Mathematisches Forschungsinstitut Oberwolfach

Free Probability Theory

Organised by
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7 June – 13 June 2015

ABSTRACT. The workshop brought together leading experts, as well as promising young researchers, in areas related to recent developments in free probability theory. Some particular emphasis was on the relation of free probability with random matrix theory.

Introduction by the Organisers

The workshop *Free Probability Theory*, organised by Alice Guionnet (MIT), Roland Speicher (Saarland University), and Dan Voiculescu (UC Berkeley), was held June 7 - 13, 2015. This meeting was well attended with over 50 participants with broad geographic representation from Austria, Canada, Denmark, France, Germany, Israel, Ireland, Japan, Luxembourg, Mexico, Poland, Switzerland, USA.

Free probability theory is a line of research which parallels aspects of classical probability, in a non-commutative context where tensor products are replaced by free products, and independent random variables are replaced by free random variables. It grew out from attempts to solve some longstanding problems about von Neumann algebras of free groups. In the almost thirty years since its creation, free probability has become a subject in its own right, with connections to several other parts of mathematics: operator algebras, the theory of random matrices, classical probability, the theory of large deviations, and algebraic combinatorics. Free probability also has connections with some mathematical models in theoretical physics and quantum information theory, as well as applications in statistics and wireless communications.

Free probability is certainly a very active area, with many unsolved problems ahead, as well as various recent new exciting developments. The Oberwolfach workshop brought together various mathematical backgrounds and was strong on the connections of free probability with other fields, with particular emphasis on the random matrix perspective. The diversity of the participants and the ample free time left in the programme stimulated a lot of fruitful discussions.

The programme consisted of 19 lectures of 50 minutes, and four lectures of 30 minutes. Because of the various backgrounds of the participants much emphasis was put on making the lectures accessible to a broad audience; most of them provided a survey on the background as well as highlighting some recent developments in connection with free probability.

In the following we want to highlight examplarily some of the topics which where covered in the talks: new results on the theory of bifreeness (free probability for pairs of faces); relations between braid group problems and free probability; aspects of asymptotics of representations; various aspects of eigenvalues of different classes of random matrices; applications of free probability in quantum information theory; regularity questions for polynomials in free variables and free stochastic integrals; relations between classical and free stochastic analysis; non-commutative distributions, traffic distribution, and free transport; non-commutative rational functions

Instead of going into more detail we will let the following abstracts speak for themselves.

Workshop: Free Probability Theory

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Abstracts

Free probability for pairs of faces
DAN-VIRGIL VOICULESCU

We recently introduced an extension of free probability to systems with left and right non-commutated random variables, based on a notion of bi-free independence. Half of the talk dealt with general properties of the emerging bi-free probability theory, like the relation to free and classical non-commutative independence, bi-free convolution operations, existence and uniqueness of additive bi-free cumulants, the bi-free Gaussian distributions and central limit theorem. The second part of the talk dealt with the explicit formulae we have found in the simple case of a pair of one left and one right variable for partial transforms for bi-additive, bi-multiplicative and additive-multiplicative convolution operations as well as for the bi-free max-convolution in the case of commuting variables. For a commuting pair of variables this settles the problems of finding the distribution of sums, products, sum-products, and max-es of two bi-free pairs. In particular, the result about bi-free max-convolution reduces the questions about finding the bi-free, max-stable and max-infinitely divisible laws to analysis questions in the classical context for bi-variable distributions.

The approach in the work covered essentially analytic, the recent combinatorial developments this has triggered were presented in talks by Paul Shoufranis and by Ian Charlesworth.

REFERENCES

[1] D. V. Voiculescu, *Free probability for pairs of faces I*, Communication Math. Phys. **332** (2014), 955–980.
[2] D. V. Voiculescu, *Free probability for pairs of faces II: 2 variables bi-free partial R-transform*, arXiv: 1308.2035.
[3] D. V. Voiculescu, *Free probability for pairs of faces II: 2 variables bi-free partial S- and T-transforms*, arXiv: 1504.03765.
[4] D. V. Voiculescu, *Free probability for pairs of faces IV: bi-free extremes in the plane*, arXiv: 1505.05020.

Enumeration of braids and free cumulants
PHILIPPE BIANE
(joint work with Patrick Dehornoy)

A Garside structure on a group is a system of generators such that each element of the group has a nice (called "S-normal") decomposition into a product of these generators. We consider the dual Garside structure of braid groups which is closely related to noncrossing partitions. The question is to enumerate the number $b_{n,d}$ of S-normal decompositions having length d in the braid group B_n. It turns out that this computation involves free cumulants: for $d = 2$, the sequence $b_{n,2}$ is the sequence of free cumulants of a product of independant commuting variables

distributed as the the square of semi-circular variables. We show that this is actually a particular case of a general formula for computing the free cumulants of a product of independent commuting random variables.

REFERENCES

[1] Philippe Biane, Patrick Dehornoy *Dual Garside structure of braids and free cumulants of products*, Séminaire Lotharingien de Combinatoire, B72b (2014), 15 pp.

Between random matrices, irreducible representations and lozenge tilings

VADIM GORIN

The talk is about Laws of Large Numbers and Central Limit Theorems for the global fluctuations in the study of random matrices, decompositions of linear representations of Lie groups into irreducible components, random lozenge tilings, and discrete log–gases.

A 25 years old result of Voiculescu [Vo] describes the Law of Large Numbers for the empirical measure of the eigenvalues of the sum C of two independent uniformly random $N \times N$ Hermitian random matrices A and B with prescribed eigenvalues $\{a_i\}$, $1 \le i \le N$ and $\{b_i\}$, $1 \le i \le N$, respectively. The result says that if the empirical measures of the eigenvalues of A and B converge as $N \to \infty$ to probability measures μ_A and μ_B, respectively, then the (a priory random) empirical measure of the eigenvalues of $C = A + B$ converges to a deterministic measure μ_C, which is the *free convolution* of μ_A and μ_B, $\mu_C = \mu_A \boxplus \mu_B$. The quantization of this problem deals with the asymptotics of decomposition into irreducible components for the tensor product of two irreducible representations of the unitary group $U(N)$. One important difference with random matrix case is that while the spectrum of a matrix is continuous, the labels of irreducible representations are discrete. We show that in this quantization a similar Law of Large Numbers is valid with free convolution replaced by its deformation — *quantized free convolution.*

Instead of adding two independent matrices, one can cut corners from a single matrix. This operation also results in a Law of Large Numbers, which is now related to the notion of *free projection.* The representation–theoretic analogue of this setup is decomposition of the restriction of an irreducible representation of $U(N)$ onto a subgroup $U(\alpha N) \subset U(N)$, $0 < \alpha < 1$. There is also an equivalent statistical–mechanics formulation where one considers sections of uniformly random lozenge tilings of planar domains. This analogue again leads to a Law of Large Numbers; it is related to a deformation (quantization) of the free projection.

We further study the second order corrections to the Laws of Large Numbers, which result in a form of Central Limit Theorem, as the global fluctuations turn out to be Gaussian. An interesting question arises on the identification and conceptual explanation of the covariance structure of the resulting one–dimensional

Gaussian field. In the random matrix context the covariance can be computed in the framework of *second order freeness*, but the resulting formulas still ask for simple explanations. Such explanation exists in the context of restrictions of the representations and lozenge tilings — in this case the limiting random field becomes $2d$ (the parameter α adds a second dimension) and can be identified with the pullback of $2d$ *Gaussian Free Field* with respect to a map constructed in a relatively simple way. For the tensor products we do not yet know any conceptual explanation and the covariance (which is structurally similar to the random matrix case, but again involves a certain deformation) still remains somewhat mysterious.

In this part our proofs rely on analysis of generating functions in terms of symmetric polynomials via contour integral representations and differential operators.

Another instance of the Central Limit Theorems in the random matrix context is that for general β log–gases, i.e. distributions on N–tuples of reals with density proportional two

$$\prod_{1 \le i < j \le N} |x_i - x_j|^{\beta} \prod_{i=1}^{N} \exp(NV(x_i)),$$

where $V(x)$ is called the *potential.* Starting from the pioneering work of Johansson [Jo] it was shown that under weak assumptions on *analytic* potential $V(x)$ the global fluctuations are always Gaussian and, moreover, the resulting covariance structure possesses certain universality (it depends only on the *support* of the equilibrium measure describing the Law of Large Numbers in the system).

The Johansson's proof (as well as all the following ones) is based on exploiting certain equations for the observables of log–gases, which are known as *loop* or Schwinger–Dyson equations. The $N \to \infty$ limit of these equations attracted attention in the free probability theory in relation with the notion of *conjugate variables.*

Despite numerous potential applications, the extension of this CLT to *discrete* log–gases was out of reach for a long time because of the absence of an appropriate analogue of the loop equations. It turns out that the key to finding such equations lies in the correct definition of the discretization. We let the discrete log–gas to be a probability measure on N—tuples of integers $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$ defined in the coordinates $\ell_i = \lambda_i + i\theta$ (where $\theta > 0$ is a parameter playing the role of $\beta/2$ in continuous log–gases) via

$$\mathbb{P}(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z} \prod_{1 \le i < j \le N} \frac{\Gamma(\ell_j - \ell_i + 1)\Gamma(\ell_j - \ell_i + \theta)}{\Gamma(\ell_j - \ell_i)\Gamma(\ell_j - \ell_i + 1 - \theta)} \prod_{i=1}^{N} w(\ell_i),$$

where Z is a normalization constant. We prove that under certain technical assumptions on the weight $w(x)$ the Central Limit Theorem is valid for the global fluctuations for such discrete log–gases. Moreover, the limiting covariance turns out to be the same as in the random–matrix case. This looks unexpected compared to the aforementioned results, where discreteness always resulted in a deformation of the answer.

In the second part our proofs rely on the use of the novel *discrete loop equations*; the idea for building such equations originates in the articles [NS], [NP], [N].

The results reported in this talk are based on three papers: [BG1], [BGG], [BG2].

REFERENCES

[BGG] A. Borodin, V. Gorin, A. Guionnet, Gaussian asymptotics of discrete β-ensembles, arXiv:1505.03760
[BG1] A. Bufetov, V. Gorin, Representations of classical Lie groups and quantized free convolution (joint paper with A. Bufetov), to appear in *Geometric and Functional Analysis (GAFA)*. arXiv:1311.5780
[BG2] A. Bufetov, V. Gorin, Fluctuations of particle systems determined by Schur generating functions, in preparation.
[Jo]   K. Johansson, *On Fluctuations of eigenvalues of random Hermitian matrices,* Duke Mathematical Journal, **91,** no. 1 1998.
[NS]   N. Nekrasov, V. Pestun, S. Shatashvili, Quantum geometry and quiver gauge theories, *High Energy Physics - Theory*, 2013, 1-83, arXiv:1312.6689.
[NP]   N. Nekrasov, V. Pestun, Seiberg-Witten geometry of four dimensional $N = 2$ quiver gauge theories. arXiv:1211.2240
[N]   N. Nekrasov, Non-perturbative Dyson–Schwinger equations and BPS/CFT correspondence, in preparation.
[Vo]   D. Voiculescu, *Addition of certain non-commuting random variables,* J. Funct. Anal., **66** (1986), 323–346.

Outliers in the Single Ring Theorem

FLORENT BENAYCH-GEORGES

(joint work with Jean Rochet)

This talk is about spiked models of non-Hermitian random matrices. More specifically, we consider matrices of the type $\mathbf{A} + \mathbf{P}$, where the rank of $\mathbf{P}$ stays bounded as the dimension goes to infinity and where the matrix $\mathbf{A}$ is a non-Hermitian random matrix, satisfying an isotropy hypothesis: its distribution is invariant under the left and right actions of the unitary group. The macroscopic eigenvalue distribution of such matrices is governed by the so called *Single Ring Theorem*, due to Guionnet, Krishnapur and Zeitouni. We first prove that if $\mathbf{P}$ has some eigenvalues out of the maximal circle of the single ring, then $\mathbf{A} + \mathbf{P}$ has some eigenvalues (called *outliers*) in the neighborhood of those of $\mathbf{P}$, which is not the case for the eigenvalues of $\mathbf{P}$ in the inner cycle of the single ring. Then, we study the fluctuations of the outliers of $\mathbf{A}$ around the eigenvalues of $\mathbf{P}$ and prove that they are distributed as the eigenvalues of some finite dimensional random matrices. Such kind of fluctuations had already been shown for Hermitian models. More surprising facts are that outliers can here have very various rates of convergence to their limits (depending on the Jordan Canonical Form of $\mathbf{P}$) and that some correlations can appear between outliers at a macroscopic distance from each other (a fact already noticed by Knowles and Yin in [1] in the Hermitian case, but only for non Gaussian models, whereas spiked Gaussian matrices belong to our model and can

have such correlated outliers). Our first result generalizes a result by Tao proved specifically for matrices with i.i.d. entries, whereas the second one (about the fluctuations) is new.

REFERENCES

[1] A. Knowles, J. Yin *The outliers of a deformed Wigner matrix*, Ann. Probab. 42 (2014), no. 5, 1980–2031.

On the Combinatorics of Bi-Freeness

Paul Skoufranis

Free probability has been an important area of Operator Algebras since its inception by Voiculescu in [10]. Although originally motivated by analytic techniques, a combinatorial approach to free probability was developed by Speicher in [9] via the theory of non-crossing partitions. These two approaches to free probability each have their own advantages and many results may be demonstrated using either approach.

Approximately two years ago, Voiculescu introduced the notion of bi-free pairs of algebras in [13]. Roughly speaking, a collection of pairs of unital algebras is bi-freely independent if their joint distribution may be computed by representing the pairs of algebras on a reduced free product vector space such that each pair of algebras is represented on one portion of the free product space with one algebra acting via the left regular representation and the other algebra acting via the right regular representation. Although the work of Muraki in [4] demonstrates that there are only five natural notions of independence for algebras, bi-free independence does not fit into this theory as it is a notion for independence for pairs of algebras.

Voiculescu was able to generalize many concepts in free probability to the bi-free setting (see [13], [14], [15], [16]). The goal of this talk is to discuss the combinatorial aspects of bi-free independence. In particular, although bi-free independence appears very different from free independence on the surface, the combinatorial aspects are not very different. However, although the similarities in the combinatorial structures directly imply that free and bi-free probability may not be substantially different, some surprising results can be obtained via combinatorial techniques.

The main difference between the combinatorial structures of free and bi-free independence come from handling permutations. First, given a product of operators from the left and right algebras, a map $\chi : \{1, \ldots, n\} \to \{\ell, r\}$ is used to designate whether the kth operator in a sequence of n operators should be considered a left operator (when $\chi(k) = \ell$) or a right operator (when $\chi(k) = r$). If

$$\chi^{-1}(\{\ell\}) = \{i_1 < \cdots < i_p\} \quad \text{and} \quad \chi^{-1}(\{r\}) = \{i_{p+1} > \cdots > i_n\},$$

the necessary permutation on $\{1, \ldots, n\}$ is s_χ which defined by $s_\chi(k) = i_k$. The analogue of non-crossing partitions for bi-free independence are the bi-non-crossing partitions with respect to χ, which are all partitions π on $\{1, \ldots, n\}$ such that

$s_\chi^{-1} \cdot \pi$ (the partition formed by applying s_χ^{-1} to each entry of each block of π) is non-crossing. These partitions have origins in the paper [3] of Mastnak and Nica and were motivated by two-sided queues.

In joint work with Charlesworth and Nelson in [2], the notion of bi-non-crossing partitions was connected with bi-free independence. This was done by describing the universal bi-free moment polynomials via bi-non-crossing partitions and verifying that mixed cumulants corresponding to this partition lattice vanishing is equivalent to bi-free independence. The results of [2] enable a combinatorial approach to bi-free probability and many results which are described below.

The R-transform and S-transform are important objects in free probability which were introduced in [11] and [12] respectively. Using analytic techniques, Voiculescu constructed partial bi-free R- and S-transforms in [14] and [15] respectively. Using the notion of bi-non-crossing partitions, one can use combinatorial techniques to construct the partial bi-free R- and S-transforms, which was done in [6] and [7] respectively.

In [13], Voiculescu noticed that both free independence and classical independence occur from bi-freely independent pairs of algebras. Perhaps surprisingly, it was demonstrated in [6] that Boolean independence and (anti-)monotonically independence also arise from bi-freely independent pairs of algebras. Consequently, all five natural notions of independence for algebras can be studied through bi-free pairs of algebras. One interesting question to ask is, "Is it possible that other notions of independence, such as conditionally free independence and type B free independence, can be realized through bi-free pairs of faces?" Such results would give evidence that bi-free independence may be a universal independence theory.

In joint work with Charlesworth and Nelson in [1], the combinatorics of bi-free independence was extended to the operator-valued setting. Although there are many technicalities and restrictions that occur due to the necessity of having two copies of the amalgamation algebra, there again is a connection with operator-valued free probability. Furthermore [1] demonstrated that bi-freeness naturally occurs from freeness whenever all left operators commute with all right operators. Consequently, one must be careful that bi-free results where all left and right algebras commute are not simply free probability results in disguise. In addition, most concrete examples of bi-free pairs of algebras occur in this commutation framework, so one may ask, "Are there any unexpected examples of bi-free pairs of algebras?"

One interesting result in free probability is that if two algebras A_1 and A_2 are free with respect to φ, then the matrix algebras $M_N(A_1)$ and $M_N(A_2)$ are free with amalgamation over $M_N(\mathbb{C})$ with respect to φ_N where $\varphi_N([x_{i,j}]) = [\varphi(x_{i,j})]$. The bi-free analogue of this result was obtained in [6] where one needs to consider pairs of matrices where the left matrices act via left matrix multiplication whereas the right matrices at via right matrix multiplication with a certain twist. Using these actions, bi-matrix models were investigated in [8]. Unfortunately, only specific bi-free central limit distributions can be obtained using matrices of self-adjoint

Gaussian random variables due to commutativity. However, if left and right creation and annihilation operators on a Fock space are used, then all bi-free central limit distributions may be obtain. Furthermore, [8] demonstrates that matrices of left and right q-deformed creation and annihilation operators asymptotically tend to left and right creation and annihilation operators on a Fock space in distribution thereby generalizing the results of [5]. Despite these results, it is still natural to ask, "Are there honest random matrices that can realize all bi-free central limit distributions?"

REFERENCES

[1] I. Charlesworth, B. Nelson, and P. Skoufranis, *Combinatorics of Bi-Free Probability with Amalgamation*, Comm. Math. Phys. **338** (2015), no. 2, 801–847.

[2] I. Charlesworth, B. Nelson, and P. Skoufranis, *On two-faced families of non-commutative random variables*, to appear in Canad. J. Math. (2015), 26 pages.

[3] M. Mastnak and A. Nica, *Double-ended queues and joint moments of left-right canonical operators on full Fock space*, preprint at arXiv:1312.0269 (2013), 28 pages.

[4] N. Muraki, *The five independences as natural products*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **6** (2003), no. 3, 337–371.

[5] D. Shlyakhteno, *Limit distributions of matrices with bosonic and fermionic entries*, Free probability theory (Waterloo, ON, 1995) **12** (1995), 241–252.

[6] P. Skoufranis, *Independences and Partial R-Transforms in Bi-Free Probability*, to appear in Ann. Inst. Henri Poincaré Probab. Stat. (2015), 31 pages.

[7] P. Skoufranis, *A Combinatorial Approach to Voiculescu's Bi-Free Partial Transforms*, preprint at arXiv:1504.06005 (2015), 19 pages.

[8] P. Skoufranis, *Some Bi-Matrix Models for Bi-Free Limit Distributions*, preprint at arXiv:1506.01725 (2015), 26 pages.

