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Non-Archimedean Geometry and Applications

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Abstract. The workshop focused on recent developments in non-Archimedean analytic geometry with various applications to other fields, in particular to number theory and algebraic geometry. These applications included Mirror Symmetry, the Langlands program, p-adic Hodge theory, tropical geometry, resolution of singularities and the geometry of moduli spaces. Much emphasis was put on making the list of talks to reflect this diversity, thereby fostering the mutual inspiration which comes from such interactions.

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Introduction by the Organisers

The workshop on Non-Archimedean Analytic Geometry and Applications, organized by Vladimir Berkovich (Rehovot), Walter Gubler (Regensburg), Peter Schneider (Münster) and Annette Werner (Frankfurt) had 53 participants. Non-Archimedean analytic geometry is a central area of arithmetic geometry. The first analytic spaces over fields with a non-Archimedean absolute value were introduced by John Tate and explored by many other mathematicians. They have found numerous applications to problems in number theory and algebraic geometry. In the 1990s, Vladimir Berkovich initiated a different approach to non-Archimedean analytic geometry, providing spaces with good topological properties which behave similarly as complex analytic spaces. Independently, Roland Huber developed a similar theory of adic spaces. Recently, Peter Scholze has introduced perfectoid spaces as a ground breaking new tool to attack deep problems in p-adic Hodge theory and representation theory.

Recent years have seen a growing interest in such spaces since they have been used to solve several deep questions in arithmetic geometry. The goal of the workshop was to bring together researchers from different areas for an exchange of ideas which may facilitate future developments. Meanwhile, applications of non-Archimedean spaces have become so diverse that the workshop filled a gap in the recent list of conferences by providing a platform to exchange new results, ideas and open problems between the different branches of the subject. In fact, during the months before the workshop the organizers received numerous requests, also from some internationally renowned mathematicians, to be included in the list of participants.

We had 19 one hour talks in this workshop. A summary of the topics can be found below. All talks were followed by lively discussions, in the form of plenary questions and also in the form of blackboard discussions in smaller groups. Several participants explained work in progress or new conjectures or promising techniques to attack open conjectures. The workshop provided a lively platform to discuss these new idea with other experts.

During the workshop, we saw new structure results for affinoid spaces over the ring of integers (Poineau) and recent progress regarding skeleta of Berkovich spaces (Ducros and Loeser). Skeleta are polyhedral substructures which are deformation retracts, and which can be used to investigate the topology of Berkovich spaces. Loeser reported on his model-theoretic approach to skeleta (jointly with Hrushovsky) which leads to the proof of local contractibility of Berkovich spaces associated to varieties over non-Archimedean fields.

A surprising application of the non-Archimedean theory of skeleta to an important problem in diophantine geometry was presented in Rabinoff's talk. In joint work with Katz and Zureick-Brown, partial very explicit solutions of the uniform Mordell conjecture and of the uniform Manin-Mumford conjecture were proved. The power of non-Archimedean geometry to give classical problems a new point of view was also seen in Chambert-Loir's talk on a non-Archimedean Ax–Lindemann theorem and in Zhang's conjecture of a non-Archimedean Poisson formula.

Several talks dealt with progress in tropical geometry and tropical moduli spaces (Payne, Nicaise, Tyomkin, Ulirsch). Moreover, applications to mirror symmetry were presented, in particular regarding a new and very promising theory of intersections to deal with Gromov-Witten invariants (Yu). Geometric applications of non-Archimedean geometry for resolutions of singularities in positive characteristics are given via some precise analysis of de Jong's alterations (Temkin). Applications in positive characteristic included new results on p-adic curvature (Esnault).

A very influential recent development is Scholze's theory of perfectoid spaces, which is based on adic spaces and which has become a crucial tool in p-adic Hodge theory. In this area we have seen spectacular recent progress in a possible reduction of the local Langlands program in number theory to a purely geometric analog of the geometric Langlands conjectures (Fargues). Related areas are relative p-adic Hodge theory (Kedlaya), p-adic representations (Strauch), Fourier transformations on $\mathbb{Q}_p$ (Baldassari) and overconvergent modular forms (Hansen). Nizioł presented interesting results with Colmez for p-adic nearby cycles using syntomic cohomology.

Apart from the plenary talks, the participants had many discussions in small groups. The organizers made a specific effort to invite Phd students and Postdocs. Altogether we had 14 participants from this group. For most of them it was the first Oberwolfach workshop they ever attended. The unique Oberwolfach atmosphere provided a singular opportunity of meeting the international leaders of the subject and of keeping track with current developments. During the breaks and in the evenings many informal mathematical discussions took place, in which the young participants played an active role.

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Workshop: Non-Archimedean Geometry and Applications

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Abstracts

Affinoid spaces over Z

Jérôme Poineau

When developing analytic geometry over $\mathbf{Q}_p$, the first objects to consider are the so-called Tate algebras $\mathbf{Q}_p\{T_1, \dots, T_n\}$. They contain the power series with coefficients in $\mathbf{Q}_p$ that converge on the closed unit disk of center 0 in $\mathbf{Q}_p^n$. Remark that this last condition behaves well thanks to the non-archimedean triangle inequality.

In the archimedean setting, one needs to modifiy it and is led to consider instead "overconvergent Tate algebras" made of power series that converge in some arbitrary neighborhood of the closed unit disk. The same construction actually works over the ring of integers $\mathbf{Z}$. Generalizing slightly, for $r_1, \dots, r_n > 0$, we define $\mathbf{Z}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}^\dagger$ to be the ring of power series with coefficients in $\mathbf{Z}$ that converge in some neighborhood of the closed disk $\bar{D} = \bar{D}(0, (r_1, \dots, r_n)) \subset \mathbf{C}^n$.

To develop p-adic analytic geometry, one starts by studying the algebraic properties of those Tate algebras and, in particular, showing that they are noetherian. In order to do so, techniques that are quite specific to the non-archimedean setting are used, most notably the reduction map that enables to pass from a ring of power series over k to a ring of polynomials over the residue field $\tilde{k}$. Over $\mathbf{Z}$ (and over $\mathbf{C}$ too), such methods do not exist and the noetherianity result appears to be much more challenging. To the best of the knowledge of the author, until very recently, the only available result in this direction was the following theorem of D. Harbater, for $n = 1$.

Theorem 1 ([Har84, theorem 1.8]). *For every $r > 0$, the ring $\mathbf{Z}\{r^{-1}T\}^\dagger$ is noetherian.*

The proof is quite technical and relies on explicit descriptions. It is very unlikely that such a strategy can be made to work for a larger number of variables.

1. The complex setting

When replacing $\mathbf{Z}$ by $\mathbf{C}$, the analogous result is known, as a consequence of the following theorem of J. Frisch.

Theorem 2 ([Fri67, théorème I, 9]). *Let X be a complex analytic space and K be a compact subset of X that is semi-analytic and Stein. Then, the ring $\mathscr{O}(K)^\dagger$ of analytic functions that converge in some neighborhood of K is noetherian.*

Recall that a subset K of a complex analytic space $X$is said to be *semi-analytic* if it is locally defined by a finite number of inequations involving analytic functions and that it is said to be *Stein* if, for every coherent sheaf $\mathscr{F}$ defined in a neighborhood of K, we have

- for every $x \in K$, the stalk $\mathscr{F}_x$ is generated by the set of global sections $H^0(K, \mathscr{F})^\dagger$ (Cartan's theorem A);
- for every $q \geqslant 1$, $H^q(K, \mathscr{F}) = 0$ (Cartan's theorem B).

The proof is very geometric and makes a crucial use of the following properties:

(1) the local rings $\mathscr{O}_x$ are noetherian;
(2) the structure sheaf $\mathscr{O}$ is coherent;
(3) every compact semi-analytic space has finitely many connected components.

When applied to the compact $K = \bar{D}(0,(r_1,\dots,r_n)) \subset \mathbf{C}^n$, the theorem shows that the ring $\mathbf{C}\{r_1^{-1}T_1,\dots,r_n^{-1}T_n\}^\dagger$ is noetherian.

2. Berkovich analytic spaces over Z

In order to use a strategy that is similar to the one used in the complex setting, one needs to have analytic spaces with good properties on which the rings $\mathbf{Z}\{r_1^{-1}T_1,\dots,r_n^{-1}T_n\}^\dagger$ naturally appear as rings of functions on some compact sets. Berkovich analytic spaces over $\mathbf{Z}$ meet all those requirements.

Those spaces have been defined by V. Berkovich at the end of the first chapter of the monograph [Ber90]. Without going into the details, let us recall that the affine analytic space $\mathbf{A}^{n,\mathrm{an}}_{\mathbf{Z}}$ of dimension n over $\mathbf{Z}$ is defined as the set of multiplicative semi-norms on $\mathbf{Z}[T_1,\dots,T_n]$, endowed with the topology of pointwise convergence. An analytic function on this space is defined to be locally a uniform limit of rational functions without poles.

Since the absolute values over $\mathbf{Z}$ can be archimedean or not, the spaces $\mathbf{A}^{n,\mathrm{an}}_{\mathbf{Z}}$ contain fibers that are non-archimedean (and look like p-adic Berkovich analytic spaces) and others that are archimedean (and look like complex analytic spaces, possibly modulo complex conjugation). Moreover, one may define a relative closed disk $\bar{\mathbf{D}} = \bar{\mathbf{D}}(0,(r_1,\dots,r_n))$ around the 0 section in $\mathbf{A}^{n,\mathrm{an}}_{\mathbf{Z}}$ and the ring of functions that converge in some neighborhood of this disk is exactly the ring $\mathbf{Z}\{r_1^{-1}T_1,\dots,r_n^{-1}T_n\}^\dagger$ defined above. We refer to [Poi10b, annexe B] for a gentle introduction and to [Poi10a] for more details, including a complete study of the affine line.

3. Local properties

The article [Poi13] is devoted to the local study of analytic spaces over $\mathbf{Z}$. The main tool is a quite general local Weierstrass division theorem for the affine line over a Banach ring $(\mathscr{A},\|\cdot\|)$ with mild conditions on $\mathscr{A}$ (that are automatically met if $\mathscr{A}$ is $\mathbf{Z}$ endowed with the usual absolute value or the completion of the ring of functions on a relative disk over $\mathbf{Z}$ for instance).

Theorem 3 ([Poi13, théorème 8.3])**.** *Denote by $\pi\colon X = \mathbf{A}^{1,\mathrm{an}}_{\mathscr{A}} \to B = \mathbf{A}^{0,\mathrm{an}}_{\mathscr{A}}$ the projection morphism. Let $b \in B$. Let $P \in \mathscr{H}(b)[T]$ be an irreducible polynomial and let x be the point of the fiber $X_b = \pi^{-1}(b)$ such that $P(x) = 0$. Let G be an element of $\mathscr{O}_{X,x}$ whose image in $\mathscr{O}_{X_b,x}$ is not zero.*

Then, there exists a non-negative integer m such that every element F of $\mathscr{O}_{X,x}$ may be written uniquely in the form $F = QG + R$, with $Q \in \mathscr{O}_{X,x}$ and $R \in \mathscr{O}_{B,b}[T]_{<m}$.

With this result at hand, one may deduce many local properties of the space $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$ by using a strategy that is close to the one used in the complex analytic setting.

Corollary 4.
- *For every point $x \in \mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$, the local ring $\mathscr{O}_x$ is henselian, noetherian, regular and excellent.*
- *The structure sheaf $\mathscr{O}$ on $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$ is coherent.*

Thanks to the Weierstrass division theorem, one may also prove a sort of noetherianity result for coherent sheaves on $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$.

Corollary 5. *Let U be an open subset of $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$. Let $\mathscr{F}$ be a coherent sheaf on U and let $(\mathscr{F}_m)_{m\geqslant 0}$ be an increasing subsequence of subsheaves of $\mathscr{F}$. Then, every point x of U admits a neighborhood V in U such that the sequence $(\mathscr{F}_{m,|V})_{m\geqslant 0}$ is eventually constant.*

4. Global properties

In order to adapt the strategy of Frisch's proof, we also need to know that disk are Stein spaces. This is indeed the case.

Theorem 6. *Let $r_1, \ldots, r_n > 0$. Set $\bar{\mathbf{D}} = \bar{\mathbf{D}}(0, (r_1, \ldots, r_n)) \subset \mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$. Let $\mathscr{F}$ be a coherent sheaf defined in a neighborhood of $\bar{\mathbf{D}}$. Then, we have*
- *for every $x \in \bar{\mathbf{D}}$, the stalk $\mathscr{F}_x$ is generated by the set of global sections $H^0(\bar{\mathbf{D}}, \mathscr{F})^\dagger$ (theorem A);*
- *for every $q \geqslant 1$, $H^q(\bar{\mathbf{D}}, \mathscr{F}) = 0$ (theorem B).*

Let us explain some consequence of there results for affinoid spaces over $\mathbf{Z}$. Let us first give a definition in the spirit of the classical definition of affinoid spaces in rigid geometry. Consider a disk $\bar{\mathbf{D}} = \bar{\mathbf{D}}(0, (r_1, \ldots, r_n)) \subset \mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$ and a finite number of functions $f_1, \ldots, f_m \in \mathscr{O}(\bar{\mathbf{D}})^\dagger$. Set

$$V = V(f_1, \ldots, f_m) = \{x \in \bar{\mathbf{D}} \mid \forall i \in [\![1, m]\!], f_i(x) = 0\}$$

and denote by j_V the inclusion of V in $\bar{\mathbf{D}}$. Let $\mathscr{I}$ be the sheaf of ideals on $\bar{\mathbf{D}}$ generated by $(f_1, \ldots, f_m)$. An *overconvergent affinoid space over* $\mathbf{Z}$ is defined to be a space isomorphic to $(V, j_V^{-1}(\mathscr{O}_{\bar{\mathbf{D}}}/\mathscr{I}))$. It is easy to deduce from theorem 6 that theorems A and B still hold for such spaces.

We would like to point out that those results are very similar to classical results in rigid analytic geometry: theorem A is analogous to Kiehl's theorem whereas theorem B resembles Tate's acyclicity theorem. For the former, this is clear. For the later, let us remark that a short argument involving the exact sequence $\mathscr{O}^m \xrightarrow{(f_1,\ldots,f_m)} \mathscr{O} \to \mathscr{O}/\mathscr{I} \to 0$ and theorem B ensures that the global sections on V are exactly those one might expect:

$$\mathscr{O}(V) \simeq \mathbf{Z}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}^\dagger/(f_1, \ldots, f_m).$$

This means that, if one would like to follow Tate's original construction and define a presheaf on an affinoid space by its global sections on its affinoid domains, then one would recover the structure sheaf we started with. In particular, this presheaf is a sheaf, which is one important part of Tate's acyclicity theorem.

5. Noetherianity

Let us finally go back to the noetherianity question we started with. The classical proof of Frisch's theorem uses a topological argument: compact semi-analytic subsets have only finitely many connected components. This is unknown in the theory of Berkovich analytic spaces over $\mathbf{Z}$, where the topological aspects are not well developed. (Let us however mention that T. Lemanissier recently proved that those spaces are locally arcwise connected in [Lem15]).

However, by using corollary 5 and the noetherianity of $\mathbf{C}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}^\dagger$, one is able to prove the expected result. Let us mention that this strategy is close to the one Langmann used in his proof of Frisch's theorem (see [Lan77]).

Theorem 7. *For every $r_1, \ldots, r_n \in (0,1)$, the ring $\mathbf{Z}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}^\dagger$ is noetherian.*

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Overconvergent modular forms: A perfectoid point of view

David Hansen

(joint work with Przemyslaw Chojecki, Christian Johansson)

Let $N \geqslant 5$ be an integer, and let $Y = Y_1(N) \subset X = X_1(N)$ be the usual modular curves over $\mathbf{Q}$. A holomorphic modular form weight k and level N admits two rather distinct interpretations, which one might call the *algebraic* and *analytic* points of view:

Algebraic: It's a global section $\omega(f)$ of the line bundle $\omega^{\otimes k}$ on $X_{\mathbf{C}}$.

Analytic: It's a holomorphic function f on the upper half-plane $\mathfrak{h}$ of moderate growth, satisfying the transformation rule $f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$ for all $\gamma \in \Gamma_1(N)$.

How do we pass between these points of view? The key is that:

i. Writing $\tilde{Y}$ for the universal cover of the complex analytic space $Y_{\mathbf{C}}^{\mathrm{an}}$, there is a *canonical* isomorphism $\tilde{Y} \cong \mathfrak{h}$ of complex analytic spaces, and
ii. The pullback of ω under the induced map $\pi : \mathfrak{h} \to Y_{\mathbf{C}}^{\mathrm{an}}$ is trivialized by a canonical differential η_{can} which satisfies $\eta_{\mathrm{can}}(\gamma z) = (cz+d)^{-1}\eta_{\mathrm{can}}(z)$.

The objects f and $\omega(f)$ are then related by the identity

$$\pi^*\omega(f) = f(z) \cdot \eta_{\mathrm{can}}^{\otimes k}.$$

Now fix a prime $p \nmid N$, and let $\mathcal{Y} = Y_{\mathbf{Q}_p}^{\mathrm{ad}} \subset \mathcal{X} = X_{\mathbf{Q}_p}^{\mathrm{ad}}$ be the associated analytic modular curves. For any open subgroup $K \subset \mathrm{GL}_2(\mathbf{Z}_p)$ we have associated coverings $\mathcal{Y}_K \subset \mathcal{X}_K$ of $\mathcal{Y} \subset \mathcal{X}$.

Theorem (Katz, Coleman, Coleman-Mazur, Andreatta-Iovita-Stevens, Pilloni, [1, 3]**).** *For $L/\mathbf{Q}_p$ finite and any continuous character $\kappa : \mathbf{Z}_p^\times \to L^\times$, there is a natural space $M_\kappa^\dagger = M_\kappa^\dagger(N)$ of "p-adic overconvergent modular forms of weight κ and level N". This is an (ind-)Banach space over L with a natural action of the Hecke algebra, and the association $\kappa \mapsto M_\kappa^\dagger$ varies analytically as a function of κ.*

The definition of $M_\kappa^\dagger$ mirrors the *algebraic* definition of classical modular forms. More precisely, Andreatta-Iovita-Stevens and Pilloni define a line bundle ω^κ on a certain family of open subsets $\{\mathcal{X}(v) \subset \mathcal{X}_{K_0(p)}\}_{0<v<\epsilon}$, and then set

$$M_\kappa^\dagger = \lim_{v \to 0^+} H^0(\mathcal{X}(v), \omega^\kappa).$$

When $\kappa(x) = x^k, k \in \mathbf{Z}$, there is a natural isomorphism $\omega^\kappa \cong \omega^{\otimes k}|_{\mathcal{X}(v)}$, and the assignment $\kappa \mapsto \omega^\kappa$ is analytic as a function of κ; these two properties essentially characterize ω^κ uniquely.

