

MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Report No. 6/2016

DOI: 10.4171/OWR/2016/6

Mini-Workshop: Operator Spaces and Noncommutative Geometry in Interaction

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7 – 13 February 2016

ABSTRACT. In recent years, operator space theory has made remarkable appearances in noncommutative geometry, notably in the study of C^* -algebras of real reductive groups and the unbounded picture of Kasparov theory. In both these developments, a central rôle is played by operator modules and the Haagerup tensor product. This workshop brought together experts in the aforementioned fields to deepen this interaction.

Mathematics Subject Classification (2010): Primary: 47L05, 58B34, 22D25, Secondary: 46L08, 19K35, 46A32.

Introduction by the Organisers

The ideas developed in operator space theory are sometimes described as ‘noncommutative analysis’ since they go beyond what we think of as classical analysis; they replace positive real numbers and their estimates by positive operators and operator estimates. Similarly, noncommutative geometry goes beyond classical geometry by replacing topological spaces by C^* -algebras and Riemannian metrics by self-adjoint ‘Dirac-type’ operators.

In this workshop, we brought together experts from operator space theory and noncommutative geometry to explore how their specific research fields could be mutually beneficial. The motivation for organising the workshop was the appearance of operator space techniques in the study of the C^* -algebras of real reductive groups and in the construction of the Kasparov product in unbounded KK -theory.

The first day of the meeting saw a program of four introductory lectures, the purpose being to acquaint participants from both sides with the main ideas behind these research fields. Gilles Pisier gave a lecture on the development of operator space theory, giving both the historical context and explaining the underlying philosophy. The day continued with a talk by David Blecher on operator modules and their tensor products, then by Nigel Higson on the representation theory of real reductive groups and an operator space point of view on the Plancherel theorem. The last talk of the day was given by Adam Rennie, who provided an overview of KK -theory, explaining how operator spaces appear in the construction of the unbounded Kasparov product.

In the early evening of the Monday there was a general opening session, in which each participant presented themselves and their research in a three minute mini-talk. This part of the workshop had an informal and light character. Its main purpose was to ‘break the ice’ and to acquaint the participants with one another’s research areas.

For the remainder of the week we scheduled each day a total of three research talks by participants. All talks took place before lunch, so as to have free afternoons open for conducting mathematical discussion and exploring common interests. While some participants engaged in discussions in smaller groups, we also organised more spontaneous events in the remaining afternoons, which were attended by the majority of participants. We made a concerted effort to maintain a coherent theme on each day.

Tuesday morning saw talks by Matthew Kennedy, Alex Bearden and Tatiana Shulman. On Tuesday afternoon we held a Q&A-session on the finer aspects of the operator space techniques in KK -theory that appeared in Rennie’s talk on Monday.

Matthias Lesch opened the session on Wednesday, followed by talks from Tyrone Crisp and Nigel Higson, who presented their recent work on representation theory and the rôle of operator spaces therein. After lunch we held an informal discussion session whose main focus was upon exploring specific examples important to the respective fields. Attendance to this session was high, lasting until dinner time.

On Thursday morning, David Blecher and Martijn Caspers gave talks, followed by Gunther Cornelissen, who gave an overview of his work on the use of operator space theory in problems related to number theory. The afternoon was free and those interested went on a walk to Oberwolfach Kirche. Friday had talks by Iain Forsyth, Francesca Arici and Adam Rennie on topics related to noncommutative geometry.

Acknowledgments: The MFO and the workshop organisers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, “US Junior Oberwolfach Fellows”. The organisers are especially grateful to the advisory scientific committee David Blecher, Alan Carey, Nigel Higson and Gilles Pisier for their active engagement prior to the workshop.

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Abstracts

Introduction to operator spaces

GILLES PISIER

The theory of “operator spaces” is usually described as starting with the 1987 PhD thesis of Z.J. Ruan who gave an “abstract” characterisation of operator spaces. Soon after, Blecher-Paulsen and Effros-Ruan independently discovered that this characterisation allows for the introduction of a duality in the category of operator spaces; they systematically developed the theory from that point on. The more recent books [1] and [2] describe its developments in the last 30 years since then.

The notion of an operator space sits inbetween that of a Banach space and that of a C^* -algebra. They could also be called “noncommutative Banach spaces” (although the commutative case should also be included here) or else “quantum Banach spaces” (although the term “quantum” seems already overused).

An operator space is simply a closed subspace $E \subset B(H)$ of the space $B(H)$ of all bounded operators on a Hilbert space H . This definition is slightly disconcerting: every Banach space E admits (for a suitable H) an isometric copy $\tilde{E} \subset B(H)$, therefore all Banach spaces can appear as operator spaces. But the novelty is in the *morphisms* (and the isomorphisms) which are not those of the category of Banach spaces. Instead of bounded linear maps, we use as morphisms the *completely bounded* (in short c.b.) maps, which appeared as a powerful tool in the early 1980s but were already implicit in the pioneering work of Stinespring (1955) and Arveson (1969) on completely positive maps.

The underlying idea is the following. Given two operator spaces

$$E_1 \subset B(H_1), \quad E_2 \subset B(H_2),$$

we want morphisms which *respect* the realisations of the Banach spaces E_1 and E_2 as operator spaces. For instance, if there exists a representation $\pi: B(H_1) \rightarrow B(H_2)$ (i.e. we have $\pi(xy^*) = \pi(x)\pi(y)^*$ and $\pi(1) = 1$, whence $\|\pi\| = 1$) such that $\pi(E_1) \subset E_2$, then the “restriction” $\pi|_{E_1}: E_1 \rightarrow E_2$ must clearly be accepted among morphisms, whence a first type. Of course, the drawback is that this class of morphisms does not form a vector space. Yet there is a second natural type of morphism that can correct this defect: suppose we are given two bounded operators $a: H_1 \rightarrow H_2$ and $b: H_1 \rightarrow H_2$ and consider the mapping $M_{ab}: B(H_1) \rightarrow B(H_2)$ given by $M_{ab}x = axb^*$. Then once again, if $M_{ab}(E_1) \subset E_2$, it is natural to accept the restriction of M_{ab} to E_1 as a morphism.