[9] R. Speicher, *Multiplicative functions on the lattice of non-crossing partitions and free convolution*, Math. Ann. **298** (1994), 611–628.

[10] D. Voiculescu, *Symmetries of some reduced free product C*-algebras*, Operator Algebras and Their Connection with Topology and Ergodic Theory, Springer Lecture Notes in Mathematics, Springer Verlag, **1132** (1985), 556–588.

[11] D. Voiculescu, *Addition of certain non-commuting random variables*, J. Funct. Anal. **66** (1986), no. 3, 323–346.

[12] D. Voiculescu, *Multiplication of certain non-commuting random variables*, J. Operator Theory **18** (1987), 223–235.

[13] D. Voiculescu, *Free Probability for Pairs of Faces I*, Comm. Math. Phys. **332** (2014), 955–980.

[14] D. Voiculescu, *Free Probability for Pairs of Faces II: 2-Variable Bi-Free Partial R-Transform and Systems with Rank ≤ 1 Commutation*, preprint at arXiv:1308.2035, (2013), 21 pages.

[15] D. Voiculescu, *Free Probability for Pairs of Faces III: 2-Variables Bi-free Partial S- and T-Transforms*, preprint at arXiv:1504.03765, (2015), 16 pages.

[16] D. Voiculescu, *Free Probability for Pairs of Faces IV: Bi-free Extremes in the Plane*, preprint at arXiv:1505.05020, (2015), 12 pages.

k-positive maps and free probability

BENOIT COLLINS

(joint work with Patrick Hayden and Ion Nechita)

1. INTRODUCTION

This report is an abridged version of the manuscript [1]. Completely positive maps play an important role in Quantum Information Theory. The Stinespring theorem provides a useful classification of these maps.

Maps with weaker positivity properties are of interest in quantum information theory because they fail in subtle ways to be physically realizable. Specifically, positive but not completely positive maps acting on entangled states may fail to preserve positivity, mapping states to non-positive operators. Since such maps will always preserve positivity for separable quantum states, positive but not completely positive maps can be used to detect the presence of entanglement. The most famous such test is partial transposition; the states with no entanglement detectable this way are known as the PPT states.

Our goal is to use free probability and random matrix techniques to construct new families of positive but not completely positive maps and apply these maps to the study of entanglement. The basic idea is to model the Choi map of a linear map instead of the map itself, and assume that the Choi map is largely independent of its blocks (in the free or asymptotically free sense).

2. POSITIVE MAPS AND ENTANGLEMENT DETECTION

Let $\mathcal{A}, \mathcal{B}$ be two C^*-algebras. Positive elements in a C^* algebra are elements x that are self-adjoint and that can be written as $x = yy^*$ for some y in the C^*-algebra. A *positive* linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is a map that sends positive elements to positive elements. If Φ is positive, then $\Phi_k = \mathrm{id}_k \otimes \Phi : \mathcal{M}_k(\mathbb{C}) \otimes \mathcal{A} \to \mathcal{M}_k(\mathbb{C}) \otimes \mathcal{B}$ is not necessarily positive, unless $\mathcal{A}$ or $\mathcal{B}$ are commutative. For example if $\mathcal{A} = \mathcal{B} = \mathcal{M}_n(\mathbb{C}), n \geq 2$ and Φ is the transpose map, then, Φ_k is not positive as soon as $k \geq 2$.

However if Φ_k is positive, then for $l \leq k$, the map Φ_l is clearly positive too. A map Φ such that Φ_k is positive is said to be k-positive. A completely positive map is one that has this property for all integers k. Note that if $\mathcal{A} = \mathcal{M}_n(\mathbb{C})$ and $\mathcal{B} = \mathcal{M}_d(\mathbb{C})$, $\min(n, d)$-positive is equivalent to completely positive. There are many examples of maps that are k positive but not $(k + 1)$-positive, but the classification of positive, or even k-positive maps is far from complete.

Next, we recall the terminology of Choi matrices. We denote by $\mathcal{M}_n(\mathbb{C})$ the algebra of $n \times n$ complex matrices. Let $E = (E_{ij})_{i,j\in\{1,\ldots n\}}$ be the basis of matrix units, i.e. $E_{ij}E_{kl} = E_{il}\delta_{jk}$ and $E_{ij}^* = E_{ji}$. For a C^*-algebra $\mathcal{A}$, let $\Phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{A}$ be a linear map. Its Choi matrix in the basis E, denoted by C_Φ is an element of

$\mathcal{M}_n(\mathbb{C}) \otimes \mathcal{A}$ defined as

$$C_\Phi = \sum_{i,j=1}^n E_{ij} \otimes \Phi(E_{ij}).$$

According to a celebrated result of Choi, Φ is completely positive if and only if C_Φ is positive in $\mathcal{M}_n(\mathbb{C}) \otimes \mathcal{A}$. We recall the following result, characterizing k-positivity of maps.

Proposition 1. *Consider a linear map $\Phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. The following are equivalent:*

(1) *The map Φ is k-positive.*
(2) *The operator $(P \otimes 1_\mathcal{H})C_\Phi(P \otimes 1_\mathcal{H})$ is positive semidefinite for any rank k orthogonal projection $P \in \mathcal{M}_n(\mathbb{C})$ ($1_\mathcal{H}$ is the identity operator in $\mathcal{B}(\mathcal{H})$).*

In the sequel, we elaborate a new systematic method to obtain k-positive maps from $\mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$, based on free probability techniques. In the next section, we describe the free probability version, and the last section describes its random matrix avatar.

3. Positive maps through free probability

We assume in the context of this conference that the reader is familiar with basic notions of free probability. We refer to references in [1] for further details, and in particular to the book of Nica and Speicher for basics about free probability. Here, we just recall the following important lemma

Lemma 2. *Let a, p be free elements in a non-commutative probability space $(\mathcal{A}, \phi)$ and assume that p is a self-adjoint projection of rank $t \in (0, 1)$ and that a is a self-adjoint random variable having distribution μ. Then, the distribution of $t^{-1}pap$ in $(p\mathcal{A}p, \phi(p\cdot))$ is $\mu^{\boxplus 1/t}$.*

We now introduce the main idea of this paper, a construction of a linear map depending on a probability measure μ. We start with a compactly supported probability measure μ on $\mathbb{R}$ and fix an integer n. Consider the space $L^\infty(\mathbb{R}, \mu)$. Note that this space is spanned as a von Neumann algebra by the operator $x \mapsto x$, which we will denote by X. By construction, X is a self-adjoint operator and its spectrum is the support of μ. Next, consider the free product

$$(\tilde{\mathcal{M}}, \mathrm{tr} * \mathbb{E}) := (\mathcal{M}_n(\mathbb{C}), \mathrm{tr}) * (L^\infty(\mathbb{R}, \mu), \mathbb{E})$$

and the contracted von Neumann probability space $(\mathcal{M}, \tau)$ where $\mathcal{M} = E_{11}\tilde{\mathcal{M}}E_{11}$ is the contracted algebra of $\tilde{\mathcal{M}}$ and τ is the restriction of $\mathrm{tr} * \mathbb{E}$ appropriately normalized (by a factor n). Defining the map $\Phi_\mu : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}$ given by

$$\Phi_\mu(E_{ij}) = E_{1i}XE_{j1},$$

our main result is as follows:

Theorem 3. *Let μ be a compactly supported probability measure. The map Φ_μ defined above is k-positive if and only if $\mathrm{supp}(\mu^{\boxplus n/k}) \subseteq [0, \infty)$.*

4. Positive maps through random matrix theory

The above result yields a very systematic method to obtain many maps that are k-positive but not $k + 1$ positive from $\mathcal{M}_n(\mathbb{C})$ to a von Neumann algebra for any integers k, n such that $1 \leq k < n$. However, the domain space of this example is not perfectly well understood. It turns out however that the whole idea can be translated into the random matrix theory context. The drawback is that one requires randomness and an additional approximation parameter which makes the estimates less quantitative. But the advantage is that the algebras do not depend on the choice of μ: they are just large matrix algebras.

Instead of describing the whole translation (for which we refer to [1]), we focus on the specific case of GUE random matrices and show important applications in this context. We first start with the following definition:

Definition 4. A random matrix $Z \in \mathcal{M}_d(\mathbb{C})$ is said to have the GUE_d distribution if its entries are as follows:

$$Z_{ij} = \begin{cases} X_{ii}/\sqrt{d}, & \text{if } i = j \\ (X_{ij} + \mathrm{i}Y_{ij})/\sqrt{2d}, & \text{if } i < j \\ \bar{Z}_{ji}, & \text{if } i > j, \end{cases}$$

where $\{X_{ij}, Y_{ij}\}_{i,j=1}^d$ are i.i.d. centered, standard real Gaussian random variables.

Given this definition, we say that $\tilde{Z}$ is a shifted GUE_d of mean a and variance b iff it can be written $\tilde{Z} = aId_d + \sqrt{b}Z$ where Z is a standard GUE_d as defined above. With these definitions at hand, we can show the following theorem.

Theorem 5. *Let Z_d be a shifted GUE random matrix in $\mathcal{M}_n(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C})$ of mean 2 and variance $\alpha \in [0, 1)$. Then there is a linear map $\Phi_d : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ – that is itself a shifted GUE – such that its Choi matrix is again a shifted GUE matrix (which depends on Z_d), of mean $(2 + \varepsilon)/\sqrt{n}$ and variance 1. With probability one as $d \to \infty$:*

   (1) *The matrix Z_d is positive and PPT.*
   (2) *The map Φ_d is positive. However, Φ_d is not completely positive, provided that $2 + \varepsilon < \sqrt{n}$.*
   (3) *As soon as $2(2 + \varepsilon) < \alpha\sqrt{n}$, the map Φ_d detects the entanglement present in Z_d.*

This theorem shows that it is possible to choose a mean, a variance and a d large enough such that, *typically*, linear maps $\mathcal{M}_n(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C})$ whose Choi matrix is a shifted GUE_{nd} are positive, not completely positive, and detect much more efficiently entanglement than PPT (in particular, these maps are far from being decomposable).

References

[1] B. Collins, P. Hayden, I. Nechita, *Random and free positive maps with applications to entanglement detection*, ArXiv:1505.08042

A new proof of Friedman's second eigenvalue Theorem and its extension to random lifts

Charles Bordenave

This talk is based on the preprint [5]. Consider a finite graph $G = (V, E)$ with $n = |V|$ vertices. Its adjacency matrix A is the matrix indexed by V and defined for all $u, v \in V$ by $A_{uv} = \mathbf{1}(\{u, v\} \in E)$. The matrix A is symmetric, its eigenvalues are real and we order them non-increasingly,

$$\mu_n \leq \ldots \leq \mu_1.$$

We assume further that, for some integer $d \geq 3$, the graph G is d-regular, that is, all vertices have degree d. We then have that $\mu_1 = d$, that all eigenvalues have absolute value at most d, and $\mu_n = -d$ is equivalent to G being bipartite. The absolute value of the largest non-trivial eigenvalues of G is denoted by $\mu = \max\{|\mu_i| : |\mu_i| < d\}$. Classical statements such as Cheeger's isoperimetric inequality or Chung's diameter inequality relate small values of μ or μ_2 with good expanding properties of the graph G, we refer for example to [7, 11]. It turns out that μ cannot be made arbitrarily small. Indeed, a celebrated result of Alon-Boppana implies that for any d-regular graph with n vertices,

$$(1) \qquad\qquad\qquad \mu_2 \geq 2\sqrt{d-1} - \varepsilon_d(n),$$

where, for some constant $c_d > 0$, $\varepsilon_d(n) = c_d / \log n$, see the above references and [17]. Following [16, 14], one may try to construct graphs which achieve the Alon-Boppana bound. A graph is called Ramanujan if $\mu \leq 2\sqrt{d-1}$. Proving the existence of Ramanujan graphs with a large number of vertices is a difficult task which has been solved for arbitrary $d \geq 3$ only recently [15]. On the other end, it was conjectured by Alon [2] and proved by Friedman [9] that most d-regular graphs are weakly Ramanujan. More precisely, for integer $n \geq 1$, we define $\mathcal{G}_d(n)$ as the set of d-regular graphs with vertex set $\{1, \ldots, n\}$. If nd is even and $n \geq n_0(d)$ large enough, this set is non-empty. A uniformly sampled d-regular graph is then a random graph whose law is the uniform distribution on $\mathcal{G}_d(n)$.

Theorem 1 (Friedman's second eigenvalue Theorem [9])**.** *Let $d \geq 3$ be an integer and nd be even with $n \geq n_0(d)$. If G is uniformly distributed on $\mathcal{G}_d(n)$, we have for any $\varepsilon > 0$,*

$$\lim_{n \to \infty} \mathbb{P}\Big(\mu_2 \vee |\mu_n| \geq 2\sqrt{d-1} + \varepsilon\Big) = 0,$$

where the limit is along any sequence going to infinity with nd even.

The first aim of this talk is to give a new proof of this important result. The argument simplifies substantially the original proof and it allows to take, in this above statement, $\varepsilon = c \log \log n / \log n$ for some large constant $c > 0$ depending on d. The method is quite robust and it has already been recently applied in [6] to random graphs with structures (stochastic block model).

The second aim of this talk is to apply this method to study similar questions on the eigenvalues of random lifts of graphs. This class of models sheds a new light on Ramanujan-type properties, and, since the work of Amit and Linial [3, 4] and

Friedman [8], it has attracted a substantial attention [12, 1, 13, 18, 10]. We will give a simpler proof of a recent result of Friedman and Kohler [10] and establish a weak Ramanujan property for the non-backtracking eigenvalues of a random lift of an arbitrary graph.

References

[1] L. Addario-Berry and S. Griffiths. The spectrum of random lifts. arXiv:1012.4097, 2010.
[2] N. Alon. Eigenvalues and expanders. *Combinatorica*, 6(2):83–96, 1986. Theory of computing (Singer Island, Fla., 1984).
[3] A. Amit and N. Linial. Random graph coverings. I. General theory and graph connectivity. *Combinatorica*, 22(1):1–18, 2002.
[4] A. Amit and N. Linial. Random lifts of graphs: edge expansion. *Combin. Probab. Comput.*, 15(3):317–332, 2006.
[5] C. Bordenave. A new proof of friedman's second eigenvalue theorem and its extension to random lifts. arXiv:1502.04482, 2015.
[6] C. Bordenave, M. Lelarge, and L. Massoulié. Non-backtracking spectrum of random graphs: community detection and non-regular ramanujan graphs. arXiv:1501.06087, 2015.
[7] F. R. K. Chung. *Spectral graph theory*, volume 92 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1997.
[8] J. Friedman. Relative expanders or weakly relatively Ramanujan graphs. *Duke Math. J.*, 118(1):19–35, 2003.
[9] J. Friedman. A proof of Alon's second eigenvalue conjecture and related problems. *Mem. Amer. Math. Soc.*, 195(910):viii+100, 2008.
[10] J. Friedman and D.-E. Kohler. The relativized second eigenvalue conjecture of alon. arXiv:1403.3462, 2014.
[11] S. Hoory, N. Linial, and A. Wigderson. Expander graphs and their applications. *Bull. Amer. Math. Soc. (N.S.)*, 43(4):439–561 (electronic), 2006.
[12] N. Linial and D. Puder. Word maps and spectra of random graph lifts. *Random Structures Algorithms*, 37(1):100–135, 2010.
[13] E. Lubetzky, B. Sudakov, and V. Vu. Spectra of lifted Ramanujan graphs. *Adv. Math.*, 227(4):1612–1645, 2011.
[14] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261–277, 1988.
[15] A. Marcus, D. A. Spielman, and N. Srivastava. Interlacing families i: Bipartite ramanujan graphs of all degrees. available on `http://arxiv.org/abs/1304.4132`.
[16] G. A. Margulis. Explicit group-theoretic constructions of combinatorial schemes and their applications in the construction of expanders and concentrators. *Problemy Peredachi Informatsii*, 24(1):51–60, 1988.
[17] A. Nilli. On the second eigenvalue of a graph. *Discrete Math.*, 91(2):207–210, 1991.
[18] D. Puder. Expansion of random graphs: New proofs, new results. arXiv:1212.5216, 2012.

Regularity properties for polynomials in free variables

Dimitri Shlyakhtenko

(joint work with I. Charlesworth)

Let $P \in \mathscr{A} = \mathbb{C}[t_1, \ldots, t_n]$ be a non-constant self-adjoint non-commutative polynomial in n variables (the self-adjoint condition means that $P = P^*$, where $*$ is the natural involution on $\mathscr{A}$ determined by $t_j^* = t_j$, and P is non-constant in the

sense that $P \notin \mathbb{C}1$). Let $X_1, \ldots, X_n$ be self-adjoint variables in a non-commutative probability space (M, τ), and let $Y = P(X_1, \ldots, X_n)$. Our aim is to study the question of regularity of the distribution μ_Y of Y under various conditions on the variables $X_1, \ldots, X_n$.

Theorem 1. [SS13] Assume that $X_1, \ldots, X_n$ are free semicircular variables. Then (i) The law of Y has no atoms. (ii) The Cauchy transform $G_Y(z) = \int (t-z)^{-1} d\mu_Y(t)$ is an algebraic function. (iii) The free entropy $\chi(Y) = \iint \log|s - t| d\mu_Y(s) d\mu_Y(t)$ is finite.

This theorem leads us to the following rather bold conjecture: *Assume that some variant of Voiculescu's free entropy $\chi(X_1, \ldots, X_n)$ is finite. Then $\chi(Y)$ is also finite.*

While we cannot prove the conjecture at this point, the results below may be considered as steps towards it.

Theorem 2. [MSW, Sh14] Assume that the free entropy dimension $\delta^\star(X_1, \ldots, X_n) = n$. Then the law μ_Y is non-atomic.

In [Sh14] we have given a proof of this theorem based on ideas from L^2 (co)-homology developed in [CS03] (in the process we have weakened the hypothesis of Theorem 2, the assumptions of which were considerably stronger in the original version of [MSW], whose later version upgraded their proof to give the stronger statement presented here). The idea is to use the implication [CS03]:

(1) $$\delta^*(X_1, \ldots, X_n) = n \implies W(X_1, \ldots, X_n) = 0$$

where

$$W(X_1, \ldots, X_n) = \{(T_1, \ldots, T_n) \in \mathcal{L}^1(L^2(M, \tau))^n : \sum [T_j, J X_j J] = 0\},$$

$\mathcal{L}^1(H)$ denotes the space of trace-class operators on a Hilbert space H and $J : L^2(M, \tau) \to L^2(M, \tau)$ is the Tomita conjugation operator. If μ_Y has an atom (which by subtracting a constant we may assume to be at zero), for some nonzero projections p, q we have $pYq = 0$, which imply

$$p[JYJ, P_1]q = [JYJ, pP_1q] = 0$$

we denote by P_1 the rank one projection onto $1 \in L^2(M, \tau)$. Let now $\partial_j : \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ be the j-th Voiculescu free difference quotient, determined by the Leibnitz property that $\partial_j t_k = \delta_{j=k} 1 \otimes 1$. Denote by R_j the value of $\partial_j P$ evaluated at $J X_1 J, \ldots, J X_n J, J X_1 J, \ldots, J X_n J$. For any derivation δ we have $\delta(JYJ) = \sum_j R_j \# \delta(JX_j J)$, if we use the notation $(a \otimes b) \# T = aTb$. Applying this to $\delta = [\cdot, P_1]$ we obtain

$$0 = p[JYJ, P_1]q = \sum_j (p \otimes q) \# R_j \# [JX_j J, P_1] = \sum [JX_j J, (p \otimes q) \# R_j \# P_1] = 0.$$

Noting that $Q_j = (p \otimes q) \# R_j \# P_1 \in \mathcal{L}^1(L^2(M))$ allows us to conclude that $Q_j = 0$ for all j. A recursive argument from [MSW] then allows us to conclude that P is a scalar multiple of 1, a contradiction.

