Mathematisches Forschungsinstitut Oberwolfach

Report No. 35/2015

DOI: 10.4171/OWR/2015/35

Interplay of Analysis and Probability in Applied Mathematics

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26 July – 1 August 2015

Abstract. This workshop brought together analysts and probabilists working on problems at some of the many interfaces of these two fields. Most of the problems discussed during the meeting have their origin in physics or chemistry. The workshop was grouped around the four themes (1) condensation in random structures, (2) disordered systems, (3) discrete-to-continuum transitions, (4) atomistic and molecular systems.

Mathematics Subject Classification (2010): 60xx, 70xx, 74xx, 82xx.

Introduction by the Organisers

Stochastic effects play an increasingly important rôle in the mathematical modeling of many physical systems. On the mathematical side, this creates the need to combine analytic and stochastic techniques. In addition, a number of challenging and important problems only become tractable when methods from analysis and probability are combined.

In recent decades, more and more researchers in both of these areas became aware of the increasing need to interact with and learn from each other, and many of them took up serious attempts to contribute. The organisers feel that this development is very promising, is likely to bear many qualitatively new results and research directions in future, and should therefore be strongly supported. This is why already in 2005, when the need for collaboration between analysis and probability was less noticed than it is now, the organisers (in a different composition) applied for the first Oberwolfach workshop focused on this aspect. This

and forthcoming applications lead to the organisation of two such workshops in 2008 and 2012. We like to think that these workshops helped significantly to promote collaboration between the two fields. Their positive effects became especially clear during the present workshop, the third one on the interplay of analysis and stochastics in Oberwolfach. It had a very lively and inspiring atmosphere and clearly showed that in the meantime the two communities have got accustomed with each other, and that there is nowadays a lot of high-level exchange between them. Very helpful in this respect was the partial substitution of some of the proposers for the present workshop, which brought in new aspects and research directions.

The aim of this workshop was to bring together analysts and probabilists to foster an exchange of expertise between statistical physics, analysis and probability. There were a number of themes where connections between the different disciplines arise naturally. An example is stochastic averaging, where analytic methods (Γ-convergence and Young measures) are used to describe the effective macroscopic behaviour of systems with micro- or mesoscopic stochasticity (Cicalese and Sandier). These approaches were complemented by quantitative estimates for stochastic homogenisation (Neukamm). The treatment of many particle quantum mechanics using statistical mechanics is another example (Sütő, Lukkarinen).

A further natural link between analysis and probability is provided by diffusion processes. On this workshop, various aspects were covered, both on the level of Brownian motions (Berger) and on the level of the Fokker-Planck equations and their numerical treatment (Pavliotis). The treatment of viscid flows via an optimal transport problem involving the Nelson derivative broadened the perspective to fluid mechanics (Léonard). The metric structure given by the Wasserstein distance also proved essential in the convergence result of Kac's model of elastic collisions to the homogeneous Boltzmann equation (Norris).

Large deviations underpinned a number of talks, including compactness questions for the mean-field polaron model (Mukherjee), rate functionals for chemical reactions (Renger), and a new passage from particles to hydrodynamic limits (Fathi). The latter approach also relies on Γ-converging, thus bridging methodologically to some of the homogenisation approaches discussed above. Two recent and technically quite sophisticated Wulff shape results for an important model from statistical mechanics (bond percolation on the square lattice) were discussed (Biskup). Applications of the large deviation theory by Freidlin and Wentzell to rare events for turbulent atmosphere jets were presented (Bouchet). In many, if not all, of these examples, the structure of the variational formula and the characteristic equations for the zeros of the rate function, particularly non-gradient terms, play a central rôle.

A different approach to study metastability and transition probabilities relies on capacity estimates (Schlichting). This is one of several examples where similar problems were studied by different parts of the community, and the intention of the workshop was to foster the exchange. Further work on metastability, in the context

of Kawasaki dynamics in Ising spin systems with Kac potentials, was presented (DE MASI).

A number of talks were devoted to the aim of understanding the emergence of structures in multi-agent systems. Examples are condensation in stochastic networks (DEREICH, MÖRTERS), crystallisation in the low-temperature limit (JANSEN), emergence of phases in complex networks (RADIN), ground state analysis of interaction energies appearing in particle dynamics such as flock formation (CAÑIZO).

Combinatorial problems were behind a number of ingredients of arguments, with entropy being only the most obvious example. These problems also were discussed as challenges in their own right, often with application from physics in mind (AURZADA, GNEDIN, ZEINDLER). The formulation often was in term of fundamental limit laws of probability distributions, like central limit theorems and point process convergence; another example was such a convergence result for the extremes of bivariate point processes, derived by means of Stein's method (CIPRIANI).

There were a number of talks by young participants (DUHART, EINAV, FLEGEL, MÖNCH, STAMATAKIS, TAGGI, WILLIAMS); the quality of their results and talks was impressive. The stimulating environment of Oberwolfach helped foster a number collaborations during the workshop, and new results were obtained during the meeting.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Marek Biskup in the "Simons Visiting Professors" program at the MFO.

Workshop: Interplay of Analysis and Probability in Applied Mathematics

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Abstracts

Emergence of condensation in a self-organised growth model

Peter Mörters
(joint work with Steffen Dereich and Cécile Mailler)

The general purpose of our project is to study a range of population models, in which every individual carries a fitness chosen according to a distribution μ on the interval $(0,1)$. Across a range of model parameters the population grows in such a way that a positive proportion of the appropriately weighted population have fitness converging to one, forming a *condensate*. The remaining positive proportion have an asymptotic fitness distribution absolutely continuous with respect to μ, we call those the *bulk*. The simultaneous occurence of condensate and bulk constitutes a phenomenon of self-organisation. Models fitting our description occur in the context of preferential attachment networks, random permutations, and branching processes. Our interest is focussed on the *emergence of condensation*, in particular the question how the system behaves at large finite times.

My talk focusses on a branching process model with selection and mutation. It has two parameters

- a *mutation probability* $\beta \in [0,1]$,
- a *mutant fitness distribution* μ, that is a probability measure on $(0,1)$ with essential supremum equal to one.

The model is a branching process in continuous time, such that

- the initial particle has a random fitness chosen according to μ;
- particles with fitness f live forever and produce single offspring at rate f;
- every particle born either
  - inherits the fitness of the parent with probability $1-\beta$, or
  - mutates with probability β in which case its fitness is drawn from μ.

This is a stochastic *house-of-cards* model for a population with a balance of genetic *selection* and *mutation*. We let

$$N(t) \;=\; \#\{\text{particles alive at time } t\}$$

and Ξ_t be the *empirical fitness distribution* at time t given by

$$\Xi_t(A) \;=\; \frac{\#\{\text{particles with fitness in } A \text{ at time } t\}}{\#\text{particles alive at time } t}.$$

The long-time behaviour of the process (Ξ_t) depends on the condition

$$\beta \int \frac{1}{1-x}\,\mu(\mathrm{d}x) \;>\; 1.$$

If it holds, then there exists a unique $\lambda^* \in (1-\beta, 1]$ such that

$$\beta \int \frac{x}{\lambda^* - (1-\beta)x}\,\mu(\mathrm{d}x) \;=\; 1,$$

which is a Malthusian parameter for the model. By general branching process arguments one can see that $N(t)$ grows exponentially with rate λ^*, and the empirical fitness distribution Ξ_t converges to a limit measure p, which is absolutely continuous with respect to μ. There is no condensation.

We are interested in the condensation case, which occurs if the mutations have a small ability to produce fit particles, i.e.

$$\beta \int \frac{1}{1-x}\,\mu(\mathrm{d}x) \;<\; 1.$$

We let $\lambda^* := 1-\beta$, which is the exponential rate of growth of the system, but the growth is not strictly exponential and finding the corrections necessary to prove a limit theorem is a rather difficult *open problem*. The empirical fitness distribution now converges to the measure

$$p(\mathrm{d}x) \;=\; \frac{\beta}{1-x}\,\mu(dx) + \gamma(\beta)\delta_1(dx),$$

where $\gamma(\beta) := 1-\beta\int(1-x)^{-1}\mu(\mathrm{d}x) > 0$ is the size of the condensate. This can be proved by approximation using processes without condensation.

From now on we make a regularity assumption on the mutant fitness distribution, i.e. that

$$\mu(1-\epsilon,1) \;=\; \epsilon^{\alpha}\ell(\epsilon),$$

for some parameter $\alpha > 0$ and function ℓ slowly varying at zero. We split the population alive at time t into families, each consisting of a mutant and all its not mutated offspring. This allows us to write the empirical fitness distribution as

$$\Xi_t \;=\; \sum_{n=1}^{M(t)} Z_n(t)\delta_{F_n},$$

where

- $M(t)$ is the number of *families* or mutants,
- $Z_n(t)$ is the size of the nth born familiy at time t,
- F_n is the fitness of all individuals in the nth born family.

Following van den Berg, Lewis and Pulé [2] we say that there is *macroscopic occupancy* by the nth family if

$$\frac{Z_n(t)}{N(t)} \;\longrightarrow\; X \;>\; 0,$$

and if there is condensation but no macroscopic occupancy, this is called *non-extensive*. The problem of macroscopic occupancy versus non-extensive condensation has been discussed in the physics literature on a non-rigorous level using a network model equivalent to our branching process. Bianconi and Barabási [1] claim that there is macrocopic occupany by the largest family, and no other family is macroscopic. Godrèche and Luck [4] argue that there is macroscopic occupancy by an infinite number of families. By contrast to both, we believe that no state is macrosopically occupied and there is extensive condensation. In the talk I sketch

the argument that the size of the largest family is negligible relative to the overall population size, i.e.

$$\lim_{t\to\infty} \frac{\max_{n=1}^{M(t)} Z_n(t)}{N(t)} \;=\; 0, \qquad \text{in probability.}$$

As a byproduct we get asymptotic results for the size, fitness and birth time of the largest family.

An interesting open problem is to identify the shape of the wave becoming the condensate. We conjecture for our model that

$$\lim_{t\uparrow\infty} \Xi_t\Big(1-\frac{x}{t},1\Big) \;=\; \frac{\gamma(\beta)}{\Gamma(\alpha)} \int_0^x \mathrm{e}^{-y} y^{\alpha-1}\,\mathrm{d}y,$$

i.e. the shape of the wave follows a gamma distribution. This conjecture is based on the analysis of a simple mean-field model, which is carried out in [3]. I summarise this analysis and conclude by briefly mentioning two current developments in the context of this project. On the one hand this is work in progress with Betz, Dereich and Ueltschi aiming to analyse the shape of the condensing wave for a range of differential equations exhibiting condensation, and on the other hand this is recent work of Dereich analysing a stochastic growth model where condensation does not self-organise but can be enforced by chosing critical parameters, see his contribution to this volume.

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Large deviation theory and the Eyring–Kramers formula for non gradient dynamics. Applications to abrupt transitions for turbulent atmosphere jets

Freddy Bouchet

(joint work with Julien Reygner, Eric Simonnet and Tomas Tangarife)

Many natural and experimental turbulent flows display a bistable behavior: rare and abrupt dynamical transitions between two very different subregions of the phase space. The most prominent natural examples are probably the Earth magnetic field reversals (over geological timescales), the Kuroshio bistability, or the Dansgaard-Oeschger events that have affected the Earth climate during the last glacial period, and are probably due to several attractors of the turbulent ocean

dynamics. Recent results show that similar bistability occur also in the turbulent dynamics of atmosphere jets. Those abrupt transitions are extremely rare events that change drastically the nature of the flow and are thus of paramount importance.

The first part of the talk will be mainly mathematical. We will review the Freidlin-Wentzell theory that describes large deviations for dynamical systems with weak noise, and the Eyring-Kramers relation for the transition rates between two attractors for gradient dynamics. We will present our recent generalization of the Eyring-Kramers relation for non gradient dynamics. This result will be useful for any non-equilibrium dynamics, including turbulent flows, in the regimes of rare transitions. The second part will focus on turbulent flows, including experimental and numerical studies. Most of this part will focus on theoretical, mathematical, and numerical works in the framework of the quasi-geostrophic barotropic model. This is the simplest turbulence model to set up the theoretical and numerical tools to study these phenomena. From a numerical point of view, those events can not be studied directly because they are too rare. We will first discuss the use of a rare event algorithm, Adaptive Multilevel Splitting, in order to sample from direct numerical simulation such rare transitions.

In quasigeostrophic models, the classical eddy-mean flow interactions are involved to explain the evolution of the large scale flow. The issue is then to understand and predict those eddy-mean flow configurations that lead to the rare fluctuations that trigger rare transitions. We will present the proper mathematical framework to analyze those rare events: large deviation theory for dynamical systems with a separation of time scales. We will discuss the development of this framework and the computation of rare transitions for the barotropic quasi-geostrophic model.

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Low-temperature Lennard-Jones chains

Sabine Jansen
(joint work with Wolfang König, Bernd Schmidt, Florian Theil)

We consider a chain of N atoms on a line, with positions $0 \le x_1 \le \cdots \le x_N \le L$. They interact via a pair potential $v(|x_i - x_j|)$ which can be, for example, the Lennard-Jones potential $v(r) = r^{-12} - r^{-6}$. Let $\beta > 0$, the *inverse temperature*. In the canonical ensemble, the distribution of particles is

$$\frac{1}{Z_N(\beta, L)} \exp\Big(-\beta \sum_{1 \le i < j \le N} v(|x_i - x_j|)\Big) \mathrm{d}x_1 \dots \mathrm{d}x_N \tag{1}$$

where the canonical partition function $Z_N(\beta, L)$ ensures that the measure (1) is a probability measure on $0 \le x_1 \le \cdots \le x_N \le L$. We take the thermodynamic limit $N, L \to \infty$ at fixed β and fixed particle density $\rho = N/L$ and wish to understand the limiting measure at low temperature $\beta \to \infty$. We also analyze the free energy $f(\beta, \rho) = -\lim_{N,L\to\infty} (\beta L)^{-1} \log Z_N(\beta, L)$.

If we took the low-temperature limit before the thermodynamic limit, the measure (1) concentrates on minimizers of the energy. Let

$$E_N = \min_{x_1, \dots, x_N \in \mathbb{R}} \sum_{1 \le i < j \le N} v(|x_i - x_j|), \quad e_0 = \lim_{N\to\infty} \frac{E_N}{N}. \tag{2}$$

For the Lennard-Jones potential it is known that the ground state energy per particle e_0 exists and is given by the unique minimizer a of $r \mapsto \sum_{k=1}^{\infty} v(kr)$, and energy minimizers approach, in the bulk, a regular lattice of spacing a [1].

At small but non-zero temperature, it is common to approximate the energy by a quadratic form around the periodic lattice, i.e., write $x_j = ja + \varphi_j$ with small displacement φ_j and replace the measure (1) by

$$\frac{1}{\tilde{Z}} \exp\Big(-\frac{\beta}{2} \sum_{k=1}^{N} \sum_{j=1}^{N-k} v''(ka)(\varphi_{j+k} - \varphi_j)^2\Big) \, \mathrm{d}\varphi_1 \cdots \mathrm{d}\varphi_N, \tag{3}$$

leading to the topic of Gibbs (gradient) fields [2]. The sum over k is often truncated so that only finitely many neighbors interact, sometimes higher-order (anharmonic) terms are taken into account.

A systematic comparison of the measures (1) and (3) has recently been undertaken by Shapeev and Luskin [4], however their work applies to next nearest neighbor interactions with fast increase at infinity only. The condition of increase is quite common in atomistic models of elasticity and contrasts the decay conditions usually imposed in the statistical mechanics of particles [3]. As a consequence the measures (1) and (3) may exhibit quite different behavior and it becomes interesting to study anharmonic effects.

We focus on systems large enough to accommodate the optimum grid of spacing a, i.e., low densities $\rho < a^{-1}$. At positive temperature entropic effects cause the chain to break up: most bonds $x_{j+1} - x_j$ will be close to a but some of them will be very large. We wish to understand the precise distribution broken bonds.

The results come in three parts and concern the surface energy at zero temperature, the defect-free chain at positive temperature, and the distribution of broken bonds. We change variables and work with the spacings $z_j = x_{j+1} - x_j$, $z_1 = x_1$, $z_{N+1} = x_N - L$ rather than particle positions. Let

$$\mathcal{E}_{\mathrm{surf}}\big((z_j)_{j\in\mathbb{N}}\big) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\big(v(z_j + \cdots + z_{j+k-1}) - v(ka)\big)$$

for square-summable strains $\sum_{j=1}^{\infty}(z_j - a)^2 < \infty$, and $\mathcal{E}_{\mathrm{surf}}((z_j)) = \infty$ otherwise. $\mathcal{E}_{\mathrm{surf}}$ represents the energy of a semi-infinite chain, normalized so that the grid $z_j \equiv a$ has energy 0.

Theorem 1. *The limit $e_{\mathrm{surf}} = \lim_{N\to\infty}(E_N - Ne_0)$ exists and is given by*

$$e_{\mathrm{surf}} = 2\min\,\mathcal{E}_{\mathrm{surf}} - \sum_{k=1}^{\infty} kv(ka).$$

The surface functional $\mathcal{E}_{\mathrm{surf}}$ has a unique minimizer.

Let $R > a$. Consider the restricted Gibbs measure

$$\mathbb{Q}_N^{\beta}(A) = \frac{1}{Q_N(\beta)}\int_{[0,R]^{N-1}} \mathbf{1}_A(\boldsymbol{z})\,\mathrm{e}^{-\beta\sum_{k=1}^{N-1}\sum_{j=1}^{N-k} v(z_j+\cdots+z_{j+k-1})}\mathrm{d}z_1\cdots\mathrm{d}z_{N-1}.$$

Theorem 2. *Under suitable assumptions on the pair potential,*

(1) *The limits*

$$e_0(\beta) = -\lim_{N\to\infty}(\beta N)^{-1}\log Q_N(\beta)$$

and

$$e_{\mathrm{surf}}(\beta) = \lim_{N\to\infty}(-\beta^{-1}\log Q_N(\beta) - Ne_0(\beta))$$

exist.

(2) *The measure $\mathbb{Q}_N^{\beta}$ converges in a suitable sense to a uniquely defined probability measure $\mathbb{Q}_\beta$ on $[0,R]^{\mathbb{N}}$.*

(3) *As $\beta\to\infty$, we have $e_0(\beta)\to e_0$ and $e_{\mathrm{surf}}(\beta)\to e_{\mathrm{surf}}$.*

(4) *As $\beta\to\infty$, the measure $\mathbb{Q}_\beta$ satisfies a large deviations principle with rate β and rate function $\mathcal{E}_{\mathrm{surf}} - \min\mathcal{E}_{\mathrm{surf}}$.*

Items (1) and (2) follow standard techniques of statistical mechanics, items (3) and (4) are new.

Finally we come back to the full Gibbs measure (1), again for the spacings z_j rather than positions x_j, at low density $\rho < a^{-1}$. We call a bond *broken* if $z_j > R$. Let M be the total number of broken bonds, Γ_r the number of broken bonds with length $z_j \geq r$, and

$$q(\beta) = \sqrt{\rho^{-1} - a}\,\exp\Big(-\frac{\beta}{2}e_{\mathrm{surf}}(\beta)\Big).$$

Write $\mathbb{P}_{\beta,L,N}$ for the Gibbs measure.

Theorem 3. *Fix $\rho \in (0, a^{-1})$. Under suitable assumptions on the pair potential $v(r)$ and sufficiently large (but β-independent) truncation parameter R, the following holds:*

(1) *As $\beta \to \infty$, $f(\beta, \rho)/\rho = e_0(\beta) - (1 + o(1))\beta^{-1}q(\beta)$.*
(2) *There is a function $\delta(\beta)$ going to zero as $\beta \to \infty$ such that*

$$\lim_{N,L\to\infty} \mathbb{P}_{\beta,L,N}\Big(\Big|\frac{M/N}{q(\beta)} - 1\Big| \le \delta(\beta)\Big) = 1,$$

$$\lim_{N,L\to\infty} \mathbb{P}_{\beta,L,N}\Big(\Big|\frac{\Gamma_{r/\lambda(\beta)}}{M} - \int_r^\infty \mathrm{e}^{-u}\mathrm{d}u\Big| \le \delta(\beta)\Big) = 1, \quad \lambda(\beta) = \frac{q(\beta)}{\rho^{-1} - a}.$$

So in particular the fraction of broken bonds is exponentially small in β—how small is determined by the surface energy e_{surf}—and the length of broken bonds is approximately exponentially distributed with large expectation $1/\lambda(\beta)$.

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Convergence of gradient flows: probabilistic applications

Max Fathi
(joint work with Marielle Simon)

In this talk, I explained how the Sandier-Serfaty approach to convergence of gradient flows can be combined with the gradient flow structure of reversible Markov chains on finite spaces discovered by Maas and Mielke to prove convergence to the hydrodynamic limit for the symmetric simple exclusion process on the discrete torus.

A gradient flow on $\mathbb{R}^d$ is an ODE of the form

$$\dot{x}_t = -\nabla V(x_t).$$

Solutions to this ODE are the only curves satisfying

$$V(x_T) - V(x_0) + \frac{1}{2}\int_0^T |\dot{x}_t|^2 \,\mathrm{d}t + \frac{1}{2}\int_0^T |\nabla V(x_t)|^2 \,\mathrm{d}t = 0. \tag{1}$$

De Giorgi pointed out in [2] how such functionals could be given a meaning in the setting of metric spaces, and hence one could use this structure to *define* gradient flows in metric spaces.

