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Model Theory: groups, geometry, and combinatorics

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Abstract. This conference was about recent interactions of model theory with combinatorics, geometric group theory and the theory of valued fields, and the underlying pure model-theoretic developments. Its aim was to report on recent results in the area, and to foster communication between the different communities.

Mathematics Subject Classification (2010): 03C45, 03C98, 03C95, 05C75, 05E99, 20F65.

Introduction by the Organisers

The workshop *Model Theory: Groups, geometry, and combinatorics*, organised by Katrin Tent (Münster), Frank O. Wagner (Lyon) and Martin Ziegler (Freiburg) was well attended with over 50 participants of various backgrounds: Model theory, but also geometric group theory and combinatorics. There were a total of 20 talks: thirteen 50-minute talks and seven 40-minute talks, plus a tutorial (three lectures) by Zlil Sela on elimination of imaginaries in the free group, and a tutorial (two lectures) by Pierre Simon and Sergei Starchenko on applications of model theory in combinatorics.

Stability theory, created by Shelah and by now a classical subject, exhibits the structure of models of *stable* theories by looking at *forking*-independence and the interaction of types. For instance, in the case of the theory of algebraically closed fields forking independence coincides with algebraic independence and types are simply prime ideals in polynomial rings over subfields. While being a rich and successful method, various important and model-theoretically well-understood theories are unstable and *a priori* lie outside the scope of stability theory, such as pseudofinite fields, real closed fields or henselian valued fields.

Neostability began when the methods of stability theory were extended in two directions in order to include a broader range of applications:

(1) To the class of *simple* theories. This class contains the pseudofinite fields and, more generally, all bounded pseudo-algebraically closed fields.
(2) To the class of *NIP*-theories (theories without the independence property). It has long been known that o-minimal theories are NIP; other important examples are the p-adic numbers and algebraically closed valued fields. But only recently an interesting model theory has been developed in general.

These two extensions of the class of stable theories are orthogonal in the sense that a theory is stable if and only if it is simple and NIP. Nevertheless, there is a third class which contains the previous two, namely the NTP_2-theories; certain classes of henselian valued fields (ultraproducts of p-adic fields, for example) have been shown to be strictly NTP_2. The model theory of NTP_2-theories is still in its infancy, but it is expected that it will yield a unification of methods.

Even though forking independence is defined combinatorially, for a long time there has been little interaction between model theory and combinatorics. This has changed recently, due to two developments:

(1) The study of pseudofinite structures, i.e. the study of asymptotic limits of finite structures, via the pseudofinite counting measure. Hrushovski has realized that this yields a dimension theory, and that certain tools from stability theory, such as the stabilizer theorem, can be generalized to this context. This has been an important input in the classification of approximate groups by Breuillard, Green and Tao; there are a number of very recent results using the group configuration theorem from stability theory.
(2) The interaction with graph and hypergraph theory. In fact, combinatorial principles underlying NIP and stability can be used to prove certain cases of the Erdös-Hajnal conjecture or the Szemerédi regularity lemma.

Finally, a new and important example of a stable theory whose behaviour is quite different from the standard examples has arisen with Sela's proof of the stability of non-abelian free groups (as well as torsion-free hyperbolic groups). This gave rise to new interactions between model theory and geometric group theory, where a lot of basic questions are still open. For instance, there is currently no full description of forking independence. Almost all work is done in the standard models of the theory, and the saturated models are hardly understood. Moreover, a good comprehension of the definable sets will have a great impact on other questions about the free group.

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Workshop: Model Theory: groups, geometry, and combinatorics

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Abstracts

Categoricity of exponential maps of algebraic groups

Martin Bays
(joint work with Jonathan Kirby)

For G a complex algebraic group and LG its Lie algebra, consider the exponential map $\exp_G : LG(\mathbb{C}) \to G(\mathbb{C})$. We consider the structure $\mathbb{C}_{\exp_G}$ of the complex field expanded by $\exp_G$. For $G = \mathbb{G}_m$ the multiplicative group, Zilber's pseudo-exponentiation [1] provides a conjectural categorical infinitary axiomatisation of this structure. For G a linear group, $\exp_G$ is matrix exponentiation, and $\mathbb{C}_{\exp_G}$ is interpretable in $\mathbb{C}_{\exp_{\mathbb{G}_m}}$. In the work discussed, we consider the case of G a simple semiabelian variety. We develop a "generic" version of pseudo-exponentiation for such G.

Fix a simple semiabelian variety G, i.e. $G = \mathbb{G}_m$ or G an abelian variety. Let $\mathcal{O} := \mathrm{End}(G)$, acting on G and, via differentials, on LG, and hence on $LG \times G$. Suppose G and its endomorphisms are over a field k_0.

Definition 1. A Γ-*field* comprises a field $K \geq k_0$ and a divisible $\mathcal{O}$-submodule $\Gamma(K) \leq (LG \times G)(K)$, such that $K = k_0(\Gamma(K))$.

An embedding of Γ-fields $K \hookrightarrow F$ is an embedding $\Gamma(K) \hookrightarrow \Gamma(F)$ which induces a field embedding.

A Γ-field K is *full* if K is algebraically closed as a field and $\pi_1(\Gamma(K)) = LG(K)$ and $\pi_2(\Gamma(K)) = G(K)$, where π_i are the projection maps of the product.

Consider $\mathbb{C}$ and k_0 as Γ-fields by defining $\Gamma(\mathbb{C})$ to be the graph of $\exp_G$ and $\Gamma(k_0) := \{0\}$.

Our aim is to identify $\mathbb{C}$ as a Γ-field.

Using the topology on $\mathbb{C}$, we can define a natural notion of closure for Γ-subfields:

Definition 2. For $K \leq \mathbb{C}$ a Γ-subfield, let $\Gamma\mathrm{cl}^{\mathbb{C}}(K)$ be the Γ-subfield generated by isolated points of $V \cap \Gamma(\mathbb{C})^n$ for $V \subseteq LG^n \times G^n$ an irreducible subvariety defined over K.

Let $C_0 := \Gamma\mathrm{cl}^{\mathbb{C}}(k_0)$. Note that C_0 is countable. More generally, $\Gamma\mathrm{cl}^{\mathbb{C}}$ satisfies the *countable closure property* (CCP): the closure of a countable set is countable.

For our purposes, we need a notion of closure which works for abstract Γ-fields, with no mention of the topology:

Definition 3. Let F be a full Γ-field. A Γ-subfield $K \leq F$ is Γ-*closed in* F if for any tuple $\gamma \in \Gamma(F)^m \setminus \Gamma(K)^m$, $\delta(\gamma/K) := \mathrm{trd}(\gamma/K) - \dim(G)\,\mathrm{ld}_{\mathcal{O}}(\gamma/\Gamma(K)) > 0$.

For a Γ-subfield $K \leq F$, define $\Gamma\mathrm{cl}^F(K)$ to be the smallest Γ-closed intermediate Γ-subfield.

δ here is a Hrushovski predimension, and it follows by standard techniques that $\Gamma\mathrm{cl}^F$ is a pregeometry.

It follows from Ax's theorem of 1972 [2] that the two definitions of $\Gamma\mathrm{cl}^{\mathbb{C}}$ agree.

Definition 4. A Γ-field F is *generically* Γ*-closed* if F is full and for any finitely generated Γ-field extension $k \geq \Gamma\mathrm{cl}^F(k_0)$ within F and any irreducible variety $V \subseteq LG^n \times G^n$ defined over k, if $\dim(V) = n$, and no translate of V by an element of $\Gamma(k)$ is defined over $\Gamma\mathrm{cl}^F(k_0)$, and for every connected proper algebraic subgroup $H < G^n$ the image $V/(LH \times H)$ under the quotient map satisfies $\dim(V/(LH \times H)) > n - \dim(H)$ and $\dim(\pi_i(V/(LH \times H)) > 0$ for $i = 1, 2$, then there exists $\gamma \in V(F) \cap \Gamma(F)^n$ which is $\mathcal{O}$-linearly independent over $\Gamma(k)$.

Conjecture 5. *$\mathbb{C}$ is generically Γ-closed.*

Theorem 6. *There exists a unique Γ-field B^{gen} such that*

- *B^{gen} is generically Γ-closed;*
- *$\Gamma\mathrm{cl}^{B^{\mathrm{gen}}}(k_0) \cong \Gamma\mathrm{cl}^{\mathbb{C}}(k_0)$ as Γ-fields;*
- *$|B^{\mathrm{gen}}| = 2^{\aleph_0}$*
- *$\Gamma\mathrm{cl}^{B^{\mathrm{gen}}}$ satisfies (CCP).*

If $\mathbb{C}$ is generically Γ-closed, then $\mathbb{C} \cong B^{\mathrm{gen}}$.

Moreover, B^{gen} is quasiminimal*: for any definable (or even invariant) subset $X \subseteq B^{\mathrm{gen}}$, either X is countable or $B^{\mathrm{gen}} \setminus X$ is countable.*

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Erdös geometry and the group configuration

Emmanuel Breuillard

(joint work with H. Wang)

Erdös geometry studies finite sets within algebraic geometry. Landmark samples of this field include the *sum-product phenomenon* of Erdös and Szemerédi, or the Szemerédi-Trotter theorem about the number of incidences of a finite set of points and lines in the plane. Often the cardinality of the finite set tends to infinity, and asymptotic statements are sought after. Classical statements of Erdös geometry can then be reformulated in terms of pseudo-finite dimension in an appropriate saturated model, shedding a new light on these questions. We refer the reader to Hrushovski's text [6] for an extensive description of this bridge between combinatorics and model theory and a wealth of fascinating material for further investigation.

1. The *sum-product phenomenon* states that there there exists a constant $\epsilon_0 > 0$ such that

$$|AA| + |A + A| \geq |A|^{1+\epsilon_0}$$

for every finite subset A of real numbers of cardinality $|A|$. Here $A + A$ is the sum set and AA the product set of all pairs of elements from A.

A conjecture of Erdös and Szemerédi asserts that ϵ_0 can be taken as close to 2 as one may wish, provided $|A|$ is sufficiently large.

2. A non-commutative analogue, in fact a generalization, of the sum-product phenomenon was proven in [1] in the context of *approximate subgroups.*

Theorem 1 (Breuillard-Green-Tao)**.** *Given d, for every $\delta_0 > 0$ there is $\epsilon_0 > 0$ such that if $A \subset GL_d(\mathbb{C})$ is a finite set, then either*

$$|AA| > |A|^{1+\epsilon_0},$$

or A is contained in at most $|A|^{\delta_0}$ left cosets of a nilpotent subgroup of $GL_d(\mathbb{C})$.

3. Following [6] these statements admit a pseudofinite formulation, which has the aesthetical advantage of removing the epsilons. For example fix a sequence of finite sets B_i (in $\mathbb{R}$ say in case **1.**, or in $GL_d(\mathbb{C})$ in case **2.**) with $|B_i| \to +\infty$ and set $\mathbb{B}$ to be the ultraproduct of the B_i's. For any other family of finite subsets A_i, we may form their ultraproduct $\mathbb{A}$ and define the *coarse pseudofinite dimension* $\delta(\mathbb{A})$ of $\mathbb{A}$ as the standard part

$$\delta(\mathbb{A}) := \mathrm{st}\Big(\frac{\log|\mathbb{B}|}{\log|\mathbb{A}|}\Big) = \lim_{\mathcal{U}} \frac{\log|B_i|}{\log|A_i|}.$$

Now the sum-product phenomenon of Erdös-Szemerédi can be reformulated simply as $\max\{\delta(\mathbb{A}\mathbb{A}), \delta(\mathbb{A}+\mathbb{A})\} > \delta(\mathbb{A})$, while their conjecture is equivalent to $\max\{\delta(\mathbb{A}\mathbb{A}), \delta(\mathbb{A}+\mathbb{A})\} = 2\delta(\mathbb{A})$. Similarly, the Breuillard-Green-Tao theorem above says that $\delta(\mathbb{A}\mathbb{A}) > \delta(\mathbb{A})$, unless there is a subset $\mathbb{X}$ of zero coarse dimension, and a nilpotent subgroup N, such that $\mathbb{A} \subset \mathbb{X}N$.

