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Cohomology of Finite Groups: Interactions and Applications

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Abstract. The cohomology of finite groups is an important tool in many subjects including representation theory and algebraic topology. This meeting was the fourth in a series that has emphasized the interactions of group cohomology with other areas.

Mathematics Subject Classification (2010): 20Jxx, 20Cxx, 55Nxx, 57Nxx.

Introduction by the Organisers

The aim of the workshop was to bring together mathematicians from several areas of algebra and topology for the purpose of fostering interactions on projects of overlapping interests. This is the fourth workshop in the series. An Oberwolfach workshop with this same title has been held every five years since 2000. The meetings have contributed substantially to the development of interactions between fields such as commutative algebra, homological algebra, homotopy theory, modular representation theory and transformation groups. The common theme has been the use and application of techniques from the cohomology theory of finite groups, but the emphasis has been on applications and interactions with a host of diverse subjects.

The meeting had 53 participants from ten different countries. In total, there were 24 lectures, almost all of which were for 50 minutes. The schedule allowed plenty of time for discussions and collaborations. The weather at the meeting was excellent, particularly for the traditional Wednesday afternoon hike to St. Roman. In addition to the stimulating mathematics, the participants were treated to a concert of classical music and jazz presented by four of the particapants: Serge Bouc, Bernhard Hanke, Markus Linckelmann and Peter Webb.

There were many scientific highlights of the meeting. Included were the presentations of Srikanth Iyengar and Julia Pevtsova on different aspects of their very recent proof, in joint work with Benson and Krause, of the classification on localizing subcategories of the stable module category for a finite group scheme. The crucial new concept in the proof is the notion of π-cosupport and its relationship with the cohomological cosupport. This work continues a thread started about 25 years ago in homotopy theory and commutative algebra by Hopkins and Neeman, and was imported into cohomology and representation theory of finite groups 20 years ago by Benson, Carlson and Rickard.

Dave Hemmer presented a lecture on recent work with Dan Nakano and Fred Cohen on the complexity of the Lie module, a module over the symmetric group on n letters. The collaboration of Hemmer, Nakano and Cohen began at the Oberwolfach workshop with this title in 2005 which produced an earlier publication. Hemmer and his collaborators settle a conjecture of Erdmann, Lim and Tan in their most resent work. The proof is a mixture of methods from representation theory and topology.

Various functor categories are used when representations and cohomology of groups are studied. The talk of Nick Kuhn was devoted to functors on the category of finite dimensional vector spaces over a finite field. He coined the term 'generic representation theory', because these functors descibe the representations of the general linear groups $\mathrm{GL}(n)$ simultaneously for all n. In fact, he analysed the functor category in nondescribing characteristic, while Steven Sam presented some exciting structural results in the equi-characteristic case. Homological stability provided the general framework for Sam's talk. Motivated by questions from algebraic topology, Aurelien Djament presented his work on polynomial functors in the study of stable homology of congruence groups.

Beren Sanders spoke on joint work with Paul Balmer in which they compute the spectrum of the G-equivariant homotopy theory for G a finite group. This follows work of Mike Hopkins and Jeff Smith in the case $G = \{1\}$. Sanders and Balmer are able to determine the set of all prime thick tensor ideal subcategories, thereby obtaining the spectrum of the category as a set. Sanders also reported on some progress on finding the topology of the spectrum.

The notion of a p-local finite group, and the associated fusion system and linking system, have played a major role in the cohomology of groups in recent years. Chermak's recent proof of the existence and uniqueness of linking systems associated with a fusion system has put the subject on a firmer footing, and this is reflected in the fact that three of the talks at this conference were on this topic. Natàlia Castellana's talk concentrated on cellular properties: when can the classifying space of a p-local finite group be built out of that for its Sylow p-subgroup? Carles Broto talked about the fusion systems of finite groups of Lie type, and their automorphisms. Ran Levi talked about his joint work with Libman on the more general concept of a p-local compact group, and the existence and uniqueness of Adams operations on their classifying spaces.

There were a number of lectures where methods from group cohomology were applied to interesting problems in topology. These included the talk by Ian Hambleton on his joint work with Ergün Yalçın, in which they described conditions on a finite group G for it to act on a sphere with periodic isotropy. It is an important step forward in developing geometric models for homotopy actions. The lecture by Bernhard Hanke on inessential Brown-Peterson homology and bordism of elementary abelian groups has important applications to the Gromov-Lawson-Rosenberg conjecture on scalar curvature. The presentation by Ben Williams on his joint work with Ben Antieau on Azumaya algebras made significant advances on the period-index problem and the cohomology of PGL_n. Ignasi Mundet spoke on a conjecture of Ghys related to the Jordan condition for diffeomorphism groups of manifolds. The lecture by Hans-Werner Henn reviewed important applications of group cohomology to current computations in homotopy theory. This sampling illustrates the broad scope of the meeting across many areas of algebra and topology.

Another highlight of the meeting was a report by Jesper Grodal on his progress towards a solution to Linckelmanns gluing conjecture in p-local group theory for the case of a finite group of Lie type. This conjecture had been proposed at the first meeting with this title some 15 years ago.

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Workshop: Cohomology of Finite Groups: Interactions and Applications

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Abstracts

On a question of Rickard on tensor product of stably equivalent algebras

Serge Bouc
(joint work with Alexander Zimmermann)

Let K be a field, and let A, B, C and D be finite dimensional K-algebras. Rickard showed in [9] that if A and B are derived equivalent, and if C and D are derived equivalent, then also $A \otimes_K C$ and $B \otimes_K D$ are derived equivalent. Rickard asks in [10, Question 3.8] if this still holds when replacing derived equivalence by stable equivalence of Morita type.

First it is clear that we have to suppose that all algebras involved have no semisimple direct factor. A result due to Liu [5] shows that then we may suppose that all algebras are indecomposable. In [7] Liu, Zhou and Zimmermann showed that the question has a negative answer in case A, B are not necessarily self-injective.

However, a derived equivalence between self-injective algebras A and B induces a stable equivalence of Morita type between A and B ([4], [8]). If A and B are not self-injective, then this implication is no longer valid. Hence, the natural playground for Rickard's question consists of self-injective algebras.

The purpose of this talk is to describe such an example for which the answer to Rickard's question is negative: for an algebraically closed base field K of characteristic $p > 0$, we construct symmetric K-algebras A and B which are stably equivalent of Morita type, but $A \otimes_K K[X]/X^p$ and $B \otimes_K K[X]/X^p$ are not stably equivalent of Morita type.

Our example ([1]) is the principal p-block B of the group $G = PSU(3, p^r)$ and its Brauer correspondent b, i.e. the group algebra over K of the normalizer N in G of a Sylow p-subgroup S of G. After observing that since the Sylow p-subgroups of G have trivial intersection (i.e. $S \cap S^g = 1$ if $g \in G - N$), the algebras B and b are stable equivalent of Morita type, we compute the radical series of the center of b. To do this, we first determine the structure constants of the centre of the group algebra of N over the integers. Next, using GAP ([3]), we compute the radical series of the center of B, in the cases $p^r \in \{3, 4, 5, 8\}$. The results are as follows (where $q = p^r$ and $\gamma = \gcd(q + 1, 3)$):

G	$PSU(3,3)$		$PSU(3,4)$		$PSU(3,5)$		$PSU(3,8)$		$PSU(3,q)$
	$Z(B)$	$Z(b)$	$Z(B)$	$Z(b)$	$Z(B)$	$Z(b)$	$Z(B)$	$Z(b)$	$Z(b)$
$\dim_K J^0$	13	13	21	21	13	13	27	27	$\frac{q^2+q}{\gamma} + \gamma$
$\dim_K J^1$	12	12	20	20	12	12	26	26	$\frac{q^2+q}{\gamma} + \gamma - 1$
$\dim_K J^2$	4	3	5	4	2	1	3	2	$\frac{q+1}{\gamma} - 1$
$\dim_K J^3$	0	0	0	0	0	0	0	0	0

This shows that when $q \in \{3,4,5,8\}$, the center of $B \otimes_K K[X]/X^p$ is not isomorphic to the center of $b \otimes_K K[X]/X^p$, as their radical square have different dimensions. Now by a theorem of Liu, Zhou and Zimmermann ([6]), the projective centers of $B \otimes_K K[X]/X^p$ and $b \otimes_K K[X]/X^p$ are equal to 0. It follows that the stable center of $B \otimes_K K[X]/X^p$ (resp. $b \otimes_K K[X]/X^p$) is isomorphic to its ordinary center.

Hence the stable centers of $B \otimes_K K[X]/X^p$ and $b \otimes_K K[X]/X^p$ are not isomorphic, so a theorem of Broué ([2]) implies that these two algebras cannot be stably equivalent of Morita type.

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Inessential Brown-Peterson homology and bordism of elementary abelian groups

Bernhard Hanke

Let p be an odd prime. We study $\Omega_*^{SO}(B(\mathbb{Z}/p)^n)$, oriented bordism of the classifying space $B(\mathbb{Z}/p)^n$, as a module over the oriented bordism ring Ω_*^{SO}. This is equivalent to bordism of free oriented $(\mathbb{Z}/p)^n$-manifolds. In the following we drop the superscript SO from our notation.

For $n = 1$ the structure of the reduced module has been described by Conner and Floyd [3]. It can be derived from the Gysin sequence for the fibration

$$S^1 \hookrightarrow B\mathbb{Z}/p \to \mathbb{C}P^\infty,$$

which splits, in the case under consideration, into short exact sequences

$$0 \to \widetilde{\Omega}_{*+2}(\mathbb{C}P^\infty) \stackrel{-\cap [p]t}{\to} \widetilde{\Omega}_*(\mathbb{C}P^\infty) \to \widetilde{\Omega}_{*+1}(B\mathbb{Z}/p) \to 0.$$

Here $t \in \Omega^2(\mathbb{C}P^\infty)$ is a generator (Conner-Floyd class) and $[p]t \in \Omega^2(\mathbb{C}P^\infty)$ is the image of t under the map induced by $B(z \mapsto z^p) : BS^1 \to BS^1$. This is determined by the formal group law associated to oriented bordism localized at p, or, equivalently, Brown-Peterson theory for the prime p.

For $n > 1$ we consider the Künneth type exact sequence of reduced oriented bordism groups

$$0 \to \widetilde{\Omega}_*(\wedge^{n-1} B\mathbb{Z}/p) \otimes_{\Omega_*} \widetilde{\Omega}_*(B\mathbb{Z}/p) \to \widetilde{\Omega}_*(\wedge^n B\mathbb{Z}/p) \to$$
$$\to (\mathrm{Tor}_{\Omega_*}(\widetilde{\Omega}_*(\wedge^{n-1} B\mathbb{Z}/p), \widetilde{\Omega}_*(B\mathbb{Z}/p)))_{*-1} \to 0$$

due to Landweber [7]. Elements in the torsion product can be pulled back to $\widetilde{\Omega}_*(\wedge^n B\mathbb{Z}/p)$ by a (matrix) Toda bracket construction. But this is noncanonical, and indeed it has been an open problem whether the map

$$\widetilde{\Omega}_*(\wedge^n B\mathbb{Z}/p) \to (\mathrm{Tor}_{\Omega_*}(\widetilde{\Omega}_*(\wedge^{n-1} B\mathbb{Z}/p), \widetilde{\Omega}_*(B\mathbb{Z}/p)))_{*-1}$$

appearing in the Künneth-Landweber sequence splits Ω_*-linearly.

Theorem 1. *There is an Ω_*-linear map*

$$\Psi_n : \widetilde{\Omega}_*(\wedge^n B\mathbb{Z}/p) \to \widetilde{\Omega}_*(B\mathbb{Z}/p) \otimes_{\Omega_*} \cdots \otimes_{\Omega_*} \widetilde{\Omega}_*(B\mathbb{Z}/p)$$

which splits the iterated Künneth map

$$\Phi_n : \widetilde{\Omega}_*(B\mathbb{Z}/p) \otimes_{\Omega_*} \cdots \otimes_{\Omega_*} \widetilde{\Omega}_*(B\mathbb{Z}/p) \to \widetilde{\Omega}_*(\wedge^n B\mathbb{Z}/p).$$

The construction of Ψ_n is independent of any choices.

From this we derive the following assertions:

- The Ω_*-module $\Omega_*(B(\mathbb{Z}/p)^n)$ splits as a direct sum of suspensions of multiple tensor products of $\Omega_*(B\mathbb{Z}/p)$. This sharpens a result of Johnson-Wilson [6], who proved a corresponding statement for the graded module associated to a certain filtration of $\Omega_*(B(\mathbb{Z}/p)^n)$ by Ω_*-submodules.
- The Künneth-Landweber sequence splits Ω_*-linearly.
- The Ω_*-module $\Omega_*(B(\mathbb{Z}/p)^n)$ is generated by generalized products of lens spaces, i.e. images of k-fold products of standard lens spaces in $\Omega_*(B(\mathbb{Z}/p)^k)$ under maps $\Omega_*(B(\mathbb{Z}/p)^k) \to \Omega_*(B(\mathbb{Z}/p)^n)$ induced by injective group homomorphisms $(\mathbb{Z}/p)^k \to (\mathbb{Z}/p)^n$, $0 \leq k \leq n$.

Note that the last assertion does not hold for ordinary homology.

The main idea for the proof of Theorem 1 is to study the *inessential Brown-Peterson group homology* of $(\mathbb{Z}/p)^n$. In our context this amounts to the Ω_*-submodule of $\Omega_*(B(\mathbb{Z}/p)^n)$ generated by elements coming from proper subgroups of $(\mathbb{Z}/p)^n$. Let $K_* \subset \widetilde{\Omega}_*(\wedge^n B\mathbb{Z}/p)$ be the image of this submodule in the reduced theory. Note that this definition is canonical (i.e. independent of any choices).

Because the map Φ_n in the iterated Künneth sequence is injective [6], Theorem 1 following from the following two facts, which show that K_* is a direct complement of $\operatorname{im}\Phi_n$ in $\widetilde{\Omega}_*(\wedge^n B\mathbb{Z}/p)$:

- $\operatorname{im}\Phi_n + K_* = \widetilde{\Omega}_*(\wedge^n B\mathbb{Z}/p)$,
- $\operatorname{im}\Phi_n \cap K_* = 0$.

The first assertion is proven by an explicit calculation, examining the Pontryagin product on $\Omega_*(B(\mathbb{Z}/p)^n)$ induced by the group structure on $(\mathbb{Z}/p)^n$. The second assertion follows from methods developped in Ravenel-Wilson's solution of the Conner-Floyd conjecture [8]. We remark that the second assertion is no longer true for $p = 2$.

As an application of our results we prove the Gromov-Lawson-Rosenberg conjecture for atoral manifolds with elementary abelian fundamental groups of odd order. In the following we call a closed oriented manifold M^d *p-atoral*, if

$$f^*(c_1) \cup \cdots \cup f^*(c_d) = 0 \in H^d(M; \mathbb{F}_p)$$

for all $c_1, \ldots, c_d \in H^1(B\pi_1(M); \mathbb{F}_p)$. Here $f : M \to B\pi_1(M)$ is the classifying map of M.

Theorem 2. *Assume that M is a p-atoral manifold of dimension $d \geq 5$ with fundamental group $(\mathbb{Z}/p)^n$, where p is an odd prime. Then the following assertions hold.*

- *If M admits a spin structure, then M admits a Riemannian metric of positive scalar curvature, if and only if $\alpha(M) = 0 \in \mathrm{KO}_d$, where α is the index invariant introduced by Hitchin* [5] *with values in the coefficients of real K-theory.*
- *If M does not admit a spin structure, then M admits a Riemannian metric of positive scalar curvature.*

These results already appeared in [2, Theorem 5.8] and [1, Theorem 2.3], but the proofs in these references contain a gap (in the proof of [2, Theorem 5.6]), which is filled by our methods.

For more details concerning our results the reader is referred to the preprint [4].

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On good $(\mathbf{p}, \mathbf{r})$-filtrations for rational G-modules

Daniel K. Nakano
(joint work with Tobias Kildetoft)

Let k be an algebraically closed field k of characteristic $p > 0$, and G be a simple, simply connected algebraic group scheme defined over $\mathbb{F}_p$. A class of modules of central interest are the induced modules $\nabla(\lambda) = \text{ind}_B^G \lambda$ where $\lambda \in X_+$ (dominant integral weight) and B is a Borel subgroup (corresponding to the negative roots). The characters of $\nabla(\lambda)$ are given by Weyl's character formula, and $\nabla(\lambda)$ has a simple socle $L(\lambda)$ where each finite-dimensional simple G-module is isomorphic to a unique such $L(\lambda)$. The category of rational G-modules is not semisimple and one of the major open problems is to determine multiplicities of composition factors in these induced modules which naturally arise from the characteristic zero through reduction modulo p.

The modules $\nabla(\lambda)$ also form the building blocks for studying injective modules. It is natural to consider modules which admit filtrations whose sections are $\nabla(\lambda)$ for suitable $\lambda \in X_+$. These filtrations are called good filtrations. For each $\lambda \in X_+$ with unique decomposition $\lambda = \lambda_0 + p^r \lambda_1$ with $\lambda_0 \in X_r$ (p^rth restricted weights) and $\lambda_1 \in X_+$, one can define

$$\nabla^{(p,r)}(\lambda) = L(\lambda_0) \otimes \nabla(\lambda_1)^{(r)}$$

where (r) denotes the twisting of the module action by the rth Frobenius morphism. A G-module M has a good (p, r)-filtration if and only if M has a filtration with factors of the form $\nabla^{(p,r)}(\lambda)$ for suitable $\lambda \in X_+$. Let $\text{St}_r = L((p^r - 1)\rho)$ be the rth Steinberg module. The following conjecture, introduced by Donkin at an MSRI lecture in 1990, interrelates good filtrations with good (p, r)-filtrations via the Steinberg module:

Conjecture. *Let M be a finite-dimensional G-module. Then M has a good (p, r)-filtration if and only if $\text{St}_r \otimes M$ has a good filtration.*

Let h be the Coxeter number of the underlying root system Φ of G. When $p \geq 2h - 2$, Andersen [And01] showed that if M has a good (p, r)-filtration then $\text{St}_r \otimes M$ has a good filtration. The verification of the other direction of the conjecture appears to be much harder. A special case is that for any $\lambda \in X_+$, the module $\nabla(\lambda)$ has a good p-filtration because tensor products of modules with good filtrations again have good filtrations. Parshall and Scott [PS12] have proved that $\nabla(\lambda)$ has a good p-filtration when $p \geq 2h - 2$ and the Lusztig character formula holds for all composition factors of $\nabla(\lambda)$.

The talk will primarily focus on issues related to the "$\Rightarrow$" direction of the conjecture. Our results expand on the work of Andersen by proving that when M has a good (p, r)-filtration then $\text{St}_r \otimes M$ has a good filtration, provided a suitable inequality holds between p, r, h and the weights occurring in the good (p, r)-filtration of M. As a special case, we recover the results of Andersen, though our method of proof is markedly different. Our method of proof involves the use of the Donkin/Scott cohomological criterion for the existence of a good filtration, and a careful analysis of the vanishing of extension groups with appropriate conditions on weights.

In order to prove the "⇒' direction of the conjecture, it is clearly enough to prove that $\mathrm{St}_r \otimes \nabla^{(p,r)}(\lambda)$ has a good filtration for any $\lambda \in X_+$. However, due to a result of Andersen (with an argument also included our work), it turns out that the "⇒" direction is equivalent $\mathrm{St}_r \otimes L(\lambda)$ having a good filtration for any $\lambda \in X_r$. The inequality obtained allows us to prove that $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration with smaller restrictions on p provided that the weight λ is also suitably smaller. This still leaves weights $\lambda \in X_+$ for which we do not know whether $\mathrm{St}_r \otimes \nabla^{(p,r)}(\lambda)$ has a good filtration when p is small. However, if $\lambda = \lambda_0 + p^r\lambda_1$ with $\lambda_0 \in X_r$ and if λ_1 is large enough compared to λ_0, then we can still show that $\mathrm{St}_r \otimes \nabla^{(p,r)}(\lambda)$ has a good filtration, even if λ_0 is not small enough to satisfy the inequality we get with respect to p, r and h.

A natural question is for which $\lambda \in X_+$ does $\mathrm{St}_r \otimes L(\lambda)$ have a good filtration? When $p \geq 2h-2$ and if $\langle \lambda, \alpha_0^\vee \rangle \leq (p^r-1)(h-1)$ (where α_0 is the highest short root of Φ), $L(\lambda) \simeq \nabla^{(p,r)}(\lambda)$ so in these cases it does hold. However, we also show that this is close to being the best bound of this type possible. Namely, we show that if $p = 2h-5$ and R is of type A, then there is a λ with $\langle \lambda, \alpha_0^\vee \rangle \leq (p-1)(h-1)$ and such that $\mathrm{St}_1 \otimes L(\lambda)$ does not have a good filtration. Furthermore, we demonstrate that our results are strong enough to prove the "⇒" direction of the (p,r)-filtration conjecture for root system of type A_2, A_3, and B_2 over fields of arbitrary characteristic, as well as for the root system of type G_2 as long as $p \neq 7$.