When using this notion in the space of probability measures on $\mathbb{R}^d$ equipped with the optimal transport (or Wasserstein) distance, this notion can be used to

reformulate Fokker Planck equations, which govern the evolution in time of the law of diffusion processes as the gradient flow of the entropy ([9, 1]). An analogous structure for Markov chains was discovered independently by Maas and Mielke ([6, 8]).

The Sandier-Serfaty approach to proving convergence of a sequence of gradient flows to some limit ([10, 11]) consists in passing to the limit in (1) by proving Γ-convergence of each term, and then identifying the limit as the gradient flow of an asymptotic energy functional with respect to an asymptotic metric structure.

The symmetric simple exclusion process on the discrete torus $\mathbb{Z}/N\mathbb{Z}$ is a system of K particles performing simple random walks independently, but with the added constraint that if a particle attempts to jump on a site that is already occupied, the jump is cancelled. It is a reversible Markov chain with respect to the uniform measure on the space of configurations of K particles. To go to the scaling limit as N goes to infinity and the density K/N converges to some $\alpha \in (0,1)$, we scale the system diffusively, i.e. we speed up the time-scale by a factor N^2.

If we denote by $\eta_i \in \{0,1\}$ the number of particles on site i, we say that a sequence of (random) configurations weakly converges to a deterministic profile $m \in L^2(\mathbb{T})$ if, for any smooth test function $J : \mathbb{T} \longrightarrow \mathbb{R}$ and any $\epsilon > 0$ we have

$$\mathbb{P}\left[\left|\frac{1}{N}\sum_{i=1}^{N} \eta_i\, J(i/N) \;-\; \int_{\mathbb{T}} m(\theta)\, J(\theta)\, \mathrm{d}\theta\right| > \epsilon\right] \;\longrightarrow\; 0.$$

The gradient flow approach allows us to recover a classical result on convergence to the hydrodynamic limit for this model: if the initial data converges to a deterministic profile m_0 and is well-prepared, then at any later time the systems converges to m_t, and the evolution of m_t is given by the heat equation

$$\partial_t m = \Delta m.$$

The gradient flow structure for the heat equation that arises in the limit is given by the functional

$$\int_{\mathbb{T}} h(m_T)\,\mathrm{d}\theta \;-\; \int_{\mathbb{T}} h(m_0)\,\mathrm{d}\theta \\ +\; \frac{1}{2}\int_0^T ||\partial_t m||^2_{-1,m_t}\,\mathrm{d}t \;+\; \frac{1}{2}\int_0^T \int_{\mathbb{T}} m(1-m)\left(\partial_\theta h'(m)\right)^2\,\mathrm{d}\theta\,\mathrm{d}t$$

where $h(m) = m\log m + (1-m)\log(1-m) - \alpha\log\alpha - (1-\alpha)\log(1-\alpha)$ and the metric structure is given by

$$||u||^2_{-1,m} = \sup_J\; 2\int Ju\,\mathrm{d}\theta - \int m(1-m)(\partial_\theta J)^2\,\mathrm{d}\theta.$$

For a general function h, this functional corresponds to the gradient flow structure for the parabolic equation

$$\partial_t m = \partial_\theta(m(1-m)\,\partial_\theta h'(m)).$$

The fact that this structure yields the heat equation follows from the relation $m(1-m)h''(m) = 1$.

The technique is fairly general, and can be adapted to many other interacting particle systems. Recent results obtained using it include the large-volume limit for systems of chemical reaction equations [7], convergence of mean field particle systems on graphs [3] and large deviations for continuous spin systems [5].

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The order of large random permutations with cycle weights

Dirk Zeindler

(joint work with Julia Storm)

This talk considers the order $O_n(\sigma)$ of a random permutation and bases on two joint papers with Julia Storm, see [1, 2]. The order $O_n(\sigma)$ of a permutation σ of n objects is the smallest integer $k \geq 1$ such that the k-th iterate of σ gives the identity. A remarkable result about the order of a uniformly chosen permutation is due to Erdős-Turán who proved in 1965 that $\log O_n$ satisfies a central limit theorem. We show that the Erdős-Turán Law can be extended to random permutations chosen according to the so-called *generalized Ewens measure* and to a *generalized weighted measure* with polynomially growing cycle weights. Furthermore, we establish for the *generalized Ewens measure* a local limit theorem as well as, under some extra moment condition, a precise large deviation estimate and also show that the expectation of the logarithm of the order has a remarkable connection

with the Riemann hypothesis. In addition, we provide a precise large deviation estimate for random permutations with polynomial growing cycle weights.

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An entropic interpolation problem for incompressible viscid fluids

Christian Léonard

(joint work with Marc Arnaudon, Ana Bela Cruzeiro and Jean-Claude Zambrini)

Notation. The sets of all probability measures on a set $\mathcal{Z}$ is denoted by $\mathrm{P}(\mathcal{Z})$. The state space $\mathcal{X}$ of the fluid is the flat torus $\mathcal{X} = \mathbb{T}^n := \mathbb{R}^n/[0,1]^n$, the set of all continuous paths on $\mathcal{X}$ is denoted by $\Omega := C([0,1], \mathcal{X})$ and Ω_a is the subset of all absolutely continuous paths. The canonical process is: $X_t(\omega) = \omega_t \in \mathcal{X}$, $\omega = (\omega_t)_{0\le t\le 1} \in \Omega$. We also denote $P_t := (X_t)_\# P \in \mathrm{P}(\mathcal{X})$ the law of the position at time t and $P_{01} = (X_0, X_1)_\# P$ the joint law of the initial and final positions.

Incompressibility, inviscidity. The incompressibility property equally translates in terms of

- the vanishing of the divergence of the velocity field v: $\nabla \cdot v = 0$, or
- the evolution of a flow $(g_t)_{0\le t\le 1}$ of diffeomorphisms which preserve the volume, i.e. $(g_t)_\# \mathrm{vol} = \mathrm{vol}$, for all t, or
- the evolution of a stochastic process whose path measure $P \in \mathrm{P}(\Omega)$ is such that its time marginals satisfy: $P_t = \mathrm{vol}$, for all t.

The inviscidity property is reflected in the Euler equation (Eu)-(i) below by the absence of viscosity force. The viscosity term $\Delta v/2$ will be introduced later in the Navier-Stokes equation (NS)-(i).

Euler equations. The system of Euler equations is

$$\text{(Eu)} \qquad \begin{cases} \text{(i)} & (\partial_t + v\cdot\nabla)v(t,z) = -\nabla p_t(z), \quad 0\le t\le 1,\ z\in\mathcal{X} \\ \text{(ii)} & \nabla\cdot v = 0, \qquad\qquad\qquad\qquad\quad 0\le t\le 1,\ z\in\mathcal{X} \end{cases}$$

where the unknown is the couple (v, p). The first identity (i) is Newton's equation which identifies the acceleration of any element of fluid with the gradient of the pressure p and (ii) is the incompressibility condition.

Brenier's problem. As the Cauchy problem associated with (Eu) is very difficult, one can think of some easier to investigate geodesic variant. After Arnold's paper [1] in 1966 who proposed to represent the evolution of an incompressible fluid as the geodesic flow in the manifold of volume preserving diffeomorphisms, in 1989 Brenier addressed in [2] the following relaxation of Arnold's problem.

Let us fix some endpoint configuration $\pi \in \mathrm{P}(\mathcal{X})X$ with

$$\pi_0 = \pi_1 = \mathrm{vol}. \tag{1}$$

Such a π is a coupling of the volume measure with itself. Brenier's problem is

$$\text{(Br)} \qquad E_P \int_0^1 |\dot{X}_t|^2/2 \,\mathrm{d}t \;\to\; \min; \quad P \in \mathrm{P}(\Omega) : [P_t = \mathrm{vol}, \forall\, t],\ P_{01} = \pi.$$

It is proved in [2] that any $P \in \mathcal{P}(\Omega)$ satisfying the marginal constraints $P_t = \mathrm{vol}$ for all t, $P_{01} = \pi$, and such that the Newton equation

$$\ddot{X}_t + \nabla p_t(X_t) = 0, \forall\, t \ P\text{-}a.e. \tag{2}$$

is satisfied for some pressure field p, solves (Br). Since (2) is the analogue of (Eu)-(i) and the constraint $P_t = \mathrm{vol}, \forall t$ is the analogue of (Eu)-(ii), Brenier's problem is very close to the Euler equations (Eu).

Navier-Stokes equations. Adding the viscosity force $\Delta v/2$ on the right hand side of Euler equation (Eu)-(i) gives the system of Navier-Stokes equations

$$\text{(NS)} \quad \begin{cases} \text{(i)} & (\partial_t + v \cdot \nabla)v(t,z) = \frac{1}{2}\,\Delta v_t(z) - \nabla p_t(z), \quad 0 \le t \le 1,\ z \in \mathcal{X} \\ \text{(ii)} & \nabla \cdot v = 0, \qquad\qquad\qquad\qquad\qquad\quad 0 \le t \le 1,\ z \in \mathcal{X} \end{cases}$$

where the unknown is the couple (v, p). The viscosity term $\Delta v/2$ suggests that replacing the absolutely continuous paths of previous section by Brownian sample paths could be of some help for deriving a variational problem related to (NS) instead of (Eu).

Relative entropy, backward velocity. We need a stochastic analogue of the kinetic action $E_P \int_{[0,1]} |\dot{X}_t|^2/2 \,\mathrm{d}t$ which appears in (Br). Let $R \in \mathrm{P}(\Omega)$ be the Markov path measure with initial marginal $R_0 = \mathrm{vol}$ and generator $\Delta/2$. It is the law of the reversible Brownian motion on $\mathcal{X} = \mathbb{T}^n$. Girsanov's theory ensures that any probability measure $P \in \mathrm{P}(\Omega)$ which is absolutely continuous with respect to R is the solution of the backward martingale problem with final measure $P_1 \ll \mathrm{vol}$ and

$$\mathrm{d}^* X_t = \overleftarrow{v}_t^P(X_{[0,t]})\,\mathrm{d}t + \mathrm{d}^*\overleftarrow{M}_t^P,\ 0 \le t \le 1, \quad P\text{-a.s.}$$

where $\mathrm{d}^*\overleftarrow{M}^P$ is the increment of a local backward P-martingale and

$$\overleftarrow{v}_t^P(X_{[t,1]}) = \lim_{h \to 0^+} \frac{1}{h} E_P(X_t - X_{t-h} \mid X_{[t,1]}) \tag{3}$$

is the backward velocity of P. For any $P \in \mathrm{P}(\Omega)$ with $P_1 = \mathrm{vol}$, the relative entropy $H(P|R)$ of P with respect to the reference path measure R is

$$H(P|R) := E_P \log\left(\frac{dP}{dR}\right) = E_P \int_{[0,1]} |\overleftarrow{v}_t^P|^2/2 \, \mathrm{d}t.$$

An entropy minimization problem. This suggests to introduce the following analogue of Brenier's problem:

$$H(P|R) \ \to \ \min; \quad P \in \mathrm{P}(\Omega) : [P_t = \mathrm{vol}, \forall\, t], \ P_{01} = \pi. \tag{4}$$

Note that as in (Br), π must verify (1) for the second constraint to be consistent with the first incompressibility constraint. Remark that both (Br) and (4) are convex minimization problems. But unlike (Br), the entropy minimization problem (4) is *strictly* convex. Therefore, if it admits some solution, it is unique.

Applying formally the method of Lagrange multipliers to (4), one expects that its solution is

$$P = \exp\left(\eta(X_0, X_1) \ + \int_{[0,1]} p_t(X_t) \, \mathrm{d}t\right) R \tag{5}$$

for some measurable functions $\eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ and $p : [0,1] \times \mathcal{X} \to \mathbb{R}$. It is not Markov in general, but it is reciprocal. In particular, for any $y \in \mathcal{X}$, the conditioned path measure $P^y := P(\cdot \mid X_1 = y)$ is Markov. With (3) and the reciprocal property we have

$$\begin{aligned} \overleftarrow{v}_t^P(X_{[t,1]}) &= \lim_{h \to 0^+} \frac{1}{h} E_P\big[X_t - X_{t-h} \ \mid X_{[t,1]}\big] \\ &= \lim_{h \to 0^+} \frac{1}{h} E_{P^{X_1}}\big[X_t - X_{t-h} \mid X_t\big] = \overleftarrow{v}_t^{P^{X_1}}(X_t). \end{aligned}$$

The main formal statements are the following.

- For P_1-almost all $y \in \mathcal{X}$ and for all $0 \le t < 1$,

$$\overleftarrow{v}_t^{P^y}(z) = \nabla \theta_t^y(z), \quad \text{for } P_t^y\text{-almost all } z \in \mathcal{X},$$

  where $\theta_t^y(z) := -\log E_{R^y}\Big[\exp\big([\eta(X_0, y) + \int_0^t p_s(X_s)\,\mathrm{d}s\big) \mid X_t = z\Big]$ and

$$\partial_t \overleftarrow{v}^{P^y} \ + \ \overleftarrow{v}^{P^y} \cdot \nabla \overleftarrow{v}^{P^y} = \Delta \overleftarrow{v}^{P^y}/2 - \nabla p, \tag{6}$$

  where the pressure p is the same for all y.
- The average vector field $\widetilde{v}_t^P(z) := E_P[\overleftarrow{v}_t^P(X_{[t,1]}) \mid X_t = z]$ satisfies

$$\widetilde{v}_t^P(z) = \int_{\mathcal{X}} \overleftarrow{v}_t^{P^y}(z) \, P(X_1 \in \mathrm{d}y \mid X_t = z) = 0, \quad \forall t, z.$$

Let us conclude with some comments.

- The identity (6) is the Newton equation (NS)-(i) of the system of Navier-Stokes equations, but of course $\overleftarrow{v}^{P^y}$ does not satisfy the incompressibility

condition (NS)-(ii): $\nabla \cdot \overleftarrow{v}^{P^y} \neq 0$. Nevertheless, the incompressibility is recovered by integrating the conditioned path measures P^y since

$$P(\cdot) = \int P^y(\cdot)\,\mathrm{vol}(\mathrm{d}y) \quad \text{and} \quad P_t = \mathrm{vol},\ \forall\, t.$$

- The average backward velocity field $\widetilde{v}^P$ coincides with the backward velocity field $\overleftarrow{v}^R = 0$ of the reference measure R. Of course it is a trivial solution of the Navier-Stokes system of equations (NS).
- This is a work in progress. Everything is granted except the existence of a *function* p in the representation (5).

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Wick polynomials and time-evolution of cumulants

Jani Lukkarinen

(joint work with Matteo Marcozzi and Alessia Nota)

The question of time evolution of a random field is central to applications of statistical and stochastic methods, for instance, in many-particle systems in Physics. If the initial state of the system is chaotic, in the precise sense that field values in far-apart spatial regions are essentially independent, it is natural to use the cumulants of the fields, the *correlation functions*, to characterize the time-evolution. Namely, if $\psi(x)$ and $\psi(x')$ are independently distributed, their joint cumulants will vanish but their joint moments are typically not zero.

The above observation allows to quantify the "chaoticity" of a random field on an infinite lattice in terms of the speed of decay of the corresponding correlation functions. If the decay leads to uniform ℓ_p-summability, we call the state ℓ_p-*clustering*: the precise condition is

$$\sup_{x\in\mathbb{Z}^d} \sum_{y\in(\mathbb{Z}^d)^{n-1}} |\kappa[\psi(x), \psi(x+y_1), \ldots, \psi(x+y_{n-1})]|^p < \infty, \quad \text{for all } n \in \mathbb{N}.$$

For instance, high-temperature thermal equilibrium Gibbs states often are ℓ_1-clustering; see [1], and the references therein, for explicit examples.

Consider now an ℓ_p-clustering random initial data for $\psi_0(x)$ and for any time $t > 0$ define a random field $\psi_t(x)$ as the unique solution to a deterministic evolution equation. (We will use the discrete nonlinear Schrödinger equation later as an explicit example of such a system.) How will the cumulants evolve in this case?

This question is addressed in [3], a joint work with M. Marcozzi. It turns out that polynomials of the field values are naturally replaced by so called Wick polynomials when studying the time-evolution of the correlation functions. The

following argument highlights the relation in the simplest situation where generating functions can be used. Consider some finite sequence J of points x_j, $j = 1, \dots, n$. For notational simplicity, let $\psi(x)_J := (\psi(x_1), \dots, \psi(x_n))$ and $\psi(x)^J := \prod_{j=1}^n \psi(x_j) = \prod_{x\in J} \psi(x)$. If the random variables in $\psi(x)_J$ have joint exponential moments, then moments, cumulants, and Wick polynomials of subsequences of $\psi(x)_J$ can be generated by using the generating functions

$$G_{\mathrm{m}}(\lambda) := \mathbb{E}[\mathrm{e}^{\lambda\cdot\psi}], \quad g_{\mathrm{c}}(\lambda) := \ln G_{\mathrm{m}}(\lambda) \quad \text{and} \quad G_{\mathrm{w}}(\lambda;\psi) := \frac{\mathrm{e}^{\lambda\cdot\psi}}{\mathbb{E}[\mathrm{e}^{\lambda\cdot\psi}]} = \mathrm{e}^{\lambda\cdot\psi - g_{\mathrm{c}}(\lambda)},$$

where $\lambda\cdot\psi := \sum_{x\in J} \lambda_x \psi(x)$. Namely, then for any subsequence I of J we have

$$\mathbb{E}[\psi(x)^I] = \partial_\lambda^I G_{\mathrm{m}}(0), \quad \kappa[\psi(x)_I] = \partial_\lambda^I g_{\mathrm{c}}(0) \quad \text{and} \quad :\psi(x)^I: = \partial_\lambda^I G_{\mathrm{w}}(0;\psi),$$

where "∂_λ^I" is a shorthand notation for $\prod_{x\in I} \partial_{\lambda_x}$. Here $:\psi^I:$ denotes the Wick polynomial associated to the random variable sequence $\psi(x)_I$. (Its highest order term is always given by $\psi(x)^I$, and the lower order terms have coefficients which depend on the distribution of $\psi(x)^I$. Hence the "powerlike" notation $:\psi(x)^I:$ which was introduced in quantum field theory. For further details about the standard definition and basic properties of Wick polynomials, see [2]. In [3], we show how Wick polynomials can also be defined combinatorially, as the unique choice which removes all "internal contractions" from the related cumulant expansions.)

These relations immediately imply that, for instance, whenever $t \mapsto \psi_t(x)$ is a continuously differentiable function at every x, we have

$$\begin{aligned}\partial_t \kappa[\psi_t(x)_J] &= \partial_\lambda^J \partial_t g_{\mathrm{c}}(\lambda)\big|_{\lambda=0} = \partial_\lambda^J \mathbb{E}[\lambda\cdot\partial_t\psi_t\, G_{\mathrm{w}}(\lambda;\psi_t)]\big|_{\lambda=0} \\ &= \sum_{y\in J} \mathbb{E}[\partial_t\psi_t(y)\, \partial_\lambda^{J\setminus y} G_{\mathrm{w}}(\psi_t,0)] = \sum_{y\in J} \mathbb{E}[\partial_t\psi_t(y)\, :\psi_t(x)^{J\setminus y}:].\end{aligned}$$

Therefore, Wick polynomials naturally appear in the study of time evolution of cumulants. In [3], we apply them to derive an evolution hierarchy for cumulants, and present three alternative forms for an evolution hierarchy for the Wick polynomials.

The rest of the talk concerns more speculative applications of these methods to evolution by the discrete nonlinear Schödinger equation (DNLS) with ℓ_1-clustering and gauge invariant initial data. We first recast the evolution equation in a regularized form, called "pairing truncated" form in [5]. In terms of Wick polynomials, the evolution equations for $\psi_t(x,1) = \psi_t(x)$ and $\psi_t(x,-1) = \psi_t(x)^*$ are

$$\begin{aligned}&\mathrm{i}\sigma\partial_t\psi_t(x,\sigma) \\ &\quad = \sum_y \alpha(x-y) :\psi_t(y,\sigma): + 2\lambda\rho_t(x) :\psi_t(x,\sigma): + \lambda :\psi_t(x,-1)\psi_t(x,\sigma)\psi_t(x,1):.\end{aligned}$$

Here $\lambda > 0$ denotes the nonlinear coupling, and the function $\alpha : \mathbb{Z}^d \to \mathbb{R}$ is called the *hopping amplitude* which we assume to be symmetric and exponentially decreasing. The function $\rho_t(x)$ is equal to the ℓ_2-density of the field, $\rho_t := \mathbb{E}[\,|\psi_t(x)|^2]$.

In [5], the simplest field-field time-correlator was considered assuming that the initial data is given by a canonical Gibbs measure. This case leads to important

simplifications since the initial data is then both invariant under spatial translations and stationary in time. In this case, a suitable "renormalization" of the field ψ allows taking a kinetic scaling limit of the time-correlator yielding an explicit evolution equation for time-scales $O(\lambda^{-2})$.

