 0\}$$

for each $d=(d_1,\ldots,d_m)\in\mathbb{Z}^m_{\geq 0}$, $b\in\mathbf{B}$, where the constants $c_{d,b}\in\mathbb{C}$ are determined by $z_1^{d_1}\cdots z_m^{d_m}=\sum_{b\in\mathbf{B}} c_b b$.

Definition. Given an injective valuation ν of $\mathbb{C}[z_1,\ldots,z_m]$ and an optimal basis $\mathbf{B}$ adapted to ν, we say that $(\nu,\mathbf{B})$ is an *algebraic RSK* if $K_{\nu,\mathbf{B}}$ is a bijection $\mathbb{Z}^m_{\geq 0}\widetilde{\to}C_\nu$.

In a similar, yet dual fashion, we introduce generalized geometric RSK.

Denote by $\mathbb{T}_m$ the m-dimensional algebraic torus defined over $\mathbb{Z}$, i.e., $\mathbb{T}_m=(\mathbb{T}_1)^m$, where $\mathbb{T}_1=\mathbb{G}_{\mathbf{m}}$ is the multiplicative group.

Definition. Given a Laurent polynomial f in m variables (which we view as an element of the coordinate algebra $\mathbb{R}[\mathbb{T}_m]$) and a birational isomorphism $\Phi : \mathbb{T}_m \to \mathbb{T}_m$, we say that (f, Φ) is a *(generalized) geometric RSK* if

$$f(\Phi(t_1, \ldots, t_m)) = t_1 + \cdots + t_m$$

for all $t_1, \ldots, t_m \in \mathbb{G}_{\mathbf{m}}$.

We will justify this definition by attaching a certain piecewise-linear map $\mathbb{R}^m_{\geq 0} \to \mathbb{R}^m$ (a "combinatrial RSK") to each such Φ as follows.

Recall that for any any subring R of $\mathbb{R}$ one can assign to each non-zero Laurent polynomial $f \in R[\mathbb{T}_m]$ its (min) tropicalization $Trop_R(f) : R^m \to R$ by:

$$(Trop_R(\sum_{a \in \mathbb{Z}^m} c_a t^a))(\tilde{t}_1, \ldots, \tilde{t}_m) = \min\{a_1\tilde{t}_1 + \cdots a_m\tilde{t}_m \,|\, a \in \mathbb{Z}^m : c_a \neq 0\}$$

for all $\tilde{t}_1, \ldots, \tilde{t}_m \in R$.

Furthermore, for a rational morphism $\Phi : \mathbb{T}_m \to \mathbb{T}_n$ one defines its tropicalization $Trop_R(\Phi) : R^m \to R^n$ as follows. If we write $\Phi(t_1, \ldots, t_m) = (\frac{f_1}{g_1}, \ldots, \frac{f_n}{g_n})$, where all f_i and g_i are non-zero Laurent polynomials in $R[\mathbb{T}_m]$, then

$$Trop_R(\Phi) = (Trop_R(f_1) - Trop_R(g_1), \ldots, Trop_R(f_n) - Trop_R(g_n)) \ .$$

$Trop_R(\Phi)$ is well-defined, i.e., it does not depend on the choice of functions f_i, g_i.

We say that a rational morphism $\Phi : \mathbb{T}_m \to \mathbb{T}_n$ is positive if $\Phi(\mathbb{R}^m_{>0}) \subset \mathbb{R}^n_{>0}$

We say that a geometric RSK (f, Φ) is *positive* if $\Phi, \Phi^{-1} : \mathbb{T}_m \to \mathbb{T}_m$ and $f \in [\mathbb{T}_m]$ are positive.

Proposition. *For each positive geometric RSK the restriction of $Trop_{\mathbb{Z}}(\Phi)$ to $\mathbb{Z}^m_{\geq 0}$ is a bijection*

$$K_{f,\Phi} : \mathbb{Z}^m_{\geq 0} \to C(trop_{\mathbb{Z}}(f)) \tag{1}$$

where we abbreviated $C(\tilde{f}) = \{a \in \mathbb{Z}^m : \tilde{f}(a) \geq 0\}$ for any function $\tilde{f} : \mathbb{Z}^m \to \mathbb{Z}$.