The inequality (1) has its origin in the following computation. Assume that $X_1, \ldots, X_n$ have a *dual system* in the sense of Voiculescu [Voi95]; in other words, for some bounded operators $D_j : L^2(M, \tau) \to L^2(M, \tau)$ one has $[D_j, JX_k J] = \delta_{j=k} P_1$. Then assuming the identity $\sum [R_j, JX_j J] = 0$ with R_j trace-class we can conclude that

$$Tr(\sum [R_j, JX_j J] D_k) = \sum_j Tr(R_j [JX_j J D_k]) = Tr(R_k P_1)$$

showing that $R_k \perp P_1$ as a Hilbert-Schmidt operator. Replacing D_k with $\sum a_j D_k b_j$ with $a_j, b_j \in M$ allows one to conclude that actually $R_k = 0$. This argument admits a quantitative version: it similarly shows that if $\| \sum [R_j, JX_j J] \|_1 \leq \varepsilon$, then $Tr(R_k P_1)$ must be at most $\varepsilon \| D_k \|_\infty$. Using this observation and the fact that the spectral measure of Y is singular with respect to Lebesgue measure if and only if for some sequence of positive finite rank contractions T_k with $T_k \uparrow 1$, $\| [T_k, Y] \|_1 \to 0$ [Voi79], we are able to deduce the following:

Theorem 3. Assume that $X_1, \ldots, X_n$ have a dual system. Then the spectral measure of Y cannot be singular.

References

[CS03] A. Connes and D. Shlyakhtenko, *L^2-homology for von Neumann algebras*, J. Reine Angew. Math., **586** (2005) 125–168.

[MSW] T. Mai, R. Speicher and M. Weber, *Absence of algebraic relations and of zero divisors under the assumption of full non-microstates free entropy dimension*, Preprint arXiv:1502.06357.

[SS13] D. Shlyakhtenko and P. Skoufranis, *Freely independent random variables with non-atomic distributions*, Preprint arXiv:1305.1920 (to appear in Trans. AMS).

[Sh14] D. Shlyakhtenko, *Free Entropy Dimension and Atoms*, Preprint arXiv:1408.0580.

[Voi79] D. Voiculescu, *Some results on norm-ideal perturbations of Hilbert space operators*, J. Operator Theory **2** (1979), 3–37.

[Voi95] D. Voiculescu, *The analogues of entropy and of Fisher's information measure in free probability, V*, Invent. Math., **132** (1998) 189-227.

Liberation questions in noncommutative geometry

Teodor Banica

We investigate liberation questions for the noncommutative analogues of the compact real algebraic manifolds $X \subset \mathbb{C}^N$. These noncommutative manifolds are by definition the duals of the universal C^*-algebras defined with generators $z_1, \ldots, z_N$, subject to noncommutative polynomial relations, as follows:

$$C(X) = C^* \left(z_1, \ldots, z_N \Big| P_i(z_1, \ldots, z_N) = 0 \right)$$

Observe that, by the Gelfand theorem, this construction covers all the compact real algebraic manifolds $X \subset \mathbb{C}^N$. In general, the axiomatization of the universal algebras at right is quite a tricky problem. Let us just say that the family of polynomials $\{P_i\}$ must be by definition such that the biggest C^*-norm on the universal $*$-algebra $< z_1, \ldots, z_N | P_i(z_1 \ldots, z_N) = 0 >$ is bounded.

Now let X be as above, and consider its classical version $X_{class} \subset \mathbb{C}^N$, obtained by dividing the algebra $C(X)$ by its commutator ideal:

$$C(X_{class}) = C^*_{comm}\left(z_1, \ldots, z_N \middle| P_i(z_1, \ldots, z_N) = 0\right)$$

We can think then of X as being a "liberation" of X_{class}, and the problem is that of understanding how the correspondence $X_{class} \to X$ can appear.

This latter question was recently solved in the quantum group case, cf. joint work with Bichon-Collins, Curran-Speicher, and others, plus more recent (and ongoing) work by Bichon-Dubois-Violette, Raum-Weber-Freslon, and others. Among the main findings was the fact that, for liberation purposes, the usual commutation relations $ab = ba$ can be succesfully replaced by the half-commutation relations $abc = cba$ ("half-liberation"), or by nothing at all ("liberation").

Some of these quantum group ideas apply to the general algebraic manifold case, and we therefore have a "half-liberation" problem to be solved, and a "liberation" question as well. Some preliminary work in this direction was done in the recent papers [1], [2], [3], [4], [5], [6], with the conclusion that the half-liberation question is basically within reach, and that the liberation question needs some new ideas.

References

[1] T. Banica, The algebraic structure of quantum partial isometries, preprint 2014.
[2] T. Banica, Liberations and twists of real and complex spheres, *J. Geom. Phys.* **96** (2015), 1–25.
[3] T. Banica, Quantum isometries of noncommutative polygonal spheres, *Münster J. Math.*, to appear.
[4] T. Banica, A duality principle for noncommutative cubes and spheres, *J. Noncommut. Geom.*, to appear.
[5] T. Banica, Half-liberated manifolds, and their quantum isometries, preprint 2015.
[6] T. Banica, Liberation theory for noncommutative homogeneous spaces, preprint 2015.

Some new limit theorems on free chaoses

Giovanni Peccati

(joint work with Solesne Bourguin, Ivan Nourdin, Guillaume Poly, and Rosaria Simone)

The aim of this presentation is to discuss some new central limit theorems (that is, limit results where the target distribution is given by a semicircular law) for sequences of homogeneous sums involving products of freely independent random variables. In particular, our main task is to explain the following result, taken from [5] (all non-commutative random variables considered in the sequel are defined on an adequate free probability space $(\mathcal{A}, \varphi)$).

Theorem 1. *Let $d \geq 2$, and consider a sequence of symmetric kernels vanishing on diagonals $f_N : [N]^d \to \mathbb{C}$, $N \geq 1$, such that*

$$\lim_{N \to \infty} \sum_{i_1, \ldots, i_d} |f_N|^2(i_1, \ldots, i_d) = 1.$$

Let $Y = \{Y_i : i \geq 1\}$ be a sequence of freely independent and identically distributed random variables such that $\varphi(Y_1) = 0$, $\varphi(Y_1^2) = 1$ and $\varphi(Y_1^4) \geq 2$. Then, the following two assertions are equivalent, as $N \to \infty$:

(i) *the sequence*

$$F_N := \sum_{i_1,...,i_d} f_N(i_1,...,i_d)Y_{i_1} \cdots Y_{i_d}, \quad N \geq 1,$$

converges in distribution to a standard semicircular random variable;

(ii) $\varphi(F_N^4) \to 2$.

Theorem 1 extends parts of the results proved in [3], that apply to the case where Y is composed of freely independent standard semicircular random variables (note, however, that the results of [3] concern general subsequences contained in the *free Wigner chaos* associated with a free Brownian motion – see [1]). The proof of Theorem 1 is based on a novel combinatorial analysis of the free cumulant of order four associated with the random variable F_N.

A connection with *universality statements* is also discussed. In this respect, a crucial role is played by the so-called *influence function*

$$i \mapsto \sum_{i_1,...,i_{d-1}} |f_N|^2(i, i_1,...,i_{d-1}),$$

that we use in a spirit similar to [4].

Finally, a brief introduction to *transfer principles* on the free semicircular and Poisson chaoses is provided, with special emphasis on the counterexamples studied in [2].

REFERENCES

[1] Ph. Biane and R. Speicher, *CStochastic calculus with respect to free Brownian motion and analysis on Wigner space*, Probability Theory and Related Fields **112** (1998), 373–409.
[2] S. Bourguin and G. Peccati, *Semicircular limits on the free Poisson chaos: counterexamples to a transfer principle*, Journal of Functional Analysis **267**(4) (2014), 963–997.
[3] T. Kemp, I. Nourdin, G.. Peccati and R. Speicher, *Wigner chaos and the fourth moment*, The Annals of Probability **40**(4) (2012), 1577–1635.
[4] E.. Mossel, R. O'Donnel and K. Olezkiewicz, *Noise stability of functions with low influences: invariance and optimality*, The Annals of Mathematics **171**(1) (2010), 295–341.
[5] I. Nourdin, G. Peccati, G. Poly and R. Simone, *Classical and free fourth moment theorems: universality and thresholds*, Journal of Theoretical Probability (2015), to appear.

Spectra of regular random graphs and related structures

IOANA DUMITRIU

For the last several decades, the spectra of regular random graphs and associated regular structures have made the object of an intense line of study; in addition to the fact that random graphs are connected to the field of random matrix theory, the potential applications from studying their spectra lie in areas like theoretical computer science, network science, electrical engineering, social sciences, etc. The

past few years have seen tremendous progress in the understanding of how these spectra behave for various densities of the graph (i.e., degrees of the vertices).

We present here a few results in this direction obtained by the author in collaboration with students and colleagues, noting that, to date, many are still state-of-the-art in certain regimes.

For simple regular graphs (graphs for which the degree of each vertex is the same), the adjacency matrix is symmetric and records the presence (1) or absence (0) of edges between vertices. Bipartite biregular graphs, on the other hand, have a two-class partition of the vertices (with sizes m, n) such that the degree of a vertex in a given class is the same (d_1, respectively, d_2), and there are no intra-class edges (hence $md_1 = nd_2$). The random models considered for either type of graph include uniform (for both), as well as configuration and permutation for the regular graph. For regular graphs, all three models are contiguous if the degrees are finite, but the author and colleagues studied also the case when the degrees grow slowly.

The first example of result obtained was laws of large numbers, both on macroscopic and microscopic (short) scales, for the uniformly random regular graph.

Theorem 1. ([7]) *If the degree d of the graph grows at most polylogarithmically in n ($d \lesssim (\log n)^\alpha$ for $\alpha > 0$), given any $\epsilon > 0$, on intervals I for which $|I| \lesssim \max\{\frac{1}{d}, \frac{1}{\log n}\}$, for n large enough, with high probability,*

$$\left| N_I - \int_I s(x)dx \right| \ < \ \epsilon|I| .$$

Here N_I is the number of eigenvalues of the scaled adjacency matrix falling in the interval I, and $s(x)$ is the density of the semicircular distribution.

This theorem has been improved on first by [13], who obtained better scales and better probabilities for $d \gtrsim (\log n)^{10}$, and by [2] in the range $d \gtrsim (\log n)^4$. It is an open question of whether it can be improved when $d \lesssim (\log n)^4$.

For bipartite biregular graphs, a similar theorem was obtained.

Theorem 2. ([5]) *If the two degrees d_1 and d_2 are roughly polylogarithmic in n, such that $d_1/d_2 \to \gamma \in (0,1)$, then, using the same notation, given $\epsilon > 0$ and m, n large enough, with high probability, on intervals I for which $|I| \lesssim \max\{\frac{1}{d_1}, \frac{1}{\log n}\}$,*

$$\left| N_I - \int_I m_\gamma(x)dx \right| \ < \ \epsilon|I| .$$

Here $m_\gamma(x)$ is a simple transformation of the well-known Marčenko-Pastur law.

The next result concerns fluctuations from the semicircle law for the random regular graph, in the case of the permutation model (the adjacency matrix $A = P_1 + P_1^T + P_2 + P_2^T + \ldots + P_d + P_d^T$ for uniformly and independently chosen permutation matrices $P_1, \ldots, P_d$). For both d fixed and $d = n^{o(1)}$ (i.e., d growing slower than any fixed power of n), if one chooses a function f that's smooth enough

(analytical plus some technical conditions), the centered linear statistic

$$\sum_{i=1}^{n} f(\lambda_i) - \mathbb{E}\left(\sum_{i=1}^{n} f(\lambda_i)\right)$$

converges to an infinitely divisible variable (if d fixed) or to a normal distribution (if d grows to ∞). In both cases the variance depends on f ([6]).

Remarkably, for the d growing case, the result is similar to the result for Gaussian ensembles and β-Gaussian ensembles, in the sense that the variance expression for the limiting normal variable is the same.

Recently, the author and colleagues have considered a related, but different regular structure, the Regular Stochastic Block Model (RSBM). The classical Stochastic Block Model (SBM), in its simplest form, has two classes of vertices, each of size n. Each class comes with an independent Erdős-Rényi $G(n,p)$ random graph placed on its vertices, and the two classes are joined by a bipartite Erdős-Rényi $G(n,n,q)$. The problem studied here is whether one can recover, partially or completely, or even detect the presence of the partition (the two classes), starting only from the adjacency matrix of the graph. This problem has been intensely studied, and completely solved recently in a series of papers ([1, 9, 10, 11, 8, 12, 14, 4]).

We have replaced the $G(n,p)$ and $G(n,n,q)$ by a configuration-model random regular graph $G(n,d_1)$, respectively, by a bipartite random configuration-model random graph $G(n,n,d_2)$ ([3]). Note that since d_1 and d_2 are fixed, the model is contiguous to the case when the configuration models were replaced by uniform ones.

The rigidity of the new graph model RSBM allows us to obtain exact recovery in regimes given by a certain technical condition on the degrees. By contrast, in similar degree regimes, all that can be obtained for the classical SBM is detection of the partition, or, if the degrees are very high, partial recovery (recovery of up to a determined fraction of the vertices). The method is spectral and adapts and hybridizes the tools of [8] to the new setup of no independent edges, but much more rigid graph structure.

We note that we also proved that, provided that the degrees are high enough, the partition is unique with high probability, and hence the graph is *theoretically* checkable (which means that the problem of whether or not the partition exists, given the adjacency matrix, is decidable with high probability). However, unlike the polynomial-time spectral algorithm we proposed in the case of the extra technical condition, we emphasize that this last result is not practical.

Finally, [3] also shows that the rigidity of the model allows us to use the Majority Rule to transform a sufficiently accurate partial labeling into a exact one.

Current work involves extending this to k-frames, graphs where one starts out with a "frame" (small, finite graph), then duplicates each vertex a number of times, keeping some fixed proportions between vertices, and then places a uniformly random bipartite biregular structure between the classes of vertices. This represents a generalization of lifts.

References

[1] Abbe, Emmanuel and Bandeira, Afonso S and Hall, Georgina, *Exact recovery in the stochastic block model*, preprint 2014, arXiv:1405.3267

[2] Bauerschmidt, Roland and Knowles, Antti and Yau, Horng-Tzer, *Local semicircle law for random regular graphs*, preprint 2015, arXiv:1503.08702v3

[3] Brito, Gerandy and Dumitriu, Ioana and Hoffman, Cristopher and Ganguly, Shirshendu and Tran, Linh V., *Recovery and Rigidity in a Regular Stochastic Block Model*, preprint 2015, arXiv:1507.00930

[4] Coja-Oghlan, Amin, *Graph partitioning via adaptive spectral techniques*, Combinatorics, Probability and Computing 19 (2010), no. 2, 227–284

[5] Dumitriu, Ioana and Johnson, Tobias, *The Marchenko-Pastur law for sparse random bipartite biregular graphs*, Random Structures & Algorithms (2014)

[6] Dumitriu, Ioana and Johnson, Tobias and Pal, Soumik and Paquette, Elliot, *Functional limit theorems for random regular graphs*, Probab. Theory Related Fields 156 (2013), 921–975

[7] Dumitriu, Ioana and Pal, Soumik, *Sparse regular random graphs: Spectral density and eigenvectors*, Ann. Probab. 40 (2012), no. 5, 2197–2235

[8] Massoulié, Laurent, *Community detection thresholds and the weak Ramanujan property*, preprint 2013, arXiv:1311.3085

[9] Mossel, Elchanan and Neeman, Joe and Sly, Allan, *Stochastic block models and reconstruction*, preprint 2012, arXiv:1202.1499

[10] Mossel, Elchanan and Neeman, Joe and Sly, Allan, *Belief Propagation, Robust Reconstruction, and Optimal Recovery of Block Models*, preprint 2013, arXiv:1309.1380

[11] Mossel, Elchanan and Neeman, Joe and Sly, Allan, *A proof of the block model threshold conjecture*, preprint 2013, arXiv:1311.4115

[12] Mossel, Elchanan and Neeman, Joe and Sly, Allan, *Consistency thresholds for binary symmetric block models*, preprint 2014, arXiv:1407.1591

[13] Tran, Linh V. and Vu, Van H. and Wang, Ke, *Sparse random graphs: Eigenvalues and eigenvectors*, Random Structures & Algorithms 42 (2013), no. 1, 110–134

[14] Yun, Se-Young and Proutiere, Alexandre, *Community Detection via Random and Adaptive Sampling*, preprint 2014, arXiv:1402.3072

Outliers in perturbed matrix models

Hari Bercovici

(joint work with Ş. Belinschi, M. Capitaine, and M. Février)

We report on joint work with Ş. Belinschi, M. Capitaine, and M. Février. Computer simulations were kindly supplied by C. Bordenave.

Consider a tracial W* probability space $(\mathcal{A}, \varphi)$ tracial W* and two selfadjoint random variables $a, b \in \mathcal{A}$ that are freely independent. Using the notation $G_a(\lambda) = \varphi((\lambda - a)^{-1})$ for the Cauchy transform of the distribution of a, it is known that G_{a+b} is subordinate to G_a and G_b. That is, we have $G_{a+b}(\lambda) = G_a(\omega_1(\lambda)) = G_b(\omega_2(\lambda))$, where ω_1, ω_2 are analytic self maps of the upper half-plane which satisfy the additional relation $\omega_1(\lambda) + \omega_2(\lambda) = \lambda + 1/G_{a+b}(\lambda)$. The maps ω_j extend continuously to the real line (provided we allow infinite values). Suppose that γ is an eigenvalue of $a + b$, so G_{a+b} behaves roughly as if it had a pole at γ. It follows that G_a (respectively, G_b) behaves roughly as if it had a pole at $\alpha = \omega_1(\gamma)$ (respectively $\beta = \omega_2(\gamma)$) and $\alpha + \beta = \gamma$. It follows that α and β are eigenvalues for a and b, respectively (and the sum of the two masses exceeds 1). These facts have

counterparts in the behavior of independent random matrices. Suppose given, for every positive integer N, selfadjoint, independent, random matrices A_N and B_N whose distribution is invariant under unitary conjugation. Assume that $A_N \to a$ and $B_N \to b$ in distribution as $N \to \infty$, in which case we also have $A_N + B_N \to a+b$ in distribution. Suppose that $t > 0$ and $A_N + B_N$ has at least tN eigenvalues arbitrarily close (as $N \to \infty$) to γ. Then it follows that A_N (respectively, B_N) must have eigenvalues arbitrarily close to $\alpha = \omega_1(\gamma)$ (respectively, $\beta = \omega_1(\gamma)$).

This is, of course, a statement about eigenvalues in the *bulk* of the spectrum of these matrices. We are interested in outlying eigenvalues, that is eigenvalues which do not belong to the spectrum of the limiting elements of $\mathcal{A}$. We work under the following basic assumption:

(A) The eigenvalues of A_N (respectively, B_N) belong to the bulk with the exception of p (respectively, q) fixed eigenvalues.