In our work, we given an *analytic* definition of $M_\kappa^\dagger$. The replacement for the upper half-plane $\mathfrak{h}$ turns out to be given by (certain subspaces of) Scholze's *infinite level modular curve:*

Theorem (Scholze, [4]**).** *There is a natural perfectoid space $\mathcal{X}_\infty$ such that*

$$\mathcal{X}_\infty \cong \lim_{\leftarrow n} \mathcal{X}_{K(p^n)}$$

in the category of stably uniform adic spaces over $\mathbf{Q}_p$, equivariantly for natural (right) actions of $\mathrm{GL}_2(\mathbf{Q}_p)$ on both sides, and there is a natural Hodge-Tate period map

$$\pi_{\mathrm{HT}} : \mathcal{X}_\infty \to \mathbf{P}^1.$$

Using this theorem, we were able to prove the following result.

Theorem (Chojecki-H.-Johansson, [2]**).** *There is a natural family of $K_0(p)$-stable open affinoid perfectoid subsets $\mathcal{X}_{\infty,w} \subset \mathcal{X}_\infty$ indexed by $w \in \mathbf{Q}_{>0}$, with*

$\mathcal{X}_{\infty,w'} \subseteq \mathcal{X}_{\infty,w}$ for $w' \geqslant w$, together with a natural global section $\mathfrak{z} \in \mathcal{O}(\mathcal{X}_{\infty,w})$ compatible with changing w and such that

$$\gamma^*\mathfrak{z} = \frac{a\mathfrak{z} + c}{b\mathfrak{z} + d}$$

for all $\gamma \in K_0(p)$. For any κ as above and any $w \gg_\kappa 0$, the space

$$M_{\kappa,w} = \left\{ f \in \mathcal{O}(\mathcal{X}_{\infty,w}) \otimes_{\mathbf{Q}_p} L \mid \gamma^* f = \kappa(b\mathfrak{z} + d)^{-1} f \ \forall \gamma \in K_0(p) \right\}$$

is well-defined, and $M_\kappa^\dagger \cong \lim_{w\to\infty} M_{\kappa,w}$ compatibly with all structures.

So $M_{\kappa,w}$ = "functions on $\mathcal{X}_{\infty,w}$ satisfying a transformation law" gives a definition of $M_\kappa^\dagger$ parallel to the analytic definition of classical modular forms. Aside from its aesthetic pleasure, this interpretation of $M_\kappa^\dagger$ also gives a new approach to the construction of "overconvergent Eichler-Shimura maps."

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Desingularization by char(X)-alterations

Michael Temkin

1. The main result

1.1. **Desingularization.** Let X be an integral algebraic variety. The famous desingularization conjecture asserts that there exists a proper birational morphism $f\colon X' \to X$ such that the variety X' is regular. In addition, one conjectures that given a closed subset $Z \subsetneq X$ one can arrange that $Z' = f^{-1}(Z)$ is an snc divisor. Also, it was conjectured by Grothendieck and is widely believed that the same desingularization result holds for any quasi-excellent integral scheme X. The conjecture was proved in characteristic zero by Hironaka (schemes of finite type over a local quasi-excellent ring), see [1], and was extended to all quasi-excellent schemes over $\mathbf{Q}$ by Temkin, see [4]. Also, it was proved very recently for quasi-excellent threefolds by Cossart and Piltant, see [6]. Already for varieties of positive characteristic the conjecture is widely open and very difficult in dimensions starting with 4.

1.2. **de Jong's altered desingularization.** de Jong found a very successful weakening of the desingularization conjecture: its proof is relatively simple (e.g. when comparing with [1] or [6]), and yet, it has numerous applications. Namely, de Jong proved in [2, Theorem 4.1] that for any integral scheme X of finite type over a quasi-excellent base of dimension 2 (using [6] this can be pushed to dimension 3) there exists an alteration $f\colon X' \to X$, i.e. a proper dominant generically finite morphism between integral schemes, such that X' is regular. In addition, if $Z \subsetneq X$ is closed then one can arrange that $Z' = f^{-1}(Z)$ is an snc divisor.

1.3. **Gabber's l'-altered desingularization.** de Jong's theorem covers various cohomological applications with coefficients containing $\mathbf{Q}$. In order to deal with cohomology theories where a prime l is not inverted, e.g. $\mathbf{Z}/l\mathbf{Z}$ or $\mathbf{Z}_l$-cohomology, Gabber strengthened de Jong's theorem as follows: keep the assumptions of the de Jong's theorem and assume that l is a prime number invertible on X, then the desingularizing alteration $f\colon X' \to X$ can be chosen so that l does not divide the degree $\deg(f) = [k(X') : k(X)]$, see [5, Theorems 2.1]. Such alterations are called l'-alterations.

1.4. $\mathrm{char}(X)$**-altered desingularization.** It is a natural question if Gabber's theorem can be strengthened so that $\deg(f)$ is not divisible by two (or more) fixed primes invertible on X. In my recent work [7] I answer this affirmatively, in fact, I prove that one can avoid all invertible primes simultaneously. By a $\mathrm{char}(X)$-alteration we mean an alteration $X' \to X$ whose degree is only divisible by primes non-invertible on X. The main result of [7] is that if X is of finite type over a quasi-excellent threefold and $Z \subsetneq X$ is closed then there exists a $\mathrm{char}(X)$-alteration $f\colon X' \to X$ such that X' is regular and $f^{-1}(Z)$ is an snc divisor. In particular, if X is of characteristic zero then f is a desingularization, and if X is of characteristic p then $\deg(f) = p^n$.

2. The method

2.1. **l'-altered desingularization.** de Jong refined his theorem in [3] as follows: the altered desingularization $f\colon X' \to X$ can be chosen so that the alteration $g\colon X'/\mathrm{Aut}_X(X') \to X$ is generically radicial (in particular, $\deg(g) = p^n$ where p is the exponential characteristic of $k(X)$). Gabber observed that the l-Sylow subgroup G_l of $G = \mathrm{Aut}_X(X')$ acts tamely on X' whenever l is invertible on X and proved a general difficult theorem on tame actions implying that there exists a G_l-equivariant modification $X'' \to X'$ such that $Y = X''/G_l$ is regular. In particular, $Y \to X$ is an l'-altered desingularization of X.

2.2. **Tame distillation.** If there exists a subgroup $H \subseteq G$ acting tamely on X' and $|H/G|$ is only divisible by primes non-invertible on X then the same argument as above works with G_l replaced by H. In general, such an H does not have to exist and the main new tool of [7] is the following result that asserts that such an H exists if one enlarges the alteration $X' \to X$. Tame distillation theorem, see [7, Theorem 3.3.6]: for any alteration $X' \to X$ of quasi-excellent schemes there exists an alteration $Y' \to Y$ such that the composition $Y' \to X$ factors into a composition of a tame Galois covering $Y' \to Y$ and a char(X)-alteration $Y \to X$.

2.3. char(X)**-altered desingularization.** The tame distillation does not apply directly to Gabber's argument since in order to construct a large enough tamely acting group H we have to replace the regular scheme X' with its alteration Y' and one cannot ensure that Y' is also regular. However, Illusie and Temkin discovered in [5, Section 3] a more flexible proof of Gabber's theorem which is also based on division by l-Sylow subgroups (the main motivation was to extend Gabber's theorem to morphisms of finite type, see [5, Theorem 3.5]). Once one replaces l-Sylow subgroups by the subgroups provided by the tame distillation theorem, the argument of Illusie-Temkin applies almost verbatim and yields a proof of the char(X)-alteration theorem. We refer to [7, Theorem 4.3.1] and its proof for details.

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Derived non-archimedean analytic spaces

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(joint work with Mauro Porta)

Motivations. Derived algebraic geometry is a far-reaching enhancement of classical algebraic geometry. We refer to Toën-Vezzosi [20, 21] and Lurie [11, 13] for foundational works. The prototypical idea of derived algebraic geometry originated from intersection theory: Let X be a smooth complex projective variety. Let Y, Z be two smooth closed subvarieties of complementary dimension. We want to compute their intersection number. When Y and Z intersect transversally, it suffices to count the number of points in the set-theoretic intersection $Y \cap Z$. When Y and Z intersect non-transversally, we have two solutions: the first solution is to perturb Y and Z into transverse intersection; the second solution is to compute the Euler characteristic of the derived tensor product $\mathcal{O}_Y \otimes^{\mathrm{L}}_{\mathcal{O}_X} \mathcal{O}_Z$ of the structural sheaves. The second solution can be seen as doing perturbation in a more conceptual and algebraic way. It suggests us to consider spaces with a structure sheaf of derived rings instead of ordinary rings. This is one main idea of derived algebraic geometry.

Besides intersection theory, motivations for derived algebraic geometry also come from deformation theory, cotangent complexes, moduli problems, virtual fundamental classes, homotopy theory, etc. (see Toën [19] for an excellent introduction). All these motivations apply not only to algebraic geometry, but also to analytic geometry. Therefore, a theory of derived analytic geometry is as desirable as derived algebraic geometry.

We propose to define a notion of derived space in non-archimedean analytic geometry and then study their basic properties. By non-archimedean analytic geometry, we mean the theory of Berkovich spaces over a non-archimedean field k with nontrivial valuation (cf. [1, 2]). Our approach is mainly based on the works of Lurie [13, 14, 15, 12] on derived algebraic geometry and derived complex analytic geometry.

A more direct motivation of our study on derived non-archimedean analytic geometry comes from mirror symmetry. Mirror symmetry is a conjectural duality between Calabi-Yau manifolds (cf. [23, 22, 4, 8]). More precisely, mirror symmetry concerns degenerating families of Calabi-Yau manifolds instead of individual manifolds. An algebraic family of Calabi-Yau manifolds over a punctured disc gives rise naturally to a non-archimedean analytic space over the field $\mathbf{C}(\!(t)\!)$ of formal Laurent series. In [9, §3.3], Kontsevich and Soibelman suggested that the theory of Berkovich spaces may shed new light on the study of mirror symmetry. Progresses along this direction are made by Kontsevich-Soibelman [10] and by Tony Yue Yu [25, 24, 27, 26]. The works by Gross, Hacking, Keel, Siebert [7, 6, 5] are in the same spirit.

In [26], a new geometric invariant is constructed for log Calabi-Yau surfaces, via the enumeration of holomorphic cylinders in non-archimedean geometry. These invariants are essential to the reconstruction problem in mirror symmetry. In order to go beyond the case of log Calabi-Yau surfaces, a general theory of virtual fundamental classes in non-archimedean geometry must be developed. The situation here resembles very much the intersection theory discussed above, because moduli spaces in enumerative geometry can often be represented locally as intersections of smooth subvarieties in smooth ambient spaces. The virtual fundamental class is then supposed to be the set-theoretic intersection after perturbation into transverse situations. However, perturbations do not necessarily exist in analytic geometry. Consequently, we need a more general and more robust way of constructing the virtual fundamental class, whence the need for derived non-archimedean geometry.

Basic ideas and main results. Our previous discussion on intersection numbers suggests the following definition of a derived scheme:

Definition 1 (cf. [19])**.** A *derived scheme* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf $\mathcal{O}_X$ of commutative simplicial rings on X, satisfying the following conditions:

(i) The ringed space $(X, \pi_0(\mathcal{O}_X))$ is a scheme.

(ii) For each $j \geqslant 0$, the sheaf $\pi_j(\mathcal{O}_X)$ is a quasi-coherent sheaf of $\pi_0(\mathcal{O}_X)$-modules.

In order to adapt Definition 1 to analytic geometry, we need to impose certain analytic structures on the sheaf $\mathcal{O}_X$. For example, we would like to have a notion of norm on the sections of $\mathcal{O}_X$; moreover, we would like to be able to compose the sections of $\mathcal{O}_X$ with convergent power series. A practical way to organize such analytic structures is to use the notions of pregeometry and structured topos introduced by Lurie [13].

We define a pregeometry $\mathcal{T}_{\mathrm{an}}(k)$ which will help us encode the theory of non-archimedean geometry relevant to our purpose. After that, we are able to introduce our main object of study: derived k-analytic spaces. It is a pair $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ consisting of an ∞-topos $\mathcal{X}$ and a $\mathcal{T}_{\mathrm{an}}(k)$-structure $\mathcal{O}_{\mathcal{X}}$, satisfying analogs of Definition 1 Conditions (i)-(ii).

Our goal is to study the basic properties of derived k-analytic spaces and to compare them with ordinary k-analytic spaces. Here are our main results:

Below all k-analytic spaces are assumed to be strict.

Theorem 2. *The category of k-analytic spaces embeds fully faithfully into the ∞-category of derived k-analytic spaces.*

Theorem 3. *The ∞-category of derived k-analytic spaces admits fiber products.*

Let $(\mathrm{An}_k, \tau_{\text{qét}})$ denote the category of k-analytic spaces endowed with the quasi-tale topology (cf. [3, §3]) and let $\mathbf{P}_{\text{qét}}$ denote the class of quasi-tale morphisms. The triple $(\mathrm{An}_k, \tau_{\text{qét}}, \mathbf{P}_{\text{qét}})$ constitutes a geometric context in the sense of [18]. The associated geometric stacks are called *higher k-analytic Deligne-Mumford stacks.*

Theorem 4. *The ∞-category of higher k-analytic Deligne-Mumford stacks embeds fully faithfully into the ∞-category of derived k-analytic spaces. The essential image of this embedding is spanned by n-localic discrete derived k-analytic spaces.*

Further developments. In order to apply derived non-archimedean geometry to enumerative geometry, mirror symmetry as well as other domains of mathematics, we must show that derived non-archimedean analytic spaces arise naturally in these contexts. The key to the construction of derived structures is to prove a representability theorem in derived non-archimedean geometry. This will be the main goal of our subsequent works.

Important ingredients in the proof of the representability theorem will include the theories of analytification and deformation. Their counterparts in derived complex geometry are studied by Mauro Porta in [16, 17] and his upcoming works.

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A non-archimedean Ax-Lindemann theorem

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(joint work with François Loeser)

1. Introduction

The classical Lindemann-Weierstrass theorem states that if algebraic numbers $\alpha_1, \ldots, \alpha_n$ are $\mathbf{Q}$-linearly independent, then their exponentials $\exp(\alpha_1), \ldots, \exp(\alpha_n)$ are algebraically independent over $\mathbf{Q}$. More generally, if $\alpha_1, \ldots, \alpha_n$ are complex numbers which are no longer assumed to be algebraic, Schanuel's conjecture predicts that the field $\mathbf{Q}(\alpha_1, \ldots, \alpha_n, \exp(\alpha_1), \ldots, \exp(\alpha_n))$ has transcendence degree at least n over $\mathbf{Q}$. In [1], Ax established power series and differential field versions of Schanuel's conjecture.

Theorem 1 (Exponential Ax-Lindemann)**.** *Let* $\exp\colon \mathbf{C}^n \to (\mathbf{C}^\times)^n$ *be the morphism* $(z_1, \ldots, z_n) \mapsto (\exp(z_1), \ldots, \exp(z_n))$. *Let* V *be an irreducible algebraic subvariety of* $(\mathbf{C}^\times)^n$ *and let* W *be an irreducible component of a maximal algebraic subvariety of* $\exp^{-1}(V)$. *Then* W *is geodesic, that is,* W *is defined by a finite family of equations of the form* $\sum_{i=1}^n a_i z_i = b$ *with* $a_i \in \mathbf{Q}$ *and* $b \in \mathbf{C}$.

In the breakthrough paper [6], Pila succeeded in providing an unconditional proof of the André-Oort conjecture for products of modular curves. One of his main ingredients was to prove an hyperbolic version of the above Ax-Lindemann theorem, which we now state in a simplified version.

Let $\mathbf{H}$ denote the complex upper half-plane and $j : \mathbf{H} \to \mathbf{C}$ the elliptic modular function. By an algebraic subvariety of $\mathbf{H}^n$ we shall mean the trace in $\mathbf{H}^n$ of an algebraic subvariety of $\mathbf{C}^n$. An algebraic subvariety of $\mathbf{H}^n$ if said to be geodesic if it is defined by equations of the form $z_i = c_i$ and $z_k = g_{k\ell} z_\ell$, with $c_i \in \mathbf{C}$ and $g_{k\ell} \in \mathrm{GL}_2^+(\mathbf{Q})$.

Theorem 2 (Hyperbolic Ax-Lindemann)**.** *Let* $j : \mathbf{H}^n \to \mathbf{C}^n$ *be the morphism* $(z_1, \ldots, z_n) \mapsto (j(z_1), \ldots, j(z_n))$. *Let* V *be an irreducible algebraic subvariety of* $\mathbf{C}^n$ *and let* W *be an irreducible component of a maximal algebraic subvariety of* $j^{-1}(V)$. *Then* W *is geodesic.*

Pila's method to prove this Ax-Lindemann theorem is quite different from the differential approach of Ax. It follows a strategy initiated by Pila and Zannier in their new proof of the Manin-Mumford conjecture for abelian varieties [9]; that approach makes crucial use of the bound on the number of rational points of bounded height in the transcendental part of sets definable in an o-minimal structure obtained by Pila and Wilkie in [8]. Recently, still using the Pila and Zannier strategy, Klingler, Ullmo and Yafaev have succeeded in proving a very general form of the hyperbolic Ax-Lindemann theorem valid for any arithmetic variety ([5], see also [10] for the compact case).

In this work, we establish a non-archimedean analogue of theorem 2.

2. Statement of the non-archimedean Ax-Lindemann theorem

Let p be a prime number and let F be a finite extension of $\mathbf{Q}_p$. In this work, we make use of Berkovich's notion of F-analytic spaces, see [2].

The group $\mathrm{PGL}(2, F)$ acts by homographies on the F-analytic projective line $(\mathbf{P}_1)^{\mathrm{an}}$, and on its F-rational points $\mathbf{P}_1(F)$.

Recall (see [4]) that a Schottky subgroup of $\mathrm{PGL}(2, F)$ is a discrete subgroup which is finitely generated and free. We say that such a subgroup Γ is arithmetic if there exists a number field $K \subset F$ such that $\Gamma \subset \mathrm{PGL}(2, K)$.

A Schottky subgroup Γ of $\mathrm{PGL}(2, F)$ has a limit set $\mathcal{L}_\Gamma$ which is a non-empty compact Γ-invariant subset of $\mathbf{P}_1(F)$; if the rank g of Γ is $\geqslant 2$, then it is a perfect set. Let then $\Omega_\Gamma = (\mathbf{P}_1)^{\mathrm{an}} \backslash \mathcal{L}_\Gamma$; the group Γ acts freely on Ω_Γ and the quotient space Ω_Γ/Γ is naturally a F-analytic space so that the projection $p_\Gamma \colon \Omega_\Gamma \to \Omega_\Gamma/\Gamma$ is topologically étale. Moreover, Ω_Γ/Γ is the F-analytic space associated with a smooth, geometrically connected, projective F-curve X_Γ of genus g.

