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Mini-Workshop: Mathematical Foundations of Isogeometric Analysis

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Abstract. Isogeometric Analysis (IgA) is a new paradigm which is designed to merge two so far disjoint disciplines, namely, numerical simulations for partial differential equations (PDEs) and applied geometry. Initiated by the pioneering 2005 paper of one of us organizers (Hughes), this new concept bridges the gap between classical finite element methods and computer aided design concepts.

Traditional approaches are based on modeling complex geometries by computer aided design tools which then need to be converted to a computational mesh to allow for simulations of PDEs. This process has for decades presented a severe bottleneck in performing efficient simulations. For example, for complex fluid dynamics applications, the modeling of the surface and the mesh generation may take several weeks while the PDE simulations require only a few hours.

On the other hand, simulation methods which exactly represent geometric shapes in terms of the basis functions employed for the numerical simulations bridge the gap and allow from the beginning to eliminate geometry errors. This is accomplished by leaving traditional finite element approaches behind and employing instead more general basis functions such as B-Splines and Non-Uniform Rational B-Splines (NURBS) for the PDE simulations as well. The combined concept of Isogeometric Analysis (IgA) allows for improved convergence and smoothness properties of the PDE solutions and dramatically faster overall simulations.

In the last few years, this new paradigm has revolutionized the engineering communities and triggered an enormous amount of simulations and publications mainly in this field. However, there are several profound theoretical issues which have not been well understood and which are currently investigated by researchers in Numerical Analysis, Approximation Theory and Applied Geometry.

Mathematics Subject Classification (2010): 65xx, 41xx, 51xx.

Introduction by the Organisers

Isogeometric Analysis (IgA) is a new paradigm which has been mainly established in the engineering sciences over the past eleven years. It merges two so far disjoint disciplines, namely, numerical simulations for partial differential equations (PDEs) and applied geometry. This new concept was initiated by the pioneering 2005 paper of one of us organizers (Hughes) and bridges the gap between classical finite element methods and computer aided geometric design concepts.

Traditional approaches to solve PDEs on complicated domains are based on modeling complex geometries by computer aided geometric design tools. These need to be converted to a computational mesh to allow for simulations of PDEs. This process has for many decades presented a severe bottleneck in performing efficient simulations, even though computers have become more and more powerful. For example, for applications involving complex fluid dynamics, the modeling of the surface and the mesh generation ("by hand") may take several weeks while the PDE simulations require only a few hours.

The main idea of Isogeometric Analysis which overcomes this bottleneck is the following. For exactly representing geometric shapes, one typically employs piecewise polynomials or rational functions as basis functions. If the same functions are used for the numerical simulations of the PDEs, one eliminates geometry errors. The new paradigm consists of leaving traditional finite element approaches behind and employing instead more general basis functions such as B-Splines and Non-Uniform Rational B-Splines (NURBS) for the PDE simulations as well. The combined concept of Isogeometric Analysis (IgA) allows for improved convergence and smoothness properties of the PDE solutions and dramatically faster overall simulations.

In the last few years, this new paradigm has revolutionized the engineering communities and triggered an enormous amount of simulations and publications mainly in this field. However, there are several profound theoretical issues which have not been well understood and which are currently investigated by researchers in Numerical Analysis, Approximation Theory and Applied Geometry. These problems firstly concern multiscale techniques for variational problems discretized by B-splines and NURBS, namely,

- multilevel solvers;
- hierarchical spaces, adaptivity;
- construction of non-tensor product functions;
- error estimation, convergence and complexity estimates;
- quadrature.

For example, the issue of constructing optimal preconditioners independent of the polynomial degree of the basis functions is except for involved auxiliary space methods a hot topic. A-posteriori error estimation and the possibility to develop adaptive methods with respect to both the mesh and the polynomial degree is not

mathematically understood in the IgA framework. A bottleneck for computations for variational formulations of PDEs even on uniform grids is currently the set-up of linear systems of equations which require highly efficient quadrature rules for B-splines and NURBS.