4. Another way to generalize the sum-product phenomenon is to pick a polynomial P in several indeterminates and ask for the number of values it can take when the variables belong to A. We expect that this number is much greater than $|A|$ unless the polynomial has a special form. For example take $P(a,b,c,d) = ab + c + d$ to recover the sum-product theorem. Bukh and Tsimerman, then Tao considered this problem for finite fields in the regime when A is large. In characteristic zero a theorem of Elekes and Ronyai says that a 2-variate polynomial $P(x,y)$ must expand (namely $|P(A,A)| > |A|^{1+\epsilon_0}$), unless it has the form $Q(f(x)+g(y))$ or the form $Q(f(x)g(y))$ for certain rational functions Q, f, g. A stronger result was obtained afterwards by Elekes and Szabó:

Theorem 2 (Elekes-Szabó [3])**.** *Let A, B, C be irreducible k-dimensional complex algebraic varieties and $F \subset A \times B \times C$ an irreducible subvariety whose projection to each pair $A \times B$, $A \times C$ and $B \times C$ is dominant and generically finite. Suppose we are given three finite subsets X, Y, Z of large size $n \geq n_0(\deg F, k)$ in A, B and C respectively. Assume further that they are in general position (that is no proper subvariety of bounded degree contains more than a bounded number of points). Suppose that*

$$|X \times Y \times Z \cap F| > n^{2-\epsilon_0}, \tag{1}$$

then F is related *to the graph of multiplication on an algebraic group $\mathbb{G}$.*

Related here means that there are multifunctions α, β, γ from $\mathbb{G}$ to A,B and C respectively, such that F is a component of the image of the graph $\{(x,y,z) \in \mathbb{G}; xy = z\}$. Note that $\dim \mathbb{G} = k$. The constant ϵ_0 here depends only on the dimension k, and not on F. The case of curves is particularly interesting, as for example, the theorem applies to the case when $A = B = C = \mathcal{C}$ and F is the graph $\{(x,y,z) \in \mathbb{C}^3; z = P(x,y)\}$ and allows to recover the Elekes-Ronyai result.

Another example is when F is the collinearity relation on a planar curve $\mathcal{C}$ and $A = B = C = \mathcal{C}$. In this case, one can use the above theorem to establish that there cannot be more than $n^{2-\epsilon_0}$ collinear triples from a given subset of $\mathcal{C}$ of size n, unless the curve $\mathcal{C}$ itself is a line or an elliptic curve ([4, 8]).

5. Here is our contribution, which complements the Elekes-Szabó theorem:

Theorem 3 (B.+Wang [2])**.** *In the conclusion of the Elekes-Szabó theorem, the group* $\mathbb{G}$ *must be abelian. Furthermore, we can take* $\epsilon_0 = \frac{1}{Ck}$ *for some absolute constant* $C > 0$.

The proof relies on the observation (due to Balog-Szemerédi and Gowers) that if a large subset A of a group has many triples a, b, c with $ab = c$, then there is a large subset of it, which does not grow much under multiplication. This then allows to apply Theorem 1, in the case when $\mathbb{G}$ is a linear algebraic group. In general one needs to generalize Theorem 1 appropriately. This forces $\mathbb{G}$ to be nilpotent. Finally the general position assumption on X, Y, Z forces $\mathbb{G}$ to be abelian. On the other hand genuine abelian examples exists: for example take $\mathbb{G}$ to be a simple abelian variety and X a large arithmetic progression. The explicit bound on ϵ_0 relies on the use of a recent incidence theorem [5]. For curves (i.e. $k = 1$) one can take $\epsilon_0 = \frac{1}{6}$, see [9] and [8].

6. We sketch the proof of the Elekes-Szabó theorem. It has three steps.

Step 1. One considers the fiber product

$$\mathcal{F} := F \times_B F = \{(a, a', c, c') \in \mathcal{A} \times \mathcal{C}; \exists b \in B \text{ s.t. } (a,b,c) \in F \text{ and } (a', b, c') \in F\},$$

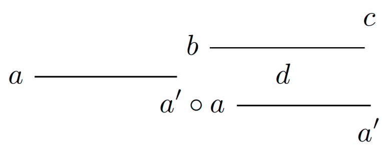
where $\mathcal{A} = A \times A$ and $\mathcal{C} = C \times C$. Then (1) together with Cauchy-Schwarz implies

$$|\mathcal{F} \cap X \times X \times Z \times Z| \gg n^{3-2\epsilon_0}. \tag{2}$$

The projection of $\mathcal{F}$ on each triple is dominant and generically finite, and the fibers $\mathcal{F}^{\mathbf{a}}$ and $\mathcal{F}_{\mathbf{c}}$ for $\mathbf{a} := (a, a')$ and $\mathbf{c} = (c, c')$ are k-dimensional subvarieties of $\mathcal{C}$ and $\mathcal{A}$ respectively (i.e. multifunctions $C \overset{a}{\to} B \overset{a'}{\to} C$ and $A \overset{c}{\to} B \overset{c'}{\to} A$). The left hand side in (2) is the number of *incidences* $\mathbf{a} \in \mathcal{F}_{\mathbf{c}}$.

Step 2. One now seeks to apply an *incidence theorem* à la Szemerédi-Trotter to the above situation. A recent result of Fox et al. [5] gives such a bound: given n points in $\mathbb{R}^d$ and n subvarieties of bounded degree, *provided any two distinct points lie on at most boundedly many common subvarieties*, the number of incidences is at most $n^{\frac{3}{2}-\eta(d)}$ for some explicit $\eta(d) > 0$. Here $d = 4k$.

Step 3. The last step consists in checking the assumption highlighted in italics in Step 2. Thanks to the general position hypothesis on X, Y, Z, this will hold unless we are in the following *group configuration*, where $b = a(c)$ and $d = a'(b)$. Hrushovski's group configuration theorem (see [7, ch. 5]) ends the proof.



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Groups geometrically representable in profinite groups

Zoé Chatzidakis
(joint work with Özlem Beyarslan)

We study the automorphism group of the algebraic closure of a substructure A of a pseudo-finite field F, or more generally, of a bounded perfect PAC field F. This paper answers some of the questions of [1], and in particular that any finite group which is geometrically represented in a pseudo-finite field must be abelian.

Notation. If F is a field, then F^s denote the separable closure of F, and $G(F)$ the absolute Galois group of F, $\mathrm{Gal}(F^s/F)$. If p is a prime different from the characteristic of F, then ζ_p and μ_{p^∞} denote a primitive p-th root of unity and the group of all primitive p^n-th roots of unity. We say that p divide $\#G(F)$ if p divides the order of some finite quotient of $G(F)$.

Definition 1. Let T be a complete theory. We say that a finite group G is geometrically representable in T if there is a model M of T, and a subset A of M containing an elementary substructure of M, and such that G is a quotient of the profinite group $\mathrm{Aut}(\mathrm{acl}(A)/\mathrm{dcl}(A))$.

The proof of the main result starts with two easy observations, both of them classical:

Lemma 2. *Let F be a field, $A = \mathrm{dcl}(A) \subseteq F$. Then $\mathrm{acl}(A) = A^{sep} \cap F$, and is a Galois extension of A. We have $\mathrm{Aut}(\mathrm{acl}(A)/A) = \mathrm{Gal}(\mathrm{acl}(A)/A)$.*

Lemma 3. *Let $A = \mathrm{dcl}(A)$ be contained in a pseudo-finite field F and contain an elementary substructure F_0 of F. Then $G(A) \simeq \mathrm{Gal}(\mathrm{acl}(A)/A) \times G(F_0)$.*

The main tool of the proof is the a result of Koenigsmann:

Theorem 4. (Thm 3.3 in [4]) *Let K be a field with $G(K) \simeq G_1 \times G_2$. If a prime p divides $(\#G_1, \#G_2)$, then there is a non-trivial Henselian valuation v on K, $\mathrm{char}(K) \neq p$, and $\mu_{p^\infty} \subset K(\zeta_p)$. Furthermore, if Kv denotes the residue field of v and $\pi : G(K) \to G(Kv)$ the canonical epimorphism, then $G(K)$ is torsion-free and $(\#\pi(G_1), \#\pi(G_2)) = 1$.*

Theorem 5. *Let F be a pseudo-finite field, and $A = \mathrm{dcl}(A)$ containing an elementary substructure F_0 of F, $G = \mathrm{Aut}(\mathrm{acl}(A)/A)$. If a prime p divides $\#G$, then $\mathrm{char}(F_0) \neq p$, F_0 contains all primitive p^n-th roots of 1 and G is abelian.*

Sketch of proof. By Lemma 3, we know that $G(A) \simeq G \times G(F_0)$. By Theorem 4, A has a non-trivial Henselian valuation v, and its characteristic is not divisible by any prime dividing $\#G$. As F_0 is relatively algebraically closed in A, the restriction of v to F_0 is Henselian. But a PAC field with a non-trivial Henselian valuation must be separably closed (by Cor 11.5.6 in [3]), and this implies that v is trivial on F_0, and that $\mathrm{Gal}(A^s/AF_0^s)$ is contained in the inertia subgroup of v. As $G \simeq \mathrm{Gal}(A^s/AF_0^s)$, and the characteristic does not divide $\#G$, it follows that $\mathrm{Gal}(A^s/AF_0^s)$ is abelian (see Theorem 5.3.3 and §5.3 in [2]), and therefore so is G. Hence $G(A)$ is abelian. Let p divide $\#G$. Then some element γ of the valuation group of A is not divisible by p, and therefore if $v(a) = \gamma$, then a does not have a p-th root in A. As $G(A)$ is abelian, the field $F(a^{1/p})$ is a Galois extension of A, and therefore A must contain ζ_p. This implies that $\mu_{p^\infty} \subset F$ and finishes the proof.

Remark 6. The proof generalises with almost no change to the case of perfect bounded PAC fields, i.e., PAC fields with only finitely many separably algebraic extension of degree n for every n. On the other hand, Beyarslan and Hrushovski [1] have shown the converse: the theory of a perfect PAC field of characteristic $\neq p$ and containing μ_{p^∞} represents every abelian p-group. The class of groups geometrically representable in the theory of a PAC field is closed under direct products.

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A census of homogeneous finite dimensional permutation structures After Sam Braunfeld

Gregory Cherlin

The talk is based on recent work by Sam Braunfeld, a graduate student at Rutgers.

Cameron [1] classified the homogeneous structures with two linear orders, which may be called *permutation structures*, and raised the question of a similar classification for structures with an arbitrary finite number of linear orders, which we will refer to as higher dimensional permutation structures.

One asks first what the natural examples are. This is already unclear.

In Cameron's case one has the following.

- The trivial permutation, on one point.
- Nontrivial primitive permutations: a pair of orders of type $\mathbb{Q}$, with the 2nd either equal to the first up to reversal, or independent from it (generic, or random).
- Imprimitive examples derived from the lexicographic order of $\mathbb{Q}^2$.

The lexicographic order is most naturally represented as $(\mathbb{Q}^2, <, E)$ with E a convex equivalence relation (i.e., its classes are convex for the order $<$). However a change of language presents this as a permutation $(\mathbb{Q}^2, <_1, <_2)$ with $<_1$ the given order $<$ and $<_2$ obtained from $<_1$ by reversing it either on the equivalence classes, or between the equivalence classes.

Evidently $\mathbb{Q}^k$ can also be represented as a homogeneous higher dimensional permutation structure; the natural language for this involves the lexicographic order $<$ together with a nested sequence of equivalence relations $E_1 \subseteq \cdots \subseteq E_{k-1}$. However this may also be presented in a language with n linear orders, for $k \leq 2^{n-1}$.

This may possibly exhaust the obvious examples of homogeneous finite dimensional permutation structures. However one may add a product construction illustrated by the following.

Example 1. Let $\mathbb{Q}^2$ be viewed as a structure equipped with the equivalence relations

$$E_i(a, b) \iff a_i = b_i$$

and the quasi-orders $<_i^*$ defined by

$$a <_i^* b \iff a_i < b_i$$

We can represent $\mathbb{Q}^2$ as a permutation structure by replacing $E_1, E_2, <_1^*, <_2^*$ by the four lexicographic orders $<_1, <_2, <_3, <_4$ derivable from $<_1, <_2$ and their reversals.

This raises the general question as to what the lattice of $\emptyset$-definable equivalence relations may look like in a homogeneous finite dimensional permutation structure.

Sam Braunfeld has proved the following [2].

Theorem 2. *1. Let Λ be a finite distributive lattice. Then there is a homogeneous finite dimensional permutation structure whose lattice of ∅-definable equivalence relations is isomorphic with Λ.*

2. Let Γ be a homogeneous finite dimensional permutation structure. Let Λ be the lattice of ∅-definable equivalence relations in Γ. Suppose that the reduct of Γ to the language Λ is homogeneous. Then Λ is distributive.