Let us now consider a finite family $(\Gamma_i)_{1\leqslant i\leqslant n}$ of Schottky subgroups of $\mathrm{PGL}(2, F)$ of rank $\geqslant 2$. Let us set $\Omega = \prod_{i=1}^n \Omega_{\Gamma_i}$ and $X = \prod_{i=1}^n X_{\Gamma_i}$, and let $p \colon \Omega \to X^{\mathrm{an}}$ be the morphism deduced from the morphisms $p_{\Gamma_i} \colon \Omega_{\Gamma_i} \to X_{\Gamma_i}^{\mathrm{an}}$.

We say that a closed subspace W of Ω is *irreducible algebraic* if there exists an F-algebraic subvariety Y of $(\mathbf{P}_1)^n$ such that W is an irreducible component of the analytic space $\Omega \cap Y^{\mathrm{an}}$.

We say that W is *flat* if it can be defined by equations of the following form:

(1) $z_i = c$, for some $i \in \{1, \ldots, n\}$ and $c \in \Omega$;
(2) $z_j = g \cdot z_i$, for some pair (i, j) of elements of $\{1, \ldots, n\}$ and $g \in \mathrm{PGL}(2, F)$.

We say that W is *geodesic* if, moreover, the elements g in (2) can be chosen such that $g\Gamma_i g^{-1}$ and Γ_j are commensurable (ie, their intersection has finite index in both of them).

Here is the main result of this paper.

Theorem 3 (Non-archimedean Ax-Lindemann theorem)**.** *Let F be a finite extension of $\mathbf{Q}_p$ and let $(\Gamma_i)_{1\leqslant i\leqslant n}$ be a finite family of* arithmetic *Schottky subgroups of* $\mathrm{PGL}(2, F)$ *of rank $\geqslant 2$. As above, let us set $\Omega = \prod_{i=1}^n \Omega_{\Gamma_i}$ and $X = \prod_{i=1}^n X_{\Gamma_i}$, and let $p \colon \Omega \to X^{\mathrm{an}}$ be the morphism deduced from the morphisms $p_{\Gamma_i} \colon \Omega_{\Gamma_i} \to X_{\Gamma_i}^{\mathrm{an}}$.*

Let V be an irreducible algebraic subvariety of X and let $W \subset \Omega$ be an irreducible component of a maximal algebraic subvariety of $p^{-1}(V^{\mathrm{an}})$. Then W is geodesic.

3. Sketch of the proof

The basic strategy we use is strongly inspired by that of Pila [6] (see also [7]), though some new ideas are required in order to adapt it to the non-archimedean setting. In particular, we have to replace the theorem of Pila-Wilkie [8] by the non-archimedean analogue recently proved by Cluckers, Comte and Loeser [3]. The role of the o-minimal structure $\mathbf{R}_{\mathrm{an,exp}}$ is now played by the subanalytic sets (in F^n) of Denef and van den Dries, and the rigid subanalytic sets of Lipshitz and Robinson (in $\mathbf{C}_p^n$). Analytic continuation and monodromy arguments are replaced by more algebraic ones and explicit matrix computations by group theory considerations. We also take advantage of the fact that Schottky groups are free and of the geometric description of their fundamental domains.

Let V and W be are as the statement of theorem 3. Let Y be the Zariski closure of W and let m be its dimension. Similarly as in Pila's approach one starts by working on some neighborhood of the boundary of our space (which, instead of a product of Poincaré upper half-planes, is a product of open subsets of the Berkovich projective line). We reduce to the case where, locally around some rigid point $\xi \in \Omega$, W is the image of a section ϕ of the projection to the first m coordinates, and that $\xi_1 \in \mathcal{L}_{\Gamma_1}$.

We consider good fundamental domains $\mathfrak{F}_j$ for the groups Γ_j and their product $\mathfrak{F}$; let $\Gamma = \prod \Gamma_j$. We then consider the subset G_0 of $\mathrm{PGL}(2, F)$ consisting of points $(g_1, \dots, g_n)$ such that $g_2 = \dots = g_m = 1$, and its subset R defined by the condition $\dim(gW \cap \mathfrak{F} \cap p^{-1}(V)) = m$. One proves that R is a subanalytic set. Studying the action of Γ_j on a neighborhood of the limit set $\mathcal{L}_{\Gamma_j}$, one proves that every element of Ω can be moved to an element of $\mathfrak{F}_j$ by applying an element of Γ_j of controlled length in some fixed generators. Since the groups Γ_j are arithmetic and free non-abelian, this allows to prove that for every real number T, R contains $\gg T^c$ algebraic points of bounded degree and height $\leqslant T$. Applying the p-adic Pila-Wilkie theorem of [3], and making use of the maximality of W, we then prove that the stabilizer of W inside $G_0 \cap \Gamma$ is infinite. This furnishes non-trivial functional equations for the coordinates ϕ_j of the section ϕ. From these functional equations, we deduce that the Schwarzian derivative of ϕ_j is constant, hence zero, because ϕ_j is algebraic. This implies that W is flat. A degree argument, relying on the maximality of W again, allows then to conclude that W is geodesic.

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p-curvature and connections

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(joint work with Mark Kisin)

This is **work in progress** with Mark Kisin.

Grothendieck p-curvature conjecture predicts that if $X/\mathbb{C}$ is a smooth variety, (E, ∇) is an integrable connection over $\mathbb{C}$, then if for all closed points $s \in S$ of a non-emtpy open of definition S of a model $(X_S, (E, \nabla)_S)$, the p-curvature $\psi((E, \nabla)_s)$ is trivial, then (E, ∇) has finite monodromy, or equivalently, (E, ∇) is isotrivial, that is trivialized by a finite étale cover $Y \to X$, or equivalently its Tannaka group in the category $MIC(X/\mathbb{C})$ of integrable connections is finite.

By André-Hrushovski [And04] one may replace $\mathbb{C}$ by a number field k, by [Kat72] one may assume X is projective. Going up again to $\mathbb{C}$, the topological Lefschetz theorem reduces then the problem to X/k a smooth projective curve over a number field k, and applying Belyi's theorem [Bel80], one may also reduce the problem to $X = \mathbb{P}^1 \backslash \{0, 1, \infty\}$ over a number field k. The latter viewpoint had led Chudnosvkys [Chu85] and André [And04] to prove the abelian version of the conjecture using p-adic analysis and criteria à la Dwork for rationality of a power series in $k[[x]]$, k being a number field.

Furthermore, Katz [Kat72] proves that the Kodaira-Spencer class in characteristic p of a Gauß-Manin connection $(E, \nabla)_s$ dies if the p-cuvature dies, a fact one can't show for a $\mathbb{Z}$-polarized variation of Hodge structure which is not coming from geometry. This enables him to show that Gauß-Manin connections verify the conjecture.

Using the more geometric reduction, that is X a smooth projective curve over a number field k, we observe that if the conjecture is true, then necessarily E is a direct sum of degree 0 stable bundle.

Let X be smooth projective over a number field k. One defines $MIC^0(X/k)$ to be the full subcategory of $MIC(X/k)$ consisting of integrable connections (E, ∇) having a model $(X_S, (E, \nabla)_S)$, for which $(E, \nabla)_s$ lifts to $\mathcal{D}(X_s/k(s))$ for *infinitely many char.* $k(s)$. One has full embeddings $\mathrm{FinConn}(X/k) \subset MIC^0(X/k) \subset MIC(X/k)$, where $\mathrm{FinConn}(X/k)$ is the category of finite connections. Here $\mathcal{D}(X_s/k(s))$ is the sheaf of rings of relative differential operators. Let $SS(X)$ be the category of semi-stable vector bundles of degree 0, $S(X) \subset SS(X)$ be the full subcategory of polystable bundles. One defines $MIC^{0,pol}(X/k) \subset MIC^0(X/k)$ to be the category of polystable objects.

Proposition 1.1. *Let $(E_s, \nabla_s) \in \mathcal{D}(X_s/k(s))$, $k(s)$ finite field. Then E_s^{∇} and E_s are semi-stable of degree 0. If in addition, (E_s, ∇_s) is stable in $MIC(X_s)$, then both E'_s and E_s are stable of degree 0.*

Theorem 1.2. *Let $(E, \nabla) \in MIC^0(X/k)$. Then E is semi-stable of degree 0. If (E, ∇) is irreducible, then E is stable of degree 0.*

So one has the functor

$$\varphi : MIC^0(X/k) \to SS(X),\ (E,\nabla) \mapsto E$$

sending $MIC^{p,pol}(X)$ to $S(X)$. All four categories are Tannakian. We say E is finite if its Tannaka group (after choosing a rational point to define a neutralization) in the corresponding category is finite.

Theorem 1.3. *For $(E,\nabla) \in MIC^{0,pol}(X/k)$, the functor φ induces an isomorphism $\pi(\langle E\rangle) \to \pi(\langle (E,\nabla)\rangle)$. In particular, if E is finite, so is (E,∇).*

Theorem 1.4. *Assume X smooth projective over $\mathbb{C}$ and let $(E,\nabla) \in MIC^0(X)$ be a $\mathbb{Z}$-polarized variation of Hodge structure. Then (E,∇) is finite.*

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On non-archimedean Poissan's equation

Shou-Wu Zhang

Let us recall the classical Poisson's equation on a compact Kähler manifold (X,ω) which is the foundation for the Hodge theory. We take a normalization $\int \omega^n = 1$ where $n = \dim X$. Then we have a Poisson's equation:

$$\Delta f = g, \qquad f, g \in C^\infty(X).$$

The uniqueness and existence of this equation are summarized by the following exact sequence:

$$0 \longrightarrow \mathbb{C} \longrightarrow C^\infty(X) \xrightarrow{\Delta} C^\infty(X) \xrightarrow{\int \cdot \omega^n} \mathbb{C} \longrightarrow 0.$$

One important application is the existence of metrics on line bundles on X with harmonic curvature form.

In this talk, we try to formulate a non-archimedean analogue of Poisson's equation on a variety X over an algebraically closed field C with a complete and nontrivial absolute value $|\cdot|$. We first need to define a notion of a curvature form and an operator $\frac{\partial\bar\partial}{\pi i}$. We use the notations in our paper [YZ] on Hodge index theorem with a modification: we change the notion "integrable metrized line bundle" to "dsp metrized line bundles", where dsp is the abbreviation of "difference of semi-positive". On X, we have a vector space $\widehat{\mathrm{Pic}}(X)_{\mathrm{dsp},\mathbb{R}}$ of dsp metrized $\mathbb{R}$-line bundles which includes a subspace $\widehat{\mathrm{Pic}}^0(X)_{\mathbb{R}}$ of flat metrized line bundles. We define the dsp metrized Neron–Severi group by

$$\widehat{\mathrm{NS}}(X)_{\mathbb{R}} := \widehat{\mathrm{Pic}}(X)_{\mathrm{dsp},\mathbb{R}}/\widehat{\mathrm{Pic}}^0(X)_{\mathbb{R}}.$$

For a metrized line bundle $\bar L \in \widehat{\mathrm{Pic}}(X)_{\mathrm{dsp},\mathbb{R}}$, define its first Chern form $c_1(\bar L)$ to be its class in $\widehat{\mathrm{NS}}(X)_{\mathbb{R}}$. We will work on the space $C(X)_{\mathrm{dsp}}$ of dsp functions f on $X(C)$ defined by requiring that the metrized line bundle $\widehat{O}(f) := (O_X, ||1|| = e^{-f})$ is dsp. For an dspfunction f on X, we define the operator

$$\frac{\partial\bar\partial}{\pi i} : C(X)_{\mathrm{dsp}} \longrightarrow \widehat{\mathrm{NS}}(X)_{\mathbb{R}}, \qquad \frac{\partial\bar\partial}{\pi i} f := c_1(\widehat{O}(f)).$$

Let $C(X)$ denote the completion of $C(X)_{\mathrm{dsp}}$ with respect to the L^∞-norm. The intersection theory on the models of X over O_C defines to multilinear and continuous pairing:

$$C(X) \times \widehat{\mathrm{NS}}(X)^n \longrightarrow \mathbb{R}, \qquad (f, \bar L_1, \cdots, \bar L_n) \mapsto \int_X f c_1(\bar L_1)\cdots(\bar L_n).$$

By Gubler, $C(X)$ can be naturally identified with the space $C(X^{\mathrm{an}})$ continuous function on the Berkovich space X^{an}. Thus the above pairing define a so called Chambert–Loir measure $c_1(\bar L_1)\cdots c_1(\bar L_n)$ on X^{an}.

Let $\mathrm{Pic}(X)_+$ denote the positive cone in $\widehat{\mathrm{Pic}}(X)_{\mathrm{dsp},\mathbb{R}}$, namely $\mathbb{R}_+$-combinations of ample line bundles with semipositive metrics and let $\widehat{\mathrm{NS}}(X)_+$ denote its image in $\widehat{\mathrm{NS}}(X)_{\mathbb{R}}$. We take a Kähler form ω on X as an element in $\widehat{\mathrm{NS}}(X)_+$ with a normalization $\int \omega^n = 1$. Now we define a Lapace operator as

$$\Delta : \qquad C(X)_{\mathrm{dsp}} \longrightarrow L^1(X, \omega^n), \qquad \Delta(f) := \frac{\frac{\partial\bar\partial}{\pi i} f \omega^{n-1}}{\omega^n}.$$

Without further restriction to both spaces, it is hard to say anything meaningful about the kernel and the image of this operator. A *key point* of this talk is to put a condition so called *ω-boundedness* on both sides: we say a form $\alpha \in \widehat{\mathrm{NS}}(X)_{\mathbb{R}}$ is ω-bounded if there is an $\epsilon > 0$ such that both $\omega \pm \epsilon\alpha \in \widehat{\mathrm{NS}}(X)_+$; and we say a function $f \in C(X)_{\mathrm{dsp}}$ is ω-bounded if $\frac{\partial\bar\partial}{\pi i} f$ is ω-bounded. Let $\widehat{\mathrm{NS}}(X)_\omega$ denote the space of ω-bounded forms, $L^\infty_\omega(X)$ the space of ω-bounded functions, and $L^1_\omega(X)$ the space $\widehat{\mathrm{NS}}(X)_\omega \wedge \omega^{n-1}$. Then we have a restricted Laplacien operator

$$\Delta : \qquad L^\infty_\omega(X) \longrightarrow L^1_\omega(X).$$

Conjecture 1. *For a given* $g \in L^1_\omega(X)$*, the Poisson equation* $\Delta f = g$ *has a solution* $f \in L^\infty_\omega(X)$ *if and only if* $\int g\omega^n = 0$*.*

Notice that the uniqueness of the Poisson equation has already been established as a consequence of the local Hodge index theorem in [YZ]. Equivalently, we show that the Laplacian equation $\Delta f = 0$ has only constant solutions. One consequence of the above conjecture is the existence of some canonical metric on any line bundle on X:

Conjecture 2. *For any line bundle* M *on* X*, there is an dspmetrization* $\bar{M}$ *(unique up scale multiple) such that the curvature* $c_1(\bar{M})$ *is* ω*-harmonic in the following sense:* $c_1(\bar{M})$ *is* ω*-bounded, and satisfies the following equation of measures:*

$$c_1(\bar{M})\omega^{n-1} = \lambda(M)\omega^n$$

where $\lambda(M)$ *is a constant defined by* $c_1(M) \cdot [\omega]^n$ *with* $[\omega]$ *the class in* $\mathrm{NS}(X)_\mathbb{R}$ *under the map* $\widehat{\mathrm{NS}}(X)_\mathbb{R} \longrightarrow \mathrm{NS}(X)_\mathbb{R}$*.*

The following are some results about these conjectures:

(1) Conjecture 1 (and then 2) holds for curves X. In fact in this case, we can solve Poisson's equation using a Green's fucntions $g(x, y)$ for the volume form ω. We can start with a green function $g_0(x, y)$ for any volume form ω_0 (for example one associate to the admissible metrics), and define

$$g(x, y) = g_0(x, y) - \int g_0(x, y)\omega(x) - \int g_0(x, y)\mu(y) + \int g_0(x, y)\omega(x)\omega(y).$$

(2) Conjecture 1 (and then 2) holds for the case ω is a model metric. In fact, in this case $L^\infty_\omega(X)$ and $L^1_\omega(X)$ are both finite dimensional with same dimension, the quadratic form $\langle f, \Delta f\rangle_{L^2}$ is positive definite on $L^\infty_\omega(X)/\mathbb{C}$ by local Hodge index theorem. Thus Δ is bijective.

(3) Conjecture 1 (and then 2) holds for the case residue characteristic of C is 0, and ω is supported on a dual complex, by method of Bouckson–Favre–Jonsson.

(4) Conjecture 2 holds when ω comes from a polarized dynamical system in the sense that there is an endomorphism $f : X \longrightarrow X$ such that $f^*\omega = q\omega$ with q a constant > 1. This follows from the construction of admissible metrics for any line bundle in [YZ].

We would like to give an application of Conjecture 2 to a variety X over a global field K, including the function field $K = k(C)$ for projective curve over another field k with a fixed positive adelic metrized line bundle $\bar{L}$ on X. We write $\omega = c_1(\bar{L})/\deg L^{1/n}$ with $n = \dim X$.

Conjecture 3. *Any line bundle* M *on* X *has an admissible merization* $\bar{M}$ *in the sense that at each place* v *of* K *the bundle* $\bar{M}_v$ *has harmonic curvature form* $c_1(\bar{M}_v)$*, and that*

$$\bar{M} \cdot \omega^n = \lambda(M)\omega^{n+1}.$$

Moreover such a metrization is unique up to multiples from $\widehat{Div}(K)$ *with degree* 0*.*

If we write $\widetilde{\mathrm{Pic}}(X)_{\mathbb{R}} = \widehat{\mathrm{Pic}}(X)_{\mathbb{R}}/\widehat{\mathrm{Pic}}(K)_{\deg=0}$. The above conjecture gives a section to the projection $\widetilde{\mathrm{Pic}}(X)_{\mathbb{R}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{R}}$.

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Arithmetic differential operators and representations of p-adic groups

Matthias Strauch
(joint work with Christine Huyghe, Deepam Patel, and Tobias Schmidt)

Let $L/\mathbb{Q}_p$ be a finite extension, and let $\mathbf{G}_0$ be a smooth reductive group scheme over the ring of integers O_L of L. The purpose of the work [5] is to study locally analytic representations of $G = \mathbf{G}_0(L)$ in terms of sheaves of modules for (suitably defined) arithmetic differential operators on formal models of the rigid analytic flag variety of $\mathbf{G} = \mathbf{G}_0 \times \mathrm{Spec}(L)$.

