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Mathematical Aspects of Hydrodynamics

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Abstract. The workshop dealt with the partial differential equations that describe fluid motion, namely the Euler equations and the Navier-Stokes equations. This included topics in both inviscid and viscous fluids in two and three dimensions. A number of the talks were connected with issues of turbulence. Some talks addressed aspects of fluid dynamics such as magnetohydrodynamics, quantum and high energy physics , liquid crystals and the particle limit governed by the Boltzmann equations.

Mathematics Subject Classification (2010): 76xx, 35xx.

Introduction by the Organisers

The workshop "Mathematical Aspects of Hydrodynamics" was held at MFO from August 9-15, 2015. The scientific program consisted of 26 main talks of 45 minutes with 15 minutes for discussions. There were 4 poster presentations on Tuesday evening preceded by 5 minute "advertisements". There was ample time at the workshop for general discussions and work in smaller groups.

The emphasis of the meeting was various aspects of incompressible fluid dynamics. This included topics in both inviscid and viscous fluids in two and three dimensions. A number of the talks were connected with issues of turbulence. Some talks addressed aspects of fluid dynamics such as magnetohydrodynamics, quantum and high energy physics , liquid crystals and the particle limit governed by the Boltzmann equations.

There were 40 participants from 16 different countries. There were 6 women participants, including one woman organizer. There were 10 young researchers.

The organizers thank the MFO staff for their great hospitality and support before and during the conference which was very well run.

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Workshop: Mathematical Aspects of Hydrodynamics

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Abstracts

Contrast between Lagrangian and Eulerian regularity properties of Euler equations

Vlad Vicol
(joint work with Peter Constantin, Igor Kukavica)

The Euler equations for ideal incompressible fluids have two formulations, the Eulerian and the Lagrangian one (apparently both due to Euler in 1757. In the Eulerian formulation the unknown functions are velocity and pressure, recorded at fixed locations in space. Their time evolution is determined by equating the rates of change of momenta to the forces applied, which in this case are just internal isotropic forces maintaining the incompressible character of the fluid. In the Lagrangian formulation the main unknowns are the particle paths, the trajectories followed by ideal particles labeled by their initial positions. The Eulerian and Lagrangian formulations are equivalent in a smooth regime in which the velocity is in the Hölder class $\mathcal{C}^s$, where $s > 1$. The particle paths are just the characteristics associated to the Eulerian velocity fields.

In recent years it was proved [1, 5, 8, 9, 10, 7, 4, 3] that the Lagrangian paths are time-analytic, even in the case in which the Eulerian velocities are only $\mathcal{C}^s$, with $s > 1$. In contrast, if we view the Eulerian solution as a function of time with values in $\mathcal{C}^s$, then this function is everywhere discontinuous for generic initial data [2, 6]. This points to a remarkable difference between the Lagrangian and Eulerian behaviors, in the not-too-smooth regime.

In this paper we describe a simple but astonishing difference of behaviors in the analytic regime. First, the radius of analyticity is locally in time conserved in the Lagrangian formulation, i.e., the equations are locally well-posed in spaces with *fixed real-analyticity radius* (more generally, a fixed Gevrey-class radius). In contrast, the analyticity radius may deteriorate instantaneously in the Eulerian formulation, as may be seen by considering a special shear-flow solution. Second, the Lagrangian formulation allows solvability in *highly anisotropic spaces*, e.g. functions which have analyticity (more generally, Gevrey-class regularity) in one variable, but are not analytic in the others. In contrast, the Eulerian formulation is ill-posed in such functions spaces.

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Small scale creation and finite time blow up in fluids

Alexander Kiselev
(joint work with Kyudong Choi, Tom Hou, Guo Luo, Vladimir Sverak, Yao Yao)

The two dimensional Euler equation for the motion of an inviscid, incompressible fluid is given in vorticity form by

$$\partial_t \omega + (u \cdot \nabla)\omega = 0, \quad \omega(x,0) = \omega_0(x). \tag{1}$$

Here ω is the vorticity of the flow, and the fluid velocity u is determined from ω by the appropriate Biot-Savart law. If we consider fluid in a smooth bounded domain D, we impose a no flow condition at the boundary: $u(x,t) \cdot n(x) = 0$ for $x \in \partial D$. This implies that $u(x,t) = \nabla^\perp \int_D G_D(x,y)\omega(y,t)\,dy$, where G_D is the Green's function for the Dirichlet problem in D and $\nabla^\perp = (\partial_{x_2}, -\partial_{x_1})$.

The global regularity of solutions to two-dimensional Euler equation is known since the work of Wolibner [14] and Hölder [6], see also for example [10] for more modern and accessible proofs. The two-dimensional Euler equation is critical in the sense that the estimates needed to obtain global regularity barely close. The best known upper bound on the growth of the gradient of vorticity and higher order Sobolev norms is double exponential in time.

The question of whether such upper bounds are sharp has been open for a long time. Yudovich [15] provided an example showing infinite growth of the vorticity gradient at the boundary of the domain, by constructing an appropriate Lyapunov functional. These results were further improved and generalized in [11], with interesting connection to classical stability questions. Nadirashvili [12] proved a more quantitative linear in time lower bound for a "winding" flow in an annulus. Bahouri and Chemin [1] provided an example of singular stationary solution of the 2D Euler equation which produces a flow map whose Hölder regularity decreases in time. This example also has a fluid velocity which is just log-Lipschitz in the spatial variables, the lack of Lipschitz regularity that is exactly related to the possibility of double exponential growth.

In recent years, there has been a series of works by Denisov on this problem. In [4], he constructed an example with superlinear growth in its vorticity gradient in the periodic case. In [5], he showed that the growth can be double exponential for any given (but finite) period of time. We also refer to a discussion at Terry Tao's blog [13] for more information on the problem and related questions.

The first result I would like to describe in this talk is a joint work with Vladimir Sverak [8]. We were able to construct an example of initial data in the disk such that the corresponding solution for the 2D Euler equation exhibits double exponential growth in the gradient of vorticity. We do not require any force or controlled stirring in the equation. Namely, we will prove

Theorem. [Kiselev-Sverak] *Consider two-dimensional Euler equation on a unit disk D. There exists a smooth initial data ω_0 with $\|\nabla\omega_0\|_{L^\infty}/\|\omega_0\|_{L^\infty} > 1$ such that the corresponding solution $\omega(x,t)$ satisfies*

$$\frac{\|\nabla\omega(x,t)\|_{L^\infty}}{\|\omega_0\|_{L^\infty}} \geq \left(\frac{\|\nabla\omega_0\|_{L^\infty}}{\|\omega_0\|_{L^\infty}}\right)^{c\exp(c\|\omega_0\|_{L^\infty}t)} \tag{2}$$

for some $c > 0$ and for all $t \geq 0$.

The theorem shows that the double exponential upper bound is in general optimal for the growth of vorticity gradient of solutions to the two-dimensional Euler equation. The growth in our example happens at the boundary. We do not know if such growth is possible in the bulk of the fluid.

The motivation for our work came from numerical simulations of Luo and Hou [7], which suggest a new scenario for singularity formation in solutions of the 3D Euler equation. The question whether solutions to 3D Euler, or 3D Navier-Stokes equations are globally regular or can form singularities in finite time is one of the major open questions of modern applied analysis. The Hou-Luo scenario is axi-symmetric, and the fast vorticity growth occurs near a hyperbolic point of the flow at boundary of the domain. The scenario is effectively two-dimensional, and is well modelled by 2D inviscid Boussinesq equation (see [9] for s description of the link between axi-symmetric 3D Euler and 2D Boussinesq):

$$\begin{aligned} \partial_t\omega + (u\cdot\nabla)\omega &= \partial_{x_1}\rho \\ \partial_t\rho + (u\cdot\nabla)\rho &= 0, \end{aligned} \tag{3}$$

along with the usual 2D Bio-Savart law $u = \nabla^\perp(-\Delta_D)^{-1}\omega$, where $\nabla^\perp = (\partial_2, -\partial_1)$ and $-\Delta_D$ is the Dirichlet Laplacian. In the second part of my talk, I would like to describe the attempt to use insight gained in [8] to better understand (3). Handling 2D geometry in the presence of nonlinear coupling as in (3) is challenging, so two 1D models have been developed to help better understand the Hou-Luo scenario. They are both given by

$$\begin{aligned} \partial_t\omega + u\partial_x\omega &= \partial_x\rho \\ \partial_t\rho + u\partial_x\rho &= 0. \end{aligned} \tag{4}$$

But for one model, studied by Choi, Yao and myself, the Biot-Savart law is given by $u(x) = -x\int_x^1 \frac{\omega(y)}{y}\,dy$, while for another, proposed by Hou and Luo, $u_x = H\omega$ where H denotes the Hilbert transform, $Hf(x) = \frac{1}{\pi}P.V.\int_R \frac{f(y)}{x-y}\,dy$. The Hou-Luo

(HL) model is derived from (3) under assumption that the vorticity is concentrated in a boundary layer near the $x_2 = 0$ axis, and is independent of x_2. The natural setting for it is periodic. The CKY model is simpler, as its Biot-Svart law is in some sense "less nonlocal" than for the HL Model. The form of its Biot-Savart law is motivated by the estimates in [8]. It is set on a finite interval (say $[0, 1]$) with compactly supported initial data. Recently, we were able to prove the following results.

Theorem. [Choi-Kiselev-Yao] *The CKY model is locally well-posed in sufficiently high order Sobolev spaces, that is for $(\omega_0, \rho_0) \in (H_0^m, H_0^{m+1})$ with m sufficiently large.*

There exist smooth initial data for which the solution develops a finite time singularity. In particular, $\int_0^T \|\omega(\cdot,t)\|_{L^\infty} \to \infty$ as $t \to T$ for some $T < \infty$.

Theorem. [Choi-Hou-Kiselev-Luo-Sverak-Yao] *The HL model is locally well-posed in sufficiently high order Sobolev spaces, that is for $(\omega_0, \rho_0) \in (H^m, H^{m+1})$ with m sufficiently large.*

There exist smooth initial data for which the solution develops a finite time singularity. In particular, $\int_0^T \|u_x(\cdot,t)\|_{L^\infty} \to \infty$ as $t \to T$ for some $T < \infty$.

The proof of the first theorem is based on a fairly hands-on argument tracing the characteristics of fluid particles and reducing effectively to a differential inequality of the type $F'' \geq e^F$ [2]. The proof of the second theorem is more subtle and is based on estimating appropriate Lyapunov-type functional [3]. In all three results, a key role is played by certain hidden monotone and/or sign-definite quantities and expressions, which allow to maintain control and stability of the growth or blow up constructions.

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Weak notions of solution for inviscid fluids

Emil Wiedemann

The use of weak concepts of solution for partial differential equations of fluid mechanics appears necessary for a variety of reasons: First, certain particular types of flows, like shear flows or, in the compressible case, flows forming shock waves, are inherently discontinuous; second, expected effects of turbulent flows, like anomalous dissipation, necessarily require a certain degree of irregularity; and third, from a mathematical perspective no satisfactory well-posedness theory is available for the compressible or incompressible Euler equations or related models. Therefore several different concepts of weak solution have been proposed in order to overcome various difficulties in the analysis of inviscid fluids.

In the sequel I give a very brief description of these notions and sketch the considerable recent progress in understanding the relations between them. Nevertheless the picture is far from complete, especially in the compressible case.

Incompressible Euler equations. Consider the incompressible Euler equations,

$$\partial_t v + (v \cdot \nabla)v + \nabla p = 0, \qquad \operatorname{div} v = 0. \tag{1}$$

The most standard way to define a weak solution is in the sense of distributions, so that for the velocity field only $v \in L^2_{loc}$ is required (a more natural choice is the energy space $L^\infty_t L^2_x$).

Since not even the existence of distributional solutions from arbitrary data with finite energy was known until very recently [12], one had to weaken the notion of solution (although the existence issue was not the only reason for this). Indeed, the most straightforward attempt to construct solutions is to consider a sequence of approximate solutions, like Leray-Hopf solutions to the Navier-Stokes equations with vanishing viscosity, and pass to the weak limit. Of course, due to the nonlinearity of the equations and the possibility of oscillation and concentration formation in the approximate sequence, it is not obvious (and in fact an outstanding open question) whether the weak limit solves Euler in the sense of distributions.

DiPerna and Majda [6] used and generalised the notion of *Young measures* to overcome this difficulty. Such a (generalised) Young measure can be thought of as giving, at each point in time and space, a probability distribution for the velocity, rather than a deterministic velocity. The loss of information that comes with this relaxation leads in particular to a high degree of non-uniqueness, as the Euler equations restrict only the first two moments of the measure.

P.-L. Lions [9] therefore introduced his *dissipative solutions*. He suggested that any reasonable concept of solution should meet the requirements of global existence and *weak-strong uniqueness*: If there exists a smooth solution, then every weak solution with the same initial data has to coincide with it. The definition of dissipative solutions immediately ensures existence and weak-strong uniqueness. Moreover, dissipative solutions have an energy that never exceeds the initial energy,

$$\int_\Omega |v(x,t)|^2 dx \le \int_\Omega |v_0(x)|^2 dx \qquad \text{for all } t > 0. \tag{2}$$

The latter observation turned out to be crucial for weak-strong uniqueness in other contexts. Following [5], we call a vector field *admissible* if it satisfies (2). The following theorem is a compilation of various results from the previously cited works and [5, 2, 1, 11].

Theorem 1. (1) *Global existence for arbitrary initial data with finite energy on, say, $\mathbb{T}^d$ ($d \ge 2$) holds for distributional, admissible measure-valued, and dissipative solutions, but is unknown for admissible distributional solutions.*

(2) *Weak-strong uniqueness holds for admissible weak, dissipative, and admissible measure-valued solutions on $\mathbb{R}^d$ or $\mathbb{T}^d$, but not on domains with boundaries.*

(3) *Uniqueness (in absence of a smooth solution) holds for neither of these concepts.*

(4) *Every (admissible) measure-valued solution can be approximated in a suitable sense by a sequence of (admissible) distributional solutions.*

Compressible Euler equations. Consider now the *isentropic Euler equations*,

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla p(\rho) = 0, \qquad \partial_t \rho + \operatorname{div}(\rho v) = 0. \tag{3}$$

The concepts of weak solution discussed above can be adjusted to the isentropic Euler system as well (although I am not aware of a definition of dissipative solutions for compressible models in the literature). The analogue of the admissibility criterion (2) is a *global* energy inequality as opposed to a *local* energy condition. The latter is an instance of an entropy inequality commonly used for conservation laws. It can also be formulated for the incompressible case [5].

One of the main differences between (1) and (3) is that the former system features the pressure merely as a Lagrange multiplier, whereas in the latter system it is constitutively given as a function of the density. This more "rigid" role of the pressure in compressible theories makes it much harder to construct solutions via the convex integration method of De Lellis–Székelyhidi, as for instance in [4, 5, 12, 11]. Nevertheless, we have [10, 3, 8]

Theorem 2. (1) *There exist admissible measure-valued solutions for* (3) *for any initial data with finite energy.*
(2) *Even under the local entropy condition, distributional solutions of* (3) *can be non-unique.*
(3) *Weak-strong uniqueness holds in the class of measure-valued solutions that satisfy the global energy inequality.*

Finally, let us remark that similar statements are true for the *Savage-Hutter system of granular flow*, which describes the motion of avalanches:

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla(\rho^2) = -\rho B(v), \qquad \partial_t \rho + \operatorname{div}(\rho v) = 0, \tag{4}$$

where $B(v)$ is a maximal monotone set-valued map. As shown in [8], the friction term $-\rho B(v)$ is strong enough to guarantee complete dissipation of momentum in finite time for *admissible* measure-valued solutions. Yet, remarkably, it is possible to construct by convex integration non-admissible distributional solutions which do not dissipate momentum [7].

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The equations of MHD with zero magnetic resistivity

James C. Robinson

(joint work with Jean-Yves Chemin, Charles L. Fefferman, David S. McCormick, Jose L. Rodrigo)

In 1985 Moffatt proposed a scheme to generate stationary solutions of the Euler equations with non-trivial topology, by means of 'magnetic relaxation'. This uses the equations of magnetodhydrodynamics (MHD)

$$\partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = (B\cdot\nabla)B \qquad \nabla\cdot u = 0$$
$$\partial_t B - \varepsilon\Delta B + (u\cdot\nabla)B = (B\cdot\nabla)u \qquad \nabla\cdot B = 0$$

with $\varepsilon = 0$, i.e. with zero magnetic resistivity.

In this case the energy equation (obtained by taking the inner product of the first equation with u, the second with B, and adding) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2 + \|B\|^2\right) + \|\nabla u\|^2 = 0.$$

In particular, $\|u\|^2 + \|B\|^2$ decreases while $\nabla u \neq 0$. Given a domain in which a Poincaré inequality holds for u, the heuristic argument is that this must imply that u tends to zero as $t \to \infty$. If a Poincaré inequality also holds for B then one can use the fact that the 'magnetic helicity', defined as $\int A\cdot B$, where $B = \nabla\times A$, is conserved to show that $\|B\|$ is bounded below. Assuming, therefore, that $B(t)$ converges to a non-zero limit, one should be able to set $u = 0$ in the u equation to deduce that $(B\cdot\nabla)B = \nabla p$: up to a change of sign for p, B is a stationary solution of the Euler equations.

There is much to do to place this argument on a rigorous footing. Núñez (2007) assumed the global uniqueness of smooth solutions with $\|B\|_\infty$ globally bounded; in this case (by taking the inner product of the u equation with $\mathrm{e}^{rt}u$ for some suitably chosen r) he was able to show that $u(t) \to 0$ as $t \to \infty$; but the existence of a unique limit for $B(t)$ as $t \to \infty$ requires further conditions not known to hold (e.g. $u \in L^1(0,\infty;L^1)$). [His argument requires boundary conditions to be chosen 'appropriately'; periodic with zero average for u and B is sufficient, for example.]