As discussed in [3], the time-evolution of spatially *inhomogeneous* initial data requires careful consideration of timescales $O(\lambda^{-1})$ which cannot be handled by a renormalization with simple compensating factors, unlike in the spatially homogeneous case. In a forthcoming work [4], we consider the time-evolution of the Wigner function of the field assuming that the state has only weak spatial variation. Explicitly, if we define

$$W_t(x,k) = \sum_{y\in\mathbb{Z}^d} \mathrm{e}^{-\mathrm{i}2\pi k\cdot y}\kappa[\psi_t(x)^*, \psi_t(x+y)],$$
$$W_t^{(4)}(x,k) = \sum_{y\in(\mathbb{Z}^d)^3} \mathrm{e}^{-\mathrm{i}2\pi k\cdot y}\kappa[\psi_t(x)^*, \psi_t(x+y_1), \psi_t(x+y_2)^*, \psi_t(x+y_3)]\,,$$

then

$$\begin{aligned}\partial_t W_t(x,k) &= -\mathrm{i}\sum_z \alpha(z)\mathrm{e}^{\mathrm{i}2\pi k\cdot z}\left(W_t(x,k) - W_t(x-z,k)\right)\\ &-\mathrm{i}2\lambda\sum_z \left(\rho_t(x+z)-\rho_t(x)\right)\int \mathrm{d}k'\,\mathrm{e}^{\mathrm{i}2\pi z\cdot(k'-k)}W_t(x,k')\\ &-\mathrm{i}\lambda\int \mathrm{d}\,k_1'\mathrm{d}k_2'\left(W_t^{(4)}(x,k_1',k_2',k-k_1'-k_2') - W_t^{(4)}(x,k_1',k_2',k)\right).\end{aligned}$$

Here $\omega(k)$ denotes the Fourier transform of α which yields also the *dispersion relation* of the unperturbed discrete wave equation.

The equation can be integrated and, if one keeps only the lowest order terms in coupling and in spatial derivatives also in the corresponding equation for $W_t^{(4)}$, it yields an approximate evolution equation

$$\begin{aligned}&\partial_t W_t(x,k) + \frac{1}{2\pi}\nabla_k\omega(k)\cdot\nabla_x W_t(x,k) - 2\lambda\nabla_x\rho_t(x)\cdot\frac{1}{2\pi}\nabla_k W_t(x,k)\\ &\quad = \lambda^2\mathcal{C}[W_t(x,\cdot)](k)\,,\end{aligned}$$

where now $x\in\mathbb{R}^d$. Here $\mathcal{C}[W]$ is the phonon Boltzmann collision operator associated with DNLS evolution equation (an explicit form for the collision operator is given for instance in [3, Equation (5.14)]). The main difference compared to the standard kinetic scaling limits is the appearance of a Vlasov-type term on the left hand side. This term is lower order than the collision operator if the spatial variation occurs only at kinetic length scales $O(\lambda^{-2})$, but it becomes stronger than the collision term if the variation is faster than $O(\lambda)$.

Finally, we also briefly mentioned a possibility of estimating the accuracy of scaling limit models, such as Boltzmann equations, by establishing a stability of the limit equation under time-dependent perturbations of the source terms, tailored to match with known *a priori* bounds for the correction terms. More details and comments about the procedure can be found in [3, Section 6.2]. Its

main benefit would be to circumvent the taking of a scaling limit of the solution, and hence it could be used in situations for which the limit either does not exist or is difficult to control. These cases include also the above mentioned inhomogeneous Boltzmann-Vlasov equation.

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Can metastable states be stable?

Anna De Masi

(joint work with Errico Presutti and Dimitrios Tsgkarogiannis)

This talk is about the stationary Dirichlet problem for a gradient conservative dynamics derived by Giacomin and Lebowitz, [4], from Kawasaki dynamics in Ising spin systems with Kac potentials. I present the results in [1] and work in progress. We study the solution $m(r) \in [-1,1]$, $r \in [-\epsilon^{-1}\ell, \epsilon^{-1}\ell]$, $\ell \geq 1$, $\epsilon > 0$ of

$$\frac{\mathrm{d}}{\mathrm{d}x} I = 0, \qquad I = -\kappa(m)\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\delta F^{\mathrm{LP}}}{\delta m}\right), \qquad m(\pm\epsilon^{-1}\ell) = u_{\pm} \tag{1}$$

the coefficient $\kappa(m)$ is the mobility that in our model is given by $\kappa(m) = \beta(1-m^2)$. The functional F^{LP}, known as the Lebowitz-Penrose functional, is a non local version of the scalar Ginzburg-Landau functional.

$$F^{\mathrm{LP}}(m) = \int \phi_\beta(m) + \frac{1}{4}\int\int J(|x-x'|)\,[m(x)-m(x')]^2$$

where $J(|r|) \geq 0$ is smooth and normalized $\int J(|r|) = 1$,

$$\phi_\beta(m) = -\frac{1}{2}\,m^2 - \frac{1}{\beta}\,S(m), \qquad \beta > 0,$$

$$-S(m) = \frac{1+m}{2}\,\log\left(\frac{1+m}{2}\right) + \frac{1-m}{2}\,\log\left(\frac{1-m}{2}\right).$$

For $\beta < 1$, ϕ_β is a single well with minimum at 0. For $\beta > 1$, ϕ_β is a double well with minima at the points $\pm m_\beta$ where $m_\beta > 0$ solves the "mean field equation" $m_\beta = \tanh\{\beta m_\beta\}$. For $\beta > 1$, $\phi''_\beta < 0$ in the interval $(-m^*, m^*)$ with $m^* = \sqrt{1-1/\beta}$. The interval $(-m^*, m^*)$ is called unstable region and the set

$(-m_\beta, -m^*) \cup (m^*, m_\beta)$ is the metastable region. In the thermodynamic free energy all the values inside the interval $(-m_\beta, m_\beta)$ are "forbidden".

By choosing $u_- < -m_\beta$ and $u_+ > m_\beta$ we force the solution m of (1) to have values also in the metastable and unstable phases. However we prove in [1] that as $\epsilon \to 0$ m converges to the stationary solution u of the heat equation in $[-\ell, \ell]$ with boundary conditions $u(\pm\ell) = u_\pm$ and diffusion coefficient

$$\begin{aligned} &D^0_\beta(u) = 1 - \beta(1-u^2) = \kappa(u)\,\phi''_\beta(u), && \text{for } u^2 \geq m^2_\beta, \\ &D^0_\beta(u) = 0, && \text{for } u^2 < m^2_\beta. \end{aligned}$$

This solution increases from u_- to $-m_\beta$ then it jumps to m_β at a point r_0 and then increases again from m_β to u_+. The current is constant for all $r \neq r_0$:

$$j = -D^0_\beta(u)\,\frac{\mathrm{d}u}{\mathrm{d}r}, \qquad \forall r \neq r_0 \tag{2}$$

The physics described by the model is about an open system in contact with two magnetization reservoirs which fix the values at the boundaries equal to $u_\pm$. Since $u_- < u_+$ there is a negative current j given in (2) constant in the whole interval, in agreement with the Fourier law. The metastable and unstable values do not appear in the macroscopic profile since u has a discontinuity from $-m_\beta$ to m_β at a point r_0. u is the stationary solution of the Stefan problem, the free boundary r_0 does not move because the current, being constant, does not jump at r_0. At the mesoscopic level the profile m is smooth since at the transition it is the instanton. Indeed we prove that the solution m of (1) with $u_- < -m_\beta$ and $u_+ > m_\beta$ is close to a profile obtained by expressing the macroscopic profile u in mesoscopic units and putting the instanton at the discontinuity.

We come now to the problem mentioned in the title namely what happens if we take $u_\pm$ in the metastable region. For instance $u_- \in (-m_\beta, -m^*)$ and $u_+ = -u_-$. Our first result proved in [1] is: there exists a stationary solution m of (1) which converges as $\epsilon \to 0$ to a stationary solution u of the heat equation with boundary conditions $u(\pm\ell) = u_\pm$ and with diffusion

$$\begin{aligned} &D^*_\beta(u) = 1 - \beta\,(1-u^2) = \kappa(u)\,\phi''_\beta(u), && \text{for } u^2 \geq (m^*)^2, \\ &D^*_\beta(u) = 0, && \text{for } u^2 < (m^*)^2. \end{aligned}$$

u decreases from u_- to $-m_\beta$ then it jumps to m_β at 0 and then decreases again from m_β to u_+. The current j is a positive constant equal to

$$j = -D^*_\beta(u)\,\frac{\mathrm{d}u}{\mathrm{d}r}, \qquad \forall r \neq 0$$

As opposite to the previous case the whole region is metastable except the point 0. There is a problem with the Fourier law because the current is positive even though $u_- < u_+$. This is not the only paradox because we have also other "strange" solutions. Take $u_- \in (-m_\beta, -m^*)$ and $u_+ \in (m^*, m_\beta)$, then there exists a solution m of (1) which converges as $\epsilon \to 0$ to a stationary solution u of the heat equation

with diffusion $D^*_\beta(u)$ as above and boundary conditions $u(-\ell) = u_-$ but with

$$\lim_{r\to\ell^-} u(r) = m_- \in (-m_\beta, -m^*)$$

u is monotone in $(-\ell, \ell)$, the value of m_- is determined by the bump q solution of the equation in $(-\infty, 0)$

$$q = \tanh\{\beta J \star q + \beta h^*\}, \qquad \lim_{x\to-\infty} q(x) = m_-$$

$q(0) = u_+$ for some positive $h^\star$. Using the results in [2] and [3] we prove (work in progress) the existence for some values of u_+ and we conjecture that for all $u_+ \in (m^*, m_\beta)$, h^* and m_- are uniquely determined.

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On minimisers of the interaction energy

José A. Cañizo

(joint work with José A. Carrillo and Francesco Patacchini)

We study minimisers of the *energy functional* $E\colon \mathcal{P}(\mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$, $d \geq 1$, defined on the set $\mathcal{P}(\mathbb{R}^d)$ of probability measures on $\mathbb{R}^d$ by

$$E(\rho) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} W(x-y)\, \mathrm{d}\rho(x)\, \mathrm{d}\rho(y), \qquad \rho \in \mathcal{P}(\mathbb{R}^d), \tag{1}$$

where $W\colon \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a pointwise defined and measurable function called the *interaction potential*, which is allowed to take the value $+\infty$. Our motivation for studying these minimisers has mainly come from the recent interest in the field of collective behaviour regarding the steady states of the aggregation equation

$$\partial_t \rho = \nabla \cdot (\rho(\nabla W * \rho)), \tag{2}$$

where $\rho = \rho(t, x)$ is a function (possibly a measure) defined for $t \geq 0$ and $x \in \mathbb{R}^d$. Since E is a Lyapunov functional for Equation (2) (in fact, (2) is the gradient flow of E with respect to the Wasserstein transport distance) its minimisers (if they exist) are natural candidates for a steady state of (2), and they are also natural candidates to represent the typical asymptotic behaviour of (2) as $t \to +\infty$. Equations of the form (2) appear in granular flow and as mentioned before in swarming models (see [3] and the references therein).

It is easy to see that if a minimiser ρ happens to be regular enough then it must satisfy the corresponding *Euler-Lagrange equation*, namely

$$W * \rho = C \quad \text{on } \operatorname{supp} \rho,$$

for some $C \in \mathbb{R}$. Consequently, it must be a stationary state of (2), again assuming that ρ is regular enough for the right-hand side of (2) to be meaningful.

Under the following hypotheses we are able to show existence of minimisers of the energy (1):

(1) W is bounded from below by a finite constant $W_{\min} < 0$.
(2) W is locally integrable (that is, $\int_B |W| < +\infty$ for any open ball $B \subset \mathbb{R}^d$).
(3) W is symmetric (that is, $W(x) = W(-x)$ for all $x \in \mathbb{R}^d$).
(4) W is lower semi-continuous.
(5) There is $R' \geq 0$ such that W is strictly increasing on $\mathbb{R}^{k-1} \times [R', \infty) \times \mathbb{R}^{d-k}$ as a function of its k-th variable, for all $k \in \{1, \dots, d\}$.
(6) The following limit exists:

$$E_\infty := \frac{1}{2} \lim_{|x| \to \infty} W(x)$$

and there exists $\rho \in \mathcal{P}(\mathbb{R}^d)$ such that $E(\rho) < E_\infty$.

We can the latter condition *instability*, in agreement with naming in statistical mechanics literature. In [6] a discrete collective behaviour model was studied, and it was noticed in numerical simulations that its large-time asymptotics seemed to depend on whether the potential satisfies a classical condition known as *stability* or *H-stability* in classical statistical mechanics, which is the complementary of the last condition above. This was one of our original motivations for studying this problem.

Our main result is the following:

Theorem 1. *Assume that $W : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ satisfies Hypothese (1)–(5) above. Then there exists a global minimiser $\rho \in \mathcal{P}(\mathbb{R}^d)$ of the energy E. In addition, there exists $K > 0$ (depending only on W and the dimension d) such that every minimiser of E has compact support with diameter at most K.*

We point out that an explicit estimate on the size of the support can be recovered from the proof of the above theorem, though we do not expect it to be sharp.

Simione, Slepčev & Topaloglu [11] have independently proved a similar result by a different method based on Lions' concentration compactness principle while this paper was being prepared. Their method does not give any estimate on the support or properties of the minimisers; on the other hand, their conditions for existence of minimisers are slightly sharper than the ones in Theorem 1.

Local minimisers of the energy (1) in several transport distances (i.e., minimisers in a small ball around them) were studied in [1], where bounds on the dimension of their support were given under some assumptions controlling the strength of the repulsion at the origin.

In order to place the problem in context, let us briefly mention some of the known rigorous results on the existence, uniqueness, and other properties of these minimisers in some remarkable cases. An often studied case is that in which the potential W is a sum of powers:

$$W(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b}, \qquad x \in \mathbb{R}^d,$$

for some $a, b \in \mathbb{R}$ with $-d < b < a$, and the understanding that $|x|^0/0 \equiv \log|x|$. Here the term $|x|^a/a$ is the attractive one (being an increasing function of $|x|$, regardless of the sign of a), and $|x|^b/b$ is the repulsive one (since it is a decreasing function of $|x|$). For $b = 2-d$ the repulsive term is called the *Newtonian potential*. It is not difficult to check that this class of potentials satisfies our hypotheses.

The case $a = 2$ simplifies the problem a lot since the attractive part of the interaction can be reduced to an external quadratic confinement by expanding the square. The case $b = 2 - d$, $a = 2$ is actually relatively well-known among probabilists: up to translations, the unique global minimiser is the characteristic of a ball with an appropriate radius. A closely related result with a compact confinement is proved in [9], and the 2-dimensional case can be found for example in [10, Theorem 6.1, p. 245]. The interest in this problem on these references comes from its links to the capacity of sets and applications to random matrix theory.

The case $a = 2$, $b = 2s - d$ for $0 < s < 1$ was studied in [4] in relation to the asymptotic behaviour of eq. (2), referred to as the *fractional porous medium equation*. The authors there showed that there is a unique steady state (up to translations) to (2), which they called a *modified Barenblatt profile*.

On the other hand, there have been many works devoted to the study of the steady states and long-time behaviour of eq. (2) (see [7, 8, 1] and the references therein). Steady states for the case $b = 2 - d$, $a > 0$ were studied in [8], where it was proved that there exists a unique radial compactly supported steady state (up to translations). The asymptotic behaviour of (2) in the case $b = 2 - d$, $a = 2$ was studied in [8, 2], and the case $-d < b < 2 - d$, $a = 2$, as already remarked, was considered in [4].

Finally, other particular interesting potentials are Morse-like potentials [6, 3], for which there is a huge numerical evidence of the existence of compactly supported global minimisers. To our knowledge, a proof of this existence was not previously available.

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Atomistic-to-continuum variational analysis of random lattice systems

Marco Cicalese
(joint work with Roberto Alicandro and Matthias Ruf)

In this talk we focus on the coarse-graining of a model for magnetic polymer composite materials commonly used for bio-magnetic separations processes. At a microscopic level these materials can be described as magnetic particles embedded into a polymer matrix. At a macroscopic level they are characterized by the formation of magnetic Weiss domains, whose structural properties are affected by the intrinsic randomness of the geometry of the polymeric matrix. The aim of the talk, which describes the results contained in [3], is to rigorously relate the micro and macro scales energetic models by a variational analysis.

We model a magnetic polymer composite materials by defining first a polymer matrix and then an interaction energy between the magnetic particles. We consider the polymer matrix to be a random network having the cross-linked molecules as nodes and we suppose that the set of the nodes of the network is an *admissible stochastic lattice* according to the following definition. A countable set of points $\Sigma = \{x_i\}_{i\in\mathbb{N}}$ in $\mathbb{R}^n$ is said to be admissible if

(i) there exists $R > 0$ such that $\inf_{z\in\mathbb{R}^n} \#(\Sigma \cap B(z, R)) \geq 1$ (i.e., arbitrarily big empty regions are forbidden),
(ii) there exists $r > 0$ such that $\inf\{|x-y|,\ x, y \in \Sigma,\ \ x \neq y\} \geq r$ (i.e., clusters are forbidden).

Then, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a random variable $\mathcal{L} : \Omega \to (\mathbb{R}^n)^{\mathbb{N}}$ is called an *admissible stochastic lattice* if, uniformly with respect to $\omega \in \Omega$, $\mathcal{L}(\omega)$ is an admissible set of points.

To every stochastic lattice $\mathcal{L}(\omega)$ we associate a Voronoi tesselation $\mathcal{V}(\mathcal{L}(\omega))$ and define the set of nearest neighboring points, namely $\mathcal{NN}(\omega)$, as the set of those pairs of points of the stochastic lattice $\mathcal{L}(\omega)$ which share a $(n-1)$-dimensional edge of the associated Voronoi tessellation. Let now $D \subset \mathbb{R}^n$ be a bounded open set, and $\varepsilon > 0$ be a small parameter (the limit $\varepsilon \to 0$ will be referred to as the continuum limit). We assume that the magnetic state of the particles in D is described by a classical spin variable $u : \varepsilon\mathcal{L}(\omega) \cap D \to \{\pm 1\}$ and we model the interactions between the spins via an Ising type energy. Our energy model takes into account the interactions among all the particles and may distinguish between short-range (between the nearest-neighbors particles) and long-range interactions. The total energy of the system for a given configuration u has the form

$$F_\varepsilon(\omega)(u) := F_{nn,\varepsilon}(\omega)(u, D) + F_{lr,\varepsilon}(\omega)(u, D),$$

where

$$F_{nn,\varepsilon}(\omega)(u, D) = \sum_{\substack{(x,y)\in\mathcal{NN}(\omega)\\ \varepsilon x,\varepsilon y\in D}} \varepsilon^{n-1} c^\varepsilon_{nn}(x, y)|u(\varepsilon x) - u(\varepsilon y)|,$$

$$F_{lr,\varepsilon}(\omega)(u, D) = \sum_{\substack{(x,y)\notin\mathcal{NN}(\omega)\\ \varepsilon x,\varepsilon y\in D}} \varepsilon^{n-1} c^\varepsilon_{lr}(x, y)|u(\varepsilon x) - u(\varepsilon y)|.$$

For $c^\varepsilon_{nn}, c^\varepsilon_{lr} : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ we assume that there exist $C > 0$ and a decreasing function $J_{lr} : [0, +\infty) \to [0, +\infty)$ with

$$\int_{\mathbb{R}^n} J_{lr}(|x|)|x| \, \mathrm{d}x = J < +\infty$$

such that, for all $\varepsilon > 0$ and all $x, y \in \mathbb{R}^n$,

$$\frac{1}{C} \le c^\varepsilon_{nn}(x, y) \le C \quad \text{and} \quad c^\varepsilon_{lr}(x, y) \le J_{lr}(|x - y|).$$

As the average distance between the nodes of the network $\varepsilon\mathcal{L}(\omega)$ is of order ε, the prefactor ε^{n-1} in the energy means that $F_\varepsilon(\omega)(u)$ is the magnetic energy per unit surface area of the network $\varepsilon\mathcal{L}(\omega) \cap D$ for a given magnetization u.

We identify the field u with its piecewise-constant interpolation taking the value $u(x)$ on the Voronoi cell centered at x, we regard the energies as defined on $L^1(D, \{\pm 1\})$ and we perform the Γ-limit as $\varepsilon \to 0$ in this space. We first prove a deterministic compactness result, asserting that, for fixed $\omega \in \Omega$, up to subsequences, the family $F_\varepsilon(\omega)$ Γ-converges with respect to the $L^1(D)$- topology to a continuum energy $F : L^1(D) \to [0, +\infty]$ which is finite only on $BV(D, \{\pm 1\})$ where it takes the form

$$F(\omega)(u) = \int_{S(u)\cap D} \phi_{\mathcal{L}(\omega)}(x, \nu_u) \, \mathrm{d}\mathcal{H}^{n-1},$$

where $S(u)$ denotes the jump set of u, $\nu_u \in S^{n-1}$ its measure theoretic inner normal and $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure. The result is proved by the abstract methods of Gamma-convergence and makes use of the integral representation theorem in [6]. We explore the dependence of the continuum energy

on the randomness induced by the stochastic lattice assuming further that the stochastic lattice is stationary, that is, for all $z \in \mathbb{Z}^n$, $\mathcal{L}(\omega)$ and $\mathcal{L}(\omega) + z$ have the same statistics and that there exist two functions $c_{nn}, c_{lr} : \mathbb{R}^n \to [0, +\infty)$ such that

$$c^{\varepsilon}_{nn}(x, y) = c_{nn}(y - x), \quad c^{\varepsilon}_{lr}(x, y) = c_{lr}(y - x). \tag{1}$$

These assumptions, which play the same role as periodicity in the case of the deterministic periodic lattice considered in [4], turn the problem of the characterization of the continuum limit energy into a stochastic homogenization problem.