The exceptional eigenvalues of A_N and B_N are called *spikes* and the goal is to determine whether $A_N + B_N$ has any identifyable outliers (eigenvalues *not* in the spectrum of $a + b$). To illustrate the nature of the results, we consider a simple example. Suppose that $\sigma(A_N) = \{1, 2, 3, 4\}$ and each eigenvalue has multiplicity $\sim N/4$ while B_N is a projection of rank 1. It is easy to check that a has eigenvalues $1, 2, 3, 4$ with eigenprojections of trace $1/4$, $b = 0$, $\omega_1(\lambda) = \lambda$, and $\omega_2(\lambda) = 1/G_a(\lambda)$. With the above convention we have $p = 0$ (no spikes for A_N), $q = 1$, and 1 is the unique spike of B_N. Clearly, $A_N + B_N$ *surely* has outliers $\rho_1, \dots, \rho_4$, one each in $[1, 2), [2, 3), [3, 4), [4, 5)$. *Almost* surely, we have $\omega_2(\rho_j) \sim 1$, while $\omega_1(\rho_j)$ are not eigenvalues of A_N.

Our main results shows that this is a general phenomenon, independent of the distributions of a and b. In the following statement, $E_A(\sigma)$ denotes the orthogonal projection onto the space generated by the eigenvectors of A corresponding to eigenvalues in the set σ.

Theorem 1. *Suppose that $\rho \notin \sigma(a + b)$ and denote by P_N (respectively, Q_N) the orthogonal projection onto the space generated by the spike eigenvectors of A_N (respectively, B_N). Then $\omega_1(\rho) \notin \sigma(a)$ and, for small $\varepsilon > 0$,*

$$P_N E_{A_N + B_N}((\rho - \varepsilon, \rho + \varepsilon)) P_N \sim \frac{1}{\omega_1'(\rho)} E_{A_N}(\{\omega_1(\rho)\})$$

almost surely as $N \to +\infty$. A similar statement holds with Q_N, B_N, and ω_2 in place of P_N, A_N, and ω_1, respectively. In particular, ρ is an outlier precisely when $\omega_1(\rho)$ is a spike for A_N or (not exclusive) $\omega_2(\rho)$ is a spike for B_N.

The special case in which there are no spikes (that is, $p = q = 0$) was treated earlier by Collins and Male and it is an essential ingredient in the proof. We try now to explain why the functions ω_j appear in this result. The intuition comes from a stronger property, discovered by Biane, that these functions satisfy. Denote by $\{a\}''$ the subalgebra of $\mathcal{A}$ generated by a, and let $\mathbb{P} : \mathcal{A} \to \{a\}''$ denote the trace-preserving conditional expectation. Then the identity

$$\mathbb{P}[(\lambda - (a + b))^{-1}] = (\omega_1(\lambda) - a)^{-1}$$

holds. The natural analog of $\mathbb{P}$ in the context of random matrices is the conditional expectation $\mathbb{E}[\cdot] = \mathbb{E}[\cdot|A_N]$. We can then define the (random analytic) function

$$C_N(\lambda) = \mathbb{E}[(\lambda - (A_N + B_N))^{-1}]$$

and hope that the relation $C_N(\lambda) = (\omega_1(\lambda) - A_N)^{-1}$ holds. Given $\rho \in \mathbb{R}$ such that $\omega_1(\lambda)$ is a spike for A_N, it follows that the function $C_N(\lambda)$ has a pole at ρ, and thus ρ is an eigenvalue for $A_N + B_N$.

An approximate version of this argument does in fact hold. Denote by $R_N(\lambda) = (\lambda - (A_N + B_N))^{-1}$ the resolvent of $A_N + B_N$. In most calculations, λ is in the upper half plane.

Lemma 2. *For every constant $N \times N$ matrix X we have*

$$C_N(\lambda)X - XC_N(\lambda) = \mathbb{E}[R_N(\lambda)(XA_N - A_N X)R_N(\lambda)].$$

Proof. It suffices to consider selfadjoint matrices X. Observe that $C_N(\lambda) = \mathbb{E}[(\lambda - (A_N + e^{i\varepsilon X}B_N e^{-i\varepsilon X}))^{-1}]$ (by unitary invariance). Differentiate at $\varepsilon = 0$ to obtain the desired identity. $\square$

It follows immediately that $C_N(\lambda)$ belongs to $\{A_N\}''$ so it is a function of A_N in the sense of functional calculus.

Lemma 3. *For every $N \times N$ matrix X we have $X(C_N(\lambda)^{-1} + A_N) - (C_N(\lambda)^{-1} + A_N)X = C_N(\lambda)^{-1}\mathbb{E}[(R_N(\lambda) - C_N(\lambda))(XA_N - A_N X)(R_N(\lambda) - C_N(\lambda))]C_N(\lambda)^{-1}.$*

This is merely a reformulation of the preceding lemma. Note that the right hand side contains the difference $R_N(\lambda) - C_N(\lambda)$ between the resolvent and its conditional expectation. This allows us to estimate the left hand side using known concentration results. The conclusion, obtained by applying the lemma to operators X of rank one, is that $C_N(\lambda) + A_N$ is close (with high probability as $N \to \infty$) to a scalar multiple of the identity matrix:

$$C_N(\lambda)^{-1} \sim \omega_1^{(N)}(\lambda) - A_N.$$

Next we can use asymptotic freeness to conclude that $\omega_1^{(N)}$ converge uniformly on compact subsets of $\mathbb{C}^+$ to ω_1.

Suppose now that the spikes of A_N are $a_1, \ldots, a_p$ with corresponding eigenvectors $e_1, \ldots, e_p$.

Corollary 4. *For every λ in the upper half plane, we have*

$$P_N C_N(\lambda) P_N \to Diag((\omega_1(\lambda) - a_1)^{-1}, \ldots, (\omega_p(\lambda) - a_1)^{-1})$$

almost surely as $N \to \infty$.

The remaining arguments, in case B_N has no spikes (that is, $q = 0$) are roughly as follows.

(1) Reduction to the study of a random matrix of fixed size p, as done earlier by Benaych-Georges and Nadakuditi.
  (a) Write $A_N = A'_N + A''_N$ where A'_N has e-values close to $\sigma(a)$ and A''_N has rank p.

(b) By results of Collins and Male, $A'_N + B_N$ has no outliers.

(c) Apply Sylvester's identity to reduce the study of outliers to that of

$$\det F_N(\lambda) = \det(I - P_N(\lambda - (A'_N + B_N))^{-1} P_N T)$$

where T is a $p \times p$ diagonal (in the basis $e_1, \ldots, e_p$).

(2) Note that F_N converges uniformly on the compact sets of $\mathbb{C} \setminus \sigma(a + b)$ to

$$\mathrm{Diag}\left(1 - \frac{a_1 - \alpha}{\omega_1(\lambda) - \alpha}, \ldots, 1 - \frac{a_p - \alpha}{\omega_1(\lambda) - \alpha}\right)$$

for a proper choice of A'_N.

(3) Use the Hurwitz theorem to deduce that the proper number of outliers are present.

(4) Treat outlying eigenvectos with methods developed by M. Capitaine, first in case spikes have no multiplicity.

(a) Assume that ρ is a spike such that $\omega_1(\rho) = a_1$. We need to show

$$P_N E_{A_N + B_N}((\rho - \varepsilon, \rho + \varepsilon)) P_N \sim \frac{1}{\omega'_1(\rho)} E_{A_N}(\{a_1\}),$$

and this is the same as

$$\mathrm{Tr}[E_{A_N}(\{a_i\}) E_{A_N + B_N}((\rho - \varepsilon, \rho + \varepsilon))] \sim \frac{\delta_{i1}}{\omega'(\rho)}.$$

(b) Here E_A is the spectral measure, a characteristic function of A. Replace it by a smooth function to evaluate

$$\mathrm{Tr}[f_i(A_N) h(A_N + B_N)].$$

This is done using concentration inequalities, turning the trace into a contour integral involving $C_N(\lambda) \sim (\omega_1(\lambda) - A_N)^{-1}$.

(c) Multiple spikes are separated by small perturbations.

(d) When both A_N and B_N have spikes, the process must be done twice (switching the roles of A and B.)

Similar results are proved for multiplicative models, both on $\mathbb{T}$ and on $\mathbb{R}_+$. Details, bibliography, simulations and historical remarks are in our preprint [1].

References

[1] S. T. Belinschi, H. Bercovici, M. Capitaine, and M. Février. Outliers in the spectrum of large deformed unitarily invariant models, 2014. Preprint, arXiv:1412.4916v2.

Free transport in free probability with amalgamation

Yoann Dabrowski

(joint work with A. Guionnet and D. Shlyakhtenko)

Free transport is a tool inspired by classical optimal transport of measures to obtain von Neumann algebras isomorphisms between algebras generated by free Gibbs states. More precisely, a free transport map from τ_0 to τ, two traces on non-commutative polynomials $\mathcal{C} := \mathbb{C}\langle X_1, ..., X_n\rangle$ is a collection of elements $F_1, ..., F_n \in W^*(\tau_0)$ the von Neumann algebra generated in the GNS-representation, such that for any non-commutative polynomial $P \in \mathcal{C}$:

$$\tau_0(P(F_1, ..., F_n)) = \tau(P).$$

Then $\alpha(P) = P(F_1, ..., F_n)$ extends to a trace preserving $*$-homomorphism $\alpha : W^*(\tau) \to W^*(\tau_0)$, that is often an isomorphism provided that τ_0 and τ can be exchanged.

Guionnet and Shlyakhtenko [5] first solved a free Monge-Ampère equation :

$$(1) \qquad (1 \otimes \tau + \tau \otimes 1)Tr \log \partial \mathcal{D}g = \{W(\mathcal{D}g(X))\} - \frac{1}{2}\sum X_j^2$$

to obtain as cyclic gradient $\mathcal{D}g$ of the solution such a transport map between free semicircular variables and free Gibbs states with potential W small analytic perturbations of quadratic potentials. Recall that this non-linear non-commutative PDE uses two differential operators, the cyclic gradient $\mathcal{D}$ and the free difference quotients ∂ (and the mixed Hessian $\partial\mathcal{D}$) defined respectively for monomials P by:

$$\mathcal{D}_i P = \sum_{P=AX_iB} BA \in \mathcal{C}, \ \ \partial_i P = \sum_{P=AX_iB} A \otimes B \in \mathcal{C}\otimes\mathcal{C}, \ \ \partial\mathcal{D}g := (\partial_j \mathcal{D}_i g)_{ij} \in M_n(\mathcal{C}\otimes\mathcal{C}).$$

In this way, as a noticeable application, [5] obtained an isomorphism of von Neumann algebras generated by n q-gaussian variables for $|q| \leq q_0(n)$ with the free group factor case $q = 0$.

Among desirable extensions of their results, such as dealing with infinitely many variables (note that their $q_0(n) \to_{n\to\infty} 0$ and does not give any result when $n = \infty$) and the non-tracial analogue obtained in [7], this work focuses on dealing with more general potentials W and more general reference trace τ_0 to be able to transport semicircular variables with covariance map $\eta : B \to B$ a unital completely positive map with $\tau(\eta(a)b) = \tau(a\eta(b))$. However, we only consider finite von Neumann algebras in the sequel. Recall that in this context of free probability with amalgamation over B, the relevant Schwinger-Dyson equation was introduced in [8] as definition of a conjugate variable ξ_i relative to η (m is the multiplication map):

$$\tau(\xi_i P) = \tau(m[(1 \otimes \eta E_B) \circ \partial_i P]), \qquad \forall P \in B\langle X_1, ..., X_n\rangle.$$

In the scalar case the choice $\xi_i = \mathcal{D}_i W$ corresponds to a potential W and we write τ_W the solution when it is unique, this generalizes [9]. In the general case, $\xi_i = X_i$ gives a characterization of semicircular variables with covariance η.

We explain the main steps to achieve this goal of obtaining transport maps between such τ_W relative to η. One first needs to capture the relations between B and a semicircular system $(S_1, ..., S_n)$ with covariance η as done in [2] and consider transport maps between variables satisfying the same relations. For instance, if $\eta = E_D$ for a von Neumann subalgebra D, the variables commute with D. In general, for $B \subset M$ finite and $\eta : B \to B$ a covariance map, these relations are captured by a quotient of the normal Haagerup tensor product $M \otimes_{w^*h\eta} M := M \otimes_{\sigma h} M / Ker(p_\eta)$, Haagerup tensor product being used to have available a multiplication map $p_\eta = .\#S_1$ extending $(a \otimes b)\#c = acb$. Then, if one considers $\eta' \cap M \otimes_{w^*h\eta} M$ a subspace of elements x such that $d\#x = 0$ if $d \in Ker(p_\eta)$, one obtains an algebra for the multiplication extending $(a \otimes b)\#(c \otimes d) = ac \otimes db$. In the case $\eta = E_D$, one recovers $D' \cap M \otimes_{ehD} M$ the commutant in the extended Haagerup tensor product extensively studied in [6]. We also define in this paper a space of non-commutative polynomials capturing these relations $B\langle X_1, ..., X_n : \eta \rangle$. This will be the space we will complete to get appropriate C^k function spaces in which we solve a non-commutative PDE.

The second step is then to replace the cyclic gradient Ansatz for the transport map $F_i = \mathcal{D}_i g$ originally used. At this stage, we rather understand when it can be used. The most general Ansatz would be to take transport maps F_i having properties similar to ξ_i that may depend on η. For instance, in the scalar case, the main property of cyclic gradient used is the relation $\partial_j \mathcal{D}_i g = \sigma(\partial_i \mathcal{D}_j g)$ for $\sigma(a \otimes b) = b \otimes a$. This suggests to consider a subalgebra $\eta' \cap M \otimes_{w^*h\eta} M$ stable by an extension of σ, we call this symmetric version $M \otimes_{w^*sh\eta} M$. It is explained in [3] that one can find such an algebra which becomes a $*$-algebra for an extension of $(a \otimes b)^* = a^* \otimes b^*$ and for which one can use interpolation to make it act by bounded operators on an Hilbert space $H_\eta = \overline{(M \otimes_{w^*sh\eta} M)\#S_1} \subset L^2(W^*(M, S_1))$. We found in [4] that the cyclic gradient Ansatz can be used when $\tau(U) = \langle U\#S_{n+1}, S_{n+1}\rangle$ is a trace on $N \otimes_{w^*sh\eta} N$ for $N = W^*(M, S_1, ..., S_n)$, in the basic semi-circular starting case. As proved in [3], this is for instance the case when $\eta = E_D$ and $B = D \rtimes \Gamma$ the cross-product of a countable discrete group Γ by a trace preserving action on a finite von Neumann algebra D.

Finally, one needs to consider appropriate classes of non-commutative functions having cyclic gradients and free difference quotients in n-ary cyclic variants of $N \otimes_{w^*sh\eta} N$. The use of variants of Haagerup tensor products is also crucial to define a convexity assumption on the potential W. Moreover, one uses free probability with amalgamation analogues of Cébron's trace polynomials [1] that were already implicitly used in [5]. This enables to obtain a semigroup for the free diffusion with drift $\mathcal{D}W$ with generator we call Δ_W. This semigroup is the starting point for solving a non-commutative PDE

Our main result in [4] is thus the following : Assume given a covariance map $\eta : B \to B$ as above and

(1) V, W are in an appropriate C^3-completion of $B\langle X_1, ..., X_n : \eta \rangle$
(2) (V, W) are generalized (c, M) convex for $c > 0$ and M much huger than $||X_i||$ under τ_V, τ_W i.e. for all $Y \in \eta' \cap N, N = W^*(Y_1, ..., Y_n, B)$ with

$||Y_i|| \leq M$ for T among V, W

$$((\partial_i \mathcal{D}_j T)(Y))_{ji} - cI_n \geq 0$$

in $M_n(N \otimes_{w^*sh\eta} N)$ in the sense they are self-adjoint and generate contraction semigroups.

(3) $\tau(U) = \langle U \# S_{n+1}, S_{n+1} \rangle$ is a trace on $N \otimes_{w^*sh\eta} N$ for $N = W^*(M, S_1, ..., S_n)$ (e.g. this is the case if $\eta = E_D$ and $B = D \rtimes \Gamma$)

Then there exists $F_1, ..., F_n \in W^*(X_1, ...X_n, B)$ such that $\tau_V(P(F_1, ..., F_n)) = \tau_W(P)$ for $P \in B\langle X_1, ..., X_n \rangle$ and we deduce that the von Neumann algebras generated by $B, X_1, ..., X_n$ in the corresponding GNS representations $W^*(\tau_W) \simeq W^*(\tau_V)$ are isomorphic.

Note that the convexity assumptions is for instance satisfied for small C^3 perturbations of $V_0 = \frac{1}{2} \sum_{i=1}^n X_i^2$ which gives back the semicircular case with covariance η.

More specifically, instead of solving a generalization of (1) we rather consider a linearized variant. One considers a path $V_\alpha = \alpha W + (1 - \alpha)V$ and find a solution for $\alpha \in [0, 1]$ of

$$\Delta_{V_\alpha} g_\alpha = W - V - \tau_{V_\alpha}(W - V).$$

The solution of this PDE is obtained using free diffusions.

We then produce a transport map F_α between τ_V and τ_{V_α} in solving at least for more regular V, W:

$$\frac{d}{d\alpha} F_\alpha = \mathcal{D}g_\alpha(F_\alpha) = (\mathcal{D}_1 g_\alpha(F_\alpha), ..., \mathcal{D}_n g_\alpha(F_\alpha)).$$

REFERENCES

[1] G. Cébron, *Free Convolution Operators and Free Hall Transform*, J. Func. Anal., **265** (2013) 2645–2708.

[2] Y. Dabrowski, *Analytic functions relative to a covariance map η : I. Generalized Haagerup tensor products and analytic relations*, Preprint, arXiv:1503.05515.

[3] Y. Dabrowski, *Analytic functions relative to a covariance map η : II. cyclic symmetry and planar algebraic operations*, (in preparation).

[4] Y. Dabrowski, A. Guionnet and D. Shlyakhtenko, *Free transport for convex potentials* (in preparation).

[5] A. Guionnet and D. Shlyakhtenko, *Free Monotone transport* Invent. Math. **197** (2014), 613–661.

[6] B. Magajna, *Duality and normal parts of operator modules,* J. Funct. Anal. **219** (2005), 306–339.

[7] B. Nelson, *Free Monotone transport without a trace,* Commun. Math. Phys. **334** (2015), 1245–1298.

[8] D. Shlyakhtenko. *Free entropy with respect to a completely positive map.* American Journal of Mathematics, **122** (2000), 45–81.

[9] D. Voiculescu: *The analogues of entropy and of Fisher's information measure in free probability,* V, Invent. Math. **132** (1998), 189–227.

Reaization theory for noncommutative rational functions around a matrix point

VICTOR VINNIKOV

(joint work with Dmitry S. Kaliuzhnyi-Verbovetskyi)

It is well known that noncommutative (nc) rational functions regular at the origin admit a good realization (or linearization) theory. This is very useful both conceptually and for a variety of applications since it often essentially reduces the study of these rational functions to a study of linear pencils. By translation the method can be applied to nc rational functions that are regular at some scalar point, but not beyond. The purpose of the talk is to use the general Taylor–Taylor series of nc function theory [6] to describe the power series expansion of a nc rational function around an arbitrary matrix (rather than scalar) centre, and to discuss a work in progress that provides a full generalization of the realization theory for nc rational functions regular at an arbitrary given matrix point, thus allowing to handle all noncommutative rational functions.