1. Arithmetic differential operators. Denote by $\mathbf{X}_0$ the flag scheme for $\mathbf{G}_0$. Let $x_1, \ldots, x_d$ be coordinates on an affine open $U \subset \mathbf{X}_0$. Let ϖ be a uniformizer of L. For non-negative integers m and k denote by $\mathcal{D}^{(m)}_{\mathbf{X}_0,k}(U)$ the O_L-module of differential operators

$$\sum_{n=(n_1,\ldots,n_d)} a_n \varpi^{k|n|} \frac{q_n^{(m)}!}{n!} \partial_{x_1}^{n_1} \cdot \ldots \cdot \partial_{x_d}^{n_d} \;,$$

where $a_n \in \mathcal{O}_{\mathbf{X}_0}(U)$, $n! = \prod_j n_j!$, $q_n^{(m)}! = \prod_j q_{n_j}^{(m)}!$, $q_{n_j}^{(m)} = \lfloor \frac{n_j}{p^m} \rfloor$, and $|n| = n_1 + \ldots + n_d$. These rings glue together to give a sheaf $\mathcal{D}^{(m)}_{\mathbf{X}_0,k}$ on $\mathbf{X}_0$. Let now $pr : \mathbf{X} \to \mathbf{X}_0$ be an admissible blow-up, i.e., the blow-up of an ideal sheaf $\mathcal{I}$ on $\mathbf{X}_0$ which contains a power of ϖ.

Key Lemma. *Suppose $\mathcal{I}$ contains ϖ^N. Then, for all $k \geqslant N$ the sheaf of rings $pr^{-1}\mathcal{D}^{(m)}_{\mathbf{X}_0,k}$ acts naturally on $\mathcal{O}_{\mathbf{X}}$. Therefore, for these k, the $\mathcal{O}_{\mathbf{X}}$-module $pr^*\mathcal{D}^{(m)}_{\mathbf{X}_0,k}$ carries a structure of a sheaf of rings.*

In the following we let $k_{\mathbf{X}}$ be the minimal k such that ϖ^k is contained in $\mathcal{I}$. For $k \geqslant k_{\mathbf{X}}$, put

$$\mathcal{D}^{(m)}_{\mathbf{X},k} = pr^*\mathcal{D}^{(m)}_{\mathbf{X}_0,k} \;.$$

Denote by $\mathfrak{X}$ the completion of $\mathbf{X}$ along its special fiber, and let $\mathscr{D}^{(m)}_{\mathfrak{X},k}$ be the p-adic completion of $\mathcal{D}^{(m)}_{\mathbf{X},k}$, which we consider as a sheaf on $\mathfrak{X}$. We put $\mathscr{D}^{(m)}_{\mathfrak{X},k,\mathbb{Q}} = \mathscr{D}^{(m)}_{\mathfrak{X},k} \otimes \mathbb{Q}$, and let $\mathscr{D}^{\dagger}_{\mathfrak{X},k,\mathbb{Q}}$ denote the inductive limit over all $\mathscr{D}^{(m)}_{\mathfrak{X},k,\mathbb{Q}}$.

2. Wide open congruence subgroups. Let $\mathbf{G}(k) = {}''\ker\big(\mathbf{G}_0 \to \mathbf{G}_0(O_L/\varpi^k)\big)''$ be the congruence subgroup scheme over O_L of level k, and denote by $\widehat{\mathbf{G}}(k)^\circ$ the completion of $\mathbf{G}(k)$ along the unit section. Let $\mathbb{G}(k)^\circ$ be the rigid analytic generic fiber of $\widehat{\mathbf{G}}(k)^\circ$. Then we consider

$$\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ) = \mathrm{Hom}_L^{\mathrm{cont}}\Big(\mathcal{O}(\mathbb{G}(k)^\circ), L\Big) ,$$

which is the *analytic distribution algebra* introduced by M. Emerton. Denote by $\mathfrak{g}$ the Lie algebra of $\mathbf{G}$, and let $\mathfrak{z}$ be the center of the enveloping $U(\mathfrak{g})$. We let θ_0 be the central character of the trivial representation and put $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_{\theta_0} = \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ) \otimes_{\mathfrak{z},\theta_0} L$.

Theorem I. *Let $pr : \mathfrak{X} \to \mathfrak{X}_0$ be an admissible formal blow-up of the smooth formal scheme $\mathfrak{X}_0$ which is the formal completion of $\mathbf{X}_0$ along its special fiber. Let $k \geqslant k_{\mathfrak{X}}$.*

(i) $\mathfrak{X}$ is $\mathscr{D}^\dagger_{\mathfrak{X},k,\mathbb{Q}}$-affine. That means that any coherent module $\mathscr{E}$ over $\mathscr{D}^\dagger_{\mathfrak{X},k,\mathbb{Q}}$ is generated by its global sections (as a $\mathscr{D}^\dagger_{\mathfrak{X},k,\mathbb{Q}}$-module), and that $H^i(\mathfrak{X},\mathscr{E}) = 0$ for all $i > 0$.

(ii) The ring $H^0(\mathfrak{X},\mathscr{D}^\dagger_{\mathfrak{X},k,\mathbb{Q}})$ is canonically isomorphic to $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_{\theta_0}$.

(iii) The functor $\mathscr{E} \rightsquigarrow H^0(\mathfrak{X},\mathscr{E})$ is an equivalence from the category of coherent $\mathscr{D}^\dagger_{\mathfrak{X},k,\mathbb{Q}}$-modules to the category of finitely generated $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_{\theta_0}$-modules. A quasi-inverse is given by sending a finitely generated $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_{\theta_0}$-module M to

$$\mathscr{L}oc^\dagger_{\mathfrak{X},k}(M) = \mathscr{D}^\dagger_{\mathfrak{X},k,\mathbb{Q}} \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_{\theta_0}} M .$$

□

This result generalizes previous work of C. Huyghe [4].

3. Locally analytic representations. For a locally analytic representation V of $G_0 = \mathbf{G}_0(O_L)$ on a vector space over a finite extension K of L, we denote by $V_{\mathbb{G}(k)^\circ-\mathrm{an}}$ the subspace of rigid analytic vectors for $\mathbb{G}(k)^\circ$, and we put

$$M_k(V) = \mathrm{Hom}_K^{\mathrm{cont}}\Big(V_{\mathbb{G}(k)^\circ-\mathrm{an}}, K\Big) .$$

This is naturally a module over the distribution algebra

$$D(\mathbb{G}(k)^\circ, G_0) = \mathrm{Hom}_K^{\mathrm{cont}}\Big(C^{\mathrm{an}}(G_0,K)_{\mathbb{G}(k)^\circ-\mathrm{an}}, K\Big) = \bigoplus_{g\in G_0/G_{k+1}} \delta_g * \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ) ,$$

where $G_{k+1} = \mathbb{G}(k)^\circ(L) = \mathbf{G}(k+1)(O_L)$ for $k \geqslant 0$, and δ_g denotes the delta distribution at g. It is shown in [3] that the representation V is admissible in the sense of Schneider and Teitelbaum if and only if for all $k \gg 0$ the $D(\mathbb{G}(k)^\circ, G_0)$-module $M_k(V)$ is finitely generated, and if the canonical map

$$D(\mathbb{G}(k)^\circ, G_0) \otimes_{D(\mathbb{G}(k+1)^\circ, G_0)} M_{k+1}(V) \longrightarrow M_k(V)$$

is an isomorphism of $D(\mathbb{G}(k)^{\circ}, G_0)$-modules. A locally analytic representation of $G = \mathbf{G}(L)$ is admissible if it is admissible as a G_0-representation.

4. Localization of admissible G_0-representations. Let $\mathcal{F}_0$ be the system of all admissible formal blow-ups of $\mathfrak{X}_0$, and put $\mathfrak{X}_\infty = \varprojlim_{\mathfrak{X}\in\mathcal{F}_0} \mathfrak{X}$. This is the Zariski-Riemann space of the rigid analytic flag variety of $\mathbb{G}$, and it is also isomorphic to the corresponding adic space. For a formal scheme $\mathfrak{X}$ in $\mathcal{F}_0$ we set

$$\mathscr{D}^{\dagger}_{\mathfrak{X}} := \mathscr{D}^{\dagger}_{\mathfrak{X},k_{\mathfrak{X}},\mathbb{Q}} .$$

In the following we consider systems of sheaves $\Big(\mathscr{M}_{\mathfrak{X}}\Big)_{\mathfrak{X}\in\mathcal{F}_0}$, where $\mathscr{M}_{\mathfrak{X}}$ is a coherent $\mathscr{D}^{\dagger}_{\mathfrak{X}}$-module with a G_0-action which extends the natural action of $G_{k_{\mathfrak{X}}+1}$.

Definition. A *G_0-equivariant coadmissible module on $\mathfrak{X}_\infty$*, is a system of sheaves $\Big(\mathscr{M}_{\mathfrak{X}}\Big)_{\mathfrak{X}}$ as above, together with isomorphisms

$$\mathscr{D}^{\dagger}_{\mathfrak{X}} \otimes_{\mathscr{D}^{\dagger}_{\mathfrak{X}'},G_{k_{\mathfrak{X}}+1}} pr_*\Big(\mathscr{M}_{\mathfrak{X}'}\Big) \xrightarrow{\simeq} \mathscr{M}_{\mathfrak{X}}$$

for any morphism $pr : \mathfrak{X}' \to \mathfrak{X}$ in $\mathcal{F}_0$. This system of isomorphisms is assumed to satisfy the obvious transitivity condition for any sequence $\mathfrak{X}'' \to \mathfrak{X}' \to \mathfrak{X}$ of morphisms in $\mathcal{F}_0$.

For the precise definition of the tensor product on the left we refer to [5]. Given an admissible G_0-representation V with infinitesimal central character θ_0, we consider the system $\mathscr{L}oc^{\dagger}(V) = \Big(\mathscr{M}_{\mathfrak{X}}(V)\Big)_{\mathfrak{X}}$ where

$$\mathscr{M}_{\mathfrak{X}}(V) = \mathscr{L}oc^{\dagger}_{\mathfrak{X},k_{\mathfrak{X}}}\Big(M_{k_{\mathfrak{X}}}(V)\Big) .$$

Proposition. *(i) For any admissible G_0-representation V the system $\mathscr{L}oc^{\dagger}(V)$ is a G_0-equivariant coadmissible module on $\mathfrak{X}_\infty$.*

(ii) Via the functor $\mathscr{L}oc^{\dagger}$, the category of admissible locally analytic G_0-representations (with infinitesimal central character θ_0) is (anti-)equivalent to the category of G_0-equivariant coadmissible modules on $\mathfrak{X}_\infty$. □

5. Passage to the limit: sheaves of $\mathscr{D}^{\dagger}_{\infty}$-modules. Let $\mathrm{sp}_{\mathfrak{X}} : \mathfrak{X}_\infty \to \mathfrak{X}$ be the projection map. Given an open subset $U \subset \mathfrak{X}_\infty$ of the form $\mathrm{sp}_{\mathfrak{X}}^{-1}(U_{\mathfrak{X}})$, for some open $U_{\mathfrak{X}} \subset \mathfrak{X}$, we have that $\mathrm{sp}_{\mathfrak{X}'}(V) = pr^{-1}(U_{\mathfrak{X}}) \subset \mathfrak{X}'$ whenever $pr : \mathfrak{X}' \to \mathfrak{X}$ is a morphism in $\mathcal{F}_0$. We then put

$$\mathscr{D}^{\dagger}_{\infty}(U) = \varprojlim_{\mathfrak{X}'\to\mathfrak{X}} \mathscr{D}^{\dagger}_{\mathfrak{X}'}\Big(\mathrm{sp}_{\mathfrak{X}'}(U)\Big) .$$

The open subsets of the form $\mathrm{sp}_{\mathfrak{X}}^{-1}(U_{\mathfrak{X}})$ form a basis for the topology of $\mathfrak{X}_\infty$, and we thus obtain a sheaf $\mathscr{D}_\infty$ on $\mathfrak{X}_\infty$[1]. Similarly, when $\mathscr{M} = \left(\mathscr{M}_{\mathfrak{X}}\right)_{\mathfrak{X}}$ is a G_0-equivariant coadmissible module one can form the sheaf $\mathscr{M}_\infty$ with the property that

$$\mathscr{M}_\infty(U) = \varprojlim_{\mathfrak{X}' \to \mathfrak{X}} \mathscr{M}_{\mathfrak{X}'}\Big(\mathrm{sp}_{\mathfrak{X}'}(U)\Big) .$$

This is a module for $\mathscr{D}_\infty^\dagger$, and it is G_0-equivariant.

Proposition. *The functor $\mathscr{M} \rightsquigarrow \mathscr{M}_\infty$ just described from G_0-equivariant coadmissible modules on $\mathfrak{X}_\infty$ to G_0-equivariant $\mathscr{D}_\infty^\dagger$-modules is a fully faithful embedding.* □

In particular, we have the functor

$$V \rightsquigarrow \mathscr{L}oc_\infty^\dagger(V) = \mathscr{L}oc^\dagger(V)_\infty$$

which is a (contravariant) fully faithful embedding of the category of admissible locally analytic G_0-representations (with infinitesimal central character θ_0) to the category of G_0-equivariant $\mathscr{D}_\infty^\dagger$-modules. We call the objects in the essential image of this functor *coadmissible G_0-equivariant $\mathscr{D}_\infty^\dagger$-modules.*

6. Localization of admissible G-representations. It is easy to see that the sheaf $\mathscr{D}_\infty^\dagger$ is not only G_0-equivariant but actually G-equivariant. Furthermore, if V is an admissible G-representation, then the sheaf $\mathscr{L}oc_\infty^\dagger(V)$ is also G-equivariant. A coadmissible G_0-equivariant $\mathscr{D}_\infty^\dagger$-module whose equivariant structure extends to the full group G, will simply be called a coadmissible G-equivariant $\mathscr{D}_\infty^\dagger$-module.

Theorem II. *The functor $V \rightsquigarrow \mathscr{L}oc_\infty^\dagger(V)$ is an anti-equivalence from the category of admissible G-representations (with infinitesimal central character θ_0) to the category of coadmssible G-equivariant $\mathscr{D}_\infty^\dagger$-modules.* □

For an application of the localization of locally analytic representations to particular representations furnished by an étale covering of the p-adic upper half plane we refer to [6].

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A p-adically entire function with integral values on $\mathbb{Q}_p$, Fourier transform of distributions, and automorphisms of the perfectoid open unit disc.

Francesco Baldassarri

We deal with the *formal perfectoid open unit disk* $\mathbb{D} = \mathrm{Spa}\,\mathscr{D}$, where $\mathscr{D} = \mathbb{Z}_p[[T^{1/p^\infty}]]$ complete in the (p,T)-adic topology. For any embedding $\mathbb{Q}_p \hookrightarrow K$ in a perfectoid field set $t := p^\flat$. The extension $\mathbb{D}_{K^\circ} = \mathbb{D} \times_{\mathrm{Spa}\,\mathbb{Z}_p} \mathrm{Spa}\,\mathrm{K}^\circ$ is a formal scheme whose generic fiber is the perfectoid open unit disc $\mathbb{D}_K = \mathbb{D}_{K^\circ} - \{p = 0\}$ over K with tilt

$$\mathbb{D}_{K^\flat} = \mathrm{Spa}\,\mathrm{K}^\flat[[\mathrm{T}^{1/\mathrm{p}^\infty}]] - \{\mathrm{t} = 0\}\ .$$

We interpret $\mathscr{D}$ as the algebra $\mathscr{D}_0(\mathbb{Q}_p, \mathbb{Z}_p)_{\mathrm{pc}}$ of $\mathbb{Z}_p$-valued measures on $\mathbb{Q}_p$ which vanish at infinity, via the identification $(1+T)^q = \delta_q$, the Dirac mass centered at $q \in \mathbb{Q}_p$. Notice that

$$(1+T)^q = \lim_{n\to\infty} (1 + T^{1/p^n})^{qp^n} \in \mathscr{D}\ .$$

Then $\mathscr{D}$ is weak dual of the space $\mathscr{C} := \mathscr{C}^0_{\mathrm{unif}}(\mathbb{Q}_p, \mathbb{Z}_p)$ of uniformly continuous functions $\mathbb{Q}_p \to \mathbb{Z}_p$, equipped with the supnorm. We regard $\mathbb{D}$ as the universal covering of the p-divisible torus over $\mathbb{Z}_p$, so that $\Delta_{\mathscr{D}}(\delta_q) = \delta_q \widehat{\otimes} \delta_q$, $\forall q \in \mathbb{Q}_p$.

We also consider the $\mathbb{Z}_p$-algebra $\mathscr{E}$ consisting of p-adically entire functions f with coefficients in $\mathbb{Q}_p$ such that $f(\mathbb{Q}_p) \subset \mathbb{Z}_p$, equipped with the topology of the valuations w_r, for $r \in \mathbb{Z}$, where

$$w_r(f) = \inf_{x \in p^{-r}\mathbb{C}_p} v_p(f(x))\ .$$

We set $S = \mathbb{Z}[1/p] \cap \mathbb{R}_{\geqslant 0} = S' \dot\cup \{0\}$. For $c > 0$ and $N = 1, 2, \ldots$, we consider $\mathbb{Z}_p$-subalgebras $\mathscr{E}_{c,N}$ of $\mathscr{E}\widehat{\otimes}_{\mathbb{Z}_p}\mathscr{D}$, consisting of the $\sum_{q\in S} a_q(x) T^q$ such that

1. $a_{pq}(px) = a_q(x)$, for any $q \in S$;
2. for any $r, v \in \mathbb{Z}$ and $C \in \mathbb{R}$,

$$w_r(a_q) \geqslant C - c(\max(qp^r, 1)^N - 1)\ ,\ \text{for almost all } q \text{ with } v(q) \leqslant v\ .$$

In particular, $a_0 \in \mathbb{Q}_p$. Notice that any element of $\mathscr{E}\widehat{\otimes}_{\mathbb{Z}_p}\mathscr{D}$ may be viewed as an endomorphism of $\mathbb{D}$.