Completely bounded maps can be described as compositions of a morphism of the first type followed by one of the second type. We will denote by $CB(E, F)$ the set (now a vector space) of all such maps from E into F . It turns out it can be equipped with a norm, the c.b.-norm, with which it becomes a Banach space in the following way. Let $E \subset B(H)$ an operator space; we denote by $M_n(E)$ the

space of $n \times n$ matrices with coefficients in E , equipped with the norm

$$\forall a = (a_{ij}) \in M_n(E),$$

$$(1) \quad \|a\|_{M_n(E)} = \sup \left\{ \left(\sum_i \left\| \sum_j a_{ij} h_j \right\|^2 \right)^{\frac{1}{2}} \mid h_j \in H \sum \|h_j\|^2 \leq 1 \right\}.$$

In other words, we view the matrix a as acting on $H \oplus \cdots \oplus H$ and we compute its usual norm.

Let $E \subset B(H)$ and $F \subset B(K)$ be two operator spaces. A linear map $u: E \rightarrow F$ is completely bounded in the above sense if and only if the mappings $u_n: M_n(E) \rightarrow M_n(F)$ defined by $u_n((a_{ij})) = u((a_{ij}))$ are uniformly bounded in the usual sense for the norm defined in (1). Moreover if we define $\|u\|_{cb} = \sup_{n \geq 1} \|u_n\|$ then we have

$$\|u\|_{cb} = \min \|a\| \|b\|$$

where the infimum is over all π, a, b that yield a factorisation as above.

The completely positive maps correspond to the case when $a = b$. Thus if $F = B(K)$ then the polarisation identity shows that any c.b. map is a linear combination of four completely positive ones.

In our presentation, we described Arveson's version of the Hahn-Banach theorem for c.b. maps, the minimal tensor product and various examples related to the C^* -algebras (full or reduced) of the free group. We explained why, in certain cases, the structure of the operator space *linearly* spanned by the unitary generators of a C^* -algebra and the unit carries important information about the algebra itself, such as exactness, residual finite-dimensionality, or the coincidence of certain C^* -tensor products.

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Operator modules and their tensor products; positivity in operator algebras

DAVID P. BLECHER

This is an extended abstract describing my contribution to the workshop and my experience and impressions of the workshop and the Institute.

Description of the two one-hour talks I gave: The first talk I gave was titled *Operator modules and their tensor products*. It was an introductory one-hour talk on the first day. The slides for the talk may be found at the address <http://www.math.uh.edu/~dblecher/mfo1.pdf> and material for this lecture was taken mostly from our works [2, 3, 4].

It began with a survey of operator space tensor products, then discussed operator algebras and operator modules. We showed that the most important examples

of modules over C^* -algebras are operator modules and focused on the operator space view of C^* -modules. Then we surveyed the module Haagerup tensor product and its properties, applying this to C^* -modules and their generalisation over non-self-adjoint operator algebras. A main point is that the operator space/module Haagerup tensor product approach is supposed to allow one to treat theories involving C^* -modules much more like ring theory in pure algebra. For example there is a ‘calculus’ of algebraic formulae involving the module Haagerup tensor product that is very useful and mostly requires the operator space setting to even make sense. Thus a sub-theme of the talk was to illustrate why operator spaces are necessary.

These ideas are currently being used in the spectacular work of Mesland [12], Kaad, Higson and their collaborators (see e.g. [11, 10]) and others in noncommutative geometry (the works just cited are just examples, there are many more in the current literature).

The second talk was on new research, entitled *Recent advances in operator algebras: positivity, approximate identities and conditional expectations*. Its subtitle was *The quest for positivity in (non-self-adjoint) operator algebras*.

With Charles Read we have introduced and studied a new notion of (real) positivity in operator algebras, with an eye to extending/generalizing certain C^* -algebraic results and theories to more general algebras [6, 7]. As motivation, note that the completely real positive maps on C^* -algebras or operator systems are precisely the completely positive maps in the usual sense [1]. However, with real positivity one may develop a useful order theory for more general spaces and algebras.

As another motivation note that operator algebras, unlike C^* -algebras, need not have approximate identities and it is often important (for example in many of the applications to noncommutative geometry mentioned in the last paragraph) to know when they do. We showed that the existence of contractive approximate identities is explainable precisely in terms of our new positivity. We have continued this work together with Read, and also with Matthew Neal and Narutaka Ozawa (see e.g. [8, 9]).

Simultaneously, we are developing applications, for example to noncommutative topology (e.g. noncommutative Urysohn and Tietze for general operator algebras), noncommutative peak sets and related noncommutative function theory, lifting problems, peak interpolation, comparison theory, conditional expectations, approximate identities and to new relations between an operator algebra and the C^* -algebra it generates (see the papers cited here and references therein). We described some recent aspects of this work and then focused on some current applications to conditional expectations ([5] and work in progress).

Experience and impressions of the workshop and the Institute: The workshop I attended was a ‘courting’ of two fields or communities, namely operator spaces and noncommutative geometry. The conference organisers are extremely strong young mathematicians who in their own research are all doing very exciting and totally new things in noncommutative geometry, using operator spaces deeply

and creatively in their approach to KK -theory. They did a great job in bringing together experts in the two areas. Their personal warmth and enthusiasm set a very productive, encouraging and positive tone to the conference. Indeed, to some extent it felt like a family; there was real family bonding going on throughout the week. I personally felt sad to leave at the end—a feeling I have not felt for decades at a conference.

The organisers were careful to ensure that the introductory talks were appropriate. For example, they discussed the contents of my introductory talk at length and in great detail beforehand, which I really appreciated and found improved it greatly. I had not met most of the participants before. Indeed, I had not been exposed much to noncommutative geometry (NCG) before, so the conference was a great learning experience both consciously and subconsciously. The speakers were very careful to speak in a way that would be accessible to those from the other subject. I noticed and was very appreciative that the speakers were able to transmit a huge amount of the ‘culture’ of their subject; this is very hard to get from books or papers.

Indeed, there were many fruitful discussions throughout the week on the topics of the conference. The most immediately valuable came in the form of questions that have arisen in people’s work, some of which I was able to solve during the time of the conference. Potential collaborations were initiated: with Matthew Kennedy on noncommutative Choquet theory; Tyrone Crisp and Nigel Higson, on questions like the exactness of the module Haagerup tensor product over a C^* -algebra and on when the flip map on a module Haagerup tensor product of commutative algebras is completely bounded; and with Bram Mesland and Jens Kaad on involutive operator algebras and modules. I am continuing to work on some of these questions in the weeks after the conference with these new collaborators and am making progress.