In the final part of the talk, I indicate how we recast Donkin's (p,r)-Filtration Conjecture via tilting modules. This allows us to establish a cohomological criteria (analogous to the one for good filtrations) for $\mathrm{St}_r \otimes M$ to admit a good filtration. This cohomological criteria is independent of the characteristic of the field. As a corollary of this result we show that if Donkin's Tilting Modules Conjecture holds then the "⇒" direction of Donkin's (p,r)-Filtration Conjecture holds. Since the tilting module conjecture is valid when $p \geq 2h-2$, this yields a second proof of the "⇒" direction of the (p,r)-filtration conjecture. We note that if both directions of the (p,r)-conjecture are true then our cohomological criteria is equivalent to a module M-admitting a good (p,r)-filtration.

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Rational Cohomology and Support Varieties for Algebraic Groups

Eric M. Friedlander

In two recent papers [1] and [2], we initiated the study of support varieties for rational G-modules M for a linear algebraic group G of exponential type. This approach utilizes constructions of earlier work with Andrei Suslin and Christopher Bendel [3] and [4] employing 1-parameter subgroups $\mathbb{G}_a \to G$. Although the resulting theory of support varieties has many good properties, the construction is somewhat subtle and thereby confusing because of a necessary twist of parameters.

In this talk, we explained how these support varieties in the special case $G = \mathbb{G}_a$ can be reformulated in terms of rational cohomology, thereby relating them to more classical constructions for finite groups. The talk presented some elementary and concrete aspects of the representation theory and cohomology of $\mathbb{G}_a$. Time did not permit the discussion of more general unipotent algebraic groups for which there appears to be a parallel theory which relies on certain refinements of computations of [3] and [4].

The talk discussed "mock injectives" for an affine algebraic group G: the rational G-module L is a mock injective if its restriction to every Frobenius kernel $G_{(r)}$ is injective.

The talk also considered infinite dimensional rational G-modules Q_ζ for a rational cohomological class $\zeta \in H^{2n}(G, k)$ which are natural analogues of Jon Carlson's important L_ζ-modules constructed for finite group schemes. Because one does not have projective rational G-modules, these are constructed using truncated injective resolutions of the trivial module.

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An attempt on the classification of unstable Adams operations for p-local compact groups.

Ran Levi
(joint work with Assaf Libman)

A p-local compact group is an algebraic object modelled on the homotopy theory associated with p-completed classifying spaces of compact Lie groups and p-compact groups. In particular p-local compact groups give a unified framework in which one may study p-completed classifying spaces from an algebraic and homotopy theoretic point of view. Like compact Lie groups and p-compact groups, p-local compact groups admit "unstable Adams operations", i.e. certain self equivalences of their algebraic structure which give rise to self homotopy equivalences of their classifying spaces, and are characterised by their effect on p-adic cohomology. Similarly to the classical case, unstable Adams operations are considered to be a very useful and important family of maps. For instance, their existence was used by Gonzalez to express p-local compact groups as colimits of certain finite approximations with some important consequences. However, for a given p-local compact group and a given p-adic degree, the question whether an unstable Adams operation of that degree exists, and if it does whether it is unique up to homotopy, is not well understood. This talk is a report on recent progress in this subject, and some remaining questions.

We start by recalling the concepts involved. Fix a prime p. A *discrete p-toral group* is a group S containing a normal subgroup $T \cong (\mathbb{Z}/p^\infty)^r$, called the *maximal torus* of S, with p-power index. A *p-local compact group* is a triple $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$, where S is a discrete p-toral group, $\mathcal{F}$ is a *saturated fusion system* over S, and $\mathcal{L}$ is a *centric linking system* associated to $\mathcal{F}$. The fusion and linking systems are categories whose objects are certain subgroups of S. The morphisms in $\mathcal{F}$ are group monomorphisms between objects that satisfy a certain set of conditions. The *classifying space* of a p-local compact group $\mathcal{G}$, denoted $B\mathcal{G}$ is the p-completed nerve $|\mathcal{L}|^\wedge_p$. For complete definitions of all these concepts see [4].

Let ζ be a p-adic unit and let S be a discrete p-toral group with maximal torus T. An *Adams automorphism* of S of degree ζ is an automorphism of S which restricts to the ζ power map on T. Such an automorphism is called *normal* if the induced map on S/T is the identity map. If T is self centralising in S, then all Adams automorphisms on S are normal.

An automorphism φ of S is said to be *fusion preserving*, if for any $\alpha\colon P \to Q$ in $\mathcal{F}$, the homomorphism $\varphi|_Q \circ \alpha \circ (\varphi|_P)^{-1}$ is a morphism in $\mathcal{F}$.

An unstable Adams operation on $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ of degree ζ is a pair (Ψ, ψ), where ψ is a fusion preserving Adams automorphism of S of degree ζ and Ψ is a self equivalence of $\mathcal{L}$ that is compatible with ψ in the appropriate sense (see [6]) for details). In [5] it was shown that the p-adic cohomology of a p-local compact group is the invariants under the action of the Weyl group of the p-adic cohomology of its maximal torus. Thus one can show that an unstable Adams operation of degree ζ on a p-local compact group induces multiplication by ζ^i on $H^{2i}_{\mathbb{Q}_p}(B\mathcal{G})$. In fact under some mild hypotheses it is shown that every self map of $B\mathcal{G}$ with this cohomological effect induces an unstable Adams operation of degree ζ on $\mathcal{G}$.

Unstable Adams operations on p-local compact groups can thus be defined in three different ways. The original definition in [6] is algebraic, starting off with an Adams automorphism ψ on S, and using the algebraic structure of $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ to define Ψ. The second definition, also from [6] is topological: a self map of $B\mathcal{G}$ which restricts to an Adams automorphism of S of the prescribed degree. In [6] we show that there is an epimorphism from the group of algebraic Adams operations on $\mathcal{G}$ to the group of geometric Adams operations. The third definition is the cohomological criterion discussed above.

In the current project, we start by analysing the difference between the algebraic and the geometric operations. In particular we compute the kernel of the epimorphism from the algebraic to the geometric operations to be a subgroup $\mathrm{Ad}_{\mathcal{L}}(S)$ of the automorphism group of S as an object in the linking system $\mathcal{L}$. This kernel contains the maximal torus T as a normal subgroup, and the quotient, denoted $D(\mathcal{F})$ is an invariant of the p-local group and plays an important role in our study.

The two questions we attempt in this project are existence and uniqueness. First, finding conditions for the existence of unstable Adams operations of a given degree on a p-local compact group $\mathcal{G}$. Second is the question of uniqueness up to homotopy of an unstable operation of a given degree.

As may be expected, to answer the uniqueness question a certain concept of connectivity is required. We say that a p-local compact group $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ is *connected* if for every $x \in S$, there is some $\varphi \in \mathcal{F}$, such that $\varphi(x) \in T$. Let W denote the Weyl group of $\mathcal{G}$, defined as the automorphism group of T in $\mathcal{F}$. We prove the following:

Theorem 1. *Suppose that $\mathcal{G}$ is connected and let W be its Weyl group, and assume that $H^1(W, T) = 0$. If p is odd, then the degree map $\mathrm{Ad}_g^{\mathrm{out}}(\mathcal{G}) \to \mathbb{Z}_p^\times / D(\mathcal{F})$ is injective, and if $p = 2$, then its kernel is given by a certain $\lim^1$ term.*

This gives many examples of p-local compact groups where degree determines an unstable operation up to homotopy. We show:

Theorem 2. *Let $\mathcal{F}$ be a connected saturated fusion system. Assume that either one of the following conditions holds:*

a) *p is odd and the Weyl group $W = \mathrm{Out}_{\mathcal{F}}(T)$ is a pseudo-reflection group, or*
b) *p is odd and $D(\mathcal{F}) \neq 1$.*
c) *$p = 2$, $D(\mathcal{F}) \neq 0$ and $H^1(W/D(\mathcal{F}), T^{D(\mathcal{F})}) = 0$.*

Then $H^1(W, T) = 0$.

Notice in particular that this gives a new proof that, for odd primes, unstable Adams operations on p-compact groups, when they exist are unique up to homotopy. Of course for p-compact groups unstable Adams operations are already well understood by works of Andersen, Grodal, Moller and Viruel [2, 1, 3].

We turn to the existence question. We consider $\mathcal{L}$ as an extension of a quotient category $\mathcal{L}/_0$ obtained from $\mathcal{L}$ by dividing out each morphism set in $\mathcal{L}$ by the maximal torus of the target object. There is a functor Φ from $\mathcal{L}/_0$ to abelian groups given by taking an object to its maximal torus, and the linking system $\mathcal{L}$ can be regarded as an extension of $\mathcal{L}/_0$ by Φ, with an extension class $[\mathcal{L}] \in H^2(\mathcal{L}/_0, \Phi)$. We prove the following:

Proposition 3. *Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p-local compact group, and let ζ be a p-adic integer such that $\zeta \cdot [\mathcal{L}] = [\mathcal{L}]$. Then there exists an unstable Adams operation on $\mathcal{G}$ of degree ζ.*

It turns out that the unstable operations constructed this way have certain additional properties, we omit the technical details and call such operations "special" for lack of a better name. The converse of Proposition 3 is true if a special operation on $\mathcal{G}$ exists. Precisely,

Proposition 4. *Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p-local compact group, and let ζ be a p-adic integer such that $\mathcal{G}$ admits a special unstable Adams operation of degree ζ. Then $\zeta \cdot [\mathcal{L}] = [\mathcal{L}]$.*

Finally, we show that there are examples of unstable Adams operations on p-compact groups which are not special. Thus the general question of existence remains open. A second question that remains open is whether uniqueness can be completely characterised by properties of the p-local group in question. In particular we are not aware of an example where $\mathcal{G}$ is connected in the sense defined above, and $H^1(W, T)$ is nontrivial, but we do not have enough evidence to state a conjecture.

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Group actions on spheres with rank one isotropy

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(joint work with Ergün Yalçın)

Actions of finite groups on spheres can be studied in various different settings. The fundamental examples come from the unit spheres $S(V)$ in a real or complex G-representation V, and natural questions arise for these examples about the dimensions of the non-empty fixed sets $S(V)^H$, $H \leq G$, and the structure of the isotropy subgroups.

A useful way to measure the complexity of the isotropy is the *rank*. We say that G has *rank* k if it contains a subgroup isomorphic to $(\mathbb{Z}/p)^k$, for some prime p, but no subgroup $(\mathbb{Z}/p)^{k+1}$, for any prime p. In recent joint work we answer the following question:

Question. For which finite groups G, does there exist a finite G-CW-complex $X \simeq S^n$ with all isotropy subgroups of rank one ?

By P. A. Smith theory, the rank one assumption on the isotropy subgroups implies that G must have $\text{rank}(G) \leq 2$ (see [4, Corollary 6.3]). Since every rank one finite group can act freely on a finite complex homotopy equivalent to a sphere (Swan [7]), we can restrict our attention to rank two groups. Here are three natural settings for the study of finite group actions on spheres:

(1) smooth G-actions on closed manifolds homotopy equivalent to spheres;
(2) finite G-homotopy representations;
(3) finite G-CW-complexes $X \simeq S^n$.

In contrast to G-representation spheres $S(V)$, the non-linear smooth G-actions on a smooth manifold $M \simeq S^n$ exhibit more flexibility. For example, in the linear case, the fixed sets $S(V)^H$ are always linear subspheres. For smooth actions, the fixed sets are smoothly embedded submanifolds but may not even be integral homology spheres.

Well-known general constraints on smooth actions arise from P. A. Smith theory: if H is a subgroup of p-power order, for some prime p, then M^H is an $\mathbb{F}_p$-homology sphere. In addition, even if the fixed sets are diffeomorphic to spheres, they may be knotted or linked as embedded subspheres in M. One can also consider topological G-actions, usually with the assumption of local linearity, otherwise the fixed sets may not be locally flat submanifolds.

In the setting (B) of G-homotopy representations, the objects of study are finite (or more generally finite-dimensional) G-CW-complexes X satisfying the property that for each $H \leq G$, the fixed point set X^H is homotopy equivalent to a sphere $S^{n(H)}$ where $n(H) = \dim X^H$. We could also consider a version of this setting where $\dim X^H$ is the same as its homological dimension, and X^H is an $\mathbb{F}_p$-homology $n(H)$-sphere, for H of p-power order.

The third setting (C) is the most flexible of all. Here we suppose that $X \simeq S^n$ is a finite G-CW-complex homotopy equivalent to a sphere, but do not require that $\dim X = n$. Moreover, we make no initial assumptions about the homology of the fixed sets X^H, although the conditions imposed by P. A. Smith theory with $\mathbb{F}_p$-coefficients still hold. In the setting (C), we will see that $\dim X^H$ must be (much) higher in general than its homological dimension, and this provides new obstructions to understanding our motivating question in setting (A) or (B).

We provide a complete answer for the existence question in setting (C). Our construction produces G-CW-complexes with prime power isotropy.

Theorem A. *Let G be a finite group of rank two. There exists a finite G-CW-complex $X \simeq S^n$ with rank one isotropy if and only if G is* $\mathrm{Qd}(p)$*-free.*

The group $\mathrm{Qd}(p)$ is defined as the semidirect product of $(\mathbb{Z}/p\times\mathbb{Z}/p)$ and $SL_2(p)$ with the obvious action of $SL_2(p)$ on $\mathbb{Z}/p \times \mathbb{Z}/p$. We say $\mathrm{Qd}(p)$ is *p'-involved in* G if there exists a subgroup $K \leq G$, of order prime to p, such that $N_G(K)/K$ contains a subgroup isomorphic to $\mathrm{Qd}(p)$. If a group G does not p'-involve $\mathrm{Qd}(p)$ for any odd prime p, then we say that G is $\mathrm{Qd}(p)$*-free.*

The necessity of the $\mathrm{Qd}(p)$-free condition was established in [9, Theorem 3.3] and [4, Proposition 5.4]. In the other direction, if G is a rank two finite group which is $\mathrm{Qd}(p)$-free then G has a p-effective representation $V_p\colon G_p \to U(n)$ which can be used to construct finite G-CW-complexes $X \simeq S^n$ with rank one isotropy. The existence of these p-effective representations was proved by Jackson [6, Theorem 47] and they were also one of the main ingredients for the constructions in Hambleton-Yalçın [4].

In our earlier work [3] and [4], we studied this problem in the setting (B) of *G-homotopy representations*, introduced by tom Dieck (see [8, Definition 10.1]).

Definition. A finite group G has the *rank one intersection property* if for every pair $H, K \leq G$ of rank one 2-subgroups such that $H \cap K \neq 1$, the subgroup $\langle H, K\rangle$ generated by H and K is a 2-group. We say that G is *2-regular* if (i) $\Omega_1(Z(G_2))$ is strongly closed in G_2 with respect to G, or (ii) G has the rank one intersection property.

Let $\mathcal{P}(G)$ denote the set of primes p such that $\mathrm{rank}_p(G) = 2$. Let $\mathcal{H}_p$ denote the family of all rank one p-subgroups $H \leq G$, for $p \in \mathcal{P}(G)$, and let $\mathcal{H} = \bigcup\{H \in \mathcal{H}_p \parallel p \in \mathcal{P}(G)\}$. Our main result in setting (B) is the following:

Theorem B. *Let G be rank two finite group satisfying the following two conditions:*

1. *G is 2-regular if $2 \in \mathcal{P}(G)$, and G is $\mathrm{Qd}(p)$-free for all $p \in \mathcal{P}(G)$ with $p > 2$;*
2. *If $1 \neq H \in \mathcal{H}_p$, then $\mathrm{rank}_q(N_G(H)/H) \leq 1$ for every prime $q \neq p$.*

Then there exists a finite G-homotopy representation X with isotropy in $\mathcal{H}$.

As an application, we studied the rank two simple groups in detail.

Theorem C. *Let G be a finite simple group of rank two. Then there exists a finite G-homotopy representation with rank one isotropy of prime power order if and only if G is one of the following: (i) $PSL_2(q)$, $q \geq 5$, (ii) $PSL_2(q^2)$, $q \geq 3$, (iii) $PSU_3(3)$, or (iv) $PSU_3(4)$.*

We remark that $G = PSL_3(q)$, q odd, and $G = PSU_3(q)$, with $9 \mid (q+1)$, are the rank two simple groups that are not Qd(p)-free at some odd prime. The remaining simple groups $G = PSU_3(q)$, $q \geq 5$, are eliminated by the Borel-Smith conditions. The groups $PSU_3(3)$ and $PSU_3(4)$ have a linear actions on spheres with rank one prime power isotropy. We note that the group $G = PSU_3(3)$ does not satisfy the rank one intersection property.

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Algebras without (Fg)

Karin Erdmann

Assume A is a finite-dimensional selfinjective algebra. Then A-modules have support using $HH^*(A)$ with properties similar to support varieties defined via group cohomology, for group representations, provided A satisfies

(Fg) $HH^*(A)$ is noetherian and $\mathrm{Ext}^*(A/\mathbf{r}, A/\mathbf{r})$ is finitely generated over $HH^*(A)$

(see [5] and [10]). This condition is difficult to verify directly, but it has some consequences. In particular

Theorem [5]**.** *Assume A is selfinjective and satisfies (Fg). Then*
(1) A non-projectie indecomposable A-module M is Ω-periodic if and only if it has complexity $= 1$.
(2) If M is an A-module such that $\mathrm{Ext}^*_A(M, M)$ is finite-dimensional then M is projective.

We call M a *criminal* if $cx(M) = 1$ but M is not Ω-periodic, and we say M is *ext finite* if M is not projective and $\mathrm{Ext}^*(M, M)$ is finite-dimensional. Hence if an algebra has a criminal, or has an ext finite module then (Fg) fails.

In [2] it is proved that the four-dimensional local algebra

$$\Lambda = K\langle x, y\rangle/(x^2, y^2, xy + qyx)$$

for q not a root of 1 has finite-dimensional Hochschild cohomology, and hence it does not satisfy (Fg). As well, this algebra has criminals and these are ext finite; this had been discovered a while ago, see [8]. Similarly [7] investigate a class of weakly symmetric algebras with radical cube zero, which have a deformation parameter. They show show that if this parameter is not a root of unity then as well the algebra has finite-dimensional Hochschild cohomology, and hence does not satisfy (Fg). One can show directly that these algebras also have criminals, and these are ext finite: in fact the algebras of [2] and [7] belong to the class of weakly symmetric algebras with radical cube zero, studied in [1], [6], and in [6] has a classification of which of these satisfy (Fg).

For the algebras in this list, of finite complexity, it turns out that (Fg) fails if and only if the algebra is either the algebra of [2], or an algebra of [7]. Furthermore, these are precisely the weakly symmetric algebras with radical cube zero which have criminals, and which have ext finite modules.

We consider a class of weakly symmetric special biserial algebras, details are for example in [4] or [3]. These include the algebras above, but many others which occur in various contexts. One typical example is

$$\Gamma_q = k\langle x, y\rangle/(x^2, y^2, (xy)^2 + q(yx)^2)$$

with $q \neq 0$. If the field has characteristic 2 and $q = -1$ then Γ is isomorphic to the group algebra of the dihedral group of order 8. A socle deformation of a selfinjective algebra A is an algebra A' with $A'/\mathrm{soc}A' \cong A/\mathrm{soc}A$. For example, Γ_q is a socle deformation of Γ_{-1} and Γ_{-1} is symmetric.

A socle deformation of A has the same indecomposable non-projective modules, but the actions of Ω can be very different. We show that most special biserial symmetric algebras have socle deformations with criminals.

Theorem [3]. *Assume A is symmetric and special biserial with no simple periodic module. Then there is a socle deformation $A_{\mathbf{q}}$ which has criminals, unless possibly A is commutative, or one of a few exceptions.*

One would like to know what goes wrong with (Fg). For a typical example we have the answer.

Theorem. *Let $A := K\langle x, y\rangle/(x^2, y^2, (xy)^2 + q(yx)^2)$ with q non-zero.*
(i) If q is not a root of 1 then A has criminals.
(ii) $HH^n(A)$ is 1-dimensional for $n \geq 3$.
(iii) Let $\mathcal{N}$ be the largest homogeneous nilpotent ideal of $HH^*(A)$, then $HH^n(A) \subseteq \mathcal{N}$ for all $n \geq 1$.

Part (i) is a special case of [3]. To prove parts (ii) and (iii), we construct an explicit minimal bimodule resolution; this is not much more complicated than the construction in [2]. For (iii), it suffices to show that for $n \geq 2$, any homomorphism $\theta : \Omega^n(A) \to A$ maps into the radical of A, by Proposition 4.4 in [9].

Proving part (iii) for the algebras $A_{\mathbf{q}}$ with criminals in general is feasible, one can use the bimodule exact sequence in [4] to construct a part of a minimal bimodule resolution, and do induction.