NC rational functions. We consider the ring of nc polynomials (the free ring) $\mathbb{K}\langle x_1, \ldots, x_d \rangle$ over a field $\mathbb{K}$. Here $x_1, \ldots, x_d$ are nc indeterminates, and $p \in \mathbb{K}\langle x_1, \ldots, x_d \rangle$ is of the form $p = \sum_{w \in \boldsymbol{\mathcal{G}}_d} p_w x^w$, where $\boldsymbol{\mathcal{G}}_d$ denotes the free monoid on d generators (letters) $g_1, \ldots, g_d$ with identity $\emptyset$ (the empty word), $p_w \in \mathbb{K}$, x^w are nc monomials in $x_1, \ldots, x_d$ ($x^w = x_{j_1} \cdots x_{j_l}$ for $w = g_{j_1} \cdots g_{j_l} \in \boldsymbol{\mathcal{G}}_d$ and $x^\emptyset = 1$), and the sum is finite. p can be evaluated in an obvious way on d-tuples of square matrices of all sizes over $\mathbb{K}$: for $X = (X_1, \ldots, X_d) \in (\mathbb{K}^{n \times n})^d$, $p(X) = \sum_{w \in \boldsymbol{\mathcal{G}}_d} p_w X^w = \sum_{w \in \boldsymbol{\mathcal{G}}_d} X^w p_w \in \mathbb{K}^{n \times n}$.

The skew field of nc rational functions $\mathbb{K}\lparen\!\langle x_1, \ldots, x_d \rangle\!\rparen$ over a field $\mathbb{K}$ (the free skew field) is the universal skew field of fractions of the ring of nc polynomials over $\mathbb{K}$. This involves some non-trivial details since unlike the commutative case, a nc rational function does not admit a canonical coprime fraction representation; see [7, Chapter 8] and [3] for good expositions and background, including the original constructions of Amitsur and Cohn. The following is most natural from the point of view of nc function theory and is a version of Amitsur's construction except that we use evaluation on d-tuples of square matrices of all sizes over $\mathbb{K}$ instead of evaluation on a "large" auxilliary skew field; see [4, 5] for details and further references. We first define (scalar) nc rational expressions by starting with nc polynomials and then applying successive arithmetic operations — addition, multiplication, and inversion. A nc rational expression r can be evaluated on a d-tuple X of $n \times n$ matrices in its *domain of regularity*, $\operatorname{dom} r$, which is defined as the set of all d-tuples of square matrices of all sizes such that all the inverses involved in the calculation of $r(X)$ exist. (We assume that $\operatorname{dom} r \neq \emptyset$, in other words, when forming nc rational expressions we never invert an expression that is nowhere invertible.) Two nc rational expressions r_1 and r_2 are called *equivalent* if $\operatorname{dom} r_1 \cap \operatorname{dom} r_2 \neq \emptyset$ and $r_1(Z) = r_2(Z)$ for all d-tuples $Z \in \operatorname{dom} r_1 \cap \operatorname{dom} r_2$. We define a *nc rational function* $\mathfrak{r}$ to be an equivalence class of nc rational expressions; notice that it has a well-defined evaluation on $\operatorname{dom} \mathfrak{r} = \bigcup_{R \in \mathfrak{r}} \operatorname{dom} R$ (here R denotes a

1×1 matrix-valued rather than scalar nc rational expression, i.e., some of the intermediate expressions may involve matrices of scalar nc rational expressions, cf. below). We set $(\mathrm{dom}\,\mathfrak{r})_n = \mathrm{dom}\,\mathfrak{r} \cap (\mathbb{K}^{n \times n})^d$.

It is clear that the evaluation of a nc rational function respects direct sums and simultaneous similarities, so that a nc rational function $\mathfrak{r}$ defines a nc function [6] on $\mathrm{dom}\,\mathfrak{r}$ (technically, on an apriori somewhat larger set called the extended domain of regularity of $\mathfrak{r}$ obtained by evaluating $\mathfrak{r}$ on d-tuples of generic matrices). In particular, nc rational functions admit a difference-differential calculus, see [5].

Power series expansion around a scalar point. A nc rational expression which is regular at 0 determines a nc formal power series. This correspondence is defined recursively using addition and multiplication of nc formal power series and inversion of a nc formal power series with an invertible constant term (the coefficient of $z^\emptyset$). Furthermore, r_1 and r_2 are equivalent if and only if the corresponding nc formal power series coincide. By translation, if $\lambda = (\lambda_1, \ldots, \lambda_d) \in (\mathrm{dom}\,\mathfrak{r})_1 \subseteq \mathbb{K}^d$ we obtain, for $X = (X_1, \ldots, X_d) \in \mathbb{K}^{n \times n}$,

$$(1) \qquad \mathfrak{r}(X) \sim \sum_{w \in \mathcal{G}_d} (X - I_n \lambda)^w \mathfrak{r}_w.$$

Here $\mathfrak{r}_w \in \mathbb{K}$ are the coefficients, and $X - I_n \lambda$ stands for $(X_1 - I_n \lambda_1, \ldots, X_d - I_n \lambda_d)$.

From the point of view of nc function theory, (1) is the Taylor–Taylor (TT) power series expansion of $\mathfrak{r}$ around λ. In particular, the coefficients $\mathfrak{r}_w$ can be calculated by means of the nc difference-differential calculus: $\mathfrak{r}_w = \Delta^w \mathfrak{r}(\lambda, \ldots, \lambda)$, Also, the series (1) actually converges to $\mathfrak{r}(X)$ in the following cases: (a) if $X - I_n \lambda$ is a jointly nilpotent d tuple of matrices so that the sum is finite; (b) in the case $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, the series converges normally on any open nc ball $\coprod_{n=1}^\infty \left\{ X \in (\mathbb{K}^{n \times n})^d : \|X - I_n \lambda\| < r \right\}$ (with respect to any operator space norm on $\mathbb{K}^d$, e.g., $\|Z\| = \|Z_1^* Z_1 + \cdots + Z_d^* Z_d\|$ for $Z = (Z_1, \ldots, Z_d)$) contained in the (extended) domain of regularity of $\mathfrak{r}$.

Fact of life if a nc rational function $\mathfrak{r}$ is regular at 0 it admits a unique (up to unique similarity) minimal (controllable and observable) state space realization:

$$(2) \qquad \mathfrak{r}(x) = D + C(I - A_1 x_1 - \cdots - A_d x_d)^{-1}(B_1 x_1 + \cdots + B_d x_d),$$

where $A_1, \ldots, A_d \in \mathbb{K}^{L \times L}$ for some integer L, $B_1, \ldots, B_d \in \mathbb{K}^{L \times 1}$, $C \in \mathbb{K}^{1 \times L}$, and $D = \mathfrak{r}(0)$. Furthermore,

$$\mathrm{dom}\,\mathfrak{r} = \coprod_{n=1}^\infty \left\{ X = (X_1, \ldots, X_d) \in \left(\mathbb{K}^{n \times n}\right)^d : \right.$$

$$\left. \det(I_{Ln} - X_1 \otimes A_1 - \cdots - X_d \otimes A_d) \neq 0 \right\}.$$

Here a realization is called *minimal* if the state space dimension L is as small as possible, *controllable* if $\mathrm{span}_{i=1,\ldots,d,\, w \in \mathcal{G}_d} \mathrm{im}\, A^w B_i = \mathbb{C}^L$, and *observable* if $\cap_{w \in \mathcal{G}_d} \ker C A^w = \{0\}$. The minimal realization can be either (a) constructed recursively by synthesis, starting with polynomuals (or even just the basic monomials $x_1, \ldots, x_d$) and using sum, product, and inversion formulae, and compressing

to achieve minimality; or (b) constructed in one step using the columns space of the infinite $\boldsymbol{\mathcal{G}}_d \times \boldsymbol{\mathcal{G}}_d$ *Hankel matrix* $\mathbb{H} = [\mathfrak{r}_{uv}]_{u,v \in \boldsymbol{\mathcal{G}}_d}$, which is of finite rank if and only if the nc power series $\sum_{w \in \boldsymbol{\mathcal{G}}_d} \mathfrak{r}_w x^w \in \mathbb{K}\langle\langle x_1, \ldots, x_d \rangle\rangle$ is the power series expansion of a nc rational function at a scalar point. We refer to [1] and [5] for details and further references, including the original work of Kleene, Schützenberger, and Fliess, see [2] for a good survey.

Power series expansion around a matrix point. Some notation: for $P = [P_{ij}]_{i,j=1,\ldots,m}, Q = [Q_{ij}]_{i,j=1,\ldots,m} \in \mathbb{K}^{sm \times sm} \cong (\mathbb{K}^{s \times s})^{m \times m}$, we let $P \odot_s Q$ denote the product of P and Q viewed as $m \times m$ matrices over the tensor algebra of $\mathbb{K}^{s \times s}$:

$$P \odot_s Q = \left[\sum_{j=1}^m P_{ij} \otimes Q_{jk} \right]_{i,k=1,\ldots,m} \in \left(\mathbb{K}^{s \times s} \otimes \mathbb{K}^{s \times s} \right)^{m \times m}.$$

For $Z = (Z_1, \ldots, Z_d) \in (\mathbb{K}^{sm \times sm})^d$ and $w = g_{j_1} \cdots g_{j_l} \in \boldsymbol{\mathcal{G}}_d$, we let $Z^{\odot_s w} = Z_{i_1} \odot_s \cdots \odot_s Z_{i_l} \in \left((\mathbb{K}^{s \times s})^{\otimes l} \right)^{m \times m}$.

The power series expansion around $Y \in (\mathrm{dom}\,\mathfrak{r})_s$ is now given by, for $X \in (\mathbb{K}^{sm \times sm})^d$,

$$(3) \qquad \mathfrak{r}(X) \sim \sum_{w \in \boldsymbol{\mathcal{G}}_d} (X - I_m \otimes Y)^{\odot_s w} \, \mathfrak{r}_w.$$

Here, the coefficient $\mathfrak{r}_w$ is a l-linear mapping $(\mathbb{K}^{s \times s})^l \longrightarrow \mathbb{K}^{s \times s}$, where l is the length of the word w, or alternatively a linear mapping $(\mathbb{K}^{s \times s})^{\otimes l} \longrightarrow \mathbb{K}^{s \times s}$. Notice that $(X - I_m \otimes Y)^{\odot_s w} \in \left((\mathbb{K}^{s \times s})^{\otimes l} \right)^{m \times m}$, hence we can apply $\mathfrak{r}_w$ to every entry of this matrix yielding a matrix in $(\mathbb{K}^{s \times s})^{m \times m} \cong \mathbb{K}^{sm \times sm}$ — which is where the value $\mathfrak{r}(X)$ lies.

NC formal power series with a matrix centre Y, of the form (3), form a ring with an obvious convolution product. It is clear that any nc polynomial can be written as a (finite) nc power series with centre Y, and the power series expansion of a nc rational expression r regular at Y can be obtained recursively using addition and multiplication of nc formal power series with centre Y and inversion of a nc formal power series with an invertible constant term. From the point of view of nc function theory, (1) is the TT power series expansion of $\mathfrak{r}$ around a matrix centre Y. One important difference with the case of a scalar centre is that the coefficients $\mathfrak{r}_w$ are not arbitrary multilinear mappings; they have to satisfy certain ccompatibility conditions with respect to Y, see [6, (4.14)–(4.17)].

We conjecture that if a nc rational function $\mathfrak{r}$ is regular at $Y \in (\mathbb{K}^{s \times s})^d$, it admits a unique (up to unique similarity) minimal (controllable and observable) state space realization with centre Y: for $X \in (\mathbb{K}^{sm \times sm})^d$,

$$(4) \quad \mathfrak{r}(X) = I_m \otimes D + (I_m \otimes C)\big(I_{Lsm} - (X_1 - I_m \otimes Y_1)\mathbf{A}_1 - $$
$$\cdots - (X_d - I_m \otimes Y_d)\mathbf{A}_d\big)^{-1}\big((X_1 - I_m \otimes Y_1)\mathbf{B}_1 + \cdots + (X_d - I_m \otimes Y_d)\mathbf{B}_d\big).$$

Here $\mathbf{A}_1, \ldots, \mathbf{A}_d \colon \mathbb{K}^{s \times s} \to \mathbb{K}^{Ls \times Ls}$ for some integer L and $\mathbf{B}_1, \ldots, \mathbf{B}_d \colon \mathbb{K}^{s \times s} \to \mathbb{K}^{Ls \times s}$ are linear mappings, $C \in \mathbb{K}^{s \times Ls}$, and $D = \mathfrak{r}(Y) \in \mathbb{K}^{s \times s}$. Furthermore,

$$(\mathrm{dom}\,\mathfrak{r})_{sm} = \left\{ X = (X_1, \ldots, X_d) \in \left(\mathbb{K}^{sm \times sm} \right)^d : \right.$$
$$\left. \det \left(I_{Lsm} - (X_1 - I_m \otimes Y_1)\mathbf{A}_1 - \cdots - (X_d - I_m \otimes Y_d)\mathbf{A}_d \right) \neq 0 \right\}.$$

Some open problems:

- The currently envisioned proof of the realisation (4) involves synthesis using sum, production, and inversion formulae. It is important to obtain necessary and sufficient "finiteness" conditions on the sequence of multilinear mappings $\mathfrak{r}_w$ satisfying the compatibility conditions [6, (4.14)–(4.17)] with respect to Y so that the nc power series (3) with centre Y is the power series expansion at Y of a nc rational function. This should then provide a one step construction of the minimal realization, analogous to the Hankel matrix realization.
- To find necessary and sufficient compatibility conditions with respect to Y on the coefficients $\mathbf{A}_1, \ldots, \mathbf{A}_d$, $\mathbf{B}_1, \ldots, \mathbf{B}_d$, C, D for the corresponding sequence of coefficients $\mathfrak{r}_w$ to satisfy the compatibility conditions [6, (4.14)–(4.17)] and for (4) to define a nc rational function.
- Use power series expansions and realizations around an arbitrary matrix point to provide a direct construction of the free skew field.

REFERENCES

[1] J.A. Ball, G. Groenewald, and T. Malakorn, *Structured noncommutative multidimensional linear systems*, SIAM J. Control Optim. **44** (2005), 1474–1528.

[2] J. Berstel and C. Reutenauer, *Rational series and their languages*, EATCS Monographs on Theoretical Computer Science **12**, Springer-Verlag, Berlin (1988).

[3] P.M. Cohn, *Free ideal rings and localization in general rings*, New Mathematical Monographs **3**, Cambridge University Press, Cambridge (2006).

[4] D.S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov, *Singularities of Noncommutative Rational Functions and Minimal Factorizations*, Lin. Alg. Appl. **430** (2009), 869–889.

[5] D.S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov, *Noncommutative rational functions, their difference-differential calculus and realizations*, Multidimens. Syst. Signal Process. **23** (2012), 49–77.

[6] D.S. Kaliuzhnyi-Verbovetzkyi and V. Vinnikov, *Foundations of Free Noncommutative Function Theory*, Math. Surveys and Monographs **199**, Amer. Math. Society (2014).

[7] L.H. Rowen, *Polynomial identities in ring theory*, Pure and Applied Mathematics **84**, Academic Press Inc. (Harcourt Brace Jovanovich Publishers), New York (1980).

Traffic spaces and unitarily invariant variables

Guillaume Cébron

(joint work with Antoine Dahlqvist and Camille Male)

In this talk, we wanted to describe the algebraic structure of a space of traffics, the convergence of the distribution of traffic of random matrices which are invariant by unitary conjugations, and how this result can be used to turn any non-commutative space into a space of traffic.

Thanks to the fundamental work of Voiculescu [4], it is now understood that, for large independent $N \times N$ matrices $A^{(N)}$ and $B^{(N)}$ which are unitarily invariant and which converge separately in non-commutative distribution (in the sense that the normalized trace of the powers of the matrices converges as N tends to ∞), the joint non-commutative distribution of $A^{(N)}$ and $B^{(N)}$ converges as well. If the matrices are invariant by conjugations by matrices of permutation instead of unitary conjugations, this is not necessarily true. In [2], Male introduced the concept of space of traffics in order to describe this new situation. Roughly speaking, a *space of traffics* is a non-commutative probability space where one can consider not only the usual operations of algebras, but also more general n-ary operations that we will denote by $(Z_g)_{g \in \mathcal{G}}$. At the level of matrices, each Z_g takes n matrices of size $N \times N$ and produces a new $N \times N$ matrix. He was able to prove the following: for large independent $N \times N$ matrices $A^{(N)}$ and $B^{(N)}$ which are invariant by conjugations by matrices of permutation and which converge separately in distribution of traffic (in the sense that the normalized trace of every $Z_g(A^{(N)}, \ldots, A^{(N)})$ converges as N tends to ∞), the joint non-commutative distribution of $A^{(N)}$ and $B^{(N)}$ converges as well.

1. Spaces of traffics and operad $\mathcal{G}$

For all $n \geq 0$, we call *n-graph operation* any connected graph g with n oriented and ordered edges, and two distinguished vertices (one input and one output, not necessarily distinct). The set $\mathcal{G}$ of graph operations is the set of all n-graph
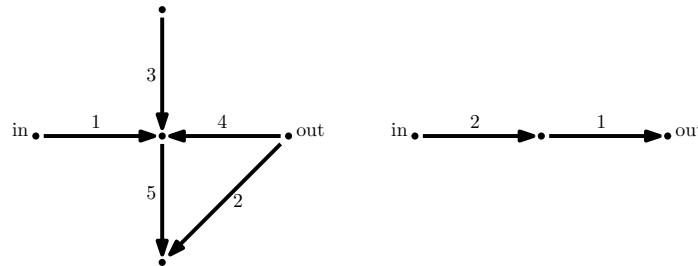

Figure 1. One 5-graph operation and one 2-graph operation

operations for all $n \geq 0$. A n-graph operation g has to be thought as an operation that accepts n objects and produces a new one. In particular, a n-graph operation g

can produce a new graph operation $g\circ(g_1,\ldots,g_n)$ from n different graph operations $g_1,\ldots,g_n$ in the following way: replace the i-th edge of g by the graph operation g_i. Endowed with those composition maps, the set of graph operations $\mathcal{G}$ is an *operad*, in the sense that $\circ$ is *associative*, has a identity (given by $(\cdot_{in} \to \cdot_{out}) \in \mathcal{G}$, which consists in two vertices and one edge from the input to the output), and is *equivariant* for the right action of the symmetric group which consists in interchanging the edges of a graph operation.

Let us fix $N \geq 1$. For each n-graph operation $g \in \mathcal{G}$, we can also define a n-linear map $Z_g : M_N(\mathbb{C}) \times \cdots \times M_N(\mathbb{C}) \to M_N(\mathbb{C})$ in the following way. Denoting by V the vertices of g, by $(v_1,w_1),\ldots,(v_n,w_n)$ the edges of g, and by $E_{k,l}$ the matrix unit $(\delta_{ik}\delta_{jl})_{i,j=1}^N \in M_N(\mathbb{C})$, we set, for all $M^{(1)},\ldots,M^{(n)} \in M_N(\mathbb{C})$,

$$Z_g(M^{(1)},\ldots,M^{(n)}) = \sum_{k:V\to\{1,\ldots,N\}} \left(M^{(1)}_{k(w_1),k(v_1)} \cdots M^{(n)}_{k(w_n),k(v_n)} \right) \cdot E_{k(out),k(in)}.$$

Following [3], we can think of $Z_g(M^{(1)},\ldots,M^{(n)})$ as an algorithm, where we are feeding a vector into the input vertex and then operate it through the graph, each edge doing some calculation thanks to the corresponding matrix $M^{(i)}$, and each vertex acting like a logic gate, doing some compatibility checks. Endowed with the sequence $(Z_g)_{g\in\mathcal{G}}$, the vector space $M_N(\mathbb{C})$ is an algebra over the operad $\mathcal{G}$, in the following sense: an *algebra over the operad* $\mathcal{G}$ is a vector space $\mathcal{A}$ with a collection of multilinear map $(Z_g)_{g\in\mathcal{G}}$ such that $Z_g : \mathcal{A}^n \to \mathcal{A}$ if g is a n-graph operation, and subject to some natural conditions which implement the idea of interpreting n-graph operations as n-ary operations on $\mathcal{A}$ (compatibility, for $g \mapsto Z_g$, of the composition maps $\circ$, of the identity of $\mathcal{G}$ and of $End(\mathcal{A})$, and of the right action of the symmetric group). Remark that the product graph operation $(\cdot \overset{1}{\leftarrow} \cdot \overset{2}{\leftarrow} \cdot)$ (which consists in three vertices and two successive edges from the input to the output) induces a bilinear map $Z_{\cdot\overset{1}{\leftarrow}\cdot\overset{2}{\leftarrow}\cdot} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ which gives to $\mathcal{A}$ a structure of associative algebra: every algebra over the operad $\mathcal{G}$ is in particular a unital algebra.