It frequently turns out that the additive monoids C_ν and $C(Trop_{\mathbb{Z}}(f))$ coincide, moreover, the bijection $K_{f,\Phi} : \mathbb{Z}^m \to C(Trop_{\mathbb{Z}}(f))$ frequently coincides with a generalized RSK $K_{\nu,\mathbf{B}}$. We believe that this is a manifestation of some kind of mirror duality, which motivates the following definition.

Definition. We say that a positive function $f \in \mathbb{Z}[T_m]$ *mirrors* an algebraic RSK $(\nu, \mathbf{B})$ if $C_{\nu,\mathbf{B}} = C(Trop_{\mathbb{Z}}(f))$. Moreover, we say that a positive geometric RSK (f, Φ) *mirrors* a an algebraic RSK $(\nu, \mathbf{B})$ if $K_{f,\Phi} = K_{\nu,\mathbf{B}}$.

Let A be an $I \times I$-matrix generalized (symmetrizable) Cartan matrix. For a Kac-Moody group $G(A)$ and the Weyl group W, denote by U the subgroup of G generated by $\overline{w}x_i(t)\overline{w}^{-1}$ for all $i \in I$, $t \in \mathbb{C}$, $w \in W$ such that $\ell(ws_i) = \ell(w) + 1$. Also set $U^- := U^T$ and B (resp. B^-) to be the subgroup of G generated by U

(resp. U^-) and T. Recall, that for each $i \in I$ there is a group homomorphism $\phi_i : SL_2 \to G$ such that

$$\overline{s}_i := \phi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x_{-i}(a) := \ \phi_i \begin{pmatrix} a^{-1} & 0 \\ 1 & a \end{pmatrix} \in B \subset B^- \ ,$$

$$x_i(a) := \ \phi_i \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in U \ , \phi_i \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} = \alpha_i^\vee(c)$$

For $u, v \in W$ denote $C^{u,v} := U\overline{u}U \cap B^- v^{-1} B^-$. Denote by $[\cdot]_+$ and $[\cdot]_-$ the natural projections $B^- \cdot U \to U$ and $B^- \cdot U \to B^-$ respectively, i.e., for each $g = b_- u \in B^- U$ we have $[g]_+ := u$ and $[g]_- := b_-$. Define a map $\eta^{u,v} : C^{u,v} \to G$ by

$$\eta^{u,v}(x) = [\overline{u}^{-1} x]_+ \cdot x^{-1} \cdot [x\overline{v}]_-$$

for $x \in C^{u,v}$. For any sequence $\mathbf{i} = (i_1, \ldots, i_m) \in I^m$ we write $w(\mathbf{i}) = s_{i_1} \cdots s_{i_m}$. We say that $\mathbf{i} \in I^m$ is *reduced* if $\ell(w(\mathbf{i})) = m$. For any $w \in W$ we denote by $R(w)$ the set of all reduced sequences $\mathbf{i}$ such that $w(\mathbf{i}) = w$.

Let $\Delta_{\omega_i,\omega_i}$ be the function on G given by

$$\Delta_{\omega_i,\omega_i}(u_- \overline{w} t u_+) = \delta_{1,w} \cdot \omega_i(t)$$

for all $u_- \in U^-$, $u_+ \in U$, $t \in T$.