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Automorphisms of fusion systems of finite groups of Lie type

Carles Broto

(joint work with Jesper M. Møller & Bob Oliver)

When p is a prime, G is a finite group, and $S \in \mathrm{Syl}_p G$, the *fusion system* of G over S is the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S, and whose morphisms are those homomorphisms between subgroups induced by conjugation in G. The fusion system $\mathcal{F}_S(G)$ can also be obtained from the p-complete classifying space $BG^\wedge_p$, up to equivalence of categories (cf. [2]). Then, we can define natural homomorphisms

$$\mathrm{Out}(G) \xrightarrow{\kappa_G} \mathrm{Out}(BG^\wedge_p) \xrightarrow{\mu_G} \mathrm{Out}(S, \mathcal{F}_S(G)) \qquad \text{and} \qquad \bar{\kappa}_G = \mu_G \circ \kappa_G\,,$$

where $\mathrm{Out}(BG^\wedge_p)$ stands for the group of homotopy classes of self-homotopy equivalences of $BG^\wedge_p$, and $\mathrm{Out}(S, \mathcal{F}_S(G)) \overset{\text{def}}{=} \mathrm{Aut}(S, \mathcal{F}_S(G))/\mathrm{Aut}_{\mathcal{F}_S(G)}(S)$, is the group of fusion preserving automorphisms of S, modulo those included in the fusion system.

We will use the above homomorphisms in order to compare $\mathrm{Out}(G)$, $\mathrm{Out}(BG^\wedge_p)$, and $\mathrm{Out}(S, \mathcal{F}_S(G)$, in case of finite groups of Lie type. The motivation comes from topology and from the theory of fusion systems, in particular the search of exotic fusion systems. Abstract (saturated) fusion systems where originally defined by Lluís Puig (cf. [3]). An abstract fusion system is called exotic if it is not the fusion system of a finite group. In [1], Andersen-Oliver-Ventura establish some technics in order to locate exotic fusion systems.

Definition. *An abstract fusion system $\mathcal{F}$ is called* tame *if*

(i) *There is a finite group G and $S \in \mathrm{Syl}_p(G)$ such that $\mathcal{F} \cong \mathcal{F}_S(G)$ (i.e. $\mathcal{F}$ is realized by G), and*
(ii) *κ_G is split surjective (we say that $\mathcal{F}$ is tamely realized by G).*

It is shown in [1] that if $\mathcal{F}$ is not tame, then there is an exotic fusion system $\tilde{\mathcal{F}}$ related to $\mathcal{F}$ by a number of extensions. As an outcome of the results discussed below we obtain:

Corollary. *Fusion systems of finite groups of Lie type are tame.*

By finite group of Lie type, defined in characteristic q, we understand a group G for which there exists a pair $(\overline{G}, \sigma)$ where $\overline{G}$ is a simple algebraic group over the algebraic closure $\bar{\mathbb{F}}_q$, and σ is a Steinberg endomorphism of $\overline{G}$ (i.e. an algebraic endomorphism with finite fixed subgroup), such that

$$G \cong O^{q'} C_{\overline{G}}(\sigma)\,,$$

that is, the maximal normal subgroup of index prime to q of the finite subgroup, $C_{\overline{G}}(\sigma)$, of $\overline{G}$ fixed by σ. G is called universal (resp. adjoint) if $\overline{G}$ is universal (resp. adjoint). The adjoint forms are simple groups with a few exceptions.

For a fixed prime p, our results must be stated separately in cases where the finite group of Lie type G is defined in characteristic $p = q$, or $q \neq p$.

Theorem A. *Let p be a prime. Assume that G is a finite group of Lie type, universal or adjoint, defined in characteristic p. Then, both κ_G and μ_G are isomorphisms, with the exceptions $G = Sz(2)$ and $PSL_3(2)$.*

In case $G = PSL_3(2)$, $p = 2$, we have $\mathrm{Out}(G) \cong C_2$, $\mathrm{Out}(BG^\wedge_p) \cong C_2 \times C_2$, and $\mathrm{Out}(S, \mathcal{F}_S(G)) \cong C_2$. $\bar{\kappa}_G$ is an isomorphism. If $G = Sz(2) \cong C_4 \rtimes C_4$, $p = 2$, then we have $\mathrm{Out}(G) \cong 1$, $\mathrm{Out}(BG^\wedge_p) \cong \mathrm{Out}(S, \mathcal{F}_S(G)) \cong C_2$. μ_G is an isomorphism.

The situation is different if the group G is defined in characteristic different from p. Among this class of groups, one can easily find examples of different groups G and H, with different outer automorphisms, but having homotopy equivalent p-complete classifying spaces, and therefore equivalent fusion systems over the respective Sylow p-subgroups. We write $G \sim_p H$ if this is the case (see [4]). The known coincidences of mod p cohomology rings for different finite groups of Lie type suggest many of these equivalences.

Theorem B. *Fix a pair of distinct primes p and q, and a group G of Lie type, universal or adjoint, defined in characteristic q. Assume that the Sylow p-subgroups of G are nonabelian. Then there is a prime $q^* \neq p$, and a group G^* of Lie type, universal or adjoint, respectively, defined in characteristic q^* such that*

(a) *$G^* \sim_p G$ and*
(b) *κ_{G^*} is split surjective.*

If, furthermore, p is odd or G^* *has universal type, then* μ_{G^*} *is an isomorphism, and hence* $\bar{\kappa}_{G^*}$ *is also split surjective.*

In order to complete the proof of the above Corollary one must handle separately the exceptions to Theorem A and the general case where G has abelian Sylow p-subgroups.

We present an example that illustrates Theorem B. Assume $p = 2$ and $G = PSL_2(17)$. In this case $\kappa_G\colon \mathrm{Out}(G) \longrightarrow \mathrm{Out}(BG^\wedge_p)$ is not surjective. For $G^* = PSL_2(81)$, we have that G and G^* have equivalent fusion systems at the prime 2, $G \sim_2 G^*$, and κ_{G^*} is an isomorphism, with $\mathrm{Out}(G^*) \cong C_2 \times C_4$ generated by diagonal and field automorphisms [2]. μ_{G^*} has kernel of order two, generated by a field automorphism so it is not split surjective. However, for the universal form $\tilde{G}^* = SL_2(81)$, both $\kappa_{\tilde{G}^*}$ and $\mu_{\tilde{G}^*}$ are isomorphisms.

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Cohomology of groups and $K(2)$-local homotopy theory

Hans-Werner Henn

This was a survey talk which aimed to highlight the role of group cohomology in chromatic stable homotopy theory.

In chromatic stable homotopy theory one localizes the stable homotopy category of spectra first at a natural prime p and then at a "chromatic prime n" where $n \geq 0$ is an integer. In case $n = 0$ this amounts to rationalization, for $n = 1$ one localizes with respect to complex K-theory modulo p. The case $n = 2$ has been under much investigation in the last 30 years with major advances in the last 10 years related to a better appreciation of the subject via group cohomology. Very little is known for $n > 2$.

The general picture is that for a fixed prime p the $K(n)$-local homotopy category is largely controlled by the category of continuous modules with an action of a profinite group $\mathbb{G}_n$ of dimension n^2. The group $\mathbb{G}_n$ is also known under the name extended Morava stabilizer group. It is a p-adic Lie group of dimension n^2; more precisely it is an extension of the group of units $\mathbb{S}_n$ in the central division algebra over $\mathbb{Q}_p$ of dimension n^2 and Hasse invariant $\frac{1}{n}$. This group acts on the Morava module $(E_n)_*X$ of a spectrum X and there is a spectral sequence (the $K(n)$-local Adams Novikov spectral sequence)

$$E_2^{s,t} = H^s(\mathbb{G}_n, (E_n)_t) \Longrightarrow \pi_{t-s}(L_{K(n)}S^0) \ .$$

starting from the continuous cohomology of $\mathbb{G}_n$ with coefficients in a certain profinite ring $(E_n)_*$ and converging towards the homotopy groups of X localized with respect to the n-th Morava K-theory at the prime p. If we fix n and vary p then this spectral sequence collapses at E_2 if p is sufficiently large and in this case calculating homotopy groups is the same as calculating group cohomology.

The case $n = 1$ has been well understood for a long time by work of Bousfield and Ravenel. In this case the group $\mathbb{G}_1$ is the group of units in the p-adics, i.e. $\mathbb{G}_1 = \mathbb{Z}_p^\times$, $E_1 = \mathbb{Z}_p[u^{\pm 1}]$ is p-completed complex K-theory and $\mathbb{G}_1$ acts by algebra maps via Adams operations. Furthermore the group cohomology calculation is fairly straightforward in this case.

The case $n = 2$ is significantly more complicated. The main problem is to calculate the continuous cohomology $H^*(\mathbb{S}_2^1, (E_n)_*X)$ where $\mathbb{S}_2^1$ is the kernel of a canonical homomorphism from $\mathbb{S}_2$ to the additive group of the p-adics. The group $\mathbb{S}_2^1$ is a virtual Poincaré duality group of dimension 3, and even a genuine Poincaré duality group of dimension 3 if $p > 3$. The trivial module $\mathbb{Z}_p$ for the group $\mathbb{S}_2^1$ therefore admits a projective resolution of length 3 if $p > 3$. At the small primes $p = 2$ and $p = 3$ the cohomological dimension is infinite and therefore no finite projective resolution can exist. However, in analogy to the case of discrete groups one can hope for finite resolutions whose terms are permutation modules on finite subgroups of $\mathbb{S}_2^1$. In fact, explicit resolutions of such a form can be constructed generalizing the resolution of the trivial $\mathbb{G}_1$-module $\mathbb{Z}_p$ in the case $n = 1$, given by

$$0 \to \mathbb{Z}_p[[\mathbb{Z}_p^\times/\mu]] \to \mathbb{Z}_p[[\mathbb{Z}_p^\times/\mu]] \to \mathbb{Z}_p \ .$$

Here μ is the finite subgroup of groups of unity in $\mathbb{Z}_p^\times$ and the map $\mathbb{Z}_p[[\mathbb{Z}_p^\times/\mu]] \to \mathbb{Z}_p[[\mathbb{Z}_p^\times/\mu]$ is given by multiplication with T, after identifying $\mathbb{Z}_p[[\mathbb{Z}_p^\times/\mu]$ with the Iwasawa algebra $\mathbb{Z}_p[[T]]$.

The following result has been proved in the case of the primes 2 and 3 by using information about the cohomology of suitable finite index subgroups of $\mathbb{S}_2^1$ which are themselves genuine Poincaré duality groups of dimension 3.

Theorem. *Let p be any prime. Then there is an exact complex of $\mathbb{Z}_p[[\mathbb{S}_2^1]]$-modules of the form*

$$0 \to C_3 \to C_2 \to C_1 \to C_0 \to \mathbb{Z}_p \to 0$$

such that $C_3 = C_0 = \mathbb{Z}_p[[\mathbb{S}_2^1]] \otimes_{\mathbb{Z}_p[G]} \mathbb{Z}_p$ and $C_2 = C_1 = \mathbb{Z}_p[[\mathbb{S}_2^1]] \otimes_{\mathbb{Z}_p[H]} M$ for suitable finite subgroups G and H of $\mathbb{S}_2^1$ and a suitable $\mathbb{Z}_p[H]$-module M. More precisely,

a) *if $p > 3$ then $G = H \cong \mathbb{F}_{p^2}^\times \rtimes Gal(\mathbb{F}_{p^2} : \mathbb{F}_p)$ is "the" unique (up to conjugation) maximal finite subgroup of $\mathbb{S}_2^1$ and M is a certain 2-dimensional $\mathbb{Z}_p$-free module for G. (*[L]*)*
b) *if $p = 3$ then $G \cong C_3 \rtimes Q_8$ and $H = SD_{16}$ are the two unique (up to conjugacy) maximal subgroups of $\mathbb{S}_2^1$ and M is a certain 1-dimensional $\mathbb{Z}_p$-free module for H. (*[GHMR], [HKM]*)*
c) *If $p = 2$ then $G = Q_8 \rtimes C_3$, $H = C_6$ and $M \cong \mathbb{Z}_2$ with trivial action of H. (*[GHMR] *unpublished,* [Be]*).*

Remark. In case (a) the resolution is a projective resolution.

These resolutions have been very successfully used for calculations with the Adams Novikov spectral sequence for $p \geq 3$ [L], [HKM], and most recently they have been used by Beaudry [Be] and Bobkova [Bo] to make interesting progress at the very difficult prime $p = 2$. The details of the calculation are quite involved.

However, some interesting structural insight has been gained in the course of these calculations.

- For example, for any n and any p the groups $H^s(\mathbb{G}_2, (E_n)_t)$ are finite abelian p-groups unless $t = 0$.
- For $t = 0$ the inclusion of the constants $\mathbb{Z}_p \subset (E_2)_0$ induces an isomorphism

$$H^*(\mathbb{S}_2, \mathbb{Z}_p) \cong H^*(\mathbb{S}_2, (E_2)_0) \ .$$

Unfortunately we can establish this isomorphism only because we can explicitly calculate the source and the target of this map and check by hand that the map is an isomorphism. The source is not very hard to calculate but the calculation of the target is quite involved. It would be very interesting to have a conceptual proof of this isomorphism and then deduce the result for the target from that for the source.

In any case, as a corollary one gets the rational homotopy of the $K(2)$-local sphere as an exterior algebra, namely

$$\pi_*(L_{K(2)}S^0) \otimes \mathbb{Q} \cong \Lambda_{\mathbb{Q}_p}(e_{-1}, e_{-3}) \ ,$$

at least if $p > 2$.

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Low dimensional cohomology of finite groups of Lie type and Linckelmann's gluing conjecture

Jesper Grodal

In my talk I presented a solution to Linckelmann's gluing conjecture for blocks [7, Conj. 4.2] (Problem 4 in [2, §IV.7]), when the block fusion system comes from a finite group of Lie type in defining characteristic. Our approach is to take a new look at the central p'–extensions of finite groups of Lie type in characteristic p, and then deduce the results about fusion systems from our results about groups; the results about groups should be of independent interest.

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Cosupport and stratification

Srikanth B. Iyengar

(joint work with Dave Benson, Henning Krause, Julia Pevtsova)

This is a progress report on a long-running collaboration between Dave Benson, Henning Krause, and myself aimed at understanding modular representations of finite groups using methods from commutative algebra and (abstract) homotopy theory. In recent years, we have also got interested in finite group schemes; Julia Pevtsova is now part of this endeavor. The fundamental object of our interest is $\mathsf{StMod}(kG)$, the stable module category of all (finite and infinite dimensional) modules over a finite group (or finite group scheme) G over a field k. This is a triangulated category with suspension $\Omega^{-1}(-)$, the inverse of the syzygy functor. The tensor product $M \otimes_k N$ of kG-modules, with the usual diagonal G-action, is inherited by $\mathsf{StMod}(kG)$, making it a tensor triangulated category. This category is compactly generated, and the subcategory of compact objects is equivalent to $\mathsf{stmod}(kG)$, the stable module category of finite dimensional kG-modules. Even if one is interested only in this subcategory it pays to work in the larger one, for there are natural constructions that result in infinite dimensional modules. And it is only in $\mathsf{StMod}(kG)$ that methods from homotopy theory can be readily applied.

Many questions that arise concerning the structure of $\mathsf{StMod}(kG)$ boil down to the following: *Given kG-modules M, N, when is M built out of N?* When I say N builds M, I mean that one can get to M from N using the operations in $\mathsf{StMod}(kG)$: taking infinite direct sums, mapping cones, and (this is important) tensor products with arbitrary kG-modules; in short, M is in $\mathrm{Loc}^{\otimes}(N)$, the tensor-ideal localising subcategory of $\mathsf{StMod}(kG)$ generated by N. When M and N are finite dimensional, if M is built out of N, then it can be *finitely* built out of N; this is by a standard argument which I learnt from Neeman [11], the key point being that M, N are compact in $\mathsf{StMod}(kG)$. The adjective “finitely” means that only finite direct sums are required; however, one has then to allow also for retracts.

One answer to the question above is in terms of $\operatorname{supp}_G M$, the *support* of M introduced by Benson, Carlson, and Rickard [1], and says: M is built out of N if (and only if) $\operatorname{supp}_G M \subseteq \operatorname{supp}_G N$; this is the main result of [4]. Another way to state this result is that assigning a subcategory $\mathcal{C}$ of $\mathsf{StMod}(kG)$ to the subset $\cup_{M\in\mathcal{C}} \operatorname{supp}_G M$ of $\operatorname{Proj} H^*(G,k)$ induces a one-to-one correspondence between the tensor-ideal localising subcategories of $\mathsf{StMod}(kG)$ and subsets of $\operatorname{Proj} H^*(G,k)$.

The crucial step in establishing this classification is to verify that for any $\mathfrak{p}$ in $\operatorname{Proj} H^*(G,k)$ and kG-modules M, N with $\operatorname{supp}_G(M) = \{\mathfrak{p}\} = \operatorname{supp}_G(N)$ one has $\operatorname{Loc}^{\otimes}(M) = \operatorname{Loc}^{\otimes}(N)$; equivalently, there is a non-zero map $M \otimes_k W \to N$, for some kG-module W; equivalently, the kG-module $\operatorname{Hom}_k(M,N)$ is not projective. In short, $\mathsf{StMod}(kG)$ is *stratified* by the action of the cohomology ring, $H^*(G,k)$.

The proof of the stratification of $\mathsf{StMod}(kG)$ in [4] is rather involved, and it goes through various triangulated categories of differential graded modules over differential graded algebras. Subsequently we realised that there is another invariant kG-modules, introduced in [2] and called *cosupport*, that plays a crucial role in all this, though it is somewhat hidden behind the scenes. Its relevance springs from the fact [2] that the property that $\mathsf{StMod}(kG)$ is stratified by $H^*(G,k)$ is equivalent (there is a precise statement to this effect applying to tensor triangulated categories) to cosuuport having the following properties:

$$\operatorname{cosupp}_G M = \varnothing \iff M \text{ is projective} \tag{1}$$

$$\operatorname{cosupp}_G \operatorname{Hom}_k(M,N) = \operatorname{supp}_G M \cap \operatorname{cosupp}_G N\,. \tag{2}$$

For the notion of cosupport defined in [2], and which is based on cohomology, the first property is clear but the second seems only accessible as a consequence of the stratification of $\mathsf{StMod}(kG)$.

This brings me to the preprint [5] where we introduce a notion of π-cosupport for modules over finite group schemes, based on the theory of π-points developed by Friedlander and Pevtsova [9]. Even for finite groups it extends Carlson’s theory of rank varieties for group algebras of elementary abelian groups [6] so that it applies to all finite groups, and gives an approach to studying representations that is based more on linear algebra than in cohomology. With π-support, introduced in [9], and π-cosupport replacing their cohomological counterparts, it is not hard to establish formula (2), even for arbitrary finite group schemes; see [5].

For *finite groups*, we have been able to verify also (1) for π-cosupport directly: Chouinard’s theorem allows us to focus on the case when G is an elementary abelian p-group and then the argument for (1) is a variant of the proof of Dade’s Lemma (for infinite dimensional modules) from [1]. As a consequence one gets a much shorter proof than the one in [4] of the stratification of $\mathsf{StMod}(kG)$.

This brings me to work in progress, of which [10] is a preview: Property (1) for π-cosupport holds also for any finite group scheme G. However, we have only been able to verify this by first establishing that $\mathsf{StMod}(kG)$ is stratified by $H^*(G,k)$. The proof of the stratification in this context is more delicate (read also 'interesting') than for finite groups, and weaves ideas and techniques from theory of π-points from [9] and the cohomological one from [3, 4]. To return to where we started: What all this means is that, even over finite group schemes, we have a satisfactory answer to the question: when does one kG-module build another? Another consequence of the stratification is that the π-versions of cosupport and support coincide with the cohomological ones, and we have now two different perspectives, and different sets of tools, to bring to bear on the representation theory for finite group schemes.

In summary, what I want to emphasise is that, like support, cosupport is an interesting and important invariant of kG-modules, and not only from the perspective of stratification. In fact, like support, there is a notion of cosupport for objects in any compactly generated triangulated category with a ring action [3, 4]. In the special case of the derived category of a commutative noetherian ring, cosupport is connected to completions, just as support is related to local cohomology, and the close connection between these two invariants is a manifestation of the adjointness relation between completions and local cohomology discovered by Greenlees and May [8], and further elaborated in the work of Dwyer and Greenlees [7].

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Localizing subcategories for finite group schemes

Julia Pevtsova

(joint work with Dave Benson, Srikanth B. Iyengar, Henning Krause)

Let k be a field of positive characteristic. An affine group scheme G over k is a representable functor from the category of commutative k-algebras to groups. The coordinate algebra, denoted $k[G]$, is a commutative Hopf k-algebra. An affine group scheme G is *finite* if the coordinate algebra is finite dimensional over k. In that case we define the *group algebra* kG to the the linear dual of $k[G]$. Hence, kG is a finite dimensional cocommutative Hopf algebra. Likewise, starting with any finite dimensional cocommutative Hopf algebra, its linear dual is a coordinate algebra of a finite group scheme. One therefore has an equivalence of categories:

$$\left\{ \begin{array}{c} \text{finite group} \\ \text{schemes} \end{array} \right\} \sim \left\{ \begin{array}{c} \text{finite dimensional co-} \\ \text{commutative Hopf algebras} \end{array} \right\}$$

Via this equivalence, one can identify representations of G with kG-modules; for the rest of this note we shall refer to representations of G as *G-modules*. Examples of finite group schemes include finite groups, restricted Lie algebras and Frobenius kernels of algebraic groups. A finite group scheme is *unipotent* if the group algebra kG is local and is *abelian* if kG is commutative.