A second thematic focus of the Mini-Workshop addressed the recent revival of collocation methods, which provide significant advantages by employing higher order B-Splines and NURBS in combination with an IgA framework. These methods have been popular some decades ago in the numerical PDE community but have then been essentially abandoned, due to a largely missing mathematical foundation with respect to, e.g., error estimates. They are still used in applications, e.g., in boundary element methods. Collocation methods are based on point evaluations of the PDE in strong form. In principle, this is a dramatic advantage over variational formulations which require efficient quadrature rules. Collocation schemes also allow for quick evaluations of nonlinearities. These advantages have recently triggered new and promising work in all fields of computational mechanics.

So far, an error analysis for collocation methods exists only for PDEs on one-dimensional domains or under very high smoothness assumptions on the PDE solution. For higher dimensions, the issue how to select the collocation points to derive corresponding estimates is not understood. This entails that also a convergence and complexity theory is not yet available.

The goal of the Mini–Workshop *Mathematical Foundations of Isogeometric Analysis* organised by Thomas J.R. Hughes (Austin), Bert Jüttler (Linz), Angela Kunoth (Köln) and Bernd Simeon (Kaiserslautern) was to bring together some leading scientists from IgA and the mathematically relevant fields. We wanted to start with brainstorming in an atmosphere of a small workshop with not too many participants who are a nice blend of researchers with various backgrounds. The Mini-Workshop was well attended with 17 participants with broad geographic representation.

The participants were experts who are strong in approximation theory (Lyche, Oswald), numerical analysis and multiscale methods (Demlow, Kunoth, Langer, Mantzaflaris, Sangalli, Simeon), applied geometry and geometric design (Harbrecht, Jüttler, Manni, Mourrain, Peters) together with researchers in the engineering sciences with a strong mathematical background in modeling and numerics (Evans, Hughes, Reali). In addition, a young Bachelor student (Akpinar) presented promising first results for approximations of high-dimensional integrals.

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Mini-Workshop: Mathematical Foundations of Isogeometric Analysis

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Abstracts

Numerical Integration: Quasi-Monte-Carlo Methods for Integrands in Besov Spaces with Dominating Mixed Smoothness

Nur Sema Akpinar
(joint work with Angela Kunoth and Tino Ullrich)

The field of Quasi-Monte-Carlo methods for functions from spaces with characteristic smoothness, specifically, Besov spaces with dominating mixed smoothness, has gained more and more attention over the last few years. I started studying this topic in preparation for my bachelor thesis [1] (supervised by Angela Kunoth) based on the recent paper [6]. The problem is to construct cubature formulae for integrals in d space dimensions,

$$I(f) := \int_A f(x)dx, \tag{1}$$

where $A \subseteq \mathbb{R}^d$ and $f : A \to \mathbb{R}$ is a given multivariate function. Prominent examples stem from financial mathematics where $d = 360$ occurs from certain financial instruments over a period of thirty years, or solving partial differential equations with stochastic/parametric coefficients by means of Galerkin-type schemes for the parameters (see the abstract by Angela Kunoth).

Typical approximations are based on quadrature rules of the form

$$I_N(f) := \frac{1}{N} \sum_{x_i \in X} \alpha_i \, f(x_i) \tag{2}$$

with pre-determined points x_i from a point set $X \subset A$ with cardinality N and some given weights α_i. Classical integration formulae like Newton-Cotes' rules which are based on approximating the integrand by polynomials of degree $r-1$ in every space dimension yield for 'smooth' integrands with a total of N cubature points an error

$$|I(f) - I_N(f)| \lesssim N^{-r/d}, \tag{3}$$

i.e., the error can be bounded by a constant depending on the smoothness of f times $N^{-r/d}$. This detereorating rate as d tends to infinity is commonly called the "curse of dimensions". A simple remedy is to employ instead Monte-Carlo methods which yield (in expectation) an error of order $N^{-1/2}$ independent of the dimension d by evaluating the integrand at N randomly chosen points. This approach works for any integrand that allows for point evaluations. Due to its simplicity, it is widely used for many practioneers in Computational Physics or Chemistry. On the other hand, this rate is often considered to be unacceptibly low, specifically in the case of PDEs with possible countable infinite stochastic parameters which would require N evaluations of the PDE in each parameter to reach an error rate of $N^{-1/2}$ with respect to this parameter. In fact, one can reach with Quasi-Monte-Carlo methods faster convergence rates of the worst-case error over all possible point sets X, for functions f belonging to a certain class with smoothness $r > 0$,