Furthermore, for $u, v \in W$, $i \in I$ define the *generalized minor* $\Delta_{u\omega_i, v\omega_i}$ to be the function on G given by

$$\Delta_{u\omega_i, v\omega_i}(g) = \Delta_{\omega_i,\omega_i}(\overline{u}^{-1} g \overline{v}) \ .$$

Theorem 1. *For any $u, v \in W$ and any $\mathbf{i} = (i_1, \ldots, i_m) \in R(u, v^{-1})$ the generalized geometric RSK $K_{\mathbf{i}} : (\mathbb{G}_m)^m \widetilde{\to} (\mathbb{G}_m)^m$ factors as:*

$$(2) \qquad (\mathbb{G}_m)^m \xrightarrow{x'_{\mathbf{i}}} C^{u,v} \xrightarrow{\eta^{u,v}} C^{v,u} \xrightarrow{(x'_{-\mathbf{i}^{op}})^{-1}} (\mathbb{G}_m)^m$$

That is, $K_{\mathbf{i}}(\mathbf{t}) = \mathbf{p}$, where $x'_{-i_m}(p_m) \cdots x'_{-i_1}(p_1) = \eta^{u,v}(x'_{i_1}(t_1) \cdots x'_{i_m}(t_m))$.

For each $\mathbf{i} = (i_1, \ldots, i_m) \in R(u, v^{-1})$ denote $u_{<\ell} = \prod\limits_{k \le \ell, i_k \in -I} s_{i_k}$, $v^{-1}_{\le \ell} = \prod\limits_{k \le \ell, i_k \in I} s_{i_k}$.

Theorem 2. *Let $u, v \in W$, $\mathbf{i} = (i_1, \ldots, i_m) \in R(u, v^{-1})$ and let $K_{\mathbf{i}}(\mathbf{t}) = \mathbf{p}$. Then*

$$p_k = \begin{cases} \frac{M_k}{M_{k^+}} & \text{if } i_k \in I \\ \frac{1}{M_{k^+} M_k} \prod\limits_{\ell:\ \ell^- < k < \ell} M_\ell^{-a_{|i_\ell|,|i_k|}} & \text{if } i_k \in -I \end{cases}$$

for $k = 1, \ldots, m$ where we abbreviated $x := x'_{\mathbf{i}}(\mathbf{t}) = x'_{i_1}(t_1) \cdots x'_{i_m}(t_m) \in C^{u,v}$ and

$$M_\ell := \Delta_{u_{<\ell}\omega_{|i_\ell|}, v^{-1}_{\le \ell}\omega_{|i_\ell|}}(x) \ .$$

Corollary 1. *In the assumptions of Theorem 2, the complement* $C^{u,v} \setminus x_{\mathbf{i}}((\mathbb{C}^\times)^m)$ *is the set of all* $x \in C^{u,v}$ *such that*

$$\Delta_{u_{<k}\omega_{|i_k|}, v_{\geq k}^{-1}\omega_{|i_k|}}(x) = 0$$

for some $k \in \{1, \ldots, m\}$.

The following result allows for an efficient computation geometric RSK of each $K_{\mathbf{i}}$ and corresponding Laurent potential $f_{\mathbf{i}}$, $\mathbf{i} \in R(w)$ in terms of generalized minors.

Corollary 2. *Let* $w \in W$, $\mathbf{i} = (i_1, \ldots, i_m) \in R(w)$ *and let* $K_{\mathbf{i}}(\mathbf{t}) = \mathbf{p}$. *Then*

(a) $p_k = \frac{\Delta_{\omega_{i_k}, s_{i_1} \cdots s_{i_k} \omega_{i_k}}(x)}{\Delta_{\omega_{i_k}, s_{i_1} \cdots s_{i_{k^-}} \omega_{i_k}}(x)}$ *for* $k = 1, \ldots, m$ *where* $x := x_{i_m}(t_m) \cdots x_{i_1}(t_1) \in U^w$.

(b) $t_k = \frac{1}{M_{k^-} M_k} \prod_{\ell:\ \ell < k < \ell^+} M_\ell^{-a_{i_\ell, i_k}}$ *for* $k = 1, \ldots, m$ *where*

$$M_\ell := \Delta_{s_{i_m} \cdots s_{i_{\ell^+}} \omega_{i_\ell}, \omega_{i_\ell}}(b),\ b := x_{-i_m}(p_m) \cdots x_{-i_1}(p_1) \in B^-_{w^{-1}}\ .$$

(c) $\sum_{k:\ i_k = i} t_k = \Delta_{w^{-1}\omega_i, s_i\omega_i}(x_{-i_m}(p_m) \cdots x_{-i_1}(p_1))$ *for all* $i \in I$.