Let G will be a finite group scheme defined over k. Since the group algebra kG is Frobenius (see, for example, [16, I.6]), the projective modules are injective and, moreover, one can construct the stable module category $\mathsf{StMod}\, G$. Recall that the objects of $\mathsf{StMod}\, G$ are G-modules, whereas the Hom-sets are defined as follows:

$$\underline{\mathrm{Hom}}(M,N) := \frac{\mathrm{Hom}_G(M,N)}{\mathrm{PHom}_G(M,N)}$$

with $\mathrm{PHom}_G(M,N)$ being the subset of all G-maps between M and N which factor through a projective G-module. The category $\mathsf{StMod}\, G$ is a compactly generated tensor triangulated category with the compact objects being the finite dimensional G-modules. This subcategory is denoted $\mathsf{stmod}\, G$.

A subcategory $\mathcal{C}$ of $\mathsf{StMod}\, G$ is *localizing* if it is a full triangulated subcategory closed under set-indexed direct sums. It is *tensor ideal* if for any $M \in \mathsf{StMod} G$, $C \in \mathcal{C}$, we have $M \otimes C \in \mathcal{C}$. A subcategory $\mathcal{C}$ of $\mathsf{stmod}\, G$ is thick (or épaisse) if it is a full triangulated subcategory closed under taking direct summands.

The cohomology ring $H^*(G,k) = \mathsf{Ext}^*_G(k,k)$ is a graded commutative k-algebra which is finitely generated by a fundamental result of Friedlander and Suslin [15]. The following is the main theorem of this note:

Theorem 1. *For any finite group scheme G, there is a one-to-one correspondence*

$$\left\{ \begin{array}{c} \textit{Localizing tensor-ideal} \\ \textit{subcategories of } \mathsf{StMod}\, G \end{array} \right\} \sim \left\{ \begin{array}{c} \textit{subsets of} \\ \mathsf{Proj}\, H^*(G,k) \end{array} \right\}$$

which restricts to one-to-one correspondence

$$\left\{ \begin{array}{c} \textit{Thick tensor-ideal} \\ \textit{subcategories of } \mathsf{stmod}\, G \end{array} \right\} \sim \left\{ \begin{array}{c} \textit{specialization closed} \\ \textit{subsets of } \mathsf{Proj}\, H^*(G,k) \end{array} \right\}$$

The correspondence is given explicitly as follows:

$$\mathcal{C} \longmapsto V = \bigcup_{M \in \mathcal{C}} \operatorname{supp} M$$

$$\mathcal{C} = \{M \in \mathsf{StMod}\, G \mid \operatorname{supp} M \subset V\} \longleftarrow V$$

This theorem generalizes the main result in [8] where it was proved for finite groups. An essential feature of the argument in [8] was the fact that various properties of modules for finite groups, such as projectivity, are detected upon restriction to elementary abelian p-subgroups. Unfortunately, this does not generalize to arbitrary finite group schemes. The approach we use to prove Theorem 1 for any finite group scheme is substantially different and relies heavily on the notion of cosupport introduced in [6]. In particular, it yields a completely new proof of the classification theorem even for finite groups. In fact, it yields two new proofs! In this note we'll sketch the strategy which yields the theorem in full generality. A simpler, and conceptually very pleasing, new proof which works only for finite groups is alluded to in S. Iyengar's note in the same volume.

The support of M, $\operatorname{supp} M$, is a geometric invariant associated to any G-module M which we now describe. In fact, to prove Theorem 1, we need to develop two notions of support, and parallel notions of cosupport. The first theory of support and cosupport is due to Benson-Iyengar-Krause [7], [9], [8], [6], building on the earlier work of Rickard in representation theory [19]. To each homogeneous prime ideal $\mathfrak{p}$ (strictly contained in the irrelevant ideal) of $H^*(G,k)$ we associate a universal module (usually infinite dimensional) $\Gamma_{\mathfrak{p}}(k)$ (see [7]). Then the *cohomological* support and cosupport are defined as follows:

Definition 2 ([7], [6]).

$$\operatorname{supp}(M) := \{\mathfrak{p} \in \mathsf{Proj}\, H^*(G,k) \mid \Gamma_{\mathfrak{p}}(k) \otimes_k M \text{ is not projective}\}.$$

$$\operatorname{cosupp}(M) := \{\mathfrak{p} \in \mathsf{Proj}\, H^*(G,k) \mid \mathsf{Hom}_k(\Gamma_{\mathfrak{p}}(k), M) \text{ is not projective}\}.$$

The general philosophy captured beautifully by Balmer in [1] prescribes that to classify tensor ideal subcategories in a tensor triangulated category one needs "good" theory of supports. Benson-Iyengar-Krause support and cosupport defined above satisfy many of the properties expected of a "good" theory but their cohomological nature renders them unsuitable for testing behavior with respect to tensor products and function objects. To repair this, we introduce another theory, that of π-supports and π-cosupports. For a field extension K/k, we denote by G_K the finite group scheme over K with the coordinate algebra $K[G_K] := K \otimes_k k[G]$.

Definition 3 ([14], [13]). A *π-point* of G, defined over a field extension K of k, is a morphism of K-algebras

$$\alpha : K[t]/(t^p) \to KG_K$$

which factors through the group algebra of a unipotent abelian subgroup scheme C of G_K, and such that KG_K is flat when viewed as a left (equivalently, as a right) module over $K[t]/(t^p)$ via α.

We say that a pair of π-points $\alpha : K[t]/(t^p) \to KG_K$ and $\beta : L[t]/(t^p) \to LG_L$ are equivalent if they satisfy the following condition: *for any finite dimensional kG-module M, the module $\alpha^*(K \otimes_k M)$ is projective if and only if $\beta^*(L \otimes_k M)$ is projective.* The set of equivalence classes of π-points is denoted $\Pi(G)$; it has a naturally defined Zariski topology. By [14, Theorem 3.6], there is a natural homeomorphism $\Pi(G) \simeq \mathsf{Proj}\, H^*(G,k)$, which allows us to identify these two spaces. Via this identification, we associate to each homogeneous prime ideal $\mathfrak{p} \subset H^*(G,k)$, $\mathfrak{p} \neq H^{*>0}(G,k)$, a π-point $\alpha_\mathfrak{p}$ whose equivalence class in $\Pi(G)$ coincides with the point $\mathfrak{p}$ on $\mathsf{Proj}\, H^*(G,k)$. By [13, 4.6], [11, 2.1], the definition given below is independent of which representative we choose.

Definition 4. The *π-support* of M is the subset of $\mathsf{Proj}\, H^*(G,k)$ defined by

$$\pi\text{-}\operatorname{supp}(M) := \{\mathfrak{p} \in \mathsf{Proj}\, H^*(G,k) \mid \alpha_\mathfrak{p}^*(K \otimes_k M) \text{ is not projective}\}.$$

The *π-cosupport* of M is the subset of $\mathsf{Proj}\, H^*(G,k)$ defined by

$$\pi\text{-}\operatorname{cosupp}(M) := \{\mathfrak{p} \in \mathsf{Proj}\, H^*(G,k) \mid \alpha_\mathfrak{p}^*(\mathsf{Hom}_k(K,M)) \text{ is not projective}\}.$$

The usefulness of π-support and π-cosupport is postulated in the following theorem.

Theorem 5. *Let M and N be G-modules. Then there are equalities*

$$\pi\text{-}\operatorname{supp}(M \otimes_k N) = \pi\text{-}\operatorname{supp}(M) \cap \pi\text{-}\operatorname{supp}(N),$$
$$\pi\text{-}\operatorname{cosupp}(\mathsf{Hom}_k(M,N)) = \pi\text{-}\operatorname{supp}(M) \cap \pi\text{-}\operatorname{cosupp}(N).$$

To prove Theorem 1, we need to identify cohomological and π-supports. This can be done formally following the strategy developed in [5] once we know the following detection result.

Theorem 6. *Let G be a finite group scheme, and M be a G-module. Then M is projective if and only if $\pi\text{-}\operatorname{supp}(M) = \emptyset$.*

This detection theorem is an ultimate generalization of the famous Dade's lemma [12]. It builds on the work of many authors, see [5], [2], [17], [18]. In this generality the result was stated in [13] but the proof contained an error. The complete proof is to appear in [10].

Theorem 6 implies the following two properties which constitute an essential step in the proof of Theorem 1.

Corollary 7. (1) $\pi\text{-}\operatorname{supp} \Gamma_\mathfrak{p}(k) = \mathfrak{p}$;
(2) *For any G-module M, $\pi\text{-}\operatorname{supp}(M) = \operatorname{supp}(M)$.*

By the work of Benson-Iyengar-Krause [7], Theorem 1 follows from the *stratification* of $\mathsf{StMod}\, G$ by the action of the cohomology ring $H^*(G,k)$. Explicitly, one needs to show the following:

For any point $\mathfrak{p} \in \mathsf{Proj}\, H^*(G,k)$, the tensor ideal localizing subcategory

$$\Gamma_\mathfrak{p}(\mathsf{StMod}\, G) = \{M \in \mathsf{StMod}\, G \mid \operatorname{supp}(M) \subseteq \mathfrak{p}\}$$

is minimal, that is, does not contain any non-trivial proper tensor ideal localizing subcategories. This is equivalent to showing that for any non-zero objects $M, N \in$

$\Gamma_{\mathfrak{p}}(\mathsf{StMod}\, G)$, the G-module $\mathsf{Hom}_k(M,N)$ is not projective. It is at this point that the function object formula for cosupport (5) becomes of utmost importance.

With these ingredients in place, the proof of Theorem 1 proceeds in two steps. First, we show that for any closed point $\mathfrak{m} \in \mathsf{Proj}\, H^*(G,k)$, the subcategory $\Gamma_{\mathfrak{m}}(\mathsf{StMod}\, G)$ is minimal. To reduce the problem from any point $\mathfrak{p}$ on $\mathsf{Proj}\, H^*(G,k)$ to a closed point on $\mathsf{Proj}\, H^*(G_K,K)$ for some field extension K/k, we use a commutative algebra calculation with Carlson modules (or, equivalently, Koszul objects) to show the following:

Theorem 8. *Let $\mathfrak{p}$ be a point on $\mathsf{Proj}\, H^*(G,k)$. Let K be the residue field at $\mathfrak{p}$ and let $\mathfrak{m}$ be a closed point in $\mathsf{Proj}\, H^*(G_K,K)$ "lying over" $\mathfrak{p}$. Then $\Gamma_{\mathfrak{p}}(k) \in \mathrm{Loc}^{\otimes}(\Gamma_{\mathfrak{m}}(K){\downarrow_G})$, where $\mathrm{Loc}^{\otimes}(\Gamma_{\mathfrak{m}}(K){\downarrow_G})$ is the minimal tensor ideal localizing subcategory containing $\Gamma_{\mathfrak{m}}(K){\downarrow_G}$.*

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Techniques for studying modules of constant Jordan type

Shawn Baland
(joint work with Kenneth Chan)

Let p be a prime number, $E \cong (\mathbb{Z}/p)^r$ an elementary abelian p-group of rank r and k an algebraically closed field of characteristic p. The purpose of this talk was to further investigate a curious functorial relationship between kE-modules of constant Jordan type and vector bundles on the projective space $\mathbb{P}_k^{r-1}$ due to Benson and Pevtsova [4]. The general setup is as follows: Choosing a collection of pairwise commuting generators $g_1, \ldots, g_r$ for E, one sets $X_i = g_i - 1 \in kE$. It is easy to verify that the elements X_i generate $\mathsf{Rad}(kE)$. For any r-tuple $\alpha = (\lambda_1, \ldots, \lambda_r) \in k^r$, one then defines the element $X_\alpha = \lambda_1 X_1 + \cdots + \lambda_r X_r \in kE$. The fact that k has characteristic p forces $X_\alpha^p = 0$. If α is not the zero r-tuple, it follows that $1 + X_\alpha$ has order p in the multiplicative group of units $kE^\times$, hence $\langle 1 + X_\alpha \rangle \cong \mathbb{Z}/p$. The subalgebra $k\langle 1 + X_\alpha \rangle$ is called a *cyclic shifted subgroup* of kE.

Now let M be a finite dimensional kE-module. Because each X_α (for $\alpha \neq 0$) is p-nilpotent, the Jordan canonical form of X_α acting as a k-linear endomorphism of M consists of Jordan blocks whose eigenvalues are all zero and whose lengths are at most p. The *Jordan type* of X_α on M is the partition

$$\mathsf{JType}(X_\alpha, M) = [p]^{a_p}[p-1]^{a_{p-1}} \ldots [1]^{a_1}$$

of $\dim_k(M)$, where X_α acts on M via a_j Jordan blocks of length j. We remark that this is simply the isomorphism type of $M{\downarrow}_{k\langle 1+X_\alpha \rangle}$ as a $k(\mathbb{Z}/p)$-module.

Definition (Carlson, Friedlander, Pevtsova [5])**.** A finite dimensional kE-module M has *constant Jordan type* if the partition $\mathsf{JType}(X_\alpha, M)$ is independent of the choice of non-zero $\alpha \in k^r$. In this case, if $\mathsf{JType}(X_\alpha, M) = [p]^{a_p} \ldots [1]^{a_1}$ for each non-zero α, then we call $[p]^{a_p} \ldots [1]^{a_1}$ the *(constant) Jordan type* of M.

As shown in [5], the modules of constant Jordan type form a full subcategory $\mathsf{cJt}(kE)$ of $\mathsf{mod}(kE)$ closed under direct sums, direct summands, tensor products over k, k-linear duals and Heller shifts. One of the main objectives in studying these modules is to determine which partitions are realised as the Jordan types of kE-modules of constant Jordan type. This turns out to be a very hard problem. Amongst the limited information we have in this regard, Benson [3] proved that if $r \geq 2$ and $p \geq 5$, then there does not exist a kE-module of constant Jordan type $[p]^n[a]$ for $2 \leq a \leq p-2$ and $n \geq 0$.

In [1], the current author used the theory of Chern classes for vector bundles on $\mathbb{P}_k^{r-1}$ to study modules of constant Jordan type with only two blocks of length less than p. This was based on the work of Benson and Pevtsova [4], who constructed functors

$$\mathcal{F}_i\colon \mathsf{mod}(kE) \longrightarrow \mathsf{coh}(\mathbb{P}_k^{r-1})$$

for $1 \leq i \leq p$. These are defined as follows:

Let $V = \mathrm{span}_k\{X_1, \dots, X_r\}$ and let $Y_i = X_i^{\#}$ be the corresponding elements in the dual vector space $V^{\#}$. The Y_i then act as coordinate functions on V, hence we consider $\mathbb{P}_k^{r-1}$ as $\mathsf{Proj}\, k[Y_1, \dots, Y_r]$. For a finite dimensional kE-module M, we then define the coherent sheaf $\widetilde{M} = M \otimes_k \mathcal{O}_{\mathbb{P}_k^{r-1}}$ on $\mathbb{P}_k^{r-1}$. For $n \in \mathbb{Z}$, Friedlander and Pevtsova [6] constructed the sheaf morphisms

$$\theta_M\colon \widetilde{M}(n) \longrightarrow \widetilde{M}(n+1)$$

defined locally via $m \otimes f \mapsto \sum X_i m \otimes Y_i f$, where $m \in M$ and f is a homogeneous rational function of degree n in the Y_i. The idea in considering these maps is that, for a closed point $\overline{\alpha} = [\lambda_1 : \dots : \lambda_r] \in \mathbb{P}_k^{r-1}$, the fibre of θ_M at $\overline{\alpha}$ is (up to a scalar multiple) the k-linear map $X_\alpha\colon M \to M$. Benson and Pevtsova [4] later defined the subquotients

$$\mathcal{F}_i(M) = \frac{\mathsf{Ker}\,\theta_M \cap \mathsf{Im}\,\theta_M^{i-1}}{\mathsf{Ker}\,\theta_M \cap \mathsf{Im}\,\theta_M^{i}}$$

of $\widetilde{M}$. In [4], those authors showed that a kE-module M has constant Jordan type $[p]^{a_p} \dots [1]^{a_1}$ if and only if $\mathcal{F}_i(M)$ is a vector bundle of rank a_i on $\mathbb{P}_k^{r-1}$ for each $1 \leq i \leq p$. They also proved that the restricted functor $\mathcal{F}_1\colon \mathsf{cJt}(kE) \to \mathsf{vec}(\mathbb{P}_k^{r-1})$ is essentially surjective up to a Frobenius twist.

The goal of the work being presented was to understand how certain geometric concepts translate into the world of representation theory under the functors $\mathcal{F}_i$. For example, a common technique of the algebraic geometer is to take a vector bundle on $\mathbb{P}_k^{r-1}$ and study its restriction to a line $L \subseteq \mathbb{P}_k^{r-1}$. A consequence of our main result is that restricting the vector bundle $\mathcal{F}_i(M)$ to L is equivalent, in an appropriate sense, to restricting the module M to a corresponding rank two shifted subgroup of kE.

In this direction, we recall that for $s \leq r$, a *rank s-shifted subgroup* of kE is an embedding $\phi\colon kE' \hookrightarrow kE$ of k-algebras, where kE' is the group algebra of an elementary abelian p-group of rank s. We let $T_1, \dots, T_s$ be a choice of generators of $\mathsf{Rad}(kE')$. In order to study linear subvarieties of $\mathbb{P}_k^{r-1}$ (e.g., a line L), it suffices to consider embeddings ϕ for which $\phi(T_j) = \sum_{i=1}^{r} a_{ij} X_i$, the a_{ij} being scalars in k. Similar to our notation for kE, we define $U = \mathrm{span}_k\{T_1, \dots, T_s\}$, and we also let $Z_j = T_j^{\#}$ denote the dual elements in $U^{\#}$. The upshot of this setup is that the matrix $\mathbf{A} = (a_{ij})$ induces an injective linear map $\mathbf{A}\colon U \hookrightarrow V$, which then induces a surjective linear map $\mathbf{A}^t\colon V^{\#} \twoheadrightarrow U^{\#}$. This, in turn, gives rise to a surjective homomorphism of graded rings $k[Y_1, \dots, Y_r] \twoheadrightarrow k[Z_1, \dots, Z_s]$. Applying the Proj functor then induces the desired closed immersion $f\colon \mathbb{P}_k^{s-1} \hookrightarrow \mathbb{P}_k^{r-1}$. We emphasise that any linear subvariety of $\mathbb{P}_k^{r-1}$ may be constructed in this way.

Our main result answers the following question: If M is a kE-module of constant Jordan type, then we know that $\mathcal{F}_i(M)$ is a vector bundle on $\mathbb{P}_k^{r-1}$, whence $f^*\mathcal{F}_i(M)$ is a vector bundle on $\mathbb{P}_k^{s-1}$. On the other hand, $M\downarrow_{kE'}$ is a kE'-module of constant Jordan type, so $\mathcal{F}_i(M\downarrow_{kE'})$ is also a vector bundle on $\mathbb{P}_k^{s-1}$. One is left to wonder whether or not these vector bundles are one and the same.

Theorem (Baland, Kenneth Chan [2]). *The diagram of functors*

$$\begin{array}{ccc} \mathsf{cJt}(kE) & \xrightarrow{\mathcal{F}_i} & \mathsf{vec}(\mathbb{P}_k^{r-1}) \\ {\scriptstyle\downarrow_{kE'}}\Big\downarrow & & \Big\downarrow{\scriptstyle f^*} \\ \mathsf{cJt}(kE') & \xrightarrow[\mathcal{F}_i]{} & \mathsf{vec}(\mathbb{P}_k^{s-1}) \end{array}$$

commutes up to natural isomorphism.

Our long term goal is to use this technology to (hopefully) answer various open questions about modules of constant Jordan type using techniques from algebraic geometry, such as the realisation problem for Jordan types stated above.

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Ghys's conjecture on finite group actions on manifolds

Ignasi Mundet i Riera

A group G is said to be Jordan if there is some constant C such that any finite subgroup Γ of G contains an abelian subgroup whose index in Γ is at most C. This terminology comes from a classic theorem of Camille Jordan, which states that $\mathrm{GL}(n, \mathbb{C})$ is Jordan for every n. Jordan's theorem implies that any finite dimensional Lie group with finitely many connected components is Jordan (by the existence and uniqueness up to conjugation of maximal compact subgroups and Peter–Weyl's theorem).

Around twenty years ago, Étienne Ghys conjectured that the diffeomorphism group of any smooth compact manifold is Jordan.

1. The diffeomorphism group of smooth manifolds is known to be Jordan in the following cases.

(A) Compact 2-dimensional manifolds (easy exercise).
(B) Compact 3-dimensional manifolds (Zimmermann, [9]).
(C) Closed n-dimensional manifolds M with cohomology classes $\alpha_1, \dots, \alpha_n \in H^1(M;\mathbb{Z})$ such that $\alpha_1 \cup \dots \cup \alpha_n \neq 0$ (M., [5]).
(D) Compact manifolds, possibly with boundary, with nonzero Euler characteristic (M., [3, 4]).
(E) Open contractible manifolds (M., [4]).
(F) Homology spheres (M., [4]).