I also wanted to express my regard and good impressions of the beautiful Oberwolfach Institute (MFO), its fantastic staff and the particular workshop I attended. My room was extremely comfortable and quiet; the food and service were excellent. I also enjoyed the interaction with the other groups who were there, over meals and elsewhere. On the Wednesday I went on the lovely hike to San Roman with another group (not my own). I made several good contacts amongst this friendly and courteous group and discussed various mathematical professional matters, which was very helpful. Similar comments apply to our group hike on Thursday. It was a great bonding time which initiated several mathematical discussions (for example discussions with Jens Kaad and Simon Brain about, respectively, involutive operator algebras, Tomita-Takesaki theory and noncommutative L^p -spaces).

Most of all I was struck by the extremely high level of mathematics and the serious, rarified and stimulating atmosphere throughout the Institute, suffused with the love of mathematics. My graduate student Alex Bearden came to the conference too, which was a great experience for him both mathematically (he got several good suggestions and leads) and in terms of making professional contacts.

He obtained some financial help from the MFO, for which he was very grateful, since he would not otherwise have been able to attend.

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 C^* -envelopes of operator systems

MATTHEW KENNEDY

Arveson conjectured in [1] the existence of an operator system X of a unique minimal C^* -algebraic extension $C_e^*(X)$, called the C^* -envelope of X . Hamana proved this conjecture using the existence of the injective envelope of X . Specifically, he realized that Arveson’s conjecture was equivalent to the statement that every operator system has a minimal C^* -algebraic injective extension. Below we briefly summarize these facts.

Recall that an operator system is a unital self-adjoint subspace of a C^* -algebra. Let X and Y be operator systems and let $\phi : X \rightarrow Y$ be a linear map. The map ϕ is positive if $\phi(x) \geq 0$ whenever $x \geq 0$. For $n \geq 1$, define $\phi_n : M_n \otimes X \rightarrow M_n \otimes Y$ by $\phi_n := \text{id}_n \otimes \phi$. Then ϕ is completely positive if each ϕ_n is positive. Similarly, ϕ is completely isometric if each ϕ_n is isometric.

Definition 1. *An operator system I is injective if for every pair of operator systems X, Y with a unital completely isometric map $\iota : X \rightarrow Y$ and a completely positive map $X \rightarrow I$, there is a completely positive map $\psi : Y \rightarrow I$ such that $\phi = \psi \circ \iota$.*

Consider Definition 1 where $X \subset Y$ and the embedding ι is simply the inclusion map. In this case, ψ is a completely positive extension of ϕ with the property that its range is also a subset of I .

Definition 2. For an operator system X , an extension of X is an operator system Y with a unital completely isometric map $\iota : X \rightarrow Y$. The extension (Y, ι) is essential if for every operator system Z , a unital completely positive map $\phi : Y \rightarrow Z$ is completely isometric whenever the restriction $\phi|_X$ is completely isometric.

Theorem 1 (Hamana [2]). Every operator system X has a unique minimal injective extension $I(X)$ called the injective envelope of X . Moreover, $I(X)$ is simultaneously the unique maximal essential extension of X . Here uniqueness is up to isomorphism.

A result of Choi and Effros showed that every injective operator system is isomorphic (as an operator system) to a C^* -algebra. In particular, this implies that every operator system has a minimal C^* -algebraic injective extension.

Theorem 2 (Hamana [2]). Every operator system X has a unique (up to $*$ -isomorphism) minimal C^* -algebraic extension $C_e^*(X)$ called the C^* -envelope of X that is characterized by the property that for any C^* -algebra A and any unital completely isometric map $\phi : X \rightarrow A$, there is a surjective $*$ -homomorphism $C^*(\phi(X)) \rightarrow C_e^*(X)$.

Proof. Let X be an operator system. Since every essential extension of X embeds completely isometrically into the injective envelope, $C_e^*(X)$ is precisely the C^* -algebra generated by the embedding of X in its injective envelope. \square

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Completely positive maps in zero-error quantum information theory

TATIANA SHULMAN

(joint work with M. Shirokov)

The effect of superactivation of quantum channel capacities is one of the main recent discoveries in quantum information theory. It means that the particular capacity of tensor product of two quantum channels may be positive despite the same capacity of each of these channels being zero.

This effect was originally observed by G. Smith and J. Yard, who gave examples of two channels Φ and Ψ with zero quantum capacity such that the channel $\Phi \otimes \Psi$ has positive quantum capacity.

The same phenomenon for (one-shot and asymptotic) zero-error classical capacities was established by T. Cubitt, J. Chen and W. A. Harrow in [1]. Simultaneously and independently, R. Duan presented a simple example of two low-dimensional channels demonstrating superactivation of one-shot zero-error classical capacities [5].

The extreme form of superactivation of zero-error capacities was observed by T. Cubitt and G. Smith in [2], who proved existence of two channels Φ and Ψ with zero (asymptotic) zero-error classical capacity such that the channel $\Phi \otimes \Psi$ has positive zero-error quantum capacity.

In this talk we presented explicit examples of low-dimensional quantum channels which demonstrate different forms of superactivation of one-shot zero-error capacities, in particular the extreme form of superactivation mentioned above. Although the existence of such channels in sufficiently high dimensions follows from the results in [2], their explicit low-dimensional examples were unknown.

We also discussed relations between the superactivation of one-shot zero-error capacities and the results on transitive and reflexive subspaces of operators [4, 6]. It is essential that these results can be used for analysis of superactivation effects for infinite dimensional quantum channels.

The results concerning transitive and reflexive subspaces of operators directly show that the superactivation of one-shot zero-error classical capacities does not hold for two channels Φ and Ψ if one of them, say Φ , is of certain type and the second one Ψ is arbitrary. The result in this direction was obtained recently by J. Park and S. Lee in [7] for the class of qubit channels. Our approach [8] gives a very simple proof of this result and makes it possible to prove the analogous assertion for several other classes of channels.

I would like to mention the fruitful discussions on topics related to my talk that arose during my stay at MFO.

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Hilbert C^* -modules for Σ^* -algebras

ALEX BEARDEN

A C^* -subalgebra of $B(H)$ is called a Σ^* -algebra if it is closed under limits of weak*-convergent sequences. These algebras were first studied by Davies in [4] and a similar class of algebras was studied in detail by Pedersen in several papers (see [7, Section 4.5] for results and more references).

Every von Neumann algebra is evidently a Σ^* -algebra. The basic commutative example is $\text{Bor}(X) \subseteq B(\ell^2(X))$, the space of bounded Borel-measurable functions on a locally compact Hausdorff space X with its canonical representation. For a noncommutative example, take the ideal of operators in $B(H)$ with separable range, i.e. the Σ^* -algebra generated by the compact operators.