Real and tropical double Hurwitz numbers

Johannes Rau
(joint work with Mathieu Guay-Paquet, Hannah Markwig)

Classical Hurwitz numbers count the number of ramified covers of a given Riemann surface with fixed degree, genus and ramification data. New interest in these numbers and their relations arose in the last few decades when deep connections to various fields such as intersection theory on the moduli space of curves, matrix models in probability theory and mathematical physics were discovered. Among these developments are the study of new types of Hurwitz numbers, with a focus on the combinatorial and asymptotic behavior of the collection of these numbers.

In my talk, we study the particular case of real double Hurwitz numbers. This refers to the following: We fix $\mathbb{CP}^1$ with its standard conjugation $\text{conj} : \mathbb{CP}^1 \to \mathbb{CP}^1$ such that $\text{Fix(conj)} = \mathbb{RP}^1$. Fix numbers d, g and two partitions σ, γ of d. Let $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the two arcs of $\mathbb{RP}^1 \setminus \{0, \infty\}$. Then we additionally fix n_+ points on $\mathbb{R}^+$ and n_- points on $\mathbb{R}^-$ such that the Riemann-Hurwitz condition

$$2g - 2 = n_+ + n_- - l(\sigma) - l(\gamma) \tag{1}$$

is satisfied. Here l denotes the length of the partition. Then we denote by $H_g^{\mathbb{R}}(\sigma, \gamma, n_+)$ the number of real equivariant ramified covers of $\mathbb{CP}^1$ with given ramification. More precisely, we count triples (C, i, π), where C is a orientable topological surface, i is a orientation-reversing involution of C and $\pi : C \to \mathbb{CP}^1$ is a ramified cover satisfying

- π is real equivariant, i.e. $\text{conj} \circ \pi = \pi \circ i$,
- the ramification profile of π over 0 is σ,

- the ramification profile of π over ∞ is γ,
- π has simple ramification over the $n_+ + n_-$ marked points in $\mathbb{RP}^1 \setminus \{0, \infty\}$.

Obviously, these triples are counted up to "real isomorphisms", i.e. equivariant cover homeomorphisms. Note that the problem is of topological nature, in particular, the numbers do not depend on the explicit choice of simple branch points on each of the two arcs. However, the numbers do depend on n_+ (n_- is determined from the given data by equation (1)).

The main goal of my talk is to explain a tropical approach to determine these numbers. This a generalization of tropical correspondence formulas for classical Hurwitz numbers (cf. [2] and [1]) and is based on our papers [3] and [4]). In the tropical picture, $\mathbb{CP}^1$ gets replaced by a real line $\mathbb{R}$ with $n_+ + n_-$ marked points. We choose n_+ of these points to be called *positive*, the others *negative*. A *real* tropical cover is given by certain map from a *coloured* metric graph Γ to $\mathbb{R}$, with its 3-valent vertices being mapped one-to-one to the marked points. The colouring has to satisfy certain restrictions, namely for each vertex, depending on whether it maps to a positive or negative marked point, we give a list of 4 allowed colourings. To each such real tropical cover we assign a multiplicity by an explicit formula in terms of the number of "even" edges and some of the weights. Then we show:

Theorem. *The number of real tropical covers counted with multiplicity (and up to isomorphism) is equal to* $H_g^{\mathbb{R}}(\sigma, \gamma, n_+)$.

In my talk I discuss some applications and examples of this statement. I will then focus on the dependence of these numbers on n_+, i.e. on the number of simple branch points on the positive real axis. For polynomials, i.e. covers $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ with one point of total ramification, Itenberg and Zvonkine defined a *signed* count of covers (in the spirit of Welschinger signs for rational curve counting) and showed that this count does not depend on the position of branch points. We consider the intersection of both set-ups, i.e. double Hurwitz numbers with $g = 0$ and $\sigma = (d)$, and discuss how the signs and the invariance can be seen tropically. In particular, we see that for the collection of actual real covers corresponding to a single real tropical cover, only two cases can occur: Either all covers in this collection have the same Itenberg-Zvonkine sign, or the number of positive and negative covers is equal and therefore the collection contributes 0 to the signed count.