We show that, for any prime number p, there exists a power series

$$\Psi = \Psi_p(T) \in T + T^2\mathbb{Z}[[T]]\ , \tag{1}$$

which trivializes the addition law of the formal group of Witt p-covectors $\widehat{\mathrm{CW}}_{\mathbb{Z}_p}$, is p-adically entire, and assumes values in $\mathbb{Z}_p$ all over $\mathbb{Q}_p$. So, we have

$$(\ldots, \Psi(px+py), \Psi(x+y)) = (\ldots, \Psi(px), \Psi(x)) + (\ldots, \Psi(py), \Psi(y)) \tag{2}$$

in the sense of covectors, and therefore Ψ satisfies the functional equation

(3)
$$\sum_{j=0}^{\infty} p^{-j}\Psi(p^j T)^{p^j} = T \ .$$

We extend the formula of Dieudonné

$$\prod_{i=0}^{\infty} AH(x_i T^{p^i}) = \exp \sum_{i=0}^{\infty} x^{(i)} T^{p^i} = 1 + \sum_{i=1}^{\infty} g_i(x_0, x_1, \dots, x_{\lfloor \log_p i \rfloor}) T^i \ ,$$

where

$$AH(T) = \exp(\sum_{i=0}^{\infty} T^{p^i}/p^i) \in \mathbb{Z}_{(p)}[[T]]$$

is the Artin-Hasse exponential series, and

$$x^{(i)} = \sum_{n=0}^{i} p^{n-i} x_n^{p^{i-n}}$$

is the usual ghost component of the Witt vector $(x_0, x_1, \dots)$ *divided by* p^i to the following identity, holding in a suitable completion $\widehat{\mathscr{P}}$ of $\mathbb{Z}_{(p)}[\dots, x_{-1}; x_0, \dots][T]$,

$$\prod_{i=-\infty}^{\infty} AH(x_i T^{p^i}) = \exp \sum_{i=-\infty}^{\infty} x^{(i)} T^{p^i} = 1 + \sum_{q \in S'} g_q(\dots, x_{\lfloor \log_p q \rfloor - 1}, x_{\lfloor \log_p q \rfloor}) T^q$$

where now

$$x^{(i)} = \sum_{n \leqslant i} p^{n-i} x_n^{p^{i-n}} \ .$$

Theorem 1.1. *The specialization* $\widehat{\mathscr{P}} \to \mathscr{E}\widehat{\otimes}\mathscr{D}$, $x_i \mapsto \Psi(p^{-i}x)$, $\forall i \in \mathbb{Z}$, *produces an element* $\varepsilon(x,T) \in \mathscr{E}_{\frac{p}{p-1},1}$, *namely*

$$\varepsilon(x,T) = \prod_{i=-\infty}^{\infty} AH(\Psi(p^{-i}x)T^{p^i}) = \exp(x \sum_{i=-\infty}^{\infty} p^{-i}T^{p^i}) = 1 + \sum_{q \in S'} G_q(x) T^q \ ,$$

where, for any $q \in S$, $G_q(x) \in \mathscr{E}$ *and*

(4)
$$G_q(x+y) = \sum_{q_1+q_2=q} G_{q_1}(x)\, G_{q_2}(y) \ ,$$

where the sum is convergent, along the filter of cofinite subsets of its index set, in the usual Fréchet topology of $\mathcal{O}(\mathbb{A}^2_{\mathbb{Q}_p})$.

Our main result is

Theorem 1.2. *The endomorphism of* $\mathbb{D}$ *induced by* $\varepsilon(x,T)$ *is a group automorphism* $\epsilon : \mathbb{D} \xrightarrow{\sim} \mathbb{D}$.

The following result was suggested to us by Jared Weinstein.

Theorem 1.3. *For any perfectoid extension* $K/\mathbb{Q}_p$ *the automorphism* ϵ_K *of* $\mathbb{D}_K$ *is the untilted form of the automorphism of* $\mathbb{D}_{K^\flat}$ *induced by the Artin-Hasse function taken modulo p, namely* $\overline{AH}(T) \in 1 + T\mathbb{F}_p[[T]]$. *That is* $\epsilon_K = \overline{AH}^\sharp_{K^\flat}$.

The calculation of the action of ϵ_K of K-points of $\mathbb{D}_K$ can be performed as follows. Any K-valued point of $\mathbb{D}_K(0)$ identifies with a character

$$\begin{aligned} \chi : (S', +) &\longrightarrow (K^{\circ\circ}, \cdot) \\ q &\longmapsto \chi(q) \ . \end{aligned}$$

The sum

$$\pi(\chi) := \sum_{i \in \mathbb{Z}} \chi(p^{-i}) p^i$$

converges in K. The image of χ in $\mathbb{D}_K(1)$ is the additive character

$$\begin{aligned} (S, +) &\longrightarrow (1 + K^{\circ\circ}, \cdot) \\ x &\longmapsto 1 + \sum_{q \in S'} G_q(x)\chi(q) \ , \end{aligned}$$

which converges in a neighborhood of $x = 0$ to the K-analytic function $x \mapsto \exp(\pi(\chi)x) = (\exp \pi(\chi))^x$ (resp. converges uniformly on $\mathbb{Q}_p$) along the filter of cofinite subsets of S, and is a character because of (4).

The general formula producing the map $(\overline{AH}_{K^\flat})^\sharp$ according to the theory of perfectoids is

$$(\overline{AH}_{K^\flat})^\sharp(\chi)(x) = \lim_{n\to\infty} AH(\chi(x/p^n))^{p^n} = \lim_{n\to\infty} AH(\chi(1/p^n))^{xp^n} = (\exp \pi(\chi))^x \ ,$$

for any χ as before, and any $x \in \mathbb{Q}_p$. This coincides with the effect of ϵ_K of χ.

In fact, if we allow calculations which exit the algebra $\mathscr{E}_{\frac{p}{p-1},1}$ and involve more general locally analytic functions and distributions on $\mathbb{Q}_p$, we have

$$\begin{aligned} \varepsilon(x,T) = \exp(x \sum_{i=-\infty}^{\infty} p^{-i} T^{p^i}) &= \lim_{n\to\infty} \exp(x \sum_{i=-n}^{\infty} p^{-i} T^{p^i}) = \\ \lim_{n\to\infty} \exp(\sum_{i=0}^{\infty} p^{-i} T^{p^{n+i}})^{xp^n} &= \lim_{n\to\infty} AH(T^{p^n})^{xp^n} \ , \end{aligned}$$

a calculation that avoids using $\Psi_p(x)$. But it seems hard to deduce the fact that $\varepsilon(x,T) \in \mathscr{E}_{\frac{p}{p-1},1}$, and our uniform description of $(\overline{AH}_{K^\flat})^\sharp(\chi)(x)$, from the previous calculation.

The existence of a universal untilted form $\epsilon : \mathbb{D} \xrightarrow{\sim} \mathbb{D}$ of $\overline{AH}$ seems to go beyond the expected properties of the tilting correspondence.

A more extended presentation can be found at
http://gaatp.gaati.org/slides/Baldassarri.pdf
http://people.math.unipr.it/andrea.bandini/Baldassarri.pdf
A paper will soon be made accessible.

Pseudocoherent sheaves and applications

Kiran S. Kedlaya
(joint work with Ruochuan Liu)

In the theory of adic spaces developed in [1], many results are restricted to spaces arising from strongly noetherian Banach rings; however, such results are not adequate for modern applications to the theory of perfectoid spaces. Consequently, in our previous paper [3], we were forced to develop some aspects of nonnoetherian adic spaces from scratch, such as the theory of vector bundles.

In [4], we have been further forced to develop a replacement for the theory of coherent sheaves. Our point of departure is the notion of a *pseudocoherent module* over a ring R in the sense of [2], i.e., a module admitting a projective resolution (not necessarily of finite length) consisting of finite projective R-modules. We show that pseudocoherent modules over stably uniform adic Banach rings (such as perfectoid rings) satisfy analogues of the classical theorems of Tate and Kiehl in rigid analytic geometry. An important intermediate result is a weak flatness theorem for rational localization maps of stably uniform adic Banach rings, which are not known to be flat as morphisms of bare rings.

Our principal motivation for this work is to provide an ambient category containing the relative (φ, Γ)-modules associated to rigid analytic spaces in [3], but which is better suited to homological methods. Recall that for any affinoid space X over a nonarchimedean field K of mixed characteristics, we define the pro-étale topology in the sense of [5]. For this topology, the extended Robba ring forms a sheaf of rings with φ-action, which is acyclic on perfectoid subdomains; the relative (φ, Γ)-modules over X are locally finite free sheaves over the extended Robba ring equipped with semilinear φ-actions. (There is no explicit action of a group Γ; this role is instead played by the sheaf axiom.) In a similar vein, we may define pseudocoherent (φ, Γ)-modules; we show that these form an abelian category satisfying the ascending chain condition. The main subtlety here is that we do not know whether this category admits projective resolutions, so some care is required; we ultimately reduce to the corresponding assertion for pseudocoherent modules over the completed structure sheaf, where the analysis is somewhat easier.

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Algebraic-tropical correspondence for rational curves

Ilya Tyomkin

Enumeration of curves in algebraic varieties is a classical problem that has a long history going back to Ancient Greeks. Many tools have been developed to approach enumerative problems including Schubert calculus, intersection theory, degeneration techniques, quantum cohomology etc.

In late 90s, Kontsevich proposed to use combinatorial objects such as skeleta of Berkovich analytifications in the Gromov-Witten theory and other enumerative problems, and in early 2000s, Mikhalkin [2] introduced the notion of parameterized tropical curves and used them to enumerate complex curves of a given genus in a given linear system on toric surfaces. The tropical approach turned out to be a powerful tool also in mirror symmetry and in real algebraic geometry, where it was a break-through, and in particular, led to the calculation of Welschinger invariants in many interesting cases, see e.g., [1, 2, 6].

Since 2005, few algebraic proofs of various versions of Mikhalkin's correspondence have been obtained by Nishinou-Siebert [3], Shustin [5], the author [7], Ranganathan [4] and others. However, the proofs are relatively complicated, involve techniques such as deformation theory, log-geometry, stacks, rigid analytic spaces etc.; and assume the ground field to be of characteristic zero, or at least, of big enough characteristic (cf. [7]).

In the talk, we discuss our recent results [8] about the algebraic-tropical correspondence in the case of rational curves in toric varieties, having prescribed tangencies to the toric boundary divisor, passing through given orbits, and satisfying multiple cross-ratio constraints. Namely, the setting is as follows:

We fix a complete discretely valued field F with algebraically closed residue field k, and its algebraic closure $\overline{F}$. We fix a pair of dual lattices N and M, a collection $n_1, \dots, n_r \in N$ such that $\sum_{i=1}^r n_i = 0$ (notice that n_i are allowed to coincide, be non-primitive or zero); a collection of sublattices $n_i \in L_i \subseteq N$, $1 \leqslant i \leqslant r$, such that N/L_i are torsion-free; a collection of T_{L_i}-orbits $\zeta_i \in (T_N/T_{L_i})(\overline{F})$ for all $1 \leqslant i \leqslant r$; a collection of cross-ratios $\boldsymbol{\lambda} \in (\overline{F}^\times)^s$, and a collection of ordered quadruples of indices J_i in $\{1, \dots, r\}$ for $1 \leqslant i \leqslant s$. We consider the fan $\Sigma \subset N_\mathbb{R}$ generated by the rays $\rho_i := \mathrm{Span}_{\mathbb{R}_+}(n_i)$ for $1 \leqslant i \leqslant r$, and set $X := X_\Sigma$ to be the corresponding toric variety. We set O_i to be the closure in X of the T_{L_i}-orbit corresponding to ζ_i. Finally, we set $\boldsymbol{\lambda}^{tr} := \mathrm{val}(\boldsymbol{\lambda}) \in \mathbb{Q}^s$ and $O_i^{tr} := \{m \mapsto \mathrm{val}(x^m(p)) \,|\, p \in \zeta_i\} \subseteq N_\mathbb{Q}$.

The goal of the talk is to describe a natural relation between the following:

The set $\mathcal{W}$ of morphisms $f\colon (C; \boldsymbol{q}) \to X$, where $(C; \boldsymbol{q})$ is a smooth projective irreducible rational curve with r marked points such that

Degree and tangency profile:	$div(f^*x^m) = \sum (n_i, m) q_i$,
Toric constraint:	$f(q_i) \in O_i$ for all $i \leqslant r$,
Cross-ratio constraint:	$\lambda(C; \boldsymbol{q}_{J_i}) = \lambda_i$ for all $i \leqslant s$;

and *the set $\mathcal{W}^{tr}$ of stable rational $N_\mathbb{Q}$-parameterized $\mathbb{Q}$-tropical curves $h\colon (\Gamma; \boldsymbol{e}) \to N_\mathbb{Q}$ with r unbounded ends for which*

Degree and multiplicity profile:	$h(u_i) = n_i$ for all $i \leqslant r$,
Affine constraint:	$h(v_i) \in O_i^{tr}$ for all $i \leqslant r$,
Tropical cross-ratio constraint:	$\lambda^{tr}(\Gamma; \boldsymbol{e}_{J_i}) = \lambda_i^{tr}$ for all $i \leqslant s$

where u_i is the i-th infinite vertex of Γ, and v_i the finite vertex attached to it.

As a first step, we explain how to construct a natural tropicalization map $Tr\colon \mathcal{W} \to \mathcal{W}^{tr}$ that associates to an algebraic curve with marked points $(C; \boldsymbol{q})$ the dual graph of the stable reduction equipped with a natural metric, or equivalently, the minimal skeleton $(\Gamma; \boldsymbol{e})$ of the punctured Berkovich analytification; and to the morphism f the parameterization h (see [7, 8] for details).

Then we introduce the notion of G-regularity for an abelian group G. To do so, we consider the natural two-term complex associated to a constrained parameterized tropical curve $h\colon (\Gamma; \boldsymbol{e}) \to N_{\mathbb{Q}}$:

$$L^{\bullet}_{(\Gamma,h;\boldsymbol{O}^{tr},\boldsymbol{\lambda}^{tr})}: \bigoplus_{w \in V^f(\Gamma)} N \oplus \bigoplus_{\gamma \in E^b(\Gamma)} \mathbb{Z} \xrightarrow{\theta} \bigoplus_{\gamma \in E^b(\Gamma)} N \oplus \bigoplus_{i=1}^{r} (N/L_i) \oplus \bigoplus_{i=1}^{s} \mathbb{Z};$$

where the map is defined combinatorially in a natural way (see [8] for details). We say that $(\Gamma, h; \boldsymbol{O}^{tr}, \boldsymbol{\lambda}^{tr})$ is *G-regular* if $H^1(L^{\bullet}_{(\Gamma,h;\boldsymbol{O}^{tr},\boldsymbol{\lambda}^{tr})} \otimes_{\mathbb{Z}} G) = 0$, and *$G$-superabundant* otherwise. Our main results assert the following:

Theorem (Realization)**.** *Let $h\colon (\Gamma; \boldsymbol{e}) \to N_{\mathbb{Q}}$ be an element of $\mathcal{W}^{tr}$, and $K \subset \overline{F}$ a complete discretely valued subfield of definition of $\boldsymbol{O}$, $\boldsymbol{\lambda}$, and (Γ, h). Assume that $\lambda_i^{tr} \neq 0$ for all i, and $(\Gamma, h; \boldsymbol{O}^{tr}, \boldsymbol{\lambda}^{tr})$ is $\mathbb{Q}$-regular. Then*

(1) $h\colon (\Gamma; \boldsymbol{e}) \to N_{\mathbb{Q}}$ belongs to the image of $Tr\colon \mathcal{W} \to \mathcal{W}^{tr}$.

(2) If $(\Gamma, h; \boldsymbol{O}^{tr}, \boldsymbol{\lambda}^{tr})$ is k-regular, Γ is three-valent, and $H^0(L^{\bullet}_{(\Gamma,h;\boldsymbol{O}^{tr},\boldsymbol{\lambda}^{tr})}) = 0$ then the fiber of the tropicalization map Tr over $h\colon (\Gamma; \boldsymbol{e}) \to N_{\mathbb{Q}}$ consists of exactly $|H^1(L^{\bullet}_{(\Gamma,h;\boldsymbol{O}^{tr},\boldsymbol{\lambda}^{tr})})|$ morphism $f\colon (C; \boldsymbol{q}) \to X$, and all morphisms in the fiber are defined over K.

Theorem (Correspondence)**.** *Assume that the constraints $\boldsymbol{O}$ and $\boldsymbol{\lambda}$ are such that $\boldsymbol{O}^{tr}$ and $\boldsymbol{\lambda}^{tr}$ are tropically general, and*

$$s + \sum_{i=1}^{r} \operatorname{rank}(N/L_i) = r - 1.$$

If the characteristic of k is big enough then the map $Tr\colon \mathcal{W} \to \mathcal{W}^{tr}$ is surjective and the size of the fiber over $h\colon (\Gamma; \boldsymbol{e}) \to N_{\mathbb{Q}}$ is $|H^1(L^{\bullet}_{(\Gamma,h;\boldsymbol{O}^{tr},\boldsymbol{\lambda}^{tr})})|$. Moreover, all curves in the fiber are defined over any field of definition of (Γ, h).

Finally, we indicate the strategy of the proofs, which are surprisingly short, elementary, and involve no deformation theory, log-geometry, stacks, or rigid analytic spaces. Similarly to [4], we do not use the degeneration of the target, but unlike the other proofs we use only the standard scheme theory.

Roughly speaking the proof of the Realization theorem consist of the following steps: First, we introduce convenient coordinates, and express the moduli space of stable maps that tropicalize to a given parameterized tropical curve and satisfy the constraints as the set of integral points in a fiber of an explicitly defined map Θ of algebraic tori. Then we show that the reduction of Θ is the map of algebraic tori associated to the homomorphism θ in the complex $L^{\bullet}_{(\Gamma,h;\boldsymbol{O}^{tr},\boldsymbol{\lambda}^{tr})}$, which allows us to prove flatness of Θ, and to deduce the result from Mumford's theorem on the existence of quasi-sections (in this case the existence is easy, and can be achieved directly without referring to Mumford's theorem). The Correspondence theorem then follows from the Realization theorem and a combinatorial lemma asserting that if the constraints are tropically general then all curves in $\mathcal{W}^{tr}$ are regular enough. The latter is rather standard and straightforward.

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Refined curve counting, tropical geometry, and motivic Euler characteristics

Sam Payne

(joint work with Johannes Nicaise and Franziska Schroeter)

This project investigates the relationship between two different approaches to counting curves, one using Euler characteristics of relative Hilbert schemes of points and the other using tropical geometry.

Suppose that $\mathcal{C} \to B$ is a family of reduced and irreducible curves of genus g with finitely many δ-nodal fibers, in which all other fibers have geometric genus greater than $g - \delta$. Such a condition is satisfied in many natural geometric situations, including, for instance, in the case of a general δ-dimensional linear series of sections of a sufficiently ample line bundle on a smooth projective surface. Then the number of δ-nodal fibers can be computed from Euler charactersitcs of relative Hilbert schemes of points, as follows. The generating function

$$q^{1-g} \sum_{i=0}^{\infty} \chi(\mathrm{Hilb}^i(\mathcal{C}/B))q^i$$

can be expressed uniquely as a sum

$$\sum_{r=g-\delta}^{g} n_r q^{1-r}(1-q)^{2r-2},$$

where the coefficients n_r are positive integers, and $n_{g-\delta}$ is the number of δ-nodal fibers in the family.

Göttsche and Shende have proposed to study refined curve counting invariants, defined similarly, but with the Euler characteristic replaced by χ_y genus. More precisely, the generating function

$$q^{1-g}\sum_{i=0}^{\infty}\chi_y(\mathrm{Hilb}^i(\mathcal{C}/B))q^i$$

can be expressed uniquely as a sum

$$\sum_{r=0}^{\infty} N_r\, q^{1-r}(1-q)^{r-1}(1-qy)^{r-1},$$

where the coefficients N_r are polynomials in the formal variable y that specialize to the ordinary curve counting invariants n_r by setting $y=1$, and these coefficients N_r are defined to be the refined invariants. (We follow the convention that $n_r=0$ for r outside the interval $[g-\delta, g]$.)