Some of the ingredients in the proofs of (D), (E), (F):

(I) Given a group G, define $\mathcal{P}(G) = \{P \leq G \mid P \text{ finite } p\text{-group, any } p\}$ and

$$\mathcal{T}(G) = \{\Gamma \leq G \mid \Gamma \text{ finite, } \Gamma \simeq P \rtimes Q,\ P \text{ abelian } p\text{-group, } Q \text{ abelian } q\text{-group, } p \neq q\}.$$

Theorem (M., Alexandre Turull). Let G be a group and suppose there exist C, d such that any $\Gamma \in \mathcal{P}(G) \cup \mathcal{T}(G)$ has an abelian subgroup $A \leq \Gamma$ satisfying $[\Gamma : A] \leq C$ and $\operatorname{rk} A \leq d$. Then G is Jordan.

The proof of this theorem uses the classification of finite simple groups.

(II) By a theorem of Mann and Su [2], for any manifold M satisfying $\sum b_j(M) < \infty$ we have

$$d(M) = \max\{r \mid \exists \text{ prime } p \text{ and monomorphism } (\mathbb{Z}/p\mathbb{Z})^r \hookrightarrow \operatorname{Diff}(M)\} < \infty.$$

So for any finite abelian subgroup $A < \operatorname{Diff}(M)$ we have $\operatorname{rk} A \leq d(M)$.

(III) To deal with $\mathcal{P}(\operatorname{Diff}(M))$ in (D) and (E) combine: (1) fixed point theorems for finite p-group actions, (2) Lemma: if $H < \operatorname{Diff}(M)$ is finite, $M^H \neq \emptyset$ and M connected, then $H \hookrightarrow \operatorname{GL}(\dim M, \mathbb{R})$, (3) Jordan's theorem. To deal with $\mathcal{P}(\operatorname{Diff}(M))$ in (F), use a theorem of Dotzel and Hamrick.

(IV) To deal with $\mathcal{T}(\operatorname{Diff}(M)) \ni \Gamma \simeq P \rtimes Q$ study the action of Q on the normal bundle of $M^P \hookrightarrow M$.

2. Suppose that M is a compact manifold, possibly with boundary, such that $H^*(M;\mathbb{Z})$ has no torsion and is supported in even degrees. Then [4] there exists a constant C so that any finite group Γ acting effectively and smoothly on M has an abelian subgroup $A \leq \Gamma$ with the properties that $[\Gamma : A] \leq C$ and $\chi(M^A) = \chi(M)$.
Corollary. For any n there is some constant C such that for any smooth action of a finite group Γ on the n-dimensional closed disk D^n there exists some $x \in D^n$ such that $[\Gamma : \Gamma_x] \leq C$.

Note that for big enough n there exist smooth finite group actions on D^n without fixed points (Floyd–Richardson 1959, Oliver 1975).

3. Popov [8] found in 2013 an open connected 4-manifold whose diffeomorphism group is not Jordan. Csikós, Pyber and Szabó [1] proved in 2014 that $\operatorname{Diff}(T^2 \times S^2)$ is not Jordan, thus giving the first counterexample to Ghys's conjecture. The construction in [1] can be generalized to yield.
Theorem. If M is a smooth manifold supporting an effective action of $\mathrm{SU}(2)$ or $\mathrm{SO}(3,\mathbb{R})$ then $\operatorname{Diff}(T^2 \times M)$ is not Jordan.

Combining this with case (C) in §**1** and using the fact that any compact connected nonabelian Lie group contains a subgroup isomorphic to SU(2) or $SO(3,\mathbb{R})$ we deduce.

Corollary. If M is a smooth compact n-dimensional manifold admitting classes $\alpha_1, \dots, \alpha_n \in H^1(M;\mathbb{Z})$ such that $\alpha_1 \cup \dots \cup \alpha_n \neq 0$ then any compact connected Lie group acting smoothly and effectively on M is abelian.

Using a result of Olshanskii we prove.

Theorem (M., [6]). For any $\epsilon > 0$ there exist a, b such that $T^a \times S^b$ admits effective smooth actions of arbitrarily large p-groups Γ all of whose abelian subgroups have at most $|\Gamma|^\epsilon$ elements.

4. Choose elements $t \in T^2$, $s \in S^2$ and orientations of T^2 and S^2. Define for any symplectic form ω on $T^2 \times S^2$

$$\alpha(\omega) = \int_{T^2 \times \{s\}} \omega, \qquad \beta(\omega) = \int_{\{t\} \times S^2} \omega,$$

$$\lambda(\omega) = \max\left\{ \left(2\mathbb{Z} \cap \left(-\infty, \left| \frac{2\alpha(\omega)}{\beta(\omega)} \right| \right) \right) \cup \{1\} \right\}.$$

Theorem (M., [7]). Let ω be a symplectic form on $T^2 \times S^2$. Any finite subgroup $\Gamma < \mathrm{Symp}(T^2 \times S^2, \omega)$ contains an abelian subgroup $A \leq \Gamma$ such that

$$[\Gamma : A] \leq \max\{144, 6\lambda(\omega)\}.$$

If $\lambda(\omega) \geq 8$ then there exists a finite subgroup $\Gamma < \mathrm{Symp}(T^2 \times S^2, \omega)$ all of whose abelian subgroups $A \leq \Gamma$ satisfy $[\Gamma : A] \geq 6\lambda(\omega)$.

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Cellular properties of fusion systems

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(joint work with Alberto Gavira)

In 1990s, E. Dror-Farjoun and W. Chachólski generalized the concept of CW-complex, spaces build from spheres by means pointed homotopy colimits. Let A be a pointed space and let $\mathcal{C}(A)$ denote the smallest collection of pointed spaces that contains A and it is closed by weak equivalences and pointed homotopy colimits. A pointed space X is A-*cellular* if $X \in \mathcal{C}(A)$. Moreover, there exists an augmented idempotent endofunctor $CW_A \colon Spaces_* \to Spaces_*$ such that for all pointed space X, the space $CW_A X$ is A-cellular and the augmention map $c_X \colon CW_A X \to X$ is an A-equivalence, that means, it is induced a weak equivalence in pointed mapping space $(c_X)_* \colon \mathrm{map}_*(A, CW_A X) \to \mathrm{map}_*(A, X)$. Roughly speaking, $CW_A X$ is the best A-cellular approximation of X. We will say that $CW_A X$ is the A-*cellularization* of X and the map $c_X \colon CW_A X \to X$ is the A-*cellular approximation* of X. See [5] for more details about the construction and main properties of the functor CW_A.

Let p be a prime. Let G be a finite group such that p divides de order of G and fix S to be a Sylow p-subgroup of G. In the stable homotopy category, the stable transfer map $t \colon \Sigma^\infty BG^\wedge_p \to \Sigma^\infty BS$ provides a retraction showing that $\Sigma^\infty BG^\wedge_p$ is a stable retract of $\Sigma^\infty BS$. In the terminology of the previous paragraph we say that $\Sigma^\infty BG^\wedge_p \in C(\Sigma^\infty BS)$. The question we solve in this project is the one: $BG^\wedge_p \in C(BS)$?

We will approach this question in the more general context introduced by Lluís Puig and Broto-Levi-Oliver of saturated fusion systems and p-local finite groups. Given a finite p-group S, p a prime, a *fusion system* over S is a subcategory of the category whose objects are the subgroup of S and morphisms are the injective homomorphisms, containing those which are induced by conjugation of elements of S. A fusion system $\mathcal{F}$ is *saturated* if it verifies certain axioms such as would be holded if S were a Sylow p-subgroup of a finite group. These ideas were develop by L. Puig in an unpublished notes. The notion of classifying space was formulated by C. Broto, R. Levi and B. Oliver in [2], where the notion of "centric linking system" (or "p-local finite group") associated to saturated fusion systems appears. Recently, A. Chermak [4] proved the existence and uniqueness of centric linking system over saturated fusion system, that means, each saturated fusion system $\mathcal{F}$ has a unique (up to isomorphism) centric linking system $\mathcal{L}$ over $\mathcal{F}$, and so a unique (up to homotopy equivalence) classifying space $B\mathcal{F} := |\mathcal{L}|^\wedge_p$.

Question 1. Given a P a finite p-group and $(S, \mathcal{F}, \mathcal{L})$ a p-local finite group, when $B\mathcal{F} \in C(BS)$?

Previous results were obtained by R. Flores [6], R. Flores-R. Foote [7] and R. Flores-J.Scherer [8] when G is a finite group generated by elements of order p and $P = \mathbb{Z}/p$.

To approach this question we will use Chacholsky's strategy. Chachólski [3, Theorem 20.5] describes a method to compute the A-cellular aproximation of X.

Let C be the homotopy cofibre of the evaluation $\bigvee_{[A,X]_*} A \to X$. Then $CW_A X$ is the homotopy fibre of $X \to P_{\Sigma A} C$.

Let C be the homotopy cofibre of the evaluation map $ev\colon \bigvee_{[BP,B\mathcal{F}]_*} BP \to B\mathcal{F}$. Then $CW_{BP}(B\mathcal{F})$ is homotopy equivalent to the homotopy fibre of the composite $r\colon B\mathcal{F} \to C \to P_{\Sigma BP} C$, where $P_{\Sigma BP}$ denotes the ΣBP-nullification functor defined by A. K. Bousfield in [1]. We proved that $CW_{BP}(B\mathcal{F}) \simeq B\mathcal{F}$ if and only if the map $r^\wedge_p$ is null-homotopic. Therefore, the BP-cellularity of $B\mathcal{F}$ is equivalent to the homotopy nullity of the map $r^\wedge_p$. To do that, we study the kernel of $r^\wedge_p$ in the sense of D. Notbohm introduced in [9] for maps from classifying space of compact Lie groups.

Definition 2. Let $(S, \mathcal{F}, \mathcal{L})$ be a saturated fusion system and let Z be a p-complete and $\Sigma B\mathbb{Z}/p$-null space. Let $f\colon B\mathcal{F} \to Z$ be a pointed map. Then

$$\ker(f) := \{g \in S \mid f|_{B\langle g\rangle} \simeq *\}.$$

Let $\mathcal{F}$ be a fusion system over a finite p-group S. Then a subgroup $K \leq S$ is *strongly $\mathcal{F}$-closed* if for all $P \leq K$ and all morphism $\varphi\colon P \to S$ in $\mathcal{F}$ we have $\varphi(P) \leq K$.

Since the intersection of strongly $\mathcal{F}$-closed subgroups is again strongly $\mathcal{F}$-closed, given a finite p-group P, we can define $Cl_{\mathcal{F}}(P)$ to be the smallest strongly $\mathcal{F}$-closed subgroup of S that contains $f(P)$ for all $f \in \mathrm{Hom}(P, S)$.

Proposition 3. *Let $f\colon B\mathcal{F} \to Z$ be a pointed map as in Definition 2.*

(1) *The kernel* $\ker(f)$ *is a strongly $\mathcal{F}$-closed subgroup of S.*
(2) *Let Z be a p-complete and $\Sigma B\mathbb{Z}/p$-null space with abelian fundamental group. Then a map $f\colon B\mathcal{F} \to Z$ is null-homotopic if and only if* $\ker(f) = S$.
(3) *Let K be a strongly $\mathcal{F}$-closed subgroup. There is $N \geq 0$ and a map $f\colon B\mathcal{F} \to (B\Sigma_N)^\wedge_p$ such that* $\ker(f) = K$.

In order to understand when $B\mathcal{F}$ is BP-cellular we need to compute the kernel of the map $r^\wedge_p$ where $r\colon B\mathcal{F} \to C \to P_{\Sigma BP} C$ is the map in Chacholski's fibration.

Theorem 4. *Let $(S, \mathcal{F})$ be a saturated fusion system and let P be a finite p-group. Then $B\mathcal{F}$ is BP-cellular if and only if $S = Cl_{\mathcal{F}}(P)$.*

Corollary 5. *Let $(S, \mathcal{F})$ be a saturated fusion system.*

(1) *The classifying space $B\mathcal{F}$ is BS-cellular.*
(2) *Let A be a pointed connected space. If BS is $B(\pi_1 A)_{ab}$-cellular, then $B\mathcal{F}$ is A-cellular.*
(3) *Let $\Omega_{p^m}(S)$ be the (normal) subgroup of S generated by its elements of order p^i, which $i \leq m$. Then $B\mathcal{F}$ is $B\mathbb{Z}/p^m$-cellular if and only if $S = Cl_{\mathcal{F}}(\Omega_{p^m}(S))$. In particular, there is a non-negative integer $m_0 \geq 0$ such that $B\mathcal{F}$ is $B\mathbb{Z}/p^m$-cellular for all $m \geq m_0$.*

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The spectrum of the equivariant stable homotopy category

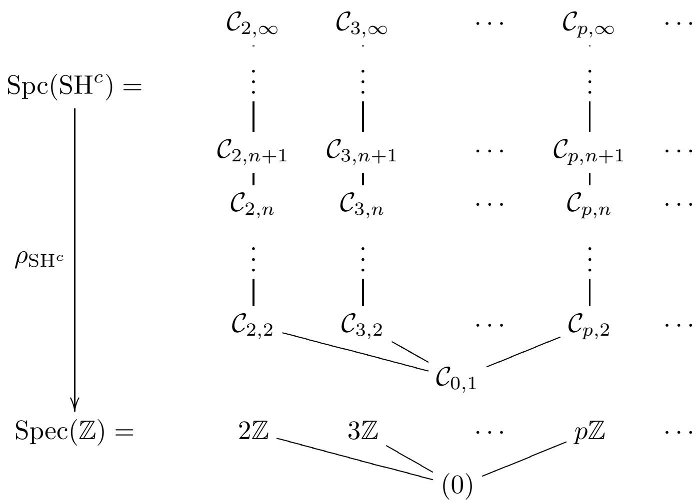
Beren Sanders

(joint work with Paul Balmer)

Let G be a finite group and let $\mathrm{SH}(G)$ denote the G-equivariant stable homotopy category. The aim of this work – joint with Paul Balmer – is to compute the spectrum (in the sense of [3]) of the subcategory of compact objects, $\mathrm{Spc}(\mathrm{SH}(G)^c)$, and thereby classify the thick $\otimes$-ideals of $\mathrm{SH}(G)^c$.

The Hopkins-Smith classification theorem [6] solves the problem for $G = 1$ (i.e. for the nonequivariant stable homotopy category):

(†)



Here $\mathcal{C}_{p,n}$ is the kernel of the $(n-1)$th Morava K-theory (at the prime p). In particular, $\mathcal{C}_{p,1} = \mathrm{SH}^{c,\mathrm{tor}} =: \mathcal{C}_{0,1}$ is the subcategory of finite torsion spectra, independently of p, while $\mathcal{C}_{p,\infty} = \cap_{n\geq 1}\mathcal{C}_{p,n}$ is the subcategory of finite p-acyclic spectra. The closure of a point is everything displayed above it, so that each $\mathcal{C}_{p,\infty}$ is a closed point, while $\mathcal{C}_{0,1}$ is a dense point.

The projection $\mathrm{Spc}(\mathrm{SH}^c) \to \mathrm{Spec}(\mathbb{Z})$ displayed above is a manifestation of a general construction from [2]: For any tensor triangulated category $\mathcal{K}$, there exists an inclusion-reversing continuous map $\rho_{\mathcal{K}} : \mathrm{Spc}(\mathcal{K}) \to \mathrm{Spec}(\mathrm{End}_{\mathcal{K}}(\mathbb{1}))$ to the affine scheme associated to the endomorphism ring of the unit object $\mathbb{1}$. For $\mathcal{K} = \mathrm{SH}(G)^c$, the endomorphism ring $\mathrm{End}_{\mathcal{K}}(\mathbb{1}) = A(G)$ is the Burnside ring and the map $\mathrm{Spc}(\mathrm{SH}(G)^c) \to \mathrm{Spc}(A(G))$ should similarly exhibit the spectrum of $\mathrm{SH}(G)^c$ as a chromatic refinement of the spectrum of the Burnside ring (the latter of which has been completely described by Dress [5]).

We are able to completely describe $\mathrm{Spc}(\mathrm{SH}(G)^c)$ *as a set* for any finite group:

Theorem. *Every prime ideal of* $\mathrm{Spc}(\mathrm{SH}(G)^c)$ *is of the form*

$$\mathcal{P}(H,p,n) := (\Phi^H)^{-1}(\mathcal{C}_{p,n})$$

for some subgroup $H \leq G$*, prime number* p*, and "integer"* $1 \leq n \leq \infty$*, where*

$$\Phi^H : \mathrm{Spc}(\mathrm{SH}(G)^c) \to \mathrm{Spc}(\mathrm{SH}^c)$$

denotes the geometric H*-fixed point functor. Moreover,* $\mathcal{P}(H,p,n) = \mathcal{P}(K,q,m)$ *iff* $H \sim_G K$*,* $n = m$*, and, if* $n = m > 1$ *then* $p = q$*.*

In particular, every prime ideal of $\mathrm{Spc}(\mathrm{SH}(G)^c)$ is obtained by pulling back the nonequivariant primes via the geometric fixed point functors. Interestingly, the fact that the prime $\mathcal{P}(H,p,n)$ is uniquely specified by the conjugacy class of H, the prime p, and the number $1 \leq n \leq \infty$ shows that the height 1 collisions in the spectrum of the Burnside ring do not occur in the spectrum of $\mathrm{SH}(G)^c$. In this way, the spectrum of the category of G-spectra is not only a chromatic refinement of the spectrum of the Burnside ring – it is also a group-theoretic refinement.

The two crucial ingredients in the proof are:

(1) The fact that the geometric fixed point functor $\Phi^G : \mathrm{SH}(G) \to \mathrm{SH}$ is a finite Bousfield localization. Although this appears already in [8], it does not seem to be much exploited in the literature.
(2) A result of [4] which asserts that restriction $\mathrm{Res}^G_H : \mathrm{SH}(G) \to \mathrm{SH}(H)$ is a separable extension (a.k.a. a finite étale extension). More precisely, $A^G_H := \Sigma^\infty G/H_+$ has the structure of a commutative separable ring object in $\mathrm{SH}(G)$ and there is an equivalence $\mathrm{SH}(H) \cong A^G_H\text{-}\mathrm{Mod}_{\mathrm{SH}(G)}$ such that restriction becomes the extension-of-scalars functor. This enables us to utilize the results of [1] on the tensor-triangular geometry of separable extensions.

In any case, having described $\mathrm{Spc}(\mathrm{SH}(G)^c)$ as a set, it remains to determine its topology. This boils down to determining the inclusions among the primes: $\mathcal{P}(H,p,n) \subset \mathcal{P}(K,q,m)$. For this, the inclusion-reversing map $\rho : \mathrm{Spc}(\mathrm{SH}(G)^c) \to$

Spec($A(G)$) is extremely useful since it greatly reduces the possibilities. Nevertheless, the problem of understanding the inclusions among the primes turns out to be related to so-called "blue-shift" phenomena in Tate cohomology. By utilizing blue-shift results of Hovey-Sadofsky and Kuhn (following on from work of Greenlees-Sadofsky) we can completely determine the topology for finite groups whose order is square-free (such as $G = C_p$ or $G = S_3$). For a general finite group, we reduce the problem to the case of p-groups. However, for p-groups there remains a slight indeterminacy in the topology that we have not yet been able to resolve. This investigation has led us to conjecture a new form of blue-shift phenomena for the Tate construction which, if true, would resolve this indeterminacy and complete the determination of the topology (and hence complete the classification of thick $\otimes$-ideals) for all finite groups. This conjecture may be stated as follows:

Conjecture. Let G be a p-group and let $X \in \mathrm{SH}$ be a nonequivariant spectrum. If $X \otimes \mathcal{C}_{p,n} = 0$ then $\Phi^G(t_G(\mathrm{triv}(X))) \otimes \mathcal{C}_{p,n-\log_p(|G|)} = 0$.

Remark. The problem of classifying the thick $\otimes$-ideals of $\mathrm{Spc}(\mathrm{SH}(G)^c)$ has also been considered by Strickland using other methods. Some of these results have been written up in the dissertation of Ruth Joachimi [7].

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The period-index problem and the cohomology of $\mathbf{PGL}_n$

Ben Williams
(joint work with Ben Antieau)

Overview of Problem. We begin with a local ring object R in a category of sheaves— of one of the following two kinds: the category of sheaves on a CW complex, X, locally ringed by $\mathbb{C}$ or X the étale site of a scheme, locally ringed by $\mathcal{O}_X$. As a special case of the latter we have Spec$k$. An Azumaya algebra A is a sheaf of R-algebras, locally isomorphic to $\mathrm{Mat}_n(R)$. Assuming X is connected, the number n is constant and is called the *degree* of A. An Azumaya algebra is equivalent to an $\mathrm{Aut}_{R\text{–alg}}(\mathrm{Mat}_n(R))$-bundle, which is a $\mathrm{PGL}_n(R)$ bundle since R is a local ring object.