Motivated by this theory, this talk discussed local existence and uniqueness for this canonical model. First, in the scale of Sobolev spaces, it is possible to obtain local existence for $u_0, B_0 \in H^s(\mathbb{R}^3)$ when $s > 3/2$ (Fefferman et al., 2014). Using the fact that H^s is an algebra when $s > 3/2$, energy estimates on the u and B equations yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^s}^2 + \|u\|_{H^{s+1}}^2 \le c\|u\|_{H^s}^2\|\nabla u\|_{H^s} + \|B\|_{H^s}^2\|\nabla u\|_{H^s}$$
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|B\|_{H^s}^2 \le \langle (u\cdot\nabla)B, B\rangle_{H^s} + c\|\nabla u\|_{H^s}\|B\|_{H^s}^2.$$

In order to deal with the first term on the right-hand side of the B equation we proved a variant of the classical Kato–Ponce commutator estimates:

$$\|\Lambda^s[(u\cdot\nabla)B] - (u\cdot\nabla)(\Lambda^s B)\|_{L^2} \le c\|\nabla u\|_{H^s}\|B\|_{H^s}, \qquad s > 3/2,$$

where $(\Lambda^s f)\hat{}(\xi) = |\xi|^s \hat{f}(\xi)$. Given this estimate the required a priori estimates follow easily; a rigorous local existence result requires some standard approximation argument (e.g. Fourier truncation).

To deal with the case $s = 3/2$ we switched from Sobolev to Besov spaces, obtaining local existence for $u_0 \in B^{1/2}_{2,1}$ and $B_0 \in B^{3/2}_{2,1}$. To define these spaces, we denote by $\Delta_k u$ the localisation of u in Fourier space in an annulus around wavenumbers 2^k, and let $B^s_{2,1}$ be the collection of all $u \in \mathcal{S}'$ such that

$$\|u\|_{B^s_{2,1}} := \sum_k 2^{ks} \|\Delta_k u\|_{L^2} \qquad \text{is finite.}$$

The advantage here is the extra regularity gained in terms of time integrability of 'two more derivatives'. Even if one considers only the heat equation $u_t - \Delta u = 0$, if $u_0 \in H^s$ then $u \notin L^1(0,T;H^{s+2})$ in general; but if $u_0 \in B^s_{2,1}$ then we gain some temporal integrability and can show that $u \in L^1(0,T;B^{s+2}_{2,1})$. To see this, apply the Littlewood–Paley operator Δ_k to the equation and take the inner product with $2^{k/2}\Delta_k v$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} 2^{k/2}\|\Delta_k v\| + c 2^{5k/2}\|\Delta_k v\| \le 0.$$

Integrating from 0 to t it follows that

$$2^{k/2}\|\Delta_k v(t)\| + c\int_0^T 2^{5k/2}\|\Delta_k v(\tau)\|\,\mathrm{d}\tau \le 2^{k/2}\|\Delta_k v(0)\|.$$

We can now sum in k to deduce that

$$\|v(t)\|_{B^{1/2}_{2,1}} + c\int_0^T \|v(\tau)\|_{B^{5/2}_{2,1}}\,\mathrm{d}\tau \le \|v_0\|_{B^{1/2}_{2,1}}.$$

Using this observation it is possible to combine Littlewood–Paley type estimates for the equations with conventional Navier–Stokes estimates for the solutions of the u equation with the term $(B\cdot\nabla)B$ considered as a forcing to prove a priori estimates on the solutions. Turning this into a rigorous proof of local existence requires some care given the coupled nature of the equations; the details are given in Chemin et al. (2015).

Given that even such limited local existence results are delicate, using the 'true' equations of MHD to construct stationary Euler solutions may not be the right way to proceed. Recent work of Brenier (2014) replaces the u equation with $u = \mathbb{P}[(B\cdot\nabla)B]$, where $\mathbb{P}$ is the Leray projector onto divergence-free fields; other more *ad hoc* choices may prove more tractable and provide an alternative to the recent more geometric methods of Enciso & Peralta-Salas (2012).

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Dynamics near the subcritical transition of the 3D Couette flow

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(joint work with Pierre Germain, Nader Masmoudi)

We study the 3D Navier-Stokes equations near the Couette flow in the domain $(x, z) \in \mathbb{T}^2$ and $y \in \mathbb{R}$ where $\nu = \mathbf{Re}^{-1}$ denotes the inverse Reynolds number.

Understanding the stability of laminar flows and the transition to turbulence is one of the main objectives of hydrodynamic stability theory (see e.g. the texts [1, 2, 3] and the references therein). One of the first and most influential experiments in the field were those of Reynolds in 1883 [9], which demonstrated the instability and spontaneous transition to turbulence of laminar flow in a pipe for sufficiently high Reynolds number (in fact, this work is the origin of the name *Reynolds number*). Since the work of Reynolds, many other influential experiments and computer simulations have shown that essentially all laminar 3D flows are unstable at sufficiently high Reynolds number; see [4] and the above texts for an extensive list of references. However, such instabilities appear inconsistent with theoretical studies, as many of these flows are spectrally stable at all Reynolds number and this spectral stability can indeed translate to nonlinear asymptotic stability, as has been shown in some cases (see [4] for references). Even for those flows which do have spectral instabilities at high Reynolds number, the instabilities observed appear at much lower Reynolds number and are different than what linear theory predicts. This behavior is ubiquitous in 3D hydrodynamics and is often referred to as *subcritical transition* or *by-pass transition* in the fluid mechanics literature.

It is natural to suggest that while the flow is technically stable for all finite Reynolds number, the set of stable perturbations shrinks as the Reynolds number increases, leading to transition in any real system at some finite Reynolds number (this suggestion goes back to Lord Kelvin [8], or arguably Reynolds [9]). A reasonable goal could be, given a norm N, find a $\gamma = \gamma(N)$ such that (of course we do not know a priori that the threshold needs to be a power law):

$$\epsilon := \|u_{in}\|_N \leq c\nu^\gamma \quad \Rightarrow \quad \text{no transition}$$
$$\epsilon := \|u_{in}\|_N \geq C\nu^\gamma \quad \Rightarrow \quad \text{possible transition.}$$

The power γ is called the *transition threshold.* It is also of practical interest to determine what kinds of instabilities are possible. For contrast, we emphasize that for sufficiently regular perturbations, the 2D Couette flow does *not undergo subcritical transition*, and instead is nonlinearly, asymptotically stable (in a suitable sense) uniformly at high Reynolds number [7] and also infinite Reynolds number [6].

A great deal of effort has been spent on trying to determine the transition threshold and the nature of the instabilities for simple laminar flows (see e.g. the texts [2, 3] and the references therein). The work of Trefethen et. al. [10] forwarded the idea that the nonlinearity could interact poorly with the non-normal behavior; the authors discussed a low-dimensional toy model meant to capture certain aspects of this idea and used it to conjecture $\gamma > 1$. A number of works used variations of this idea to understand the threshold via combinations of simplified ODE models, asymptotic analysis, and computation (see [4, 2] for references). Various predictions have been made, ranging generally from $1 \leq \gamma \leq 7/4$; for the infinite channel, the mathematically rigorous bound $\gamma \leq 4$ is known [11]. We also would like to emphasize that not all of these works consider exactly the same problem. For example, some consider boundaries in y, others do not, and some consider a domain which is unbounded in x and others take periodic conditions in x. Both could potentially alter the answers.

In [4], we prove that there exists a universal constant $c_0 > 0$ such that if the initial data is of size $\epsilon < c_0\nu$ (in a sufficiently regular sense), then the solution is global in time and converges back to the Couette flow as $t \to \infty$. Further, we demonstrate that perturbations which are $O(\epsilon)$ initially can grow to be as large as $O(c_0)$ before eventually decaying due to the lift-up effect (a 3D non-normal effect [12]). Note that the supremum in time of these solutions remains $O(c_0)$ uniformly as $\nu \to 0$. Hence, for sufficiently regular pertubations, we are essentially proving that $\gamma = 1$. That we can still obtain global solutions despite of this large growth depends crucially on the stabilizing effects of the mixing combined with a detailed weakly nonlinear study. Due to this mixing, the x-dependence of the solution is damped for $t \gg \nu^{-1/3}$ and all solutions converge to the set of x-independent solutions, referred to as "streaks". Furthermore, due to the mixing and vortex stretching, the solutions can also exhibit a direct cascade of energy. To our knowledge, this kind of behavior in the 3D Navier-Stokes equations has not previously been confirmed in a mathematically rigorous setting. The class of initial data we consider is the sum of a sufficiently smooth function (Gevrey-$\frac{1}{s}$ for $s > 1/2$) and a much smaller (relative to ν) H^3 function.

In [5], our goal was to characterize possible instabilities. We prove that there is a universal constant c_0 with $0 < c_0 \ll 1$ such that for sufficiently regular initial data (in the same sense as [4]) of size ϵ, if $\epsilon \leq \nu^{2/3+\delta}$ for $\delta > 0$, then the solution exists until at least time $t = c_0\epsilon^{-1}$ and is rapidly attracted to the class of x-independent solutions known as *streaks* for times $t \gg \nu^{-1/3}$. Due to the lift-up effect, the streaks (and hence all solutions) will in general grow linearly as $O(\epsilon t)$ and by the final time can be $O(c_0)$ (which is independent of ν). While our previous analysis in [4] did include solutions which get $O(c_0)$ from the Couette flow, all solutions never deviate farther from the Couette flow and are demonstrably not involved in any transition processes. To contrast, the streaks in [5] are expected to trigger a secondary instability known as streak breakdown, which is well-documented as one of the primary routes towards turbulent transition observed experimentally (see [5] for references). While we cannot take our solutions through the secondary instability, we prove that solutions above the threshold (but not too far above) can in general converge to unstable streaks and that this is the only instability possible, which is suggestive of the genericity (for *sufficiently smooth* perturbations) of the multi-step "lift-up effect ⇒ streak growth ⇒ streak breakdown ⇒ transition" process forwarded in the applied mathematics and physics literature.

In [4, 5], the stability mechanisms which make our results possible by suppressing the plethora of fully 3D nonlinear effects are the *mixing-enhanced dissipation* and *inviscid damping*. Both effects arise from the mixing effect of the background Couette flow. The primary stability mechanism is enhanced dissipation; it was first derived in the context of Couette flow by Lord Kelvin [8] (at least in 2D) and has been subsequently observed or studied by numerous authors in fluid mechanics in various settings – see [4, 7] for references. The general intuition is that as information is mixed to smaller scales, the effectiveness of the viscosity is greatly enhanced in streamwise dependent modes. Here it will imply that the x dependence of the perturbation is damped out for $t \gg \nu^{-1/3}$, Inviscid damping was first derived on the linear level by Orr [13] in 1907 and was later noticed to be the hydrodynamic analogue of Landau damping in plasma physics (see [6] for references and more information on the connection). In 2D, the effect leads to the asymptotic stability (in the correct sense) of the 2D Couette flow even with no viscosity at all [6]. However in 3D, due to the vortex stretching, it only results in the decay of u^2, the second component of the velocity. Instead of providing the main stability mechanism, due to the special structure of the nonlinearity in the Navier-Stokes equations, it is key for suppressing some leading order nonlinear effects that would otherwise destroy any hope of estimates.

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On axisymmetric vortex rings

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(joint work with Vladimír Šverák)

A *vortex ring* is a flow in which the vorticity is essentially concentrated in a solid torus, so that the fluid particles spin around an imaginary line that forms a closed loop. Such flows are ubiquitous in nature, and appear to be very stable. For the incompressible Euler equations with rotational symmetry, uniformly translating vortex rings can be constructed by variational techniques, see e.g. [1]. Using the conserved quantities of the system, it is also possible to show that, for general initial data with sufficiently concentrated vorticity, the vortex ring structure is preserved over a finite time interval, see [2]. In viscous fluids, uniformly translating vortex rings do not exist, since all vortical structures are eventually destroyed by diffusion. In that case it is natural to consider the Cauchy problem for the axisymmetric Navier-Stokes equations and to assume that the initial vorticity is a *vortex filament*, namely a divergence-free vector measure supported by a closed circle. General well-posedness results for the Navier-Stokes equations in critical function spaces guarantee that such a solution exists and is unique if the circulation Reynolds number (namely, the ratio of the filament's circulation to the kinematic viscosity) is sufficiently small, see e.g. [5]. Using more specific techniques, existence of vortex rings with arbitrary large circulation has been established recently [3], and uniqueness is the subject of the present talk.

We consider the three-dimensional incompressible Navier-Stokes equations, and we restrict our attention to *axisymmetric solutions without swirl*. In cylindrical coordinates (r, θ, z), the velocity field takes the form

$$u(x,t) \,=\, u_r(r,z,t)e_r + u_z(r,z,t)e_z \ ,$$

where e_r, e_θ, e_z denote the unit vectors in the radial, toroidal, and vertical directions, respectively. For such flows, the vorticity $\omega = \operatorname{curl} u$ has only one nonzero component:

$$\omega(x,t) \,=\, \omega_\theta(r,z,t)e_\theta \ .$$

The evolution equation for ω_θ reads

$$(1) \qquad \begin{cases} \partial_t\omega_\theta + u\cdot\nabla\omega_\theta - \dfrac{u_r}{r}\omega_\theta \,=\, \nu\Big(\Delta\omega_\theta - \dfrac{\omega_\theta}{r^2}\Big) \ , \\ \operatorname{div} u \,=\, 0 \ , \quad \operatorname{curl} u = \omega_\theta \ , \end{cases}$$

where $u\cdot\nabla = u_r\partial_r + u_z\partial_z$, $\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2$, $\operatorname{div} u = \partial_r u_r + \frac{1}{r}u_r + \partial_z u_z$, and $\operatorname{curl} u = \partial_z u_r - \partial_r u_z$. In (1), the parameter $\nu > 0$ denotes the kinematic viscosity of the fluid. Equation (1) is posed in the half-plane $\Omega = \{(r,z)\,|\,r > 0,\, z \in \mathbb{R}\}$, with homogeneous Dirichlet condition at the boundary $r = 0$.

We consider solutions of (1) with *vortex filaments* as initial data. More precisely, we assume that

$$\omega_\theta\big|_{t=0} \equiv \omega_\theta^0 = \Gamma\,\delta_{(R,0)}\ , \tag{2}$$

where $\Gamma > 0$ is a parameter (the circulation of the filament) and $\delta_{(R,0)}$ denotes the Dirac mass located at the point $(R,0) \in \Omega$. In more geometric terms, the initial vorticity $\omega^0 = \omega_\theta^0 e_\theta$ is a *1-current* of intensity Γ supported by the circle $\mathcal{C} = \{(R\cos\theta, R\sin\theta, 0)\,|\,0 \le \theta \le 2\pi\}$. This means that, if $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ is any continuous (vector-valued) function, we have the formula

$$\langle \omega^0, \phi\rangle = \Gamma \int_{\mathcal{C}} \phi(x)\cdot \mathrm{d}x\ .$$

Let u^0 be the velocity field associated with ω^0 via the Biot-Savart formula in $\mathbb{R}^3$. It is easily verified that $u^0(x)$ diverges like $\mathrm{dist}(x,\mathcal{C})^{-1}$ as $\mathrm{dist}(x,\mathcal{C}) \to 0$, hence $u^0 \notin L^2(\mathbb{R}^3)$. One can also check that u^0 does not belong to the critical Besov space $\dot{B}^{-1+3/p}_{p,q}(\mathbb{R}^3)$ unless $p = q = \infty$. However u^0 does belong to the space $BMO^{-1}(\mathbb{R}^3)$ defined in [6], as well as to the critical Morrey spaces used in [4, 7]. Thus, applying the results established in [4, 6, 7], one concludes that equation (1) with initial data (2) has a unique global solution (in some appropriate class) provided the circulation satisfies the following smallness condition

$$\Gamma \le C\nu\ , \tag{3}$$

where $C > 0$ is a universal constant. Unfortunately, condition (3) is quite restrictive: it means that the vortex ring will be destroyed by diffusion before having time to travel a distance comparable to its radius R.

Existence of vortex rings of arbitrarily large circulation was recently established by H. Feng and V. Šverák:

Theorem 1 [3]**.** *Fix $\Gamma > 0$, $R > 0$, and $\nu > 0$. Then the axisymmetric vorticity equation* (1) *has a nonnegative global solution such that $\omega_\theta(t) \rightharpoonup \omega_\theta^0$ as $t \to 0$. Moreover, this solution satisfies, for all $t > 0$,*

$$\int_0^\infty\!\!\int_{\mathbb{R}} \omega_\theta(r,z,t)\,\mathrm{d}z\,\mathrm{d}r \le \Gamma\ , \qquad \int_0^\infty\!\!\int_{\mathbb{R}} r^2\omega_\theta(r,z,t)\,\mathrm{d}z\,\mathrm{d}r \le \Gamma R^2\ .$$

Theorem 1 is proved by an approximation procedure. Instead of a vortex filament, the authors consider smooth initial data for which the vorticity $\omega_\theta \ge 0$ is concentrated in a disk of radius $\epsilon > 0$ centered at $(R,0)$ and has circulation equal to Γ. By classical results [8], these data give rise to a global smooth solution of (1), which satisfies nice a priori estimates that are independent of the regularization parameter $\epsilon > 0$. A solution of the original problem can then be obtained by taking the limit $\epsilon \to 0$.

Albeit nice, Theorem 1 is still unsatisfactory, because there is no evidence that the solution constructed by the approximation procedure is unique. A fortiori, this result gives very little information on the qualitative behavior of the solution, especially for small times. Addressing some of these issues is the purpose of our main result:

Theorem 2. *Fix $\Gamma > 0$, $R > 0$, and $\nu > 0$. Then the axisymmetric vorticity equation* (1) *has a unique global solution ω_θ such that*

i) $\sup_{t>0}\Big(\|\omega_\theta(t)\|_{L^1(\Omega)} + \|r^2\omega_\theta(t)\|_{L^1(\Omega)} + t\|\omega_\theta(t)/r\|_{L^\infty(\Omega)}\Big) < \infty$;

ii) $\omega_\theta(t) \rightharpoonup \omega_\theta^0$ *as* $t \to 0+$, *where* ω_θ^0 *is given by (2).*

Moreover we have the estimate

$$\int_\Omega\Big|\omega_\theta(r,z,t) - \frac{\Gamma}{4\pi\nu t}\,e^{-\frac{(r-R)^2+z^2}{4\nu t}}\Big|\,\mathrm{d}r\,\mathrm{d}z \,\le\, C\,\Gamma\,\frac{\sqrt{\nu t}}{R}\,\log\frac{R}{\sqrt{\nu t}}\,, \tag{4}$$

as long as $\sqrt{\nu t} \ll R$.