It is well-known that nodal curves on toric surfaces can also be counted tropically. If $\mathcal{C}\to B$ is the universal curve over the locus of reduced and irreducible curves in a general δ-dimensional linear series in the complete linear series of an ample line bundle, then the number of δ-nodal fibers can be computed as the number of parametrized tropical curves of genus $g-\delta$, with unbounded edge directions specified by the ample line bundle, and passing through $n-\delta$ points in general position, counted with combinatorially defined multiplicities. Block and Göttsche have proposed refined tropical curve counting invariants, which are combinatorially defined polynomials in a formal variable y that specialize to the ordinary tropical multiplicities by setting $y=1$, and conjecture that the refined curve counting invariant $N_{g-\delta}$ for $\mathcal{C}\to B$ should be recovered as a sum of these tropical refined multiplicities over the same set of tropical curves.

The main goal of our project is to give a natural geometric interpretation for the combinatorially defined refined tropical curve counting multiplicities of Block and Göttsche. We observe that, for a given tropical curve Γ, the locus in B parametrizing curves with tropicalization Γ is a semialgebraic set B_Γ. We write $\mathcal{C}_\Gamma\to B_\Gamma$ for the universal family, and show that all of the curves in this family are reduced and irreducible. Our strategy, then, is to define a geometric invariant analogous to that of Göttsche and Shende, but for this semialgebraic family. The generating function

$$q^{1-g}\sum_{i=0}^{\infty}\chi_y(\mathrm{Hilb}^i(\mathcal{C}_\Gamma/B_\Gamma))q^i,$$

where $\chi_y(\mathrm{Hilb}^i(\mathcal{C}_\Gamma/B_\Gamma))$ denotes the χ_y-specialization of the motivic measure of this semialgebraic set, in the framework of Hrushovski and Kazhdan, can be expressed uniquely as a sum

$$\sum_{r=0}^{\infty} N_r\, q^{1-r}(1-q)^{r-1}(1-qy)^{r-1},$$

in which the coefficients N_r are polynomials in the formal variable y. We conjecture that the Block–Göttsche refined tropical curve counting multiplicity of Γ is equal to the polynomial $N_{g-\delta}$. We prove this conjecture in the case $g = 1$. We also show that the conjecture is correct after specializing to the ordinary Euler characteristic. In other words, $N_{g-\delta}(1)$ is the ordinary tropical multiplicity of the curve Γ.

Geometrization of the local Langlands correspondence

Laurent Fargues

Given a quasisplit reductive group G defined over a p-adic field E we first define the moduli stack Bun_G of G-bundles over the curve we defined and studied in our joint work with Fontaine. This is a "perfectoid stack" in characteristic p over $\mathbb{F}_q$ the residue field of E. The points of $\mathrm{Bun}_G \otimes \overline{\mathbb{F}}_q$ are identified with Kottwitz set $B(G)$ of σ-conjugacy classes in $G(L)$ where L is the completion of the maximal unramified extension of E. There is a dictionnary between Kottwitz description of $B(G)$ and reduction theory. In particular basic in Kottwitz sens is equivalent to semi-stable for a G-bundle.

This stack has a nice Harder-Narasimhan stratification, in particular the semi-stable locus is open. Its connected components are parametrized by $\pi_1(G)_\Gamma$ where $\Gamma = \mathrm{Gal}(\overline{E}|E)$. In each of those components there is a unique semi-stable point given by some $b \in G(L)$ basic. This is given by Kottwitz bijection

$$\kappa : B(G)_{basic} \xrightarrow{\sim} \pi_1(G)_\Gamma.$$

The associated semi-stable stratum is then the classifying stack

$$\big[\mathrm{Spa}(\overline{\mathbb{F}}_q)/\underline{J_b(E)}\big]$$

where J_b is an inner form of G (all inner forms of G are reached in this way when the center of G is connected, for example for GL_n). This is the classifying stack of pro-étale $J_b(E)$-torsors.

Choose $\ell \neq p$ and let ${}^LG$ be the corresponding ℓ-adic Langlands dual over $\overline{\mathbb{Q}_\ell}$. Consider a Langlands parameter $\varphi : W_E \to {}^LG$. Note S_φ for the group of automorphisms of φ

$$S_\varphi = \{g \in \widehat{G} \mid g\varphi g^{-1} = \varphi\}.$$

Suppose φ is discrete that is to say $S_\varphi/Z(\widehat{G})^\Gamma$ is finite.

We conjecture the existence of a "perverse Weil sheaf" $\mathscr{F}_\varphi$ on $\mathrm{Bun}_G \otimes \overline{\mathbb{F}}_q$ equipped with an action of S_φ satisfying the following properties:

- the action of $Z(\widehat{G})^\Gamma \subset S_\varphi$ on the connected component given by $\alpha \in \pi_1(G)_\Gamma$ is given by α via the identification $\pi_1(G)_\Gamma = X^*(Z(\widehat{G})^\Gamma)$.
- For b basic, via the inclusion of the corresponding semi-stable component $x_b : [\mathrm{Spa}(\overline{\mathbb{F}}_q)/\underline{J_b(E)}] \hookrightarrow \mathrm{Bun}_G \otimes \overline{\mathbb{F}}_q$, the decomposition of the action of S_φ

$$x_b^* \mathscr{F}_\varphi = \bigoplus_{\substack{\rho \in \widehat{S_\varphi} \\ \rho_{|Z(\widehat{G})^\Gamma} = \kappa(b)}} \mathscr{F}_{\varphi,\rho}$$

  as smooth representations of $J_b(E)$ defines an L-packet $\{\mathscr{F}_{\varphi,\rho}\}_\rho$ for a local Langlands correpondence for the inner form J_b of G. When $b = 1$, and thus $J_b = G$, $\mathscr{F}_{\varphi,1}$ is the unique generic element of the L-packet (and thus the construction of $\mathscr{F}_\varphi$ has to depend on the choice of a Whittaker datum).
- There are Hecke correspondences defined between Bun_G and $\mathrm{Bun}_G \times \mathrm{Spa}(E)^\diamond$. They are parametrized by element $\mu \in X_*(A)^+$ where A is a maximal split torus in G and $X_*(A)^+$ is the positive Weyl chamber relative to the the choice of a Borel subgroup containing A. Then $\mathscr{F}_\varphi$ is an eigenvector for those Hecke correspondences with eigenvalue $r_\mu \circ \varphi$ seen as an ℓ-adic Weil local system on $\mathrm{Spa}(E)^\diamond$.
- $\mathscr{F}_\varphi$ has to satisfy a local global compatibility with Caraiani-Scholze sheaf $R\pi_{HT*}\overline{\mathbb{Q}_\ell}$ where π_{HT} is the Hodge-Tate period map associated to a Hodge type Shimura variety.

This conjecture implies Kottwitz conjectural description of the discrete part of the cohomology of Rapoport-Zink spaces. It is checked for GL_1 where this is equivalent to local class field theory.

On skeleta

François Loeser
(joint work with E. Hrushovski)

Let val : $K \to \Gamma_\infty$ be a valued field. Here $\Gamma_\infty = \Gamma \cup \{\infty\}$ with Γ an ordered abelian group (no restriction on the rank of Γ is assumed). Let V be an algebraic variety over K. In [2] we introduced the stable completion $\widehat{V}$ of V, which is a model-theoretic version of the Berkovich analytification. Points in $\widehat{V}$ are definable types on V that are dominated by their stable part. $\widehat{V}$ is naturally endowed with a topology coming from the order topology on Γ. A key feature of $\widehat{V}$ is that it is *pro-definable* in the geometric language of [1]. A subset of $\widehat{V}$ is called *iso-definable* resp. *iso-definable* Γ*-internal* if it is pro-definably isomorphic to a definable set, resp. to a definable subset of Γ^n, for some n.

An important role is played by those types in $\widehat{V}$ that satisfy a form of Abhyankar equality, namely those definable types p on V such that there exists a definable map $f : V \to W$ with W defined over the residue field and such that the Zariski dimension of the support of p and of $f_*(p)$ are equal. We call such types *strongly stably dominated* and denote the set of those types by $V^\#$. When $\dim(V) \leqslant 1$, $V^\# = \widehat{V}$, but the inclusion is strict as soon as $\dim(V) \geqslant 2$ as shown by the next example. An important property of $V^\#$ is that it naturally endowed with the structure of an *ind-definable* space.

Example: Take $K = F((t))$ with F trivially valued and $\mathrm{val}(t) = 1$. Consider a non-algebraic power series $\varphi(x) = \sum_{i \geqslant 0} a_i x^i$, with $a_i \in F$, and for any non negative integer n, set $\varphi_n(x) = \sum_{0 \leqslant i \leqslant n} a_i x^i$. For $\gamma \in \Gamma_\infty$ consider the complete type p_γ in (x, y) generated by the generic type of the closed ball $\mathrm{val}(x) \geqslant 1$ and the formulas

$$\mathrm{val}(y - \varphi_n(x))) \geqslant \min(n + 1, \gamma).$$

One can check that p_γ belongs to $(\mathbb{A}^2)^\#$ if and only γ is finite, i.e. smaller than n_0 for some integer n_0. Furthermore, the mapping $g : \Gamma_\infty \to \widehat{\mathbb{A}^2}$ sending γ to p_γ is continuous and pro-definable but its image is **not** iso-definable in $\widehat{\mathbb{A}^2}$.

By a *generalized interval* we mean a definable set which is obtained by glueing end-to-end a finite number of intervals in Γ_∞. We say an iso-definable Γ-internal subset $\widehat{V}$ is *topologically* Γ-internal if it is pro-definably homeomorphic to a definable subset of Γ_∞^n, for some n.

Call a subset $\Upsilon \subset \widehat{V}$ a *skeleton* if Υ is topologically Γ-internal, is contained in $V^\#$ and for any irreducible component V_i of V, $\Upsilon \cap \widehat{V_i}$ if of o-minimal dimension $\dim(V_i)$ everywhere. The main result in [2] is the following theorem:

Theorem 1. *Let V be a quasi-projective variety over a valued field. There exists a continuous prodefinable map $h : I \times \widehat{V} \to \widehat{V}$, with I a generalized interval, which is a strong deformation retraction onto a subset $\Upsilon \subset \widehat{V}$ with Υ a skeleton. Furthermore, given a finite number of definable functions $\alpha_i : V \to \Gamma_\infty$ one may require h to respect the α_i.*

A first connection between $V^\#$ and o-minimal geometry is provided by the following proposition:

Proposition 1. *Let V be a variety of dimension n, and let $W \subset \widehat{V}$ be iso-definable Γ-internal. If W is of pure o-minimal dimension n, then $W \subset V^\#$.*

In view of the following rigidity statement it explains the importance of the space $V^\#$ in the proof of Theorem 1:

Proposition 2 (rigidity). *Let V be a variety of dimension n, and let $W \subset \widehat{V}$ be iso-definable Γ-internal. If W is of pure o-minimal dimension n, and $\alpha : \widehat{V} \to \Gamma_\infty$ is pro-definable and finite-to-one on W, then any $h : I \times \widehat{V} \to \widehat{V}$, continuous pro-definable with I a generalized interval respecting α, fixes pointwise W.*

We ended the talk by sketching the proof of the following recent result which is included in the latest versions of [2].

Theorem 2. *Let V be a quasi-projective variety over a valued field. Then $V^\#$ is exactly the union of all skeleta inside $\widehat{V}$.*

Sketch of proof: One has to prove that any point p in $V^{\#}$ belongs to a skeleton. By increasing the basis, one reduces to the case p is a realized type (recall $V^{\#}$ is ind-definable). The proof then proceeds by descending induction on the o-minimal dimension, the case of maximal dimension being consequence of Proposition 2.

One deduces from Theorem 2 the following topological characterisation of the points satisfying the equality in the Abhyankar inequality:

Corollary. *The set $V^{\#}$ is exactly the locus in $\widehat{V}$ of points having local o-minimal dimension the Zariski dimension of V; e.g., if V is of pure dimension n, $V^{\#}$ is the locus of points of o-minimal dimension n, namely those contained in an iso-definable Γ-internal set of o-minimal dimension n.*

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Geometric invariants for non-archimedean semi-algebraic sets

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(joint work with Franziska Schroeter, Sam Payne)

Let K be an algebraically closed real-valued field of equal characteristic zero, with valuation ring R and residue field k. The prime example to keep in mind is the field of complex Puiseux series $K = \cup_{n>0}\mathbb{C}((t^{1/n}))$, which is an algebraic closure of the field of complex Laurent series $\mathbb{C}((t))$. A *semi-algebraic* subset of an algebraic K-variety X is a subset of $X(K)$ that can locally be defined by Boolean operators and inequalities of the form $v(f) \leqslant v(g)$ where f, g are algebraic functions on X and v denotes the valuation on K. The aim of my talk was to show how one can attach geometric invariants to semi-algebraic sets over the field K using the theory of motivic integration developed by Hrushovski and Kazhdan [3]. The motivation for this construction is twofold:

(1) Semi-algebraic sets occur naturally in tropical and non-archimedean geometry. For instance, given a family of subvarieties of an algebraic torus, the locus of fibers of the family with fixed tropicalization is semi-algebraic. This follows from Robinson's quantifier elimination for algebraically closed valued fields.
(2) Even if one is ultimately interested in computing invariants for algebraic varieties, it is often useful to know that one can compute these invariants on semi-algebraic decompositions of the variety, for instance to obtain tropical formulas.

Both motivations play a role in our project, which aims to give a geometric interpretation of the refined tropical multiplicities of Block and Göttsche [1] and to obtain a tropical correspondence theorem for the refined curve counting invariants of Göttsche and Shende [2]. The first results will appear in [7].

The central tool in our approach is the *motivic volume* defined by Hrushovski and Kazhdan. This is a morphism

$$\mathrm{Vol}: K_0(\mathrm{VF}_K) \to K_0(\mathrm{Var}_k)$$

from the Grothendieck ring of semi-algebraic sets over K to the Grothendieck ring of algebraic varieties over the residue field k. With the help of this morphism we can extend all the classical motivic invariants in algebraic geometry to semi-algebraic sets, by composing Vol with the motivic invariant on $K_0(\mathrm{Var}_k)$. For instance, this allows us to define the Hodge-Deligne polynomial, the χ_y-genus and the Euler characteristic of a semi-algebraic set.

A common feature of all the theories of motivic integration is that they try to understand the structure of semi-algebraic objects over K in terms of data living over the residue field k (that is, algebraic k-varieties) and over the value group $|K^*|$ (polyhedra). This aim is realized in the theory of Hrushovski and Kazhdan by a complete description of the Grothendieck ring of semi-algebraic sets $K_0(\mathrm{VF}_K)$ as a tensor product of certain Grothendieck rings of k-varieties and polyhedra, respectively. They show that $K_0(\mathrm{VF}_K)$ is generated by the classes of the following types of semi-algebraic sets:

- inverse images of closed $|K^*|$-rational polyhedra Γ in $\mathbb{R}^n$ under the tropicalization map $\mathrm{trop}: (K^*)^n \to \mathbb{R}^n$;
- tubes around subvarieties Y of the special fibers of smooth R-schemes $\mathcal{X}$ of finite type.

Moreover, they express in a simple and elegant way all the relations that exist between these classes. The motivic volume

$$\mathrm{Vol}: K_0(\mathrm{VF}_K) \to K_0(\mathrm{Var}_k)$$

is fully characterized by its values on the generators above:

- $\mathrm{Vol}(\mathrm{trop}^{-1}(\Gamma)) = [\mathbb{G}^n_{m,k}]$ for every $n \geqslant 1$ and every *closed* $|K^*|$-rational polyhedron Γ in $\mathbb{R}^n$;
- the volume of the tube around Y in $\mathcal{X}$ equals $[Y]$.

The motivation for the first expression is that we can think of $\mathrm{trop}^{-1}(\Gamma)$ as a $(R^*)^n$-torsor over Γ, and that the volume of R^* according to the second expression is $[\mathbb{G}^n_{m,k}]$. In many situations, the invariants of semi-algebraic sets defined in this way have a natural geometric meaning. For instance, one can deduce from work by Martin [6] and Hrushovski & Loeser [4] that the Euler characteristic of a semi-algebraic set coincides with the one obtained from Berkovich's theory of étale cohomology for K-analytic spaces.

In order to be able to compute all of these motivic invariants in practice, we have established a tropical expression for the classes in $K_0(\mathrm{VF}_K)$ of schön subvarieties X of algebraic K-tori. Every tropical polyhedral decomposition Σ of the tropicalization of X gives rise to a semi-algebraic decomposition of $X(K)$ whose pieces are the inverse images of the cells of Σ under the tropicalization map. This leads to an expression for $[X(K)]$ in terms of the generators of $K_0(\mathrm{VF}_K)$ given above, involving the cells of Σ on the polyhedral level and the strata of the special fiber of a certain model for X induced by Σ on the residue field level. As a corollary, we find the expression

$$\mathrm{Vol}(X(K)) = \sum_{\gamma}(-1)^{\dim(\gamma)}[\mathrm{in}_{\gamma}X]$$

where γ runs over the bounded open cells of Σ and $\mathrm{in}_{\gamma}X$ denotes the corresponding initial degeneration of X. A similar expression for the motivic nearby fiber of Denef and Loeser was given by Katz and Stapledon in [5].

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Syntomic complexes and p-adic nearby cycles

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(joint work with Pierre Colmez)

We compute syntomic cohomology of semistable affinoids in terms of cohomology of (φ, Γ)-modules which, thanks to work of Fontaine-Herr, Andreatta-Iovita, and Kedlaya-Liu, is known to compute Galois cohomology of these affinoids. For a semistable scheme over a mixed characteristic local ring this implies a comparison isomorphism, up to some universal constants, between truncated sheaves of p-adic nearby cycles and syntomic cohomology sheaves. This generalizes the comparison results of Kato, Kurihara, and Tsuji for small Tate twists (where no constants are necessary) as well as the comparison result of Tsuji that holds over the algebraic closure of the field. As an application, we combine this local comparison isomorphism with the theory of finite dimensional Banach Spaces and finitness of étale cohomology of rigid analytic spaces proved by Scholze to prove a Semistable conjecture for formal schemes with semistable reduction.

Let $\mathscr{O}_K$ be a complete discrete valuation ring with fraction field K of characteristic 0 and with perfect residue field k of characteristic p. Let $\mathscr{O}_F = W(k)$ and

$F = \mathscr{O}_F[\frac{1}{p}]$ so that K is a totally ramified extension of F; let $e = [K : F]$ be the absolute ramification index of K. Let $\overline{\mathscr{O}}_K$ denote the integral closure of $\mathscr{O}_K$ in $\overline{K}$. Set $G_K = \mathrm{Gal}(\overline{K}/K)$, and let $\phi = \phi_{W(\overline{k})}$ be the absolute Frobenius on $W(\overline{k})$. For a log-scheme X over $\mathscr{O}_K$, X_n will denote its reduction mod p^n, X_0 will denote its special fiber.