The goal of our work is to find a theory for modules over Σ^* -algebras analogous to the well known C^* -module theory and the lesser known theory of W^* -modules (see [6] for the former and [3, Section 8.5] for the latter).

For motivation, recall that a C^* -module Y over a W^* -algebra M is called a W^* -module if it is self-dual, i.e. if every bounded M -module map $Y \rightarrow M$ is of the form $\langle y | \cdot \rangle$ for some $y \in Y$. There are several beautiful characterisations of W^* -modules (e.g. they are the C^* -modules with preduals) but the characterisation most useful for our purposes is the following: a C^* -module Y over a von Neumann algebra $M \subseteq B(H)$ is a W^* -module if and only if the canonical copy of Y in $B(H, Y \otimes_M H)$ is weak*-closed. Thus we arrive at our definition.

Definition 1. *A C^* -module \mathfrak{X} over a Σ^* -algebra $\mathfrak{B} \subseteq B(H)$ is a Σ^* -module if its canonical copy in $B(H, \mathfrak{X} \otimes_{\mathfrak{B}} H)$ is weak*-sequentially closed.*

This definition allows all of the natural analogues of the basic C^* - and W^* -module results to come through. Here is a sample.

Theorem 1. *If \mathfrak{X} is a Σ^* -module over $\mathfrak{B} \subseteq B(H)$, then $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X}) \subseteq B(\mathfrak{X} \otimes_{\mathfrak{B}} H)$ is a Σ^* -algebra, and \mathfrak{X} is a left Σ^* -module over the Σ^* -algebra in $B(\mathfrak{X} \otimes_{\mathfrak{B}} H)$ generated by $\mathbb{K}_{\mathfrak{B}}(\mathfrak{X})$.*

Similarly, one may define Morita equivalence in a way analogous to that of both C^* - and W^* -Morita equivalence and obtain analogues of the basic results from those theories.

In C^* -module theory, one often has to make additional countability assumptions (e.g. separability of the algebra or the existence of a countable set of generators for the module) to obtain interesting results. There are analogous assumptions one can make in the context of Σ^* -algebras that yield interesting results. For example, if $\{x_n\}$ is a countable set in a C^* -module \mathfrak{X} over a Σ^* -algebra \mathfrak{B} such that the weak*-sequentially closed subspace of $B(H, \mathfrak{X} \otimes_{\mathfrak{B}} H)$ generated by $\{x_n b : b \in \mathfrak{B}, n \in \mathbb{N}\}$ contains \mathfrak{X} , then \mathfrak{X} is a Σ^* -module if and only if it is self-dual.

This concludes the material covered in my talk during the workshop. In relation to this project, I would like to thank Jens Kaad for pointing out a missing piece in the Σ^* -Morita equivalence theory, Tyrone Crisp for a helpful general question

about Σ^* -algebras, and Matthew Kennedy for several helpful suggestions and for leading me to the related work in Robert Hart's thesis [5].

Beyond this project, the organised lectures in the workshop were quite informative and inspiring. I found it astonishing to learn the very *different* ways that operator space theory has found applications in other areas of noncommutative analysis. For me, the result of the organised portion of the workshop was to open my eyes to the likelihood of further interactions of operator space theory with noncommutative geometry. This realisation will certainly affect the direction of my studies and hopefully lead to more research into these areas and their connections.

Another mathematical highlight of the week for me was discussing aspects of noncommutative Choquet theory with Matthew Kennedy and my advisor David Blecher. I am hopeful and excited for these discussions to continue and lead to collaboration and future projects in this beautiful area of mathematics.

Overall, my week in Oberwolfach was inspiring and extremely helpful mathematically. I would like to offer my sincere thanks to the organisers for inviting me and allowing me to present my work, to the NSF and the University of Houston mathematics department for providing me with the funds to make this trip possible, and to the administration and staff at the MFO for making the stay extremely pleasant and conducive to mathematical thinking.

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The resolvent expansion for second order elliptic differential multipliers

MATTHIAS LESCH

(joint work with H. Moscovici)

In my talk I reported on some technical aspects of the recent preprint [8]. A talk of the same title had already been given at the MFO conference no 1525, see [9] for a detailed report. I will not duplicate [9] here. I will rather give a brief informal summary of the content of my talk.

Let (M, g_0) be a closed oriented surface and denote by Δ the Laplacian on functions. Then for any smooth function $f \in C^\infty(M)$ there is an asymptotic expansion

$$\mathrm{Tr}(f \cdot e^{-t\Delta}) \sim_{t \searrow 0} \sum_{j=0}^{\infty} a_{2j}(f, P) \cdot t^{j - \dim M/2},$$

where $a_{2j}(f, P) = \int_M f(x) \cdot \tilde{a}_{2j}(x) d \mathrm{vol}(x)$ and \tilde{a}_{2j} is a smooth function depending only on the jets of the metric at x .

Under a conformal change of metric $g = e^{-h}g_0$, h smooth real valued, the new Laplacian becomes $\Delta_g = e^h \Delta_{g_0}$. Furthermore, one has the following explicit formula for the second heat coefficient:

$$a_2(f, \Delta_g) = \frac{1}{24\pi} \int_M f(\Delta_{g_0} h + 2K_{g_0}) d \mathrm{vol}_{g_0}.$$

As a consequence one obtains the *Polyakov formula* for the zeta-regularized determinant:

$$\log \det_{\zeta} \Delta_g = \log \det_{\zeta} \Delta_{g_0} + \frac{1}{24\pi} \int_M h(\Delta_{g_0} h + 2K_{g_0}) d \mathrm{vol}_{g_0} + \log \mathrm{vol} g_h + C.$$

Osgood, Phillips and Sarnak [10] used this to prove that for fixed volume within a conformal class $\det_{\zeta} \Delta_g$ takes its extremum at the constant curvature metric.

In [3], Connes and Moscovici extended the results outlined above to the conformal Laplacian on the noncommutative torus. Their method made heavy use of symbolic computer calculations which were verified independently by Fathizadeh and Khalkhali [6]. In the arXiv version of *loc. cit.* the printout of the expanded version of the second heat coefficient, obtained by symbolic calculations, fills about 20 pages.

In our paper [8] we achieve the following modest improvements of the previous results mentioned above.

We proved a complete asymptotic expansion of the heat respective resolvent trace of Laplace type operators on vector bundles over the noncommutative torus (Heisenberg modules). Moreover we computed the second heat coefficient *without* computer assistance. The second coefficient contains significant geometric information, as in the case of classical Riemann surfaces. As discovered in [3] the noncommutativity of the symbol exhibited a completely new phenomenon, namely the appearance of universal entire functions in the expression for the second heat coefficients. This has no counterpart in the commutative situation. The main technical device which we developed is a pseudodifferential calculus adapted to twisted C^* -dynamical systems, extending the well known calculi due to Connes [2] and Baaĵ [1].