This gives us a "census" of homogeneous finite dimensional permutation structures which may be supposed as a working hypothesis to be representative of the general case. This suggests various concrete conjectures.

Conjecture 3. *1. Any primitive homogeneous finite dimensional permutation structure is generic, modulo a set of relations of the form*

$$<_i=<_j^{\pm}$$

requiring some pairs of orders to coincide up to reversal.

2. The minimal forbidden substructures of a homogeneous finite dimensional permutation structure are of order at most 3. Furthermore, any such constraint of order 3 will either be part of the requirement that a particular definable relation be an equivalence relation, or will impose a convexity condition on the classes of some definable equivalence relation, with respect to one of the orders.

It is less clear how constraints of order 2 fit into this picture, but Braunfeld has proved the following.

Proposition 4. *Let Γ be a finite dimensional permutation structure whose minimal forbidden substructures all have order 2. Then Γ is primitive and is of the expected form (part (1) of the conjecture).*

We add a few words about the proof of the main theorem. This uses the method of amalgamation classes (Fraïssé).

Realization of a given distributive lattice Λ.

1. A structure equipped with equivalence relations E_λ ($\lambda \in \Lambda$) may be viewed as a generalized ultrametric space with values in Λ: the triangle inequality becomes

$$d(a,b) \leq d(a,c) \vee d(b,c)$$

The usual amalgamation procedure for metric spaces can be seen to work for Λ-ultrametric spaces if the lattice is distributive.

This gives a template structure, a realization of the lattice Λ by a homogeneous structure in the language of equivalence relations E_λ ($\lambda \in \Lambda$).

2. The structure may be expanded generically by linear orders such that for some generating set for Λ, each of the corresponding equivalence relations is convex for one of the corresponding orders. This is the main step. It goes by a direct construction if the minimal element 0, representing the relation of equality, is meet irreducible; otherwise an indirect approach must be taken.

3. Once sufficiently many equivalence relations have been made convex, they may be replaced by definable orders in the language, and the structure is then presented as a finite dimensional permutation structure.

Proof of distributivity

1. Use the orderings to prove an *infinite index property:* each E-class splits into infinitely many F-classes for $F < E$. This may be viewed as a kind of Neumann's Lemma (no equivalence class for one relation is covered by finitely many proper subclasses for other relations).

2. For each instance of the distributive law, one can build an amalgamation diagram whose factors are furnished by Neumann's Lemma, and whose completion requires that instance of the distributive law.

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Computing certain invariants of topological spaces of dimension three

Javier de la Nuez González

(joint work with Chloé Perin, Rizos Sklinos)

1. Motivating example

The rank R^∞ was introduced by Shelah [1]. Its value for a certain definable set X can be defined as either the foundational rank for the forking preorder on the class of non-empty definable sets contained in X or the exceptional value ∞ when said pre-order is not well-founded (see [2]). Recall that in a stable theory with elimination of imaginaries a definable set $\emptyset \neq X$ is said to fork over another set Y, which contains it, whenever for some $k \in \mathbb{N}$, a family of k-inconsistent conjugates of X over some defining parameters of Y can be found.

This notion is mainly relevant for superstable theories, which are precisely those for which R^∞ has only ordinal values. This is not the case for the common theory T_{fg} of nonabelian free groups. A fairly elementary proof can be found in [3] and this was known long before Sela's proof that T_{fg} is stable (see [4]). In the work in progress presented in the talk, lower bounds for the value of R^∞ on particular definable sets are given. Note that despite some recent progress (see [5]), the question of which sets have ordinal rank is still open.

The starting point is the following example. Let $\mathbb{F}$ be the free group in two generators, α, β, and consider the formula:

$$\phi(x, y, [\alpha, \beta]) \equiv [x, y] = [\alpha, \beta]$$

Now, it is a well known fact ([6], Theorem 3.14) that the automorphisms $\sigma : \alpha \mapsto \alpha$, $\beta \mapsto \alpha\beta$ and $\tau : \alpha \mapsto \alpha\beta$, $\beta \mapsto \beta$ of $\mathbb{F}$, which fix $[\alpha, \beta]$, generate a free subgroup of $Aut(\mathbb{F})$. Let $S = \langle \sigma \rangle$. For any m, the set $X_n = S \circ (\tau \circ S)^n \cdot (\alpha, \beta)$ is invariant under any element of S and contains infinitely many conjugates by elements of S of the orbit $Y_n = (\tau \circ S)^n \cdot (\alpha, \beta)$. Said fact implies that those conjugates are pairwise disjoint. Using that $S \cdot (\alpha, \beta)$ is definable, one can prove that X_n and Y_n are definable. It follows that $R^\infty(\phi) \geq \omega$.

One can generalize this by showing that $R^\infty(\phi_g) \geq \omega^{2g-1}$, for

$$\phi_g(\bar{x}) \equiv [x_1, y_1][x_2, y_2] \cdots [x_g, y_g] = [\alpha_1, \alpha_2] \cdots [\alpha_{2g-1}, \alpha_{2g}]$$

defined over the free group $\mathbb{F}_{2g}$ with base $\alpha_1, \cdots \alpha_{2g}$. The interpretation of F_{2g} as the free group of an orientable surface Σ_g of genus g with one boundary component (which corresponds to c_g) and that of the automorphisms of $\mathbb{F}_g$ fixing c_g as elements of the modular group of Σ plays a fundamental role in generalizing the idea used in the previous example. Roughly speaking, we can regard $Mod(\Sigma_j)$ as a subgroup of $Mod(\Sigma_{j+1})$ via the obvious embedding of Σ_j into Σ_{j+1}. Definability concerns aside, the key is to show that for $1 \leq j \leq g-1$, there is an element $\tau \in Mod(\Sigma_{j+1})$ such that $\langle Mod(\Sigma_j), \tau \rangle \cong Mod(\Sigma_j) * \langle \tau \rangle$. This involves a careful analysis of the action of $Mod(\Sigma_{j+1})$ onto the complex of curves of Σ_{j+1}. This is a hyperbolic simplicial complex with vertices in correspondence to homotopy classes of essential simple closed curves in Σ_j and edges in correspondence to those pairs of classes which can be realized disjointly. Complexes of curves of surfaces have been the object of intense study on the part of geometric group theory and are very closely linked to the Nielsen-Thurston classification of homeomorphisms on a surface (see [7] for a survey). Our hope is to be able to generalize this method to provide lower bounds for the Shelah rank of all those varieties associated to towers (see [8] for a definition) in the free group. Apart from the case of centralizers, which have rank 1, no upper bounds for R^∞ in T_{fg} are available yet.

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Dense-codense predicates in NTP1 theories

Jan Dobrowolski
(joint work with Hyongjoon Kim)

The goal of the talk is an exposition of the results of [5]. The main result states that the NTP_1 property is preserved when passing to the theory of lovely pairs or to the theory of independent pairs of a geometric theory (i.e., a theory which eliminates the symbol $\exists^\infty$ and in which the algebraic closure has the exchange principle).

Definition 1. A formula $\phi(x; y)$ has TP_1 (in a fixed theory T) if there is a collection of tuples $(a_\eta)_{\eta\in\omega^\omega}$ such that:
(1) For all $\eta \in \omega^\omega$, the set $\{\phi(x; a_{\eta|_n}) : n < \omega\}$ is consistent,
(2) If $\eta, \nu \in \omega^{<\omega}$ are incomparable (with respect to inclusion), then $\phi(x; a_\eta) \wedge \phi(x; a_\nu)$ is inconsistent.
A theory T has TP_1 if some formula does. Otherwise, we say that T has NTP_1.

After a brief overview of the history of results on unary expansions (for example in stable, simple and NIP contexts) we focus on the context of geometric theories. We are concerned with two types of unary expansions, namely lovely pairs and independent pairs in the sense of [2] and [3].

Definition 2. Given a geometric complete theory T in a language $\mathcal{L}$ and a model $M \vDash T$, add a new unary predicate symbol H to form an extended language $\mathcal{L}_H := \mathcal{L}\cup\{H\}$. Let $(M, H(M))$ denote an expansion of M to $\mathcal{L}_H$, where $H(M) := \{x \in M \mid H(x)\}$.

1. $(M, H(M))$ is called a *dense/co-dense* expansion if, for any non-algebraic $\mathcal{L}$-type $p(x) \in S_1(A)$ where $A \subseteq M$ has a finite dimension, $p(x)$ has realizations both in $H(M)$ and in $M \setminus \mathrm{acl}_T(A \cup H(M))$.
2. A dense/co-dense expansion $(M, H(M))$ is called a *lovely pair* if $H(M)$ is an elementary substructure of M.
3. A dense/co-dense expansion $(M, H(M))$ is called an *H-structure* if $H(M)$ is an algebraically independent subset of M.

It was proved in [2] and [3] that, for any geometric theory T, all of its lovely pairs (resp. H-structures) are elementarily equivalent (and it is well-known that they form nonempty classes). Hence, one can define:

Definition 3. T_P and T^{ind} denote the common complete theories of the lovely pairs and H-structures, respectively, associated with a given geometric theory T. By T^*, we mean either T_P or T^{ind}.

We present some basic tools used in investigation of T^*. The following fundamental theorem (proved in [2],[3]) is of a particular importance.

Fact 4. *For any H-independent tuples a, b,*

$$\mathrm{tp}_H(a) = \mathrm{tp}_H(b) \;\Leftrightarrow\; \mathrm{tp}_T(aH(a)) = \mathrm{tp}_T(bH(b)).$$

We mention some tools that are used in our proof, in particular, (a local version of) the theorem from [4] stating the TP_1 property is always witnessed by a formula in a single variable.

In [1], it was proved that if a geometric theory T has the NTP_2 property then so does T^*. Our main result states the same holds for NTP_1.

Theorem 5. *If T has NTP_1, then so does T^*.*

Finally, we present a certain construction of a class of examples of geometric NTP_1 non-simple theories. Such theories can be obtained from theories of Fraisse limits which are simple, by applying first the 'imaginary cover' and then the 'pfc' construction described in Subsection 6.3 of [4].

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A new notion of minimality in valued fields

Immanuel Halupczok
(joint work with Raf Cluckers, Silvain Rideau)

In this talk, I presented a new analogue of o-minimality in valued fields. More precisely, suppose we are given a valued field K in some language L expanding the language of valued fields. We aim for a simple set of axioms about (K, L) such that:

(1) The axioms hold in any (K, L) where model theory is known to behave well. In particular, we would like them to hold in the following settings:
- The field K can be any henselian valued field of characteristic 0 (and arbitrary residue characteristic).
- The language L can be the pure valued field language, or an expansion by analytic functions (e.g. in the very general sense of [1]), and an additional arbitrary expansion on RV.

RV: RV is the quotient $K/{\sim}$, where $a \sim b \iff v(a-b) > v(a)$; we write $\mathrm{rv}\colon K \to \mathrm{RV}$ for the canonical map. If there exists an angular component map $\mathrm{ac}\colon K \to k$, then $\mathrm{rv}(a) = \mathrm{rv}(b) \iff (v(a) = v(b) \wedge \mathrm{ac}(a) = \mathrm{ac}(b))$. In particular, if the language contains ac, then we have a definable bijection $\mathrm{RV} \setminus \{0\} \to \Gamma \times k^{\times}$, where Γ is the value group and k is the residue field.

(2) The axioms should imply a good understanding of definable sets and functions. In particular, we would like to obtain the following:

- a notion of dimension
- some form of cell decomposition
- that definable functions are almost everywhere continuous and even differentiable
- the Jacobian property (see e.g. [1])

The Jacobian property: Since valued fields are totally disconnected, being almost everywhere differentiable is not strong enough for many applications. The Jacobian property provides an appropriate strengthening, stating that even on rather big sets, definable functions have good approximations by linear functions.

Various notions of minimality in valued fields already exist, but most of them only work in rather specific contexts, and not all of them imply everything we want.

- The notions of p-minimality [4] and (to my knowledge) t-minimality [6] only work in $\mathbb{Q}_p$ and finite extensions.
- The notions of c-minimality [3] and v-minimality [5] only work in algebraically closed valued fields. (With a trick, v-minimality can also be exploited in a more general class of valued fields, but it still does not cover the full generality we would like to cover.)
- The notion of b-minimality [2] comes very close to our desires, but that set of axioms is rather complicated and does not feel very natural. (In particular, the Jacobian property has to be imposed as a separate axiom.)