We prove that the functionals $F_\varepsilon(\omega)$ Γ-converge with respect to the $L^1(D)$-topology to the functional $F_{\rm hom}(\omega) : L^1(D) \to [0, +\infty]$ which is finite on $BV(D, \{\pm 1\})$ and it takes the form

$$F_{\rm hom}(\omega)(u) = \int_{S(u)\cap D} \phi_{\rm hom}(\omega; \nu_u)\, \mathrm{d}\mathcal{H}^{n-1}$$

where, for $\mathbb{P}$-almost every ω and for all $\nu \in S^{n-1}$, $\phi_{\rm hom}(\omega; \nu_u)$ is given by an asymptotic homogenization formula. In the case $\mathcal{L}$ is ergodic the limit energy is proven to be deterministic and its energy density $\phi_{\rm hom}(\nu)$ is obtained by averaging over the probability space:

$$\phi_{\rm hom}(\nu) = \int_\Omega \phi_{\rm hom}(\omega; \nu_u)\, \mathrm{d}\mathbb{P}(\omega).$$

The proof of this last result uses two main ingredients: the abstract methods of Γ-convergence and the subadditive ergodic theorem by Ackoglu and Krengel in [1]. The combination of these two results in the framework of discrete-to-continuum limits is already contained in [2] and draws ideas from the pioneering paper [7]. It consists in proving that the sequence of minimum problems characterizing the energy density of the Γ-limit at a certain point in the space and in a given direction agrees (up to lower order terms) with a sequence of subadditive stochastic processes for which the main result in [1] applies. It is at this point that one strongly uses the assumptions on the stationarity of the lattice together with (1). This step of the proof is the most delicate one and requires new arguments with respect to the Sobolev case in [2] and in particular the generalization to higher dimensions of the translation invariance of the first passage percolation formula in [5].

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Persistence of activity in critical scale free Boolean networks

Christian Mönch

Introduction. *Scale free Boolean networks (SFBNs)* generalise Kauffmann's model [7] of gene regulation to networks with power law degree distribution. SFBNs can be viewed as deterministic dynamical systems with random initial configuration: The dynamics run on a random directed graph $G_N = ([N], E_N)$ with vertex set $[N] = \{1, \dots, N\}$. Every vertex $x \in [N]$ is in a certain state $\eta_t(x) \in \{0, 1\} = \mathbb{F}_2$ for times $t = 0, 1, 2, \dots$. At $t = 0$ we assign to each vertex x an intial state $\eta_0(x)$ and a randomly chosen *update rule* $\sigma(x)$ which computes the state $\eta_t(x)$ at times $t \geq 1$ from the states $\{\eta_{t-1}(y) : (y, x) \in E_N\}$. Updating occurs simultaneously for all $x \in [N]$. After the intial setup the graph and the rules are fixed and only the configurations $\eta_t = \{\eta_t(x), x \in [N]\}$ change over time. One basic question in the study of SFBNs is to determine the asymptotic size, as $N \to \infty$, of $\mathcal{N}_{\mathrm{nf}} = \{x \in [N]; \forall\, t \in \mathbb{N}\ \exists\, s > t : \Delta\eta_s(x) = 1\}$, where $\Delta\eta_t = \eta_{t+1} - \eta_t \in \mathbb{F}_2^N$. The set $\mathcal{N}_{\mathrm{nf}}$ is usually referred to as the *non-frozen core* of the network.

In the literature (see e.g. [4] for a survey) the model is usually studied under the following assumptions:

- the underlying network has a power law in- and outdegree distribution with respective exponents $\tau^{\mathrm{in}}, \tau^{\mathrm{out}} > 2$,
- in- and outdegree of a vertex are (asymptotically) independent.

If the dynamics additionally satisfy a certain criticality assumption, simulations suggest that for the canonical distribution on p-biased update rules and independently and uniformly chosen $\eta_0 \in \mathbb{F}_2^N$, with high probability as $N \to \infty$,

$$|\mathcal{N}_{\mathrm{nf}}| = N^{\rho + o(1)}, \text{ where } \rho = \begin{cases} 1 + \frac{1}{\tau^{\mathrm{out}} \wedge 3} - \frac{2}{\tau^{\mathrm{in}} \wedge 3}, & \text{if } \tau^{\mathrm{in}} \leq \tau^{\mathrm{out}}; \\ 1 - \frac{1 - \frac{3 - (\tau^{\mathrm{out}} \wedge 3)}{\tau^{\mathrm{out}}}}{2(\tau^{\mathrm{in}} \wedge 3) - 3}, & \text{if } \tau^{\mathrm{in}} > \tau^{\mathrm{out}}. \end{cases} \tag{1}$$

In [5] these exponents have also been derived by a non-rigorous analysis of a "container algorithm" which simultaneously constructs the frozen part of the network and the network itself.

Due to their quenched nature, the dynamics of SFBN are hardly tractable by rigorous means. It is thus common practice to work with the "annealed approximation" to $(\eta_t)_{t=0}^{\infty}$, introduced in [3]. The fixed rule ensemble $\sigma = \{\sigma(x), x \in [N]\}$ is here replaced by an i.i.d. sequence $(\sigma_t)_{t=0}^{\infty}$. In this approximation the process $(\Delta\eta_t)_{t=0}^{\infty}$ becomes a (discrete time) *threshold contact process (TCP)* $(\xi_t)_{t=0}^{\infty}$ and $\mathcal{N}_{\mathrm{nf}}$ is well approximated by the *range of infection* $\mathcal{R} = \{x \in [N]; \exists\, t > 0 : \xi_t(x) = 1\}$, given that ξ_0 contains enough well-connected infected sites. Persistence of the supercritical TCP on a uniform random graph with fixed indegree $r \geq 3$ has been

studied in [2, 9] and recently the analysis has been extended to random graphs with very general joint in- and outdegree distributions [1].

Our motive is twofold: firstly, to rigorously verify relationship (1) with $\mathcal{N}_{\mathrm{nf}}$ replaced by $\mathcal{R}$. Secondly, to obtain new critical scaling exponents for SFBNs with dependent in- and outdegrees from the TCP via the annealed approximation. Our approach is similar in spirit to the one adopted in [1, 2, 9]. We aim to couple the growth of $\mathcal{R}$ to a branching process. However, a different choice of the underlying graph model simplifies the analysis considerably and enables us to use results about the component sizes in critical random graphs, see [6], via explicit couplings.

Model and results. The actual random graph model we use is a *Poissonian inhomogeneous random graph* with i.i.d. weights. The graph is parameterised by weights $(\Lambda_x)_{x\in\mathbb{N}}$, which are i.i.d. copies of some generic bivariate r.v. $\Lambda = (\Lambda^{\mathrm{in}}, \Lambda^{\mathrm{out}}) \in \mathbb{N}^2$ with $1 < \lambda = \mathbb{E}\Lambda^{\mathrm{in}} = \mathbb{E}\Lambda^{\mathrm{out}} < \infty$. Conditionally on the weights, we obtain $G_N = ([N], E_N)$ by introducing each off-diagonal edge $(x, y) \in [N]^2$ into E_N independently of all other edges with probability

$$1 - \exp\Big(-\frac{\Lambda_x^{\mathrm{out}}\Lambda_y^{\mathrm{in}}}{L_N}\Big), \text{ where } L_N = \sum_{x=1}^{N} \frac{\Lambda_x^{\mathrm{in}} + \Lambda_x^{\mathrm{out}}}{2}.$$

Thus the limiting in-/outdegree distribution of G_N is mixed Poisson($\Lambda^{\mathrm{in/out}}$) and follows a power law with exponent $\tau^{\mathrm{in/out}}$ if and only if $\Lambda^{\mathrm{in/out}}$ does. We further denote, for $x \in [N]$, by $C^{\rightarrow}(x)$ the set of all vertices which can be reached from x following edges in E_N.

The TCP on G_N is defined as follows:

$$\xi_0 = \delta_{\arg\max_{x\in[N]} |C^{\rightarrow}(x)|},$$

i.e. we infect the "most connected" vertex and, for $t \geq 1$,

$$\xi_t(x) = \begin{cases} 1 \text{ with probability } q \in (0,1), & \text{if } \exists\, (y,x) \in E_N : \xi_{t-1}(y) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

independently for all $x \in [N]$. The special choice of ξ_0 is for the purpose of exposition. As previously stated, ξ approximates the Boolean activity $\Delta\eta$ for the special choice of p-biased update functions, where p, q satisfy $q = 2p(1-p)$. To obtain a *critical* TCP/SFBN q has to be chosen as

$$q = \frac{\sum_{k,l} k\mathbb{P}(\Lambda^{\mathrm{in}} = k, \Lambda^{\mathrm{out}} = l)}{\sum_{k,l} lk\mathbb{P}(\Lambda^{\mathrm{in}} = k, \Lambda^{\mathrm{out}} = l)}.$$

In this setup, a perfect coupling between ξ and a local exploration process in a suitably chosen random graph allows us to infer variants of (1) for two special cases:

Theorem 1 (Range of critcal TCP [8])**.** *Let ξ be the critical TCP on G_N and let $\mathcal{R} = \{x \in [N]; \exists\, t > 0 : \xi_t(x) = 1\}$. Then the following holds, with high probability as $N \to \infty$:*

- *if Λ^{in} and Λ^{out} are independent and follow the same power law with exponent $\tau > 2$, then*

$$|\mathcal{R}| = N^{\frac{(\tau\wedge 3)-1}{\tau\wedge 3}+o(1)},$$

- *if $\Lambda^{in} = \Lambda^{out}$ follows a power law with exponent $\tau > 3$, then*

$$|\mathcal{R}| = N^{\frac{(\tau\wedge 4)-2}{(\tau\wedge 4)-1}+o(1)}.$$

Observe that the first statement of the theorem coincides with (1) and that the second statement applies to a situation which is not covered by (1), since in- and outdegree are not independent. It is most likely that the techniques used to establish the theorem can also be applied to the case in which $\Lambda^{\text{in}}, \Lambda^{\text{out}}$ have different distributions and are indepedent. This would completely recover (1) in the annealed approximation.

The second statement together with the results of [1] make it very plausible that the analysis may be extendend to obtain scaling exponents even in the situation where the in- and outdegree of the underlying network are arbitrarily correlated.

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Capacitary inequalities in discrete setting and application to metastable Markov chains

André Schlichting
(joint work with Martin Slowik)

Capacitary inequalities were introduced by V. Ma'zya [4] to prove many results and fine properties about Sobolev functions in $\mathbb{R}^n$. We carry over one of these inequalities to discrete state spaces and use them to explicitly characterize Poincaré and logarithmic Sobolev constants of Markov chains in terms of capacities.

We consider a Markov process on a countable state space $\mathcal{S}$ with transition probabilities $p : \mathcal{S} \times \mathcal{S} \to [0,1]$. Its generator is given for any bounded function $f : \mathcal{S} \to \mathbb{R}$ by

$$(Lf)(x) \;:=\; \sum_{y \in \mathcal{S}} p(x,y)\,(f(y) - f(x))\,.$$

We assume this process to be positive recurrent and reversible with respect to the probability measure μ. Further, the equilibrium potential $h_{A,B}$ is defined for any two disjoint subsets $A, B \subset \mathcal{S}$ as the solution of

$$\begin{cases} Lh_{A,B} \;=\; 0 \quad , & \text{on } (A \cup B)^c \\ h_{A,B} \;=\; \mathbf{1}_A, & \text{on } A \cup B, \end{cases}$$

where $\mathbf{1}_A$ is the indicator function on A. Therewith, the capacity between two disjoint sets $A, B \subset \mathcal{S}$ is defined by

$$\operatorname{cap}(A,B) \;:=\; \langle -Lh_{A,B}, h_{A,B} \rangle = \mathcal{E}(h_{A,B}),$$

where $\mathcal{E}$ is the Dirichlet form associated to the generator L. The capacitary inequality can now be stated in the form

Theorem 1 (Capacitary inequality [8]). *For $f : \mathbb{R} \to \mathcal{S}$ define by $A_t \subset \mathcal{S}$ its super-level sets $A_t := \{|f| > t\}$. Moreover, assume, that $f|_B \equiv 0$, then it holds*

$$\int_0^\infty 2t \, \operatorname{cap}(A_t, B) \, \mathrm{d}t \;\le\; 4\mathcal{E}(f).$$

From this result, we can deduce Poincaré inequalities for certain Orlicz norms. These norms can be compared to the variance and relative entropy and the following statements can be deduced.

Proposition 1 (Poincaré inequality [8]). *Let $\nu \in \mathcal{S}$ and $b \in \mathcal{S}$. Then, there exist $C_{\mathrm{var}}, C_{\mathrm{PI}} > 0$ satisfying*

$$\nu[b]\, C_{\mathrm{var}} \;\le\; C_{\mathrm{PI}} \;\le\; 4\, C_{\mathrm{var}}$$

such that the following statements are equivalent:

a) *For all $A \subset \mathcal{S} \setminus \{b\}$ it holds the inequality*

$$\nu[A] \;\le\; C_{\mathrm{var}} \operatorname{cap}(A,b).$$

b) *It holds the mixed Poincaré inequality*

$$\mathrm{var}_\nu[f] \;\le\; C_{\mathrm{PI}}\,\mathcal{E}(f).$$

Proposition 2 (Logarithmic Sobolev inequality [8])**.** *Let $\nu \in \mathcal{P}(\mathcal{S})$ and $b \in \mathcal{S}$. Then, there exist $C_{\mathrm{Ent}}, C_{\mathrm{LSI}} > 0$ satisfying*

$$\frac{\nu[b]}{\ln(1+e^2)}\,C_{\mathrm{Ent}} \;\le\; C_{\mathrm{LSI}} \;\le\; 4\,C_{\mathrm{Ent}}$$

such that the following statements are equivalent:

a) *For all $A \subset \mathcal{S} \setminus \{b\}$ it holds the inequality*

$$\nu[A]\,\ln\left(1+\frac{e^2}{\nu[A]}\right) \;\le\; C_{\mathrm{Ent}}\,\mathrm{cap}(A,b).$$

b) *It holds the logarithmic Sobolev inequality*

$$\mathrm{Ent}_\nu[f^2] \;\le\; C_{\mathrm{LSI}}\,\mathcal{E}(f).$$

The above results can be seen as a the generalization of the Muckenhoupt criterion which is used to characterize the Poincaré and logarithmic Sobolev constant in the one dimensional case [7, 2, 6, 1].

Moreover, the results can be especially applied in the setting of metastable dynamics. In the potential theoretic approach, see [3, 9], metastability can be characterized in terms of hitting probabilities.

Definition 1 (ϱ-metastable Markov chain)**.** *The Markov process $\{X(t) : t \ge 0\}$ is called ϱ-metastable with respect to a* set of metastable points *$\mathcal{M}$, if*

$$\frac{\max_{m\in\mathcal{M}} \mathbb{P}_m\big[\tau_{\mathcal{M}\setminus m} < \tau_m\big]}{\min_{A\subset\mathcal{S}\setminus\mathcal{M}} \mathbb{P}_{\mu_A}\big[\tau_{\mathcal{M}} < \tau_A\big]} \;\le\; \varrho \;\ll\; 1,$$

where $\mu_A[x] := \mu[x|A]$ denotes the conditional probability.

Hitting probabilities are connected to capacities by

$$\mathrm{cap}(A,B) = \mu[A]\mathbb{P}_{\mu_A}[\tau_B < \tau_A].$$

A set of metastable points induces a partition into metastable valleys. A point $x \in \mathcal{S}$ belongs to the metastable valley $\mathcal{V}_m$ of the metastable point m, if its hitting probability is the largest among all other metastable points. Therewith, we can decompose the variance into local variances inside the valleys and global ones between the valleys and analogously for the entropy (cp. [5]).

The Propositions above allow us to show that the contribution of the local variances and relative entropies to the total Poincaré and logarithmic Sobolev constant is negligible. The global one can be reduced to capacities between single points. Therewith, the final result stated for simplicity in the case $|\mathcal{M}| = 2$ only reads

Theorem 2 (Poincaré and logarithmic Sobolev constants for metastable Markov chains [8])**.** *The Poincaré constant* C_{PI} *for a metastable Markov chain satisfies*

$$C_{\mathrm{PI}} \;=\; \frac{\mu[\mathcal{V}_m]\mu[\mathcal{V}_{\tilde{m}}]}{\mathrm{cap}(m,\tilde{m})}\;\Big(1+O\big(\sqrt{\varrho|\mathcal{M}|}\big)\Big)\,.$$

Moreover, the logarithmic Sobolev constant C_{LSI} *satisfies*

$$C_{\mathrm{LSI}} \;\le\; \frac{\mu[\mathcal{V}_m]\mu[\mathcal{V}_{\tilde{m}}]}{\Lambda\left(\mu[\mathcal{V}_m],\mu[\mathcal{V}_{\tilde{m}}]\right)\;\mathrm{cap}(m,\tilde{m})}\;\Big(1+O\big(\sqrt{\varrho|\mathcal{M}|}\big)\Big)\,,$$

where $\Lambda(a,b)=\int_0^1 a^s b^{1-s}\,\mathrm{d}s$ *is the logarithmic mean for* $a,b\ge 0$.

A matching lower bound for the logarithmic Sobolev constant can be obtained under further assumptions on the mass distribution among the metastable valleys. A test function leading to a universal matching lower bound has still to be obtained.

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Applied Probability, Functional Inequalities and Kacs Model

Amit Einav

(joint work with Kleber Carrapatoso)

One of the most influential equations in the kinetic theory of gases is the so-called Boltzmann equation, describing the time evolution of the probability density of a particle in dilute gas. While widely used, a rigorous validation to the equation, one that is true for mesoscopic time scales, is still unavailable and is one of the hardest open problems in kinetic theory. In his 1956 work, [8], Marc Kac presented an attempt to do the above for the spatially homogeneous equation by considering a linear stochastic model of N indistinguishable particles, with one-dimensional velocities, that undergo a random binary collision process. Under the property

of 'chaoticity', which means that the particles become 'more and more independent' as the number of particles goes to infinity, Kac managed to show that the equation of the first marginal of the $N-$particle density function converges to a one-dimensional variant of the Boltzmann equation, the so-called Boltzmann-Kac equation, as N goes to infinity.

Kac was very hopeful that this mean field approach will lead to new understanding in the study of the Boltzmann equation. He suggested that one can investigate the simple, many particle, *linear* equation and attempt to take the limit as N goes to infinity. One of the biggest problem he hoped to resolve in this manner was finding the rate of convergence to equilibrium.

In our short talk we will introduce Kac's model and the question of trends to equilibrium, and will see how functional inequalities and local central limit theorems are connected to them. To begin with we will investigate the question of *existence* of so-called chaotic states, and will show how that problem is intimately connected to the study of Gaussian and Lévy type local central limit theorems that measure possible concentration of the sum of i.i.d variables on an appropriate sphere. This part of the talk is based on the work done in [3, 4].

We will then continue with the convergence to equilibrium problem and introduce the associated spectral gap problem, its inadequacy to deal with the problem and the associated entropy method solution. We will discuss the invalidity of the desired Cercignani's conjecture by mentioning the works of Villani and the Speaker (see [5, 6, 9]). We will show that the aforementioned conjecture is strongly related to a subadditivity type inequality for the entropy on the sphere (see [1]) and will discuss known results by Carlen, Lieb and Loss ([2]) on which the above relies. We will finish by presenting recent new improvements to the 'subadditivity' inequality ([7]) that may hold the key to finding the right subspace under which one can gain a uniform in N exponential rate of convergence for the entropy. The new 'almost' subadditivity of the entropy on the sphere draws much of its intuition from kinetic related quantities, such as moments, as well as the right choice of distances - involving optimal transportation quantities in the mix.

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Two recent Wulff-shape theorems

Marek Biskup
(joint work with Oren Louidor, Eviatar Procaccia, Ron Rosenthal)

Limit shapes are a topic of recurring interest in both probability and statistical mechanics. One successful approach to these came up in the 1990s under the banner of the *Wulff construction*. There one was interested in the asymptotic shape of a "droplet" of one equilibrium phase (e.g., the minus phase of the Ising model at inverse temperature β) immersed in another (the plus Ising phase) subject to a restriction on the overall order parameter (the magnetization, or the total number of plus spins, in the Ising case). In this case, the asymptotic shape is determined by a solution of an *isoperimetric problem*; namely,

$$\inf\bigl\{\mathcal{P}(U)\colon U\subset\mathbb{R}^d \text{ open, } \mathrm{Leb}(U)=1\bigr\} \tag{1}$$

where $\mathrm{Leb}(U)$ is the Lebesgue measure of U while $\mathcal{P}(U)$ is a suitable "perimeter functional" given as an integral with respect to the $(d-1)$-dimensional Hausdorff measure on the boundary ∂U of U of a normal-vector dependent quantity called the surface tension.

The work to be reported here deals with other instances where a closely related limit theorem appeared in recent years. There are actually two separate projects to be discussed; one dealing with isoperimetric shapes in random environment, the other with random walks interacting through their boundary.