In my talk I focused on the global pseudodifferential calculus on the ordinary torus to motivate and explain the corresponding calculi on C^* -dynamical systems.

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Frobenius reciprocity and the Haagerup tensor product

TYRONE CRISP

(joint work with P. Clare, N. Higson)

One of the cornerstones of the representation theory of finite groups is the Frobenius reciprocity theorem [5], asserting that induction and restriction of representations are adjoint functors:

$$(1) \quad \mathrm{Hom}_G(\mathrm{Ind}_H^G X, Y) \cong \mathrm{Hom}_H(X, \mathrm{Res}_H^G Y)$$

whenever Y is a representation of a group G , and X is a representation of a subgroup H . In view of the many important applications of this relation – Frobenius’s computation of the character table of the symmetric groups being a notable early example – one would naturally like to extend (1) to more general representation-theoretic settings.

One setting familiar to many at this meeting is that of unitary representations of locally compact groups on Hilbert spaces. In this category the functors of induction and restriction are not adjoint to one another and, indeed, neither of these functors admits any adjoint at all. In this talk, I explained how enlarging the category to include representations on more general *operator spaces* sheds new light on this and other adjoint functor problems for representations of groups and C^* -algebras.

To set the problem in a fairly general context, let A and B be C^* -algebras and let F be a C^* -correspondence from A to B (i.e. a Banach A - B bimodule whose

norm is given by a positive-definite B -valued inner product). Tensor product with F , in the sense of Rieffel [7], defines a functor

$$(2) \quad F \otimes_B^{\text{Rieffel}} : * \text{Rep}(B) \longrightarrow * \text{Rep}(A)$$

from the category of $*$ -representations of B to the category of $*$ -representations of A . Unitary induction of group representations, for example, is a functor of this kind. In contrast to purely algebraic settings, where tensor product functors always have right adjoints, the functor (2) need not possess any adjoint.

Blecher [1] has shown that C^* -correspondences are examples of a more general kind of bimodule, namely operator bimodules, and that the functor (2) extends to a functor

$$(3) \quad F \otimes_B^{\text{Haagerup}} : \text{OpMod}(B) \longrightarrow \text{OpMod}(A)$$

between the categories of operator modules over A and B , using the *Haagerup tensor product* [6]. Now, the bimodule F has an operator-theoretic adjoint F^* , defined by representing F concretely as a space of Hilbert space operators and applying the usual adjoint operation. Haagerup tensor product with this operator B - A -bimodule gives a functor

$$(4) \quad F^* \otimes_A^{\text{Haagerup}} : \text{OpMod}(A) \longrightarrow \text{OpMod}(B)$$

which turns out, in many cases of interest, to be an adjoint functor in the categorical sense:

Theorem (cf. [4]). *If the functor (3) has a left adjoint, then the left adjoint is (4). This happens if and only if the action of A on F is by B -compact operators.*

Returning to group representations:

Corollary. *Let H be a closed subgroup of a locally compact group G . The unitary induction functor*

$$\text{Ind}_H^G : \text{OpMod}(C^*(H)) \longrightarrow \text{OpMod}(C^*(G))$$

admits a left adjoint if and only if G/H is compact.

In some cases, the operator bimodule F^* is itself a C^* -correspondence – i.e. its operator space structure comes from an A -valued inner product. In this case, the functor (4) restricts to a functor on $*$ -representations

$$(5) \quad F^* \otimes_A^{\text{Rieffel}} : * \text{Rep}(A) \longrightarrow * \text{Rep}(B)$$

which is automatically adjoint to (2). This situation arises, for example, in the representation theory of real reductive groups:

Theorem ([2], [3]). *Let G be a real reductive group, and let L be a Levi subgroup of G . The functor of parabolic induction, from the unitary representations of L to the unitary representations of G , admits a two-sided adjoint.*

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Operator spaces and the Plancherel formula

NIGEL HIGSON

(joint work with P. Clare and T. Crisp)

Let G be a real reductive group and let $P = LN$ be a parabolic subgroup (for instance let G be $SL(2, \mathbb{R})$, let N be the subgroup of unipotent upper triangular matrices and let L be the diagonal matrices). In his thesis [3], Pierre Clare introduced a Hilbert C^* -correspondence from $C_r^*(G)$ to $C_r^*(L)$ that implements parabolic induction from L to G and, in doing so, raised the possibility of approaching the representation theory of G , as pioneered by Harish-Chandra and others, from the point of view of noncommutative geometry.

The first steps in this direction were taken by Pierre Clare, Tyrone Crisp and myself in [5] and [6], where we developed an analogue of Bernstein’s second adjoint theorem [1] for categories of tempered representations of real reductive groups. One unexpected aspect of this work was the appearance of operator spaces in the proofs. Operator spaces reappeared in further work with Crisp [7] and they arose yet again in unpublished work with Clare [4]. It therefore seems worthwhile to investigate potential roles for operator space theory in representation theory and noncommutative geometry more closely.

In connection with this, a first question that one might ask is this: does $L^2(G)$ carry a natural operator space structure, or matrix-normed space structure? The abstract Plancherel theorem (see for example [11, Chapter 14]) suggests a positive answer. The explicit Plancherel theorem [11, Chapter 13] of Harish-Chandra then prompts another question: can this structure be related to geometric structure (and, in this way, can we find a new route to the Plancherel formula)?

In my joint work with Clare and Crisp, the operator space *adjoint* bimodule $C_r^*(N \backslash G)$ of Clare’s correspondence $C_r^*(G/N)$ is given the structure of a Hilbert C^* -correspondence from $C_r^*(L)$ to $C_r^*(G)$. This is done from a spectral point of view (using representation theory), and it is shown that the correspondence $C_r^*(G/N) \otimes_{C_r^*(L)} C_r^*(N \backslash G)$ is very closely related to $L^2(G)$. As an operator bimodule, the tensor product agrees with the Haagerup tensor product [2], and so

can be accessed geometrically (as opposed to representation-theoretically). We hope that this might be a first step towards approaching the Plancherel formula through noncommutative geometry.