In this statement, it is understood that the half-plane Ω is equipped with the 2D measure $\mathrm{d}r\,\mathrm{d}z$. The existence part in Theorem 2 is not new, because it can be checked that the solutions constructed in Theorem 1 satisfy the bounds i). The novelty is therefore the uniqueness claim, which requires a completely different approach. In this respect, we believe that Theorem 2 is not optimal, and that the assumption on the norm $\|\omega_\theta(t)/r\|_{L^\infty(\Omega)}$ could be dropped in condition i), but that question remains to be clarified.

Estimate (4) gives precise information on the vortex ring as $t \to 0$, for a fixed viscosity $\nu > 0$: it shows that $\omega_\theta(t)$ behaves to leading order like the solution of the two-dimensional heat equation in Ω with the same initial data (2). Unfortunately, Theorem 2 cannot be used to control the solution at fixed time in the vanishing viscosity limit, because the constant C in (4) depends on the ratio Γ/ν. We expect that an estimate like (4) should remain true in the vanishing viscosity limit if the Gaussian function is centered at the moving point $(R, Z(t))$, where

$$Z(t) \,=\, \frac{\Gamma t}{4\pi R}\Big(\log\frac{R}{\sqrt{\nu t}} + \mathcal{O}(1)\Big)\,,$$

but proving that is yet another story.

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Critical regularity for energy conservation in solutions of 2D Euler

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(joint work with A. Cheskidov, M.C. Lopes Filho, R. Shvydkoy)

The incompressible Euler equations are given by:

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p, \\ \operatorname{div} u = 0. \end{cases} \tag{1}$$

This system is a model for hydrodynamic fluid flow. For simplicity we assume here that the fluid domain is a box and u satisfies periodic boundary conditions.

For smooth solutions we find, multiplying by u and integrating over $\mathbb{T}^2$:

$$\frac{d}{dt}\int_{\mathbb{T}^2} \frac{|u|^2}{2}\,dx = -\int_{\mathbb{T}^2} u \cdot [(u\cdot\nabla)u]\,dx - \int_{\mathbb{T}^2} u \cdot \nabla p\,dx = 0,$$

so that the kinetic energy $\|u(t)\|_{L^2}^2$ is a conserved quantity. The question we are concerned with is how regular does the solution u need to be for this calculation to hold true.

This is the subject of the Onsager conjecture, formulated by Onsager in 1949, see [7]. Loosely stated, u must be "more than $\frac{1}{3}$ differentiable" in order for energy to be conserved. In addition, Onsager conjectured that, if u was "less than or exactly $\frac{1}{3}$ differentiable" then there should exist "solutions" (in a weak sense) of the Euler equations which do not conserve energy. This conjecture has attracted much attention recently, with considerable progress in the field. Following the landmark results [5], and [2], the current state-of-the-art, with respect to conservation of energy, was obtained by Cheskidov, Constantin, Friedlander and Shvydkoy in [1], where it was shown that a weak solution u belonging to $C^0_w([0,T];L^2) \cap L^3((0,T);B^{1/3}_{3,c_0})$ conserves energy. The space $B^{1/3}_{3,c_0}$ is a Besov space; it is defined using the Littlewood-Paley components of u, whose definition we briefly recall.

Choose a nonnegative, radial function $\chi \in C^\infty_c(B(0;1))$, such that $\chi(\xi) \equiv 1$ if $|\xi| \le 1/2$. Set $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$. If $\widehat{u}(\xi)$ represents the Fourier transform of u, while $(u)^\vee$ is the inverse Fourier transform of u, then the q-th Littlewood-Paley component u_q is given by $u_q = (\varphi(2^{-q}\cdot)\widehat{u})^\vee$. The Besov space $B^{1/3}_{3,c_0}$ is defined as those distributions for which $\limsup_{q\to\infty} 2^q \|u_q\|^3_{L^3} = 0$.

The main regularity restriction towards energy conservation seems to be the vanishing of the energy flux, namely that $\int u \cdot (u \cdot \nabla u) = 0$. For this to hold one needs to have, in particular, that $u \cdot [(u \cdot \nabla)u] \in L^1$. In two space dimensions this condition is guaranteed by $\omega \equiv \operatorname{curl} u \in L^{3/2}$, as this in turn implies $u \in$

$W^{1,3/2} \subset L^6$, and the product of two functions in L^6 with a function in $L^{3/2}$ is integrable. We note that $W^{1,3/2} \subset W^{1/3,3} \subset B^{1/3}_{3,c_0}$ so that, in view of the result in [1], weak solutions u of the 2D incompressible Euler equations which belong to $C^0_w([0,T]; L^2) \cap L^3((0,T); W^{1,3/2})$ conserve energy. A weaker result, of similar nature, can be found in [4].

The spaces $W^{1,3/2}$ and $B^{1/3}_{3,c_0}$ are, in a certain sense, in the same *scale* of function spaces, the so-called *critical spaces* as introduced in [8]. If we fix dimensional units for velocity $[u] = U$, length $[x] = L$ and time $[t] = T$, then a (space-time) function space X is said to be critical if the unit for its norm is given by $[\|\cdot\|_X]^3 = TU^3L^{N-1}$, where N is the dimension of physical space ($N = 2$, 3). The motivation for this notion of criticality is to provide a condition for an estimate of the sort

$$\left| \int_0^T \int_{\mathbb{T}^N} u \cdot [(u \cdot \nabla)u] \, dxdt \right| \leq C\|u\|_X^3$$

to hold true with a nondimensional constant $C > 0$.

The discussion above is strongly suggestive that one should not seek conservation of energy in supercritical spaces. We drive this point deeper by exhibiting a kinematic example in a supercritical space for which a version of the energy flux does not vanish. To do so, we first introduce the upper energy flux, as follows. If u_q represents the Littlewood-Paley component of a distribution u, then $(u)_{<q} := \sum_{r=-1}^{q-1} u_r$. Now, the energy flux, if it is defined, is given by

$$\Pi = \Pi[u] = \int u \cdot [(u \cdot \nabla)u].$$

The upper energy flux is defined as $\limsup_{q\to\infty} \Pi_q$, where

$$\Pi_q = \Pi_q[u] = \int (u)_{<q} \cdot [(u \cdot \nabla)u]_{<q}.$$

Whenever Π is defined we have $\Pi = \limsup_{q\to\infty} \Pi_q$.

Theorem 1. *There exists a divergence-free vector field* $u \in L^2(\mathbb{T}^2)$ *such that its vorticity* $\operatorname{curl} u \in L^p(\mathbb{T}^2)$ *for every* $1 \leq p < 3/2$*, but* $\operatorname{curl} u \notin L^{3/2}(\mathbb{T}^2)$*, and such that*

$$\limsup_{q\to\infty} \Pi_q \neq 0.$$

This discussion is at odds with a consequence of the work done by R. DiPerna and A. Majda in the late '1980s, see for example [3], on the evolution of vortex sheets. Among their results DiPerna and Majda established the existence of weak solutions of the two-dimensional Euler equations with initial vorticity in L^p, $p > 1$, by proving that sequences of approximations u^n were compact in the strong topology of $C^0([0,T]; L^2)$. A corollary of this result is that, if u^n is an exact solution of the Euler equations with smooth initial data which approximates, in L^2, an initial velocity u_0 satsifying $\omega_0 = \operatorname{curl} u_0 \in L^p(\mathbb{T}^2)$, then any weak limit will conserve energy. Indeed, if u^n is a smooth solution then:

$$\|u^n(t)\|^2_{L^2} = \|u^n_0\|^2_{L^2}.$$

Now, $\|u_0^n\|_{L^2}^2 \to \|u_0\|_{L^2}^2$ and, since u^n is compact in $C^0([0,T];L^2)$, it follows that $\|u^{n_k}(t)\|_{L^2}^2 \to \|u(t)\|_{L^2}^2$, for some subsequence n_k. Thus,

$$\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2.$$

The role of the hypothesis $\omega_0 \in L^p$, $p > 1$, is to guarantee that $W^{1,p}$ be compactly imbedded in L^2. However, if $1 < p < 3/2$ then $W^{1,p}$ is a *supercritical* space. A summary of additional examples in supercritical spaces can be found in [6].

Existence of weak solutions to the incompressible 2D Euler equations has been established assuming vorticity belongs to L^p, for any $p \geq 1$, and also for bounded Radon measures of distinguished sign. Uniqueness, however, has only been established if the vorticity belongs to L^∞ or nearly so (see [10] and [9] for up-to-date uniqueness results). It is, hence, reasonable to investigate conservation of energy among weak solutions as obtained in the existence theorems. We have observed that weak solutions obtained as a limit of smooth, conservative, approximations conserve energy if the vorticity $\omega_0 \in L^p$, even in the supercritical case $1 < p < 3/2$. We now discuss conservation of energy for *physically realizable* weak solutions.

Definition 2. Let $u \in C(0,T;L^2(\mathbb{T}^2))$ be a weak solution of the incompressible 2D Euler equations. We say that u is a *physically realizable weak solution* with initial velocity $u_0 \in L^2(\mathbb{T}^2)$ if:

there exists a family of solutions of the incompressible 2D Navier-Stokes equations with viscosity $\nu > 0$, $\{u^\nu\}$, such that

(1) $u^\nu \rightharpoonup u \ w-*L^\infty(0,T;L^2(\mathbb{T}^2))$;
(2) $u^\nu(0,\cdot) \equiv u_0^\nu \to u_0 \ sL^2(\mathbb{T}^2)$.

With this definition we can show the following supercritical result.

Theorem 3. *Let* $u \in C(0,T;L^2(\mathbb{T}^2))$ *be a physically realizable weak solution of the incompressible 2D Euler equations. Suppose that* $u_0 \in L^2$ *is such that* $\operatorname{curl} u_0 \equiv \omega_0 \in L^p(\mathbb{T}^2)$*, for some* $p > 1$*. Then* u *conserves energy.*

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Applications of optimization and optimal control analysis to some fundamental problems in mathematical hydrodynamics

Charles R. Doering

Modern methods of optimization and dynamical control are used to investigate the accuracy of analytical estimates for solutions of basic equations of mathematical hydrodynamics. Even though individual estimates used in a sequence may each be demonstrably sharp, this does not mean that the result of a sequence of applications is sharp. We examine the classical analysis bounding enstrophy ($\dot{H}^1$ norm) and palinstrophy ($\dot{H}^2$ norm) amplification in Burgers' and the Navier-Stokes equations and discover that the best known instantaneous growth rates estimates are sharp. But the time-integrated estimates producing time-uniform bounds are not always sharp. When they are not, optimal control techniques must be brought to bear to determine the actual behavior. The question of 3D Navier-Stokes regulatiry remains unanswered although work is in progress to apply these new tools to this challenge.

Asymptotic Coupling and Applications for Nonlinear Stochastic Partial Differential Equations

Nathan Glatt-Holtz
(joint work with Peter Constantin, Juraj Foldes, Jonathan Mattingly, Geordie Richards, Vlad Vicol)

We introduce the notion of asymptotic coupling and explain how this formalism provides a simple means proving unique ergodicity in certain stochastic systems whose deterministic counterpart possesses a finite dimensional attractor.

Solution of Leray's problem for stationary Navier-Stokes equations in plane and axially symmetric spatial domains

Konstantin Pileckas
(joint work with Mikhail Korobkov, Remigio Russo)

Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$, with C^2-smooth boundary $\partial\Omega = \cup_{j=0}^{N}\Gamma_j$ consisting of $N+1$ disjoint components Γ_j, $j = 0, \dots, N$. Consider the stationary Navier–Stokes system the with nonhomogeneous boundary conditions

$$\text{(1)} \qquad \begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{a} & \text{on } \partial\Omega. \end{cases}$$

The continuity equation (1_2) implies the compatibility condition

$$\text{(2)} \qquad \int\limits_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,ds = \sum_{j=0}^{N}\int\limits_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,ds = \sum_{j=0}^{N}\mathcal{F}_j = 0,$$

which is necessary for the solvability of problem (1). Here $\mathbf{n}$ is a unit outward (with respect to Ω) normal vector to $\partial\Omega$ and $\mathcal{F}_j = \int_{\Gamma_j}\mathbf{a}\cdot\mathbf{n}\,dS$. Condition (2) means that the total flux of the fluid through $\partial\Omega$ is zero.

In his famous paper of 1933 [2] Jean Leray proved that problem (1) has a solution provided[1]

$$\text{(3)} \qquad \mathcal{F}_j = \int\limits_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,dS = 0, \qquad j = 0, 1, \dots, N.$$

The case when the boundary value $\mathbf{a}$ satisfies only the necessary condition (2) was left open by Leray and the problem whether (1), (2) admit (or do not admit) a solution is know in the scientific community as *Leray's problem*.

Leray's problem has been studied in many papers. However, in spite of all efforts, the existence of a weak solution $\mathbf{u} \in W^{1,2}(\Omega)$ to problem (1) was established only under assumption (3), or for sufficiently small fluxes $\mathcal{F}_j$[2], or under certain symmetry conditions on the domain Ω, the boundary value $\mathbf{a}$ and the external force $\mathbf{f}$.

In this talk we present the solution of Leray's problem for the plane domains with multi connected boundaries and for the axially symmetric domains in $\mathbb{R}^3$. (For axially symmetric spatial domains the boundary value $\mathbf{a}$ and the external force $\mathbf{f}$ are assumed to be axially symmetric as well.) The main result in the plane case is as follows.

Theorem. *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with C^2-smooth boundary $\partial\Omega$. If $\mathbf{f} \in W^{1,2}(\Omega)$ and $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ satisfies condition* (2)*, then problem* (1) *admits at least one weak solution* $\mathbf{u}$.

[1] Condition (3) does not allow the presence of sinks and sources.

[2] This condition does not assumes the norm of the boundary value $\mathbf{a}$ to be small.

The proof of the existence theorem is based on an a priori estimate which we derive using a *reductio ad absurdum* argument of Leray [2]. The essentially new part in this argument is the use of Bernoulli's law obtained in [3] for Sobolev solutions to the Euler equations (the detailed proofs are presented in [4]). The results concerning Bernoulli's law are based on the recent version of the Morse-Sard theorem proved by J. Bourgain, M. Korobkov and J. Kristensen [5]. This theorem implies, in particular, that almost all level sets of a function $\psi \in W^{2,1}(\Omega)$ are finite unions of C^1-curves. This allows to construct suitable subdomains (bounded by smooth stream lines) and to estimate the L^2-norm of the gradient of the total head pressure. We use here some ideas which are close (on a heuristic level) to the Hopf maximum principle for the solutions of elliptic PDEs. Finally, a contradiction is obtained using the Coarea formula.

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Towards Non-uniqueness for Vortex Sheets

Tarek M. Elgindi

(joint work with Nader Masmoudi)

We study the uniqueness of solutions to the 2d incompressible Euler equations with vorticity given by a positive measure–that is solutions in the so-called Delort class. We identify a novel mechanism for non-uniqueness: non-uniqueness via mass separation. We prove mass separation for the linearized Euler equations around a particular vortex sheet steady state. We then move on to prove non-uniqueness for a class of (non-linear) active scalar equations which are regularized versions of the Euler equations. This seems to give the first example of non-uniqueness for an active scalar equation with odd kernel.

1. Background and Delort's Theorem

The study of weak solutions for the Euler equations is, by now, a classical and indispensible part of mathematical fluid dynamics. Among the reasons for this is that weak solutions give us insight into questions of turbulence and the long-time behavior of smooth solutions. Weak solutions of the Euler equations also arise naturally in certain idealized situations. One such example is a classical vortex sheet where the velocity of the fluid is discontinuous across a curve in two dimensions. A very simple example of a stationary vortex sheet weak solution of the Euler equations is given by $\mathbf{u} = (sgn(y), 0)$ (see Figure 1).

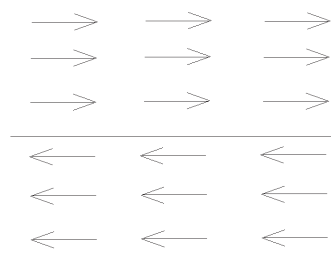

FIGURE 1. Simple example of a vortex sheet.

While weak solutions for the Euler equations have been studied for over a hundred years, many questions remain open with regards to the analysis of these solutions. In fact, it was only in the early 1990's when global existence of solutions of vortex sheet type was established by Delort under a sign condition. Specifically, Delort proved:

Theorem 1. *Let $\mathbf{u}_0 \in L^2_{loc}$ be such that $\omega_0 := curl(\mathbf{u}_0) \in BM_+$. Then, there exists a global weak solution of the 2D Euler equations belonging to the class $L^\infty([0,\infty); BM_+)$.*

Here, $\omega \in BM_+$ means that the vorticity is a positive bounded measure. For the purposes of the abstract, it isn't too important to go into the details of what sense the weak solution solves the 2D Euler equations; howevever, we would refer the reader to the paper of Schochet where a simpler proof of Delort's theorem is given. However, we should note that one of the main features of the existence proof is that, if ω is a positive measure belonging to H^{-1}, then ω cannot have a Dirac part and this can even be made quantitative in the following way:

Lemma 2. *Let $\mathbf{u} \in L^2_{loc}$ and $\omega := curl(\mathbf{u}) \in BM_+$. Then,*

$$\int_{B_r} \omega \lesssim \frac{1}{\sqrt{Log(\frac{1}{r})}}, \quad as \ \ r \to 0.$$

2. A Mechanism for Non-uniqueness

While existence theorems for weak solutions of the Euler equations are plentiful, the question of uniqueness of those weak solutions is open to a large extent with the notable exception of the results on the Onsager conjecture obtained by Scheffer, Shnirelman, De Lellis, Szekelyhidi, Isett, Buckmaster, and Daneri. Unfortunately, all of the solutions constructed (via the method of convex integration, for example) have such irregular vorticity that they do not even belong to the Delort class. Hence, it seems a new mechanism for non-uniqueness needs to be thought of before approaching the non-uniqueness question.

The basis of our discussion is the following theorem of Delort:

Theorem 3. *(Weak Stability)*
Let ω_0^ϵ be a sequence of smooth initial vorticities belonging to BM_+ and converging weakly (in the sense of measures) to $\omega_0 \in BM^+$ with velocity field $\mathbf{u}_0 \in L^2_{loc}$. Then, if we solve the 2D Euler equations with initial data ω_0^ϵ, the solutions $\omega^\epsilon(t)$ must converge weakly to a weak solution $\omega(t)$ with initial data ω_0.