Part I: Isoperimetry on supercritical percolation cluster

Consider the bond percolation on $\mathbb{Z}^2$ with parameter $p>p_{\mathrm{c}}$. It is well known that almost every realization of the percolation process will contain a unique infinite open cluster $\mathcal{C}^\infty$ with density $\theta_p>0$. Applications to mixing of the random walk on this component, and other geometrical questions in general, naturally lead to the consideration of an *anchored isoperimetric profile*,

$$\Phi(r):=\min\biggl\{\frac{|\partial^\omega U|}{|U|}\colon 0\in U\subset\mathcal{C}^\infty,\ U \text{ connected},\ |U|<r\biggr\}, \tag{2}$$

where $|\partial^\omega U|$ stands for the size of the open-edge external boundary of U and $|U|$ is the cardinality of U. (The definition assumes containment in the event $\{0\in\mathcal{C}^\infty\}$.)

Let $\mathcal{M}(r)$ denote the set of minimizers in (2). The questions we address are: (1) the asymptotic behavior of $\Phi(r)$ and (2) the asymptotic shape of the sets $U\in\mathcal{M}(r)$, both in the limit as $r\to\infty$. The salient portion of our results is summarized in:

Theorem 1. *For each $p > p_c$ there exists a norm ρ_p on $\mathbb{R}^2$ such that the following holds almost surely on the event $\{0 \in \mathcal{C}^\infty\}$: First,*

$$\lim_{r\to\infty} r^{1/2}\Phi(r) = \theta_p^{-1/2}\varphi_p, \tag{3}$$

where θ_p is as above and φ_p is the value of the infimum in (1) for $\mathcal{P}(U)$ being the arclength of ∂U with respect to the norm ρ_p.

Second, if W_p denotes a minimizer of the perimeter functional $\mathcal{P}(U)$ — the one defined using ρ_p — over open sets $U \subset \mathbb{R}^2$ with $\mathrm{Leb}(U) = 1$, then, using $\mathrm{dist}_{\mathrm{H}}$ to denote the Hausdorff distance on subsets of $\mathbb{R}^2$,

$$\max_{U\in\mathcal{M}(r)} \inf_{x\in\mathbb{R}^2} \mathrm{dist}_{\mathrm{H}}\bigl(r^{-1/2}U,\, x + \theta_p^{-1/2}W_p\bigr) \underset{r\to\infty}{\longrightarrow} 0$$

and

$$\max_{U\in\mathcal{M}(r)} \left| \frac{|U|}{r} - 1 \right| \underset{r\to\infty}{\longrightarrow} 0.$$

We remark that, by classical results in geometric measure theory, the minimizer W_p is determined uniquely up to spatial shifts. The conclusion (3) solves a version of a conjecture due to Itai Benjamini. A key point of the proof is the construction, and control, of the norm ρ_p which requires techniques from combinatorics of planar graphs, ergodic theory and concentration of measure. See the joint paper with O. Louidor, E.B. Procaccia and R. Rosenthal [1] for further details.

Part II: Limit shapes for interacting random walks

The second problem to be discussed concerns interacting random walks. Consider the continuous-time simple random walk $\{X_t\colon t \geq 0\}$ on $\mathbb{Z}^d$ with jump rate $2d$ and let P^x be its law for $P^x(X_0 = x) = 1$. Denote by $R(t)$ to be the hull — namely, the complement of the unique infinite connected component — of the set of vertices visited by the random walk by time t. For $\beta > 0$ consider the Gibbs measure $Q^x_{\beta,t}$ on the path space defined by

$$Q^x_{\beta,t}(A) := \frac{1}{Z(\beta,t)}\, E^x\bigl(1_A\, \mathrm{e}^{-\beta|\partial R(t)|}\bigr),$$

where $\partial R(t)$ is the external vertex boundary of $R(t)$ and $Z(\beta,t) := E^x\bigl(\mathrm{e}^{-\beta|\partial R(t)|}\bigr)$, with E^x denoting expectation with respect to P^x, is the normalizing constant. The problem we wish to address is the asymptotic shape of the set $R(t)$ for t large.

Random walks interacting through the shape of their support have been studied since the mid 1990s. For the case of $|\partial R(t)|$ replaced by the cardinality of the support, Bolthausen [3] proved that the asymptotic shape of the support is an Euclidean ball of radius of order $t^{\frac{1}{d+2}}$. The current setting has been addressed by Berestycki and Yadin [4] who identified the typical linear scale of the support as

$$r(t,\beta) := \Bigl(\frac{t}{\beta}\Bigr)^{\frac{1}{d+1}}.$$

Our contribution is a proof of a shape theorem. Unfortunately, as it turns out, we are only able to do this for $d = 2$ and in the limit $\beta \to \infty$. The precise statement is as follows:

Theorem 2. *Let $d = 2$. There exists a bounded open set $U_0 \subset \mathbb{R}^d$ such that the following holds: For each $\epsilon > 0$ there is $\beta_0(\epsilon) < \infty$ such that for all $\beta > \beta_0(\epsilon)$,*

$$\lim_{t\to\infty} Q^0_{\beta,t}\biggl(\inf_{x\in\mathbb{R}^d} \operatorname{dist}_{\mathrm{H}}\Bigl(r(t,\beta)^{-1}R(t),\, x+U_0\Bigr) > \epsilon\biggr) = 0.$$

Moreover, U_0 is the unique minimizer up to shifts of the map

$$U \mapsto \lambda(U) + \mathcal{P}(U) \tag{4}$$

where $\lambda(U)$ is the principal Dirichlet eigenvalue of the negative Laplacian in U and $\mathcal{P}(U)$ is the perimeter of U measured using the ℓ^∞-distance.

As it turns out, the second part of the theorem is actually hiding a far simpler conclusion: The limit shape U_0 is an ℓ^1-ball. However, this reflects rather strongly the choice of the interaction: If $\partial R(t)$ is taken to mean the external edge boundary of $R(t)$, then $\mathcal{P}(U)$ is the perimeter measured using the ℓ^1-distance and U_0 is an ℓ^∞-ball. This still sounds as if the minimizer of (4) coincided with that of (1); however, we know this to be false in general. We refer to a joint paper with E.B. Procaccia [2] for a shape theorem for rather general versions of the interaction as well as full analysis of the variational problem for the functional (4) under a general choice of the underlying norm. This is still restricted to $d = 2$ and $\beta \to \infty$; the case of finite β and/or $d \geq 3$ is at present open.

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Total momentum of quantum systems: three applications of Galilean invariance

András Sütő

Galilean invariance for N quantum particles of unit mass on a d-torus means that both the spectrum and the set of eigenvectors of the Hamiltonian are preserved, when the system is set into motion of a constant velocity $\mathbf{v}$, chosen from a discrete group. The only effect of the transformation is that it permutes the eigenvalues among the eigenvectors. This happens because the eigenvectors of the Hamiltonian are also eigenvectors of the total momentum operator, and the velocity boost only adds $N\mathbf{v}$ to the momentum of each eigenvector, mapping the group onto

itself. Galilean invariance has some interesting consequences, three of which are discussed in my talk. First, it is possible to derive some results about the probability distribution of the eigenvalues of the total momentum operator in thermal equilibrium. In one dimension, the total momentum has normal fluctuations of variance $\sim N$ about the zero mean, in two dimensions one can prove only that the tail distribution is normal of variance $\sim N$, and for $d \geq 3$ one can only derive an even weaker result. Second, one can show that Landau's criterion for excitations in moving superfluids is an in some cases correct result of an erroneous derivation. Third, the ground state of one-dimensional systems becomes highly degenerate in the limit of infinite length and, under some conditions on the interaction, these states are periodically ordered. I also speak about a possible relation between the probability distribution of the total momentum in the thermodynamic limit and the type of the thermodynamic phases. For superfluids I conjecture that the total momentum is zero with a positive asymptotic probability. Based on this hypothesis, I discuss the dissipative flow of superfluids, and derive a temperature-dependent critical velocity. Details can be found in Ref. [1].

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Quantitative stochastic homogenization of elliptic equations

Stefan Neukamm

(joint work with Antoine Gloria and Felix Otto)

We are interested in quantitative results for stochastic homogenization of

$$-\nabla \cdot (a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon) = f \qquad \text{in } H^1_0(\Omega), \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain and $a(x) \in \mathbb{R}^{d\times d}$ denotes a coefficient matrix that is uniformly elliptic (in the sense that there exists $\lambda > 0$ s.t. for all $x \in \mathbb{R}^d$ and all $\xi \in \mathbb{R}^d$ we have $|a(x)\xi| \leq \xi$ and $a(x)\xi \cdot \xi \geq \lambda|\xi|^2$). The qualitative theory of stochastic homogenization of (1), first considered 1979 by Papanicolaou & Varadhan and by Kozlov, is well understood and states: If the coefficients a in (1) are random with a stationary and ergodic law (with respect to shifts from $\mathbb{R}^d$), then there exists a deterministic, uniformly elliptic matrix a_0 such that in the limit $\varepsilon \downarrow 0$, u_ε weakly converges to $u_0 \in H^1_0(\Omega)$, the solution to

$$-\nabla \cdot (a_0 \nabla u_0) = f \qquad \text{in } H^1_0(\Omega).$$

The homogenized coefficients a_0 are defined by the average

$$\forall \xi \in \mathbb{R}^d : \qquad a_0\xi = \lim_{L\uparrow\infty} \frac{1}{|B_L|} \int_{B_L} a(\nabla\phi_\xi + \xi)\,\mathrm{d}x,$$

where the so called corrector $\phi_\xi = \phi_\xi(a,\cdot) : \mathbb{R}^d \to \mathbb{R}$ is defined as a solution to

$$-\nabla \cdot (a(\nabla\phi_\xi(a,\cdot) + \xi)) = 0 \qquad \text{in } \mathbb{R}^d \text{ and for a.e. } a, \tag{2}$$

such that $\nabla\phi_\xi$ is stationary, in the sense of $\nabla\phi_\xi(a,\cdot+z)=\nabla\phi_\xi(a(\cdot+z),\cdot)$, has finite second moment and vanishing expectation, and $\int_B \phi\,\mathrm{d}x=0$ for a.e. a.

We are interested in the following *quantitative* questions:

- *Estimates on the corrector.* By uniform ellipticity, standard energy estimates show that $\langle|\nabla\phi_\xi|^2\rangle\le C(d,\lambda)|\xi|^2$. Moreover, thanks to $\langle\nabla\phi_\xi\rangle=0$, the corrector ϕ_ξ grows sublinearly in the sense of
$$\lim_{L\to\infty} L^{-2}\frac{1}{|B_L|}\int_{B_L}\left|\phi_\xi-\frac{1}{|B_L|}\int_{B_L}\phi_\xi\right|^2=0.$$
Can we prove boundedness of higher moment bounds for $\nabla\phi_\xi$? Can we prove bounds on moments of $\phi_\xi(x)$ that are either independent of x (which would yield existence of a stationary solution to (2)) or that diverge as $|x|\to\infty$ with a precise rate (quantifying the sublinear growth of the corrector)?
- *Quantitative two-scale expansion.* A formal argument suggests that u_ε admits the two-scale expansion
$$u_\varepsilon=u_0+\varepsilon\sum_{i=1}^d \phi_i(\tfrac{\cdot}{\varepsilon})\,\partial_i u_0+z_\varepsilon$$
where ϕ_i stands short for ϕ_{e_i} and the remainder z_ε is of higher order. In the case of periodic coefficients, classical results show that $\|z_\varepsilon\|_{H^1}\lesssim\varepsilon$ (away from the boundary of the domain and for sufficiently smooth data). Can we prove a similar estimate in the case of stochastic homogenization?

These quantitative questions require to strengthen the assumption of ergodicity. For this purpose in [6], [4], inspired by [7], we assume a Spectral Gap Inequality (SG) to quantify ergodicity. More precisely, we consider a discrete setting where $\mathbb{R}^d$ is replaced by the lattice $\mathbb{Z}^d$ and prove the following statement:

Theorem 1 (see [6],[4]). *Suppose the law of $\{a(x)\}_{x\in\mathbb{Z}^d}$ satisfies (SG) in the sense that for all random variable $\zeta(a)$ with vanishing expectation:*

$$\langle\zeta^2\rangle\le\sum_{x\in\mathbb{Z}^d}\langle(\frac{\partial\zeta}{\partial a(x)})^2\rangle. \tag{SG}$$

Then for any $T\ge1$ the bounded solution ϕ_T to

$$\frac{1}{T}\,\phi_T+\nabla^*(a(\nabla\phi_T+\xi))=0\qquad\text{in }\mathbb{Z}^d,$$

(i.e. the massive term approximation to (2)) satisfies for all $p<\infty$ the estimates $\langle|\nabla\phi_T|^{2p}\rangle^{\frac{1}{2p}}\le C(d,\lambda,p)|\xi|$ and

$$\langle\phi_T^{2p}\rangle^{\frac{1}{2p}}\le C(d,\lambda,p)|\xi|\times\begin{cases}\log^{\frac12}T & \text{for } d=2,\\ 1 & \text{for } d>2.\end{cases}$$

Moreover in [4] we prove a sharp result on the decay rate of the semigroup generated by $\nabla^*(a\nabla)$, seen as an operator acting on the L^{2p}-space of stationary

random fields. These results are optimal (in terms of their dependence on the parameter T and in terms of the decay rate, respectively), and imply that under the assumption of (SG) (which is fulfilled in the case when $\{a(x)\}_{x\in\mathbb{Z}^d}$ are i.i.d.) the corrector equation (2) admits a stationary solution for $d > 2$. For $d = 2$ existence of stationary correctors is not expected. A key ingredient in the proof of these results is elliptic regularity theory of De Giorgy-Nash-Moser type. It is used to establish estimates on the gradient of the elliptic Green's function (resp. heat kernel), which hold uniformly in a. In contrast to these results, which are restricted to scalar elliptic equations that satisfy a maximum principle, in [3] we obtain the same bounds for elliptic equations without appealing to the maximum principle and thus cover the case of non-symmetric a; see also [1] for systems.

The results discussed above rely on variants of (SG) and thus are restricted to coefficients with integrable correlations. In [5] we treat elliptic systems in $\mathbb{R}^d$ with coefficients that might have arbitrarily slow-decaying correlations, for example coefficients based on Gaussian fields satisfying for some parameter $\beta \in [0, 1)$:

$$\int_{\mathbb{R}^d} |\operatorname{cov}(a(x), a(0))|\,(|x| + 1)^{-\beta d}\,\mathrm{d}x < \infty. \tag{3}$$

We get a family of estimates

- on the decay of averages of the gradient of the corrector,
- on the sublinear growth of the corrector,
- on the error of the two-scale expansion.

The scaling of these estimates is sensitive to the parameter β and the dimension. In particular, we identify a critical exponent $\bar{\beta} = 1-\frac{2}{d}$, below which we get uniform bounds on the corrector and existence of stationary solutions to (2), while at or above $\bar{\beta}$ the sublinear corrector grows logarithmically (resp. with an algebraic rate).

The analysis in [5] relies on a novel regularity theory for elliptic operators with random (stationary & ergodic) coefficients, which is of independent interest and is inspired on the one hand by very recent work of Armstrong and Smart [2] and on the hand by the philosophy of Avellaneda and Lin to gain regularity for the equation on the ε-level from the good regularity properties of the homogenized equation. We introduce a critical radius r_*, which is defined in a natural way in form of a smallness condition on the sublinear growth of the corrector. We then argue that

- on length scales above r_* the operator $-\nabla \cdot a\nabla$ has "good" regularity properties, e.g. a-harmonic functions u are Lipschitz continuous, in the sense of $\frac{1}{|B_r|}\int_{B_r} |\nabla u|^2 \lesssim \frac{1}{|B_R|}\int_{B_R} |\nabla u|^2$ for $R \geq r \geq r_*$,
- r_* is finite almost surely, if we assume ergodicity and stationarity of the coefficients, and r_* has stretched exponential moments, if correlations of the coefficients are (possibly slowly) decaying in the spirit of (3).

The above mentioned results on quantitative stochastic homogenization are then obtained by combining these regularity properties with a sensitivity estimate that monitors the effect of (localized) perturbations of the coefficients a on ϕ_ξ.

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Fluid limits for population processes with a continuum of types: an example from gas kinetics

James Norris

We discuss the combination of two classical ideas. The first is the use of hierarchichal decompositions for test-functions in estimating the Wasserstein distance of a probability measure from its sample empirical distribution. The second is the use of martingale estimates to show convergence of Markov chains to solutions of differential equations. The ideas can be combined because the techniques used for sample empirical distributions extend naturally to martingale measures associated to a Markov chain.

The driving example is Kac's stochastic mean-field model for velocity exchange by elastic collision in a dilute gas of spherical particles, say in dimension 3. In this model, a vector of N velocities $(v_1, \dots, v_N)$ evolves as follows: for each pair of velocities v, v_*, say, at rate $|v - v_*|$, we draw a sphere with poles at v, v_*, choose randomly a new axis for the sphere, and replace v, v_* by the poles of this new axis.

We are able to show a new estimate, of the following form: with probability exceeding $1 - \varepsilon$, for all $t \le T$,

$$W\big(\mu_t^N, \mu_t\big) \;\le\; C\, N^{-1/3}.$$

Here, μ_t^N is the normalized empirical distribution of velocities from Kac's model and μ_t is the solution to the spatially homogeneous Boltzmann equation with collision kernel $|v - v_*|$ starting from μ_0^N. The constant C may be chosen to depend only on $\varepsilon > 0$, $p > 8$, $T < \infty$ and any upper bound for $\frac{1}{N}\sum_{i=1}^N |v_i|^p$.

Details may be found in [1].

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Random walk in balanced random environments

Noam Berger

(joint work with Jean-Dominique Deuschel, Xiaoqin Guo, Alejandro F. Ramírez)

We consider random walk in a balanced random environment. This is defined as follows. Let $\mathcal{M}^d$ denote the space of all probability measures on the nearest neighbors of the origin $\{\pm e_i\}_{i=1}^d$ and let $\Omega = (\mathcal{M}^d)^{\mathbb{Z}^d}$. An *environment* is a point $\omega \in \Omega$, we denote by P the distribution of the environment on Ω. We restrict our attention to cases where P is translation invariant, ergodic and balanced, i.e. for every z, every $e \in \{e_1, \ldots, e_d\}$ and P-almost every ω, $\omega(z,e) = \omega(z,-e)$.

For a given environment $\omega \in \Omega$, the *Random Walk* on ω is a time-homogenous Markov chain jumping to the nearest neighbors with transition kernel

$$P_\omega\big[X_{n+1} = z + e \;\big|\; X_n = z\big] \;=\; \omega(z,e) \geq 0, \qquad \sum_e \omega(z,e) \;=\; 1.$$

The *quenched law* P_ω^z is defined to be the law on $(\mathbb{Z}^d)^{\mathbb{N}}$ induced by the kernel P_ω and $P_\omega^z[X_0 = z] = 1$. We let $\mathsf{P}^z = P \otimes P_\omega^z$ be the joint law of the environment and the walk, and the *annealed* law is defined to be its marginal

$$\mathbb{P}^z = \int_\Omega P_\omega^z \,\mathrm{d}P(\omega).$$

This model was first studied by Lawler [4], who proved a CLT in the uniformly elliptic case (i.e. $\omega(z,e)$ is bounded away from zero) using ideas which first appeared in [5]. Later, Guo and Zeitouni [3] proved a CLT in the i.i.d. elliptic case and for the general ergodic cases with some moment assumptions on $\omega(z,e)^{-1}$. Berger and Deuschel [1] proved a CLT in the general i.i.d. case. Those cases do not cover the entire spectrum. Therefore, the talk began with an example (which appears in [1]) of a mixing and elliptic environment that supports no CLT, not even a degenerate one. The example is described as follows.

For every point $z \in \mathbb{Z}^2$ we have Bernoulli variables $(X^{z,ver})_{n\in\mathbb{N}}$ and $(X^{z,hor})_{n\in\mathbb{N}}$. Those variables are all independent, and $P[X_n^{z,ver} = 1] = P[X_n^{z,hor} = 1] = 3^{-n}$. Then, for every $z \in \mathbb{Z}^2$, if $X_n^{z,ver} = 1$, then the 2^n vertices directly above z all get chance $1 - \mathrm{e}^{-2n}$ to move in the vertical direction and chance e^{-2n} to move in the horizontal direction. If $X_n^{z,hor} = 1$, then the 2^n vertices directly to the right of z all get chance $1 - \mathrm{e}^{-2n-1}$ to move in the horizontal direction and chance e^{-2n-1} to move in the vertical direction. If a point hasn't been spoken for, it gets probability $\frac{1}{4}$ to go in each direction. If a point has been spoken for more than once, it gets the highest value assigned to it.

The main result presented is an improvement of all previous results. To state it, we first need to borrow the notion of the rescaling stopping time from [1]. We

define $T_i := \inf\{n : X_n - X_{n-1} \in \{e_i, -e_i\}\}$, and the rescaling stopping time is defined as $T = \max T_1, \ldots, T_d$. The new result is that if $\mathbb{E}\left[T^{d-1}\right] < \infty$, then there is a CLT, albeit with a random diffusion matrix. This covers all known cases, and also proves a CLT in several new cases, e.g. two dimensional systems with fast enough polynomial mixing.

Now we wish to deal with the issue of whether the diffusion matrix is deterministic. This is beyond what was given in the talk, due to time limitations. In dimension 2, our condition guarantees that the diffusion matrix is indeed deterministic. In higher dimensions we have examples where the diffusion matrix is truly random. Therefore we introduce another condition, similar to the finite energy condition from Percolation Theory (see [2]), under which the diffusion matrix is deterministic. All known cases fall under this condition.