There are coboundary maps $\mathrm{H}^1(X, \mathrm{PGL}_n) \to \mathrm{H}^2(X, \mathbb{G}_m)$; the joint image

$$\coprod_n \mathrm{H}^1(X, \mathrm{PGL}_n) \to \mathrm{H}^2(X, \mathbb{G}_m)$$

is the *Brauer group* of X—the group operation being inherited from the tensor product of R-algebras. Two Azumaya algebras, A, A' are said to be *Brauer equivalent* if they have the same image in the Brauer group. In the case of $X = \mathrm{Spec}k$, one recovers the theory of central simple algebras over a field.

The Brauer group is torsion, assuming X is connected, and $\mathrm{per}([A])$, the *period*, is a name for the order of $[A] \in \mathrm{Br}(X)$. This is the first measurement of the nontriviality of $[A]$.

We have $\mathrm{per}([A]) | \deg(A)$. We ask what the degrees of the algebras $A' \sim A$ are. To this end, we define $\mathrm{ind}([A]) = \gcd_{A' \sim A}(\deg(A'))$.

Period–Index. The *period–index* problem is to give a bound on $\mathrm{ind}(\alpha)$ in terms of X and $\mathrm{per}(\alpha)$. As a special example we have the *period–index conjecture* : if X is a smooth variety of dimension d, and $\alpha \in \mathrm{Br}(k(X))$, then $\mathrm{ind}(\alpha) | \mathrm{per}(\alpha)^{d-1}$. There are examples due to Colliot-Thélène [3] where this bound is seen to be sharp. It has been proved when $d = 2$ by de Jong [4]. One passes from the period–index conjecture to the following, by work of de Jong & Starr [5], to the *unramified brauer group* of $k(X)$: that is, $\mathrm{Br}(X)$ if X is smooth & projective.

We try to understand this problem for complex varieties by taking a topological realization functor $X \mapsto X(\mathbb{C})$. Azumaya algebras may be topologized and we obtain a homomorphism: $\mathrm{Br}(X) \to \mathrm{Br}(X(\mathbb{C}))$ and relations $\mathrm{per}_{\mathrm{topo}}(\alpha) | \mathrm{per}(\alpha)$, $\mathrm{ind}_{\mathrm{topo}}(\alpha) | \mathrm{ind}(\alpha)$. We were led by the period–index conjecture to make the following 'straw-man' conjecture: if X is a $2d$ dimensional finite CW complex, and $\alpha \in \mathrm{Br}(X)$, then $\mathrm{ind}_{\mathrm{topo}}(\alpha) | \mathrm{per}_{\mathrm{topo}}(\alpha)^{d-1}$.

In [1] we disproved this conjecture in a specific case. In the talk I made a conjecture similar to the following.

Topological Period–Index Conjecture: The 'straw-man' conjecture is true unless $\mathrm{per}(\alpha) \equiv 2 \pmod 4$ and $d = 3$.

Counterexample. One has a CW complex X and a class $\alpha \in \mathrm{Br}(X) \subset \mathrm{H}^3(X, \mathbb{Z})$. We concentrate on the case where $\mathrm{per}(\alpha) = 2$. In this case α may be lifted to a class $\xi \in \mathrm{H}^2(X, \mathbb{Z}/2)$.

We are now faced with the problem of lifting a map $\xi : X \to K(\mathbb{Z}/2, 2)$ to a map $X \to \mathrm{BPGL}_{2m}(\mathbb{C})$. this case is a an unstable operation $mP_2 : \mathrm{H}^2(\cdot, \mathbb{Z}/2) \to \mathrm{H}^5(\cdot, \mathbb{Z})$. The operation P_2 can be interpreted as a Pontryagin square: $2P_2(\xi) = \beta_2(\xi^2)$, where β_2 denotes the unreduced Bockstein map.

The choice of ξ is not unique; some work is required to turn the obstruction theory into a bound on the index. This can be done ([1]):

Proposition 1. *Suppose* $\xi \in \mathrm{H}^2(X, \mathbb{Z}/2)$ *has* $\beta_2(\xi) = \alpha$. *Define* $Q(\alpha)$ *to be the class of* $P_2(\xi) \in \mathrm{H}^5(X, \mathbb{Z})/(\alpha \smile \mathrm{H}^2(X, \mathbb{Z}))$. *Then* $\mathrm{ord}(Q(\alpha))\,\mathrm{per}(\alpha) | \mathrm{ind}(\alpha)$, *and the bound is sharp if* $\dim(X) \leq 6$.

Examples. We seek examples of classes $\xi \in \mathrm{H}^2(X, \mathbb{Z}/2)$ with the property that ξ^2 is not a reduction of integral class, or better yet that the Bockstein $\beta_2(\xi^2)$ is not a multiple of $\beta_2(\xi)$— so $\mathrm{ord}(Q(\alpha)) = 4$. We are grateful to M. Kameko for pointing out to us that this behaviour is expected in the cohomology of the finite groups $\mathrm{SL}_8(\mathbb{F}_q)/\mu_2$ for q odd. We seek examples when X is a complex 3-fold, in particular.

Addenda.

- In [2], we showed that the topological period–index problem for BG, where G is a topological group, is essentially a problem in the projective representation theory of G.
- There is a family of further obstructions $\beta_2(\xi^{2^n})$, that seem to generalize the one above, but we do not understand the topology of $\mathrm{BPGL}_n(\mathbb{C})$ well enough to make precise claims.
- There is a remarkable similarity between our candidate family of obstructions and some of the obstructions obtained by [6] to the realization of cohomology classes as being induced by immersed submanifolds. I do not understand this coincidence.

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Homological epimorphisms, stable equivalences and the Lie bracket in Hochschild cohomology

Reiner Hermann

Throughout, we let K be a commutative ring and our algebras will be unital and associative, defined over K. We further put $\otimes = \otimes_K$. The following exposition is mainly based on parts of the preprint [7].

1. The loop bracket

Let A be a K-algebra, and $A^{\mathrm{ev}} = A \otimes A^{\mathrm{op}}$. In [13], Stefan Schwede described the Lie bracket in Hochschild cohomology (see [6]) in terms of bimodule extensions. More precisely, Schwede took advantage of the (asymmetric) monoidal structure of $(\mathsf{Mod}(A^{\mathrm{ev}}), \otimes_A, A)$ to produce, for given m- and n-self extensions S and T of A with Yoneda composite $S \circ T$, a loop

$$\Omega(S,T) \quad \equiv \quad \begin{array}{ccc} & S \boxtimes_A T & \\ \swarrow & & \searrow \\ S \circ T & & (-1)^{mn} T \circ S \\ \nwarrow & & \nearrow \\ & (-1)^{mn} T \boxtimes_A S & \end{array}$$

in the category $\mathcal{E}xt^{m+n}_{A^{\mathrm{ev}}}(A,A)$ of $(m+n)$-self extensions of A over A^{ev}, that is, an element in the fundamental group $\pi_1(\mathcal{E}xt^{m+n}_{A^{\mathrm{ev}}}(A,A), S \circ T)$. This loop identifies with an element in $\mathrm{Ext}^{m+n-1}_{A^{\mathrm{ev}}}(A,A)$ thanks to Vladimir Retakh (see [12], and also [10]) who proved the existence of an isomorphism

$$\mathrm{Ext}^{n-1}_R(U,V) \xrightarrow{\sim} \pi_1(\mathcal{E}xt^n_R(U,V), S') \quad \text{(for a ring } R \text{ and } U, V \in \mathsf{Mod}(R))$$

which is, in an appropriate sense, independent of the taken base point S'. Schwede's main theorem in this context is now the following.

Theorem 1 (see [13, Thm. 3.1]). *Let A be a K-projective K-algebra and $m, n \geqslant 1$ be integers. Then for all elements $\alpha \in \mathrm{HH}^m(A)$ and $\beta \in \mathrm{HH}^n(A)$, represented by extensions $S = S(\alpha)$ and $T = T(\alpha)$ respectively, the Lie bracket $(-1)^n\{\alpha, \beta\}_A$ of α and β identifies with the image of the loop $\Omega(S,T)$ in $\mathrm{Ext}^{m+n-1}_{A^{\mathrm{ev}}}(A,A)$.*

2. Homological epimorphisms and the main theorem

2.1. **A compatibility result.** In [8], we generalised Schwede's construction to "suitable" exact monoidal categories. One class of such categories are exact monoidal K-categories $(\mathsf{C}, \otimes, \mathbb{1})$ which are closed under kernels of epimorphisms (that is, the class of admissible epimorphisms conincides with the class of epimorphisms in C) such that $- \otimes X : \mathsf{C} \to \mathsf{C}$ is an exact functor for all $X \in \mathrm{Ob}\,\mathsf{C}$. In this setting, we can provide a map $[-,-]_{\mathsf{C}} : \mathrm{Ext}^m_{\mathsf{C}}(\mathbb{1},\mathbb{1}) \times \mathrm{Ext}^n_{\mathsf{C}}(\mathbb{1},\mathbb{1}) \to \mathrm{Ext}^{m+n-1}_{\mathsf{C}}(\mathbb{1},\mathbb{1})$ which specialises to Schwede's map, and hence the Lie bracket, when C is taken to be the full subcategory of $\mathsf{Mod}(A^{\mathrm{ev}})$ whose objects are A^{ev}-modules which are projective on either side. For exact and colax monoidal functors between exact monoidal categories of the above form, we proved the result below. See [1] to recall the definition of a colax monoidal functor.

Theorem 2. *Let* $F : (\mathsf{C}, \otimes_{\mathsf{C}}, \mathbb{1}_{\mathsf{C}}) \to (\mathsf{D}, \otimes_{\mathsf{D}}, \mathbb{1}_{\mathsf{D}})$ *be an exact and colax monoidal functor whose unit morphism* $F\mathbb{1}_{\mathsf{C}} \xrightarrow{\sim} \mathbb{1}_{\mathsf{D}}$ *is an isomorphism. Then the induced graded* K*-algebra homomorphism*

$$\mathrm{Ext}^*_{\mathsf{C}}(\mathbb{1}_{\mathsf{C}}, \mathbb{1}_{\mathsf{C}}) \xrightarrow{F} \mathrm{Ext}^*_{\mathsf{D}}(F\mathbb{1}_{\mathsf{C}}, F\mathbb{1}_{\mathsf{C}}) \xrightarrow{\sim} \mathrm{Ext}^*_{\mathsf{D}}(\mathbb{1}_{\mathsf{D}}, \mathbb{1}_{\mathsf{D}})$$

takes $[-,-]_{\mathsf{C}}$ *to* $[-,-]_{\mathsf{D}}$.

2.2. **Homological epimorphisms.** Recall that a ring homomorphism $f : R \to S$ is an epimorphism if, and only if, the restriction functor $f_\star : \mathsf{Mod}(S) \to \mathsf{Mod}(R)$ is full and faithful. Due to a classical result of Silver, this is the same as saying that the multiplication map $S \otimes_R S \to S$ is an isomorphism of S-bimodules.

The case where the derived restiction functor $\mathsf{D}(f_\star)$ defines a full and faithful functor $\mathsf{D}(\mathsf{Mod}(S)) \to \mathsf{D}(\mathsf{Mod}(R))$ has been studied by Geigle-Lenzing; see [5]. It is evident, that f will have to be an epimorphism in that case. By adding the condition $\mathrm{Tor}^R_i(S, S) = 0$ for all $i > 0$ one obtains a precise characterisation of this situation. Epimorphisms satisfying the latter Tor-vanishing condition are called *homological epimorphisms.*

2.3. **The main theorem.** Let us fix two K-algebras A and B which are projective when considered as K-modules. Let further $q : B \to A$ be a K-linear homological epimorphism. The induced K-algebra homomorphism $q^{\mathrm{ev}} = q \otimes q^{\mathrm{op}} : B^{\mathrm{ev}} \to A^{\mathrm{ev}}$ remains a homological epimorphism, and the left adjoint to the restriction functor $\mathsf{D}(q^{\mathrm{ev}}_\star)$ has remarkable properties, by Theorem 2:

Theorem 3. *The graded* K*-algebra homomorphism*

$$A \otimes^{\mathsf{L}}_B (-) \otimes^{\mathsf{L}}_B A : \mathrm{HH}^*(B) = \mathrm{Hom}_{\mathsf{D}(B^{\mathrm{ev}})}(B, B[*]) \longrightarrow \mathrm{Hom}_{\mathsf{D}(A^{\mathrm{ev}})}(A, A[*]) = \mathrm{HH}^*(A)$$

also preserves the Lie bracket.

Indeed, the crucial observation for the proof of the above statement is that $q : B \to A$ gives rise to a colax monoidal functor

$$\mathfrak{A} = A \otimes_B (-) \otimes_B A : (\mathsf{Mod}(B^{\mathrm{ev}}), \otimes_B, B) \longrightarrow (\mathsf{Mod}(A^{\mathrm{ev}}), \otimes_A, A).$$

To produce the desired exact and monoidal subcategories between which $\mathfrak{A}$ defines an exact functor, one starts with the category of bimodules being projective on either side, and successively removes those modules that do not contribute to the exactness of $\mathfrak{A}$, in such a way that the resulting category remains exact and closed under $\otimes_A$.

3. An application to a long exact sequence of Koenig-Nagase

3.1. **Stratifying idempotents.** Let R be a ring. In [4], Cline-Parshall-Scott introduced the notion of a *stratifying idempotent* in R. Such an idempotent $e \in R$ is defined by the multiplication map $Re \otimes_{eRe} eR \xrightarrow{\sim} ReR$ being an isomorphism, and

$$\mathrm{Tor}^{eRe}_i(Re, eR) = 0 \quad (\text{for } i > 0).$$

For $I = ReR$, $S = R/I$ and $q : R \to S$, one has $\mathrm{Tor}_1^R(S,S) = I/I^2 = 0$ and

$$\begin{aligned} \mathrm{Tor}_i^R(S,S) &\cong \mathrm{Tor}_{i-1}^R(I,S) && \text{(by a long exact sequence)} \\ &\cong \mathrm{Tor}_{i-1}^R(Re \otimes_{eRe} eR, S) && \text{(as } Re \otimes_{eRe} eR \cong ReR) \\ &\cong \mathrm{Tor}_{i-1}^{eRe}(Re, eR \otimes_R S) && \text{(by [3, Chap. IX, Thm. 2.8])} \\ &\cong 0 && \text{(as } e \text{ annihilates } S) \end{aligned}$$

for $i > 1$, whence $q : R \to R/ReR$ is a (surjective) homological epimorphism.

3.2. **A cohomological long exact sequence.** Let B be a K-projective K-algebra and $e \in B$ a stratifying idempotent such that $A = B/BeB$ is K-projective. From the adjunction isomorphism $\mathrm{Hom}_{A^{\mathrm{ev}}}(A \otimes_B \mathbb{B}B \otimes_B A, A) \cong \mathrm{Hom}_{B^{\mathrm{ev}}}(\mathbb{B}B, A)$, where $\mathbb{B}B$ is the bar resolution of B, one obtains $\mathrm{HH}^*(A) \xrightarrow{\sim} \mathrm{Ext}^*_{B^{\mathrm{ev}}}(B, A)$. Therefore, applying $\mathrm{Hom}_{B^{\mathrm{ev}}}(B, -)$ to the canonical short exact sequence

$$0 \longrightarrow BeB \longrightarrow B \xrightarrow{q} B/BeB \longrightarrow 0$$

induces a cohomological long exact sequence

$$(\dagger) \qquad \cdots \longrightarrow \mathrm{Ext}^n_{B^{\mathrm{ev}}}(B, BeB) \longrightarrow \mathrm{HH}^n(B) \xrightarrow{\gamma_n} \mathrm{HH}^n(A) \longrightarrow \cdots$$

as observed by Koenig-Nagase in [9]. The following Lemma implies, when combined with Theorem 3, that the map $\gamma_* : \mathrm{HH}^*(B) \to \mathrm{HH}^*(A)$ in the long exact sequence (†) preserves the cup product (see also [9]) and the Lie bracket.

Lemma 4. *The map* γ_* *agrees with* $A \otimes_B^{\mathsf{L}} (-) \otimes_B^{\mathsf{L}} A : \mathrm{HH}^*(B) \longrightarrow \mathrm{HH}^*(A)$.

In this context, we like to raise the following question.

Question 5. Let K be a field of characteristic 2. If B denotes the algebra

$$B = \begin{bmatrix} K(\mathbb{Z}_2 \times \mathbb{Z}_2) & K\mathbb{Z}_2 \\ 0 & K \end{bmatrix}, \text{ with stratifying idempotent } e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

what is the precise Lie algebra structure of $\mathrm{HH}^*(B)$? Can the above sequence (†) be used to determine it (the Lie structure of $\mathrm{HH}^*(B/BeB)$ is well understood)?

4. Towards a result for stable equivalences of Morita type

4.1. **Stable equivalences.** Let A and B be two K-algebras. Then a *stable equivalence of Morita type* between A and B (in the sense of Broué; see [2]) is given by a quadruple (M, N, φ, ψ), wherein M is an $A \otimes B^{\mathrm{op}}$-module, N is an $B \otimes A^{\mathrm{op}}$-module, both being finitely generated projective when considered as one-sided modules, and φ and ψ are bimodule isomorphisms $\varphi : M \otimes_B N \xrightarrow{\sim} A \oplus P$, $\psi : N \otimes_A M \xrightarrow{\sim} B \oplus Q$ for some projective A^{ev}-module P and some B^{ev}-module Q.

Clearly, each pair of Morita equivalent algebras gives rise to a stable equivalence of Morita type. However, there are algebras which are stably equivalent but not even derived equivalent. The latter circumstance already suggests, that Hochschild cohomology might not be the right cohomology theory in this setting. Indeed, $\mathrm{HH}^*(A)$ has to be replaced by its stable analogue, the *stable Hochschild cohomology ring* $\underline{\mathrm{HH}}^*(A)$, which is invariant under stable equivalences of Morita type (see [11]).

It is open whether the same holds true for the Lie algebra structure (in appropriate degrees).

4.2. **An attempt to prove invariance.** In current work in progress, we offer the following refinement of Theorem 2 as a step towards a solution of the above problem. We assume that our categories are exact monoidal with properties as described in Paragraph 2.1. Recall that the image of the unit under a bilax monoidal functor contains the unit of the target category as a direct summand; see [1].

Refinement of Thm. 2. *Let* $F : (\mathsf{C}, \otimes_{\mathsf{C}}, \mathbb{1}_{\mathsf{C}}) \to (\mathsf{D}, \otimes_{\mathsf{D}}, \mathbb{1}_{\mathsf{D}})$ *be an exact and bilax monoidal functor. Assume that there is an integer* $d \geqslant 0$ *such that the cosummand* C *of* $\mathbb{1}_{\mathsf{D}}$ *in* $F\mathbb{1}_{\mathsf{C}}$ *satisfies* $\mathrm{Ext}^i_{\mathsf{D}}(\mathbb{1}_{\mathsf{D}}, C) = 0$ *for* $i \geqslant d$. *Then the induced map*

$$\mathrm{Ext}_{\mathsf{C}}^{\geqslant d}(\mathbb{1}_{\mathsf{C}}, \mathbb{1}_{\mathsf{C}}) \xrightarrow{F} \mathrm{Ext}_{\mathsf{D}}^{\geqslant d}(F\mathbb{1}_{\mathsf{C}}, F\mathbb{1}_{\mathsf{C}}) \xrightarrow{\mathrm{can}} \mathrm{Ext}_{\mathsf{D}}^{\geqslant d}(\mathbb{1}_{\mathsf{D}}, \mathbb{1}_{\mathsf{D}})$$

takes $[-,-]_{\mathsf{C}}$ *to* $[-,-]_{\mathsf{D}}$.

The question thus is, whether, for a given stable equivalence (M, N, φ, ψ), the functor $M \otimes_B (-) \otimes_B N : \mathsf{Mod}(B^{\mathrm{ev}}) \to \mathsf{Mod}(A^{\mathrm{ev}})$ can be turned into a bilax monoidal functor with (split surjective) unit map $M \otimes_B N \xrightarrow{\sim} A \oplus P \twoheadrightarrow A$.

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Torsion endo-trivial modules

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(joint work with Jon F. Carlson)

Let k be an algebraically closed field of prime characteristic p and let G be a finite group of order divisible by p. A kG-module M is *endo-trivial* if $M \otimes M^*$ is isomorphic to the trivial module k in the stable category, that is, $M \otimes M^* \cong k \oplus (\text{proj})$, where (proj) denotes some projective module. Let $T(G)$ be the group of isomorphism classes, in the stable category, of all endo-trivial kG-modules. This is an abelian group (for tensor product) and it is known to be finitely generated (Puig). We write $T(G) = TT(G) \oplus TF(G)$, where $TT(G)$ is the torsion subgroup (a finite group) and $TF(G)$ is a torsion-free group. Thus $TF(G) \cong \mathbb{Z}^N$ for some integer N, and this is essentially known (see [2] for details). We still have to understand $TT(G)$.

Let $X(G)$ be the group of one-dimensional kG-modules. Obviously $X(G)$ is a subgroup of $TT(G)$, isomorphic to the dual group $(G/G')^*$ of the abelian group G/G', where $G' = [G, G]S$ and $S \in \mathrm{Syl}_p(G)$ (so that G/G' is the largest abelian p'-quotient of G). For p-groups, the following result was proved 10 years ago using heavy cohomological machinery.