The project of finding a notion of minimality with all desired properties is still very much work in progress. In the talk, I presented a preliminary notion which does imply all of (2), but which only works under the additional assumptions that K has equi-characteristic 0 and that the value group is elementarily equivalent to $\mathbb{Z}$.

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Almost centralizers as a useful tool in Model theory

Nadja Hempel

Given an arbitrary subgroup of a definable group which is abelian, nilpotent or solvable, one might ask if there is a definable abelian, nilpotent or solvable subgroup containing the given subgroup. We call these definable envelopes. They always exists in groups definable in stable theories. Passing to simple theories, one does not obtain definable envelopes in this strong sense. However, in groups with a simple theories, due to Milliet, one can find a definable finite-by-abelian group around any abelian subgroups (see [4]) and definable envelopes for nilpotent and solvable subgroups "up to finite index", i. e. finitely many translate of the definable subgroup cover the given subgroup (see [5]). Using those, Palacín and Wagner showed in [6] that the Fitting subgroup, i. e. the group generated by all normal nilpotent subgroups, of any group type-definable in a simple theory is nilpotent and definable.

A crucial property of groups with a simple theory is that they satisfy a descending chain condition up to finite index (DCC up to finite index) for uniformly definable subgroups. We are interested in studying groups in which the DCC up to finite index holds merely for centralizers:

Definition 1. A group G is called $\widetilde{\mathfrak{M}}_c$-*group* if for any two definable subgroups H and N, such that N is normalized by H, there exists natural numbers n_{HN} and d_{HN} such that any sequence of centralizers

$$C_{H/N}(g_0N) \geq \ldots \geq C_{H/N}(g_0N, \ldots, g_mN) \geq \ldots$$

each having index at least d_{HN} in its predecessor has length at most n_{HN}.

A useful tool and object in this context are almost centralizers and the relation "being almost contained":

Definition 2. Let G be a group and A be a parameter set. Let H, K and N be three A-invariant subgroups of G such that N is normalized by H.

- We say that H is *almost contained* in K, denoted by $H \lesssim K$, if the index $[H : H \cap K]$ is finite.
- The *almost-centralizer* of H in K modulo N is defined as follows:

$$\widetilde{C}_K(H/N) = \{g \in N_K(N) : H \sim C_H(g/N)\}$$

Whereas the proof of the abelian and solvable case is easily adaptable to $\widetilde{\mathfrak{M}}_c$-groups, the proof for nilpotent envelopes uses machinery from simple theories. I isolated the two missing ingredients of the proof of Milliet for nilpotent envelopes and gave a purely group theoretical approach. The results are summarized below:

Theorem 3 (H. [2]). *Let A be a parameter set and G be a group. For H and K two A-ind-definable subgroups (a union of a directed system of A-type-definable subgroups) of G, we obtain the following:*

- *(*symmetry*) If N is a subgroup of G which is the union of some A-definable sets and normalized by H and K, then*

$$H \lesssim \widetilde{C}_G(K/N) \quad \text{if and only if} \quad K \lesssim \widetilde{C}_G(H/N).$$

- *(*generalized Neumann theorem*) Let H and K be two A-definable subgroups of G such that H normalizes K. Suppose that*

$$K \leq \widetilde{C}_G(H) \quad \text{and} \quad H \leq \widetilde{C}_G(K).$$

Then we have that $[K, H]$ is finite.

These enabled us to obtain definable envelops for $\widetilde{\mathfrak{M}}_c$-groups as well as generalize the result of Palacín and Wagner on the Fitting subgroup.

Theorem 4 (H. [2]). *Let G be an $\widetilde{\mathfrak{M}}_c$-group and H be a subgroup of G. Then the following holds:*

(1) *If H is abelian, then there exists a definable finite-by-abelian subgroup of G which contains H.*
(2) *If H is a nilpotent subgroup of class n, then there exists a definable nilpotent subgroup N of G of class at most $2n$ which almost contains H.*
(3) *If H is a solvable subgroup of class n, then there exists a definable solvable subgroup S of G of class at most $2n$ which almost contains H.*

Theorem 5 (H. [2]). *The Fitting subgroup of any $\widetilde{\mathfrak{M}}_c$-group is nilpotent and definable.*

For groups with a dependent theory, Shelah showed in [7] that any abelian subgroup is contained in a definable abelian subgroup (in a saturated extension) and Aldama generalized this result to nilpotent and normal solvable subgroups [1].

Together with Onshuus, we generalized the results on definable envelops for simple and dependent groups to groups with an NTP_2 theory. Whereas in the proof of the abelian case we follow some of the ideas already present in the proof of de Aldama, in the nilpotent case, we use additionally symmetry of the almost centralizer and the generalized Neumann theorem presented as Theorem 3 above.

Theorem 6 (H., Onshuus [3]). *Let G be a group definable in an NTP_2 theory, H be a subgroup of G and suppose that G is $|H|^+$-saturated. Then the following holds:*

(1) *If H is abelian, then there exists a definable finite-by-abelian subgroup of G which contains H. Furthermore, if H was normal in G, the definable finite-by-abelian subgroup can be chosen to be normal in G as well.*
(2) *If H is a solvable subgroup of class n which is normal in G, then there exists a definable normal solvable subgroup of G of class at most $2n$, for which finitely many translates cover H.*
(3) *If H is a nilpotent subgroup of class n, then there exists a definable nilpotent subgroup N of G of class at most $2n$, for which finitely many translates cover H. Moreover, if H is normal in G, the group N can be chosen to be normal in G as well.*

Another application of the results on almost centralizers is a generalization of the following theorem of Neumann on bounded almost abelian groups.

Definition 7. A group H is *almost abelian* if the centralizer of any of its element has finite index in H. H is *bounded* if there is a natural number d such that for all h in H, the index $[H : C_H(h)]$ is at most d.

Fact 8 (B.H. Neumann). *Any bounded almost abelian group G is finite-by-abelian.*

Palacín and myself introduced a corresponding notion of bounded almost nilpotent groups and proved, using symmetry of the almost centralizer as well as the generalized Neumann theorem, that these are always nilpotent-by-finite.

Definition 9. A group H is *almost nilpotent* if there exists an *almost central series* of finite length, i. e. a sequence of normal subgroups of H

$$\{1\} \leq H_0 \leq H_1 \leq \cdots \leq H_n = H$$

such that H_{i+1}/H_i is a subgroup of $\widetilde{Z}(H/H_i)$ for every $i \in \{0, \ldots, n-1\}$.

We say that H is *bounded* if there is d such that for all $i < n$ and $g \in G_{i+1}$ the index $[G : C_G(g/G_i)]$ is at most d.

Theorem 10 (H., Palacín)**.** *Any bounded almost nilpotent group is nilpotent-by-finite.*

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Classification of imaginaries in separably closed valued fields

Martin Hils

(joint work with Moshe Kamensky and Silvain Rideau)

1. Introduction

In their fundamental paper [1], Haskell, Hrushovski and Macpherson classified imaginaries in the theory ACVF of algebraically closed non-trivially valued fields, thus initiating the *geometric* model theory of valued fields. It is not hard to see that if $K \models$ ACVF, then K does not eliminate imaginaries, even if sorts for the residue field k_K and for the value group Γ_K are added. Note that both k_K and Γ_K are imaginary sorts. Indeed, let $\mathcal{O}_K$ be the valuation ring, with maximal ideal $\mathfrak{m}_K$. Then $k_K = \mathcal{O}_K/\mathfrak{m}_K$ and $\Gamma_K = K^\times/\mathcal{O}_K^\times$. In [1], the authors introduce the following higher-dimensional analogues of k_K and Γ_K: for $n \geq 1$, let

$$S_n(K) := \mathrm{GL_n}(K)/\mathrm{GL_n}(\mathcal{O}_K), \text{ the set of } \mathcal{O}_K\text{-lattices in } K^n, \text{ and}$$

$$T_n(K) := \dot{\bigcup}_{s \in S_n(K)} s/\mathfrak{m}_K s.$$

These are imaginary sorts, and $\Gamma_K \cong S_1(K)$ and $k_K \subseteq T_1(K)$ canonically. The *geometric sorts* are given by $\mathcal{G} := \{\mathrm{VF}\} \cup \{S_n, T_n | n \geq 1\}$, where VF is the sort of the valued field. In [1], the authors give a natural language $\mathcal{L}_{\mathcal{G}}$ with sorts $\mathcal{G}$, in which ACVF eliminates quantifiers.

Fact 1 ([1])**.** *The theory ACVF eliminates imaginaries in $\mathcal{L}_{\mathcal{G}}$: for any $\emptyset$-definable set D and any $\emptyset$-definable equivalence relation E on D there is a $\emptyset$-definable function f inducing an injection of D/E into a finite product of sorts from $\mathcal{G}$.*

In the meantime, it has been shown that other theories of valued fields eliminate imaginaries down to the geometric sorts, e.g. for the theory RCVF of real closed fields with a convex valuation by Mellor [5], for p-adic fields as well as for ultraproducts of p-adic fields by Hrushovski, Martin and Rideau [4], and for the theory VDF of existentially closed valued differential fields of characteristic 0, satisfying $v(\partial(x)) \geq v(x)$ for every x, by Rideau [6]. We add a new item to this list, namely separably closed valued fields of finite Ershov invariant.

2. Separably closed valued fields and the main result

We fix a prime p. Let K be a separably closed field of characteristic p. Then $[K : K^p] = p^e$ for some $e \in \mathbb{N} \cup \{\infty\}$, where e is called the *Ershov invariant* of K. In what follows, we suppose that $e \geq 1$ is finite. Recall that K admits a *p-basis*, i.e. a tuple $(b_1, \ldots, b_e)$ of elements of K such that any $a \in K$ may be uniquely written as

$$a = \sum_{\nu \in p^e} x_\nu^p b^\nu, \text{ where } b^\nu := \prod_{i=1}^{e} b_i^{\nu_i} \text{ and } x_\nu \in K.$$

The functions $f_\nu(a) := x_\nu$ are called the *p-coordinates*, and they are definable in the ring language (with parameters b). The function $\lambda : K \to K^{p^e}$ given by $\lambda(x) = (f_\nu(x))_{\nu \in p^e}$ is then a definable bijection, with inverse given by a polynomial function.

Let $\mathcal{L}^\lambda := \mathcal{L}_{\mathrm{Rings}} \cup \{b_1, \ldots, b_e\} \cup \{f_\nu | \nu \in p^e\}$. The theory $\mathrm{SCF}_{p,e}$ of separably closed fields of characteristic 0 with p-basis $(b_1, \ldots, b_e)$ and p-coordinate functions f_ν is complete, eliminates quantifiers and imaginaries, and it is stable. Now let $\mathcal{L}^\lambda_{\mathrm{div}} := \mathcal{L}^\lambda \cup \{|\}$, where $|$ is a binary relation symbol.

Definition 2. Let $\mathrm{SCVF}_{p,e}$ be the $\mathcal{L}^\lambda_{\mathrm{div}}$-theory of separably closed non-trivially valued fields of characteristic p with p-basis $(b_1, \ldots, b_e)$ and p-coordinate functions f_ν, and where $|$ is interpreted as $x|y :\Leftrightarrow v(x) \leq v(y)$.

Fact 3 (Delon; Hong [3])**.** *The theory* $\mathrm{SCVF}_{p,e}$ *eliminates quantifiers. Its completions are NIP, and they are determined by the isomorphism type of the valued field $\mathbb{F}_p(b_1, \ldots, b_e)$.*

Let $\mathcal{L}^\lambda_{\mathcal{G}} := \mathcal{L}^\lambda \cup \mathcal{L}_{\mathcal{G}}$. The following is our main result.

Theorem 4. *Any completion of* $\mathrm{SCVF}_{p,e}$ *eliminates imaginaries in $\mathcal{L}^\lambda_{\mathcal{G}}$.*

We also characterise the stable stably embedded sets in $\mathrm{SCVF}_{p,e}$, as well as the stably dominated types (in terms of ACVF), and we show the following results.