Our strategy is thus to take a stationary solution of the 2D Euler equations ω_0 and then solve the Euler equations with initial data $\omega_0^\epsilon = \omega_0 + \omega_{pert}^\epsilon$ with $\omega_{pert}^\epsilon \rightharpoonup 0$. If ω_{pert}^ϵ is chosen properly, perhaps we can cause $\omega^\epsilon(t)$ to be very far from ω_0 by time 1. This will ensure the non-uniqueness.

2.1. **Mass separation.** The main idea that we will use is that weak convergence to 0 in the sense of measures can be achieved simply by the "crashing" of a positive and negative dirac mass and if it were possible to show that the nonlinear evolution forces the separation of two arbitrarily close dirac masses, then we will be able to invoke the weak stability theorem to prove non-uniqueness. For the Euler equations, we see that mass separation may be possible if the velocity field is discontinuous Indeed, the Euler equations (in vorticity form) are just given by:

$$\partial_t \omega + u \cdot \nabla \omega = 0.$$

Hence, if u has a jump such as in the case $\mathbf{u}(x, y) = (sgn(y), 0)$, then (at least in a certain linearized sense) it is possible to separate arbitrarily close masses in finite time. Of course, the difficulty in the full non-linear (or even the full linear) problem is that there is a feedback mechanism which might completely destroy the structures we are tryng to keep. Luckily, this is not the case for the full linearized problem or a certain modified active scalar equation.

Savage-Hutter model of the motion of a gravity driven avalanche flow

Eduard Feireisl
(joint work with Piotr Gwiazda, Agnieszka Świerczewska-Gwiazda)

This is a report on an ongoing research programme concerning solvability of certain hyperbolic conservation laws appearing in continuum fluid dynamics. An example of such a problem is a gravity driven avalanche flow that is qualitatively similar to the model of a compressible fluid. We consider a simple situation describing the time evolution of the flow height $h = h(t, x)$ and depth-averaged velocity $\mathbf{u} = \mathbf{u}(t, x)$ through a system of balance laws - the *Savage-Hutter system*, see [4]:

$$\partial_t h + \mathrm{div}_x(h\mathbf{u}) = 0, \tag{1}$$

$$\partial_t(h\mathbf{u}) + \mathrm{div}_x(h\mathbf{u}\otimes\mathbf{u}) + \nabla_x(ah^2) = h\left(-\gamma\frac{\mathbf{u}}{|\mathbf{u}|} + \mathbf{f}\right), \tag{2}$$

where $|\cdot|$ is an Euclidean metric, $a \geq 0$, $\gamma \geq 0$, and $\mathbf{f}$ are given (smooth) functions of the spatial coordinate $x \in \Omega \subset R^2$. We restrict ourselves to the periodic boundary conditions supposing that Ω is the flat torus

$$\Omega = \left([0,1]|_{\{0,1\}}\right)^2. \tag{3}$$

The resulting problem is completed by prescribing the initial conditions

$$h(0,\cdot) = h_0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0. \tag{4}$$

We report the following result, see [3, Theorem 2.1]:

Theorem 1. *Let $T > 0$ and the initial data h_0, $\mathbf{u}_0$ satisfying*

$$h_0 \in C^2(\Omega),\ \mathbf{u}_0 \in C^2(\Omega; R^2), h_0 > 0 \text{ in } \Omega$$

be given. Suppose that $a, \gamma \in C^2(\Omega)$, $a, \gamma \geq 0$, $\mathbf{f} \in C^1([0,T]\times\Omega; R^2)$.

Then the problem (1–4) admits infinitely many weak solutions in $(0,T)\times\Omega$. The weak solutions belong to the class

$$h,\ \partial_t h, \nabla_x h \in C^1([0,T]\times\Omega),$$

$$\mathbf{u} \in C_{\mathrm{weak}}([0,T]; L^2(\Omega; R^2)) \cap L^\infty((0,T)\times\Omega; R^2),\ \mathrm{div}_x\mathbf{u} \in C([0,T]\times\Omega).$$

The result is a consequence of the abstract theory developed in [2] on the basis of the convex integration method by DeLellis and Székelyhidi [1]. Using a similar method, we also get, see [3, Theorem 4.1]:

Theorem 2. *Under the hypotheses of Theorem 1, let $T > 0$ and*

$$h_0 \in C^2(\Omega),\ h_0 > 0 \text{ in } \Omega$$

be given.

Then there exists

$$\mathbf{u}_0 \in L^\infty(\Omega; R^2)$$

such that the problem (1–4) admits infinitely many dissipative *weak solutions in $(0,T)\times\Omega$, meaning weak solutions satisfying, in addition, the (integrated) energy inequality.*

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Exponential tails for the homogeneous Boltzmann equation

Nataša Pavlović
(joint work with Ricardo J. Alonso, Irene M. Gamba and Maja Tasković)

1. Introduction

In this note we summarize our results [2] on exponential-type moments for solutions of the spatially homogeneous Boltzmann equation without the Grad's cutoff.

1.1. **The Boltzmann Equation.** We consider the Cauchy problem for the spatially homogeneous Boltzmann equation

$$\begin{cases} \partial_t f(t,v) \;=\; Q(f,f)(t,v), \quad t \in \mathbb{R}^+, v \in \mathbb{R}^d, \quad d \geq 2 \\ f(0,v) \;=\; f_0(v). \end{cases} \tag{1}$$

for the time $t \in \mathbb{R}^+$ and velocity $v \in \mathbb{R}^d$, which describes the evolution of the density $f(t,v)$ of gas particles. $Q(f,f)$ is a quadratic integral operator that expresses the change of f due to instantaneous binary collisions of particles:

$$Q(f,f)(x,t,v) \;=\; \int_{\mathbb{R}^d} \int_{S^{d-1}} \left(f' f'_* - f f_*\right) B(|u|, \hat{u}\cdot\sigma)\, d\sigma\, dv_*, \tag{2}$$

where $f' = f(x,t,v')$, $f'_* = f(x,t,v'_*)$, $f_* = f(x,t,v_*)$ and v', v'_* denote pre-collisional velocities, v, v_* denote post-collisional velocities, and are connected via:

$$v' \;=\; \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\,\sigma, \qquad v'_* \;=\; \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\,\sigma, \qquad \sigma \in S^{d-1},$$

with σ the unit vector in the direction of the pre-collisional relative velocity. The relative post-collisional velocity is denoted by $u = v - v_*$, and $\hat{u} := u/|u|$.

The collisional kernel $B(|u|, \hat{u}\cdot\sigma)$ is assumed to take the form

$$B(|u|, \hat{u}\cdot\sigma) \;=\; |u|^\gamma\, b(\cos\theta), \tag{3}$$

where $\theta \in [0,\pi]$ is the angle between the pre and post collisional relative velocities. We consider the variable hard potentials case $0 < \gamma \leq 1$. The angular kernel is given by a positive measure $b(\hat{u}\cdot\sigma)$ over the sphere S^{d-1}. In many models, this function is non-integrable over the sphere, while its weighted integral is finite. We assume that for some $r \in (0,2]$ the following weighted integral is finite[1]

$$\int_{S^{d-1}} b(\hat{u}\cdot\sigma) \sin^r\theta\, d\sigma = V_{d-2} \int_0^\pi b(\cos\theta)\, \sin^r\theta\, \sin^{d-2}\theta\, d\theta < \infty. \tag{4}$$

(1) When $r = 0$, this condition is known as Grad's cutoff assumption, under which the collisional operator can be split into the gain and loss terms,

$$Q(f,f) = Q^+(f,f) - Q^-(f,f), \tag{5}$$

$$\begin{aligned} Q^+(f,f)(t,v) \;&=\; \int_{\mathbb{R}^d} \int_{S^{d-1}} f' f'_*\, B(|u|, \hat{u}\cdot\sigma)\, d\sigma\, dv_*, \\ Q^-(f,f)(t,v) \;&=\; f(v) \int_{\mathbb{R}^d} \int_{S^{d-1}} f_*\, B(|u|, \hat{u}\cdot\sigma)\, d\sigma\, dv_*. \end{aligned}$$

[1] Here $V_{d-2} = \frac{\pi^{(d-2)/2}}{\Gamma((d-1)/2)}$ is the volume of the $d-2$ dimensional unit sphere.

In [14] Grad proposed considering a bounded $b(\cos\theta)$, and noted that different cutoff conditions could be implemented. Since then the cutoff theory developed extensively, with the belief that removing the singularity of the angular kernel should not affect properties of the equation. Recently, however, it has been observed (e.g. [15], [7], [8], [9]) that the singularity of $b(\cos\theta)$ carries a regularization. This, and the analytical challenge, motivated further study of the non-cutoff regime.

(2) The typical non-cutoff assumption in the literature is (4) with $r = 2$.

(3) We work in the non-cutoff regime and allow parameter $r \in (0, 2]$ to vary in order to see how the strength of the singularity of b influences our result.

1.2. **Motivation.** Since the equilibrium state for the Boltzmann equation is a Gaussian distribution, one expects that the solution would be controlled by bounds of exponential decay. Thus bounds for exponential moments (expressing summability of polynomial moments, see the definition below) are of interest.

Definition 1 (Polynomial and exponential moments)**.** Polynomial moment of order q and exponential moment of order s and rate α are respectively defined by:

$$m_q(t) := \int_{\mathbb{R}^d} f(t,v)\ \langle v\rangle^q\ d(v), \tag{6}$$

$$\mathcal{M}_{\alpha,s}(t) := \int_{\mathbb{R}^d} f(t,v)\ e^{\alpha\,\langle v\rangle^s}\ dv. \tag{7}$$

Remark 2. Using the Taylor series expansion, one observes:

$$\mathcal{M}_{\alpha,s}(t)\ =\ \sum_{q=0}^{\infty} \frac{m_{qs}(t)\ \alpha^q}{q!}. \tag{8}$$

In an extensive work [10, 6, 19, 17] generation (moments are instantaneously created and stay finite for all times) and propagation (moments are finite for all times if they are initially finite) of polynomial moments was shown. The study of exponential moments in the Grad cutoff case was initiated by Bobylev [3, 4], and further developed in [5, 12, 18]. All these papers used a technique based on establishing a term-wise geometric decay for terms in (8). Recently a new type of proof was developed in [1], where estimates on the partial sums corresponding to (8) were obtained. The only result on exponential moments in the non-cutoff case was [16], where the generation of exponential moments up to the order $s \in (0, \gamma]$ was obtained, via implementing the term-by-term method. No results were known beyond the rate of the potentials, which is what motivates our work.

2. Statements of the results

2.1. **Our Set-up.** Inspired by [1] we implement the partial sum approach in the non-cutoff case to obtain generation and propagation of exponential-like moments. Due to the non-cutoff setting we need to overcome the singularity of the collision kernel which we achieve by using a cancellation property that can be identified via using a weak formulation of the collision operator, similar to [16]. To exploit decay of certain sums of Beta functions, our calculations lead to expressions related to (8), which in place of $q!$ have $\Gamma(aq+1)$, with non-integer $a > 1$. The question whether such sums still describe an exponential tail behavior motivated us to use Mittag-Leffler functions, which are a generalization of the Taylor expansion of the exponential function and are defined for some parameter $a > 0$

$$\mathcal{E}_a(x) := \sum_{q=0}^{\infty} \frac{x^q}{\Gamma(aq+1)}. \tag{9}$$

It is also well known (see e.g. [11]) that the Mittag-Leffler function $\mathcal{E}_a$ asymptotically behaves like an exponential function of order $1/a$, and consequently

$$\mathcal{E}_{2/s}(\alpha^{2/s}\, x^2) \sim e^{\alpha\, x^s}, \qquad \text{for}\, x \to \infty. \tag{10}$$

This motivates our definition of Mittag-Leffler moments:

Definition 3 (Mittag-Leffler moment)**.** Mittag-Leffler moment of order s and rate $\alpha > 0$ of a function f is introduced via

$$\int_{\mathbb{R}^d} f(t,v)\;\; \mathcal{E}_{2/s}(\alpha^{2/s}\, \langle v \rangle^2)\, dv. \tag{11}$$

2.2. **The main result.** Now we are ready to state the main result of [2]:

Theorem 4. *Suppose f is a solution to* (1) *with initial data*[2] $f_0 \in L^1_{2+}$.

(a) *If the angular kernel satisfies the non-cutoff condition* (4) *with $r = 2$, then the exponential moment of order γ is generated. More precisely, there are some positive constants C, α, depending only on b, γ and initial mass and energy, so that*

$$\int_{\mathbb{R}^d} f(t,v)\; e^{\alpha\; \min\{t,1\}\, |v|^\gamma}\, dv \;\le\; C, \quad \textit{for}\;\; t \ge 0. \tag{12}$$

(b) *Let $s \in (\gamma, 2)$ and suppose*

$$\int_{\mathbb{R}^d} f_0(v)\;\; \mathcal{E}_{2/s}(\alpha_0^{2/s}\, \langle v \rangle^2)\, dv < C_0. \tag{13}$$

For $s \in (\gamma, 1]$ assume the non-cutoff condition (4) *with $r = 2$. For $s \in (1, 2)$ assume that the angular kernel $b(\cos\theta)$ satisfies* (4) *with $r = \frac{4}{s} - 2$. Then there are some positive constants C, α, depending only on C_0, α_0, b, γ and initial mass and energy such that:*

$$\int_{\mathbb{R}^d} f(t,v)\;\; \mathcal{E}_{2/s}(\alpha^{2/s}\, \langle v \rangle^2)\, dv < C, \quad \textit{for}\;\; t \ge 0. \tag{14}$$

Recently Gamba-Pavlović-Tasković [13] employed the above result to prove propagation in time of exponentially decaying point-wise bounds for the solution, therefore generalizing the work of Gamba-Panferov-Villani [12] to the non-cutoff setting.

[2] Here we use the notation: $L^1_k = \{f \in L^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} f |v|^k dv < \infty\}$.

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Mixing and loss of regularity for transport equations

Anna L Mazzucato

We discuss the problem of optimal mixing of a passive scalar by an incompressible flow. The scalar θ satisfies the following transport equation:

$$\partial_t \theta + u \cdot \nabla \theta = 0, \qquad \theta(0) = \theta_0, \tag{1}$$

where θ_0 is assumed to be a bounded function, and u is a prescribed, divergence-free vector field. For simplicity, we impose periodic boundary conditions in two space dimensions, even though higher dimensions and other compatible boundary conditions on θ and u are allowed.

The degree of "mixedness" will be quantified in terms of the decay of certain negative Sobolev norms of θ following [9] and [7]. Specifically, we will employ the homogeneous Sobolev norm

$$\|\theta\|_{\dot{H}^{-1}} := \|(-\Delta)^{-1/2}\theta\|_{L^2},$$

henceforth called the""mix-norm" of θ. A main question is to derive lower bounds for the decay in time of the mix-norm under physically motivated constraints on u, and prove their optimality by constructing examples of flows and initial tracer configurations that realize the lower bound. In the literature, three types of constraints have been considered:

(1) Energy budget: the L^2 norm of u is uniformly bounded in time;
(2) Enstrophy or power budget: the Sobolev H^1 norm of u is uniformly bounded in time;
(3) Palinstrophy budget: the Sobolev H^2 norm of u is uniformly bounded in time.

It is clear that it is more difficult to mix efficiently the higher the constraint on the norm of u, as in absence of diffusion, the main mechanism for mixing is stirring and filamentation of the tracer configuration by the flow, that is, the creation of small scales and possibly large gradients.

Self-similar mixing, based on a rescaling strategy, gives the following bounds: finite-time perfect mixing under an energy budget, exponential decay under an enstrophy budget, and polynomial decay under a palinstrophy budget [2]. It can be shown that the first two bounds are sharp. What is the optimal bound for palinstrophy-constrained flows, whether polynomial or exponential in time, is still an open question.

For energy-constrained flow, the "slice and dice" strategy, using two orthogonal shear flows, of [8] (following arguments in [3] and [5]), gives finite-time perfect mixing. This example is optimal also with respect to the regularity of the flow, in the sense that finite-time perfect mixing is not possible if solutions to (1) are unique, due to the time-reversibility of the transport equation. By the results of [1], uniqueness holds for velocity fields that are integrable in time with values in the space BV of functions of bounded variation. The example in [8] has a velocity field that barely misses this threshold, being weak L^1 in time with values in BV.

For enstrophy-constrained flows, an exponential lower bound for the mix-norm can be rigorously established using estimates [4] on the cost of rearranging a set [6] or optimal transportation arguments [10]. Two examples of flows achieving the exponential decay are available in the literature. The first [2] is geometric in nature and assumes a specific initial configuration of the tracer, but is optimal for velocities in $W^{1,p}$ for all $1 \leq p \leq \infty$. The second [11] is analytic in nature, based on a certain combination of cellular flows, and works for all bounded θ_0, but only with a restricted range of p for the $W^{1,p}$ bound on the velocity, namely $1 \leq p \leq \bar{p}$, for some $\bar{p} > 2$.

Finally the example in [2] can be modified to show instantaneous, complete loss of regularity for solutions to the initial-value problem (1). More precisely, the following holds: there exists a velocity field $u \in L^\infty([0,T], W^{1,p})$, $T > 0$, $1 \leq p < \infty$, and a smooth θ_0 such that the solution $\theta(t)$ of (1) does not belong to any Sobolev space H^s, $s > 0$, for $t > 0$.

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Inhomogeneous NSEs, critical regularity, Lagrangian coordinates, connection to compressible model

Piotr B. Mucha

(joint work with Raphaël Danchin)

The subject of our study are equations of inhomogeneous Navier-Stokes system (INS)

$$
\begin{aligned}
&\rho_t + v \cdot \nabla v = 0, \qquad \nabla \cdot v = 0, \\
&\rho v_t + \rho v \cdot \nabla v - \mu \Delta v + \nabla p = 0.
\end{aligned} \tag{1}
$$

We aim at answering to the following question: what is the largest regularity class of initial velocity and density admitting the well posedness of the system (existence and uniqueness of solutions). The most interesting results are done in the critical function framework described by the Besov space of type $\dot{B}^s_{p,1}$. The most efficient approach allows to consider rough initial densities, even allowed to be discontinuous.