A *space of traffics* is a vector space $\mathcal{A}$ which is an algebra over the operad $\mathcal{G}$, together with an involution $*$ and a linear map $\tau : \mathcal{A} \to \mathbb{C}$ fulfilling some technical conditions which imply in particular that $(\mathcal{A},\tau)$ is a tracial non-commutative probability space (see [1, 2] for more details).

2. From non-commutative probability spaces to spaces of traffics

Let g be a graph operation. Because the map $Z_g : M_N(\mathbb{C})^n \to M_N(\mathbb{C})$ is multilinear, we can think of it as a linear map from $M_N(\mathbb{C})^{\otimes n}$ to $M_N(\mathbb{C})$. Let us fix a random matrix $A \in M_N(\mathbb{C})$, and write

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}(Z_g(A \otimes \cdots \otimes A)) \right] = \frac{1}{N}\mathrm{Tr}(Z_g(\mathbb{E}[A \otimes \ldots \otimes A])).$$

If we assume that A is unitarily invariant, it is possible to express this quantity thanks to the expectation of products of the moments $(\mathrm{Tr}(A^k)/N)_{k>0}$ (see [5]). In [1], we used this fact to prove the following result.

Theorem 1. *For all $N \geq 1$, let $A^{(N)}$ be a random matrix in $M_N(\mathbb{C})$. We assume*

  (1) *The unitary invariance: for all $N \geq 1$ and all $U \in M_N(\mathbb{C})$ which is unitary, $U A^{(N)} U^*$ and $A^{(N)}$ have the same law.*
  (2) *The convergence in distribution: for all $k > 0$, $\mathbb{E}\left[\frac{1}{N} \mathrm{Tr}\left((A^{(N)})^k\right)\right]$ converges as N tends to ∞.*
  (3) *The factorization property: for all $k_1, \ldots, k_m > 0$, we have the following convergence*

$$\lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left((A^{(N)})^{k_1}\right) \cdots \frac{1}{N}\mathrm{Tr}\left((A^{(N)})^{k_n}\right)\right]$$

$$= \lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left((A^{(N)})^{k_1}\right)\right] \cdots \lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left((A^{(N)})^{k_n}\right)\right].$$

Then, $A^{(N)}$ converges in distribution of traffic, in the sense that, for all $g \in \mathcal{G}$,

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left(Z_g(A^{(N)}, \ldots, A^{(N)})\right)\right]$$

converges as N tends to ∞.

We have also a version of this theorem for a family of random matrices. Furthermore, we have a formula for the limit of $\mathbb{E}[\mathrm{Tr}(Z_g(A^{(N)}, \ldots, A^{(N)}))/N]$ which involves only the limit of $(\mathbb{E}[\mathrm{Tr}(A^k)/N])_{k>0}$. Replacing in this formula the limiting distribution by an arbitrary distribution, we were able to prove the following result, which is a result about the positivity of the state which occurs.

Theorem 2. *Let $(\mathcal{A}, \phi)$ be a noncommutative probability space. There exists a space of traffics $(\mathcal{B}, \tau)$ such that $(\mathcal{A}, \phi)$ is included in $(\mathcal{B}, \tau)$ as a noncommutative probability space.*

The space $(\mathcal{B}, \tau)$ is canonical for at least two reasons. First, in this setting, freeness in $\mathcal{A}$ is equivalent to traffic freeness in $\mathcal{B}$. Secondly, if $A^{(N)}$ is a sequence of random matrices that converges in distribution to a random variable $a \in \mathcal{A}$ and which is unitarily invariant, then $A^{(N)}$ converges in distribution of traffics to a seen as a random variable in $\mathcal{B}$.

References

[1] Guillaume Cébron, Antoine Dahlqvist, and Camille Male. On the limiting distribution of traffics of unitarily invariant random matrices. *Work in progress*, 2015.
[2] Camille Male. The distributions of traffics and their free product. *arXiv:1111.4662*, 2011.
[3] James A. Mingo and Roland Speicher. Sharp bounds for sums associated to graphs of matrices. *Journal of Functional Analysis*, 262(5):2272 – 2288, 2012.
[4] Dan-Virgil Voiculescu. Limit laws for random matrices and free products. *Invent. Math.*, 104(1):201–220, 1991.
[5] Benoit Collins and Piotr Śniady. Integration with respect to the Haar measure on unitary, orthogonal and symplectic group *Communications in Mathematical Physics*, 264(3):773–795, 2004.

Local eigenvalue distribution of random regular graphs

ANTTI KNOWLES

(joint work with Roland Bauerschmidt, Jiaoyang Huang, and Horng-Tzer Yau)

Let A be the adjacency matrix of a random d-regular graph on N vertices. For fixed $d \geq 3$, it is well known that as $N \to \infty$ the empirical spectral measure of A converges weakly to the *Kesten-McKay law* [9, 11], with density

$$(1) \qquad \frac{d}{d^2 - x^2} \frac{1}{2\pi} \sqrt{[4(d-1) - x^2]_+} \,.$$

Thus, the rescaled adjacency matrix $(d-1)^{-1/2}A$ has asymptotic spectral density

$$(2) \qquad \varrho_d(x) := \left(1 + \frac{1}{d-1} - \frac{x^2}{d} \right)^{-1} \frac{\sqrt{[4 - x^2]_+}}{2\pi} \,.$$

Clearly, $\varrho_d(x) \to \varrho(x)$ as $d \to \infty$, where $\varrho(x) := \frac{1}{2\pi} \sqrt{[4-x^2]_+}$ is the density of Wigner's *semicircle law.* The semicircle law is the asymptotic eigenvalue distribution of a random Hermitian matrix with independent (upper-triangular) entries (correctly normalized and subject to mild tail assumptions). From (2) it is natural to expect that, for sequences of random d-regular graphs such that $d \to \infty$ as $N \to \infty$ simultaneously, the spectral density of $(d-1)^{-1/2}A$ converges to the semicircle law. This was only proved recently [12].

In the study of universality of random matrix statistics, local versions of the semicircle law and its generalizations have played a crucial role; see for instance the survey [7]. The local semicircle law is a far-reaching generalization of the weak convergence to the semicircle law mentioned above. First, the local law admits test functions whose support decreases with N so that far fewer than N eigenvalues are counted, ideally only slightly more than order 1. (In contrast, weak convergence of probability measures applies only to macroscopic test functions counting an order N eigenvalues). Second, the local law controls *individual matrix entries* of the Green function. Both of these improvements have proved of fundamental importance for applications. For Wigner matrices, i.e. Hermitian random matrices with independent identically distributed upper-triangular entries, the semicircle law is known to hold down to the optimal spectral spectral scale $1/N$, corresponding to the typical eigenvalue spacing, up to a logarithmic correction. In [12, 4, 1, 8], it was shown that the semicircle law (for $d \to \infty$) or the Kesten-McKay law (for fixed d) holds for random d-regular graphs on spectral scales that are slightly smaller than the macroscopic scale 1.

In [3], we show that d-regular graphs with degree d at least $(\log N)^4$ obey the semicircle law down to spectral scales $(\log N)^4/N$. This scale is optimal up to the power of the logarithm.

From the perspective of random matrix theory, the adjacency matrix of a random d-regular graph is a symmetric random matrix with nonnegative integer entries constrained so that all row and column sums are equal to d. These constraints impose nontrivial dependencies among the entries. For example, if the sum of the first k entries of a given row is d, the remaining entries of that row must be zero.

Our general strategy in [3] is in part inspired by that developed for random matrices with independent entries, starting with [6]. All previous works on local laws rely heavily on the independence of the matrix entries, which allows one to condition on a subset of entries to obtain precise large deviation bounds on the entries of the Green function. (See for instance [5] for a detailed account.) While the independence of the matrix entries can presumably be replaced by weak or short-range dependence, the dependence structure of the entries of random regular graphs is global, and therefore a fundamentally different approach is required. In [3], we introduce a new method to analyse random regular graphs.

For simplicity, we present the results for the uniform random d-regular graph; similar results hold for other models of random regular graphs, such as the permutation and configuration models. From now on, therefore, A denotes the adjacency matrix of the uniform random d-regular graph on N vertices. First, we note that A has the trivial eigenvector $\mathbf{e} := N^{-1/2}(1, \ldots, 1)^*$ with eigenvalue d. We consider a matrix H obtained from A by setting the trivial eigenvalue d to zero followed by a rescaling:
$$H := (d-1)^{-1/2}(A - d\,\mathbf{e}\mathbf{e}^*)\,.$$
The local law is best stated in terms of the *Green function*
$$G(z) := (H - zI)^{-1}\,,$$
where $z \in \mathbb{C}_+$.

Theorem 1 (Local semicircle law for random regular graphs [3]). *Define*
$$D := d \wedge \frac{N^2}{d^3}\,,$$
and suppose that $\xi \geq (\log N)^2$ and $D \geq \xi^2$. Fix $c > 0$. Then, with probability at least $1 - \mathrm{e}^{-\xi \log \xi}$,
$$(3) \qquad \max_{i,j} |G_{ij}(z) - \delta_{ij}m(z)| = O\left[\xi\left(\frac{1}{\sqrt{N\,\mathrm{Im}\,z}} + \frac{1}{\sqrt{D}}\right)\right]$$
uniformly for all $z \in [-2 + c, 2 - c] \times (\xi^2/N, \infty)$, where
$$m(z) := \int \frac{\varrho(x)}{x - z}\,\mathrm{d}x$$
is the Stieltjes transform of the semicircle law. A similar estimate holds in the whole domain $z \in \mathbb{R} \times (\xi^2/N, \infty)$, with a slightly more complicated error bound in (3).

Theorem 1 has a number of important consequences. For instance, it implies that all eigenvectors of A (or H) are completely delocalized with high probability.

Corollary 2. *Under the assumptions of Theorem 1, any ℓ^2-normalized eigenvector of A or H has ℓ^∞-norm of size $O(\xi/\sqrt{N})$ with probability at least $1 - \mathrm{e}^{-\xi \log \xi}$.*

Another consequence of Theorem 1 is that the bulk eigenvalue statistics of A coincide with those of the Gaussian Orthogonal Ensemble (GOE). The universality of local eigenvalue statistics is one of the central questions in random matrix theory.

Random matrix statistics are believed to apply to very general complex systems, including the zeros of the Riemann ζ-function on the critical line. However, proofs of random matrix statistics have so far been limited mostly to matrix ensembles. There are two classes of matrix ensembles for which random matrix statistics have been established under very general conditions: invariant ensembles and ensembles with independent entries. Random regular graphs are not invariant and do not have independent entries. In [2], using the local semicircle law of [3] and the rate of convergence to equilibrium of the Dyson Brownian motion from [10], we prove that the eigenvalues of A obey random matrix statistics in the bulk of the spectrum.

Theorem 3 ([2])**.** *For $d \in [N^\varepsilon, N^{2/3-\varepsilon}]$ with $\varepsilon > 0$ fixed, the local eigenvalue statistics of A in the bulk coincide with those of the GOE.*

References

[1] N. Anantharaman and E.L. Masson. Quantum ergodicity on large regular graphs, 2013. Preprint, arXiv:1304.4343.

[2] R. Bauerschmidt, J. Huang, A. Knowles, and H.-T. Yau. Bulk eigenvalue statistics for random regular graphs, 2015. Preprint, arXiv:1505.06700.

[3] R. Bauerschmidt, A. Knowles, and H.-T. Yau. Local semicircle law for random regular graphs, 2015. Preprint, arXiv:1503.08702.

[4] I. Dumitriu and S. Pal. Sparse regular random graphs: spectral density and eigenvectors. *Ann. Probab.*, 40(5):2197–2235, 2012.

[5] L. Erdős, A. Knowles, H.-T. Yau, and J. Yin. The local semicircle law for a general class of random matrices. *Electron. J. Probab.*, 18:no. 59, 58, 2013.

[6] L. Erdős, B. Schlein, and H.-T. Yau. Local semicircle law and complete delocalization for Wigner random matrices. *Comm. Math. Phys.*, 287(2):641–655, 2009.

[7] L. Erdős and H.-T. Yau. Universality of local spectral statistics of random matrices. *Bull. Amer. Math. Soc. (N.S.)*, 49(3):377–414, 2012.

[8] L. Geisinger. Convergence of the density of states and delocalization of eigenvectors on random regular graphs, 2013. Preprint, arXiv:1305.1039.

[9] H. Kesten. Symmetric random walks on groups. *Trans. Amer. Math. Soc.*, 92:336–354, 1959.

[10] B. Landon and H.-T. Yau. Convergence of local statistics of Dyson Brownian motion, 2015. Preprint, arXiv:1504.03605.

[11] B.D. McKay. The expected eigenvalue distribution of a large regular graph. *Linear Algebra Appl.*, 40:203–216, 1981.

[12] L.V. Tran, V.H. Vu, and K. Wang. Sparse random graphs: eigenvalues and eigenvectors. *Random Structures Algorithms*, 42(1):110–134, 2013.

Limiting Spectral Distributions of Non-Hermitian Random Matrices

Holger Kösters

(joint work with Friedrich Götze and Alexander Tikhomirov)

We consider matrix-valued functions $\mathbf{F}(\mathbf{X}_1, \ldots, \mathbf{X}_m)$ of independent random matrices with independent matrix entries. More precisely, we assume that $\mathbf{X}_q = (n^{-1/2} X_{jk}^{(q)})_{jk=1,\ldots,n}$, $q = 1, \ldots, m$, where the $X_{jk}^{(q)}$ are independent uniformly square-integrable random variables with zero mean and unit variance, and that $\mathbf{F} := \mathbf{F}(\mathbf{X}_1, \ldots, \mathbf{X}_m)$ is a (non-Hermitian) random matrix of dimension $n \times n$.

We investigate the asymptotic behavior of the empirical singular value and eigen-value distributions (SVD's and EVD's) as $n \to \infty$. Here, all limits are understood as weak limits in probability.

In [4], we establish the universality of the limiting SVD's and EVD's within a general framework; see also [3] for related results. Informally, with $\mathbf{I}$ the identity matrix of appropriate dimension, our main result may be summarized as follows:

Theorem. *Suppose that there exist regularized matrix functions $\mathbf{F}_t$ such that (i) for any $\alpha \in \mathbb{C}$, the matrices $\mathbf{F}_t - \alpha \mathbf{I}$ and $\mathbf{F} - \alpha \mathbf{I}$ have the same limiting SVD's as $t \to 0$, (ii) for any $\alpha \in \mathbb{C}$, the regularized matrix functions $\mathbf{F}_t(\mathbf{X}_1, \ldots, \mathbf{X}_m) - \alpha \mathbf{I}$ satisfy certain rank, smoothness and boundedness conditions, (iii) for any $\alpha \in \mathbb{C}$, the large and small singular values of the matrices $\mathbf{F} - \alpha \mathbf{I}$ can be controlled. Then the limiting SVD's and EVD's of the matrices $\mathbf{F}$ are universal, i.e. they do not depend on the distributions of the matrix entries $X_{jk}^{(q)}$.*

We apply these general results to a special class of matrix-valued functions, namely sums of products of independent random matrices and their inverses [4, 7], where the limiting SVD's and EVD's can be described more explicitly using tools from free probability theory. By universality, it is sufficient to identify the limits in the Gaussian case. Furthermore, there exists a 1–1 correspondence between (symmetrized) limiting SVD's on the real line and (rotation-invariant) limiting EVD's on the complex plane [5, 6]. One exemplary result from [7] is as follows:

Theorem. *Suppose that $\mathbf{F}(\mathbf{X}_0, \ldots, \mathbf{X}_m) := \mathbf{X}_0 \mathbf{X}_1^{-1} \cdots \mathbf{X}_m^{-1}$. Then the limiting SVD is given (after symmetrization) by the symmetric $\boxplus$-stable distribution of parameter $\frac{2}{m+1}$, and the limiting EVD is the associated rotation-invariant distribution. Moreover, for any fixed $k \in \mathbb{N}$, if $\mathbf{F}_1, \ldots, \mathbf{F}_k$ are independent copies of $\mathbf{F}$, the sums $k^{-(m+1)/2}(\mathbf{F}_1 + \ldots + \mathbf{F}_k)$ have the same limiting SVD and EVD as the matrices $\mathbf{F}$.*

Let us also mention the recent work [1, 2], which provides an algorithm for calculating the Brown measures of general polynomials in free non-commutative random variables. This leads to a wide class of further examples (e.g. sums of products of independent shifted random matrices $\mathbf{X}_q - \alpha_q \mathbf{I}$) where the limiting spectral distributions may be determined. For general matrix-valued functions, however, the control of the small singular values is an open problem.

REFERENCES

[1] S. T. Belinschi, T. Mai, R. Speicher (2013): *Analytic subordination theory of operator-valued free additive convolution and the solution of a general random matrix problem.* Preprint, arXiv:1303.3196. To appear in J. Reine Angew. Math.
[2] S. T. Belinschi, P. Śniady, R. Speicher (2015): *Eigenvalues of non-Hermitian random matrices and Brown measure of non-normal operators: Hermitian reduction and linearization method.* Preprint, arXiv:1506.02017.
[3] Ch. Bordenave (2011): *On the spectrum of sum and product of non-Hermitian random matrices.* Electron. Commun. Probab. **16**, 104–113.

[4] F. Götze, H. Kösters, A. Tikhomirov (2014): *Asymptotic spectra of matrix-valued functions of independent random matrices and free probability.* Preprint, arXiv:1408.1732. To appear in Random Matrices Theory Appl.

[5] U. Haagerup, F. Larsen (2000): *Brown's spectral distribution measure for R-diagonal elements in finite von Neumann algebras.* J. Funct. Anal. **2**, 331–367.

[6] U. Haagerup, H. Schultz (2007): *Brown measures of unbounded operators affiliated with a finite von Neumann algebra.* Math. Scand. **100**, 209–263.

[7] H. Kösters, A. Tikhomirov (2015): *Limiting spectral distributions of sums of products of non-Hermitian random matrices.* Preprint, arXiv:1506.04436.

Strong Convergence of Unitary Brownian Motion

ANTOINE DAHLQVIST AND TODD KEMP

(joint work with Benoît Collins)

Let $\mathbb{U}_N$ denote the unitary group of $N \times N$ complex matrices U satisfying $UU^* = I$. It is a (real) Lie group of dimension N^2, with Lie algebra $\mathfrak{u}(N)$ consisting of all skew-Hermitian matrices. Equip this Lie algebra with the appropriately scaled Hilbert-Schmidt inner product, $\langle X, Y \rangle \equiv -N\mathrm{Tr}(XY)$, which is $\mathrm{Ad}(\mathbb{U}_N)$-invariant. By right-translation, this gives rise to a left-invariant Riemannian metric and thus a left-invariant Laplace operator $\Delta_{\mathbb{U}_N}$ on $\mathbb{U}_N$. The Laplacian can be computed by fixing any orthonormal basis β_N for $\mathfrak{u}(N)$; then

$$\Delta_{\mathbb{U}_N} = \sum_{X \in \beta_N} \partial_X^2$$

where ∂_X is the left-invariant vector field whose value at the identity I is X: in other words, $\partial_X(f)(U) = \frac{d}{dt}\big|_{t=0} f(U \exp tX)$.