Theorem 5. *Work in a completion of* $\mathrm{SCVF}_{p,e}$ *in the language* $\mathcal{L}_{\mathcal{G}}^{\lambda}$*. Then the following holds:*

(1) *Definable types are dense, i.e. if* D *is a definable set over* $A = \mathrm{acl}(A)$*, there is a global* A*-definable type* $p(x)$ *implying "*$x \in D$*".*
(2) *Every type over an algebraically closed set* A *has a global* Aut_A*-invariant extension.*
(3) *The theory* $\mathrm{SCVF}_{p,e}$ *is* metastable *in the sense of* [2]*, with* good bases *given by models which do not admit any proper immediate separable extension.*

3. Proof sketch of Theorem 4

Let $K \models \mathrm{SCVF}_{p,e}$. It is easy to see that K^p is dense in K, and so K^{p^n} is dense in K^{alg} for every $n \in \mathbb{N}$. In particular, $S_n(K) = S_n(K^{\mathrm{alg}})$ and $T_n(K) = T_n(K^{\mathrm{alg}})$ for all n. It follows that every element from $\mathcal{G}(K^{\mathrm{alg}})$ is interdefinable (in ACVF) with an element from $\mathcal{G}(K)$. By Fact 1, finite sets in $\mathcal{G}(K^{\mathrm{alg}})$ are coded in $\mathcal{G}(K^{\mathrm{alg}})$. It is thus enough to show that any definable set $X \subseteq K^m$ is weakly coded in $\mathcal{G}$. We perform various reductions:

(1) There is $n \geq 1$ such that $\lambda^n(X)$ is semialgebraic, i.e. definable by a quantifier free $\mathcal{L}_{\mathrm{Rings}} \cup \{|\}$-formula with parameters from K. As λ is a $\emptyset$-definable bijection, we may thus assume X is definable by such a formula $\psi(x)$.

(2) By a result of Hong [3], we may find ψ as above such that $X = \psi[K]$ is dense in $Y = \psi[K^{\mathrm{alg}}]$ (in the valuation topology).

(3) Let V be the Zariski closure of X. Arguing by induction on dimension, we may reduce to the case where V is irreducible, smooth, and defined over K as an algebraic variety. It follows from Hensel's lemma (essentially) that a semialgebraic subset of $V(K^{\mathrm{alg}})$ is Zariski-dense if and only if it has non-empty interior in the valuation topology. Using this, we may reduce to the case where X is a closed subset of $V(K)$ and where Y is the closure of X in $V(K^{\mathrm{alg}})$.

Having performed all these reductions, one may show that the ACVF-code of Y (which exists in $\mathcal{G}(K^{\mathrm{alg}})$ by Fact 1 and thus in $\mathcal{G}(K)$ by what we have mentioned in the beginning) serves as a code for X in $\mathrm{SCVF}_{p,e}$.

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Pseudofinite dimensions: Proper intersections and modularity

Ehud Hrushovski

(joint work with Boris Bukh and Jacob Tsimerman)

This is a report about work in progress with Boris Bukh and Jacob Tsimerman, extending ideas from Terry Tao's paper on expanding polynomials.

Let X be a definable set, $M = \lim_{i \to u} M_i$ an ultraproduct with $X(M_i)$ finite. We say that Y is *pseudofinite*. We have $|X| \in \mathbb{R}^* = \lim_{i \to u} \mathbb{R}$. Sometimes we will use a reference cardinality $p \in \mathbb{R}^*$, $p > \mathbb{N}$.

We have a *fine valuation* on $\mathbb{R}^*$, with valuation ring

$$\mathbb{O}_{fin} = \text{ convex hull of } \mathbb{R}$$

$$\mathbb{M}_p = \{r : \bigwedge_{n \in \mathbb{N}} |r| < 1/n\}.$$

And *coarse valuation*

$$\mathbb{O}_p = \{r : \bigwedge_{n \in \mathbb{N}} |r|^n < p\},$$

$$\mathbb{M}_p = \{r : \bigvee_{n \in \mathbb{N}} |r| < p^{-1/n}\}.$$

The fine and coarse *pseudo-finite dimensions* are

$$\delta(X) = |\mathrm{val}_{fin}(|X|)|, \qquad \boldsymbol{\delta}(X) = |\mathrm{val}_p(|X|)|.$$

The value group of $\mathbb{O}_p$ begins with $\mathbb{R}$, so for $|X| < p^m$ one can also identify

$$\boldsymbol{\delta}(X) = st(\log |X| / \log p) \in \mathbb{R}, \qquad \delta(X) = \log |X| + \mathbb{O}_{fin} \in \mathbb{R}^*/\mathbb{O}_{fin}.$$

The *measure* at fine dimension α : If $\delta(X) = \delta(X') = \alpha$,

$$\mu_\alpha(X)/\mu_\alpha(X') = st(|X/X'|).$$

It is defined only up to a scalar multiple, but we will only write formulas that are well-defined.

$$\alpha \approx \beta \iff \delta|\alpha - \beta| < \delta(\alpha) \iff \delta(\alpha) = \delta(\beta) \;\&\; \mu(\alpha) = \mu(\beta).$$

$$\alpha \boldsymbol{\approx} \beta \iff \boldsymbol{\delta}|\alpha - \beta| < \delta(\alpha).$$

Definition 1. Let $X, X' \subset Y$. Then X, X' intersect *properly* on Y if $|X \cap X'| \boldsymbol{\approx} |X||X'|/|Y|$.

In particular, if X, X' intersect ($\boldsymbol{\delta}$-) properly, then they intersect δ-properly, i.e.

$$\delta(X \cap X') = \delta(X) + \delta(X') - \delta(Y),$$

and even, with $\mu = \mu_{\delta(X) + \delta(X') - \delta(Y)}$,

$$\mu((X \cap X') \times Y) = \mu(X \times X').$$

Consider a family of definable sets $X_a \subset Y$, with $\delta(X) = \delta(X_a) = \alpha$. After taking into account a *unary* partition, we have *generic proper intersections.* More precisely, decomposing $\mu_\alpha = \int_q \mu_q$ over compact Lascar strong types q,

$$\mu_q(X_a \cap X_b) = \mu_q(X_a)\mu_q(X_b).$$

More generally, for any definable $B \subset Y$ with $\delta(B) = \alpha$, for generic a we have δ-proper intersection of X_a, B on Y.

What can be said when $\boldsymbol{\delta}(X) = \boldsymbol{\delta}(X_a)$? Here we will discuss a theorem in a setting with two languages $L \subset \mathbf{L}$; take L to be the language of a *pseudo-finite field*, and $\mathbf{L}$ *an expansion by a pseudo-finite set or predicate.* The language L will be assumed to satisfy a strong form of definability of δ; but $\mathbf{L}$ is an arbitrary pseudo-finite expansion. From now on, *proper intersections* and all other notions refer to $\boldsymbol{\delta}$ unless otherwise stated. We will say *definable* to refer to L, *pseudo-finite* to refer to $\mathbf{L}$.

In the first dichotomy theorem, non-modularity will enter via a hypothesis on U of the following form: $U \leq X \times Y$ (defined over F) is purely non-modular if for generic $(a, b) \in U$, $\operatorname{acl}(a) \cap \operatorname{acl}(b) = \operatorname{acl}(F)$.

For varieties U, this is equivalent to: if $U \leq X \times_W Y \leq X \times Y$, then $\dim(W) = 0$.

This hypothesis is in line with a number of theorems of stability such as Buechler's dichotomy, unimodularity, canonical base property of Pillay and Ziegler.

By contrast U is a *purely modular* interaction if a, b are independent over $\operatorname{acl}(a) \cap \operatorname{acl}(b)$.

Proposition 2. *Let M be a pseudo-finite structure. Let X, Y, $U \subset X \times Y$, be definable. Let $B \subset Y$ be a pseudo-finite subset. Assume:*

- (1) $|U_a|/\approx$ *is constant for $a \in X$;*
  (2) $|U^b|/\approx$ *is constant for $b \in Y$.*
  (3) *For some definable $Q_\nu \subset Y^2$ ($\nu = 1, \dots, f$) for $(b, b') \in Q_\nu$, $|U^b \cap U^{b'}| \approx \alpha_\nu$*
- *Let $W = \{(x, y, y') \in X \times Y^2 : (x, y), (x, y') \in U, (y, y') \in \cup_{\nu=0}^f Q_\nu$. Then $W \approx U \times_X U$ and $W(B) \approx U \times_X U(B)$.*
- *$B \times B, Q_\nu$ meet properly on $Y \times Y$ for each $\nu = 1, \dots, f$.*

Then there exists a pseudo-finite $X' \subset X$, $X' \approx X$, such that for all $a \in X'$, B meets $U(a, y)$ properly on Y

Remark 3. The condition $W(B) \approx U \times_X U(B)$ follows from $W \approx U \times_X U$, given that $\boldsymbol{\delta}(B) = \boldsymbol{\delta}(Y)$. The requirement $\boldsymbol{\delta}(B) = \boldsymbol{\delta}(Y)$ can be replaced by an explicit lower bound on $\boldsymbol{\delta}(B)/\boldsymbol{\delta}(Y)$, below 1 and sometimes more generous.

Theorem 4. *Let X, Y be L-definable sets (say varieties over a pseudo-finite field F), $U \leq X \times Y$ purely non-modular. Then there exists an L-definable finite partition $Y = \cup_{i=1}^k Y_i$ such that whenever $B \subset Y_i$ is a pseudo-finite set with $\boldsymbol{\delta}(B) = \boldsymbol{\delta}(Y)$, for almost all $a \in X$, U_a intersects B properly.*

Corollary 5. *For any pseudofinite $A \subset X$ with $\boldsymbol{\delta}(A) = \boldsymbol{\delta}(X)$, $A \times B$ intersects U properly in $X \times Y_i$.*

Theorem 4 contains an assumption of non-modularity. On the other hand, properness of intersections can fail in either direction for an approximate subgroup.

Let $n < p/3$ and let A be the image of $A = [-n, n] \cap \mathbb{Z}$ in $\mathbb{Z}/p\mathbb{Z}$; let A' be the image of $[1, n] \cap \mathbb{Z}$. Let $U = V = W = G_a$.

Then the proper intersection dimension is $3\boldsymbol{\delta}(A) - codim(F, U) = 3\boldsymbol{\delta}(A) - 1$. But

$$\boldsymbol{\delta}(A^3 \cap F) = 2\boldsymbol{\delta}(A) > 3\boldsymbol{\delta}(A) - \boldsymbol{\delta}(U) \quad \text{too many relations}$$
$$(A')^3 \cap F = \emptyset \quad \text{too few relations}$$

There are also interesting examples of improper intersections induced by approximate subgroups, but less directly. The intersection of an approximate subgroup with a generic curve inherits the improperness of intersection with the pullback of the graph of addition; though the algebraic relation one sees on the curve is unrelated to any one-dimensional algebraic group. More generally, let G be a commutative algebraic group and let $f : C \rightsquigarrow G$. Let A be an approximate subgroup of G. Then $f^{-1}(A)^3$ can have lower or upper improper intersection with $f^{-1}(+_G)$.

With this in mind, one can ask whether approximate subgroups are the only sources of improper intersections, at maximal coarse dimension. A positive partial answer is given by the following result.

Consider $R \leq C_1 \times \cdots \times C_n$ with C_i a curve, $\dim(R) = n - d$, such that R projects onto each $n - d$-tuple.

Theorem 6. *Assume $n \geq 4d$, $B_i \subset C_i$, $\boldsymbol{\delta}(B_i) = 1$ and $\Pi_i B_i$ has improper intersection with R. Then there exist $f_i : C_i \rightsquigarrow G$, G a commutative algebraic group, such that $\Pi_i B_i$ has improper intersection with $f^{-1}(+_G)$.*

When does dependence transfer from fields to henselian expansions?

Franziska Jahnke

1. Introduction and Motivation

There are many open questions connecting NIP and henselianity, e.g.,

Question 1.
(1) Is any valued NIP field (K, v) henselian?
(2) Let K be an NIP field, neither separably closed nor real closed. Does K admit a definable non-trivial henselian valuation?

Both of these questions have been recently answered positively in the special case where 'NIP' is replaced with 'dp-minimal' (cf. Johnson's results in [5]).

The question discussed in this talk is the following:

Question 2. Let K be an NIP field and v a henselian valuation on K. Is (K, v) NIP?