Our toolbox is the following:

The Lagrangian coordinates system. The basic difference between the INS and classical homogeneous Navier-Stokes system is clearly the continuity equation. It is just a pure transport equation giving for free the point-wise bounds for the density. It causes problems with control of regularity of solutions to INS and makes complex time asymptotics. A natural solution here is to change the system of coordinates related to motion of particles

$$\frac{dx(t,y)}{dt} = v(x(t,y),t), \qquad x|_{t=0} = y. \tag{2}$$

The system above defines us the well know Lagrangian coordinates system. The advantages are: we remove the presence of the transport term $v \cdot \nabla v$, but first of all the density becomes the know function $\rho(x(t,y),t) = \rho_0(y)$. The Besov spaces. In order to control the well posedness of the solution in the largest possible function space we are trying to work in the Besov space framework. The most optimal choice is the space

$$\dot{B}^0_{d,1}(\Omega), \tag{3}$$

since it is the limit case for the Sobolev imbedding $\dot{B}^d_{d,1}(\mathbb{R}^d) \subset L_\infty(\mathbb{R}^d)$. The second important property is the following. The solution to the heat equation

$$u_t - \Delta u = 0 \ \text{ in } \ \mathbb{R}^d \times (0,T), \qquad u|_{t=0} = u_0 \ \text{ in } \ \mathbb{R}^d \tag{4}$$

are in the class prescribed by the following estimate in the maximal regularity regime

$$\sup_{t \in (0,T)} \|u(t)\|_{\dot{B}^0_{d,1}(\mathbb{R}^d)} + \|u_t, \nabla^2 u\|_{L_1(0,T;\dot{B}^0_{p,1}(\mathbb{R}^d))} \le C \|u_0\|_{\dot{B}^0_{d,1}(\mathbb{R}^d)}. \tag{5}$$

The most particular point here is the time regularity in the L_1-space.

The results. The basic goal of our programme is to analyze the systems with rough initial densities. The methods develop in [1, 2], allows to obtain solvability in the such class that initial density has the following form $\rho = 1 + \sigma\chi_A$, for sufficiently regular set A. However more challenging problems are related to the domains, the first result has been done in [3]. Its generalization on the exterior problems is much more complex [4].

Slightly compressible flows. Currently we are working on the issue concerning the limit analysis between (1) and the compressible Navier-Stokes equation

$$\rho_t + \nabla \cdot (v\rho) = 0, \qquad \rho v_t + \rho v \cdot \nabla v - \Delta v - \nu \nabla \nabla \cdot v + \nabla p(\rho) = 0. \tag{6}$$

We want to understand the connection of the above system with (1) as $\nu \to \infty$. It appears the problems here are interesting and mostly are related to nontrivial structure of linearized equations.

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Micropolar electrorheological fluid flows

Michael Růžička
(joint work with F. Ettwein, B. Weber)

The steady motion of micropolar electrorheological fluids is described by the system

$$\begin{aligned} \operatorname{div}\mathbf{E} &= 0 \qquad \text{in } \Omega\,,\\ \operatorname{curl}\mathbf{E} &= \mathbf{0} \qquad \text{in } \Omega\,, \end{aligned} \tag{1}$$

$$\begin{aligned} -\operatorname{div}\mathbf{S} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) + \nabla\pi &= \mathbf{f} && \text{in } \Omega\,,\\ \operatorname{div}\mathbf{v} &= 0 && \text{in } \Omega\,,\\ -\operatorname{div}\mathbf{N} + \operatorname{div}(\boldsymbol{\omega}\otimes\mathbf{v}) &= \boldsymbol{\ell} - \boldsymbol{\varepsilon}:\mathbf{S} && \text{in } \Omega\,, \end{aligned} \tag{2}$$

which is completed by the following boundary conditions

$$\mathbf{E}\cdot\mathbf{n} = \mathbf{E}_0\cdot\mathbf{n}\,,\quad \mathbf{v} = \mathbf{0}\,,\quad \boldsymbol{\omega} = \mathbf{0} \qquad \text{on } \partial\Omega\,. \tag{3}$$

Here $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded domain, $\mathbf{n}$ the outer normal vector of the boundary $\partial\Omega$, $\boldsymbol{\varepsilon}$ the isotropic third order tensor and $\boldsymbol{\varepsilon}:\mathbf{S}$ the vector having the components $\varepsilon_{ijk}S_{jk}$, $i = 1,\dots,d$, where the summation convention over repeated indices is used. In these equations $\mathbf{v}$ denotes the velocity, $\boldsymbol{\omega}$ the micro-rotation, π the pressure, $\mathbf{S}$ the mechanical extra stress tensor, $\mathbf{N}$ the couple stress tensor, $\boldsymbol{\ell}$ the electromagnetic couple force, $\mathbf{f} = \tilde{\mathbf{f}} + \chi^E \operatorname{div}(\mathbf{E}\otimes\mathbf{E})$ the body force, where $\tilde{\mathbf{f}}$ is the mechanical body force, χ^E the dielectric susceptibility and $\mathbf{E}$ the electric field. A representative example for a constitutive relation for the stress tensors in (2) reads (cf. [2])

$$\begin{aligned} \mathbf{S} &= (\alpha_{31} + \alpha_{33}|\mathbf{E}|^2)(1+|\mathbf{D}|)^{p-2}\mathbf{D} + \alpha_{51}(1+|\mathbf{D}|)^{p-2}\big(\mathbf{DE}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{DE}\big)\\ &\quad + \alpha_{71}|\mathbf{E}|^2(1+|\mathbf{R}|)^{p-2}\mathbf{R} + \alpha_{91}(1+|\mathbf{R}|)^{p-2}\big(\mathbf{RE}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{RE}\big)\,,\\ \mathbf{N} &= (\beta_{31} + \beta_{33}|\mathbf{E}|^2)(1+|\nabla\boldsymbol{\omega}|)^{p-2}\nabla\boldsymbol{\omega}\\ &\quad + \beta_{51}(1+|\nabla\boldsymbol{\omega}|)^{p-2}\big((\nabla\boldsymbol{\omega})\mathbf{E}\otimes\mathbf{E} + \mathbf{E}\otimes(\nabla\boldsymbol{\omega})\mathbf{E}\big)\,, \end{aligned} \tag{4}$$

with constants $\alpha_{31},\alpha_{33},\alpha_{71},\beta_{33} > 0$ and $\beta_{31} \geq 0$. The constants $\alpha_{51},\alpha_{91},\beta_{51}$ have to satisfy certain restrictions (cf. [2]), which ensure the validity of the second law of thermodynamics. In (4) we used the notation $\mathbf{D} = (\nabla\mathbf{v})^{\mathrm{sym}}$, $\mathbf{R} = \mathbf{W}(\mathbf{v}) + \boldsymbol{\varepsilon}:\boldsymbol{\omega}$, with $\mathbf{W}(\mathbf{v}) = (\nabla\mathbf{v})^{\text{skew}}$ and $\boldsymbol{\epsilon} : \mathbf{v}$ denoting the tensor with components $\varepsilon_{ijk}v_k$, $i,j = 1,\dots,d$.

From (1), (3) follows that $\mathbf{E} \in L^2(\Omega)$ is real analytic and that $|\mathbf{E}|^2$ belongs to the Muckenhoupt class A_∞. Moreover, the set $|\mathbf{E}|^{-1}(0)$ is a finite union of C^1-manifolds M_i with $\dim M_i \le d-1$. One can impose conditions on $\mathbf{E}_0$ that ensure $\mathbf{E} \in L^\infty(\Omega)$. Motivated by this we assume in the following that the electric field $\mathbf{E}$ belongs to $C^\infty(\Omega)\cap L^2(\Omega)$ satisfies $|\mathbf{E}| > 0$ a.e. in Ω. Moreover, we assume that either $|\mathbf{E}|^2 \in A_p$ or $|\mathbf{E}|^2 \in L^\infty(\Omega)$. From these assumptions, various embedding theorems, Korn's inequality, the definition of $\mathbf{R}$ and Young's inequality we easily derive the a priori estimate

$$\begin{aligned}
&\|\mathbf{v}\|^p_{1,p,|\mathbf{E}|^2} + \|\mathbf{v}\|^p_{1,p} + \|\boldsymbol{\omega}\|^p_{1,p,|\mathbf{E}|^2} + \beta_{31}\|\boldsymbol{\omega}\|^p_{1,p}\\
&\le c\Big(1+\beta_{31}+\|\mathbf{E}\|_2^2 + \|\mathbf{f}\|^{p'}_{(-1,p')+(-1,p',|\mathbf{E}|^{\frac{-2}{p-1}})} + \|\boldsymbol{\ell}\|^{p'}_{-1,p',|\mathbf{E}|^{\frac{-2}{p-1}}}\Big) \qquad (5)\\
&=: K(\beta_{31},\mathbf{E},\mathbf{f},\boldsymbol{\ell})\,.
\end{aligned}$$

In the following we only discus the degenerate case $\beta_{31} = 0$. Using the theory of pseudo-monotone operators one can prove for all $p \in (1,\infty)$ and all $n \in \mathbb{N}$ the existence of approximate solutions $(\mathbf{v}^n,\boldsymbol{\omega}^n) \in (V_p(\Omega)\cap L^q(\Omega)\cap H^{1,p}_0(\Omega;|\mathbf{E}|^2)) \times (H^{1,p}_0(\Omega)\cap L^q(\Omega)\cap H^{1,p}_0(\Omega;|\mathbf{E}|^2))$ satisfying for all $\boldsymbol{\varphi} \in V_p(\Omega)\cap L^q(\Omega)\cap H^{1,p}_0(\Omega;|\mathbf{E}|^2)$, $\boldsymbol{\psi} \in H^{1,p}_0(\Omega)\cap L^q(\Omega)\cap H^{1,p}_0(\Omega;|\mathbf{E}|^2)$

$$\begin{aligned}
(6)\quad &\big\langle \mathbf{S}\big(\mathbf{D}\mathbf{v}^n,\mathbf{R}(\mathbf{v}^n,\boldsymbol{\omega}^n),\mathbf{E}\big),\mathbf{D}\boldsymbol{\varphi}\big\rangle + \frac{1}{n}\langle|\mathbf{v}^n|^{q-2}\mathbf{v}^n,\boldsymbol{\phi}\rangle - \big\langle \mathbf{v}^n\otimes\mathbf{v}^n,\nabla\boldsymbol{\varphi}\big\rangle\\
&+\big\langle \mathbf{N}(\nabla\boldsymbol{\omega}^n,\mathbf{E}),\nabla\boldsymbol{\psi}\big\rangle + \frac{1}{n}\big\langle(1+|\nabla\boldsymbol{\omega}^n|)^{p-2}\nabla\boldsymbol{\omega}^n,\nabla\boldsymbol{\psi}\big\rangle + \frac{1}{n}\langle|\boldsymbol{\omega}^n|^{q-2}\boldsymbol{\omega}^n,\boldsymbol{\psi}\rangle\\
&-\big\langle\boldsymbol{\omega}^n\otimes\mathbf{v}^n,\nabla\boldsymbol{\psi}\big\rangle + \big\langle \mathbf{S}\big(\mathbf{D}\mathbf{v}^n,\mathbf{R}(\mathbf{v}^n,\boldsymbol{\omega}^n),\mathbf{E}\big),\mathbf{R}(\boldsymbol{\phi},\boldsymbol{\psi})\big\rangle = \big\langle\mathbf{f},\boldsymbol{\varphi}\big\rangle + \big\langle\boldsymbol{\ell},\boldsymbol{\psi}\big\rangle\,,
\end{aligned}$$

where $q > 2p'$. These solutions satisfy the uniform with respect to $n \in \mathbb{N}$ a priori estimate

$$\begin{aligned}
&\|\mathbf{v}^n\|^p_{1,p,|\mathbf{E}|^2} + \|\mathbf{v}^n\|^p_{1,p} + \frac{1}{n}\|\mathbf{v}^n\|^q_q + \|\boldsymbol{\omega}^n\|^p_{1,p,|\mathbf{E}|^2} + \frac{1}{n}\|\boldsymbol{\omega}^n\|^p_{1,p} + \frac{1}{n}\|\boldsymbol{\omega}^n\|^q_q\\
(7)\quad &\le K(0,\mathbf{E},\mathbf{f},\boldsymbol{\ell})\,.
\end{aligned}$$

To identify the limits of the nonlinear terms in (6) we use compact embedding theorems and either the Lipschitz truncation method or the L^∞-truncation method.

Let us start with the Lipschitz truncation method in the case that $|\mathbf{E}|^2 \in A_p$. This method is well-established in Sobolev spaces $H^{1,p}_0(\Omega)$ (cf. [1]) and can be generalized to weighted Sobolev spaces $H^{1,p}_0(\Omega;|\mathbf{E}|^2)$ (cf. [3]) if $|\mathbf{E}|^2 \in A_p$. Using the Lipschitz truncation method both in $H^{1,p}_0(\Omega)$ and in $H^{1,p}_0(\Omega;|\mathbf{E}|^2)$ together with the local sequential weak stability of the stress tensors $\mathbf{S}$ and $\mathbf{N}$ one can show that there exist subsequences of $\mathbf{v}^n$ and $\boldsymbol{\omega}^n$, resp., which converge a.e. in Ω to the weak limits $\mathbf{v}$ and $\boldsymbol{\omega}$ of the corresponding sequences respectively. This and the classical result that weak and a.e. limits coincide enables us to deduce that the weak limits of $\mathbf{S}\big(\mathbf{D}\mathbf{v}^n,\mathbf{R}(\mathbf{v}^n,\boldsymbol{\omega}^n),\mathbf{E}\big)$ and $\mathbf{N}(\nabla\boldsymbol{\omega}^n,\mathbf{E})$, resp., coincide with $\mathbf{S}\big(\mathbf{D}\mathbf{v},\mathbf{R}(\mathbf{v},\boldsymbol{\omega}),\mathbf{E}\big)$ and $\mathbf{N}(\nabla\boldsymbol{\omega},\mathbf{E})$, respectively. The identification of the limit in the convective term $\operatorname{div}(\mathbf{v}^n \otimes \mathbf{v}^n)$ follows with standard arguments using the compact embedding $H_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$, $q < p^*$. The identification of the limit of the other convective term $\operatorname{div}(\boldsymbol{\omega}^n \otimes \mathbf{v}^n)$ is more subtle since we only know that $\boldsymbol{\omega}^n$ is bounded in $H_0^{1,p}(\Omega; |\mathbf{E}|^2)$. Due to the open end property of the Muckenhoupt classes we know that

$$p_0 := p_0(|\mathbf{E}|^2) = \inf \left\{ q > 1 \,\middle|\, |\mathbf{E}|^2 \in A_q \right\} < p\,.$$

Thus one can show (cf. [4]) that the embedding $H_0^{1,p}(\Omega; |\mathbf{E}|^2) \hookrightarrow L^r(\Omega)$ is compact for $r \in [1, (\frac{p}{p_0})^*)$, which enables the identification of the limit of $\operatorname{div}(\boldsymbol{\omega}^n \otimes \mathbf{v}^n)$ as $\operatorname{div}(\boldsymbol{\omega} \otimes \mathbf{v})$. All these arguments result in

Theorem 1. *Let $\mathbf{E} \in A_p$. Then for $p > \frac{d(p_0+1)}{d+2}$ there exists a weak solution $(\mathbf{v}, \boldsymbol{\omega}) \in (V_p(\Omega) \cap H_0^{1,p}(\Omega; |\mathbf{E}|^2)) \times H_0^{1,p}(\Omega; |\mathbf{E}|^2)$ of problem (2), (3) satisfying for all $\boldsymbol{\varphi}, \boldsymbol{\psi} \in C_0^1(\Omega)$ with $\operatorname{div} \boldsymbol{\phi} = 0$*

$$\begin{aligned} &\left\langle \mathbf{S}\big(\mathbf{Dv}, \mathbf{R}(\mathbf{v}, \boldsymbol{\omega}), \mathbf{E}\big), \mathbf{D}\boldsymbol{\varphi} \right\rangle - \left\langle \mathbf{v} \otimes \mathbf{v}, \nabla \boldsymbol{\varphi} \right\rangle + \left\langle \mathbf{S}\big(\mathbf{Dv}, \mathbf{R}(\mathbf{v}, \boldsymbol{\omega}), \mathbf{E}\big), \mathbf{R}(\boldsymbol{\phi}, \boldsymbol{\psi}) \right\rangle \\ &+ \left\langle \mathbf{N}(\nabla \boldsymbol{\omega}, \mathbf{E}), \nabla \boldsymbol{\psi} \right\rangle - \left\langle \boldsymbol{\omega} \otimes \mathbf{v}, \nabla \boldsymbol{\psi} \right\rangle = \left\langle \mathbf{f}, \boldsymbol{\varphi} \right\rangle + \left\langle \boldsymbol{\ell}, \boldsymbol{\psi} \right\rangle, \end{aligned} \tag{8}$$

and the apriori estimate

$$\|\mathbf{v}\|_{1,p,|\mathbf{E}|^2}^p + \|\mathbf{v}\|_{1,p}^p + \|\boldsymbol{\omega}\|_{1,p,|\mathbf{E}|^2}^p \le K(0, \mathbf{E}, \mathbf{f}, \boldsymbol{\ell})\,. \tag{9}$$

The lower bound for p in the previous theorem can be improved if we weaken the notion of solution. Using the compact embedding $H_0^{1,p}(\Omega; |\mathbf{E}|^2) \hookrightarrow L^q(\Omega; |\mathbf{E}|^2)$, $q \in [1, p^\#(|\mathbf{E}|^2)\big)$ where $\frac{1}{p^\#(|\mathbf{E}|^2} = \frac{1}{p} - \frac{1}{dp_0}$ (cf. [4]) one can show that

$$\lim_{n \to \infty} \langle \boldsymbol{\omega}^n \otimes \mathbf{v}^n, \nabla \boldsymbol{\psi} \rangle = \langle \boldsymbol{\omega} \otimes \mathbf{v}, \nabla \boldsymbol{\psi} \rangle \tag{10}$$

if $\boldsymbol{\psi} \in |\mathbf{E}|^2 \times C_0^1(\Omega)$. This implies

Theorem 2. *Let $\mathbf{E} \in A_p$. Then for $p > \frac{2dp_0}{dp_0+2}$ there exists a very weak solution $(\mathbf{v}, \boldsymbol{\omega}) \in (V_p(\Omega) \cap H_0^{1,p}(\Omega; |\mathbf{E}|^2)) \times H_0^{1,p}(\Omega; |\mathbf{E}|^2)$ of problem (2), (3), i.e. $\mathbf{v}, \boldsymbol{\omega}$ satisfy the apriori estimate (9) and the identity (8) is satisfied for all $\boldsymbol{\varphi} \in C_0^1(\Omega)$ with $\operatorname{div} \boldsymbol{\phi} = 0$ and all $\boldsymbol{\psi} \in |\mathbf{E}|^2 \times C_0^1(\Omega)$.*