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Amoebas, Nonnegative Polynomials and Sums of Squares Supported on Circuits

Timo de Wolff
(joint work with Sadik Iliman)

This extended abstract is based on the article [7], which is joint work with Sadik Iliman.

Deciding nonnegativity of multivariate real polynomials is a crucial problem with various applications; see [11] for an overview. Since this problem is challenging in practice and moreover NP-hard [2], one is interested in finding *certificates* of nonnegativity. These are properties, which are both easier to check and imply nonnegativity. Since over 120 years, particularly motivated by seminal results by Hilbert [6], which lead to Hilbert's 17th problem, the standard certificates of nonnegativity are *sums of squares (SOS)*.

Namely, let $H_{n,d}$ denote the the vector space of all real homogeneous polynomials in n variables of degree d. $H_{n,2d}$ contains the cone of nonnegative polynomials and the cone of sums of squares:

$$\begin{aligned} P_{n,2d} &= \{f \in H_{n,2d} \ : \ f(\mathbf{x}) \geq 0, \text{ for all } \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n\}, \\ \Sigma_{n,2d} &= \left\{f \in P_{n,2d} \ : \ f = \sum\nolimits_i f_i^2 \text{ for some } f_i \in H_{n,d}\right\}. \end{aligned}$$

Hilbert's result [6] states that

$$P_{n,2d} = \Sigma_{n,2d} \quad \text{if and only if} \quad n = 2 \text{ or } 2d = 2 \text{ or } (n, 2d) = (3, 4).$$

An analogue version holds in the non-homogeneous case.

In particular *semidefinite programming*, a standard tool for solving polynomial optimization problems, relies on SOS [1]. But for many current polynomial optimization problems SOS certificates are not strong enough since problems become too difficult or numerical issues become too large. Thus, not only better solvers but particularly *new* certificates of nonnegativity as well as a better understanding of the cones of positive polynomials and the SOS cone are strongly desired. For an overview about nonnegative polynomials and sums of squares see for example [1, 17].

Exploiting *sparsity* in problems can reduce the complexity of solving hard problems. An important example is given by sparse polynomial optimization problems, [10]. Motivated by structural results in [14, 18] and similar statements for special cases [3, 16] we investigate *real polynomials supported on circuits*, a genuine class of sparse polynomials. Recall that $A \subset \mathbb{N}^n$ is called a *circuit*, if A is affinely dependent, but any proper subset of A is affinely independent, [5]. We write these polynomials as

$$f = \sum_{j=0}^{n} b_j \mathbf{x}^{\alpha(j)} + c\mathbf{x}^y \in \mathbb{R}[\mathbf{x}]$$

where the Newton polytope $\Delta = \mathrm{New}(f) = \mathrm{conv}\{\alpha(0), \ldots, \alpha(n)\} \subset \mathbb{R}^n$ is a lattice simplex with vertices $\alpha(j) \in (2\mathbb{Z})^n$. Moreover, $y \in \mathrm{Int}(\Delta)$, $b_j \in \mathbb{R}_{>0}$, and $c \in \mathbb{R}^*$. We denote this class of polynomials as P_Δ^y.

For $f \in P_\Delta^y$ we define the *circuit number* Θ_f as

$$\Theta_f = \prod_{j=0}^{n} \left(\frac{b_j}{\lambda_j} \right)^{\lambda_j},$$

where the λ_j are uniquely given by the convex combination $\sum_{j=0}^{n} \lambda_j \alpha(j) = y, \lambda_j \geq 0, \sum_{j=0}^{n} \lambda_j = 1$. We show via a norm relaxation strategy that nonnegativity of every polynomial $f \in P_\Delta^y$ is completely characterized by the λ_j's and its circuit number Θ_f.