Known results in this direction were obtained by Delon and Bélair (see [1] for the relevant definitions):

Theorem 3. *Let (K, v) be a henselian valued field.*

(1) [2] *If the residue field Kv is NIP (as a pure field) of characteristic* $\mathrm{char}(Kv) = 0$*, then (K, v) is NIP (as a valued field).*
(2) [1, Corollaire 7.6] *Assume that (K, v) is Kaplansky and algebraically maximal of characteristic $p > 0$. If Kv is NIP (as a pure field) then (K, v) is NIP (as a valued field).*

The aim of this talk is to show that if K is an NIP field and v a henselian valuation on K, then (K, v) is NIP or Kv is separably closed. As separably closed fields are NIP, we obtain that the residue field Kv is always NIP (as a pure field).

2. Externally definable sets

Throughout the section, let M be a structure in some language $\mathcal{L}$.

Definition 4. Let $N \succ M$ be an $|M|^+$-saturated elementary extension. A subset $A \subseteq M$ is called *externally definable* if it is of the form

$$\{a \in M^{|\bar{x}|} \mid N \models \varphi(a, b)\}$$

for some $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$ and some $b \in N^{|\bar{y}|}$.

The notion of externally definable sets does not depend on the choice of N.

Definition 5. The *Shelah-expansion* M^{Sh} is the expansion of M by predicates for all externally definable sets.

Proposition 6 (Shelah, see [7, Chapter 3])**.** *If M is NIP then so is M^{Sh}.*

Example 7. Let (K, w) be a valued field and v be a coarsening of w. Then, there is a convex subgroup $\Delta \leq wK$ such that we have $vK \cong wK/\Delta$. As Δ is externally definable in the ordered abelian group wK, the valuation ring $\mathcal{O}_v$ is definable in $(K, w)^{\mathrm{Sh}}$.

3. p-henselian valuations

Throughout this section, let K be a field and p a prime. We define $K(p)$ to be the compositum of all Galois extensions of K of p-power degree (in a fixed algebraic closure). Note that we have

- $K \neq K(p)$ iff K admits a Galois extension of degree p and
- $[K(p) : K] < \infty \implies K = K(p)$ or $p = 2$ and $K(2) = K(\sqrt{-1})$.

Definition 8. A valuation v on a field K is called *p-henselian* if v extends uniquely to $K(p)$. We call K *p-henselian* if K admits a non-trivial p-henselian valuation.

In particular, every henselian valuation is p-henselian for all primes p. Assume $K \neq K(p)$. Then, there is a canonical p-henselian valuation on K: We divide the class of p-henselian valuations on K into two subclasses,

$$H_1^p(K) = \{v \text{ } p\text{-henselian on } K \mid Kv \neq Kv(p)\}$$

and

$$H_2^p(K) = \{v \text{ } p\text{-henselian on } K \mid Kv = Kv(p)\}.$$

One can show that any valuation $v_2 \in H_2^p(K)$ is *finer* than any $v_1 \in H_1^p(K)$, i.e. $\mathcal{O}_{v_2} \subsetneq \mathcal{O}_{v_1}$, and that any two valuations in $H_1^p(K)$ are comparable. Furthermore, if $H_2^p(K)$ is non-empty, then there exists a unique coarsest valuation v_K^p in $H_2^p(K)$; otherwise there exists a unique finest valuation $v_K^p \in H_1^p(K)$. In either case, v_K^p is called the *canonical p-henselian valuation* (see [6] for more details).

The following properties of the canonical p-henselian valuation follow immediately from the definition:

- If K is p-henselian then v_K^p is non-trivial.
- Any p-henselian valuation on K is comparable to v_K^p.
- If v is a p-henselian valuation on K with $Kv \neq Kv(p)$, then v coarsens v_K^p.

Theorem 9 ([4, Theorem 3.1])**.** *Fix a prime p. Let K be a field with $K \neq K(p)$. In case* $\mathrm{char}(K) \neq p$, *assume that K contains a primitive pth root of unity. In case $p = 2$ and* $\mathrm{char}(K) = 0$, *assume further that K is not real. There exists a parameter-free $\mathcal{L}_{ring}$-formula $\phi_p(x)$ independent of K with $\phi_p(K) = \mathcal{O}_{v_K^p}$.*

4. External definability of henselian valuations

Proposition 10. *Let (K, v) be henselian such that Kv is neither separably closed nor real closed. Then v is definable in K^{Sh}.*

Proof. (Sketch) Assume Kv is neither separably closed nor real closed. Choose any prime p such that Kv has a finite Galois extension of degree divisible by p^2. We construct some finite extension (L, v') of (K, v) such that v_L^p is $\emptyset$-definable on L in $\mathcal{L}_{\mathrm{ring}}$ and such that v_L^p refines v'. The restriction of v_L^p to K is then definable in $\mathcal{L}_{\mathrm{ring}}$. Thus, v is definable in the Shelah expansion of K. □

Proposition 11. *Let (K, v) be henselian such that Kv is real closed. Then v is definable in K^{Sh}.*

Proof. (Sketch) Assume that (K, v) is henselian and Kv is real closed. Using the definability of the 2-henselian valuation, we reduce to the case that vK is 2-divisible. In this case, K is uniquely ordered. By [3, Corollary 3.6] and Beth Definability Theorem, the ordering on K is $\mathcal{L}_{\mathrm{ring}}$-definable. Let $\mathcal{O}_w \subseteq K$ be the convex hull of $\mathbb{Z}$ in K. Then, $\mathcal{O}_w$ is definable in K^{Sh}. By [3, Proposition 2.2], w is the finest henselian valuation ring on K with real closed residue field. In particular, we get $\mathcal{O}_w \subseteq \mathcal{O}_v$ and hence $\mathcal{O}_v$ is also definable in K^{Sh}. □

Corollary 12. *Let K be NIP, v henselian on K.*

(1) *If Kv is not separably closed, then (K, v) is NIP.*
(2) *Kv is NIP as a pure field.*

The question what happens in case Kv is separably closed remains. In particular it would be interesting to know an answer to the following

Question 13. Let (K, v) be henselian, Kv NIP, not perfect. Is (K, v) NIP?

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Exact saturation in simple and NIP theories

Itay Kaplan

(joint work with Saharon Shelah, Pierre Simon)

A first order theory T has exact saturation at κ if it has a κ-saturated model which is not κ^+-saturated. When $\kappa > |T|$ is regular, then any theory has exact saturation at κ, hence we are only interested in the case where κ is singular.

Possibly adding set-theoretic assumptions, we expect that for a given theory T, having exact saturation at a singular cardinal κ does not depend on κ, and that this property is an interesting dividing line within first order theories. We indeed show this for stable, simple and NIP theories.

In the stable case, it was shown by Shelah in [1, IV, Lemma 2.18] that stable theories have exact saturation at any κ.

We generalize this result to the simple case in the following sense. Let T be simple and assume that κ is singular of cofinality greater than $|T|$, $2^\kappa = \kappa^+$ and $\square_\kappa$ holds, then T has exact saturation at κ.

In the NIP case, Shelah showed that an NIP theory with an infinite indiscernible set has exact saturation at any singular κ with $2^\kappa = \kappa^+$ ([2, Claim 2.26]).

In this work we establish the precise dividing line for NIP theories: with the same assumptions on κ, an NIP theory has exact saturation at κ if and only if it is not distal. This gives a new characterization of distality within NIP theories.

The techniques used to construct models in the simple, stable and NIP non-distal cases use the notion of a D-type and D-model, for a finite diagram (a collection of types in finitely many variables over some set, in our case an indiscernible sequence). For instance in the stable case, we let I be an infinite indiscernible set of size κ, and let D_I be the collection of types over I in finitely many variables that are isolated by their restriction to a smaller set. Then the type of a new element in I is not a D_I-type hence not realized in the model. In the simple case we use the independence property to find an indiscernible sequence which witnesses it, and in the NIP non-distal case we use a non-distal indiscernible sequence.

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A non-archimedean Ax-Lindemann theorem

François Loeser

(joint work with Antoine Chambert-Loir)

1. Introduction

The classical Lindemann-Weierstrass theorem states that if algebraic numbers $\alpha_1, ..., \alpha_n$ are $\mathbf{Q}$-linearly independent, then their exponentials $\exp(\alpha_1), ..., \exp(\alpha_n)$ are algebraically independent over $\mathbf{Q}$. More generally, if $\alpha_1, \dots, \alpha_n$ are complex numbers which are no longer assumed to be algebraic, Schanuel's conjecture predicts that the field $\mathbf{Q}(\alpha_1, \dots, \alpha_n, \exp(\alpha_1), \dots, \exp(\alpha_n))$ has transcendence degree at least n over $\mathbf{Q}$. In [1], Ax established power series and differential field versions of Schanuel's conjecture.

Theorem 1 (Exponential Ax-Lindemann)**.** *Let* $\exp\colon \mathbf{C}^n \to (\mathbf{C}^\times)^n$ *be the morphism* $(z_1, \dots, z_n) \mapsto (\exp(z_1), \dots, \exp(z_n))$*. Let* V *be an irreducible algebraic subvariety of* $(\mathbf{C}^\times)^n$ *and let* W *be an irreducible component of a maximal algebraic subvariety of* $\exp^{-1}(V)$*. Then* W *is geodesic, that is,* W *is defined by a finite family of equations of the form* $\sum_{i=1}^n a_i z_i = b$ *with* $a_i \in \mathbf{Q}$ *and* $b \in \mathbf{C}$*.*

In the breakthrough paper [6], Pila succeeded in providing an unconditional proof of the André-Oort conjecture for products of modular curves. One of his main ingredients was to prove a hyperbolic version of the above Ax-Lindemann theorem, which we now state in a simplified version.

Let $\mathbf{H}$ denote the complex upper half-plane and $j : \mathbf{H} \to \mathbf{C}$ the elliptic modular function. By an algebraic subvariety of $\mathbf{H}^n$ we shall mean the trace in $\mathbf{H}^n$ of an algebraic subvariety of $\mathbf{C}^n$. An algebraic subvariety of $\mathbf{H}^n$ is said to be geodesic if it is defined by equations of the form $z_i = c_i$ and $z_k = g_{k\ell} z_\ell$, with $c_i \in \mathbf{C}$ and $g_{k\ell} \in \mathrm{GL}_2^+(\mathbf{Q})$.

Theorem 2 (Hyperbolic Ax-Lindemann)**.** *Let* $j : \mathbf{H}^n \to \mathbf{C}^n$ *be the morphism* $(z_1, \dots, z_n) \mapsto (j(z_1), \dots, j(z_n))$*. Let* V *be an irreducible algebraic subvariety of* $\mathbf{C}^n$ *and let* W *be an irreducible component of a maximal algebraic subvariety of* $j^{-1}(V)$*. Then* W *is geodesic.*

Pila's method to prove this Ax-Lindemann theorem is quite different from the differential approach of Ax. It follows a strategy initiated by Pila and Zannier in their new proof of the Manin-Mumford conjecture for abelian varieties [9]; that approach makes crucial use of the bound on the number of rational points of bounded height in the transcendental part of sets definable in an o-minimal structure obtained by Pila and Wilkie in [8]. Recently, still using the Pila and Zannier strategy, Klingler, Ullmo and Yafaev have succeeded in proving a very general form of the hyperbolic Ax-Lindemann theorem valid for any arithmetic variety ([5], see also [10] for the compact case).

In this work, we establish a non-archimedean analogue of theorem 2.

2. Statement of the non-archimedean Ax-Lindemann theorem

Let p be a prime number and let F be a finite extension of $\mathbf{Q}_p$. In this work, we make use of Berkovich's notion of F-analytic spaces, see [2]. The group $\mathrm{PGL}(2, F)$ acts by homographies on the F-analytic projective line $(\mathbf{P}_1)^{\mathrm{an}}$, and on its F-rational points $\mathbf{P}_1(F)$.

Recall (see [4]) that a Schottky subgroup of $\mathrm{PGL}(2, F)$ is a discrete subgroup which is finitely generated and free. We say that such a subgroup Γ is arithmetic if there exists a number field $K \subset F$ such that $\Gamma \subset \mathrm{PGL}(2, K)$.

A Schottky subgroup Γ of $\mathrm{PGL}(2, F)$ has a limit set $\mathcal{L}_\Gamma$ which is a non-empty compact Γ-invariant subset of $\mathbf{P}_1(F)$; if the rank g of Γ is ≥ 2, then it is a perfect set. Let then $\Omega_\Gamma = (\mathbf{P}_1)^{\mathrm{an}} \setminus \mathcal{L}_\Gamma$; the group Γ acts freely on Ω_Γ and the quotient space Ω_Γ/Γ is naturally a F-analytic space so that the projection $p_\Gamma \colon \Omega_\Gamma \to \Omega_\Gamma/\Gamma$ is topologically étale. Moreover, Ω_Γ/Γ is the F-analytic space associated with a smooth, geometrically connected, projective F-curve X_Γ of genus g.