If we assume that $\mathbf{E} \in L^\infty(\Omega)$ the Lipschitz truncation in weighted Sobolev space does not work any more. However, we can adapt the L^∞-truncation method developed in [5] to the present situation. This method also delivers that there exist subsequences of $\mathbf{v}^n$ and $\boldsymbol{\omega}^n$, resp., which converge a.e. in Ω to the weak limits $\mathbf{v}$ and $\boldsymbol{\omega}$, respectively. Thus we can proceed similarly to the above procedure and we can identify all limits of the nonlinear terms in (6). For the convective terms we use that $\langle \mathbf{v}^n \otimes \mathbf{v}^n, \nabla \boldsymbol{\varphi} \rangle = -\langle [\nabla \mathbf{v}^n]\mathbf{v}, \boldsymbol{\varphi} \rangle$ and $\langle \boldsymbol{\omega}^n \otimes \mathbf{v}^n, \nabla \boldsymbol{\psi} \rangle = -\langle [\nabla \boldsymbol{\omega}^n]\mathbf{v}, \boldsymbol{\psi} \rangle$. Using $\mathbf{E} \in L^\infty(\Omega)$, the compact embedding $H_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$, $q < p^*$ and $p' < p^*$, valid for $p > \frac{2d}{d+1}$, we obtain that $\mathbf{v}^n \widetilde{\boldsymbol{\psi}} |\mathbf{E}|^2 \to \mathbf{v} \widetilde{\boldsymbol{\psi}} |\mathbf{E}|^2$ strongly in $(L^p(\Omega; |\mathbf{E}|^2))^*$. This and $\boldsymbol{\omega}^n \rightharpoonup \boldsymbol{\omega}$ in $H_0^{1,p}(\Omega; |\mathbf{E}|^2)$ gives for $\boldsymbol{\psi} \in |\mathbf{E}|^2 \times C_0^1(\Omega)$

$$\lim_{n \to \infty} \left\langle [\nabla \boldsymbol{\omega}^n]\mathbf{v}^n, \boldsymbol{\psi} \right\rangle = \left\langle [\hat{\nabla} \boldsymbol{\omega}]\mathbf{v}, \boldsymbol{\psi} \right\rangle.$$

Thus, we proved

Theorem 3. *Let* $\mathbf{E} \in L^\infty(\Omega)$. *Then for* $p > \frac{2d}{d+1}$ *there exists a very weak solution* $(\mathbf{v}, \boldsymbol{\omega}) \in V_p(\Omega) \times H_0^{1,p}(\Omega; |\mathbf{E}|^2)$ *of problem* (2), (3), *i.e.* $\mathbf{v}, \boldsymbol{\omega}$ *satisfy the apriori estimate* (9) *and for all* $\boldsymbol{\varphi} \in C_0^1(\Omega)$ *with* $\operatorname{div} \boldsymbol{\phi} = 0$ *and all* $\boldsymbol{\psi} \in |\mathbf{E}|^2 \times C_0^1(\Omega)$

$$\begin{aligned}&\big\langle \mathbf{S}\big(\mathbf{D}\mathbf{v}, \mathbf{R}(\mathbf{v}, \boldsymbol{\omega}), \mathbf{E}\big), \mathbf{D}\boldsymbol{\varphi}\big\rangle + \big\langle [\nabla \mathbf{v}]\mathbf{v}, \nabla \boldsymbol{\varphi}\big\rangle + \big\langle \mathbf{S}\big(\mathbf{D}\mathbf{v}, \mathbf{R}(\mathbf{v}, \boldsymbol{\omega}), \mathbf{E}\big), \mathbf{R}(\boldsymbol{\phi}, \boldsymbol{\psi})\big\rangle \\ &+ \big\langle \mathbf{N}(\hat{\nabla}\boldsymbol{\omega}, \mathbf{E}), \nabla \boldsymbol{\psi}\big\rangle + \big\langle [\hat{\nabla}\boldsymbol{\omega}]\mathbf{v}, \boldsymbol{\psi}\big\rangle = \big\langle \mathbf{f}, \boldsymbol{\varphi}\big\rangle + \big\langle \boldsymbol{\ell}, \boldsymbol{\psi}\big\rangle . \end{aligned} \tag{11}$$

Missing details of all above results and additional results can be found in [3].

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Forward Self-Similar and Discretely Self-Similar Solutions of the 3D incompressible Navier-Stokes Equations

Tai-Peng Tsai

Denote $\mathbb{R}^4_+ = \mathbb{R}^3 \times (0, \infty)$. Consider the 3D incompressible Navier-Stokes equations for velocity $u : \mathbb{R}^4_+ \to \mathbb{R}^3$ and pressure $p : \mathbb{R}^4_+ \to \mathbb{R}$,

$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \quad \operatorname{div} u = 0, \tag{1}$$

in $\mathbb{R}^4_+$, coupled with the initial condition $u|_{t=0} = u_0$ with $\operatorname{div} u_0 = 0$. The system (1) enjoys a scaling property: If $u(x,t)$ is a solution, then so is

$$u^{(\lambda)}(x,t) := \lambda u(\lambda x, \lambda^2 t) \tag{2}$$

for any $\lambda > 0$. We say $u(x,t)$ is *self-similar* (SS) if $u = u^{(\lambda)}$ for every $\lambda > 0$. In that case, the value of $u(x,t)$ is decided by its value at $t = 1$. On the other hand, if $u = u^{(\lambda)}$ only for one particular $\lambda > 1$, we say u is *discretely self-similar* (DSS) with *factor* λ. Its value in $\mathbb{R}^4_+$ is decided by its value in the strip $x \in \mathbb{R}^3$ and $1 \le t < \lambda^2$. They are called *forward* because $0 < t < \infty$. We can also consider (1) for $-\infty < t < 0$ or for time independent u. For both cases the scaling law (2) still holds, and we define *backward* and *stationary* SS and DSS solutions in the same manner. If we introduce the self-similar transform (for $0 < t < \infty$)

$$u(x,t) = \frac{1}{\sqrt{2t}}\, U(y,s), \quad y = \frac{x}{\sqrt{2t}}, \quad s = \log \sqrt{2t}, \tag{3}$$

we get the *Leray system*

$$\partial_s U - \Delta U - U - y \cdot \nabla U + (U \cdot \nabla)U + \nabla P = 0, \quad \operatorname{div} U = 0. \tag{4}$$

A self-similar solution of (1) corresponds to a stationary solution of (4), while a DSS solution of (1) with factor λ corresponds to a time-periodic solution of (4) with period $\log \lambda$.

When $u(x,t)$ is either SS or DSS, then so is $u_0(x)$. Thus it is natural to assume $|u_0(x)| \le \frac{C_*}{|x|}$ for some constant $C_* > 0$ and look for solutions satisfying

$$|u(x,t)| \le \frac{C(C_*)}{|x|}, \quad \text{or} \quad \|u(\cdot,t)\|_{L^{3,\infty}} \le C(C_*). \tag{5}$$

Here by $L^{q,r}$, $1 \le q, r \le \infty$, we denote the Lorentz spaces. In such classes, with sufficiently small C_*, the unique existence of mild solutions (solutions obtained by treating the nonlinearity as a source term for Stokes system) was obtained by Giga-Miyakawa, Cannone-Meyer-Planchon and Koch-Tataru. As a consequence of the uniqueness, if $u_0(x)$ is SS or DSS with small C_*, the corresponding small mild solution is also SS or DSS.

For large C_*, the existence theory for mild solutions is not available, and one may extend the concept of weak solutions and consider local Leray solutions constructed by Lemarié-Rieusset. However, there is no uniqueness theorem for them to guarantee self-similarity.

Recently, Jia and Šverák [1] constructed SS solutions for every SS u_0 which is locally Hölder continuous. Their main tool is a local Hölder estimate of any local Leray solution near $t = 0$, assuming minimal control of u_0 in the large. This estimate enables them to prove a priori estimates of SS solutions, and then get the existence by applying the Leray-Schauder theorem. Note that it does not assert uniqueness.

Similar results were later proven in Tsai [2] for λ-DSS solutions with factor λ *close to one* where closeness is determined by the local Hölder norm of u_0 away from the origin. It is also shown in [2] that the closeness condition on λ can be eliminated if the initial data is axisymmetric with no swirl. These two results show existence of DSS solutions when *strong solutions* can be expected.

In this talk I will present two new results. The first is a joint work with Mikhail Korobkov [3]. The second is a joint work with Zachary Bradshaw [4].

In [3], the existence of self-similar solutions in the half space is established for C^1_{loc} initial data. The approach of [3] differs from [1] and [2] in that the existence of a solution to the stationary Leray equations is established directly. A new approach is needed in [3] due to lack of spatial decay estimates, which gives *global compactness* needed by the Leray-Schauder theorem in [1] and [2]. The spatial decay estimates in [1, 2] is provided by the local in time Hölder estimate of local Leray solutions, whose theory is not available in the half space. The paper [3] uses the method of invading domains. The key to this method is to prove a priori estimates of the solutions in bounded domains and in the whole space by the method of contradiction, which leads to a study of the limiting Euler equations, with different proofs for domains and the whole space. It is similar to the recent progress of Korobkov-Pileckas-Russo on the boundary value problem of stationary Navier-Stokes, but it is much simpler because of the simplicity of the domains.

The paper [4] constructs *weak* DSS solutions for general DSS initial data, and is a companion to [2] where *strong* DSS solutions are constructed for special initial data. As in [3], it constructs solutions of the Leray equations directly. Unlike [3], the method of contradiction does not give a limit solution because of lack of time compactness. The paper [4] is based on a new observation that, unlike the Navier-Stokes system, the a priori bound of the Leray system contains the L^2-norm in the left side, and that the usual trouble term, $\iint U \cdot \nabla W \cdot U$, can be absorbed when W is suitably "cut off." This observation provides a new a priori estimate that we can use to construct periodic solutions of the Leray system directly.

The same approach of [3] gives a second construction of large self-similar solutions in the whole space (the first is in [1]). The approach of [4] gives a third, and probably the easiest, construction.

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On the local existence for the 3D Euler equation with a free interface

Igor Kukavica

(joint work with A. Tuffaha and V. Vicol)

We address the local existence of solutions in low regularity Sobolev spaces for the rotational free-surface Euler equations

$$u_t + u \cdot \nabla u + \nabla p = 0 \quad \text{in } \Omega(t) \times (0, T)$$
$$\nabla \cdot u = 0 \quad \text{in } \Omega(t) \times (0, T)$$

in a time-dependent domain $\Omega(t)$. The boundary of the domain consists of two parts: the moving part $\Gamma_1(t)$ and the stationary part Γ_0. On the free boundary $\Gamma_1(t)$ we require the vanishing of the pressure, while on Γ_0 we impose the no-flow boundary condition $v \cdot N = 0$.

The earliest work to treat the local existence problem is a paper by Nalimov [N], where existence was proven in two space dimensions for small initial data. Other early works [Y, Sh] also considered the problem of local existence under a smallness assumption of the data or under the irrotationality assumption, i.e., when the initial vorticity vanishes. For the existence of solutions when the data is rotational, the Taylor stability sign condition $\partial p/\partial N < 0$ must be imposed, as was shown by Ebin [E]. Beale, Hou, and Lowengrub then proved in [BHL] the local existence of solutions to the linearized system under the Taylor sign condition. In [W1, W2], Wu established local existence of the solution without a smallness assumption on the initial data and under the general Taylor sign condition, in two and three space dimensions. In [AM1, AM2], Ambrose and Masmoudi treated the problem in the presence of surface tension. Many other important works treating the problem of local existence and regularity using different methods include [ABZ1, ABZ2, CL, EL, KT1, KT2, L, Li1, Li2, ZZ]. Notably, Coutand and Shkoller provided in [CS1, CS2] existence and uniqueness of solutions for H^3 initial velocity with the vorticity in $H^{2.5}$. A similar result but with completely different methods were at the same time obtained by Shatah and Zeng [SZ] and Zhang and Zhang [ZZ].

Consider the Euler equation on the domain $\Omega = \mathbb{R}^2 \times (0,1) \subseteq \mathbb{R}^3$ with periodic boundary conditions in x_1 and x_2 with period 1. The top $\Gamma_1 = \mathbb{R} \times \{x_3 = 1\}$ represents the free boundary, while the rigid bottom is represented by $\Gamma_0 = \mathbb{R} \times \{x_3 = 0\}$. We denote by $v(x,t) = (v^1, v^{(2)}, v^3)$ the Lagrangian velocity, while $q(x,t)$ represents the Lagrangian pressure. The Euler equation in Lagrangian coordinates may be written as

$$\begin{aligned} &v_t^i + a_i^k \partial_k q = 0 \text{ in } \Omega \times (0,T), \qquad i = 1,2,3 \\ &a_i^k \partial_k v^i = 0 \text{ in } \Omega \times (0,T) \end{aligned}$$

with the initial condition $v(0) = v_0$, where $a = (\nabla \eta)^{-1}$ with $\eta_t(x,t) = v(x,t)$ and $\eta(x,0) = x$. On the top, which represents the free boundary, we impose $q = 0$, while on the bottom boundary we assume $v^i N^i = 0$ where $N = (N^1, N^2, N^3)$ stands for the outward unit normal. The following is our main result.

Theorem 1. *Let $\delta > 0$. Assume that $v(\cdot, 0) = v_0 \in H^{2.5+\delta}(\Omega)$ is divergence-free with $v \cdot N = 0$ on Γ_0 and $\operatorname{curl} v_0 \in H^{2+\delta}(\Omega)$. Assume that the initial pressure $q(\cdot, 0)$ satisfies the Rayleigh-Taylor condition $(\partial q / \partial N)(x, 0) \leq -1/C_0 < 0$ for $x \in \Gamma_1$, where $C_0 > 0$ is a constant. Then there exists a unique solution (v, q, a, η) to the free boundary Euler system with the initial condition $v(0) = v_0$ such that*

$$\begin{aligned} &v \in L^\infty([0,T]; H^{2.5+\delta}(\Omega)) \cap C([0,T]; H^{2+\delta}(\Omega)) \\ &v_t \in L^\infty([0,T]; H^{2+\delta}(\Omega)) \\ &\eta \in L^\infty([0,T]; H^{3+\delta}(\Omega)) \cap C([0,T]; H^{2.5+\delta}(\Omega)) \\ &a \in L^\infty([0,T]; H^{2+\delta}(\Omega)) \cap C([0,T]; H^{1.5+\delta}(\Omega)) \\ &q \in L^\infty([0,T]; H^{3+\delta}(\Omega)) \\ &q_t \in L^\infty([0,T]; H^{2.5+\delta}(\Omega)) \end{aligned}$$

for $T > 0$ which depends on the initial data.

For irrotational flows, i.e., those with vanishing vorticity, the local existence with optimal regularity assumptions on the initial datum has already been established by Alazard, Burq, and Zuily in [ABZ2] in two and three space dimensions, and by

Hunter, Ifrim, and Tataru in two dimensions [HIT]. In a recent work [KT2], two of the authors provided an alternative proof of the optimal regularity for irrotational flow in three dimensions, where the initial data is assumed to be irrotational with $H^{2.5+\delta}$ Sobolev regularity.

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Determining modes for the 3D Navier-Stokes equations

Alexey Cheskidov
(joint work with Mimi Dai, Landon Kavlie)

The Navier-Stokes equations (NSE) on a torus, are given by

$$\begin{cases} u_t + (u \cdot \nabla)u - \nu\Delta u + \nabla p = f \\ \nabla \cdot u = 0, \end{cases} \tag{1}$$

where u is the velocity, p is the pressure, and f is the external force. We assume that f has zero mean, and consider zero mean solutions.

The dissipative nature of these equations is reflected in the existence of an absorbing ball in L^2. Moreover, in the two-dimensional case, there exists a compact global attractor which uniformly attracts all bounded subsets of L^2. This attracting set is, in fact, finite dimensional. The first result for the finite dimensionality of a two-dimensional fluid appeared in the work of Foias and Prodi when they showed that high modes of a solution are controlled by low modes asymptotically as time goes to infinity. The number of these low modes, called determining modes, was estimated by Foias, Manley, Temam and Treve and later improved by Jones and Titi.

In three dimensions the situation is drastically different as the equations have thus far eluded a proof for the existence of classical solutions. Even so, the existence of a global attractor for weak solutions is known in a weak sense. This weak global attractor consists of points on complete bounded trajectories and attracts all bounded subsets of L^2 in the weak topology. However, it is not known whether the solutions on the attractor are regular, unless the attractor consists of a single fixed point. Neither is it known whether the attractor is compact or finite-dimensional. Similarly, the existence of a finite number of determining modes is not known in the three-dimensional case. Nevertheless, Constantin, Foias, Manley, and Temam showed the existence of determining modes assuming that the H^1 norm of solutions is uniformly bounded. The question whether the global attractor of the 3D NSE is bounded in H^1 is open and may very well require a resolution of the regularity problem. However, even assuming regularity, this would not immediately guarantee that the H^1 bound would depend only on the size of the force (a Grashof constant), and not on the shape of the force.

With no hope of getting a finite number of determining modes for the 3D NSE, one might ask whether this can be done in some average sense. Indeed, the Kolmogorov 41 phenomenological theory of turbulence predicts that the number of degrees of freedom should be of order κ_{d}^3, where κ_{d} is Kolmogorov's dissipation wavenumber. This number is often used as the resolution needed for direct numerical simulations, so one might ask an alternative question: What is the number of determining modes for a time discretization of the 3D NSE and how does it depend on the force and time step?

Without making any assumptions regarding regularity properties of solutions or bounds on the global attractor, we prove the existence of a time-dependent determining wavenumber $\Lambda_u(t)$ defined for each individual solution u. We show that any weak solutions on the global attractor u and v that coincide below $\max\{\Lambda_u, \Lambda_v\}$ have to be identical. The wavenumber $\Lambda_u(t)$ blows up if and only if the solution $u(t)$ blows up. Nevertheless, the time average of this wavenumber is uniformly bounded on the global attractor, which we estimate in terms of the Kolmogorov dissipation number and Grashof constant.