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Emergence of phases in complex networks

Charles Radin

(joint work with Richard Kenyon, Kui Ren, Lorenzo Sadun)

There is a rich and distinguished history of combinatorial problems in which one analyzes asymptotically-large discrete systems with extreme values of certain constraints, for instance large permutations which do not include certain patterns.

It is useful to consider the relation of such discrete extremum problems with the older analysis of asymptotically large configurations of interacting particles with minimum energy, or of the densest packings of many unit spheres. This suggests how to generalize the combinatorial problems to nonextreme values of constraints by an appropriate use of entropy, as in statistical physics, with the goal of determining the structure of 'most' large systems with fixed nonextreme constraints.

In a series of papers with Rick Kenyon, Kui Ren and Lorenzo Sadun [4, 5, 6, 2] we have analyzed the asymptotics of large simple graphs with constraints on the values of the densities of one or more subgraphs (for instance edges and triangles), obtaining the structure of typical large constrained simple graphs. Furthermore

we show that these systems exhibit phases and phase transitions as the values of the densities are varied.

Our basic tool is a variational formulation of entropy, which we proved using the graphon formalism of Lovász et al. [3], and a large deviations theorem of Chatterjee/Varadhan [1]. For further detail see arXiv:1405.0599 and references therein.

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Persistence probabilities

Frank Aurzada

Introduction. Persistence concerns the behaviour of a stochastic process when it has a long excursion. The first point to analyse here is the probability of not passing a boundary, which is called persistence probability. To be more precise, let $X = (X_t)_{t\geq 0}$ be a real-valued stochastic process, say with $X_0 = 0$. Then we are interested in

$$\mathbb{P}\Big[\sup_{s\in[0,T]} X_s \leq 1\Big], \qquad \text{when } T\to\infty.$$

If the process X is self-similar (e.g. Brownian motion, fractional Brownian motion, etc.) then one expects the behaviour of the persistence probability to be polynomial of order $T^{-\theta+o(1)}$, where θ is called the persistence exponent of X. Similarly, if one deals with stationary processes X then one expects exponential decrease with rate θ, also called persistence exponent. Both cases are related via exponential time-rescaling (so-called Lamperti transform).

Apart from analysing the persistence probability one can ask for properties of the conditional law:

$$\mathcal{L}\Big((X_s)_{s\in[0,T]} \;\Big|\; \sup_{s\in[0,T]} X_s \leq 1\Big).$$

As an example, let us look at Brownian motion, which is exceptionally easy: by the reflection principle $\sup_{s\in[0,T]} X_s \stackrel{d}{=} |X_T| \stackrel{d}{=} T^{1/2}|\mathcal{N}(0,1)|$, from which the persistence probability can be worked out, leading in particular to $\theta = 1/2$ for the persistence exponent. Similarly, many things can be said about the conditional law for Brownian motion.

However, the only cases where these questions are reasonably well understood are Brownian motion, Lévy processes, random walks, and diffusions. Apart from that, the problem is wide open, and no general methods are available. In particular, even for Gaussian processes, only a few scattered results are known.

Besides being a classical and difficult problem in probability theory, the question of persistence is studied intensively in theoretical physics. Here, the point of view is often as follows: consider a physical system with a certain dynamics, started in some disordered state, and observe it on its way to equilibrium. Then one asks for the probabiliy that at a particular site e.g. the sign has not changed. For example, one can consider the heat equation $\Delta u(x,t) = \partial_t u(t,x)$, $x \in \mathbb{R}^d$, $t \geq 0$, with white noise initial condition $u(x,0) = \dot{W}(x)$. It is known that for the probability $\mathbb{P}\big[\sup_{s\in[0,T]} u(x_0,s) \leq 0\big] = T^{-\theta(d)+o(1)}$, where the exponents $\theta(d)$ are unknown and of relevance in physics, [7]. Similarly, many other physical models are considered, for example the Ising model started at infinite temperature and run with a Glauber dynamics; and one asks for the probability that the spin at some particular site has not flipped until time T, which is a persistence probability.

Persistence probabilities also turn up at a number of different occasions in statistical mechanics, for example, in wetting models. There, via a Gibbs approach one defines a process that has a certain interaction and, additionally, the constraint to be non-negative. The interaction and the constraint are competing effects that produce phase transitions in the limit. The partition function of the model is exactly the persistence probability for a process resulting from the interaction, cf. [6].

Results for Fractional Brownian motion (FBM). In the following, $(X_t)_{t\geq 0}$ will denote a Fractional Brownian motion, which is a centered Gaussian process with covariance

$$\mathbb{E}\big[X_t\, X_s\big] \;=\; \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big), \qquad t,s \geq 0,$$

where $0 < H < 1$ parametrises the family. For $H = 1/2$ this is just ordinary Brownian motion, and thus FBM generalises BM. For example, FBM is H-self-similar, i.e. $(X_{ct})_{t\geq 0}$ has the same distribution as $(c^H X_t)_{t\geq 0}$. Also, FBM has stationary increments, but no independent increments. Most importantly, FBM for $H \neq 1/2$ is not a Markov process.

For the persistence probability for FBM, the following result is valid ([8, 1]):

Theorem 1. *For FBM one has for some $c > 0$ and any $T \geq 2$,*

$$T^{-(1-H)}(\log T)^{-c} \;\leq\; \mathbb{P}\Big[\sup_{s\in[0,T]} X_s \leq 1\Big] \;\leq\; T^{-(1-H)}(\log T)^{c}.$$

For the conditional law, very little is known ([2]):

Theorem 2. *For FBM one has for some $c > 0$ and any $T \geq 2$ and $s > 0$,*

$$-s^H(\log T)^{c} \;\leq\; \mathbb{E}\Big[X_s \,\Big|\, \sup_{s\in[0,T]} X_s \leq 1\Big] \;\leq\; -s^H(\log T)^{-c}.$$

Similar results can be shown in the discrete-time setup, in particular for the discrete-time analog of FBM, which are sums of stationary Gaussian sequences with correlations exhibiting long-range dependence. Furthermore, one can consider random walks in random scenery (RWRS), see [3].

Let us give a few comments on how to prove these theorems. The main idea is to relate the persistence probability to the exponential functional

$$\mathcal{I}(T) \;:=\; \mathbb{E}\left[\frac{1}{\int_0^T \mathrm{e}^{X_s}\,\mathrm{d}s}\right].$$

The asymptotic order of $\mathcal{I}$ and the persistence probability are the same for FBM (possibly up to lower order terms), which can be proved via the following route:

$$\mathcal{I}(T) \;=\; \mathbb{E}\left[\frac{1}{\int_0^1 \mathrm{e}^{X_{Ts}}\,T\mathrm{d}s}\right] \;=\; \mathbb{E}\left[\frac{1}{\int_0^1 e^{T^H X_s} T\,\mathrm{d}s}\right].$$

Now, a careful and technical analysis of the integral shows that it is in fact a Laplace integral and thus governed by $\mathrm{e}^{T^H \sup_{s\in[0,1]} X_s}$, c.f. [1]. Therefore, one can control $\mathcal{I}$ by $\mathbb{E}[\mathrm{e}^{-T^H \sup_{s\in[0,1]} X_s}]$. This however is a Laplace transform of a non-negative random variable, and by Tauberian theorems it is related to the lower tail of that random variable: $\mathbb{P}\big[\sup_{s\in[0,1]} X_s \le T^{-H}\big] = \mathbb{P}\big[\sup_{s\in[0,T]} X_s \le 1\big]$.

On the other hand, computing the asymptotic rate of $\mathcal{I}$ turns out to be possible, [8]. Note that using a time-reversal argument:

$$\frac{\mathrm{d}}{\mathrm{d}T}\mathbb{E}\left[\log\int_0^T \mathrm{e}^{X_s}\,\mathrm{d}s\right] \;=\; \mathbb{E}\left[\frac{\mathrm{e}^{X_T}}{\int_0^T \mathrm{e}^{X_s}\,\mathrm{d}s}\right] \;=\; \mathbb{E}\left[\frac{1}{\int_0^T \mathrm{e}^{X_s}\,\mathrm{d}s}\right] \;=\; \mathcal{I}(T).$$

The integral on the the left-hand side again is a Laplace integral and after-rescaling it is governed by $\mathrm{e}^{T^H \sup_{s\in[0,1]} X_s}$. Taking logarithms of this and differtiating w.r.t. T one obtains the asymptotic rate of $\mathcal{I}$ and thus the persistence probability, which turns out to be T^{H-1}.

Open problems. Despite much progress in the last 5 years on this subject, the essential nature of persistence-type problems is still not very well understood, and there are many easy-to-formulate open questions. We refer to the surveys [4] (for the mathematics literature) and [5] (for the physics literature) for extensive discussions, a literature overview, and many open problems.

Let us outline one interesting open problem. Let B be a standard Brownian motion and define multiply integrated Brownian motion by: $A_t^{(1)} := \int_0^t B_s\mathrm{d}s$ and $A_t^{(d)} := \int_0^t A_s^{(d-1)}\mathrm{d}s$. These processes are perfectly nice Gaussian processes. It can be shown that the rate of decay of their persistence probability is polynomial with persistence exponent θ_d, respectively. The only known exponent is $\theta_d = 1/4$, and computing θ_d for $d \ge 2$ is a challenging open problem, which is of great interest for a number of related problems – in particular, those for the heat equation with random initial condition and the wetting models mentioned in the introduction, respectively – and for other problems in theoretical physics.

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Large deviations for reacting particle systems: the empirical and ensemble process

Michiel Renger
(joint work with Robert Patterson and Mark Peletier)

A recent and growing body of work has revealed a close connection between large deviations of microscopic particle systems and macroscopic gradient flow structures, see for example [1, 7]. On the other hand, the discrete counterpart of the transport distance, first discovered in [2, 4], can be fruitfully generalised to obtain a gradient structure for non-linear reaction rate equations [3]. Motivated by this, we study a microscopic system of reacting particles and prove a dynamic large deviation principle. One abstraction level higher, we study the large deviations of the empirical measure of concentrations [6]. This result is particularly useful to approximate the probability on concentrations. As a side-product, these large deviations can be used to improve the efficiency of Monte Carlo simulations via importance sampling. Both large deviation principles will be used to derive a 'natural' gradient structure for the reaction rate equation in a separate paper [5].

It turns out that the spaces of bounded variation provide natural topologies for both processes. In fact, the usage of these topologies in large deviations is not very common and therefore, hopefully, interesting in its own right.

Particle systems: the empirical and ensemble processes. For a large volume $V > 0$, we consider the empirical process $C^{(V)} : t \mapsto \frac{1}{V}\sum_i^{Vc(0)} \mathbb{1}_{Y_i(t)}$ of random particles $Y_i(t)$ in a countable space $\mathcal{Y}$ of chemical species, that can react to form new particles according to any of the given reactions:

$$\sum_{y\in\mathcal{Y}} \alpha_y^{(r)} A_y \rightleftharpoons \sum_{y\in\mathcal{Y}} \beta_y^{(r)} A_y, \qquad r \in R.$$

Assuming $\alpha^{(r)}, \beta^{(r)} \in l^1(\mathcal{Y})$ for each reaction $r \in R$, this process is Markov in $l^1(\mathcal{Y})$ with generator

$$(\mathcal{Q}^{(V)}\Phi)(c) = \sum_{r\in R} k_{\mathrm{fw}}^{(r,V)}(c)\,\big(\Phi(c - \tfrac{1}{V}\alpha^{(r)} + \tfrac{1}{V}\beta^{(r)}) - \Phi(c)\big) + k_{\mathrm{bw}}^{(r,V)}(c)\,\big(\Phi(c - \tfrac{1}{V}\beta^{(r)} + \tfrac{1}{V}\alpha^{(r)}) - \Phi(c)\big).$$

If $V^{-1}k_{\mathrm{fw,bw}}^{(r,V)}(c) \to \bar{k}_{\mathrm{fw,bw}}^{(r)}(c)$ as $V \to \infty$, then the measures $C^{(V)}(t)$ converge to the deterministic solution of the reaction rate equation,

$$\dot{c}(t) = \sum_{r\in R} \Big(\bar{k}_{\mathrm{fw}}^{(r)}\big(c(t)\big) - \bar{k}_{\mathrm{fw}}^{(r)}\big(c(t)\big)\Big)\,\Big(\beta^{(r)} - \alpha^{(r)}\Big).$$

Similarly, we study the ensemble process $\rho_t^{(V,N)} := \frac{1}{N}\sum_{i=1}^N \delta_{C^{(V,i)}(t)}$ in $\mathcal{P}\big(l^1(\mathcal{Y})\big)$ of independent copies $C^{(V,i)}$ of the empirical process. As $V, N \to \infty$, this measure converges to the solution of the Liouville equation,

$$\dot{\rho}_t(c) + \mathrm{div}_c\Big(\rho_t(c)\sum_{r\in R}\big(\bar{k}_{\mathrm{fw}}^{(r)}(c) - \bar{k}_{\mathrm{bw}}^{(r)}(c)\big)\big(\beta^{(r)} - \alpha^{(r)}\big)\Big) = 0.$$

Spaces of bounded variation. For trajectories of the empirical process we use the space of functions of bounded variation $\mathrm{BV}\big(0,T;l^1(\mathcal{Y})\big)$, equipped with the *hybrid* topology induced by:

$$c^{(V)} \rightrightarrows c \quad :\Longleftrightarrow \|c^{(V)} - c\|_{L^1(0,T;l^1(\mathcal{Y}))} \to 0 \quad \text{and} \quad \int_0^T \dot{\phi}(t)\cdot c^{(V)}(t)\,\mathrm{d}t \to \int_0^T \dot{\phi}(t)\cdot c(t)\,\mathrm{d}t \quad \forall\,\phi \in C_c^1\big(0,T;l^\infty(\mathcal{Y})\big).$$

Since $l^1(\mathcal{Y})$ is countable-dimensional, we need to generalise a well-known compactness result for functions of bounded variation. The generalised result says that for any compact family $\big(k(t)\big)_{t=0}^T \subset l^1(\mathcal{Y})$, any $c \in \mathrm{BV}\big(0,T;l^1(\mathcal{Y})\big)$ and $L > 0$, the set

$$\Big\{\hat{c} \in \mathrm{BV}\big(0,T;l^1(\mathcal{Y})\big) : \|\hat{c} - c\|_{L^1(0,T;l^1(\mathcal{Y}))} + \mathrm{epvar}(\hat{c} - c) \le L \text{ and } \hat{c}(t) \in k(t)\,\forall t\Big\}$$

is compact in the hybrid topology.

For trajectories of the ensemble process we use the space $\mathrm{BV}\big(0,T;\mathrm{BL}(l^1(\mathcal{Y}))^*\big)$, equipped with the weak-* topology induced by:

$$\rho^{(V)} \overset{*}{\rightharpoonup} \rho \;:\Longleftrightarrow\; \int_0^T \langle\dot{\Phi}(t), \rho_t^{(V)}\rangle\,\mathrm{d}t \to \int_0^T \langle\dot{\Phi}(t), \rho_t\rangle\,\mathrm{d}t \quad \forall\,\Phi \in C_{\mathrm{c}}^1\big(0,T;\mathrm{BL}(l^1(\mathcal{Y}))\big).$$

As a consequence of the Kantorovich representation theorem, the ensemble process automatically lies in this space as soon as each copy $C^{(V,i)}$ lies in $\mathrm{BV}\big(0,T;l^1(\mathcal{Y})\big)$.

Large deviations. Under suitable assumptions, we prove that the empirical process $C^{(V)}(t)$ satisfies a large-deviation principle in $\mathrm{BV}\big(0,T;l^1(\mathcal{Y})\big)$ with good rate functional $\int_0^T L\big(c(t),\dot{c}(t)\big)\,\mathrm{d}t$, where $L(c,s) = \sup_\xi c\cdot s - H(c,\xi)$ and

$$H(c,\xi) = \sum_{r=1}^R \bar{k}_{\mathrm{fw}}^{(r)}(c)\big(\mathrm{e}^{\xi\cdot(\beta^{(r)}-\alpha^{(r)})} - 1\big) + \bar{k}_{\mathrm{bw}}^{(r)}(c)\big(\mathrm{e}^{\xi\cdot(\alpha^{(r)}-\beta^{(r)})} - 1\big).$$

The proof is by a usual change-of-measure technique, where the proof of the exponential tightness is based on the generalised compactness result described above.

In the second main result, we prove that the ensemble process $\rho_t^{(V,N)}$ satisfies a large-deviation principle in $\mathrm{BV}\big(0,T;\mathrm{BL}(l^1(\mathcal{Y}))^*\big)$ with good rate functional $\int_0^T \mathbb{L}(\rho_t,\dot\rho_t)\,\mathrm{d}t$, where $\mathbb{L}(\rho,S)=\sup_\Xi \int \Xi(c)\,S(\mathrm{d}c)-\mathbb{H}(\rho,\Xi)$ and

$$\mathbb{H}(\rho,\Xi)=\int_{l^1(\mathcal{Y})} H\big(c,\nabla\Xi(c)\big)\,\rho(\mathrm{d}c).$$

To show this, we first prove a large-deviation principle for the empirical measure on trajectories $\Theta^{(V,N)}(dc(\cdot)):=\frac{1}{N}\sum_{i=1}^N \delta_{C^{(V,i)}(\cdot)}$, and then prove that the projection $\mathcal{P}(\mathrm{BV}\big(0,T;l^1(\mathcal{Y})\big)\to \mathrm{BV}\big(0,T;\mathrm{BL}(l^1(\mathcal{Y}))^*\big)$ is continuous so that we can apply the contraction principle.

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Hydrodynamic Limit of Condensing Zero Range Processes with sub-critical initial profiles

Marios G. Stamatakis

In this short talk we discuss the hydrodynamic behaviour of symmetric nearest neighbour (n.n.) condensing Zero Range Processes (ZRPs). ZRPs are interacting particle systems such that each particle X jumps at an exponential rate $g(k)$ that depends only on the number k of particles that occupy the same site as particle X through some function $g:\mathbb{Z}_+\to\mathbb{R}_+$, which is called the local jump rate. Particles that jump move to a nearest neighbour with equal probability. For particular choices of local jump rate functions g they exhibit phase transition phenomena and such ZRPs are called condensing.

We are interested in the hydrodynamic limit of ZRPs on the flat torus $\mathbb{T}^d$ and thus we consider ZRPs whose particles evolve in the finite lattice $\mathbb{T}_N^d :=$

$\{0,\dots,N-1\}^d$ with periodic boundary conditions. In this case the ZRP with jump rate $g:\mathbb{Z}_+\to\mathbb{R}_+$ and elementary step distribution p on $\mathbb{Z}^d$ is the Markov jump process on the state space

$$\mathbb{M}_N^d := \mathbb{Z}_+^{\mathbb{T}_N^d}$$

of configurations of particles on the discrete torus $\mathbb{T}_N^d$ with generator L^N given by

$$L^N f(\eta) = \sum_{x,y\in\mathbb{T}_N^d} \big(f(\eta^{x,y}) - f(\eta)\big)\, g(\eta_x)\, p(y-x),$$

where

$$\eta_z^{x,y} = \begin{cases} \eta_z, & \text{if } z\notin\{x,y\} \\ \eta_x - 1, & \text{if } z = x \\ \eta_y + 1, & \text{if } z = y \end{cases},$$

if $\eta_x\neq 0$ and, say, $\eta^{x,y}=\eta$ otherwise, and p is the n.n. kernel, i.e. $p(y-x)=\frac{1}{2}$ if $y = x\pm e_j$ for some $j=1,\dots,d$. Note the total mass of p is equal to d. This is done to obtain the probabilist's diffusion equation as the hydrodynamic limit.

For particular decreasing local jump rate functions g there exists a critical value $\rho_c=\rho_c(g)<+\infty$ of the density such that there exist product and translation equilibrium states characterized by the density ρ iff $\rho\le\rho_c$. To be more precise, solving the equation $\nu L^N=0$ that yields the equilibrium states under the assumption that ν is translation invariant and product, one obtains that ν is the product measure with common marginal $\bar\nu_\varphi^1$ on $\mathbb{Z}_+$ given by

$$\bar\nu_\varphi^1\{k\} = \frac{1}{Z(\varphi)}\,\frac{\varphi^k}{g!(k)}, \quad k\in\mathbb{Z}_+, \qquad \varphi := \int g(\eta(0))\,\mathrm{d}\nu,$$

where $g!(k) := g(1)\cdot\ldots\cdot g(k)$ and $Z(\varphi)$ is the normalizing constant. Of course in order for $\bar\nu_\varphi^1$ to be a probability measure the normalising constant $Z(\varphi)$ must be finite, i.e. the partition function $Z(\varphi)=\sum_{k=0}^\infty \frac{\varphi^k}{g!(k)}$ must converge at φ. It is known that the mean density

$$R(\varphi) = \int \eta(0)\,\mathrm{d}\bar\nu_\varphi^N$$

of the occupation variable under $\bar\nu_\varphi^1$ is smooth strictly increasing function of φ and so by reparametrizing the equilibrium distributions by its inverse $\Phi := R^{-1}$ we obtain the grand canonical ensemble

$$\nu_\rho^N = \bar\nu_{\Phi(\rho)}^N, \quad \text{for } \rho\ge 0 \text{ such that } Z(\Phi(\rho))<+\infty.$$

We will refer to ZRPs for which the radius of convergence φ_c of the partition function Z is infinite as non-condensing. ZRPs such that

$$\varphi_c<+\infty \quad\text{and}\quad \rho_c := \sup_{\varphi<\varphi_c} R(\varphi) = +\infty$$

will be called weakly condensing, while ZRPs for which $\rho_c < +\infty$ will be called strictly condensing. A particular example of a jump rate g that defines condensing ZRPs (see [2]) is the Evans jump rate g_b, $b \geq 0$, given by

$$g_b(k) = 1 + \frac{b}{k}, \quad k \geq 1, \quad g_b(0) = 0$$

with critical fugacity $\varphi_c \equiv \varphi_c(b) = 1$ for all $b \geq 0$ and critical density

$$\rho_c \equiv \rho_c(b) = \begin{cases} +\infty & \text{if } 0 \leq b \leq 2 \\ \frac{1}{b-2} & \text{if } b > 2 \end{cases}.$$

The lack of equilibrium states corresponding to densities $\rho > \rho_c$ constitutes a main problem in the description of the hydrodynamic behaviour of strictly condensing ZRPs, since the formulation via the notion of local equilibrium faces difficulty in observing densities higher than the critical density. Furthermore, even weakly condensing ZRPs exhibit pathological behaviour.