Theorem 1. (Carlson-Thévenaz [3]) *Let S be a finite p-group. Then the torsion subgroup $TT(S)$ is trivial, except if S is cyclic, generalized quaternion, or semi-dihedral.*

Note that $TT(S)$ is completely known in the three exceptional cases. In order to pass from a Sylow p-subgroup S of G to the whole group G, it is natural to introduce

$$K(G) := \mathrm{Ker}\left(\mathrm{Res}^G_S : T(G) \longrightarrow T(S)\right).$$

Thus the class of a kG-module M belongs to $K(G)$ if and only if $M{\downarrow}^G_S \cong k \oplus (\text{proj})$. Since such a module must have trivial source, there are finitely many of them and so $K(G)$ is a finite group, that is, $K(G) \subseteq TT(G)$. In fact, in view of Theorem 1, we have $K(G) = TT(G)$ if S is not one of the three exceptional cases.

We are left with the determination of $K(G)$. Many results appeared in recent years about $TT(G)$ for specific families of groups G (see the introduction of [4] for a list). In many cases, we have simply $TT(G) = K(G) = X(G)$, but there are also numerous examples where $K(G)$ is larger than $X(G)$. The starting point of the analysis is the following lemma.

Lemma. *Let S be a Sylow p-subgroup of G.*

(a) *For any nontrivial subgroup $Q \subseteq S$, we have $K(N_G(Q)) = X(N_G(Q))$.*
(b) *The restriction map $\mathrm{Res}^G_{N_G(S)} : T(G) \longrightarrow T(N_G(S))$ is injective.*

In fact, the map in part (b) is induced by the Green correspondence, which must be injective. It follows that we have an injective restriction map

$$\mathrm{Res}^G_{N_G(S)} : K(G) \longrightarrow K(N_G(S)) = X(N_G(S)),$$

and the problem is to find its image. In other words, given a one-dimensional $kN_G(S)$-module L, we need to know when its class is the restriction of some class of endo-trivial kG-modules.

For any nontrivial subgroup Q of a Sylow p-subgroup S, we define a sequence of subgroups $\{\rho^i(Q) \mid i \geq 1\}$ inductively as follows :

$$\rho^1(Q) \ := \ N_G(Q)' \, .$$

As before, $N_G(Q)'$ is the product of the commutator subgroup of $N_G(Q)$ and a Sylow p-subgroup of $N_G(Q)$. For $i \geq 2$, we let

$$\rho^i(Q) \ := \ < \ N_G(Q) \cap \rho^{i-1}(R) \ \mid \ \{1\} \neq R \subseteq S \ > \ .$$

This contains $\rho^{i-1}(Q)$, so we have a nested sequence of subgroups

$$Q \subseteq \rho^1(Q) \subseteq \rho^2(Q) \subseteq \rho^3(Q) \subseteq \ldots \subseteq N_G(Q) \, .$$

Since G is finite, the sequence eventually stabilizes and we let $\rho^\infty(Q)$ be the limit subgroup of the sequence $\{\rho^i(Q) \mid i \geq 1\}$, namely their union.

If a one-dimensional $kN_G(S)$-module L is the restriction of some endo-trivial kG-module M, that is, $M{\downarrow}^G_{N_G(S)} \cong L \oplus (\mathrm{proj})$, then it is easy to see, using the lemma above, that $\rho^\infty(S)$ must be in the kernel of L. We conjecture that this necessary condition is also sufficient.

Conjecture. *Let S be a Sylow p-subgroup of G.*

(a) *Let L be a one-dimensional $kN_G(S)$-module. There exists an endo-trivial kG-module M such that $M{\downarrow}^G_{N_G(S)} \cong L \oplus (\mathrm{proj})$ if and only if $\rho^\infty(S)$ is in the kernel of L.*

(b) *The image of the restriction map*

$$\mathrm{Res}^G_{N_G(S)} : K(G) \longrightarrow K(N_G(S)) = X(N_G(S))$$

is the dual group of $N_G(S)/\rho^\infty(S)$. Thus $K(G) \cong \big(N_G(S)/\rho^\infty(S)\big)^*$.

Note that (b) is an immediate consequence of (a), so the conjecture is actually only part (a). Our main result settles the problem when S is abelian.

Theorem 2. (Carlson-Thévenaz [4]) *Assume that the Sylow p-subgroup S is abelian. Then the conjecture holds. More precisely, $\rho^\infty(S) = \rho^2(S)$ and restriction induces an isomorphism $K(G) \cong \big(N_G(S)/\rho^2(S)\big)^*$.*

One ingredient is Burnside's well-known result which asserts that $N_G(S)$ controls fusion when S is abelian. But the main ingredient is a new method due to Balmer [1]. He provided a new characterization of the group $K(G)$ in terms of the group of weak homomorphisms.

Definition. As above, S denotes a Sylow p-subgroup of G. A map $\chi : G \to k^\times$ is called a *weak homomorphism* if it satisfies the following three conditions:

(a) If $s \in S$, then $\chi(s) = 1$.
(b) If $g \in G$ and $S \cap {}^gS = \{1\}$, then $\chi(g) = 1$.
(c) If $a, b \in G$ and if $S \cap {}^aS \cap {}^{ab}S \neq \{1\}$, then $\chi(ab) = \chi(a)\chi(b)$.

The set $A(G)$ of all weak homomorphisms is an abelian group under the usual product of maps.

Theorem 3. (Balmer [1]) *The groups $K(G)$ and $A(G)$ are isomorphic.*

Balmer's isomorphism is explicit and is described in [1]. The proof of Theorem 2 is based on the construction of a weak homomorphism $\chi : G \to k^\times$ that extends a given homomorphism $\varphi : N_G(S) \to k^\times$ which is trivial on $\rho^2(S)$.

Remark. Apart from Balmer's approach, there is another useful new method for handling endo-trivial modules, due to Lassueur-Malle [5]. They show that any endo-trivial kG-module lifts to a module in characteristic zero. Then ordinary character theory can be applied. In particular, a criterion is given for characterizing endo-trivial modules in $K(G)$ by purely character-theoretic means. This can be used for showing the existence, or the nonexistence, of endo-trivial modules for specific groups, by examination of their character table, as in [5] and [6].

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Algorithmic isomorphism classification of modular cohomology rings of finite groups

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(joint work with Bettina Eick, David Green)

For a finite group G, let $H^*(G; \mathbb{F}_p)$ denote the *modular cohomology ring* with coefficients in the finite prime field $\mathbb{F}_p$. It is well known that modular cohomology rings of finite groups are finitely presentable graded-commutative $\mathbb{F}_p$–algebras.

It is possible that the modular cohomology rings of two non-isomorphic finite groups are isomorphic as graded $\mathbb{F}_p$–algebras. In fact, Carlson [2] has shown that, for any $c \in \mathbb{N}$, the finite 2–groups of coclass c (of which there are infinitely many) only have finitely many different graded isomorphism types of modular cohomology rings. Recently, A. Dias Ramos et al. announced a similar result for p–groups of fixed coclass, for any prime $p > 2$. The proof is not constructive, hence, it is unclear how many isomorphism types actually occur. Also, it is conjectured that in each of the "coclass families" of p–groups defined in [5] all but finitely many groups have isomorphic modular cohomology rings. Again, it is not clear how many exceptions will occur.

To explore the isomorphism types by computer calculations, it is needed (1) to compute modular cohomology rings for many groups, and (2) to create an algorithm that can decide whether or not two modular cohomology rings are graded isomorphic. In [4], we classify the modular cohomology rings of all prime power groups of order at most 81 up to graded isomorphism. For example, the 340 2-groups of order at most 64 have 260 graded isomorphism classes of mod-2 cohomology rings.

For computing group cohomology, we use an optional package [11] for the free open source computer algebra system [13] SageMath. It can compute a minimal presentation of $H^*(G;\mathbb{F}_p)$ as graded–commutative $\mathbb{F}_p$–algebra, as well as some ring theoretic invariants such as Poincare series, a–invariants and depth. Experimentally, it also computes Massey products. The package can deal with induced homomorphisms and uses them to compute the nilradical and the essential respectively depth essential ideals.

It was asked by Hambleton whether the mod-2 cohomology in degree 2 is detected by metabelian groups, in the same way as the degree 1 is detected by cyclic groups. Computer experiments suggested to generalise the question for any prime p and any degree: For any mod-p cohomology class of G, does there exist a metabelian not necessarily proper subgroup of G to which the class has non-trivial restriction? During the workshop in Oberwolfach, Green found that the answer to the generalised question is negative: There is a group of order 3^{16} that is not metabelian and has essential mod-3 classes (but most likely not in degree 2).

To compute the cohomology ring, the package computes "ring approximations" in increasing degrees, and uses various completeness criteria to test whether the current ring approximation actually is isomorphic to the whole cohomology ring.

In the case of a prime power group G, the computation of approximations of $H^*(G;\mathbb{F}_p)$ is based on the construction of a minimal projective resolution. For this, a signature based non-commutative standard basis algorithm of Green [6] is used. It is work in progress to replace Green's algorithm by a potentially more efficient non-commutative version of Faugere's F_5 algorithm [9].

If G is not of prime power order, the package uses the stable element method [3, XII §10]: If $U \leq G$ is any subgroup containing a Sylow p–subgroup of G, the restriction from G to U is injective, and thus $H^*(G;\mathbb{F}_p)$ can be considered as a sub-ring of $H^*(U;\mathbb{F}_p)$. Moreover, this sub-ring can be described by *stability conditions*, that are associated to the double cosets $U \setminus G/U$ and give rise to linear equation systems in each degree.

For the third Conway group $G = Co_3$, a Sylow 2–subgroup $S \leq G$ has as many as 484,680 double cosets, which is unfeasible. It was suggested by Holt [8] to use stable elements in two or more steps: One considers a subgroup U that is strictly between G and S (in many cases, $U = N_G(Z(S))$ will do), and then uses stability to first compute $H^*(U;\mathbb{F}_p)$ as a subring of $H^*(S;\mathbb{F}_p)$ and finally $H^*(G;\mathbb{F}_p)$ as a subring of $H^*(U;\mathbb{F}_p)$. Using a certain tower of five groups, the computation of $H^*(Co_3;\mathbb{F}_2)$ only involves a total of 11 stability conditions [12].

To prove completeness of cohomology ring approximations, we use criteria from [1], [7], [14] and [10]. In all criteria, the key is to construct elements in the ring approximation over which the cohomology ring is finite, so that the degrees of these elements is as small as possible. Depending on the criterion, the elements have to have additional properties. The criteria have different advantages and disadvantages, we therefore combine them. The first part of the criterion from [10] tests whether the ring approximation contains a generating set of the whole cohomology ring, which is very useful in the stable element method: By consequence, it is enough to solve the stability conditions only in relatively small degrees.

In [4], we provide an algorithmic solution of the graded isomorphism problem for finitely presented associative unital F–algebras that are generated in positive degrees, where F is a finite field. This holds, in particular, for modular cohomology rings of finite groups, where one can additionally use Groebner basis techniques to speed-up some computations.

The basic idea of the algorithm is straight forward. Let R_1, R_2 be algebras satisfying the above hypotheses. Since R_2 is finitely generated in positive degrees, for each d, the degree-d part $R_2^{(d)}$ is a finite dimensional F vector space, and actually a finite set, since F is finite. Therefore, there are only finitely many possibilities to map a generator of R_1 to an element of the same degree in R_2. And for each choice of generator images, it is possible to test if it extends to an isomorphism.

Of course, in that basic form, the algorithm wouldn't be usable. It is essential to drastically cut down the choices. We have developed techniques to do so, so that the isomorphism classification of modular cohomology rings of all prime power groups up to order 81 is just a matter of few minutes.

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Twisted homological stability for groups via functor categories

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(joint work with Andrew Putman)

A sequence of groups and maps $G_1 \to G_2 \to \cdots$ satisfies **homological stability** if, for each $i \geq 0$, the induced map on homology $\mathrm{H}_i(G_n) \to \mathrm{H}_i(G_{n+1})$ is an isomorphism for $n \gg i$. Some sequences of groups that satisfy homological stability (the maps are the usual ones):

- Symmetric groups $G_n = S_n$ (Nakaoka [Nak]);
- For any group Γ, the wreath products $G_n = S_n \ltimes \Gamma^n$ (this seems to have been well-known – it is stated explicitly in [HW, Prop. 1.6]);
- For well-behaved rings R (such as commutative noetherian rings of finite Krull dimension), $G_n = \mathrm{GL}_n(R)$ (van der Kallen [Va]), and
- the symplectic groups $G_n = \mathrm{Sp}_{2n}(R)$ (Mirzaii–van der Kallen [MV]).

More generally, G_n-representations M_n equipped with G_n-equivariant maps $M_n \to M_{n+1}$ satisfy **twisted homological stability** if, for each $i \geq 0$, the induced map $\mathrm{H}_i(G_n; M_n) \to \mathrm{H}_i(G_{n+1}; M_{n+1})$ is an isomorphism for $n \gg i$.

The problem we consider is to determine which kinds of sequences satisfy twisted homological stability. Wahl [W] gave a general setup using the notion of *homogeneous categories* (they are monoidal categories; we omit the definition since we use a special case below). If $(\mathcal{G}, \oplus, 0)$ is a symmetric monoidal groupoid such that $\mathrm{Aut}(0) = \{1\}$ and such that the map $\mathrm{Aut}(A) \to \mathrm{Aut}(A \oplus B)$ given by $f \mapsto f \oplus 1_B$ is injective for all A, B, then there is a minimal homogeneous symmetric monoidal category $\mathcal{UG}$ containing $\mathcal{G}$ as its underlying groupoid [W, 1.4, 1.5].

Corresponding to the previous examples, we give a few cases of $\mathcal{G}$ and $\mathcal{UG}$:

- The groupoid of finite sets under disjoint union gives the category FI, whose objects are finite sets and whose morphisms are injections;
- The groupoid of free Γ-sets under disjoint union gives the category FI_Γ, whose objects are finite sets and whose morphisms are Γ-injections: an injective function $f\colon R \to S$ and a function $\rho\colon R \to \Gamma$; the composition with $(g\colon S \to T, \sigma)$ is given by (gf, τ) where $\tau(x) = \sigma(f(x)) \cdot \rho(x))$;

- The groupoid of finite rank free R-modules under direct sum gives the category $\mathrm{VIC}(R)$, whose objects are finite rank free R-modules and whose morphisms $V \to W$ are pairs of maps $V \to W \to V$ composing to 1_V;
- The groupoid of finite rank free symplectic R-modules under direct sum gives the category $\mathrm{SI}(R)$, whose objects are finite rank free symplectic R-modules and whose morphisms are linear maps preserving the form (and hence must be injective).

The above examples of $\mathcal{UG}$ are in fact complemented categories. A symmetric monoidal category is **complemented** if it satisfies the following properties:

- Every morphism is a monomorphism;
- 0 is an initial object, and so we have canonical maps $V \to V \oplus V'$ and $V' \to V \oplus V'$;
- The map $\mathrm{Hom}(V \oplus V', W) \to \mathrm{Hom}(V, W) \times \mathrm{Hom}(V', W)$ is injective;
- Every subobject $C \subset V$ has a complement, i.e., another subobject $D \subset V$ so that $V \cong C \oplus D$ and where the isomorphism identifies the inclusion $C \subset V$ with the canonical map $C \to C \oplus D$, and similarly for D.

Each one has a **generator** X, i.e., every object is isomorphic to $X^{\oplus n}$.

Fix a commutative ring $\mathbf{k}$. Given a complemented category $\mathcal{C}$ with generator X, and a functor $F\colon \mathcal{C} \to \mathbf{k}\text{-Mod}$, define $\Sigma F\colon \mathcal{C} \to \mathbf{k}\text{-Mod}$ to be the precomposition with the functor $Y \mapsto Y \oplus X$. There is a natural transformation $F \to \Sigma F$, and its kernel and cokernel are denoted $\ker F$ and $\operatorname{coker} F$. We can use this to define the **degree** of a functor:

- If $F = 0$, then its degree is -1;
- If $\ker F$ and $\operatorname{coker} F$ have degree $\leq r-1$, then F has degree $\leq r$.

Otherwise F has infinite degree. Also, for each n, define a semisimplicial set $W_n(X)$ whose p-simplices are $\mathrm{Hom}(X^{\oplus p+1}, X^{\oplus n})$.

Let $\mathcal{C}$ be a complemented category with generator X. Suppose that there is an integer $k \geq 2$ so that for all $n \geq 1$, $W_n(X)$ is $(n-2)/k$-connected. Then a special case of [W, Theorem 5.6] is that for any functor of finite degree $\leq r$, the map

$$\mathrm{H}_i(\mathrm{Aut}(X^{\oplus n}); F(X^{\oplus n})) \to \mathrm{H}_i(\mathrm{Aut}(X^{\oplus n+1}); F(X^{\oplus n+1}))$$

is an isomorphism when $i \leq (n-r)/k$. Implicitly, we always use the morphisms $X^{\oplus n} \to X^{\oplus n+1}$ as inclusion via the first n factors to define all structure maps. We will say that the functor F satisfies homological stability.

For some purposes, having finite degree is too restrictive of a condition. For example, if $\mathbf{k}$ is a field and F takes finite-dimensional values, then it implies that the function $n \mapsto \dim_{\mathbf{k}} F(X^{\oplus n})$ is a polynomial for $n \gg 0$. A basic property of complemented categories $\mathcal{C}$ with generator X is that for $n \geq r$, the permutation representation $\mathbf{k}[\mathrm{Hom}(X^{\oplus r}, X^{\oplus n})]$ is isomorphic to the induced representation $\mathrm{Ind}_{\mathrm{Aut}(X^{\oplus n-r})}^{\mathrm{Aut}(X^{\oplus n})}\mathbf{k}$. So by Shapiro's lemma, the functor $P_r\colon \mathcal{C} \to \mathbf{k}\text{-Mod}$ defined by $Y \mapsto \mathbf{k}[\mathrm{Hom}(X^{\oplus r}, Y)]$ satisfies homological stability if the same is true for the constant functor, i.e., the groups $\mathrm{Aut}(X^{\oplus n})$ satisfy homological stability. *From now on, we will make this assumption about* $\mathrm{Aut}(X^{\oplus n})$.

By Yoneda's lemma, the set of natural transformations $P_r \to F$ identifies with $F(X^{\oplus r})$, and so the P_r are a set of projective generators for the functor category $[\mathcal{C}, \mathbf{k}\text{-Mod}]$. In particular, any functor F admits a projective resolution of the form

$$\cdots \to \mathbf{P}_d \to \mathbf{P}_{d-1} \to \cdots \to \mathbf{P}_1 \to \mathbf{P}_0 \to F \to 0$$

where $\mathbf{P}_d$ is a direct sum of P_r. If we assume that each $\mathbf{P}_d$ has a decomposition as $\bigoplus_{r \leq D} P_r$ (D depending on d), then $\mathbf{P}_d$ also satisfies homological stability. Note that for each n, there is a spectral sequence

$$\mathrm{E}^1_{p,q}(n) = \mathrm{H}_p(\mathrm{Aut}(X^{\oplus n}); \mathbf{P}_q(X^{\oplus n})) \Longrightarrow \mathrm{H}_{p+q}(\mathrm{Aut}(X^{\oplus n}); F(X^{\oplus n})),$$

and spectral sequence morphisms $\mathrm{E}^1_{*,*}(n) \to \mathrm{E}^1_{*,*}(n+1)$. So with the assumption on $\mathbf{P}_d$ above, we see that for a given diagonal $p+q$, the map of spectral sequences on all relevant terms to calculate H_{p+q} is an isomorphism for $n \gg 0$, and hence F satisfies homological stability.

This motivates the following definitions. Say that F is **finitely generated** if it is a quotient of a finite direct sum $P_{r_1} \oplus \cdots \oplus P_{r_n}$, and say that F is **noetherian** if every subfunctor of F is finitely generated; $[\mathcal{C}, \mathbf{k}\text{-Mod}]$ is (locally) noetherian if every finitely generated functor is noetherian. This implies that $\mathbf{k}$ is a noetherian ring. If $[\mathcal{C}, \mathbf{k}\text{-Mod}]$ is noetherian, then every finitely generated functor has a projective resolution where each $\mathbf{P}_d$ is a finite direct sum of P_r, and hence satisfies homological stability. This is formalized in [PS, Theorem 4.2].

Some examples of when $[\mathcal{C}, \mathbf{k}\text{-Mod}]$ is noetherian (take $\mathbf{k}$ to be any noetherian ring) corresponding to the running examples:

- FI (Church–Ellenberg–Farb–Nagpal [CEFN, Theorem A])
- When Γ is virtually polycyclic, FI_Γ (Sam–Snowden [SS, Cor. 1.2.2])
- When R is a finite commutative ring, $\mathrm{VIC}(R)$ and $\mathrm{SI}(R)$ (Putman–Sam [PS, Theorems C, D])

Finally, a word about cohomology versus homology. Let $\mathbf{k}$ be a field of characteristic $p > 0$ and let $\mathfrak{h}(n) = \{(x_1, \ldots, x_n) \in \mathbf{k}^n \mid \sum_i x_i = 0\}$ be the reflection representation of S_n; note that $\{1, \ldots, n\} \mapsto \mathfrak{h}(n)$ defines a finitely generated functor $\mathrm{FI} \to \mathbf{k}\text{-Mod}$. For $n \geq 3$ we have $\mathrm{H}_0(S_n; \mathfrak{h}(n)) = 0$, whereas

$$\mathrm{H}^0(S_n; \mathfrak{h}(n)) = \mathfrak{h}^{S_n} = \begin{cases} 0 & \text{if } p \nmid n \\ \mathbf{k} & \text{if } p \mid n \end{cases}.$$

In fact, this periodic behavior is typical: Nagpal shows that if F is a finitely generated FI-module, then for each i, the function $n \mapsto \dim_{\mathbf{k}} \mathrm{H}^i(S_n; F(\{1, \ldots, n\}))$ is a periodic function of n for $n \gg 0$ with period a power of p [Nag, Theorem D].