To begin, let u be a weak solution of the 3D Navier-Stokes equations. For $r \in (2,3)$ we define a local determining wavenumber

$$\Lambda_{u,r}(t) := \min\{\lambda_q : \lambda_p^{-1+\frac{3}{r}} \|u_p\|_{L^r} < c_r\nu,\ \forall p > q \text{ and } \lambda_q^{-1}\|u_{\le q}\|_{L^\infty} < c_r\nu,\ q \in \mathbb{N}\},$$

where c_r is an adimensional constant that depends only on r. Here $\lambda_q = \frac{2^q}{L}$ with L being the size of the torus, and $u_q = \Delta_q u$ is the Littlewood-Paley projection of u. Thanks to Bernstein's inequality, we have

$$\Lambda_{u,r} \ge \Lambda_u^{\mathrm{dis}} := \min\{\lambda_q : \lambda_p^{-1}\|u_p\|_{L^\infty} < c_0\nu,\ \forall p > q,\ q \in \mathbb{N}\},$$

which is a local dissipation wavenumber introduced by Cheskidov and Shvydkoy. It defines a dissipation range where a local Reynolds number corresponding to high frequencies is small, i.e.,

$$\mathcal{R}_q^h := \frac{l_q\|u_q\|_{L^\infty}}{\nu} < c_0, \qquad \forall \lambda_q > \Lambda_u^{\mathrm{dis}},$$

where $l_q = \lambda_q^{-1}$. The dominance of the dissipation term above Λ_u^{dis} is reflected in improved Beale-Kato-Majda and Prodi-Serrin criteria where u is replaced with its projection on modes below Λ_u^{dis}. The determining wavenumber $\Lambda_{u,r}$ imposes tougher condition on high modes, as well as requires a control on low modes via the low frequency Reynolds number

$$\mathcal{R}_q^l := \frac{l_q\|u_{\le q}\|_{L^\infty}}{\nu} < c_r, \qquad \lambda_q = \Lambda_{u,r}.$$

It is also worth mentioning that a similar determining wavenumber is used to prove the existence of a finite number of determining modes for the surface quasi-geostrophic equation equation in critical and subcritical cases. Even though the determining wavenumber enjoys uniform bounds in those cases, it still proved useful to start with a time dependent wavenumber defined based on the structure of the equation only, and then study its dependence on the force using available bounds for the global attractor.

We prove the following.

Theorem 1. *Let $u(t)$ and $v(t)$ be weak solutions of the 3D Navier-Stokes equations. Let $\Lambda(t) := \max\{\Lambda_{u,r}(t), \Lambda_{v,r}(t)\}$ for some $r \in (2,3)$. Let $Q(t)$ be such that $\Lambda(t) = \lambda_{Q(t)}$. If*

$$u(t)_{\leq Q(t)} = v(t)_{\leq Q(t)}, \qquad \forall t > 0, \tag{2}$$

then

$$\lim_{t\to\infty} \|u(t) - v(t)\|_{L^2} = 0.$$

We also have the following version of this result for solutions on the global attractor.

Theorem 2. *If $u(t)$ and $v(t)$ are two Leray-Hopf solutions on the weak global attractor $\mathcal{A}$ such that*

$$u(t)_{\leq Q(t)} = v(t)_{\leq Q(t)}, \qquad \forall t < 0, \tag{3}$$

where Q is given in Theorem 1, then

$$u(t) = v(t), \qquad \forall t \leq 0.$$

It is worthwhile to note that the determining wavenumber $\Lambda_{u,r}$ depends on time and may not be bounded. Actually, it is bounded if and only if u is regular. However, the average determining wavenumber $\langle\Lambda\rangle = \frac{1}{T}\int_t^{t+T} \Lambda_{u,r}(\tau)\, d\tau$ always enjoys a uniform bound. Indeed, we establish the following pointwise bound:

$$\Lambda_{u,r}(t) \lesssim_r \frac{\|\nabla u(t)\|_{L^2}^2}{\nu^2}. \tag{4}$$

Note that this automatically provides a finite number of determining modes and recovers the results by Constantin, Foias, Manley, and Temam in the case where $\|\nabla u(t)\|_{L^2}^2$ is bounded on the global attractor, which is known for small forces (laminar regimes). On the other hand, (4) holds in general for arbitrary forces and implies that $\langle\Lambda\rangle$ is uniformly bounded for all Leray-Hopf solutions on the global attractor, i.e., complete bounded trajectories. However, the bound (4) is sharp only in the case of extreme intermittency, where on average there is only one eddy at each dyadic scale. To make a connection with Kolmogorov's turbulence theory, we have to define an intermittency dimension and analyze $\langle\Lambda\rangle$ in various intermittency regimes.

We further examine $\langle\Lambda\rangle$, comparing it to Kolmogorov's dissipation wavenumber as well as the Grashof constant, defined as

$$\kappa_{\mathrm{d}} := \left(\frac{\varepsilon}{\nu^3}\right)^{\frac{1}{d+1}}, \qquad G := \frac{\|f\|_{H^{-1}}}{\nu^2\lambda_0^{1/2}}, \qquad \varepsilon := \lambda_0^d \nu\langle\|\nabla u\|_{L^2}^2\rangle,$$

where $d \in [0,3]$ is the intermittency dimension.This parameter is defined in terms of the level of saturation of Bernstein's inequality. The case $d = 3$, where there is no intermittency and eddies occupy the whole region, corresponds to Kolmogorov's regime. In this case the bounds read

$$\langle \Lambda \rangle \lesssim_\delta \kappa_d^{2+\delta} \lambda_0^{-1-\delta}, \qquad \langle \Lambda \rangle \lesssim_\delta \lambda_0 \left(\frac{G^2}{T\nu^2\lambda_0^2} + G^2 \right)^{\frac{1}{2}+\delta}, \qquad d = 3,$$

where δ can be arbitrary small when r is chosen close to 3. On the other hand, in the case of extreme intermittency, the bounds become

$$\langle \Lambda \rangle \lesssim \kappa_d, \qquad \langle \Lambda \rangle \lesssim \frac{G^2}{T\nu^2\lambda_0} + \lambda_0 G^2, \qquad d = 0.$$

We conjecture that

$$\langle \Lambda \rangle \lesssim \kappa_d$$

for values of d near 3 as well, as Kolmogorov's turbulence theory predicts. However, this is still an open problem.

Anomalous diffusion in the transport of passive scalars by a fast cellular flow.

Gautam Iyer

(joint work with Alexei Novikov)

Consider the advection diffusion equation

$$\partial_s \varphi + u \cdot \nabla \varphi - \nu \Delta \varphi = 0, \quad \text{for } x \in \mathbb{R}^2, s > 0. \tag{1}$$

We assume throughout that u is cellular; namely,

$$u = \nabla^\perp h$$

for a smooth periodic stream function h whose critical points are non-degenerate. Further we assume that u only has bounded trajectories (i.e. all orbits of the dynamical system $\dot{X} = u(X)$ are bounded). A prototypical example of such a vector field is given by the stream function $h(x, y) = \sin(x)\sin(y)$, and models a field of opposing vortices.

Our eventual aim is to study a scaling limit of this system in an intermediate time regime. We begin, however, by describing a few well known regimes where the limiting behaviour of (1) is well known.

The fixed time, zero-viscosity limit. Without a change of coordinates, as $\nu \to 0$ it is easy to see that $\varphi \to \varphi^0$, where

$$\partial_s \varphi^0 + u \cdot \varphi^0 = 0.$$

This is neither surprising, nor particularly interesting.

The fixed viscosity, long time limit. When viscosity is held fixed, the long time behaviour of (1) is diffusive with an *enhanced* diffusion coefficient. This originated with [1] and is by now a classical result. To state it precisely let $\epsilon > 0$ and consider the change of coordinates

$$x = \frac{y}{\epsilon}, \quad s = \frac{t}{\epsilon^2}.$$

Equation (1) now reduces to

$$\partial_t \varphi + \frac{1}{\epsilon} u\Big(\frac{y}{\epsilon}\Big) \cdot \nabla\varphi - \nu\Delta\varphi = 0.$$

In the (y, t)-coordinates, $\varphi \to \varphi_{\text{eff}}$ as $\epsilon \to 0$, where φ_{eff} satisfies the effective (homogeneous) equation

$$\partial_t \varphi_{\text{eff}} - \nu_{\text{eff}} \Delta\varphi_{\text{eff}} = 0,$$

where ν_{eff} is the effective diffusivity which asymptotically behaves like $\nu_{\text{eff}} \approx O(\sqrt{\nu})$ as $\nu \to 0$.

The zero-viscosity long time regime. In '53 GI Taylor [5] studied the effective dispersion of a solute in laminar flow. In this context, he showed that the variance of φ grows linearly after time of order $1/\nu$, and obtained an explicit formula for the rate of growth on this time scale.

In the context of cellular flows, we first consider (1) with *periodic boundary conditions.* Making the coordinate change

$$x = y, \quad s = \frac{t}{\nu}, \tag{2}$$

equation (1) reduces to

$$\partial_t \varphi + \frac{1}{\nu} u \cdot \nabla\varphi - \Delta\varphi = 0.$$

Freidlin (see for instance [2]) studied the limiting behaviour of the associated SDE as $\nu \to 0$ and obtained an effective process. At the PDE level, his result guarantees $\varphi \to \bar{\varphi}$ where $\bar{\varphi}$ is constant on connected components of level sets of the stream function h. Further, in coordinates given by h, φ satisfies

$$T(h)\partial_t \bar{\varphi} - \partial_h (P(h)\partial_h \bar{\varphi}) = 0,$$

with certain gluing conditions on cell boundaries. Here T and P are defined in a cell Q by

$$T(h_0) = \int_{Q \cap h^{-1}(h_0)} \frac{1}{|\nabla h|}\, dl, \quad \text{and} \quad P(h_0) = \int_{Q \cap h^{-1}(h_0)} |\nabla h|\, dl.$$

Returning to (1) on the whole space, however, the PDE approach begins to crumble. Various heuristic arguments can be made to estimate the variance of φ grows like $\sqrt{\nu} t$ in the coordinates given by (2). This suggests the scaling

$$x = \nu^{1/4} y, \quad s = \frac{t}{\nu},$$

should yield an effective equation. Indeed, a recent result of Hairer, Koralov and Pajor-Gyulai [3] settles this for the associated SDE. Namely, let

$$dX_t = -u(X_t)\,dt + \sqrt{2\nu}\,dW_t,$$

where W is a standard 2D Brownian motion. Then [3] shows

$$\nu^{1/4} X_{t/\nu} \xrightarrow{\nu\to 0} \bar{W}_{L_t^{-1}}, \tag{3}$$

where $\bar{W}$ is an effective Brownian motion and L is an independent local time process. Since the effective process is not Markov, there is no simple PDE analogue of this result.

The zero-viscosity intermediate time regime. In '88 [6] suggested that at time scales between 1 and $1/\nu$ a stable, robust anomalous diffusive behaviour is observed for particles that start on cell boundaries. The heuristic explanation given was that a large fraction of tracer particles will be trapped in cell interiors and contribute negligibly to the average travel distance, and a small fraction of "active" particles will travel ballistically along cell boundaries.

A recent result [4] proves a super-linear bound on the variance for times between 1 and $1/\nu$. In the original (x, s) coordinates the result is as follows:

Theorem 1 (Iyer, Novikov [4]). *Suppose $\varphi_0 = \delta(x_0)$ for some x_0 on a cell boundary. Then, there exists constants c_1 and c_2 independent of ν so that*

$$\frac{c_1\sqrt{s}}{|\ln \nu|} \leq \int_{\mathbb{R}^2} |x - x_0|^2 \varphi(x, s)\,dx \leq c_2\sqrt{s}$$

This result does not provide an effective behaviour on the time scale $1 \ll s \ll 1/\nu$, however, it shows that the effective behaviour (if it exists) is not not diffusive. We believe that an effective behaviour of the form (3) can be obtained in this regime, though the process L will have qualitatively different properties. The authors of [3] and [4] are collaborating and have a preliminary result to this effect.

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Blowup criteria for 3D Navier-Stokes equations: Can the vector potential serve as a blowup criterion?

Koji Ohkitani

1. Introduction

We consider the incompressible Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + \nu\triangle \boldsymbol{u}, \ \ \nabla\cdot\boldsymbol{u} = 0$$

with smooth initial data with finite energy in [4] $\mathbb{R}^3$ or $\mathbb{R}^2$.

Our motivations are as follows. In 3D, a critical norm $\|\boldsymbol{u}\|_{L^3}$ is known to be a blowup criterion [2]. Its proof is based an a highly sophisticated use of a contradiction argument. In 2D, global regularity of the Navier-Stokes equations is well-known.

We seek alternative proofs for these facts using the vector potential $\boldsymbol{A}$ in 3D and the stream function ψ in 2D. Some preliminary results are given in this direction. In view of embedding

$$\|\boldsymbol{A}\|_{\mathrm{BMO}} \le C\|\boldsymbol{u}\|_{L^3} \ \text{ in 3D},$$

and

$$\|\psi\|_{\mathrm{BMO}} \le C\|\boldsymbol{u}\|_{L^2} \ \text{ in 2D},$$

we ask whether $\|\boldsymbol{A}\|_{\mathrm{BMO}}$ serves as a blowup criterion in 3D and $\|\psi\|_{\mathrm{BMO}}$ in 2D. In other words, we conjecture that

$$\text{blowup at } t = t_* \text{ in 3D } \implies \|\boldsymbol{A}\|_{\mathrm{BMO}} \to \infty \ \text{ as } \ t\to t_*$$

and

$$\text{blowup at } t = t_* \text{ in 2D } \implies \|\boldsymbol{\psi}\|_{\mathrm{BMO}} \to \infty \ \text{ as } \ t\to t_*.$$

2. 2D Navier-Stokes equations

Using ψ, the Navier-Stokes equation reads [6]

$$\frac{\partial\psi}{\partial t} - \nu\triangle\psi = \frac{1}{\pi}\mathrm{P.V.}\int_{\mathbb{R}^2} \frac{[(\boldsymbol{x}-\boldsymbol{x'})\times\nabla\psi(\boldsymbol{x'})]\,(\boldsymbol{x}-\boldsymbol{x'})\cdot\nabla\psi(\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|^4}\,\mathrm{d}\boldsymbol{x'},$$

or

$$\frac{\partial\psi}{\partial t} - \nu\triangle\psi = \epsilon_{jk}R_iR_j\partial_k\psi\partial_i\psi.$$

By $-\triangle\psi = \omega$ and [1, 3] the viscous term becomes singular in the sense that

$$\int_0^{t_*}\|\triangle\psi\|_{\mathrm{BMO}}dt = \int_0^{t_*}\|\omega\|_{\mathrm{BMO}}dt = \infty.$$

There is an inversion formula

$$\partial_i\psi\partial_k\psi - \frac{1}{2}|\nabla\psi|^2\delta_{ik} = -\epsilon_{kl}R_iR_l(\psi_t - \nu\triangle\psi) + \frac{1}{2}(\psi_t-\nu\triangle\psi)\epsilon_{ik},$$

from which it follows that

$$c\|\boldsymbol{u}\|^2_{\mathrm{BMO}} \le \left\|⨍(\nabla\psi)^2\right\|_{L^\infty}.$$

By $\int_0^{t_*} \|\boldsymbol{u}\|^2_{\mathrm{BMO}} dt = \infty$ [7, 3], the nonlinear term also becomes singular

$$\int_0^{t_*} \left\| ⨍ (\nabla\psi)^2 \right\|_{L^\infty} dt = \infty$$

for a blowup.

3. 3D Navier-Stokes equations

With vector potentials $\boldsymbol{A}$ ($\boldsymbol{u} = \nabla \times \boldsymbol{A}, \nabla \cdot \boldsymbol{A} = 0$), the Navier-Stokes equations read [5]

$$\frac{\partial \boldsymbol{A}}{\partial t} - \nu \triangle \boldsymbol{A} = \frac{3}{4\pi} \mathrm{P.V.} \int_{\mathbb{R}^3} \frac{\boldsymbol{r} \times (\nabla \times \boldsymbol{A}(\boldsymbol{x}'))\, \boldsymbol{r} \cdot (\nabla \times \boldsymbol{A}(\boldsymbol{x}'))}{|\boldsymbol{r}|^5} \mathrm{d}\boldsymbol{x}',$$

where $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'$. Or, in components we have

$$\frac{\partial A_i}{\partial t} - \nu \triangle A_i = \epsilon_{kpq} R_j R_k \partial_p A_q (\partial_j A_i - \partial_i A_j).$$

By $-\triangle \boldsymbol{A} = \boldsymbol{\omega}$ and [1, 3], the viscous term becomes singular in the sense that

$$\int_0^{t_*} \|\triangle \boldsymbol{A}\|_{\mathrm{BMO}} dt = \int_0^{t_*} \|\boldsymbol{\omega}\|_{\mathrm{BMO}} dt = \infty.$$

A similar inversion formula is available and if blowup, the nonlinear term becomes unbounded as

$$c \int_0^{t_*} \|\boldsymbol{u}\|^2_{\mathrm{BMO}} dt \leq \int_0^{t_*} \left\| ⨍ (\nabla \boldsymbol{A})^2 \right\|_{L^\infty} dt = \infty.$$

4. Duhamel principle and heuristics

$$\left(\frac{\partial}{\partial t} - \nu \triangle \right) \boldsymbol{A} = ⨍ (\nabla \boldsymbol{A})^2, \quad \equiv \boldsymbol{f}$$

can be recast as

$$e^{\nu t \triangle} \frac{\partial}{\partial t} e^{-\nu t \triangle} \boldsymbol{A} = \boldsymbol{f},$$

or

$$\boldsymbol{A}(t) = e^{\nu t \triangle} \boldsymbol{A}(0) + \int_0^t e^{\nu (t-s) \triangle} \boldsymbol{f}(s) ds.$$

More explicitly, we have

$$\boldsymbol{A} = \int_0^t ds \int_{\mathbb{R}^3} \frac{1}{(4\pi\nu(t-s))^{3/2}} \exp\left(-\frac{|\boldsymbol{x} - \boldsymbol{y}|^2}{4\nu(t-s)} \right) \boldsymbol{f}(\boldsymbol{y}, s) d\boldsymbol{y}.$$

For an isotropic singularity $\boldsymbol{f}(\boldsymbol{y}, s) > \frac{1}{|\boldsymbol{y}|^2 + \nu(t-s)}$, the above spatial integral is

$$I \equiv \frac{4\pi}{(4\pi\nu\tau)^{3/2}} \int_0^\infty \exp\left(-\frac{r^2}{4\nu\tau} \right) \frac{r^2 dr}{r^2 + \nu\tau} = \frac{\sqrt{2}}{\nu\tau} \left(1 - e^{1/4} \frac{\sqrt{\pi}}{2} \mathrm{Erfc}\left(\frac{1}{2} \right) \right) \simeq \frac{0.64}{\nu\tau},$$

where $\tau = t - s$ and $\mathrm{Erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-u^2} du$. For this example, $\|\boldsymbol{A}\|_{L^\infty} \to \infty$ as $t \to t_*$.