The **unitary Brownian motion** $(U^N(t))_{t \geq 0}$ is the diffusion process whose generator is $\frac{1}{2}\Delta_{\mathbb{U}_N}$. Alternatively, it can be described as the unique process with a.s. continuous paths whose left multiplicative increments $\{U^N(s)^{-1}U^N(t) \colon s < t\}$ are independent and stationary, with distribution given by the heat kernel μ_{t-s}^N (the fundamental solution of the heat equation $\partial_t \phi = \frac{1}{2}\Delta_{\mathbb{U}_N}\phi$ at time $t - s$). A still more computationally practical characterization is as the solution of a stochastic differential equation. Let $(W^N(t))_{t \geq 0}$ be a standard Brownian motion in the Lie algebra $\mathfrak{u}(N)$: $W^N(t) = \sum_{X \in \beta_N} W_X(t) \cdot X$ where $\{W_X \colon X \in \beta_N\}$ are i.i.d. standard Brownian motions on $\mathbb{R}$. If we define $X^N(t) = -iW^N(t)$, then $(X^N(t))_{t \geq 0}$ is a GUE^N Brownian motion: the entries are Brownian motions, and the matrix $X^N(t)$ is a GUE^N of total variance t. The unitary Brownian motion is the unique solution of the SDE

$$dU^N(t) = U^N(t) \circ dW^N(t) = iU^N(t)\,dX^N(t) - \frac{1}{2}U^N(t)\,dt, \qquad U^N(0) = I$$

where $\circ$ denotes the Stratonovich integral, and the right-hand-side is written as a more typical Itô integral.

In [1], Biane showed that the process $(U^N(t))_{t\geq 0}$ converges, in the sense of its finite dimensional noncommutative distributions, to the **free unitary Brownian motion** $(u(t))_{t\geq 0}$, which is analogously defined as the solution of the *free* SDE

$$du(t) = iu(t)\, dx(t) - \frac{1}{2}u(t)\, dt, \qquad u(0) = 1$$

where $(x(t))_{t\geq 0}$ is a standard free additive Brownian motion in a W^* probability space $(\mathcal{A}, \tau)$. To be clear, the notion of convergence here is as follows: for any times $t_1, \ldots, t_k \geq 0$, and any noncommutative polynomial P in $2k$ indeterminates,

$$\lim_{N\to\infty} \frac{1}{N}\text{Tr}\left[P(U^N(t_1), U^N(t_1)^*, \ldots, U^N(t_k), U^N(t_k)^*)\right]$$
$$= \tau\left[P(u(t_1), u(t_1)^*, \ldots, u(t_k), u(t_k)^*)\right] \ a.s.$$

In the case $k = 1$ of a single time $t_1 = t$, the fact that $U^N(t)$ and $u(t)$ are normal reduces Biane's result to a random matrix result: the ESD (empirical spectral distribution) of $U^N(t)$ converges, weakly a.s., to the law ν_t of $u(t)$. This law is a measure on the unit circle $\mathbb{U}_1$ which was also first computed by Biane in [1]. It has the following properties: ν_t has a continuous density ϱ_t with respect to the Haar measure on $\mathbb{U}_1$, with support

$$\text{supp}\,\nu_t = \left\{e^{i\theta}: |\theta| \leq \frac{1}{2}\sqrt{t(4-t)} + \arccos\left(1 - \frac{t}{2}\right)\right\}$$

when $0 < t < 4$, and full support for $t \geq 4$. The density ϱ_t is real analytic on the interior of its support, symmetric about 1, and is determined by $\varrho_t(e^{i\theta}) = \text{Re}\,\kappa_t(e^{i\theta})$ where $z = \kappa_t(e^{i\theta})$ is the unique solution (with positive real part) to the equation

$$\frac{z-1}{z+1}e^{\frac{t}{2}z} = e^{i\theta}.$$

Biane's convergence result is a (noncommutative generalization of a) *bulk* result: it concerns the average behavior of *all* eigenvalues, and is therefore not fine enough to characterize the behavior of the *edge* eigenvalues. That is: for $0 < r < 1$, the eigenvalue whose angle is $\lfloor rN \rfloor$th largest is known to converge, by Biane's result, to the point up to which the proportion r of the mass of the measure ν_t is distributed. But the largest and smallest angled eigenvalues (or more generally the ℓth largest and smallest, for fixed ℓ as N grows) are not determined by Biane's result.

The topic of this lecture was the dissemination of the results in the speakers' paper [2], addressing the edge eigenvalues of the unitary Brownian motion, and the appropriate noncommutative generalization thereof. Restricting to the case $k = 1$, we state the first main theorem, known as the *hard edge* theorem for the unitary Brownian motion.

Theorem 1. *For $0 < t < 4$, the largest and smallest angled eigenvalues of $U^N(t)$ converge, a.s. as $N \to \infty$, to the edges of $\text{supp}\,\nu_t$; for $t \geq 4$, they converge to -1.*

This can be restated in a more geometric fashion. Given two compact sets A and B in a metric space, their *Hausdorff distance* is defined to be $\mathrm{d}_H(A, B) \equiv \inf\{\epsilon \geq 0 : A \subseteq B_\epsilon\ \&\ B \subseteq A_\epsilon\}$, where A_ϵ is the neighborhood of all points within distance

ϵ of A. Theorem 1 can be restated thus: *For $t \geq 0$, $\mathrm{d}_H(\mathrm{spec}(U^N(t)), \mathrm{supp}\,\nu_t) \to 0$ a.s. as $N \to \infty$.* The random spectrum of $U^N(t)$, as a random compact subset of $\mathbb{R}$, converges a.s. in Hausdorff distance, to the arc that is the support of ν_t.

The proof of Theorem 1 uses harmonic analysis techniques to detect eigenvalues in any neighborhood outside the putative limit spectrum. It boils down to having sharp enough estimates on the average trace moments of the unitary Brownian motion: how fast do they converge to their limits in Biane's theorem? The sufficient answer is as follows.

Theorem 2. *For $t \geq 0$ and $n \in \mathbb{N}$,*

$$\left| \frac{1}{N} \mathbb{E}\mathrm{Tr}[(U_t^N)^n] - \int_{\mathbb{U}_1} u^n \, \nu_t(du) \right| \leq \frac{t^2 n^4}{N^2}.$$

The summable $\frac{1}{N^2}$ allows the use of the Borel-Cantelli lemma to yield a.s. convergence, while the polynomial factor n^4 gives the freedom to use Sobolev test functions (in $H_5(\mathbb{U}_1)$, in particular) to test for eigenvalues outside the bulk. The majority of the first half of the paper is devoted to the proof of Theorem 2. It uses a mixture of direct computation with SDEs for unitary Brownian motion, a subtle coupling technique to compare $U^N(t)$ with $U^{2N}(t)$ (and then sum the differences to get a geometric series comparison with the limit), and representation theory of the unitary groups, notably using the Schur-Weyl duality to calculate and estimate all involved quantities via the action of the symmetric groups.

In the remainder of the lecture, we discussed the noncommutative generalization of the hard edge theorem. The analogue of Hausdorff convergence of the spectrum is called **strong convergence**. Introduced formally in Camille Male's dissertation [5], the notion augments convergence in noncommutative distribution by measuring the mixed moments not only in trace but also in operator norm. In the case of unitary Brownian motion, the definition is that, in addition to convergence of finite dimensional noncommutative distributions, we also have

$$\lim_{N \to \infty} \left\| P(U^N(t_1), U^N(t_1)^*, \ldots, U^N(t_k), U^N(t_k)^*) \right\|$$
$$= \left\| P(u(t_1), u(t_1)^*, \ldots, u(t_k), u(t_k)^*) \right\| \ a.s.$$

for all noncommutative polynomials P and times $t_1, \ldots, t_k \geq 0$. The main theorem of the second half of our paper is as follows.

Theorem 3. *For any times $t_1, \ldots, t_k \geq 0$, $(U^N(t_1), \ldots, U^N(t_k))$ converges strongly to $(u(t_1), \ldots, u(t_k))$ as $N \to \infty$.*

In fact, in the paper [2], we prove a strong *extension* form of this result: any strongly convergent collection of ensembles, when augments with an independent unitary Brownian motion, still converges strongly.

The proof of Theorem 3 relies on earlier work of Collins and Male [3] and Haagerup and Thorbjørnsen [4] which showed that this extension property of strong convergence holds for GUE ensembles. We then, using Theorem 1 at a critical initial juncture, show how to uniformly approximate the increments of a

unitary Brownian motion by nice functions of independent GUE ensembles, yielding the proof.

REFERENCES

[1] Biane, P., *Free Brownian motion, free stochastic calculus and random matrices.* In Free probability theory (Waterloo, ON, 1995), vol. 12 of Fields Inst. Commun., Amer. Math. Soc., Providence, RI, 1997, pp. 1–19.
[2] Collins, B., Dahlqvist, A., and Kemp, T., *Strong convergence of unitary Brownian motion.* Preprint, arXiv:1502.06186.
[3] Collins, B., and Male, C., *The strong asymptotic freeness of Haar and deterministic matrices.* Ann. Sci. Éc. Norm. Supér. (4) **47**, 1 (2014), 147–163.
[4] Haagerup, U., and Thorbjørnsen, S., *A new application of random matrices:* $\mathrm{Ext}(C^*_{\mathrm{red}}(F_2))$ *is not a group.* Ann. of Math. (2) **162**, 2 (2005), 711–775.
[5] Male, C., *The norm of polynomials in large random and deterministic matrices.* Probab. Theory Related Fields **154**, 3-4 (2012), 477–532. With an appendix by Dimitri Shlyakhtenko.

The Douglas-Kazakov phase transition

THIERRY LÉVY AND MYLÈNE MAÏDA

We present a proof of the existence of a third order phase transition discovered by Douglas and Kazakov around 1993 [1]. From the point of view of theoretical physics, this transition occurs for the $U(N)$ Yang-Mills measure on the two dimensional sphere S^2, in the large N limit, as the total area of the sphere crosses the critical value π^2.

From a probabilistic point of view, the transition can be observed on the large N limit of the Brownian bridge on $U(N)$ as the total lifetime of the bridge crosses the critical value π^2.

The first mathematical study of this transition was made by Liechty and Wang in their study of non-intersecting Brownian motions on a circle [5]. Their results show that the phase transition can be understood as a qualitative difference in the large N limit of the behaviour of the eigenvalues between a Brownian bridge of short lifetime and a Brownian bridge of long lifetime.

We analyse this phase transition from a dual point of view, in Fourier space. There is a natural partition function associated with the Brownian bridge of lifetime T on $U(N)$, namely the value at the identity matrix of the heat kernel at time T on the unitary group, which can be expressed as

$$p_T(I_N) = \frac{e^{\frac{T}{24}(N^2-1)}}{(1! \ldots (N-1)!)^2} \sum_{(\ell_1 > \ldots > \ell_N) \in \mathbb{Z}^N} e^{-\frac{T}{2} \sum_{i=1}^N (\ell_i - \frac{N-1}{2})^2} \prod_{1 \leq i < j \leq N} (\ell_i - \ell_j)^2.$$

We identify, in the limit where N tends to infinity, and as a function of T, the shape of the N-tuple of integers which brings the largest contribution to the sum.

Technically, introducing the empirical measure $\hat{\mu}_\ell = \frac{1}{N} \sum_{i=1}^N \delta_{\frac{1}{N}(\ell_i - \frac{N-1}{2})}$, a mild reformulation of the expression above and an application of a principle of large

deviations proved by Guionnet and Maïda [2] leads us to the problem of finding the minimum of the functional

$$I_T(\mu) = \iint -\log|x-y|\, d\mu(x)d\mu(y) + 2\int \frac{T}{4}x^2\, d\mu(x)$$

over the set of all Borel probability measure on $\mathbb{R}$ which are absolutely continuous with respect to the Lebesgue measure, with density bounded by 1. This last condition can be understood as the continuous analogue of the fact that the integers $\ell_1 > \ldots > \ell_N$ which index Fourier modes on the unitary group are pairwise distinct.

If one ignores the constraint, the solution of this problem is well known. Indeed, I_T attains its minimum at the semi-circular distribution of parameter $\frac{1}{T}$. However, for $T > \pi^2$, the density of this absolute minimiser of I_T exceeds 1 on an open interval, and the solution of the constrained problem is another probability measure.

Using tools of potential theory (for which an excellent first reference is the book of Saff and Totik [6]), we explain how to derive a solution of this constrained problem, and how to prove that the probability measure thus obtained is indeed the unique solution to our problem.

The figure below shows the shape of the density of the measure which minimises I_T for $T = 4, 7, 10, 13, 16$. We obtain (and, more critically, explain *how* to obtain) explicit expressions for these densities, which were already present in [5], and in an older work of Gross and Matytsin [3].
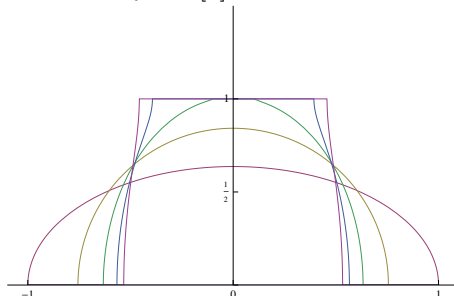
Finally, a precise formulation of the phase transition is the following: for all $T > 0$, the limit

$$F(T) = \lim_{N\to\infty} \frac{1}{N^2} \log p_T(I_N)$$

exists, and the function F thus defined is smooth on $\mathbb{R}_+^* \setminus \{\pi^2\}$, of class $\mathcal{C}^2$ on $\mathbb{R}_+^*$, and F''' has a jump discontinuity at $T = \pi^2$.

References

[1] M. R. Douglas and V. A. Kazakov. Large N Phase Transition in continuum QCD$_2$. *Phys. Lett. B*, 319, 1993.
[2] Alice Guionnet and Mylène Maïda. Character expansion method for the first order asymptotics of a matrix integral. *Probab. Theory Related Fields*, 132(4):539–578, 2005.

[3] David J. Gross and Andrei Matytsin. Some properties of large-N two-dimensional Yang-Mills theory. *Nucl. Phys., B*, 437(3):541–584, 1995.

[4] Thierry Lévy and Mylène Maïda. On The Douglas-Kazakov Phase Transition. arXiv:1503.00502.

[5] K. Liechty and D. Wang. Nonintersecting Brownian motions on the unit circle. To appear in *Ann. Prob.*, arXiv:1312.7390.

[6] Edward B. Saff and Vilmos Totik. *Logarithmic potentials with external fields*, volume 316 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1997.

The Thompson group F from the viewpoint of noncommutative probability

CLAUS KÖSTLER

This talk reports on new research results on the Thompson group F; related publications are in preparation. Our main results establish that the Thompson group F has a very rich and interesting representation theory. We illustrate this from a viewpoint of noncommutative probability.

Traditionally the Thompson group F is an infinite group generated by X_0, X_1 subject to some relations, but here it is convenient to take the infinite presentation with generators $X_0, X_1, X_2, \ldots$ satisfying the relations

$$X_n X_k = X_k X_{n+1} \qquad (0 \le k < n).$$

For our purposes it will be even more convenient to use the inverse generators $x_n := X_n^{-1}$ satisfying the relations

$$x_k x_n = x_{n+1} x_k \qquad (0 \le k < n)$$

and to work with the Thompson monoid

$$MF := \langle x_n | x_k x_n = x_{n+1} x_k; 0 \le k < n < \infty \rangle.$$

The starting point of our investigations was the following observation: 'Spreadability', a distributional invariance principle for infinite sequences of noncommutative random variables [7, 2], is implemented by maps $(\delta_n)_{n \ge 0}$ satisfying the relations

$$\delta_k \delta_n = \delta_{n+1} \delta_k \qquad (0 \le k \le n).$$

As for classical de Finetti theorems, it was shown in [7] that spreadability yields noncommutative Bernoulli shifts. Above relations are those of a quotient of the Thompson monoid MF. This initiated our studies to understand representations of MF as a 'nice' perturbation of noncommutative Bernoulli shifts: noncommutative Markov shifts. We illustrate this by a simple example for the Thompson monoid MF.

Let $\mathcal{A}$ and $\mathcal{B}$ von Neumann algebras and, for some fixed automorphism $\alpha \in \mathrm{Aut}(\mathcal{A} \otimes B)$, define on the infinite von Neumann algebraic tensor product

$$\mathcal{M} := \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \cdots$$

the coupling

$$\beta_1(a \otimes b_1 \otimes b_2 \otimes b_3 \otimes \cdots) \quad := \quad \alpha(a \otimes b_1) \otimes b_2 \otimes b_3 \otimes \cdots$$

and, for $n \geq 2$, the local flips

$$\beta_n(a \otimes b_1 \otimes \cdots \otimes b_{n-1} \otimes b_n \otimes \cdots) \quad := \quad a \otimes b_1 \otimes \cdots \otimes b_n \otimes b_{n-1} \otimes \cdots$$

Now consider the endomorphisms

$$T_n = \lim_{N \to \infty} \beta_{n+1}\beta_{n+2} \cdots \beta_{n+N}$$

where the limit is taken in the pointwise strong operator topology. Using the fact that $(T_n)_{n \geq 1}$ implements 'spreadability', it is straightforward to verify that the endomorphisms $(T_n)_{n \geq 0}$ satisfy the relations:

$$T_k T_n = T_{n+1} T_k \qquad (0 \leq k < n).$$

In other words, the multiplicative extension of the map

$$\rho \colon MF \ni x_n \mapsto T_n \in \mathrm{End}(\mathcal{M})$$

is a representation of the Thompson monoid MF in the unital endomorphisms of the von Neumann algebra $\mathcal{M}$. Equipping $\mathcal{M}$ with a normal faithful tensor product state φ, the endomorphism T_0 is a unilateral (noncommutative) Markov shift and T_1 is a unilateral (noncommutative) Bernoulli shift over $\mathcal{A}$. By construction $T_0 = \beta_1 T_1$, so the Markov shift is given as a coupling to the Bernoulli shift. As a consequence one obtains the following result.

Theorem 1. *Every classical stationary Markov chain (on a finite state space) defines a representation of the Thompson monoid MF.*

This result transfers to the noncommutative setting, for example, taking square matrices for $\mathcal{A}$ and $\mathcal{B}$ and equipping them with normalized traces. In particular one obtains for factorizable Markov maps (as introduced in [1, 5]) the following result. Let tr_d denote the normalized trace on the complex $d \times d$-matrices $M_d(\mathbb{C})$.

Theorem 2. *Any tr_d-factorizable Markov map on $M_d(\mathbb{C})$ defines a representation of the Thompson monoid MF.*

It is natural to ask if a converse of these results is also true. Note that a Bernoulli shift over some von Neumann algebra is a Markov shift with a conditional expectation as transition operator and thus the sequence of endomorphisms $(T_n)_{n \geq 0}$ can be interpreted as a system of Markov shifts. To avoid additional technical conditions we announce the following result in a tracial setting of noncommutative W*-algebraic probability spaces.

Theorem 3. *Let the von Neumann algebra $\mathcal{M}$ be equipped with a normal faithful tracial state tr. If $\rho \colon MF \to \mathrm{End}(\mathcal{M}, \mathrm{tr})$ is a representation in the tr-preserving endomorphisms of $\mathcal{M}$, then $(\rho(x_n))_{n \geq 0}$ is a system of unilateral noncommutative Markov shifts.*

As an immediate consequence, a representation of the Thompson group F in the tr-preserving automorphisms of $\mathcal{M}$ gives rise to a system of bilateral noncommutative Markov shifts.