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The Lie module and its complexity

David Hemmer
(joint work with Frederick Cohen, Daniel Nakano)

Let Σ_n be the symmetric group and k and algebraically closed field of characteristic p. In this talk we discussed our determination of the complexity of the $k\Sigma_n$ module $\mathsf{Lie}(n)$, which we define next. For any commutative ring R and positive integer n, let $\mathsf{Lie}_R(x_1, x_2, \ldots, x_n)$ be the free Lie algebra over R generated by $x_1, x_2, \ldots, x_n$ and let $\mathsf{Lie}_R(n)$ be the submodule spanned by all bracket monomials containing each x_i exactly once. Then $\mathsf{Lie}_R(n)$ is a module for the symmetric group Σ_n acting by permuting the variables.

We will be interested in $\mathsf{Lie}_k(n) := \mathsf{Lie}(n)$. This module arises naturally in topology, for example as the top degree homology of the configuration space of n points in the plane tensored by the sign representation. In characteristic zero there is a beautiful description of its complex character in terms of tableaux combinatorics, see [4, Chapter 8] for a thorough treatment. Furthermore, the representation $\mathsf{Lie}(n)$ is a direct summand of $\mathbb{Q}\Sigma_n$. In characteristic p, very little is known about the module structure of $\mathsf{Lie}(n)$ except in special cases, for example small n or when $p^2 \nmid n$. Over an arbitrary field k, $\mathsf{Lie}(n)$ has dimension $(n-1)!$ and is free over $k\Sigma_{n-1}$.

Erdmann, Lim and Tam [2] stated a conjecture for $c_{\Sigma_n}(\mathsf{Lie}(n))$. In particular they conjectured the complexity to be r, where p^r is the largest power of p dividing $n!$. Our strategy in proving this conjecture was to first employ earlier results of Hemmer-Nakano which reduce the calculation to studying $\mathrm{H}_\bullet(\Sigma_\lambda, \mathsf{Lie}(n))$ for various Young subgroups Σ_λ. This reduction lets us apply a result of Arone and Kankaanrinta [1] giving bases for these homology groups.

Specifically we used the following result, where $\lambda(V_\bullet)$ denotes the polynomial rate of growth of the dimensions of a sequence of vector spaces:

Theorem 1. *Let M be a $k\Sigma_n$-module with complexity $c_G(M)$. The following are equivalent.*

(a) $c_G(M)$
(b) $\max_{\lambda \vDash n}\{\gamma(\mathrm{H}^\bullet(\Sigma_\lambda, M))\}$
(c) $\max_{\lambda \vDash n}\{\gamma(\mathrm{H}_\bullet(\Sigma_\lambda, M))\}$

Now we apply the work of Aronke-Kankaarinta. Let

$$M_r = \mathrm{H}_\bullet(\Sigma_{p^r}, \mathsf{Lie}(p^r)).$$

Arone and Kankaanrinta gives a basis for the $(r+1)$st suspension $\Sigma^{1+r}M_r$ in terms of "completely inadmissible Dyer-Lashof words of length r". Their results are summarized in the next theorem.

Theorem 2. [1, Thm. 3.2] *The following elements constitute a basis for $\Sigma^{1+r}M_r$: if $p > 2$*

$$\{\beta^{\epsilon_1}Q^{s_1}\cdots\beta^{\epsilon_r}Q^{s_r}u \mid s_r \geq 1, s_j > ps_{j+1} - \epsilon_{j+1}\forall 1 \leq j < r\},$$

if p=2

$$\{Q^{s_1}\cdots Q^{s_r}u \mid s_r \geq 1, s_j > 2s_{j+1}\forall 1 \leq j < r\}.$$

Here u is of dimension 1, the $Q^{s_j}s$ are Dyer-Lashof operations and the βs are the homology Bocksteins. Thus Q^s increases dimension by s if $p = 2$ and by $2s(p-1)$ if $p > 2$, and β decreases dimension by one.

As a special case we remark that the basis element $Q^{s_1}Q^{s_2}\cdots Q^{s_r}u$ lies in degree $2(p-1)(s_1+s_2+\cdots+s_r)$ for p odd and $(s_1+s_2+\cdots+s_r)$ for $p=2$.

By analyzing the basis elements in Theorem 2 we obtained our main result:

Theorem 3. *For all $n \in \mathbb{N}, c_{\Sigma_n}(\mathsf{Lie}(n)) = r$ where $p^r \mid n$ and $p^{r+1} \nmid n$.*

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Polynomial functors on hermitian spaces

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(joint work with Christine Vespa)

In the early fifties, Eilenberg and Mac Lane [3] introduced the notion of *polynomial functors* between categories of modules to study the homology of topological spaces which have now their name. The interest of this notion has remained strong, because of other connections with algebraic topology (Henn-Lannes-Schwartz), representation theory, algebraic K-theory and stable homology of linear groups (Betley, Suslin, Scorichenko). This classical notion of polynomial functor can be defined in the same way for functors from a (small) symmetric monoidal category $(\mathcal{C}, +, 0)$ whose unit 0 is an *zero object* to a (nice) abelian category.

But for some purposes, this setting is not enough. For example, many recent works deal with FI-modules (functors from the category of finite sets with injections to abelian groups, see [1]); finitely generated FI-modules carry polynomial properties (for dimension functions, when the values are in finite-dimensional vector spaces over a field). Another example comes from the quotients of the lower central serie of the automorphism group of a free group — see recent works by Satoh and Bartholdi. But our main motivation to introduce a generalized notion of polynomial functors in the study of *stable homology of congruence groups.* To be more precise, let I a ring *without unit*, n a non-negative integer and

$$GL_n(I) := Ker\,(GL_n(I \oplus \mathbb{Z}) \to GL_n(\mathbb{Z}))$$

the corresponding general linear group, which is congruence group ($I \oplus \mathbb{Z}$ is the ring obtained by adding formally a unit to I). The study of the homology of these groups is known to be extremely hard and related to the problem of excision in algebraic K-theory. For a qualitative approach of this problem, let us remark that the stabilization maps $H_*(GL_n(I)) \to H_*(GL_{n+1}(I))$ and the natural action of $GL_n(\mathbb{Z})$ on $H_*(GL_n(I))$ assemble to give a functor

$$\mathbf{S}(\mathbb{Z}) \to \mathbf{Ab} \qquad \mathbb{Z}^n \mapsto H_d(GL_n(I))$$

for each $d \in \mathbb{N}$ (which will be denoted by $H_d(GL(I))$). Here, we denote by $\mathbf{S}(R)$, for any ring R, the category of finitely generated left free R-modules with split R-linear injections, the splitting being given in the structure.

Conjecture. *For any ring without unit I and any $d \in \mathbb{N}$, the functor $H_d(GL(I))$: $\mathbf{S}(\mathbb{Z}) \to \mathbf{Ab}$ is weakly polynomial of degree $\leq 2d$.*

This conjecture is inspired by the beautiful work of Suslin [4].

We will now explain the meaning of *weakly polynomial* and how wich kind of classification result we can get for this kind of polynomial functors (following [2]).

The category $\mathbf{S}(R)$ can be seen as a particular case of the category $\mathbf{H}(A)$ of hermitian spaces over a ring with involution A (the objects are the finitely generated free A-modules endowed with a non-degenerate hermitian form, the morphisms are A-linear maps which preserve the hermitian forms): $\mathbf{S}(R)$ is equivalent to $\mathbf{H}(R^{op} \times R)$, where $R^{op} \times R$ is endowed with the canonical involution. We deal with this general hermitian setting, which is not harder than $\mathbf{S}(\mathbb{Z})$.

Strongly polynomial functors

In the sequel, $(\mathcal{C}, +, 0)$ denotes a (small) symmetric monoidal category whose unit 0 is an *initial object* (as FI with the disjoint union or $\mathbf{H}(A)$ with the hermitian sum) and $\mathcal{A}$ a Grothendieck category. The category $\mathbf{Fct}(\mathcal{C}, \mathcal{A})$ of functors from $\mathcal{C}$ to $\mathcal{A}$ is also a Grothendieck category.

Definition. For any object x of $\mathcal{C}$, let $\tau_x : \mathbf{Fct}(\mathcal{C}, \mathcal{A}) \to \mathbf{Fct}(\mathcal{C}, \mathcal{A})$ denote the precomposition by the functor $- + x : \mathcal{C} \to \mathcal{C}$. We denote also by δ_x (respectively κ_x) the cokernel (resp. kernel) of the natural transformation $\mathrm{Id} = \tau_0 \to \tau_x$ induced by the unique map $0 \to x$.

A functor $F : \mathcal{C} \to \mathcal{A}$ is said *strongly polynomial of degree* $\leq d$ if $\delta_{a_0}\delta_{a_1}\dots\delta_{a_d}(F) = 0$ for any $(d+1)$-tuple $(a_0,\dots,a_d)$ of objects of $\mathcal{C}$.

This notion is not so well-behaved, because is it not stable under subfunctors.

Weakly polynomial functors

To avoid this problem, we change the definition by working in a suitable quotient category:

Proposition and definition. The full subcategory $\mathcal{SN}(\mathcal{C},\mathcal{A})$ of $\mathbf{Fct}(\mathcal{C},\mathcal{A})$ of functors F such that $F = \sum_{x\in \mathrm{Ob}\,\mathcal{C}} \kappa_x(F)$ is localizing. We denote by $\mathbf{St}(\mathcal{C},\mathcal{A})$ the quotient category $\mathbf{Fct}(\mathcal{C},\mathcal{A})/\mathcal{SN}(\mathcal{C},\mathcal{A})$.

For any object x of $\mathcal{C}$, τ_x induces an exact functor (always denoted in the same way) of $\mathbf{St}(\mathcal{C},\mathcal{A})$; in this category, the natural transformation $\mathrm{Id} \to \tau_x$ is monic. So its cokernel δ_x is exact.

An object X of $\mathbf{St}(\mathcal{C},\mathcal{A})$ is said polynomial of degree $\leq d$ if $\delta_{a_0}\delta_{a_1}\dots\delta_{a_d}(X) = 0$ for any $(d+1)$-tuple $(a_0,\dots,a_d)$ of objects of $\mathcal{C}$.

The full subcategory $\mathcal{P}ol_d(\mathcal{C},\mathcal{A})$ of $\mathbf{St}(\mathcal{C},\mathcal{A})$ of these objects is bilocalizing.

A functor $F : \mathcal{C} \to \mathcal{A}$ is said *weakly polynomial of degree* $\leq d$ if its image in $\mathbf{St}(\mathcal{C},\mathcal{A})$ belongs to $\mathcal{P}ol_d(\mathcal{C},\mathcal{A})$.

Main result

Theorem ([2]). Let A be a ring with involution and $\mathcal{A}$ a Grothendieck category. For any $d \in \mathbb{N}$, the forgetful functor $\mathbf{H}(A) \to \mathbf{F}(A)$ (category of finitely generated free A-modules, with usual morphisms) induces an equivalence of categories:

$$\mathcal{P}ol_d(\mathbf{F}(A),\mathcal{A})/\mathcal{P}ol_{d-1}(\mathbf{F}(A),\mathcal{A}) \to \mathcal{P}ol_d(\mathbf{H}(A),\mathcal{A})/\mathcal{P}ol_{d-1}(\mathbf{H}(A),\mathcal{A})$$

(the source category can be described from the wreath product of A and $\mathfrak{S}_d$).

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Generic Representations of finite fields in nondescribing characteristic

Nicholas J. Kuhn

1. Introduction and the Main Theorem

Let $\mathbb{F}$ and K be fields. Let $\mathrm{Rep}(\mathbb{F};K)$ denote the category whose objects are functors

$$F : \text{ fin. dim. } \mathbb{F}\text{–vector spaces} \longrightarrow K\text{–vector spaces},$$

and having natural transformations as morphisms. We refer to $F \in \mathrm{Rep}(\mathbb{F};K)$ as a *generic representation* of the field $\mathbb{F}$.

This is a K–linear category in the obvious way. For example, a sequence of functors

$$0 \to F \to G \to H \to 0$$

is short exact if

$$0 \to F(V) \to G(V) \to H(V) \to 0$$

is a short exact sequence of K–vector spaces for each $\mathbb{F}$–vector space V.

Structure in K–vector spaces tends to induce structure in $\mathrm{Rep}(\mathbb{F};K)$. For example, given $F, G \in \mathrm{Rep}(\mathbb{F};K)$, $F \otimes G$ is defined to be the functor $V \mapsto F(V) \otimes G(V)$.

We will focus on the case when $\mathbb{F} = \mathbb{F}_q$, a finite field of order $q = p^r$.

The category $\mathrm{Rep}(\mathbb{F};\mathbb{F})$ has been much studied since late 1980's with good effect: see, e.g., [HLS93, K00, FFSS99] (with some results reviewed below). Our new result concerns the case with *char* $\mathbb{F} \neq$ *char* K. Using a 1992 result in semigroup theory by L.G.Kovács [Ko92], we show that there is great simplification:

Theorem 1. *Let $\mathbb{F}$ be a finite field of characteristic p. If p is invertible in K, there is a natural equivalence of K–linear abelian categories*

$$\mathrm{Rep}(\mathbb{F};K) \simeq \prod_{n=0}^{\infty} K[GL_n(\mathbb{F})]\text{–}modules.$$

Some structural results about $\mathrm{Rep}(\mathbb{F};K)$ are immediate corollaries.

Corollary 2. *If K is a field of characteristic different than p, then all projectives in $\mathrm{Rep}(\mathbb{F};K)$ are also injective, and indecomposable projectives have only finitely many composition factors.*

Corollary 3. *If K is a field of characteristic 0, then $\mathrm{Rep}(\mathbb{F};K)$ is semisimple.*

2. Examples of generic representations

Examples 4. $\mathrm{Rep}(\mathbb{F};\mathbb{F})$ contains the familiar polynomial functors: T^n defined by $T^n(V) = V^{\otimes n}$, $S^n = (T^n)_{\Sigma_n}$, $\Gamma^n = (T^n)^{\Sigma_n}$, Λ^n. These all have polynomial growth, and this property characterizes functors in $\mathrm{Rep}(\mathbb{F};\mathbb{F})$ which have only a finite number of composition factors.

Examples 5. For any K, one has generic representations P_n and $\mathbb{G}r_n$ defined by $P_n(V) = K[\mathrm{Hom}(\mathbb{F}^n, V)]$ and $\mathbb{G}r_n(V) = K[Gr_n(V)]$, where $Gr_n(V)$ is the set of n–planes in V. By Yoneda's lemma, $\mathrm{Hom}_{\mathrm{Rep}(\mathbb{F};K)}(P_n, F) = F(\mathbb{F}^n)$, so the P_n form a set of small projective generators for $\mathrm{Rep}(\mathbb{F}; K)$.

Remark 6. We give a sense of what P_n looks like when $\mathbb{F} = K = \mathbb{F}_2$. First note that $P_n = P_1^{\otimes n}$. Then P_1 has an infinite number of composition factors: it is the direct sum of a one dimensional constant functor and a uniserial module with composition factors Λ^n for $n \geq 1$.

Pondering this in the late 1980's, Lionel Schwartz conjectured that the functors $P_n \in \mathrm{Rep}(\mathbb{F}; \mathbb{F})$ are always Noetherian objects. This has been recently proved by Steven Sam, Andrew Snowden, and Andrew Putman with proofs [PS14, SS14] that show this for all $\mathrm{Rep}(\mathbb{F}; K)$. Corollary 2 then shows that a much stronger result holds in the non-describing characteristic case.

3. Connection with $K[GL_n(\mathbb{F})]$–modules

Note that $F \in \mathrm{Rep}(\mathbb{F}; K)$ determines a $K[GL_n(\mathbb{F})]$–module $F(\mathbb{F}^n)$ for each n. Even better: $F(\mathbb{F}^n)$ is a module for the semigroup ring $K[M_n(\mathbb{F})]$. These modules for different n are compatible as follows. Let $e_{n-1} = [I_{n-1}] \in K[M_n(\mathbb{F})]$ where I_{n-1} is the $n \times n$ matrix which has 1's on the first $(n-1)$ diagonal entries and is zero elsewhere. Then $F(\mathbb{F}^{n-1})$ is naturally isomorphic to $e_{n-1}F(\mathbb{F}^n)$, and $\mathrm{Rep}(\mathbb{F}; K)$ is roughly the category of such compatible sequences of $K[M_n(\mathbb{F})]$–modules.

Furthermore, as described in [K94], one has a recollement diagram:

$$K[GL_n(\mathbb{F})]\text{–mod} \;\underset{\xleftarrow{\;p\;}}{\overset{\xleftarrow{\;q\;}}{\xrightarrow{\;i\;}}}\; K[M_n(\mathbb{F})]\text{–mod} \;\underset{\xleftarrow{\;r\;}}{\overset{\xleftarrow{\;l\;}}{\xrightarrow{\;e\;}}}\; K[M_{n-1}(\mathbb{F})]\text{–mod}.$$

In this diagram, e is multiplication by e_{n-1}, and the functors i and e are exact, with left adjoints q and l, and right adjoints p and r.

Thus $\mathrm{Rep}(\mathbb{F}; K)$ is built from the categories of $K[GL_n(\mathbb{F})]$–modules for all n. One consequence is that there is a bijection

$$\{\text{simple } F \in \mathrm{Rep}(\mathbb{F}; K)\} \leftrightarrow \coprod_{n=0}^{\infty} \{\text{simple } K[GL_n(\mathbb{F})]\text{–modules }\}.$$

Example 7. If K is the trivial $K[GL_n(\mathbb{F}_2)]$–module, the corresponding simple functor in $\mathrm{Rep}(\mathbb{F}; K)$ is Λ^n if $K = \mathbb{F}_2$, and $\mathbb{G}r_n$ if *char* $K \neq 2$.

4. Kovács' theorem

Let $\mathrm{Sing}_n(\mathbb{F}) \subset M_n(\mathbb{F})$ be the set of singular matrices. There is a short exact sequence

$$0 \to K[\mathrm{Sing}_n(\mathbb{F})] \to K[M_n(\mathbb{F})] \to K[GL_n(\mathbb{F})].$$

Theorem 8. [Ko92] *Let $\mathbb{F}$ be a finite field of characteristic p. If p is invertible in K, the sequence above splits as unital K–algebras. Equivalently, there exists $e_n^S \in K[\mathrm{Sing}_n(\mathbb{F})]$ which serves as a unit.*

Remark 9. e_n^S is easily seen to be the unique element in $K[\mathrm{Sing}_n(\mathbb{F})]$ that is invariant under conjugation by $GL_n(\mathbb{F})$ and satisfies $e_n^S e_{n-1} = e_{n-1}$.

5. Sketch proof of Theorem 1

Let $e_n^G = 1 - e_n^S \in K[M_n(\mathbb{F})]$. e_n^G is a central idempotent satisfying:

(1) $e_n^G K[M_n(\mathbb{F})] e_n^G \simeq K[GL_n(\mathbb{F})]$ as algebras.
(2) $e_n^G \cdot [A] = 0$ for all $A \in \mathrm{Sing}_n(\mathbb{F})$.

The algebra $\mathrm{End}_{\mathrm{Rep}(\mathbb{F};K)}(P_n)$ identifies with $K[M_n(\mathbb{F})]$, and we let $P_n^G = P_n e_n^G$.

The next two propositions follow from the properties of e_n^G.

Proposition 10. *$P_n \simeq \bigoplus_{k=0}^n gr_k(n) P_n^G$, where $gr_k(n)$ is the number of k–planes in $\mathbb{F}^n$. Thus the P_n^G form a set of small projective generators for* $\mathrm{Rep}(\mathbb{F}, K)$.

Proposition 11. $\mathrm{Hom}_{\mathrm{Rep}(\mathbb{F};K)}(P_m^G, P_n^G) = \begin{cases} K[GL_n(\mathbb{F})] & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases}$

The main theorem then follows from these using general Morita theory.

Given a sequence $M_0, M_1, M_2, \ldots$, with M_n a $K[GL_n(\mathbb{F})]$–module, the associated generic representation F is

$$F = \bigoplus_{n=0}^{\infty} P_n^G \otimes_{K[GL_n(\mathbb{F})]} M_n.$$

Conversely, given $F \in \mathrm{Rep}(\mathbb{F}; K)$, the associated sequence has

$$M_n = \mathrm{Hom}_{\mathrm{Rep}(\mathbb{F};K)}(P_n^G, F).$$

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