An interesting K -theoretic variation on this arises in my work with Clare in which, among other things, we investigate extensions of computations that Lafforgue made in [8] to connect Harish-Chandra's parametrisation of the discrete series with the Baum-Connes isomorphism. In the course of calculating a Kasparov product it becomes necessary to understand the Hilbert space $C_r^*(N \backslash G) \otimes_{C_r^*(G)} L^2(G/K)$ and relatives, and to construct *connections* on these tensor products in the sense of Skandalis [10]. Once again, the tensor product can be calculated quite easily from the spectral point of view, but the construction of connections requires a more direct, geometric perspective. For this purpose it is very helpful to note again that the tensor product is isometric with a Haagerup operator space tensor product, which is at least in principle computable in geometric terms.

Even though the context is rather different, there are interesting parallels between the above K -theory computation and the work of Mesland [9] and collaborators on the construction of unbounded representatives of the Kasparov product, where operator spaces also play an important role. This became clear during the workshop. Other conversations during the workshop, especially with David Blecher concerning exactness properties of the Haagerup tensor product and with Adam Rennie on the work of Watatani et al. [12] were extremely helpful and have already stimulated new work.

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Hecke von Neumann algebras, operator spaces and absence of Cartan subalgebras

MARTIJN CASPERS

Hecke algebras arise as q -deformations of Coxeter groups. As a $*$ -algebra they are generated by self-adjoint operators T_s , with s in some generating set S , that satisfy a relation $(T_s T_t)^{m(s,t)} = 1$ as well as the Hecke relation

$$(T_s - q)(T_s + 1) = 0$$

for a deformation parameter $q > 0$. Hecke algebras were used by Jones in his knot invariants and play important roles in representation theory. Taking a GNS-representation, these algebras generate a so-called Hecke–von Neumann algebra, say \mathcal{M}_q . These were studied earlier on by Dymara [3] and later by Davis–Dymara–Januszkiewicz–Okun [4]. In the right-angled case it was proved by Garncarek [5] that \mathcal{M}_q is a factor for a certain range of $q \in [\rho, \rho^{-1}]$ and otherwise it is a sum of a factor and \mathbb{C} . Outside the right-angled case this question is still open. We take Garncarek’s result as a starting point and assume that \mathcal{M}_q is the Hecke–von Neumann algebra of a right-angled Coxeter system.

The aim of this talk was to explain how operator spaces can be used to study the von Neumann algebra \mathcal{M}_q . In particular we showed that these algebras are non-injective, have the completely bounded approximation property (CBAP) and have the property of being strongly solid algebras. Recall that a von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$ is injective if it is the image of a conditional expectation $B(\mathcal{H}) \rightarrow \mathcal{M}$. It follows from Connes’ characterisation of injectivity that an injective II_1 -factor \mathcal{M} with trace τ has the property that for all $x_i \in \mathcal{M}$ we have

$$\left\| \sum_i x_i \otimes \bar{x}_i \right\|_{\mathcal{M} \otimes \mathcal{M}^{op}} \geq \tau \left(\sum_i x_i^* x_i \right).$$

We can violate this inequality for \mathcal{M}_q as it is possible to identify Σ_d , the linear span of T_w with w a word with letters in S of length d , with a finite sum of Haagerup tensor products of row and column Hilbert spaces

$$\mathcal{H}_c \otimes_h \mathcal{H}_c \otimes_h \mathbb{C} \otimes_h \mathcal{H}_r \otimes_h \mathcal{H}_r.$$

Details can be found in [2]. Such techniques have previously been found in different contexts in the study of free Araki-Woods factors or q -Gaussian algebras; see for instance [7].

Using explicit Stinespring decompositions for radial multipliers and word length cut-downs, we showed that \mathcal{M}_q has the CBAP. Using the Gromov boundary action of the Coxeter group generated by S we showed that \mathcal{M}_q is strongly solid.

Recall that a Cartan subalgebra of a von Neumann algebra is a maximal Abelian subalgebra whose normaliser generates the von Neumann algebra itself. Cartan subalgebras typically arise in crossed products of free ergodic probability measure-preserving actions of discrete groups on a probability measure space. Their study is central in classification programs for von Neumann algebras (for instance by Popa–Vaes). On the other hand the group von Neumann algebra of a free group does not have a Cartan subalgebra, as was shown by Voiculescu. Moreover it is strongly solid, as was shown by Ozawa–Popa. Later on, other examples of von Neumann algebras with no Cartan subalgebra have been found, see for instance [1] or [6]. The conclusion of our results above is that \mathcal{M}_q also does not have a Cartan subalgebra.

Acknowledgments. I thank the organisers of this workshop. Fruitful discussions, especially with Tyrone Crisp, Matthew Kennedy and Gilles Pisier, helped me to put some of the material above into a wider context and may lead to follow-up projects.

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Reconstruction problems in number theory in the light of C^* -algebras

GUNTHER CORNELISSEN

(joint work with Xin Li, M. Marcolli, V. Karemaker)

A global field K (a number field or a field of functions of a curve over a finite field) has several associated invariants; there is often a tension between how well-understood or computable they are and the question as to whether or not they uniquely determine K . Invariants that we understand, such as zeta functions or Abelianised Galois groups, often do not determine K ; invariants that do determine K , such as the absolute Galois group, remain mysterious.

We have described a dynamical system $I_K \curvearrowright X_K$ in which a monoid (the monoid of ideals of the ring of integers or of effective divisors) acts on a topological space X_K derived from the Abelianised Galois group, through the reciprocity map from class field theory. The main result [4, 8, 7] says that for two fields K and L , isomorphism is the same as topological conjugacy (or orbit equivalence) of the

two associated dynamical systems, which in turn is equivalent to the existence of a group isomorphism of groups of Dirichlet characters for K and L under which the L -series agree; this is equivalent to the existence of a (complex) algebra isomorphism of algebraic crossed product algebras $C(X_K) \rtimes^{\text{alg}} I_K \cong C(X_L) \rtimes^{\text{alg}} I_L$ which maps $C(X_K)$ to $C(X_L)$. The result has an analogue in Riemannian geometry, where an isometry can be detected by agreement of parametrised zeta-functions $\text{tr}(a\Delta^s)$ for smooth functions a and the Laplace-Beltrami-operator Δ [2].

The result came out of operator algebra theory. More specifically, one considers Bost-Connes-style “quantum statistical mechanical” (QSM-) systems [1] as introduced by Ha and Paugam [11], where one considers the reduced crossed product C^* -algebra $A_K := C(X_K) \rtimes I_K$ with a one-parameter group of automorphisms $\sigma_K: \mathbf{R} \rightarrow \text{Aut}(A_K)$ such that $t \in \mathbf{R}$ scales monoid-like elements \mathbf{n} by $N(\mathbf{n})^{it}$, where $N(\cdot)$ is the absolute norm map. Here, one can show [5] that two fields K and L have the same zeta function if and only if their corresponding systems are isomorphic.