An open problem is at the moment if each of these Markov shifts $\rho(x_n)$ is given as a coupling to a Bernoulli shift, a feature exhibited in above motivating

example. Moreover, since unitary representations of the infinite braid group give rise to 'spreadability' ([7, 3]), it will be of interest to investigate how above results relate to the construction of unitary representations of the Thompson group F in [6]. Finally we speculate if a better understanding of the representation theory of F creates valuable insight for the still open amenability problem of F (compare [4]).

REFERENCES

[1] C. Anantharaman-Delaroche: *On ergodic theorems for free group actions on noncommutative spaces.*Probab. Theory Rel. Fields **135**, 520–546 (2006).
[2] G. Evans, R. Gohm, C. Köstler: *Semi-cosimplicial objects and spreadability.* In preparation (2015).
[3] R. Gohm, C. Köstler: *Noncommutative independence from the braid group $\mathbb{B}_\infty$.* Commun. Math. Phys. **289**, 435–482 (2009).
[4] S. Haagerup, U. Haagerup, M. Ramirez-Solano: *A computational approach to the Thompson group F.* arXiv:1409.1486 (2014).
[5] U. Haagerup, M. Musat:*Factorization and dilation problems for completely positive maps on von Neumann algebras.* Comm. Math. Phys. **303**, 555–594 (2011).
[6] V.F.R. Jones: *Some unitary representations of Thompson's groups F and T.* arXiv:1412.7740 (2014).
[7] C. Köstler: *A noncommutative extended de Finetti theorem.* J. Funct. Anal. **258**, 1073–1120 (2010).

Tail algebras of symmetric states and de Finetti theorems

KEN DYKEMA

(joint work with Claus Köstler and John Williams)

A classical theorem of B. de Finetti [5] shows that an infinite sequence of classical random variables is exchangeable (namely, has distribution invariant under arbitrary permutations of the variables) if and only if the random variables are conditionally independent over the tail algebra. The noncommutative de Finetti theorem of Köstler and Speicher [6], in the setting of a W*-noncommutative probability space equipped with a faithful state, forms one of the many marvelous parallels between classical and free probability theories. It shows that an infinite sequence of noncommutative random variables is quantum exchangeable if and only if they are free with amalgamation over the tail algebra. Here, quantum exchangeable means invariant under the action of of S. Wang's quantum permutation group [9].

E. Størmer [8] extended the purview of the classical de Finetti theorem to the realm of C*-algebras, showing the symmetric states on the infinite tensor product $\bigotimes_1^\infty A$ of a unital C*-algebra A with itself, form a Choquet simplex and that the extreme points of this simplex are the infinite tensor product states $\otimes_1^\infty \phi$, of states ϕ on A. Here "symmetric" means invariant under the obvious permutation action of S_∞ on the above tensor product algebra.

In the paper [4], we investigate quantum symmetric states on the universal unital free product $*_1^\infty A$ of a C*-algebra A with itself infinitely many times, these

being those that are invariant under the appropriate action of Wang's quantum permutation group. It turns out that, by a proof analogous to Köstler's and Speicher's proof of the noncommutative de Finetti theorem, these states are characterized by freeness of the copies of A with amalgamation over the tail algebra. We investigate the structure of the set of these quantum symmetric states and for this it is essential to study also quantum symmetric states that yield non-faithful states on the von Neumann algebras generated by the images of their GNS representations. (We would like to point out also some similar freeness-with-amalgamation results of S. Curran [2], proved with different techniques that yield interesting results about finite sequences, though assuming faithfulness of states.) These states form a compact convex set, though they are not a simplex. In [4], we characterize the extreme points of this convex set.

We also mention that in [3], Dabrowski, Dykema and Mukherjee show that the set of tracial quantum symmetric states does form a Choquet simplex and the authors characterize its extreme points.

The main body of the talk, (after a summary of results slightly more detailed than the one found above), focused the tail algebra and questions about conditional expectations onto the tail algebra for symmetric states on the free product C*-algebra $\mathfrak{A} = *_1^\infty A$. The symmetric states are those that are invariant under the obvious permutation action of S_∞ on $\mathfrak{A}$. These include the quantum symmetric states, but are much more general. If ϕ is such a symmetric state, let π_ϕ be the GNS representation of it, let $\mathcal{M}_\phi$ denote the von Neumann algebra generated by the image of π_ϕ, and let $\hat{\phi}$ denote the normal state $\langle \cdot \hat{1}, \hat{1} \rangle$ on $\mathcal{M}_\phi$ so that $\hat{\phi} \circ \pi_\phi = \phi$. The *tail algebra* of ϕ is the von Neumann subalgebra

$$\mathcal{T}_\phi = \bigcap_{n=1}^\infty W^*\big(\bigcup_{j \geq n} \pi_\phi(A_j) \big),$$

of $\mathcal{M}_\phi$, where A_j is the j-th copy of A in $\mathfrak{A}$. An example of Weihua Liu [7] (described in [4]) shows that there need not be a normal, $\hat{\phi}$-preserving conditional expectation from $\mathcal{M}_\phi$ onto $\mathcal{T}_\phi$. However, we show that there is a $\hat{\phi}$-preserving, S_∞-invaraint conditional expectation from the C*-algebra

$$Q_\phi := C^*(\mathcal{T}_\phi \cup \pi_\phi(\mathfrak{A}))$$

onto $\mathcal{T}_\phi$, and we define the *tail C*-algebra* D_ϕ to be the the smallest unital C*-subalgebra of $\mathcal{T}_\phi$ containing $E_\phi(C^*(D_\phi \cup \pi_\phi(\mathfrak{A})))$. It is freeness with amalgamation over D_ϕ (with respect to the restriction of E_ϕ to Q_ϕ) that characterizes quantum symmetric states, and that we can use in the description of the convex structure of quantum symmetric states.

We ask two open questions about tail algebras of symmetric states:

Question 1. Do we always have $D_\phi \subseteq \pi_\phi(\mathfrak{A})$?

Question 2. Is T_ϕ generated as a von Neumann algebra by D_ϕ?

The answers to both of these questions are "yes" in the case of quantum symmetric states (for the first question, the proof depends on a version of Haagerup's

inequality for free products with amalgamation, found in [1]). These questions are relevant to investigating the extreme points of the convex set of symmetric states on $\mathfrak{A}$.

REFERENCES

[1] B. Abadie, K. Dykema, *Unique ergodicity of free shifts and some other automorphisms of* C^**-algebras*, J. Operator Theory **61** (2009), 279–294.
[2] S. Curran, *Quantum exchangeable sequences of algebras*, Indiana Univ. Math. J. **58** (2009), 1097–1125.
[3] Y. Dabrowski, K. Dykema, K. Mukherjee, *The simplex of tracial quantum symmetric states*, Studia Math. **255** (2014), 203–218.
[4] K. Dykema, C. Köstler, J. Williams, *Quantum symmetric states on free product C*-algebras*, Trans. Amer. Math. Soc. (to appear), arXiv:1305.7293.
[5] B. de Finetti, *La prevision: ses lois logiques, ses sources subjectives*, Ann. Inst. H. Poincaré **7**, (1937), 1–68.
[6] C. Köstler, R. Speicher, *A noncommutative de Finetti theorem: invariance under quantum permutations is equivalent to freeness with amalgamation*, Comm. Math. Phys. **291**, (2009), 473–490.
[7] W. Liu, *private communication*, (2014).
[8] E. Størmer, *Symmetric states of infinite tensor products of C*-algebras*, J. Funct. Anal. **3**, (1969), 48–68.
[9] S. Wang, *Quantum symmetry groups of finite spaces*, Comm. Math. Phys. **195**, (1998), 195–211.

An example of factoriality under non-tracial finite free Fisher information assumptions

Brent Nelson

Suppose M is a von Neumann algebra equipped with a faithful normal state φ, and is generated by an algebraically free finite set $G = G^*$, $|G| \geq 3$. We show that if G consists of eigenvectors of the modular operator Δ_φ and has finite free Fisher information, then the centralizer M^φ is a II_1 factor and M is a factor of type depending on the eigenvalues of G.

We first use the finite free Fisher information assumption to establish the existence of diffuse elements in M^φ. As in the tracial case (*cf.* Theorem 3.1 in [3] and Theorems 3 and 4 in [4]), a reduction argument is used that shows when $xp = 0$ for certain polynomials $x \in \mathbb{C}\langle G \rangle \cap M^\varphi$ and a non-zero projection p, then $(\varphi \otimes 1)(\delta_y(x))p = 0$ for every free difference quotient δ_y, $y \in G$. A non-tracial analogue of Theorem 4.4 in [1] is used, wherein the generators are regularized by quasi-free semicircular random variables (i.e. generators of a free Araki-Woods factor) rather than free semicircular random variables.

The construction of a diffuse element $x \in M^\varphi$ requires $|G| \geq 2$ in the non-tracial case. The requirement that $|G| \geq 3$ is necessary so that $\exists y \in G$ such that $\delta_y(x) = 0$.

Using a contraction resolvent argument of Dabrowski (*cf.* Section 1 in [2]), we show that if $z \in (M^\varphi)' \cap M$ then $\delta_y(z)$ is a Hilbert-Schmidt operator which

commutes with the diffuse element x; that is, $\delta_y(z) = 0$. An simple ad hoc argument can then be used to show $z \in \mathbb{C}$, and therefore M^φ and M are factors.

That M^φ is a II_1 factor is immediate since it contains the diffuse element x and $\varphi \mid_{M^\varphi}$ is a finite trace. The type classification of M follows by studying the Arveson spectrum of σ^φ and Connes $S(M)$ invariant.

<div align="center">REFERENCES</div>

[1] A. Connes and D. Shlyakhtenko, L^2-homology for von Neumann algebras, J. Reine Angew. Math. **586** (2005), 125–168.

[2] Y. Dabrowski, *A note about proving non-Γ under a finite non-microstates free Fisher information assumption*, J. Funct. Anal. **258** (2010), no. 11, 3662–3674.

[3] T. Mai, R. Speicher and M. Weber, *Absence of algebraic relations and of zero divisors under the assumption of finite non-microstates free Fisher information*, Preprint (2014), arXiv:1407.5715.

[4] D. Shlyakhtenko, *Free entropy dimension and atoms*, Preprint (2014), arXiv:1408.0580.

Freeness and the Partial Transposes of Haar Distributed Unitary Matrices

JAMES A. MINGO

In 2012 Guillaume Aubrun [1] showed that the partial transpose of a real or complex Wishart matrix converges to a shifted semi-circular distribution. The resulting semi-circular operator is positive when the aspect ratio of the Wishart matrix is greater than 4, otherwise the partially transposed matrix will asymptotically have negative eigenvalues.

In a recent paper with Mihai Popa, [3], we showed that unitarily invariant ensembles were asymptotically real second order free from their transposes. In particular unitarily invariant matrix was asymptotically free from its transpose. A natural question was what happens with partial transposes. Suppose that $G_1 \ldots, G_{d_1}$ are independent complex Gaussian $d_2 \times p$ random matrices whose entries are complex Gaussian random variables with mean 0 and (complex) variance 1 and let

$$W = \frac{1}{d_1 d_2} \left(\frac{\begin{array}{c} G_1 \\ \vdots \\ \hline G_{d_1} \end{array}}{} \right) \left(\begin{array}{c|c|c} G_1^* & \cdots & G_{d_1}^* \end{array} \right).$$

By our result [3], W and W^{T} our asymptotically free where W^{T} is the transpose of W. Writing W as a block matrix $W = (W(i,j))_{ij}$ with $1 \leq i, j \leq d_1$ and each $W(i,j)$ a $d_2 \times d_2$ matrix we let W^{T} be the block matrix whose (i,j) entry is $W(j,i)$ and W^Γ be the block matrix whose (i,j) entry is $W(i,j)^{\mathrm{T}}$. We showed that the family $\{W, W^{\mathrm{T}}, W^\Gamma, W^{\mathrm{T}}\}$ is an asymptotically free family. Using the method of graph sums, [4], this has been extended to the case of non-Gaussian entries with the same conclusion. Back in the Gaussian case, the second order cumulants have been computed. In the real Gaussian case $W = W^{\mathrm{T}}$ and $W^{\mathrm{T}} = W^\Gamma$, so we only get W and W^Γ becoming asymptotically free.

In this report we turn to the case of Haar distributed random unitary matrices. Let U be a $d_1 d_2 \times d_1 d_2$ Haar distributed random unitary matrix, we would like to find the joint distribution of the following eight operators

$$U, U^*, U^{\mathsf{T}}, (U^{\mathsf{T}})^*, U^{\Gamma}, (U^{\Gamma})^*, U^{\mathrm{T}}, (U^{\mathrm{T}})^*.$$

Here we define the partial transpose with respect to $M_{d_1}(\mathbf{C}) \otimes M_{d_2}(\mathbf{C})$. We write $U = (U(i, j))_{i,j}$ as a $d_1 \times d_1$ block matrix where $U(i, j)$ is a $d_2 \times d_2$ matrix. Then the (i, j) block of U^{T} is $U(j, i)$; the (i, j) block of U^{Γ} is $U(i, j)^{\mathrm{T}}$ and the (i, j) block of U^{T} is $U(j, i)^{\mathrm{T}}$.

Theorem 1. *The operators $U, U^{\mathsf{T}}, U^{\Gamma}, U^{\mathrm{T}}$ are asymptotically $*$-free and U^{T} and U^{Γ} are asymptotically circular.*

In order to keep track of all eight of these operators we introduce some notation: ϵ is left partial transpose 'bit', η is the right partial transpose 'bit', θ is the adjoint 'bit'.

$$U^{(\epsilon, \eta, \theta)} = \begin{cases} U & \epsilon = 1 & \eta = 1 & \theta = 1 \\ U^* & \epsilon = 1 & \eta = 1 & \theta = -1 \\ U^{\mathsf{T}} & \epsilon = -1 & \eta = 1 & \theta = 1 \\ (U^{\mathsf{T}})^* & \epsilon = -1 & \eta = 1 & \theta = -1 \\ U^{\Gamma} & \epsilon = 1 & \eta = -1 & \theta = 1 \\ (U^{\Gamma})^* & \epsilon = 1 & \eta = -1 & \theta = -1 \\ U^{\mathrm{T}} & \epsilon = -1 & \eta = -1 & \theta = 1 \\ (U^{\mathrm{T}})^* & \epsilon = -1 & \eta = -1 & \theta = -1 \end{cases}.$$

To prove Theorem 1 we need to compute for every n and for every $(\epsilon_1, \ldots, \epsilon_n)$, for every $(\eta_1, \ldots, \eta_n)$, and for every $(\theta_1, \ldots, \theta_n)$ the limit
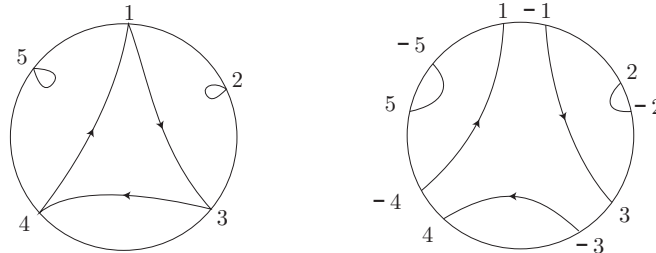
$$(1) \qquad \lim_{d_1, d_2 \to \infty} \mathrm{E}(U^{(\epsilon_1, \eta_1, \theta_1)} \cdots U^{(\epsilon_n, \eta_n, \theta_n)})$$

and show that this is the joint distribution of a $*$-free family of operators u_1, c_1, c_2, u_2 where u_1 and u_2 are Haar unitaries and c_1 and c_2 are circular operators. The asymptotic freeness is achieved by showing that mixed cumulants vanish.
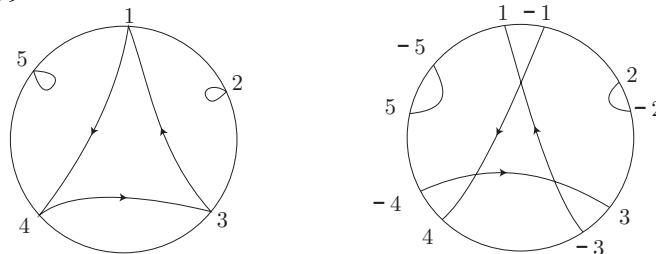
Let us elaborate on how we obtain that U^{T} and U^{Γ} converge to circular operators. Recall that P. Biane, [2], showed that the non-crossing partitions on $[n]$ can be embedded into the permutation group S_n by considering each block of a partition to be a cycle of the corresponding permutation where the elements are written in *increasing* order. The permutations that arise in this way are exactly those permutations σ that satisfy the equality $\#(\sigma) + \#(\sigma^{-1}\gamma) = n + 1$. Recall that $\#(\sigma)$ denotes the number of cycles in the cycle decomposition of σ and that γ is the permutation with the single cycle $(1, 2, 3, \ldots, n)$.

The crucial point for us is that σ is non-crossing then σ^{-1} *cannot* be non-crossing unless σ has only cycles of length 1 or 2. We illustrate this with an example. Suppose $n = 5$ and $\sigma = (1, 3, 4)(2)(5)$. Then $\#(\sigma) + \#(\sigma^{-1}\gamma) = 3 + 3 = n + 1$. On the other hand $\#(\sigma^{-1}) + \#(\sigma\gamma) = 3 + 1 = 4 = n - 1$. If we convert a permutation $\sigma \in S_n$ to a pairing π of $\{1, -1, 2, -3, \ldots, n, -n\}$ by letting

$\pi = \sigma\delta\sigma^{-1}$ then the crossing will become visible. In this case the pairing for σ is $\{(1,-4)(-1,3)(2,-2)(-3,4)(5,-5)\}$. See the figure below.

The fact that U^{T} converges in moment distribution to a circular operator c_1 arises from the fact that when writing (1) as a sum of cumulants the only noncrossing partitions that arise are the ones that are also non-crossing when the order of elements in each block is reversed.

On the other hand for σ^{-1} the corresponding pairing is $\{(1,-3)(-1,4)(2,-2)(3,-4)(5,-5)\}$.

References

[1] G. Aubrun, *Partial transposition of random states and non-centered semicircular distributions* Random Matrices Theory Appl. 1 (2012), no. 2, 1250001, 29 pp.

[2] P. Biane, *Some properties of crossings and partitions*, Disc. Math, **175** (1997), 41–53.

[3] J. A. Mingo and M. Popa, *Freeness and The Transposes of Unitarily Invariant Random Matrices*, arXiv:1411.6173, 30pp.

[4] J. A. Mingo and R. Speicher, *Sharp bounds for sums associated to graphs of matrices*, J. Funct. Anal. **262**, (2012), 2272–2288.

Special Activities

In addition to the regular talks we also scheduled two sessions of 10 minutes announcements of research results. This was mainly, but not exclusively, intended for young researchers, who could so give an idea of their work to the general audience; quite often those announcement resulted in more in depth discussions in small groups afterwards.

List of 10 minutes research announcements.

- Uwe Franz: On unitary dual groups
- Greg Anderson: Semicirculars as building blocks for all allgebraic non-commutative formal power series: a free probabilistic version of the Chomsky-Schützenberger theorem
- Tobias Mai: Regularity of distributions of Wigner integrals
- Octavio Arizemendi: On a logarithmic transform for non-classical convolutions
- Arup Bose: High dimensional time series, random matrices, and free probability
- Ian Charlesworth: Combinatorics of bi-free probability
- Franz Lehner: Spreadability, cumulants and Hausdorff series
- John Williams: B-valued free convolution of unbounded operators
- Steen Thorbjørnsen: Unimodality of the freely selfdecomposable probability laws

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