Let us now consider a finite family $(\Gamma_i)_{1\leq i\leq n}$ of Schottky subgroups of $\mathrm{PGL}(2, F)$ of rank ≥ 2. Let us set $\Omega = \prod_{i=1}^n \Omega_{\Gamma_i}$ and $X = \prod_{i=1}^n X_{\Gamma_i}$, and let $p \colon \Omega \to X^{\mathrm{an}}$ be the morphism deduced from the morphisms $p_{\Gamma_i} \colon \Omega_{\Gamma_i} \to X_{\Gamma_i}^{\mathrm{an}}$.

We say that a closed subspace W of Ω is *irreducible algebraic* if there exists an F-algebraic subvariety Y of $(\mathbf{P}_1)^n$ such that W is an irreducible component of the analytic space $\Omega \cap Y^{\mathrm{an}}$.

We say that W is *flat* if it can be defined by equations of the following form:

(1) $z_i = c$, for some $i \in \{1, \dots, n\}$ and $c \in \Omega$;
(2) $z_j = g \cdot z_i$, for some pair (i, j) of elements of $\{1, \dots, n\}$ and $g \in \mathrm{PGL}(2, F)$.

Here is our main result:

Theorem 3 (Non-archimedean Ax-Lindemann theorem)**.** *Let F be a finite extension of $\mathbf{Q}_p$ and let $(\Gamma_i)_{1\leq i\leq n}$ be a finite family of* arithmetic *Schottky subgroups of* $\mathrm{PGL}(2, F)$ *of rank ≥ 2. As above, let us set $\Omega = \prod_{i=1}^n \Omega_{\Gamma_i}$ and $X = \prod_{i=1}^n X_{\Gamma_i}$, and let $p \colon \Omega \to X^{\mathrm{an}}$ be the morphism deduced from the morphisms $p_{\Gamma_i} \colon \Omega_{\Gamma_i} \to X_{\Gamma_i}^{\mathrm{an}}$.*

Let V be an irreducible algebraic subvariety of X and let $W \subset \Omega$ be an irreducible component of a maximal algebraic subvariety of $p^{-1}(V^{\mathrm{an}})$. Then W is flat.

3. Sketch of the proof

The basic strategy we use is strongly inspired by that of Pila [6] (see also [7]), though some new ideas are required in order to adapt it to the non-archimedean setting. In particular, we have to replace the theorem of Pila-Wilkie [8] by the non-archimedean analogue recently proved by Cluckers, Comte and Loeser [3].

The role of the o-minimal structure $\mathbf{R}_{\mathrm{an,exp}}$ is now played by the subanalytic sets (in F^n) of Denef and van den Dries, and the rigid subanalytic sets of Lipshitz and Robinson (in $\mathbf{C}_p^n$). Analytic continuation and monodromy arguments are replaced by more algebraic ones and explicit matrix computations by group theory considerations. We also take advantage of the fact that Schottky groups are free and of the geometric description of their fundamental domains.

Let V and W be are as the statement of theorem 3. Let Y be the Zariski closure of W and let m be its dimension. Similarly as in Pila's approach one starts by working on some neighborhood of the boundary of our space (which, instead of a product of Poincaré upper half-planes, is a product of open subsets of the Berkovich projective line). We reduce to the case where, locally around some rigid point $\xi \in \Omega$, W is the image of a section ϕ of the projection to the first m coordinates, and that $\xi_1 \in \mathcal{L}_{\Gamma_1}$.

We consider good fundamental domains $\mathfrak{F}_j$ for the groups Γ_j and their product $\mathfrak{F}$; let $\Gamma = \prod \Gamma_j$. We then consider the subset G_0 of $\mathrm{PGL}(2, F)$ consisting of points $(g_1, \dots, g_n)$ such that $g_2 = \dots = g_m = 1$, and its subset R defined by the condition $\dim(gW \cap \mathfrak{F} \cap p^{-1}(V)) = m$. One proves that R is a subanalytic set. Studying the action of Γ_j on a neighborhood of the limit set $\mathcal{L}_{\Gamma_j}$, one proves that every element of Ω can be moved to an element of $\mathfrak{F}_j$ by applying an element of Γ_j of controlled length in some fixed generators. Since the groups Γ_j are arithmetic and free non-abelian, this allows to prove that for every real number T, R contains $\gg T^c$ algebraic points of bounded degree and height $\leq T$. Applying the p-adic Pila-Wilkie theorem of [3], and making use of the maximality of W, we then prove that the stabilizer of W inside $G_0 \cap \Gamma$ is infinite. This furnishes non-trivial functional equations for the coordinates ϕ_j of the section ϕ. From these functional equations, we deduce that the Schwarzian derivative of ϕ_j is constant, hence zero, because ϕ_j is algebraic. This implies that W is flat.

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One relator groups and some equations over $\mathbb{F}$

Larsen Louder
(joint work with Henry Wilson)

One relator groups are homologically coherent, i.e. if $H \leq G$ is a finitely generated subgroup of a one relator group G, we have $rk(H_2(H, \mathbb{Z})) < \infty$.

Profinite NIP groups

Dugald Macpherson
(joint work with Katrin Tent)

We consider a profinite group G as a 2-sorted structure $\mathcal{G}$ of form $\mathcal{G} = (G, I)$, where the group G is considered in the language of groups, and I indexes a basis of open subgroups of G; that is, the language has a relation $K \subset G \times I$ so that if $K_i = \{g : (g, i) \in K\}$ for each $i \in I$, then $\{K_i : i \in I\}$ is a basis of open neighbourhood subgroups of 1. We say (G, I) is a *full* profinite group if every open subgroup of G has the form K_i for some $i \in I$. The main theorem (see [5]) is that a full profinite group (G, I) has NIP theory if and only if G has an open normal subgroup $N = P_1 \times \ldots \times P_t$, where each P_i is a compact p_i-adic analytic group, for distinct primes $p_1, \ldots, p_t$. By classical work of Lazard and others, there are many conditions equivalent to a pro-p group G being p-adic analytic, and we may now add to these the condition that G is NIP as a full 2-sorted pro-p group.

In the proof, it follows from the work of Lazard [3], and that of du Sautoy [2], that if $\mathcal{G} = (G, I)$ is a 'uniformly powerful' pro-p group, then $\mathcal{G}$ is interpretable in the structure $\mathbb{Z}_p^{\mathrm{an}}$, the expansion of the ring of p-adic integers by convergent power series functions considered by Denef and van den Dries in [1]. The latter structure is well-known to be NIP, which rapidly yields the right-to-left direction of the main theorem.

In the other direction, we first show that any full NIP profinite group has finite rank, that is, there is a uniform bound on the number of topological generators needed for closed subgroups of G. This, together with a structural result of Colin Reid for finite rank profinite groups, and general profinite group theory, yields the group-theoretic characterisation of NIP. It appears that the weaker model-theoretic condition NTP_2 already yields this characterisation.

Without the fullness assumption, we can still show that if $\mathcal{G} = (G, I)$ is a NIP profinite group then G has an open definable normal subgroup which is pro-soluble. This is shown by finding a family $\mathcal{C}$ of uniformly definable finite quotients of G, by open normal subgroups which have trivial intersection. Any ultraproduct of

$\mathcal{C}$ will be definable in an ultrapower of $\mathcal{G}$, so will have a soluble definable normal subgroup of bounded index, by Theorem 1.2 of [4].

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On the strong canonical base property

Daniel Palacín

(joint work with Anand Pillay)

The study of geometric properties such as one-basedness has contributed to understand not only geometric aspects of stability theory, but also it has been essential for the applications of stability to other areas in mathematics. Recall that a set X which is type-definable over a small set A is *one-based* if for any tuple $\bar{a}$ from X and any set B, the canonical base of $\mathrm{stp}(\bar{a}/B)$ is contained in $\mathrm{acl}(A, \bar{a})$. Hrushovski and Pillay (see [2]) showed that if X is a type-definable one-based group then it is rigid, i.e. all its type-definable connected subgroups are type-definable over $\mathrm{acl}(A)$. Nevertheless, not every rigid group is one-based; for instance, semi-abelian varietes over algebraically closed fields are rigid.

In the past few years relative version of one-basedness have come into scene. The main example is the *Canonical Base Property* (CBP) whose model-theoretic formulation was motivated by results of Campana and Fujiki in complex geometry (see [6]) and the analogous results due to Pillay and Ziegler (see [7]) in differential and difference varieties.

The CBP, named by Moosa and Pillay in [4], states for a stable theory of finite U-rank that for any tuple $\bar{a}$ and any set B, if c is the canonical base $\mathrm{stp}(\bar{a}/B)$ then the type $\mathrm{stp}(c/\bar{a})$ is almost internal to the family of U-rank 1 types (or equivalently, to the family of non locally modular types of Morley rank 1).

This property of the finite rank context holds for the many-sorted theory CCM of compact complex spaces (see [6]) as well as for the finite Morley rank part of the theory DCF_0 of differently closed fields of characteristic zero (see [7]). In both cases, it yields the existence of a unique non locally modular strongly minimal set up to non-orthogonality. Namely, the projective line over $\mathbb{C}$ in CCM and the field of constants in the case of DCF_0. Furthermore, as pointed out in [7], a group-like version of the CBP yields an account of Mordell-Lang for function fields in characteristic zero, following Hrushovski's proof but circumventing the use of Zariski geometries.

An example of a finite rank theory, in fact $\aleph_1$-categorical, where the CBP fails is given in [3]. On the other hand, there, under suitable assumptions it is shown that rigidity of *definable Galois groups* implies a strong version of the CBP. In fact, this was generalized in [5] to arbitrary stable theories of finite rank using results from [1]. By the definable Galois group of a stationary type $p \in S(A)$ relative to a family $\mathcal{Q}$ of partial types with parameters over A, we mean a type-definable group that acts definable on the set of realizations of p, and it is naturally isomorphic to the group of permutations of realizations of p which are induced by automorphisms of the monster model that fix pointwise A as well as the realizations of the partial types from $\mathcal{Q}$. The existence of such a group is given by the general theory of stability whenever p is internal to $\mathcal{Q}$.

In this talk we shall present the equivalence between the strong canonical base property (see condition (2) in the Theorem) and the rigidity of the relevant definable Galois groups.

We assume that the ambient theory is stable of finite U-rank and that every non locally modular type is non-orthogonal to $\emptyset$. Setting $\mathcal{Q}$ to be the family of types $\mathrm{tp}(c)$ such that $\mathrm{stp}(c)$ is internal to the family of non locally modular strongly minimal sets, we obtain:

Theorem 1. *The following are equivalent:*

(1) *All definable Galois groups are rigid.*
(2) *For any tuple $\bar{a}$ and any set B, if c is the canonical base of $\mathrm{stp}(\bar{a}/B)$, then c is algebraic over $\bar{a}$ and a tuple of realizations of types in $\mathcal{Q}$.*

As a consequence, the strong version of the CBP corresponds to a one-based phenomena.

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Elimination of imaginaries over the free group

Zlil Sela

We gave 3 talks that aimed to sketch the approach towards a weak elimination of imaginaries over the free (and hyperbolic) groups.

In the first talk we talked about a geometric approach to study sets of solutions (varieties) over a free group. These include limit groups, their Grushko and JSJ decompositions. Finally we describe the Makanin-Razborov diagram that one can associate with a variety, that encodes all the points in a variety, or alternatively it encodes all the homomorphisms from a given finitely generated group into a free group.

In the second talk we surveyed some basic results on the first order theory of a free group. We discussed Merzlyakov's theorem on positive AE sentences. Then we explained how can one use the Makanin-Razborov diagram to generalize Merzlyakov theorem to positive formulas when the universal variables are restricted to a given variety.

In the last part of the second talk, we described building blocks of the Boolean algebra of the definable sets over the free groups. This uses the notions of rigid and solid limit group, that enables one to encode exceptional families in a parametric family of varieties.

In the last talk we sketched the approach towards weak elimination of imaginaries. First, we explained what is an envelope of a definable set. This is a Diophantine set that contains the definable set, and such that (properly defined) generic points in the Diophantine set are contained in the definable set. An envelope is a useful tool in studying global properties of definable sets, like stability, equationality, superstability, and definable equivalence relations.