This leads to general questions about reconstructing dynamical systems from associated operator algebras. It seems that non-involutive algebras *or* QSM-systems do the job. For example, if \mathbf{Z} acts on a topological spaces X , then the dynamics can be reconstructed from the non-involutive algebra $C(X) \rtimes \mathbf{Z}_+$ (using only positive powers of a group generator). For groups with more generators acting on connected spaces, one retrieves the action up to “piecewise conjugacy” [9]. (Observe: the systems above have countable rank acting on a totally disconnected space.) As a second example, if G is a finite multi-graph with first Betti number $b \geq 2$, universal covering tree T and fundamental group Γ (a free group of rank b), then $A_b := C(\partial T) \rtimes \Gamma$ depends only on b , but, based on Busemann functions, one can construct a QSM-system for A_b which does determine the graph uniquely [6].

Let me formulate some questions:

- (1) Is it possible to reconstruct the dynamics $I_K \curvearrowright X_K$ from the non-involutive Banach algebra which is the closure of $C(X_K) \rtimes^{\text{alg}} I_K$ in its natural representation as bounded operators on Hilbert space?
- (2) Is it possible to prove this without requiring that the MASA $C(X_K)$ is respected?
- (3) Is it possible to reconstruct K from A_K , without reference to σ_K ?

For more general dynamical systems where a (commutative) monoid I acts on a topological space X , the dynamics can sometimes be reconstructed from the non-involutive crossed product algebra $C(X) \rtimes I_+$, almost never from its C^* -envelope $C_{\text{env}}^*(C(X) \rtimes I_+)$, but sometimes there exists a suitable QSM-structure σ so that $(C_{\text{env}}^*(C(X) \rtimes I_+), \sigma)$ does the job. This prompts very general questions for pairs of unital operator spaces A_1, A_2 :

- (4) Do there exist suitable one-parameter groups of automorphisms $\sigma_i: \mathbf{R} \rightarrow \text{Aut}(C_{\text{env}}^*(A_i))$ such that $A_1 \cong A_2 \iff (C_{\text{env}}^*(A_1), \sigma_1) \cong (C_{\text{env}}^*(A_2), \sigma_2)$?
- (5) Do there exist suitable actions of \mathbf{R} on A_i such that $A_1 \cong A_2 \iff C_{\text{env}}^*(A_1) \rtimes \mathbf{R}_+ \cong C_{\text{env}}^*(A_2) \rtimes \mathbf{R}_+$.

One may find non-isomorphic operator spaces with isomorphic C^* -envelopes, for example various (non-involutive) limits of matrix algebras all of whose C^* -envelopes are isomorphic to the CAR-algebra; certain non-involutive graph algebras whose enveloping algebras are Cuntz-Krieger algebras, or the Toeplitz algebras T_q for $0 < q < 1$ with one generator. It would be interesting to answer question (4) positively for some of these examples by constructing suitable σ .

Finally, the “Langlands philosophy” roughly says that certain Galois representations in $\mathrm{GL}_n(\mathbf{C})$ should correspond to certain modules over the Hecke algebra $\mathcal{H}_n(K)$, the convolution algebra of compactly supported smooth functions on the K - or K -adelic points of GL_n . Here a dichotomy is given by the observation of Karemaker [10] that $\mathcal{H}_2(K)$ is the same up to Morita equivalence for all local fields K , whereas for two number fields K, L that are Galois over \mathbf{Q} , an isomorphism of Hecke algebras *that respects the L^1 -norm* is the same as a field isomorphism [3].

At the workshop, I had various discussion about an operator space problem that came up in work on distances between spectral triples, allowing a sharpening of our understanding of the theory. I also met some previously-unknown-to-me people with whom I will discuss Question (4) in the future. Finally, the workshop allowed me to increase my understanding of the interaction between representation theory and operator spaces in relation to Hecke algebras.

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Factorisation of equivariant spectral triples

IAIN FORSYTH

(joint work with A. Rennie)

For details of the following see [4]. Suppose M is a manifold carrying a smooth, free action by a compact Lie group G . Then the manifold M has the structure of a principal G -bundle over M/G and locally there is the decomposition of manifolds $M \cong G \times (M/G)$ as G -spaces. So the manifold M “factorises” into two parts in some sense, with one part coming from the group G and one coming from the quotient space M/G .

Suppose M carries some additional information, such as that encoded by an equivariant Dirac operator over M . What would it mean for a Dirac operator to factorise? We would like to be able to somehow recover such a Dirac operator from the Dirac operator on G and a Dirac operator on M/G . Our goal is to make precise what this factorisation is, as well as to generalise this idea to the noncommutative setting. An equivariant Dirac operator over a compact manifold M defines an equivariant spectral triple for $C(M)$, so equivariant spectral triples are the natural objects of consideration.

Let A be a separable C^* -algebra carrying an action by a compact group G , and let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an equivariant spectral triple for A . This spectral triple defines a class in the equivariant KK -group $KK_G^j(A, \mathbb{C})$, where $j = 0$ (respectively $j=1$) if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is even (respectively odd) [1]. There is the Kasparov product [6]

$$KK_G^{\dim G}(A, A^G) \times KK_G^{j-\dim G}(A^G, \mathbb{C}) \rightarrow KK_G^j(A, \mathbb{C}),$$

so we say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *factorises in KK -theory* if we can construct unbounded Kasparov modules x and y defining respective classes in $KK_G^{\dim G}(A, A^G)$ and $KK_G^{j-\dim G}(A^G, \mathbb{C})$ such that $[x] \widehat{\otimes}_{A^G} [y] = [(\mathcal{A}, \mathcal{H}, \mathcal{D})]$ under the Kasparov product. We can also say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *factorises as an unbounded cycle* if it is recovered as the unbounded Kasparov product of x and y [2, 5, 8, 9].

At this point we restrict to the case that G is Abelian. To construct the unbounded Kasparov module x we require that the action of G on A satisfies the *spectral subspace assumption*, [3]. The spectral subspace assumption is satisfied if the action is free. Provided the spectral subspace assumption is satisfied, x is constructed using the spin Dirac operator on G .

To construct the cycle y , we require a representation of the Clifford algebra $\text{Cl}_{\dim G}$ on \mathcal{H} satisfying some compatibility conditions, which accounts for the shift in KK -dimension. In the case $A = C(M)$, this representation can be constructed using the fundamental vector fields of G .