1.1. **Statement of the main results.** Let X be a (strict) semistable scheme over $\mathscr{O}_K$. For $r \geqslant 0$, let $\mathscr{S}_n(r)_X$ be the (log) syntomic sheaf modulo p^n on $X_{0,\acute{e}t}$. It can be thought of as a derived Frobenius and filtration eigenspace of crystalline cohomology or as a relative Fontaine functor. Fontaine-Messing [4] have defined a period map

$$\alpha^{\mathrm{FM}}_{r,n}: \quad \mathscr{S}_n(r)_X \to i^* Rj_* \mathbf{Z}/p^n(r)'_{X_{\mathrm{tr}}}$$

from syntomic cohomology to p-adic nearby cycles. Here $i : X_0 \hookrightarrow X$, $j : X_{\mathrm{tr}} \hookrightarrow X$, and X_{tr} is the locus where the log-structure is trivial. We set $\mathbf{Z}_p(r)' := \frac{1}{p^{a(r)}}\mathbf{Z}_p(r)$, for $r = (p-1)a(r) + b(r)$, $0 \leqslant b(r) \leqslant p-1$.

We prove that the Fontaine-Messing period map $\alpha^{\mathrm{FM}}_{r,n}$, after a suitable truncation, is essentially a quasi-isomorphism. More precisely, we prove the following theorem.

Theorem 1.1. *For $0 \leqslant i \leqslant r$, consider the period map*

$$\alpha^{\mathrm{FM}}_{r,n}: \quad \mathscr{H}^i(\mathscr{S}_n(r)_X) \to i^* R^i j_* \mathbf{Z}/p^n(r)'_{X_{\mathrm{tr}}}. \tag{1.2}$$

(i) *If K has enough roots of unity*[1] *then the kernel and cokernel of this map are annihilated by p^{Nr} for a universal constant N* (not depending on p, X, K, n or r).

(ii) *In general, the kernel and cokernel of this map are annihilated by p^N for an integer $N = N(e,p,r)$, which depends on e, r, but not on X or n.*

For $i \leqslant r \leqslant p-1$, it is known that a stronger statement is true: the period map

$$\alpha^{\mathrm{FM}}_{r,n}: \quad \mathscr{H}^i(\mathscr{S}_n(r)_X) \xrightarrow{\sim} i^* R^i j_* \mathbf{Z}/p^n(r)_{X_{\mathrm{tr}}}. \tag{1.3}$$

is an isomorphism for X a log-scheme log-smooth over a henselian discrete valuation ring $\mathscr{O}_K$ of mixed characteristic. This was proved by Kato [7], [8], Kurihara [10], and Tsuji [14], [15]. In [13] Tsuji generalized this result to some étale local systems. As Geisser has shown [5], in the case without log-structure, the isomorphism (1.3) allows to approximate the (continuous) p-adic motivic cohomology (sheaves) of p-adic varieties by their syntomic cohomology; hence to relate p-adic algebraic cycles to differential forms.

As an application of Theorem 1.1, one can obtain the following generalization of the Bloch-Kato's exponential map [2]. Let $\mathscr{X}$ be a quasi-compact formal, semistable scheme over $\mathscr{O}_K$ (for example a semi-stable affinoid). For $i \geqslant 1$, consider the composition

$$\alpha_{r,i}: \quad H^{i-1}_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}}) \to H^i(\mathscr{X}, \mathscr{S}(r))_{\mathbf{Q}} \xrightarrow{\alpha^{FM}_r} H^i_{\acute{e}t}(\mathscr{X}_{K,\mathrm{tr}}, \mathbf{Q}_p(r)).$$

[1] The field F contains enough roots of unity and for any K, the field $K(\zeta_{p^n})$, for $n \geqslant c(K)+3$, where $c(K)$ is the conductor of K, contains enough roots of unity.

If X is a proper semistable scheme X over $\mathscr{O}_K$, and $1 \leqslant i \leqslant r-1$, then the G_K-representation $V_{i-1} = H^{i-1}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r))$ is finite dimensional over $\mathbf{Q}_p$, $H^{i-1}_{\text{dR}}(X_K)$ is finite dimensional over K, and $H^{i-1}_{\text{dR}}(X_K) = D_{\text{dR}}(V_{i-1})$. The map $\alpha_{r,i}$ for the formal scheme $\mathscr{X}$ associated to X is then the Bloch-Kato's map [11]

$$D_{\text{dR}}(V_{i-1}) \to H^1(G_K, V_{i-1}) = H^i_{\text{ét}}(X_K, \mathbf{Q}_p(r)).$$

Easy comparison between de Rham and syntomic cohomologies, together with Theorem 1.1, yield the following result.

Corollary 1.4. *For $i \leqslant r-1$, the map*

$$\alpha_{r,i} : H^{i-1}_{\text{dR}}(\mathscr{X}_{K,\text{tr}}) \to H^i_{\text{ét}}(\mathscr{X}_{K,\text{tr}}, \mathbf{Q}_p(r))$$

is an isomorphism. Moreover, the map $\alpha_{r,r} : H^{r-1}_{\text{dR}}(\mathscr{X}_{K,\text{tr}}) \to H^r_{\text{ét}}(\mathscr{X}_{K,\text{tr}}, \mathbf{Q}_p(r))$ is injective (but not necessarily surjective: the case $i = r = 1$ and $\mathscr{X} = \emptyset_K^\times$ already provides a counterexample).

Recall how one shows that the period map $\alpha^{\text{FM}}_{r,n}$ from (1.3) is an isomorphism. Under the stated assumptions one can do dévissage and reduce to $n = 1$. Then one passes to the tamely ramified extension obtained by attaching the p'th root of unity ζ_p. There both sides of the period map (1.3) are invariant under twisting by $t \in \mathbf{A}_{\text{cr}}$ and ζ_p, respectively, so one reduces to the case $r = i$. This is the Milnor case: both sides compute Milnor K-theory modulo p. To see this, one uses symbol maps from Milnor K-theory to the groups on both sides (differential on the syntomic side and Galois on the étale side). Via these maps all groups can be filtered compatibly in a way similar to the filtration of the unit group of a local field. Finally, the quotients can be computed explicitly by symbols [1], [6], [10], [13] and they turn out to be isomorphic. This approach to the computation of p-adic nearby cycles goes back to the work of Bloch-Kato [1] who treated the case of good reduction and whose approach was later generalized to semistable reduction by Hyodo [6].

Our proof is of very different nature: we construct another local (i.e., on affinoids of a special type, see below) period map, that we call $\alpha^{\mathcal{L}az}_r$. Modulo some (ϕ, Γ)-modules theory reductions, it is a version of an integral Lazard isomorphism between Lie algebra cohomology and continuous group cohomology. We prove directly that it is a quasi-isomorphism and coincides with Fontaine-Messing's map up to constants as in Theorem 1.1. The (hidden) key input is the purity theorem of Faltings [3], Kedlaya-Liu [9], and Scholze [12]: it shows up in the computation of Galois cohomology in terms of (φ, Γ)-modules [9] which we use as a black box.

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Chabauty–Coleman on basic wide opens and applications to uniform boundedness

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(joint work with Eric Katz, David Zureick-Brown)

1. Uniform boundedness questions

This is a report on completed and continuing work on proving uniform boundedness statements by using p-adic integration and applying the Chabauty–Coleman method on basic wide open subdomains of Berkovich curves. By a *curve*, we will always mean a smooth, proper, geometrically connected curve C over a field. We will always denote the Jacobian of C by J.

We will consider the following (proved) conjectures, and their uniform variants.

Mordell Conjecture 1.1. *Let C be a $\mathbf{Q}$-curve of genus $g \geqslant 2$. Then C has finitely many $\mathbf{Q}$-rational points.*

The Mordell conjecture was proved by Faltings in 1983. Giving an explicit bound on the size of $C(\mathbf{Q})$ amounts to producing a number N, calculated in terms of invariants of the curve C, such that $\#C(\mathbf{Q}) \leqslant N$. One might hope that the only necessary invariant is the genus of the curve; this gives rise to the uniform version of 1.1.

Uniform Mordell Conjecture 1.2. *For $g \geqslant 2$, there exists a number $N = N(g)$ such that for every $\mathbf{Q}$-curve C of genus g, we have $\#C(\mathbf{Q}) \leqslant N(g)$.*

Perhaps the strongest currently known statement in the direction of Conjecture 1.1 is the following theorem, which relies heavily on recent work of Stoll [Sto13].

Theorem 1 ([KRZB15, Theorem 1.1])**.** *Let $g \geqslant 3$. For every $\mathbf{Q}$-curve C of genus g such that* $\mathrm{rank} J(\mathbf{Q}) \leqslant g-3$*, we have*

$$\#C(\mathbf{Q}) \leqslant 76g^2 - 82g + 22.$$

This theorem is proved with a variant of the techniques we will sketch in this note. These techniques are based on the Chabauty–Coleman method; for this reason it is not likely that the condition on the Mordell–Weil rank can be removed. We will focus mostly on the following (proved) conjecture, and its uniform variant.

Manin–Mumford Conjecture 1.3. *Let C be a curve of genus $g \geqslant 2$ defined over the complex numbers $\mathbf{C}$. Let $\alpha\colon C \hookrightarrow J$ be the Abel–Jacobi map, defined with respect to a choice of base point. Then only finitely many points of C map to torsion points of J: in symbols, $\#\alpha^{-1}(J(\mathbf{C})_{\mathrm{tors}}) < \infty$.*

The Manin–Mumford conjecture was first proved by Raynaud in 1983, with many subsequent proofs. As above, the uniform variant asserts a bound in terms of the genus alone.

Uniform Manin–Mumford Conjecture 1.4. *For $g \geqslant 2$, there exists a number $N = N(g)$ such that for every $\mathbf{C}$-curve C of genus g and every choice of Abel–Jacobi map $\alpha\colon C \hookrightarrow J$, we have $\#\alpha^{-1}(J(\mathbf{C})_{\mathrm{tors}}) \leqslant N(g)$.*

Buium [Bui96] proved that if C is defined over an unramified finite extension $K/\mathbf{Q}_p$ (for $p \geqslant 3$) and has good reduction, then

$$\#\alpha^{-1}(J(\mathbf{C}_p)_{\mathrm{tors}}) \leqslant p^{4g}\, 3^g \big[p(2g-2) + 6g\big]\, g!.$$

One can use this result to give another proof of 1.3. Notice however that the bound only depends on g (and p), and as such gives a uniform bound as in Conjecture 1.4 for curves over K with good reduction.

2. p-adic integration

There are several flavors of p-adic line integration. We mention here the two that we will employ. In this section we work entirely over $\mathbf{C}_p$.

2.1. **The abelian integral.** Using general considerations on p-adic Lie groups, one can show that there exists a unique p-adic Lie group homomorphism

$$\underline{\log}\colon J(\mathbf{C}_p) \longrightarrow \mathrm{Lie}(J) = H^0(J, \Omega^1)^*,$$

which induces the identity on Lie algebras. Using $H^0(J,\Omega^1) = H^0(C,\Omega^1)$, for $x, y \in C(\mathbf{C}_p)$ and $\omega \in H^0(C,\Omega^1)$ we define

$$\overset{\mathrm{Ab}}{\int_x^y} \omega \coloneqq \underline{\log}(\alpha(y) - \alpha(x))(\omega).$$

Noting that $\underline{\log}$ kills $J(\mathbf{C}_p)_{\mathrm{tors}}$, we have

$$(2.1) \qquad {}^{\mathrm{Ab}}\!\!\int_x^y \omega = 0 \quad \text{for all} \quad x, y \in \alpha^{-1}(J(\mathbf{C}_p)_{\mathrm{tors}}) \text{ and } \omega \in H^0(C, \Omega^1).$$

2.2. **The Berkovich–Coleman Integral.** The Berkovich–Coleman integral associates to a path $\gamma\colon x \rightsquigarrow y$ in the Berkovich analytification C^{an}, where $x, y \in C(\mathbf{C}_p)$, and to a differential $\omega \in H^0(C, \Omega^1)$, a number

$${}^{\mathrm{BC}}\!\!\int_\gamma \omega \in \mathbf{C}_p.$$

This integration theory enjoys many desirable properties, including:

(1) ${}^{\mathrm{BC}}\!\int_\gamma$ only depends on the fixed end-point homotopy class of γ.
(2) ${}^{\mathrm{BC}}\!\int_\gamma$ is intrinsic to any analytic subdomain containing γ, i.e. it can be calculated locally.
(3) ${}^{\mathrm{BC}}\!\int_\gamma df = f(y) - f(x)$ for an analytic function f defined on γ, i.e. the integral satisfies the fundamental theorem of calculus.

Properties (2) and (3) imply that ${}^{\mathrm{BC}}\!\int_\gamma \omega$ can be computed locally using formal antidifferentiation on domains in which ω is exact.

2.3. **Comparing the integrals.** The abelian and Berkovich–Coleman integrals do not necessarily coincide; however, their difference is well-controlled.

Vague Proposition 2.4. *The difference* ${}^{\mathrm{BC}}\!\int - {}^{\mathrm{Ab}}\!\int$ *is controlled by the tropical Abel–Jacobi map.*

See [KRZB15, Proposition 3.16, §3.5] for precise statements. We will give some concrete consequences of 2.4, using the following notation. Let $\Gamma \subset C^{\mathrm{an}}$ be a skeleton. This is a weighted metric graph, which we assume here and below has no loop edges; this can always be achieved by adding more vertices. Let $\tau\colon C^{\mathrm{an}} \to \Gamma$ be the retraction map, let $v \in \Gamma$, let V_v denote the union of v with all open edges adjacent to v, and let $U_v = \tau^{-1}(V_v)$. This is an open analytic domain in C^{an}.

Definition 1. An open subset of the form $U_v \subset C^{\mathrm{an}}$ is called a *basic wide open subdomain.*

It is not hard to see that Definition 1 coincides with Coleman's notion [Col89, §3]. Note that U_v is simply-connected, so it makes sense to write ${}^{\mathrm{BC}}\!\int_x^y \omega$ for $x, y \in U_v$.

Corollary 1 (to 2.4)**.** *Let $v \in \Gamma$ be a vertex, and let $d = \deg(v)$, the valency of v in Γ. Then there exists a subspace $W \subset H^0(C, \Omega^1)$ of codimension at most $d - 1$ such that ${}^{\mathrm{BC}}\!\int_x^y \omega = {}^{\mathrm{Ab}}\!\int_x^y \omega$ for all $\omega \in W$ and $x, y \in U_v(\mathbf{C}_p)$.*

It follows from Corollary 1 that ${}^{\mathrm{BC}}\!\int \omega = {}^{\mathrm{Ab}}\!\int \omega$ on open discs, which explains why the usual Chabauty–Coleman method does not require a comparison of the two integrals. On an open annulus, which is a basic wide open associated to a point

v of valency 2, there is one linear condition needed for ${}^{\mathrm{BC}}\!\!\int \omega = {}^{\mathrm{Ab}}\!\!\int \omega$; this was discovered by Stoll [Sto13, Proposition 6.1]. On a "p-adic pair of pants", i.e. a basic wide open U_v associated to a vertex v of valency 3, the equation ${}^{\mathrm{BC}}\!\!\int \omega = {}^{\mathrm{Ab}}\!\!\int \omega$ imposes two linear conditions. And so forth.

3. Overall technique

In this section we describe the overall strategy for using Chabauty–Coleman on basic wide opens to obtain uniform Manin–Mumford statements. Throughout we will work with a curve C defined over $\mathbf{C}_p$, with J its Jacobian. We proceed in the following steps.

(1) Since C is hyperbolic, there exists a *minimal* skeleton $\Gamma \subset C^{\mathrm{an}}$. This is a weighted metric graph of genus g. The combinatorics of such graphs give bounds on the number of vertices and edges of Γ, in terms of g alone.
(2) We have $C^{\mathrm{an}} = \bigcup U_v$, where the union is taken over vertices of Γ. That is, C^{an} is covered by a uniformly bounded number of basic wide open subdomains.
(3) Suppose that, for each vertex $v \in \Gamma$, we can find a nonzero global differential $\omega_v \in H^0(C, \Omega^1)$ satisfying:
    (E) $\omega_v = df_v$ for $f_v \in \mathcal{O}(U_v)$, i.e. ω_v is exact on U_v.
    (I) ${}^{\mathrm{BC}}\!\!\int_x^y \omega_v = {}^{\mathrm{Ab}}\!\!\int_x^y \omega_v$ for all $x, y \in U_v(\mathbf{C}_p)$.

We claim that conditions **(E)** and **(I)** suffice to prove a uniform Manin–Mumford statement.

(4) For a suitable choice of antiderivative f_v, all torsion points on U_v are zeros of f_v. This is a consequence of (2.1) and **(I)**.

Now we describe how to bound the number of zeros of f_v. This plays the role of the "p-adic Rolle theorem" part of the classical Chabauty–Coleman method, and is central to [KRZB15]. Some of the assertions below are simplified to the point of being not quite correct; see [KRZB15] for precise statements.

Let

$$G_v := -\log ||\omega_v|| \colon V_v \longrightarrow \mathbb{R}.$$

Here the metric $||\cdot||$ on Ω^1 comes from the canonical extension of Ω^1 to a semistable model of C defined using an integral version of Rosenlicht differentials. It can also be defined using Temkin's theory of metrization of differential pluriforms [Tem14], or using the relative dualizing sheaf if C has a semistable model over a discretely valued subfield of $\mathbf{C}_p$. In any case, G_v is piecewise linear with integer slopes.

Proposition 1. $\operatorname{div}(G_v) + K_\Gamma \geqslant 0$.

Here $\operatorname{div}(G_v)$ is the divisor of G_v in the sense of divisors and linear equivalence on metric graphs, and K_Γ is the canonical divisor of the weighted graph Γ. Proposition 1 can be phrased as saying "G_v is a section of the tropical canonical bundle." It is a result in potential theory on Berkovich curves, and can be seen as a consequence of the Poincaré–Lelong formula as applied to the metrized line bundle $(\Omega^1, ||\cdot||)$.

A combinatorial analysis of piecewise-linear functions on weighted metric graphs of genus g satisfying Proposition 1 yields the following consequence.

Corollary 2. *On any segment where G_v is linear, the slope of G_v is bounded by $2g-2$ in absolute value.*

(5) Bound the slopes of $F_v := -\log|f_v|$ in terms of the slopes of G_v.
(6) The number of zeros of f_v on U_v is equal to the sum of the incoming slopes of F_v on the edges adjacent to v.