Theorem 1 (Iliman, dW., [7]). *Let $f \in P_\Delta^y$ for a simplex Δ with even vertices $\alpha(j) \in (2\mathbb{N})^n$ for all $0 \leq j \leq n$. Then the following statements are equivalent.*

(1) *$c \in [-\Theta_f, \Theta_f]$ for $y \notin (2\mathbb{N})^n$ respectively $c \geq -\Theta_f$ for $y \in (2\mathbb{N})^n$.*
(2) *f is nonnegative.*

Furthermore, f is located on the boundary of the corresponding cone of nonnegative polynomials if and only if $|c| = \Theta_f$ for $y \notin (2\mathbb{N})^n$ respectively $c = -\Theta_f$ for $y \in (2\mathbb{N})^n$. In these cases f has at most 2^n real zeros which only differ in their signs.

Indeed these results are independent of sums of squares (see [7, Theorem 5.2] for a detailed version of the following statement):

Theorem 2 (Iliman, dW., [7]; rough version). *Let $f \in P_\Delta^y$ for a simplex Δ with even vertices $\alpha(j) \in (2\mathbb{N})^n$ for all $0 \leq j \leq n$. Let f be nonnegative. Then f is a sum of squares if and only if y is the midpoint of two even distinct lattice points contained in a particular subset of lattice points in Δ. In particular, this is independent of the choice of the coefficients b_j, c.*

Based on these characterizations we are able to define a new convex cone $C_{n,2d} \subset P_{n,2d}$ of *sums of nonnegative circuit polynomials (SONC)* in n variables of degree $2d$ serving as a nonnegativity certificate.

A crucial consequence of our results is that nonnegativity of polynomials supported on circuits are closely related to the structure of their amoeba.

Given a Laurent polynomial f in $\mathbb{C}[\mathbf{z}^{\pm 1}] = \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ the *amoeba* $\mathcal{A}(f)$ introduced by Gelfand, Kapranov, and Zelevinsky in [5] is the image of its variety $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$ under the $\mathrm{Log}\,|\cdot|$-map

$$\mathrm{Log}\,|\cdot| : (\mathbb{C}^*)^n \to \mathbb{R}^n, \quad (z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|),$$

where $|z|$ denotes the absolute value of a complex number $z \in \mathbb{C}^*$.

Amoebas have strong connections to various topics in mathematics such as complex analysis [4], the topology of real algebraic curves [12], and dimers / crystal shapes [9]. In particular they form a cornerstone of tropical geometry; see [8, 13].

The amoeba $\mathcal{A}(f)$ of a Laurent polynomial f is a closed set with finitely many convex components in the complement [5]. Furthermore, every component $E_\alpha(f)$

corresponds uniquely to a lattice point $\alpha \in \mathbb{Z}^n$ in the Newton polytope New(f) via an *order map*

$$\text{ord} : \mathbb{R}^n \setminus \mathcal{A}(f) \;\to\; \text{New}(f) \cap \mathbb{Z}^n,$$

a multivariate analogue of the classical argument principle from complex analysis [4]. The vertices of the Newton polytope are always contained in the image of the order map. An amoeba is called *solid* if the image of the order map are *exactly* the vertices of the Newton polytope. For an overview about amoebas see [13, 15].

For a fixed support set A the relation between the existence of certain components in the complement of an amoeba $\mathcal{A}(f)$ and the coefficients of the defining polynomials f is a key problem in amoeba theory. It was already stated by Gelfand, Kapranov and Zelevinsky; see [5, Remark 1.10, page 198]. We solve this problem in the case of real polynomials supported on circuits establishing a first direct connection between nonnegativity of real polynomials and amoeba theory.

Theorem 3 (Iliman, dW., [7]). *Let $f \in P_\Delta^y$ as in Theorem 1 such that $c\mathbf{x}^y$ is not a monomial square, i.e., not both $c > 0$ and $y \in (2\mathbb{N})^n$. Then the following are equivalent.*

(1) *f is nonnegative.*
(2) *The amoeba $\mathcal{A}(f)$ is solid.*

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