To check whether factorisation in KK -theory has been achieved, we test whether Kucerovsky’s criteria are satisfied [7]. In this case this reduces to checking a positivity condition. If \mathcal{D} is an equivariant Dirac operator on a compact manifold M defining an equivariant spectral triple $(C^\infty(M), L^2(E), \mathcal{D})$, then factorisation

in KK -theory is always achieved. Factorisation of $(C^\infty(M), L^2(E), \mathcal{D})$ as an unbounded cycle is achieved up to bounded perturbation if the orbits of G are embedded isometrically in M .

The generalisation of these results to the case that G is compact and non-Abelian is a work in progress but it is expected that the results will carry through without substantial difficulty.

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Gysin sequences for principal circle bundles via Cuntz-Pimsner algebras

FRANCESCA ARICI

(joint work with S. Brain, J. Kaad, G. Landi, A. Rennie)

My talk focused on the noncommutative topology of principal circle bundles as described in [4] and later in [3].

Classically, to any principal circle bundle $\pi : P \rightarrow X$ one can associate the Gysin exact sequence, a long exact sequence in (singular) cohomology which relates the topology of the total space of the bundle to that of the base space. This exact sequence admits a version in complex topological K -theory, in the form of a six-term exact sequence. There, a central rôle is played by the Euler class of the line bundle L associated to P via the regular representation.

In the dual C^* -algebraic picture, the noncommutative counterpart of a line bundle is a self-Morita equivalence bimodule over a C^* -algebra A , that is a right full Hilbert C^* -module E_A together with an isomorphism of A with the compact endomorphisms $\mathcal{K}_A(E)$. Through a natural universal construction this data gives rise to two C^* -algebras, the Toeplitz algebra \mathcal{T}_E and the (Cuntz-)Pimsner algebra

O_E , which depend only on the isomorphism class of the pair (E, ϕ) . These algebras fit into an extension by compact operators

$$(1) \quad 0 \longrightarrow \mathcal{K}_A(F_E) \longrightarrow \mathcal{T}_E \xrightarrow{\pi} O_E \longrightarrow 0 .$$

Pimsner algebras were introduced in [12] for the more general case of an injective left action $\phi : A \rightarrow \mathcal{L}_A(E)$ and later generalised to the non-injective case in [10]. They provide a unifying framework for a range of important C^* -algebras including crossed products by the integers and Cuntz-Krieger algebras. Generalised crossed products, a related notion obtained from bi-Hilbertian bimodules in the sense of [6], were independently constructed in [1]. In the case of self-Morita equivalence bimodules, the resulting algebra agrees with the Cuntz-Pimsner algebra.

The algebra O_E can be thought of as the total space of a noncommutative principal circle bundle with base space A associated to the noncommutative line bundle E . This analogy is spelled out in [4], both in the commutative and in the quantum case, together with the explicit connections with the theory of \mathbb{Z} -graded C^* -algebras.

With a Pimsner algebra come two natural six term exact sequences in KK -theory, which relate the KK -theories of the Pimsner algebra O_E with that of the C^* -algebra A . These exact sequences are noncommutative analogues of the Gysin sequence, which, as mentioned before, in the commutative case relates the K -theories of the total space and of the base space of a circle bundle. The classical cup product with the Euler class is replaced by a Kasparov product with the identity minus the class of the self-Morita equivalence bimodule E .

An interesting class of examples arises from quantum lens spaces as circle bundles over quantum projective space, both weighted and unweighted. These were studied in [2, 4], leading to a computation of the K -theory and K -homology groups of these spaces, together with the description of explicit representatives of classes. Such computations naturally lead to the observation of a striking similarity between the Cuntz-Pimsner exact sequences and the exact sequences associated to the mapping cone of the inclusion $\iota : A \rightarrow O_E$.

While it is known that every exact sequence of C^* -algebra is equivalent to a mapping cone exact sequence [11], determining the explicit form of the isomorphism between the corresponding exact sequences in KK -theory is a highly non-trivial task. In [5] we use techniques developed in [7] to lift the unbounded representative of the extension (1) constructed in [8] to the class in $KK(M(A, O_E), A)$. This isomorphism is described in the case of bi-Hilbertian bimodules of finite Jones-Watatani index in the sense of [9] subject to some additional assumption.

Comments from Magnus Goffeng and Jens Kaad about triangulated categories and KK -equivalences gave some deeper insight on the latter problem. The paper [5] will be finalised after the workshop and its final form reflects the positive influence that the meeting had on this line of research.

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The non-commutative geometry of Cuntz-Pimsner algebras

ADAM RENNIE

My talk described a range of recent results contained in the papers [1, 3, 6]. These results all centre around constructing representatives of the defining extension of a Cuntz-Pimsner algebra O_E of suitable bimodules over an algebra A . This extension is the short exact sequence

$$(1) \quad 0 \rightarrow K \otimes A \rightarrow T_E \rightarrow O_E \rightarrow 0,$$

where K is the compact operators and T_E is the Toeplitz algebra associated to the bimodule. The class of bimodules turns out to be determined via Watatani’s notion of finite right index bi-Hilbertian bimodules, as presented in [4]. For such bimodules, subject to one additional assumption, a singular operator-valued weight $\Phi : T_E \rightarrow A$ can be constructed which vanishes on $K \otimes A \subset T_E$. This allows the construction of the Kasparov module in bounded form, as described in [6]. Imposing a further technical condition allows a more refined description of the underlying C^* -module and facilitates the construction of an unbounded representative, [3]. Finally, Arici and Rennie had both observed a striking similarity between K -theory calculations using the defining exact sequence (1) and the K -theory of

the mapping cone exact sequence

$$(2) \quad 0 \rightarrow \mathcal{S}O_E \rightarrow M(A, O_E) \rightarrow A \rightarrow 0$$

for the inclusion $\iota : A \hookrightarrow O_E$. Using a lift of the representative of the defining extension to the mapping cone, provided by [2], the relationship was made precise and explicit in [1].

The workshop produced the following benefits for this line of research. Questions and comments from Magnus Goffeng and Jens Kaad sharpened some of the results of [1]. Some open questions concerning the existence of KK -equivalences and applications to K -homology exact sequences were particularly helpful.

Another striking point was that bi-Hilbertian bimodules of finite right Watatani index also appeared in the work of Tyrone Crisp and Nigel Higson. It is very likely that these two lines of research will benefit each other, if only in matters of technique, but probably also in more conceptual ways.

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