So far two main methods have been applied to prove the hydrodynamic behaviour of ZRPs. The Entropy method developed by Guo, Papanikolaou and Varadhan which and the Relative Entropy method developed by Yau. The Entropy method has been applied to non-condensing ZRPs with super-linear jump rate function $g(k) \geq a_0 k$ for some $a_0 > 0$ when starting from an initial distribution with integrable profile $\rho_0 \in L^1(\mathbb{T}^d)$, while the Relative Entropy method has been applied when starting from an initial distribution with a $C^{2+\theta}$ profile, under the additional assumption

$$\lim_{\varphi \to \varphi_c} Z(\varphi) = +\infty \tag{1}$$

which forces the critical density to be infinite. For example, in the Evans model assumption (1) is satisfied only for $b \in [0, 1]$ and our main aim in this short talk is to remove this assumption. Both methods yield the non-linear diffusion equation

$$\begin{cases} \partial_t \rho = \frac{1}{2} \Delta_x \Phi(\rho) \\ \rho(0, \cdot) \equiv \rho_0 \end{cases} \tag{2}$$

as the hydrodynamic equation. An exposition of the proofs is contained in chapters 5 and 6 of [1], respectively.

In this short talk we decribe an extension of the Relative Entropy method to n.n. condensing ZRPs with bounded jump rates, by interpreting the boundedness assumption on the initial profile as boundedness away from the critical density. This is achieved by extending the One-Block estimate, a main tool in all known approaches to the hydrodynamic limit of ZRPs, to condensing ZRPs. This extension of the One-Block estimate is made possible by the equivalence of Ensembles proved in [2, Th. 1]. As was already noted in [2, 4], the equivalence of ensembles indicates that the fugacity function $\Phi := R^{-1}$ should be extended to be equal to $\Phi(\rho_c)$ for densities $\rho > \rho_c$. With this extension of Φ the one-block estimate remains valid for condensing ZRPs with bounded jump rates:

Proposition 1 (One-Block Estimate)**.** *Suppose the local jump rate g of the ZRP is bounded and that the limit inferior $\varphi_c = \liminf_{k\to+\infty} \sqrt[k]{g!(k)}$ defining the critical fugacity exists as a limit. Let μ_0^N be any sequence of initial distributions on $\mathbb{M}_N^d$ satisfying the $O(N^d)$-entropy assumption, i.e. such*

$$\limsup_{N\to\infty} \frac{1}{N^d} H(\mu_0^N \,|\, \nu_{\rho_*}^N) < +\infty,$$

for some (and thus for any) $\rho_* \in (0, \rho_c) \subseteq \mathbb{R}_+$. Then

$$\lim_{\ell\to\infty} \limsup_{N\to\infty} \mathbb{E}^N \left| \int_0^T \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} G\Big(t, \frac{x}{N}\Big) \Big(g(\eta_t(x)) - \Phi\big(\eta_t^\ell(x) \wedge \rho_c\big)\Big)\, \mathrm{d}t \right| = 0$$

for all functions $G \in C([0,T]\times\mathbb{T}^d)$, $T > 0$, where $\mathbb{E}^N$ denotes the expectation with respect to the diffusively accelerated law of the ZRP starting from μ_0^N.

Using this version of the one-block estimate one can apply H. T. Yau's relative entropy method to conclude that when starting from a sequence of initial distributions μ_0^N of profile $\rho_0 \in C^{2+\theta}(\mathbb{T}^d)$ with values in $(0, \rho_c)$ the hydrodynamic equation (in the diffusive scaling) is still given by (2). See [5] for the precise result.

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Rates of convergence for extremes of geometric random variables and marked point processes

Alessandra Cipriani

(joint work with Anne Feidt)

We use the Stein-Chen method to study the extremal behaviour of the problem of extremes for univariate and bivariate geometric laws. It is known that for an i.i.d. sample of random variables $X_1, \ldots, X_n$ no non-degenerate limit distribution may be found for $X_{(n)}$, the maximum of the sample. Using a result from the Stein-Chen method for Poisson approximation by [1], we determine bounds on the error, in the Kolmogorov distance, of the approximation of the maximum, under different normalisations, of i.i.d. geometric random variables by the Gumbel distribution. Our results show that convergence is faster and requires no conditions on the distributional parameter when approximating by a discretised Gumbel distribution,

in comparison to [5]. We similarly determine an error bound for the approximation of the joint law of maxima of random pairs, following the bivariate Marshall-Olkin geometric distribution [2], by an appropriate discrete limit. We similarly find a rate of convergence for the law of maxima of bivariate Marshall-Olkin geometric random pairs when approximating by a discrete limit law. We further use the Stein-Chen method for Poisson process approximation to determine bounds on the error, in a suitable probability metric, of the law of a *marked point process of exceedances* (MPPE), defined by

$$\Xi_{u,n} \ := \ \sum_{i=1}^{n} I_{\{X_i \in A\}} \, \delta_{X_i},$$

by that of a Poisson process whose mean measure equals that of the MPPE, where the X_i are i.i.d. geometric random variables and A is a subset of extreme values of the marks' state space. Though the MPPE does not mark the points that exceed a threshold in the way that a *point process of exceedances* (PPE) of the form $\sum_{i=1}^{n} I_{\{X_i \in A\}} \delta_{in^{-1}}$ does, it contains more information relevant to the study of extreme values than a *marked point process* (MPP) of the form $\sum_{i=1}^{n} \delta_{X_i}$, as it is not only a random configuration of points in space, but specifically a random configuration of points exceeding a threshold. For both cases, the estimate for the actual "Poisson approximation" comes easily. The reason for this is that we use i.i.d. samples $X_1, \ldots, X_n$ and i.i.d. indicators $I_{\{X_1 \in A\}}, \ldots I_{\{X_n \in A\}}$. This allows us to apply a result by [4], which reduces the problem to the approximation of a binomial by a Poisson distribution. However, as the marks have geometric, and thereby discrete margins, the mean measure will live on a lattice and be rather tedious to work with in practical applications. We would therefore prefer to approximate by a further Poisson process with a continuous mean measure. Since the total variation distance is too strong for this kind of approximation, we use the weaker d_2-distance instead, which is not as sensitive towards small changes in the positions of the points of the point processes. Our main effort thus lies in determining error bounds on the approximation of a Poisson process by another more suitable Poisson process.

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Localization of the first Dirichlet eigenvector in the heavy-tailed random conductance model

Franziska Flegel

Our goal is to find almost-sure asymptotics for the first Dirichlet eigenvalue and eigenvector of the random walk generator $\mathscr{L}_w$ of the (variable speed) random conductance model

$$(\mathscr{L}_w f)(x) = \sum_{y\colon y\sim x} w_{xy}(f(y)-f(x)), \qquad f\colon \mathbb{Z}^d \to \mathbb{R},$$

where the conductances w_{xy} are i.i.d. with law $\mathbb{P}$, symmetric ($w_{xy} = w_{yx}$), and $\mathbb{P}[0 < w \leq 1] = 1$. We assume zero Dirichlet boundary conditions outside the box $B_n = [-n,n]^d \cap \mathbb{Z}^d$ and are concerned with the limit $n \to \infty$. The first Dirichlet eigenvalue $\lambda^{(n)}$ is given by the variational formula

$$\lambda^{(n)} = \frac{1}{2} \inf_{\substack{\operatorname{supp} f \subseteq B_n, \\ \|f\|_2 = 1}} \sum_{x\sim y} w_{xy} \left(f(x)-f(y)\right)^2, \tag{1}$$

where the minimizer is the corresponding eigenvector $\psi^{(n)}$.

We are interested in the question whether there is a dichotomy between the scenarios where $\psi^{(n)}$ behaves macroscopically or microscopically. More precisely: Under what conditions is the shape of the eigenvector $\psi^{(n)}$ homogeneous and when is it localized? In particular, does the invalidity of a quenched local central limit theorem on the entire $\mathbb{Z}^d$ imply localization of the first eigenvector and vice versa?

Here, we give an example of how a $\mathbb{P}$-a.s. counter scenario to a local CLT implies the concentration of $\psi^{(n)}$ on a sparse subset of the box B_n. For the sake of simplicity, we restrict ourselves to the special case where the conductances are Pareto-distributed near zero, i.e., $\mathbb{P}[w \leq a] = a^\gamma$ for some $\gamma > 0$ and all $a \in (0,1]$. Note that the conditions can be made more general [1]. In the Pareto case, the authors of [2] prove the validity of a local CLT for $\gamma > 1/4$ and construct a $\mathbb{P}$-a.s. counter scenario for $\gamma < 1/4$. We obtain the following theorem: If $d \geq 2$ and $\gamma < 1/4$, then

(i) $\mathbb{P}$-a.s. $\lambda^{(n)} = n^{-\frac{1}{2\gamma}+o(1)}$ and
(ii) there exists a sparse set $\mathscr{I}_n \subset B_n$ of disconnected sites such that $\mathbb{P}$-a.s.

$$\left\| \psi^{(n)}\big|_{B_n \setminus \mathscr{I}_n} \right\|_2 \to 0 \text{ as } n \to \infty.$$

The upper bound in (i) is obtained by inserting a suitable test function into Eq. (1). Here we use of the counter scenario of [2]: It can be shown that $\mathbb{P}$-a.s. there exists a site $z_n^* \in B_n$ such that all incident conductances are smaller than $n^{-\frac{1}{2\gamma}+o(1)}$. Then the test function $\delta_{z_n^*}$ gives the desired upper bound. For the proof of the lower bound we use the estimate

$$\lambda^{(n)} \geq \left(\sup_{x\in B_n} E_x^w \left[\tau_{\partial B_n}\right] \right)^{-1}$$

from [3], where $E_x^w[\tau_{\partial B_n}]$ is the quenched expected escape time from the box B_n of a random walker starting in site x. Although we do not have an upper bound for $E_x^w[\tau_{\partial B_n}]$ in the case $\gamma < 1/4$, we can apply Jensen's inequality to Eq. (1) to lift the conductances to a law with parameter $\gamma' > 1/4$. For such laws the authors of [2] proved that the expected escape time is $\mathbb{P}$-a.s. bounded by cn^2 where $c > 0$ is a constant. After some technical steps, we obtain the matching lower bound on $\lambda^{(n)}$.

Statement (ii) can be proved by contradiction. We use Eq. (1) and the upper bound on $\lambda^{(n)}$ from part (i) of our theorem. For $\delta \in (0, (17\gamma d)^{-1})$, let $\mathscr{I}_n$ be the set of sites in B_n that are surrounded by conductances smaller than $n^{-\frac{1}{2\gamma}+\delta}$. If $\psi^{(n)}$ was not asymptotically concentrated on $\mathscr{I}_n$, the upper bound on $\lambda^{(n)}$ would be false.

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Rotational Symmetry Breaking in Two Dimensions

Luke D. Williams

(joint work with Florian Theil)

In this short talk we prove a simplified version of a rigidity estimate based on the work of Müller et al. for a specific class of functions. These functions are used in a statistical mechanical model of a crystal. With this tool we demonstrate orientational ordering in the strong sense of Aumann & Heydenrich et al. an improvement on current literature where only a "weak" ordering was demonstrated.

In order do demonstrate that a system is ordered, we assume we have some field V which describes the rotation and dilation of some system at any given point. For the purposes of the abstract, a geometric rigidity theorem of the following can be stated, from [1]. It is a generalisation of work by Müller *et al* in [2]

Theorem 1 (Geometric Rigidity)**.** *Let Ω be a simply connected, Lipschitz domain. Let $V \in L^2(\Omega, \mathbb{R}^{2\times 2})$ be such that $\operatorname{curl} V \in L^2(\Omega, \mathbb{R}^2)$. Then there exists a rotation $R \in \mathrm{SO}(2)$ such that*

$$\|V - R\|_{L^2(\Omega)} \leq C_1 \,\|\operatorname{dist}(V, \mathrm{SO}(2))\|_{L^2(\Omega)} + C_2 \,\|\operatorname{curl} V\|_{L^2(\Omega)}.$$

This inequality shows that we can compare V to some specific rotation R, and bound how far V is from a rotation globally by its pointwise difference from any rotation, plus a term describing the curl of V. If a function is equal to one rotation in part of a domain, and another rotation elsewhere, if the field is continuous there

must be a region in between in which V takes on intermediate values, and in this region V has non-zero curl.

When we wish to show the ordering in a crystal, typically we consider a family of problems, roughly corresponding to making a bigger crystal each time. In the above, then the domain Ω gets larger. It is vital that, when the domain changes, it is known how the constants involved in the estimate above change.

A simple scaling argument gives the following. If $\Omega = B_N(0)$, then we can choose constants C_1 and C_2 such that

$$\|V - R\|_{L^2(\Omega)} \leq C_1 \,\|\operatorname{dist}(V, \mathrm{SO}(2))\|_{L^2(\Omega)} \,+\, NC_2 \,\|\operatorname{curl} V\|_{L^2(\Omega)}$$

as was done in [1]. However, for many defects this is not optimal. The above takes into consideration when, on a macroscopic level, V is equal to different rotations with a long strip in between. These "grains" necessitate possessing a large curl. The above is not optimal if, say, suppcurl$V = B_1(0)$, and $V = \mathrm{Id}$ everywhere outside this ball. The distance $\|V - I\|_{L^2(\Omega)} = O(1)$ regardless of domain size.

If we wish to show orientational ordering, to begin with we only grow a crystal possessing a single grain. In this case, by bounding the possible size of regions of non-zero curl and demanding they posses separation, we arrive at a class of admissible functions: for V as above, and for $r > 0$

$$\mathcal{A}_r(\Omega) = \big\{V \,:\, \operatorname{supp}\operatorname{curl} V \subset \cup_{i=1}^{M} B_r(x_i),\, |x_i - x_j| > 2r,\, \operatorname{dist}(x_i, \partial\Omega) > 2r\big\},$$

That is, the curl of V can be covered by disjoint balls of fixed size, with a minimum separation between each other and the boundary of Ω.

For this class of functions, we have the following rigidity estimate:

$$\|V - R\|_{L^2(\Omega)} \leq C_1 \,\|\operatorname{dist}(V, \mathrm{SO}(2))\|_{L^2(\Omega)} \,+\, C(r) \,\|\operatorname{curl} V\|_{L^2(\Omega)}.$$

The point here is that we only allow defects of a fixed size, and place them into large domains, an exclude the formation of grains. This removed the dependency on system size in the final term.

If $\sigma > 0$ given is some term given which penalises the formation of defects, β inverse temperature, and N the diameter of an appropriate family of domains Ω_N we can then show that in the framework of [1] that

$$\lim_{\beta\to\infty} \limsup_{N\to\infty} \sup_{\sigma>\sigma_0} \mathbb{E}_{N,\beta,\sigma}\left[\inf\nolimits_{R\in\mathrm{SO}(2)} \frac{\|V - R\|^2_{L^2(\Omega_N)}}{N^2}\right],$$

where $V \in A_r(\Omega_N)$. Expectation is taken with respect to the standard Boltzmann measure, with some Hamiltonian H satisfying $H(R) = 0$ for any rotation, and $H(M) \geq C\mathrm{dist}(M, \mathrm{SO}(2))$ for matrices "near" SO(2).

Due to the scaling argument mentioned, previous work required that $\sigma > cN^2$ for some $c > 0$: that is, defects became vanishingly unlikely for large crystals. The above estimates are uniform and represent an improvement on existing ones.

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Large deviation principle for the empirical density of a semi-infinite TASEP via matrix products

Horacio G. Duhart

(joint work with Peter Mörters and Johannes Zimmer)

In this talk I present the work of my PhD [1] which finds explicitly the rate function for a large deviation principle of the semi-infinite totally asymmetric simple exclusion process via the description found by Großkinsky [3] of the invariant measures based on the work by Derrida, Lebowitz and Speer [2].

The proof relies on finding an upper bound using functional analytic techniques and a lower bound from a combinatorial approach.

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On recent developments in compactness and large deviations

Chiranjib Mukherjee

(joint work with S. R. Srinivasa Varadhan, Erwin Bolthausen and Wolfgang König)

We are motivated by the classical mean-field polaron problem coming from statistical mechanics. The mathematical layout of the problem is to consider the tilted measure

$$\mathrm{d}\widehat{\mathbb{P}}_t = \frac{1}{Z_t} \exp\left\{tH(L_t)\right\} \mathrm{d}\mathbb{P}$$

where $\mathbb{P}$ is the three dimensional Wiener measure, Z_t is the total mass and the Hamiltonian

$$H(\mu) = \int\int \frac{1}{|x-y|}\, \mu(\mathrm{d}x)\, \mu(\mathrm{d}y)$$

acts on the normalized occupation measures $L_t = \frac{1}{t}\int_0^t \delta_{W_s}\mathrm{d}s$. The first key result is the following variational formula for the free-energy:

$$\lim_{t\to\infty}\frac{1}{t}\log Z_t = \sup_{\|\psi\|_2=1}\left\{\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\psi^2(x)\psi^2(y)}{|x-y|} - \frac{1}{2}\|\nabla\psi\|_2^2\right\}.$$

It turns out that the set of maximizing functions $\mathfrak{m}$ of the variational formula above is a singleton modulo spatial shifts. Given these two facts, one expects L_t to "localize" around $\mathfrak{m}$ with high $\widehat{\mathbb{P}}_t$ probability.

Large deviation theory plays the central role in the proof. As is well-known, in large deviation estimates, lower bound for open sets and upper bound for compact sets are essentially local estimates. However, upper bounds for all closed sets often require the compactness of the ambient space or exponential tightness, which is often absent in many cases (for example, distributions of L_t).

Motivated by the aforementioned problem, we present a robust theory of "translation-invariant compactification" of orbits of probability measures in $\mathbb{R}^d$. This enables us to prove a full large deviation principle on this compactified space. Thanks to shift-invariance of the above problem, we are able to reduce this abstract theory painlessly to probability measures in $\mathbb{R}^d$ and fully determine localization of L_t under $\widehat{\mathbb{P}}_t$ (based on [1], [2] and [3]).

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Condensation in preferential attachment models with fitness

Steffen Dereich

A popular network model is the preferential attachment model which gained popularity in the late 90's. Its success was mainly triggered by the fact that it gives a simple explanation for the appearance of power laws in real world networks. Mathematically, one considers a sequence of random graphs that is built dynamically: in each step a new vertex is added and linked randomly by a random or deterministic number of edges to the vertices already present in the system. In this process, links to vertices with high degree are preferred.

In 2001 Bianconi and Barabási [1] introduced a variant of preferential attachment that additionally assigns individual vertices independent fitnesses that effect the likelihood to be linked to by new vertices. They argue that such network models feature an intriguing condensation phenomenon which was later rigorously confirmed in [2]. In this talk we discuss explicitly the scaling window of vertices that are responsible for the condensation phenomenon in a particular Poissonized variant of *preferential attachment with fitness*. We stress that the model is closely

related to the condensation models studied in the talk by P. Mörters and we refer the reader to the corresponding abstract. This talk is based on [3].

In general the graphs under consideration are directed graphs and we denote for a vertex v of a graph G by

$$\mathrm{imp}_G(v) \;=\; 1 + \mathrm{indegree}_G(v)$$

the so called *impact* of vertex v in G. We introduce a variant of *preferential attachment with fitness* with adaptive normalisation depending on two parameters

- a distribution μ on $(0,1)$ with essential supremum equal to one (*fitness distribution*) and
- $\lambda > 0$ (*activity parameter*).

We let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ denote a sequence of independent μ-distributed random variables and construct a sequence $(\mathcal{G}_n)_{n\in\mathbb{N}}$ of labelled random graphs as follows: $\mathcal{G}_1$ consists of a single vertex labelled by 1 and no edges. Further given $\mathcal{G}_n$ (together with the whole sequence $(\mathcal{F}_n)$) the graph $\mathcal{G}_{n+1}$ is obtained from $\mathcal{G}_n$ by

- insertion of a vertex labelled $n+1$ and
- insertion of an independent Poisson-distributed number of directed edges $n+1 \to k$ with parameter

$$\frac{\mathcal{F}_k \,\mathrm{imp}_{\mathcal{G}_n}(k)}{n\,\bar{\mathcal{F}}_n}$$

for each old vertex $k \in \{1,\dots,n\}$, where

$$\bar{\mathcal{F}}_n \;=\; \frac{1}{\lambda n}\sum_{j=1}^{n} \mathcal{F}_j \,\mathrm{imp}_{\mathcal{G}_n}(j).$$

The random adaptive normalisation guarantees that each new vertex establishes an independent Poisson-distributed number of edges with parameter λ. As shown in [4] such a network shows condensation, if

$$\int \frac{x}{1-x}\,\mu(\mathrm{d}x) \;<\; \lambda.$$

In this case the *empirically weighted fitness distribution*

$$\Xi_n \;=\; \frac{1}{n}\sum_{k=1}^{n} \mathrm{imp}_{\mathcal{G}_n}(k)\,\delta_{\mathcal{F}_k}$$

converges almost surely in the weak sense to the deterministic measure Ξ with

$$\Xi(\mathrm{d}x) \;=\; \frac{1}{1-x}\,\mu(\mathrm{d}x) \;+\; \left(1+\lambda-\int \frac{1}{1-f}\,\mu(\mathrm{d}f)\right)\delta_1(\mathrm{d}x).$$

Hence as $n\to\infty$ the random measures Ξ_n accumulate mass in the essential supremum of μ although there are no vertices with fitness equal to one. This raises immediately the following question.