We currently handle (5) with a tedious Newton polygon argument. The resulting bound also depends on p and on the length of the shortest edge in Γ adjacent to v. Assertion (6) is another fact from potential theory, and can be derived from the Poincaré–Lelong formula as applied to $(\mathcal{O}_C, |\cdot|)$. Completing steps (1)–(6) in general would give a complete proof of the uniform Manin–Mumford conjecture, at least for curves defined over a finitely ramified extension of $\mathbf{C}_p$ (see Remark 1 below).

Remark 1. The entire argument presented above is carried out over $\mathbf{C}_p$. However, the bounds obtained in (5) for the slopes of F_v do depend on the shortest edge length of Γ. If C admits a split semistable model over a subfield $K \subset \mathbf{C}_p$ with finite ramification index e over $\mathbf{Q}_p$, then all edges have length at least $1/e$.

Remark 2. Our approach is a generalization of the Chabauty–Coleman method; the main new ingredient is p-adic integration on wide open subdomains. In classical Chabauty–Coleman, one only performs p-adic integration on open discs; for this purpose it is not necessary to compare the two types of p-adic integration, as mentioned above. (But see Remark 3 below.) Another interesting feature of our method is that it uses p-adic integration to give *geometric* point-counting bounds, instead of rational point bounds.

Remark 3. A variant of the argument outlined above is used to prove Theorem 1. In this context, almost all ingredients are already contained in Stoll's paper [Sto13], which was a major inspiration. Indeed, for rational points bounds, it is only necessary to integrate on open discs and open annuli (these being special cases of wide open subdomains), where the theory simplifies considerably. Stoll was only able to carry out step (5) for annuli in the case of hyperelliptic curves. From the perspective of the uniform Mordell conjecture, the solution to (5) is the key new ingredient contained in [KRZB15]; integration on more general wide opens is not needed.

4. Current results and work in progress

Carrying out the program of §3, in [KRZB15, Theorem 1.3] we prove the following theorem. We keep the notation from the previous section.

Theorem 2. *Suppose that all edge lengths in* Γ *are at least* $1/e$*, and that for every vertex* v *of* Γ*, one has*

$$g > 2g(v) + \deg(v), \tag{†}$$

where $g(v)$ *is the weight of* v *and* $\deg(v)$ *is its valency. Then there is an explicit constant* $N_p(g, e)$ *such that*

$$\#\alpha^{-1}(J(\mathbf{C}_p)_{\mathrm{tors}}) \leqslant N_p(g, e)$$

for any choice of Abel–Jacobi map $\alpha\colon C \hookrightarrow J$*.*

The condition (†) is designed to guarantee that **(I)** and **(E)** can be satisfied (along with some trick to decrease the number of degrees of freedom needed to choose the differential ω_v).

We are currently working on proving the uniform Manin–Mumford conjecture for all Mumford curves over $\mathbf{C}_p$. This would be interesting in its own right, as it would give a uniform bound on torsion packets on Shimura curves of bounded genus (using the Čerednik–Drinfel'd uniformization); this can be seen as a Shimura curve analogue of the Coleman–Kaskel–Ribet conjecture, which is a theorem of Baker [Bak00]. We are also working on extending Buium's arguments [Bui96] to give a proof of uniform boundedness for curves over finite extensions of $\mathbf{Q}_p$ with compact-type reduction.

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Logarithmic structures, Artin fans, and tropical compactifications

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The theory of *Artin fans* has emerged in [AW13] and in [ACMW14] in the context of logarithmic Gromov-Witten theory (also see [ACM+15]), but can be traced back to the work of Olsson [Ols03] studying classifying stacks of logarithmic structures. It has already found applications to the realizability problem for tropical curves (see [Ran15b]) as well as to a version of the correspondence between algebraic and tropical curve counts (see [Ran15a]).

Throughout let k be a field that is endowed with the trivial absolute value. An *Artin fan* is a fine and saturated logarithmic algebraic stack, locally of finite type over k, that is logarithmically étale over k. Despite this seemingly abstract definition, Artin fans are essentially combinatorial objects and can be described as geometric stacks over the category of Kato fans (see [Kat94]), an incarnation of the geometry over the "field with one element".

Non-Archimedean geometry of Artin fans. Every fine and saturated logarithmic scheme X comes with a canonical strict morphism $X \to \mathcal{A}_X$ into an associated Artin fan $\mathcal{A}_X$. If X is a T-toric variety, then the Artin fan $\mathcal{A}_X$ is nothing but the quotient stack $[X/T]$ and, in general, the Artin fan of a fine and saturated logarithmic scheme X is étale locally constructed by these toric quotient stacks. Applying Thuillier's [Thu07] generic fiber functor $(.)^{\beth}$ we obtain the following result.

Theorem 1 ([Uli15b]). *On the level of underlying topological spaces the analytic map $X^{\beth} \to \mathcal{A}^{\beth}$ is equal to the tropicalization map* $\mathrm{trop}_X : X^{\beth} \to \overline{\Sigma}_X$ *constructed in* [Uli13].

If X is logarithmically smooth, by [Uli13, Theorem 1.2] the analytic map $X^{\beth} \to \mathcal{A}_X^{\beth}$ factors through Thuillier's [Thu07] deformation retraction $\mathbf{p}_X : X^{\beth} \to X^{\beth}$ onto the non-Archimedean skeleton $\mathfrak{S}(X)$ of X and the skeleton $\mathfrak{S}(X)$ is naturally homeomorphic to $\mathcal{A}_X$. Note, in particular, that this procedure canonically endows the skeleton of a logarithmically smooth scheme with the structure of a non-Archimedean analytic stack.

The case of toric varieties. For a T-toric variety $X = X(\Delta)$, defined by a rational polyhedral fan Δ in the vector space $N_{\mathbb{R}}$ spanned by the cocharacter lattice N of T, one can give a generalization of Theorem 1; it describes the Kajiwara-Payne tropicalization map $\mathrm{trop}_{\Delta} : X^{an} \to N_{\mathbb{R}}(\Delta)$ from X^{an} to a partial compactification $N_{\mathbb{R}}(\Delta)$ of $N_{\mathbb{R}}$ (see [Kaj08] and [Pay09]) as a non-Archimedean analytic stack quotient.

Theorem 2 ([Uli14]). *The Kajiwara-Payne tropicalization map* $\mathrm{trop}_X : X^{an} \to N_{\mathbb{R}}(\Delta)$ *is a non-Archimedean analytic stack quotient of X^{an} by the affinoid torus T° associated to T.*

One can view Theorem 2 from a different perspective: It is the non-Archimedean version of the fact that for a complex toric variety $X = X(\Delta)$ with big torus $T \simeq \mathbb{G}_m^n$ the logarithmic complex absolute value on $\mathbb{C}$ induces a homeomorphism

$$X(\mathbb{C})/(S^1)^n \simeq N_{\mathbb{R}}(\Delta) .$$

Theorem 2 therefore adds another layer to the analogy between the tropicalization map and the complex moment map. Contrary to the Archimedean case, however, we have to work with stack quotients and not with topological quotients, since the underlying space of the affinoid torus T° is not a group.

Tropical compactifications. Artin fans were already implicit in the work of Tevelev [Tev07] on tropical compactifications and give a reinterpretation of basic concepts of this theory in terms of logarithmic geometry. For example, the compactification $\overline{Y}$ of a very affine variety $Y \subseteq T$ in a T-toric variety X is a *tropical compactification* if and only if $\overline{Y}$ is logarithmically faithfully flat. Moreover, a tropical compactification is *schön* if and only if it is also logarithmically smooth.

In this language Tevelev's existence theorem for tropical compactifications can be generalized to the following toroidal version of the Raynaud-Gruson flattening theorem (see [RG71]).

Theorem 3 ([Uli15a])**.** *Let F be a coherent sheaf on an logarithmically smooth integral scheme X of finite type over k. Then there is a toroidal modification $X' \to X$ such that the strict transform F^{st} of F is logarithmically flat over k.*

Let Y be a closed subset of the locus X_0 in X where the logarithmic structure on X is trivial. Theorem 3 implies that after a toroidal modification $X' \to X$ of X the closure of $\overline{Y}$ in X' is logarithmically flat over k and thus the intersections of $\overline{Y}$ with the logarithmic strata of X' are either empty or have the expected dimensions. If X is a toric variety, this is precisely Tevelev's result.

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Reified valuations spaces and skeletons of Berkovich spaces

Antoine Ducros

(joint work with Amaury Thuillier)

Let k be any non-Archimedean field (possibly trivially valued). For every finite family $r = (r_1, \dots r_n)$ of non-negative real numbers, we denote by η_r the point of the Berkovich space $\mathbf{A}_k^{n,\mathrm{an}}$ defined by the semi-norm $\sum a_I T^I \mapsto \max |a_I| r^I$. The map $r \mapsto \eta_r$ induces a homeomorphism between $\mathbf{R}_{\geqslant 0}$ and a closed subset S_n of $\mathbf{A}_k^{n,\mathrm{an}}$, which is usually called the *standard skeleton* of $\mathbf{A}_k^{n,\mathrm{an}}$. In a joint work with Amaury Thuillier (currently in progress, and not yet available online) we prove – among other things – the following result.

Theorem. *Let $X = \mathscr{M}(A)$ be a k-affinoid space and let $f\colon X \to \mathbf{A}_k^{n,\mathrm{an}}$ be a morphism with zero-dimensional (hence finite) fibers. Set $\Sigma = f^{-1}(S_n)$. There exists a finite family $(g_1, \dots, g_m)$ of functions belonging to A such that the following hold.*

(1) The map $(|g_1|, \dots, |g_m|)$ from X to $\mathbf{R}_{\geqslant 0}^m$ induces a homeomorphism ι between Σ and a compact subset P of $\mathbf{R}_{\geqslant 0}^m$ which is piecewise monomial. *This means that P can be defined by a boolean combination of inequalities between monomial functions with non-negative integral exponents; i.e., functions of the kind $ax_1^{e_1} \dots x_m^{e_m}$ with $a \in \mathbf{R}_{\geqslant 0}$ and the e_i's in $\mathbf{Z}_{\geqslant 0}$.*

(2) For every analytic domain V of X and every analytic function h on V, the subset $\iota(V \cap \Sigma)$ of P is piecewise-monomial, and $|h| \circ \iota^{-1}\colon \iota(V \cap \Sigma) \to \mathbf{R}_{\geqslant 0}$ is piecewise monomial with rational exponents (of course, negative exponents can only occur for non-vanishing coordinates).

The case where the f_i's are invertible. The proof is then simpler. Indeed, this assumption on the f_i's implies that every point of Σ is Zariski-generic on X (more precisely, its Zariski-closure is an n-dimensional irreducible component of X), which allows through some standard tricks to algebraize the situation around every point of Σ. Once we have done it, the above Theorem can be proved by using either de Jong's alterations or the model theory of valued fields; we refer to the author's papers [3] (for the first method) and [1] (for the second one).

Back to the general case. When the f_i's are allowed to vanish, points of Σ can be non-generic (for instance, the origin of $\mathbf{A}^{n,\mathrm{an}}$ belongs to S_n). In order to understand what can happen around such a point, we have introduced a new space of valuations. Before describing it, let us mention that Temkin has already developped in [7] valuation theoretic tools for the local study of Berkovich spaces, which have been described recently by Kedlaya [6] in a more "Huber-like" spirit in terms of what he calls *reified valuations* (see the definition below). Though those tools have proved powerful, we cannot use them here, for the following reason: they give a very nice description of the germs of *analytic domains* around a given point of a Berkovich space, but do not say anything about phenomena related to the Zariski topology – which are crucial for our purposes. For example, Temkin's space associated to a rigid point x of a Berkovich space X consists of a single point, no matter how bad the singularity of X at x is (this encodes the fact that an analytic domain of X containing x is always a neighborhood of x or, otherwise said, that every analytic function invertible at x has constant norm around x).

The space $\widehat{X}$. We are now going to describe the valuation space we have introduced. First of all, let us recall's Kedlaya definition of a *reified* valuation: if A is a commutative ring, a reified valuation on A is a valuation $A \to \Gamma_0$ (where Γ is an arbitrary ordered abelian group, with multiplicative notation, and $\Gamma_0 = \Gamma \cup \{0\}$) together with an increasing embedding $\mathbf{R}_{>0} \hookrightarrow \Gamma$; there is a natural notion of equivalence of reified valuations. If $\xi\colon A \to \Gamma_0$ is a reified valuation, its kernel is a prime ideal $\mathfrak{p}$ of A. The fraction field of $A/\mathfrak{p}$ will be denoted by $\kappa(\xi)$, and the natural map $A \to \kappa(\xi)$ will be denoted by $f \mapsto f(\xi)$. The reified valuation ξ induces a reified valuation $\kappa(\xi) \to \Gamma_0$, which is denoted by $|\cdot|$. Hence one has $\xi(f) = |f(\xi)|$ for every $f \in A$ (and we shall use the latter notation); the subgroup $\mathbf{R}_{>0} \cdot |\kappa(\xi)^\times|$ of Γ only depends (as an ordered group equipped with an embedding of $\mathbf{R}_{>0}$, up to isomorphism) on the equivalence class of ξ.

Now let $X = \mathscr{M}(A)$ be a k-affinoid space. We denote by $\widehat{X}$ the set of equivalence classes of reified valuations ξ on A that are *bounded*, that is such that $|f(\xi)| \leqslant ||f||_\infty$ for every $f \in A$, where $||\cdot||_\infty$ is the spectral semi-norm. We endow it with the topology generated by the subsets described by inequalities of the form $|f| \bowtie \lambda|g|$ with f and g in A, with λ in $\mathbf{R}_{\geqslant 0}$, and with $\bowtie$ in $\{<, \leqslant, >, \geqslant\}$ (hence this is kind of a constructible topology: any inequality, strict or non-strict, defines an open subset). The topological space $\widehat{X}$ is compact and totally disconnected.

The Berkovich space X can be identified with the subset of $\widehat{X}$ consisting of points ξ such that $|\kappa(\xi)^\times| \subset \mathbf{R}_{>0}$ (but the inclusion $X \subset \widehat{X}$ is not a topological embedding, because the topology of a Berkovich space makes the difference between strict and non-strict inequalities). This subset is dense in $\widehat{X}$: this is essentially a rephrasing of Huber's theorem [4], Th. 4.1.

Kedlaya's space of *continuous* reified valuations can be identified with the subset of $\widehat{X}$ consisting of points ξ such that for every $\lambda \in |\kappa(\xi)^\times|$, there exists $\varepsilon \in \mathbf{R}_{>0}$ with $\varepsilon < \lambda$ (there are no infinitesimal elements in the value group).

A general element of $\widehat{X}$ can be thought of as mixing a "Kedlaya" part and an algebraic, not reified valuation on Spec A (giving rise to infinitesimal elements).

There is a continuous map $c\colon \widehat{X} \to X$, which is a retraction of the natural inclusion $X \subset \widehat{X}$. If $\xi \in \widehat{X}$ then $c(\xi) = f \mapsto \inf\{\lambda \in \mathbf{R}_{\geqslant 0}, |f(\xi)| \leqslant \lambda\}$.

Example. Let R be a positive real number and set $X = \mathscr{M}(k\{T/R\})$. Let Γ be an ordered group containing $\mathbf{R}_{>0}$, and let $r \in \Gamma_0$ such that $r \leqslant R$. The map $\sum a_i T^i \mapsto \max |a_i| r^i$ is then a bounded reified valuation on $k\{T/R\}$; it therefore defines a point η_r of $\widehat{X}$, and there are three possibilities. If $r \in \mathbf{R}_{\geqslant 0}$, then η_r is the usual Berkovich point. If r is infinitesimally closed to a positive real number ρ (i.e., $r \neq \rho$ but $\lambda^{-1} < r/\rho < \lambda$ for every real number $\lambda > 1$) then η_r is a Kedlaya point, and $c(\eta_r) = \eta_\rho$. If r is infinitesimal (i.e., $r > 0$ and $r < \varepsilon$ for every positive real number ε) then $c(\eta_r) = 0$, and η_r is not a Kedlaya point; this a "new" point, encoding the unique branch starting from the rigid point 0 (on a nodal curve one would have *two* such points over every singular point). It can also be described as the composition of the vanishing order at the origin with the absolute value of k.

Functoriality. Every morphism $Y \to X$ between k-affinoid spaces gives rise to a continuous map $\widehat{Y} \to \widehat{X}$. Assume that Y is an affinoid domain of X. By Gerritzen-Grauert theorem, Y can be described by a positive boolean combination $\mathscr{S}$ of inequalities of the form $|f| \leqslant \lambda |g$ where f and g are analytic functions on X and where $\lambda \geqslant 0$. The image of the map $\widehat{Y} \to \widehat{X}$ is then the compact open subset of $\widehat{X}$ defined by the system $\mathscr{S}$, but $\widehat{Y} \to \widehat{X}$ *is not injective in general.* The problem comes from the fact that the Zariski topology of Y is, in general, strictly finer than the restriction of the Zariski topology of X; this gives rise to (algebraic) non-trivial valuations on $\mathscr{O}_X(Y)$ whose restriction to $\mathscr{O}_X(X)$ is trivial, and then (by composition with suitable Kedlaya or even Berkovich valuations) to distinct points of $\widehat{Y}$ having the same image on $\widehat{X}$.

Link with our theorem. Our general strategy consists in studying the preimage Σ of the standard skeleton through its avatar $\widehat{\Sigma}$ on $\widehat{X}$, where our totally disconnected topology allows compactness arguments even while working on Zariski-open subsets. Every point ξ of $\widehat{\Sigma}$ are *Abhyankar*, in the following sense: the sum of the rational rank of $|\kappa(\xi)^\times|/|k^\times|$ and of the transcendence degree of $\widetilde{\kappa(\xi)}/\tilde{k}$ is equal to the dimension of $\overline{\{c(\xi)\}}^{\mathrm{Zar}}$. Hence most of our work is devoted to Abhyankar points of $\widehat{X}$. Usually, we handle them by making kind of a dévissage between their "Kedlaya" part, which is often not so difficult to deal with, and their "algebraic" part, which requires more work and often relies on algebraic geometry *à la* Grothendieck. For instance, we prove that if V is an affinoid domain of X and if ξ is an Abhyankar point of $\widehat{X}$ then ξ has at most one pre-image on $\widehat{V}$ (hence the aforementioned pathologies cannot occur), and if it has one, say η, then $|\kappa(\eta)^\times| = |\kappa(\xi)^\times|$ and $\widetilde{\kappa(\eta)} = \widetilde{\kappa(\xi)}$. Key ingredients for the "algebraic" part of this proof are: the good behavior of normality under a regular map (i.e., flat with geometrically regular fibers) between noetherian schemes; and a result by Raynaud describing, being given a flat morphism $\mathscr{Y} \to \mathscr{X}$ of noetherian schemes, which are the Cartier divisors on $\mathscr{Y}$ that come from some Cartier divisor on $\mathscr{X}$ (see [5], *Errata*, Prop. 21.4.9).

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