We further described the 3 basic families of imaginaries: conjugation, left or right cosets of cyclic groups, and double cosets of cyclic groups. We stated a result that claims that these (families of) basic imaginaries are all not reals.

Then we explained the weak elimination theorem. It claims that given a definable equivalence relation $E(p, q)$, where p and q are m-tuples, it is possible to a multi-function:

$$f : (F_k)^m \to (F_k)^s \times R_1 \times \ldots \times R_\ell$$

where $R_1, \ldots, R_\ell$ are sources for some of the 3 (families) of basic relations. f is a multi-function, i.e., it sends every element in $(F_k)^m$ into a finite (in fact, bounded) set, it is a class function, and it separates between classes.

We gave a brief sketch of the proof of the weak elimination theorem. This includes Diophantine envelopes, which is a generalization of the notion of envelopes to a parametric family of definable sets, uniformization limit groups, and finally the coupling of the quantifier elimination procedure with a procedure for separation of variables, that constructs the multi-function f after finitely many steps.

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The Erdős-Hajnal property for distal theories

Pierre Simon

The class of distal theories was introduced a few years ago in [2] as an abstraction of semi-algebraic structures ($\mathbb{R}$ and $\mathbb{Q}_p$ are the prototypical examples of distal structures). By definition, a first order theory T is distal if for every formula $\phi(x; y)$, there is some formula $\psi(x; z)$ such that for any finite A, one can find instances $\psi(x; d_i)$, $i < N$ with $d_i \in A^{|z|}$ whose union covers x-space and such that any $\psi(x; d_i)$ implies a complete ϕ-type over A. One can think of this as cutting x-space into finitely many cells $\psi(x; d_i)$, having some control over ϕ on each cell.

If a theory is distal, then it is NIP (since distality implies that there are polynomially many ϕ-types over finite sets). In fact, one can think of distal theories as being NIP theories which are purely unstable in some sense. They are meant to serve as an orthogonal complement to stable theories inside the class of NIP theories.

Alon et al. proved that any bipartite graph definable by a semi-algebraic relation has the *strong Erdős-Hajnal property*: there is some $\delta > 0$ such that for any two finite sets A, B, there are $A_0 \subseteq A$ and $B_0 \subseteq B$, $|A_0| \geq \delta|A|$ and $|B_0| \geq \delta|B|$, such that the graph restricted to $A_0 \times B_0$ is either complete or empty. In a recent paper [1], Chernikov and Starchenko give a generalization of this result to all graphs definable in a distal theory. They also point out that this property does not hold for ACF_p, proving that this theory has no distal expansion. Thus, it appears that distality is a meaningful property from the point of view of finite combinatorics, which was rather unexpected.

In this talk, I presented a relatively short proof of Chernikov and Starchenko's theorem using the theory of generically stable measures in NIP as developed in [3]. A measure (in an NIP theory) is *generically stable* if it can be obtained as an ultraproduct of finite counting measures. The fundamental VC-theorem implies that for such a measure $\mu(x)$, given a formula $\phi(x; y)$ and some $\epsilon > 0$, there is a finite set A such that if b and b' have the same ϕ-type over A, then $\mu(\phi(x; b) \triangle \phi(x; b')) \leq \epsilon$. Then applying the definition of distality given above to this ϕ and A easily gives the result.

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Definable fields in non abelian free groups

Rizos Sklinos
(joint work with Ayala Byron)

Non abelian free groups have been all proven to have the same first order theory ([2],[1]). Although this theory admits quantifier elimination down to boolean combination of $\forall\exists$ formulas it is hard to determine when a subset of some finite cartesian power of a non abelian free group is definable.

On the other hand, after the seminal work of Zilber [4] in understanding uncountably categorical theories via some naturally defined pregeometries much attention has been given to questions regarding the type of groups that are definable in a first order theory or whether an infinite field is definable in it.

In this talk we presented the following theorem:

Theorem 1. *Let $\mathbb{F}$ be a non abelian free group. Then no infinite field is definable in $\mathbb{F}$.*

Our proof combines techniques from geometric group theory and geometric stability. Roughly speaking a definable set (over a non abelian free group) would either be internal to a finite set of centralizers of non trivial elements or it cannot be given definably the structure of an abelian group. To conclude we prove that centralizers of non trivial elements are 1-based (a result that had already been proved by C.Perin) and the result follows, since by [3] our definable set would be 1-based.

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Tutorial: On th Szemerédi Regularity Lemma

Sergei Starchenko

In this tutorial we discuss the Szemerédi Regularity Lemma (see [3]) and present its proof based on non-standard analysis and pseudo-finite measures.

We also consider special cases of graphs: graphs whose edge relation is stable, and graphs whose edge relation has a finite VC-dimension.

In the stable case we obtain that the in Szemerédi regularity lemma we can require in addition that there are no exceptional pairs (see [1]); and in a finite VC-dimension case we obtain a polynomial bound on the number of sets in a partition (see [2]).

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Laws for finite groups

Andreas Thom

One way to state the prime number theorem is to say that for any $n \in \mathbb{Z}$, there exists an abelian group G of size bounded by roughly $\log|n|$ and a homomorphism $\varphi\colon \mathbb{Z} \to G$ such that $\varphi(n) \neq 0$. Equivalently, there exists a subgroup $\Lambda \subset \mathbb{Z}$ of index roughly bounded by $\log|n|$, such that $n \notin \Lambda$. We study analogous questions for the group $\mathbb{F}_2$ instead of $\mathbb{Z}$ – following a programme of quantifying residual finiteness for non-commutative groups started by Bou-Rabee. We set:

$$D(w) := \min\{[\mathbb{F}_2 : \Lambda] \mid \Lambda < \mathbb{F}_2 \text{ subgroup}, w \notin \mathbb{F}_2\}$$

and

$$D^{\lhd}(w) := \min\{[\mathbb{F}_2 : \Lambda] \mid \Lambda < \mathbb{F}_2 \text{ normal subgroup}, w \notin \mathbb{F}_2\}.$$

To obtain a more tractable object to study, we define the divisibility functions:

$$d(n) := \max\{D(w) \mid 0 < |w| \leq n\},$$

and

$$d^{\lhd}(n) := \max\{D(w) \mid 0 < |w| \leq n\}.$$

A law for a group G is a word in $\mathbb{F}_2$, so that the associated word map $w\colon G \times G \to G$ is trivial, i.e., $w(g,h) = 1_G$ for all $g,h \in G$. The following elementary lemma establishes a close relationship between the existence of certain laws and the objective to bound the divisibility functions.

Lemma 1. *Let $w \in \mathbb{F}_2 \setminus \{1\}$. Then $D(w) > n$ holds if and only if w is a law for the group* $\mathrm{Sym}(n)$. *Similarly, $D^{\lhd}(w) > n$ holds if and only if w is a law for all groups of size bounded by n.*

In order to provide lower bounds for the divisibility functions, we will provide upper bounds for length of non-trivial laws for $\mathrm{Sym}(n)$ and the class of all finite groups of size bounded above by n.

We start stating our results about laws for the symmetric group.

Theorem 2 (Kozma-T.)**.** *There exists a constant C such that for all $n \in \mathbb{N}$, there exists a law for* $\mathrm{Sym}(n)$ *of length bounded by*

$$\exp(C\log(n)^4 \log\log(n)).$$

Moreover, assuming Babai's Conjecture on logarithmic diameter bounds for Cayley graphs of simple groups, there exists a law of length

$$\exp(C\log(n)\log\log(n)).$$

This improves on previous bounds of $\exp(Cn)$ by Bogopolski, $\exp(Cn^{1/2}\log(n))$ by Gimadeev-Vyalyi [3] and $\exp(C(n\log(n))^{1/2})$ by Bou-Rabee and McReynolds [2]. This result has appeared as joint work with Gady Kozma in [5] – the proof depends in various ways on the Classification of Finite Simple Groups.

The result for the class of finite groups of size bounded by n is the following:

Theorem 3. *There exists a constant C such that for all $n \in \mathbb{N}$, there exists a law for the class of all finite groups of size bounded from above by n of length bounded by*

$$\frac{Cn\log\log(n)^{9/2}}{\log(n)^2}.$$

This result improves on previous bounds of Cn^3 by Bou-Rabee [1] and $Cn^{3/2}$ by Kassabov-Matucci [4] and answers a question from [4]. This result has appeared in the preprint [6] – again, the proof depends on the Classification of Finite Simple Groups.

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Meridional rank versus bridge number

Richard Weidmann
(joint work with Michel Boileau, Ederson Dutra and Yeonhee Jang)

Given a projection of a knot, an arc is called a bridge if it overcrosses at least once. The bridge number $b(K)$ of a knot K is defined as the minimal number of bridges in all projections of K. It follows easily from inspecting the Wirtinger presentation of the knot group $\pi_1(S^3\backslash K)$ that it is generated by $b(K)$ meridians. Here we call an element a meridian if is represented by a closed curve that is freely homotopic to the meridian of the knot.

S. Cappell and J. Shaneson as well as K. Murasugi, have asked whether the minimal number of meridians needed to generate a knot group always coincides with the bridge number. An affirmative answer to this question has been established in a number of cases, in particular it is known in the case of torus knots, 3-bridge knots and generalized Montesinos knots. More recently Cornwell and Hemminger have given a positive answer for many iterated cable knots.

We greatly generalize the result of Cornwell and Hemminger by giving an affirmative answer for a class of knots that contains all knots whose exterior is a graph manifold. The class we study contains 2-bridge knots and torus knots and is closed under connected sums and satellite constructions with braid patterns. The proof uses a a folding sequence in graphs of groups to compute the minimal number of meridians needed to generate the knot group which is then shown to agree with the bridge number using classical results of Schubert. This is joint work with Michel Boileau, Ederson Dutra and Yeonhee Jang.

Actions on sets of Morley rank 2

Joshua Wiscons
(joint work with Tuna Altınel)

In [2], Borovik and Cherlin initiate a broad study of permutation groups of finite Morley rank around the problem of finding an upper bound on the Morley rank of a permutation group that depends only upon the Morley rank of the set being acted on. It is not hard to see that such a bound does not exist in general, but in the context of (definably) primitive actions, the problem becomes very interesting. Using deep results from the theory of groups of finite Morley rank, including the classification of the simple groups of even and mixed type, Borovik and Cherlin did indeed succeed in finding a bound. That is, they prove the existence of a function $\rho : \mathbb{N} \to \mathbb{N}$ that assigns to each r the maximum rank for a group of finite Morley rank that has a faithful, primitive action on a set of rank r. However, parts of their analysis are, in their words, "soft," and throughout the paper they pose many problems around the hope of finding a sharper, more natural bound on ρ.

In what follows, we will call (X, G) a permutation group if X is a set equipped with a fixed *faithful* action of the group G. A key component of Borovik and

Cherlin's proof is a relationship they prove between ρ and the so-called degree of generic transitivity. A permutation group of finite Morley rank (X, G) is said to be *generically n-transitive* if G has an orbit $\mathcal{O}$ on X^n such that $\mathrm{rk}(X^n - \mathcal{O})$ is strictly less than $\mathrm{rk}(X^n)$; the maximum n such that (X, G) is generically n-transitive will be denoted by $\mathrm{gtd}(X, G)$. The relationship mentioned above is that

$$r \cdot \mathrm{gtd}(X, G) \leq \mathrm{rk}\, G \leq r \cdot \mathrm{gtd}(X, G) + r(r-1)/2$$

whenever (X, G) is a primitive permutation group of finite Morley rank. As such, we turn our focus from bounding the rank of G to bounding $\mathrm{gtd}(X, G)$. Here, the conjecture is quite clear.

Conjecture 1. *If (X, G) is a transitive and generically $(n+2)$-transitive permutation group of finite Morley rank with G connected and $\mathrm{rk}\, X = n$, then (X, G) is equivalent to $(\mathbb{P}^n(K), \mathrm{PGL}_{n+1}(K))$ for some algebraically closed field K.*

The case of $\mathrm{rk}\, X = 1$ has been known since the 1980's as a consequence of Hrushovski's theorem about actions on strongly minimal sets. Here we present a solution to the rank 2 case.

Theorem 2 ([1, Theorem A]). *If (X, G) is a transitive and generically 4-transitive permutation group of finite Morley rank with $\mathrm{rk}\, X = 2$, then (X, G) is equivalent to $(\mathbb{P}^2(K), \mathrm{PGL}_3(K))$ for some algebraically closed field K.*

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