Which vertices form the condensate?

Unfortunately we cannot answer this question in the model with adaptive normalisation and we modify the dynamics of the network formation by replacing the random adaptive normalisation $(\bar{\mathcal{F}}_n)$ by a *deterministic* one. We consider the regime where μ has regularly varying tails in one meaning that there exists $\alpha > 0$ and a slowly varying function ℓ at zero such that

$$\mu(1-x,1) \;=\; x^{\alpha}\,\ell(x), \qquad \text{for } x > 0. \tag{1}$$

In order to prove a limit theorem for the condensation window we grow the network exponentially fast in time and assign the network of size $n \in \mathbb{N}$ the *time*

$$t_n \;:=\; \sum_{j=1}^{n-1} \frac{1}{j} \;=\; \big(1 + o(1)\big) \log n,$$

where generally o and O denote the Landau symbols. For $t \in [0,\infty)$, we denote by $N(t) \in \mathbb{N}$ the number of vertices in the network at time t, that is $N\colon [0,\infty) \to \mathbb{N}$ is the generalized right continuous inverse of $(t_n)_{n\in\mathbb{N}}$ given by

$$N(t) \;=\; \max\{n \in \mathbb{N} \,:\, t_n \le t\}.$$

The number of vertices in the network increases exponentially in the time t and there exists a constant $C_N > 0$ such that

$$N(t) \;=\; \big(C_N + o(1)\big)\, \mathrm{e}^{t}.$$

In terms of the time $t \ge 0$ we set $\bar{F}_t := \bar{\mathcal{F}}_{N(t)}$. As seen below the condensate at time t will be carried by vertices born at time of order

$$T(t) \;:=\; \max\{\alpha \log t, 1\}.$$

The choice of the normalisation $(\mathcal{F}_n)$ is very subtle and a technical assumption is that $\bar{F}_t = 1 - \alpha t^{-1} + o(t^{-1})$ with α as in (1). By [3, Thm. 1.2] existence of the limit

$$w \;:=\; \lim_{t\to\infty} T(t)\, \mathrm{e}^{\int_{T(t)}^{t} (1-\bar{F}_u)\,\mathrm{d}u - T(t)} \;\in\; [0,\infty) \tag{2}$$

implies that

$$\lim_{t\to\infty} \Xi_t \;=\; \Xi, \qquad \text{in the weak topology,}$$

in probability, where Ξ is the measure on $[0,1]$ with

$$\Xi(\mathrm{d}x) \;=\; \frac{1}{1-x}\,\mu(\mathrm{d}x) \;+\; \frac{\alpha}{\alpha-1}\,\Gamma(\alpha)\, w\, \delta_1(\mathrm{d}f)$$

and $\Gamma(\alpha)$ denotes the Gamma function with argument α. Hence we can replicate the behaviour of the condensation phase by choosing $(\bar{F}_t)$ in such a way that (2) holds with a strictly positive limit w.

To understand the type of vertices inducing the condensation phenomenon we look at the random measure

$$\Gamma_n \;=\; \frac{1}{n} \sum_{k=1}^{n} \mathrm{imp}_{\mathcal{G}_n}(k)\, \delta_{(t_k/T(t_n),\,(1-\mathcal{F}_k)t_n)}$$

on $[0,\infty)^2$ and denote by γ_α the measure on $[0,\infty)^2$ with Lebesgue density

$$\mathrm{d}\gamma_\alpha(s,x) \;=\; \alpha\,\mathbf{1}_{[1,\infty)\times[0,\infty)}(s,x)\,s^{-\alpha}\,x^{\alpha-1}\,\mathrm{e}^{-x}\,\mathrm{d}s\,\mathrm{d}x.$$

By [3, Thm. 1.4] one has

$$\lim_{n\to\infty}\Gamma_n \;=\; w\,\gamma_\alpha$$

in the weak topology induced by continuous, compactly supported test functions, in probability. The accumulated impact outside this condensation window does not contribute to the condensate.

Altogether we conclude that with increasing size of the network the contribution to the condensate stems from an increasing number of vertices whose time of birth and fitness are located in a particular scaling window. Individual vertices typically have no significant contribution and we refer the reader to the talk by P. Mörters, where the strongest vertices are analysed in a different but closely related model.

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Random Permutations and Queues

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1. When a deck of n cards is shuffled, the probability that none of the cards remain in the original position is about e^{-1}. This theorem, due to de Montmort, is one of the oldest limits of probability theory. In modern terms, the result concerns the event $A_1(n) = 0$, where $A_1(n)$ is the number of fixed points in permutation σ_n uniformly distributed over the symmetric group S_n. See [2] for recent variations on the theme.

Here, we add to de Montmort's problem a new twist. Suppose $\sigma_1, \sigma_2, \dots$ are viewed as snapshots of a permutation-valued growth process. What are the pathwise properties of A_1? For instance, how often does the event $A_1(n) = 0$ occur as n varies? Naively, one might think that the long-run frequency of the event is e^{-1}, and this clearly holds true for σ_n's *independent* uniform permutations. However, the e^{-1}-rule is not so obvious and even may fail for natural processes where permutations grow incrementally, by minor updates.

2. In the Dubins-Pitman Chinese Restaurant Process (CRP) [7], the transition from σ_n to σ_{n+1} amounts to updating the cycle notation of permutation: with probability $\theta/(\theta+n)$ a 1-cycle $(n+1)$ (fixed point) is appended to σ_n, otherwise an element of σ_n is chosen uniformly at random and integer $n+1$ is inserted clockwise next into the cycle of this element. The algorithm generates σ_n with Ewens' distribution, which is uniform on S_n for $\theta = 1$, and for $\theta \neq 1$ has the mass function depending on the number of cycles. The cycle partition of permutation under Ewens' distribution has been thoroughly studied [1], in particular it is well known that the limit law for $A_1(n)$ is Poisson(θ), so $\mathbb{P}(A_1(n) = 0) \to \mathrm{e}^{-\theta}$.

Now, in the CRP scenario the process A_1 is a nonstationary Markov chain with possible jumps

$$a_1 \to \begin{cases} a_1 + 1, \text{ with probability } \frac{\theta}{\theta+n}, \\ a_1 - 1, \text{ with probability } \frac{a_1}{\theta+n} \end{cases} \tag{1}$$

for $a_1 \in \mathbb{Z}_+$. As the permutation grows, the jumps of A_1 become more rare and sojourns longer, implying that the ratio $\#\{n' \leq n : A_1(n') = 0\}/n$ oscillates hence the $e^{-\theta}$-rule fails.

The stationarity is achieved by a simple time-change. Define X_1 to be a birth-death process with transitions

$$a_1 \to \begin{cases} a_1 + 1, \text{ at rate } \theta, \\ a_1 - 1, \text{ at rate } a_1, \end{cases} \tag{2}$$

and interpolate A_1 by a piecewise constant function. Comparing (1) and (2) it is seen that A_1 and X_1 have identical embedded jump-chains, thus to render closer similarity we only need to adjust the sojourn times.

Proposition 1. *As $\nu \to \infty$, the process $(A_1(\nu \mathrm{e}^t), t \geq 0)$ converges weakly to X_1, provided $A_1(\nu) = X_1(0)$ is some fixed initial state.*

The p.g.f. of $X_1(t)$ given $X_1(0) = a_1$ is

$$g(z,t) = (1 + (z-1)e^{-\theta t})^{a_1} \exp\big(\theta(z-1)(1-\mathrm{e}^{-t})\big)$$

(a convolution of binomial and Poisson), implying the Poisson(θ) invariant distribution, as is also easy to check directly along with reversibility. The stationarity implies that the $\mathrm{e}^{-\theta}$-rule holds for X_1, hence approximately for time-changed A_1.

The process X_1 has many interpretations, notably as the length of a infinite-server queue of the type $M/M/\infty$. Results on the behaviour of X_1 can be borrowed from the extensive literature on queueing theory and re-stated in terms of A_1 by the virtue of Proposition 1. For instance, given $A_1(\nu) = 1$ define $\tau_\nu = \inf\{t : A_1(\nu \mathrm{e}^t) = 0\}$. By continuity arguments, for ν large τ_ν is approximable by the duration τ of the queue busy period. We have $\mathbb{E}\tau = (\mathrm{e}^\theta - 1)/\theta$, and from [5] the Laplace transform is given in terms of the confluent hypergeometric function as

$$\mathbb{E}\big[\mathrm{e}^{-s\tau}\big] = \frac{s+\theta}{\theta} + \frac{1}{{}_1F_1(1, s, -\theta)}.$$

Further results on passage times and excursions of X_1 above given threshold $c \geq 0$ (congestion period) are found in [8].

3. Continuing the CRP thread, we let $A_1(n), A_2(n), \dots, A_m(n)$ to be the counts of 1-,2-,...,m-cycles in σ_n, respectively. Note that $n+1$ joining a j-cycle triggers the transition of the cycle counts $(a_1, \dots, a_m) \to (\dots, a_j - 1, a_{j+1} + 1, \dots)$ (with obvious adjustment for the edge cases $j = 1, m$), which can be regarded as migration of a j-cycle to the population of $(j+1)$-cycles. Similarly to (1), this kind of transition occurs with probability $ja_j/(\theta + n)$ (for $j > 1$)

Proposition 2. *With the time change $n = \nu \mathrm{e}^t$ and fixed state at time ν the process $(A_1, \dots, A_m)$ has a weak limit $(X_1, \dots, X_m)$, as $\nu \to \infty$.*

The limit process has interpretation as a tandem of $M/M/\infty$ queues with exponential service rate j at node $j = 1, \dots, m$. See [11] for other interpretations and survey of transient characteristics, and [10] for properties of the queue at times of new arrivals. The tandem is the simplest Jackson's network, having in equilibrium the Poisson output from each node, and with the equilibrium state being the product of Poisson(θ/j). This confirms the limit law for small-cycle counts $(A_1(n), \dots, A_m(n))$, derived in the literature by other methods [1].

Defining the busy period τ as the time needed to reach $(0, \dots, 0)$ starting from $(1, 0, \dots, 0)$, we may regard τ as the duration of busy period of a $M/G/\infty$ queue $\sum_{j=1}^m X_j(t)$ with the generic service time of the kind $\sum_{j=1}^m E_j/j$, where E_j's independent exponential variables. The distribution of τ can be derived by specialising formulas from [9]. Combinatorially, the empty queue corresponds to permutation with no cycles shorter than m (the generalised derangement [3, p. 122]).

4. Resorting to the more general Pitman [7] two-parameter CRP(α, θ), for $\alpha > 0$ we lose the Markov property of the fixed-point count and, more importantly, face very different behaviour caused by a power-like growth of $A_1(n)$. The following two variations keep closer to de Montmort's finding.

Ewens' permutations $\sigma_1, \sigma_2, \dots$ are intrinsically related to sampling from a discrete distribution with random masses $P_j = W_1 \cdots W_{j-1}(1 - W_j)$, where W_j's are i.i.d. beta$(\theta, 1)$, see [1, 7]. Replacing beta by some other distribution for the 'stick-breaking' factors W_j still results in permutations that have $(A_1(n), \dots, A_m(n))$ converging in distribution and with moments [4], so that, in particular, $\mathbb{E}A_j(n) \to j^{-1}\mathbb{E}|\log W|$. However, the nice Markovian properties inherent to the CRP construction of permutation are lost, and it is not clear if more general models of the queueing theory can be of some use to study the evolution of short cycles.

The second variation concerns yet another way to update the uniform permutation. Suppose σ_{n+1} obtains by inserting $n+1$ uniformly at random in the one-row notation of σ_n. The transition may change the number of fixed points dramatically, and A_1 is not Markovian. But the e^{-1}-rule seem to holds literally, like for the independent σ_n's.

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Two-scale Γ-convergence for Random Nonconvex Homogenization

Etienne Sandier

(joint work with Leonid Berlyand and Sylvia Serfaty)

Random homogenization of convex functionals was studied in the seminal papers of Dal Maso and Modica [2, 3] which generalized the homogenization of random linear elliptic equations. Here we are interested in the homogenization of energies where non convex lower order terms are allowed. The basis of our approach is to extend, in a way inspired by the abstract method in [4, 5] (itself following a suggestion of Varadhan), the notion of Young measure on micropatterns introduced by Alberti-Müller in [1]. Young measures on micropatterns encode the profiles (or shapes) of an oscillating sequence of functions at a certain predetermined scale.

This is done by addition to these measures the oscillating profiles of a sequence of *functionals*. In the case of a Lagrangian of the type $a_\varepsilon(x)|\nabla u(x)|^2$ this reduces to the profiles of the oscillating function a_ε. However, using the idea from Dal Maso-Modica of metrizing the space of functionals endowed with the topology of Γ-convergence, this can be done in the utmost abstract setting, leading to what we consider to be the natural lower-bound for abstract two-scale Γ-convergence.

Random Homogenization. A possible way to express the fundamental result in linear random homogenization is the following. Consider a random stationary ergodic $n \times n$ matrix $A(\omega, x)$ depending on the random parameter $\omega \in \Omega$ and $x \in G$, where G is a smooth bounded open set in $\mathbb{R}^n$ — stationary ergodic meaning that $A(\omega, x + y) = A(T_y\omega, x)$, where $\{T_y\}_y$ is a group of measure-preserving

transformations on the probability space Ω for which the only invariant sets have measure 0 or 1.

If we assume that $A(\omega, x)$ is uniformly elliptic with respect to ω, x, then there exists a matrix A_h (the h stands for *homogenized*) such that the following holds for a.e. ω:

First, for any sequence $\{\varepsilon\}$ of real numbers tending to 0 and any bounded sequence $\{u_\varepsilon\}$ in $H^1(G)$ converging weakly to u,

$$\liminf_{\varepsilon\to 0} \int_G \langle A(\omega, x/\varepsilon)\nabla u_\varepsilon, \nabla u_\varepsilon\rangle \geq \int_G \langle A_h \nabla u, \nabla u\rangle. \tag{1}$$

Second, given any $u \in H^1(G)$ and any $\{\varepsilon\}$ tending to 0, there exists $\{u_\varepsilon\}$ converging weakly to u such that

$$\limsup_{\varepsilon\to 0} \int_G \langle A(\omega, x/\varepsilon)\nabla u_\varepsilon, \nabla u_\varepsilon\rangle \leq \int_G \langle A_h \nabla u, \nabla u\rangle. \tag{2}$$

Dal Maso and Modica [2, 3] have generalized this to lagrangians of the form $f(\omega, x/\varepsilon, \nabla u_\varepsilon)$ which are convex in the last variable, stationary ergodic, and satisfy suitable growth conditions. They prove that, again, there exists a homogenized lagrangian $f_h(q)$ which is convex and satisfies the same growth conditions.

Young measures on micropatterns. The Young measures on micropatterns introduced in [1] are defined, given a sequence of functions $\{u_\varepsilon\}$ defined on $G \subset \mathbb{R}^n$, as the limit of

$$P_\varepsilon = \big(x \to (x, u_\varepsilon(x + \varepsilon\cdot))\big) \sharp \frac{\mathrm{d}x_{|G}}{|G|},$$

meaning that what is encoded in P_ε is not the *values* $u_\varepsilon(x)$ taken by u_ε but the *profiles* $u_\varepsilon(x + \varepsilon\cdot)$ at the scale ε (other ways of rescaling are allowed). If the scale of oscillations is ε, then the limit of $\{P_\varepsilon\}$ exists and is nontrivial, it is a measure P on $G \times X$, where X is a suitable function space to which u_ε belongs and on which the sequence $\{u_\varepsilon\}$ is bounded.

Let us use these measures for the following toy model. For any sawtooth function u on $\mathbb{R}$ (i.e. piecewise affine and s.t. $u' = \pm 1$) let

$$E_\varepsilon(u) = \varepsilon\|u''\| + \int_0^1 u^2,$$

where $\|u''\|$ is the total variation of the measure u''. Then, given a sequence $\{u_\varepsilon\}$, and letting $v_{\varepsilon,x}(y) = \varepsilon^{-1/3} u_\varepsilon(x + \varepsilon^{1/3} y)$, we may write for small ε, using Fubini's Theorem and a change of variables

$$E_\varepsilon(u_\varepsilon) \approx \int_0^1 \left(\fint_{x-\varepsilon^{1/3}}^{x+\varepsilon^{1/3}} {u_\varepsilon}^2 + \varepsilon |u_\varepsilon''| \right) \mathrm{d}x = \varepsilon^{2/3} \int f(v)\, \mathrm{d}P_\varepsilon(x, v), \tag{3}$$

where P_ε is the push-forward of the Lebesgue measure on $[0,1]$ by $x \to (x, v_{\varepsilon,x})$ and $f(v) = \frac{1}{2}\int_{-1}^1 |v''| + v^2$.

One may then pass to the limit in (3) to deduce that, letting $P := \lim_\varepsilon P_\varepsilon$ and using the translation-invariance of P,

(4)

$$\liminf_\varepsilon \varepsilon^{-2/3} E_\varepsilon(u_\varepsilon) \geq \int f(v)\,\mathrm{d}P(x,v) = \int \left(\limsup_{R\to+\infty} \fint_{-R}^{R} |v''| + v^2 \right) \mathrm{d}P(x,v).$$

This lower-bound is the equivalent of (1) for our toy model, we generalize it in the next section. The equivalent of (2) could be proved as usual by constructing a recovery sequence, this step is more problem-specific and we do not try to cast it in an abstract setting.

Young measures on micropatterns and functionals. We adapt the approach above as follows. First we define an *extended functional* to be a map $f : X \times \mathbb{R}^n \to \mathbb{R}_+$, where X is a space of functions on $\mathbb{R}^n$. One should think of $f(v,y)$ as the local average of a lagrangian of v around the point y. In our toy model, for instance, we would let $f(v,y) = \fint_{y-1}^{y+1} |v''| + v^2$.

Then we consider a family $\{f_\varepsilon^{\omega,x}\}$ depending on the small parameter ε (which represents the scale of oscillations, see below), on a random parameter $\omega \in \Omega$ and on the slow variable $x \in G$, where G is a smooth compact subset of $\mathbb{R}^n$. We assume this family is stationary ergodic, i.e. that there is an ergodic group of measure-preserving transformations $\{T_z\}_{z\in\mathbb{R}^n}$ on Ω such that

$$f_\varepsilon^{T_z\omega,x}(v,y) = f_\varepsilon^{\omega,x}(v(\cdot - z), y + z).$$

Then we let, given a sequence $\{u_\varepsilon\}$ in X,

$$E_\varepsilon^\omega(u_\varepsilon) = \fint_G f_\varepsilon^{\omega,x}(u_\varepsilon, x/\epsilon)\,\mathrm{d}x.$$

We have the following equivalent of (3)

Theorem 1. *Assume suitable compactness, equi-coercivity and regularity assumptions on the stationary ergodic family $\{f_\varepsilon^{\omega,x}\}$, and assume $\{E_\varepsilon^\omega(u_\varepsilon)\}$ is bounded. Then the image P_ε^ω of the normalized Lebesgue measure on G by the map*

$$x \to \left(x, f_\varepsilon^{T_{x/\varepsilon}\omega,x}, u_\varepsilon(x/\varepsilon + \cdot)\right)$$

converges as $\varepsilon \to 0$ to a probability measure P^ω on $G \times X \times \mathcal{F}$, where $\mathcal{F}$ is a compact space of functionals. Moreover,

$$\liminf_{\varepsilon\to 0} E_\varepsilon^\omega(u_\varepsilon) \geq \int \left(\limsup_{R\to+\infty} \fint_{B_R} f(u,y)\,\mathrm{d}y \right) \mathrm{d}P^\omega(x,f,u).$$

Here the Young measure P^ω shows up in the limiting process. An easy corollary of this somewhat mysterious statement is

Theorem 2. *Assuming moreover that $f_\varepsilon^{\omega,x} \to f^{\omega,x}$ as $\varepsilon \to 0$ uniformly with respect to ω, x, we have for almost every ω*

$$\liminf_{\varepsilon\to 0} E_\varepsilon^\omega(u_\varepsilon) \geq \fint_G \min_{u\in X} \left(\limsup_{R\to+\infty} \fint_{B_R} f^{\omega,x}(u,y)\,\mathrm{d}y \right) \mathrm{d}x,$$

and the right-hand side is independent of ω *up to a set of probability* 0.

The above abstract framework applies in particular to a random version of the variational model investigated in [1] (to which our toy model above is closely related) and we believe it can be effective in the homogenization of many variational problems involving singular perturbations.

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