For an isotropic singularity in 2D $\boldsymbol{f}(\boldsymbol{y}, s) > \frac{1}{|\boldsymbol{y}|^2+\nu(t-s)}$,

$$I \equiv \frac{1}{2\nu\tau}\int_0^\infty \exp\left(-\frac{r^2}{4\nu\tau}\right)\frac{rdr}{r^2+\nu\tau} = \frac{1}{4\nu\tau}e^{1/4}\mathrm{E}_1\left(\frac{1}{4}\right) \simeq \frac{0.34}{\nu\tau},$$

where $E_1(x) \equiv \int_x^\infty \frac{e^{-u}}{u}du, \ \ (x>0)$. For this example, $\|\psi\|_{L^\infty} \to \infty$ as $t \to t_*$.

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The Liouville and the unique continuation type results for the self-similar Euler system

Dongho Chae

The question of spontaneous apparition of singularity(blow-up)/global regularity in the 3D incompressible Euler equations is among the most outstanding open problems in the partial differential equations. There are many numerical/physical evidences that if the finite time blow-up happens it is highly probable that it is of the self-similar type(see e.g. [14]). In the case of the 3D Navier-Stokes equations, the question of self-similar blow-up is proposed by J. Leray in 1930[12], and answered negatively by Nečas-Ružička-Šverák[13] and Tsai[15]. The crucial tool of their proof is the maximum principle, which is originated from the ellipticity nature of the corresponding self-similar equations. In the case of Euler equations, mainly due to the lack of the elliptic structure in the self-similar equations we need to develop new methods. in this talk we review the result on this problem by myself in [2], and a series of further developments on the subject later by myself [3, 4, 5, 6, 9, 10]and my collaborators until the very recent Liouville type/the unique continuation type results on the time periodic solutions of the self-similar Euler equations[7, 8].

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Hydrodynamics in quantum and high-energy Physics

Yann Brenier

I. Madelung's transform of the Schrödinger equation. Madelung observed in 1926 that the (possibly nonlinear) Schrödinger equation (from 1925) $i\partial_t\psi + \triangle\psi = \alpha|\psi|^2\psi$ admits a hydrodynamic version:

$$(1) \qquad \partial_t\rho + \nabla\cdot q = 0, \quad \partial_t q + \nabla\cdot(\frac{q\otimes q + \beta\,\nabla\rho\otimes\nabla\rho}{\rho}) = \nabla(\beta\triangle\rho - \gamma\rho^2)$$

through the polar representation of the wave function $\psi = \sqrt{\rho}\exp(i\frac{\phi}{2}) \in \mathbb{C}$, with $\rho \geq 0$, $q = \rho\nabla\phi$, and a suitable relation between constants α, β and γ.

II: From quantum particles to Burgers' equation. The Wigner transform of the free Schrödinger equation is just the free transport equation $\partial_t f + \xi\cdot\nabla_x f = 0$ with typical initial conditions such as "wave packets" $f_0(x,\xi) = \exp(-\pi(\alpha|x|^2 + \beta|\xi|^2))$ where $\alpha\beta$ is limited by the Heisenberg principle. For a given family of

N points $(A_k \in \mathbb{R}^d)_{k=1}^N$, we consider the free transport equation in $\mathbb{R}^{2Nd}$ with symmetric initial condition

$$f(0, X, \Xi) = \sum_{\sigma \in \mathcal{S}_N} \Pi_{k=1}^N \exp(-\pi(\alpha|x_k - A_{\sigma_k}|^2 + \beta|\xi_k|^2)), \ \ (X, \Xi) = (x_k, \xi_k)_{k=1}^N,$$

with respect to the group $\mathcal{S}_N$ of all permutations of $\{1, \cdots, N\}$. Inspired by de Broglie's concept of "onde pilote", we introduce the dynamical system

$$\frac{dX}{dt} = \frac{\int_{\mathbb{R}^{Nd}} \ \Xi f(t, X, \Xi) d\Xi}{\int_{\mathbb{R}^{Nd}} \ f(t, X, \Xi) d\Xi}, \ \ X \in \mathbb{R}^{Nd}.$$

This (first order) dynamical system is very similar to the model of Kaehlerian geometry recently addressed by Berman and Onnheim in arXiv:1501.07820v2 and enjoys remarkable properties. In particular, we deduce from it, as $d = 1$ and the Planck constant is neglected, the very simple and highly dissipative mechanical model of N particles moving at constant speed with sticky collisions. Letting further N go to infinity, we may recover all "entropy solutions" of the "invisicid Burgers" equation, one of the simplest model known in Hydrodynamics. In addition, in any dimension, we may derive from the second order version of this dynamical system the so-called Monge-Ampère gravitational model discussed in Y.B. Confluentes Math. 2010 and, recently, Y.B. arXiv:1504.07583 (through a large deviation approach).

III: Born-Infeld, MHD and magnetic relaxation. The Born-Infeld model, quite popular in high energy Physics and String Theory, involves a $d+1$ dimensional Lorentzian space-time manifold of metric $g_{ij}dx^i dx^j$ and vector potentials $A = A_i dx^i$ that are critical points of the (fully covariant) "action" $\int \sqrt{-\det(g + dA)}$. Here, we only consider the 3+1 Minkowski space (as Born and Infeld did in 1934), for which the BI model admits a paradoxical Galilean hydrodynamic formulation (Y.B. Arma 2004):

$$\partial_t B + \nabla \times (B \times v + \rho^{-1} D) = 0, \ \ \partial_t D + \nabla \times (D \times v - \rho^{-1} B) = 0,$$

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \ \partial_t(\rho v) + \nabla \cdot (\rho v \otimes v - \frac{B \otimes B - D \otimes D}{\rho}) = \nabla(\rho^{-1}).$$

We observe an interesting structure after a quadratic rescaling of time:

$$t \to \theta = t^2/2, \quad (\rho, B)(t, x) \to (\rho, B)(\theta, x), \quad (v, D)(t, x) \to (v, D)(\theta, x)\frac{d\theta}{dt}$$

$$\partial_\theta \rho + \nabla \cdot (\rho v) = 0, \quad \partial_\theta B + \nabla \times (B \times v + \rho^{-1} D) = 0,$$

$$D - \nabla \times (\rho^{-1} B) = -2\theta[\partial_\theta D + \nabla \times (D \times v)],$$

$$\rho v - \nabla \cdot (\frac{B \otimes B}{\rho}) - \nabla(\rho^{-1}) = -2\theta[\partial_\theta(\rho v) + \nabla \cdot (\rho v \otimes v - \frac{D \otimes D}{\rho})].$$

This reveals two remarkable asymptotic systems as $\theta >> 1$ or $\theta << 1$. One

$$\partial_\theta \rho + \nabla \cdot (\rho v) = 0, \quad \partial_\theta D + \nabla \times (D \times v) = 0, \quad \partial_\theta(\rho v) + \nabla \cdot (\rho v \otimes v - \frac{D \otimes D}{\rho}) = 0$$

(as $\theta >> 1$) just describes the motion of a well-ordered (i.e. crossing-free) continuum of free strings (i.e. extremal surfaces in the Minkowski space). The second one (obtained as $\theta << 1$) is a sort of Darcy model of MHD, including a dissipative induction equation with magnetic resistivity

$$\partial_\theta B + \nabla \times (B \times v + \rho^{-1}\nabla \times (\rho^{-1}B)) = 0.$$

Its incompressible version, without magnetic resistivity, is nothing but one of Moffatt's magnetic relaxation models of topological Hydrodynamics, say

$$\partial_\theta B + \nabla \times (B \times v) = 0, \quad v = \nabla \cdot (B \otimes B) + \nabla p, \quad \nabla.v = 0,$$

for which the global existence of "dissipative" solutions "à la P.-L. Lions" (which are unique when smooth) has been recently proven for $d = 2$ (see Y.B. CMP 2014, as well as Y.B. arXiv:1410.0333 for a closely related problem).

A Tropical Model: Global Well-posedness and Relaxation Limit

Jinkai Li

(joint work with Edriss S. Titi)

In the context of the large-scale atmospheric and oceanic dynamics, the aspect ratio of the vertical scale to the horizontal scale is very small. Taking this advantage and adopting the Boussinesq approximation, one can derive the primitive equations (PEs), for the large-scale atmosphere and ocean, from the Navier-Stokes equations by taking the small aspect ratio limit. Such small aspect ratio limit is strongly, uniformly and globally in time, see Li–Titi [1].

It is observed in physics that the wind in the lower troposphere is of equal magnitude but with opposite sign to that in the upper troposphere, in other words, the primary effect is captured in the first baroclinic mode. However, for the tropical-extratropical interaction, where the transport of momentum between the barotropic and baroclinic modes plays an important role, it is necessary to retain both the barotropic and baroclinic modes of the velocity. Thanks to these facts, by taking the Galerkin projection to the primitive equations in the vertical variable up to the first baroclinic mode, Frierson–Majda–Pauluis derived in [2] a nonlinear interaction system between the barotropic mode and the first baroclinic mode of the tropical atmosphere with moisture. In the system, there is convective adjustment relaxation time parameter ε, which in physics is positive but very small.

We establish the global existence and uniqueness of strong solutions to this system, with any initial data in H^1, for each fixed convective adjustment relaxation time parameter $\varepsilon > 0$. Moreover, if the initial data enjoy slightly more regularity than H^1, then the unique strong solution depends continuously on the initial data. Furthermore, by establishing several appropriate ε-independent estimates, we prove that the system converges to a limiting system, as the relaxation time parameter ε tends to zero, with convergence rate of the order $O(\sqrt{\varepsilon})$. Moreover, the limiting system has a unique global strong solution, for any initial data in H^1, and such unique strong solution depends continuously on the initial data if the the initial data posses slightly more regularity than H^1. Notably, this solves the VISCOUS VERSION of an open problem proposed in the above mentioned paper of Frierson, Majda and Pauluis.

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Thermodynamics and statistical constraints in Q-tensor models of nematic liquid crystals

Arghir Dani Zarnescu

(joint work with E. Feireisl, E. Rocca, G. Schimperna)

In [2] and [3] we proposed thermodynamically consistent models of nematics within the framework proposed by M. Fremond in [4] and proved the existence of weak solutions.

The main difference between the two models consists in the way the temperature couples with the potential proposed in [1] to enforce the statistical constraints relevant to the Q-tensor description of nematics. If the temperature couples with the regular part of the potential, as in [2] then one can obtain stronger estimates then in the case when it couples with the singular part of the potential (as in [3]). In the latter case a certain specific type of convexity of the singular part is necessary in order to be able to close the estimates and obtain weak solutions.

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Turbulence in zero viscosity limit with Boundary effects

Claude Bardos

1. Introduction

This is an abstract of a talk where I reported on progresses recently made in collaboration with several colleagues: F. Golse and L. Paillard, T. Nguyen, L. Szekelyhidi, E. Titi and E. Wiedemann, mainly contained in the following papers: [1] , [2], [3] and [4].

2. Abstract

There is a strong analogy between the notion of statistical turbulence and the notion of weak convergence for deterministic regimes. Both are based on some kind of average. In particular the appearance of turbulence is related to anomalous energy dissipation while weak convergence also differs from strong convergence by energy dissipation, i.e; the lower semi continuity of the L^2 norm. Notably it turns out that this effect is the most explicit in the presence of physical boundary. This is in full agreement with the following fact: In most of the physical experiments the turbulence is generated by some boundary effect.

Therefore in the present talk I do not touch the "Clay problem" and I assume the existence of a smooth solution $u(x,t)$, $x \in \Omega$, $t \in [0,T]$, of the Euler equations subject to no-normal flow at the boundary and of smooth solutions of the Navier-Stokes equations with the same initial data. I will consider both systex with the same initial data and on a finite fixed time interval $[0,T]$

Since the energy is the only uniform estimate available, I use it to revisit the basic criteria of Kato. To the best of my knowledge this is the only deterministic scenario where one can relate anomalous dissipation of energy with appearance of turbulence. The solenoidal (incompressible) Navier-Stokes and Euler equations in $\Omega \subset \mathbb{R}^d, d=2, d=3$, are written below.

$$
\begin{aligned}
&u_\nu(x,0) = u(x,0)\,, \quad Re = \frac{UL}{\nu^{\text{fluide}}}\,, \quad \nu = Re^{-1}\,,\\
&\partial_t u_\nu + (u_\nu\cdot\nabla)u_\nu - \nu\Delta u_\nu + \nabla p_\nu = 0\,,\\
(1)\quad &\text{ in } \Omega\times[0,T] \quad \nabla\cdot u_\nu = 0\,, \text{on} \quad \partial\Omega\times(0,T) \quad u_\nu\cdot\vec n = 0\,, \text{ and } \quad (u_\nu)_\tau = 0\,,\\
&\partial_t u + (u\cdot\nabla)u + \nabla p = 0\,.\\
&\text{in } \Omega\times[0,T] \quad \nabla\cdot u = 0\,, \text{on } \partial\Omega\times(0,T) \quad u_\nu\cdot\vec n = 0\,, u_\nu(x,0) = u(x,0)\,.
\end{aligned}
$$

The obvious difficulty comes from the fact that only the impermeability condition remains at the limit as $\nu \to 0$. The relation $(u_\nu)_\tau = 0$ needs not persist. Therefore, the solution of the Navier-Stokes equation equations may become singular near the boundary. Moreover, due to the nonlinearity of the advection term $u\cdot\nabla u$ and to the effect of the pressure, such singularities may propagate inside the domain. This turn out to be the most natural effect to generate turbulence, even for homogenous turbulence observed far from the boundary.

In presence of a smooth solution u of the Euler equations a simple manipulation leads to a stability estimate:

$$\begin{aligned} &\frac{d}{dt}(\frac{1}{2}|u_\nu - u|^2_{L^2(\Omega)}) + \nu\int_\Omega |\nabla u_\nu|^2 dx \le |(u_\nu - u, \frac{\nabla u + \nabla^t u}{2}(u_\nu - u))| \\ &\qquad + \nu\int_\Omega (\nabla u_\nu \cdot \nabla u)dx - \nu\int_{\partial\Omega} \partial_{\vec{n}} u_\nu u d\sigma\,. \end{aligned} \tag{2}$$

which indicates that the contributions this balance l is the behavior of the term

$$\nu\int_{\partial\Omega} (\partial_{\vec{n}} u_\nu)_\tau u d\sigma\,.$$

This is because u is tangent to the boundary, *but also in most of the configuration not equal to zero on this boundary.* This leads to the following remarks

- In the absence of a physical boundary and in the presence of a smooth solution u of the Euler equations with the same initial data the convergence of u_ν to u holds in $L^\infty(0,T;L^2(\Omega))$.
- In the presence of a physical boundary, even in the presence of a smooth solution of the Euler equations, the only uniform estimate is the energy estimate

$$\int_\Omega \frac{|u_\nu(x,t)|^2}{2}dx + \nu\int_0^t\int_\Omega |\nabla u_\nu(x,s)|^2 dxds = \int_\Omega \frac{|u(x,0)|^2}{2}dx\,. \tag{3}$$

Hence the only available result is an updated version of a theorem of T. Kato [6]:

Theorem 1. *The following facts are equivalent:*

$$u_\nu(t) \to u(t) \text{ in } L^2(\Omega) \text{ uniformly in } t \in [0,T]\,, \tag{4}$$

$$u_\nu(t) \to u(t) \text{ weakly in } L^2(\Omega) \text{ for } t = T\,, \tag{5}$$

$$\lim_{\nu\to 0} \nu\int_0^T\int_\Omega |\nabla u_\nu(x,t)|^2 dxdt = 0\,, \tag{6}$$

$$\lim_{\nu\to 0} \nu\int_0^T\int_{\Omega\cap\{d(x,\partial\Omega)<\nu\}} |\nabla u_\nu(x,t)|^2 dxdt = 0\,, \tag{7}$$

$$\forall w \in C^1([0,T];\partial\Omega) \quad \lim_{\nu\to 0} \nu\int_0^T\int_{\partial\Omega} (\frac{\partial u_\nu}{\partial\vec{n}}(\sigma,t))_\tau w(\sigma,t))d\sigma dt = 0\,. \tag{8}$$

The fact that (4) $\Rightarrow$ (5) $\Rightarrow$ (6) was already observed by Kato as a trivial consequence of (3). In particular with (2) one observes that (8) $\Rightarrow$ (4). Eventually, one adapts the method of Kato [6] to extend the vector field $w(\sigma,t)$ defined on the boundary by a convenient divergence free vector field $w_\nu(x,t)$ with support in a layer of size ν near the boundary. Then one multiplies the Navier Stokes equation by this vector field, uses integration by part and Poincaré inequality to deduce (8) from (7).

Remark 2. The criteria (8) was introduced in [1] and systematized in [3]. It turns out to be very robust with respect to different applications. It produces a direct proof of the result given in [5]. In [1] it was used to study the limit of the solution of the Boltzmann equation in an " incompressible scaling" (cf. [7]) . In [4] it is used to study the convergence of solutions of the "compressible Navier-Stokes" equations and this indicates that in the presence on the time interval $(0;T)$ of a smooth solution of the Euler equations, there no much difference in the behavior for $\nu \to 0$ between the incompressible and the compressible point of view.

Remark 3. In the engineering literature for description boundary effects one introduces (cf. [8]) based on scaling argument and explicit computations at least three subdomains corresponding to the Prandlt-laminar regime, the laminar regime with recirculation and the turbulent boundary layer. In most cases these regimes are introduced without any mathematical justification ,however it is important to notice that they imply behaviors which are in full agreement with the Kato criteria.

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