specifically, the Besov space with dominating mixed smoothness defined by mixed differences of order 2. Functions in Besov spaces allow for isolated singularities and can, therefore, be considered as generalizations of standard Sobolev spaces.

In order to define this space, we consider from now on the situation from [6] where the integration area A in (1) equals the two-dimensional torus $\mathbb{T}^2$, a subset of the unit square $[0,1]^2$ where opposite sides are identified. We define for parameters $0 \le p, q \le \infty$ a Besov space $B^r_{p,q}(\mathbb{T}^2)$ with smoothness r on $\mathbb{T}^2$ which is embedded in the Lebesgue space $L_p(\mathbb{T}^2)$ such that its corresponding (quasi-)norm is bounded,

$$B^r_{p,q}(\mathbb{T}^2) := \left\{ f \in L_p(\mathbb{T}^2) : \ \|f\|_{B^r_{p,q}(\mathbb{T}^2)} < \infty \right\}, \tag{4}$$

where q plays the role of a 'fine-tuning' parameter. There are numerous ways to define such norms. Here we choose the definition in terms of the weighted bivariate Faber basis from [6] (or, in other terms, tensor products of the piecewise linear, continuous hierarchical basis known in the finite element community) which is as follows. Setting $\mathbb{N}_{-1} = -1, 0, 1, \ldots$, let $D_j := \left\{0, \ldots, \left\lceil 2^{j_1} \right\rceil - 1\right\} \times \left\{0, \ldots, \left\lceil 2^{j_2} \right\rceil - 1\right\}$ and define the set of functions $F^2 := \{v_{j,k} : j_1, j_2 \in \mathbb{N}_{-1}, k = (k_1, k_2) \in D_j\}$ with $v_{j,k}(x) := v_{j_1,k_1}(x_1) v_{j_2,k_2}(x_2)$. Each univariate function is defined as a scaled 'hat' function (i.e., a continuous, piecewise linear function),

$$v_{j_i,k_i}(x_i) := \begin{cases} 2^{j_i+1}\left(x_i - 2^{-j_i}k_i\right), & x_i \in \left[2^{-j_i}k_i, 2^{-j_i}k_i + 2^{-j_i-1}\right], \\ 2^{j_i+1}\left(2^{-j_i}(k_i+1) - x_i\right), & x_i \in \left[2^{-j_i}k_i + 2^{-j_i-1}, 2^{-j_i}(k_i+1)\right], \end{cases}$$

for $i = 1, 2$, and 0 elsewhere. Any function $f \in \mathcal{C}(\mathbb{T}^2)$ is one-periodic in each component, and it is known (see, e.g., the references in [6]) that f can be represented as

$$f(x_1, x_2) = \sum_{j \in \mathbb{N}^2_{-1}} \sum_{k \in D_j} d^2_{j,k}(f)\, v_{j,k}(x_1, x_2) \tag{5}$$

with expansion coefficients consisting of point evaluations,

$$d^2_{j,k}(f) := \begin{cases} f(0,0) & \text{if } j = (-1,-1), \quad k \in \mathbb{Z}^2, \\ c_{j_1} \quad f(2^{-j_1}k_1, 0) & \text{if } j = (k_1, -1),\ j_1 \in \mathbb{N}_0,\ k \in \mathbb{Z}^2, \\ c_{j_2} \quad f(0, 2^{-j_2}k_2) & \text{if } j = (-1, k_2),\ j_2 \in \mathbb{N}_0,\ k \in \mathbb{Z}^2, \\ c_{j_1,j_2} \quad f(2^{-j_1}k_1, 2^{-j_2}k_2) & \text{if } j \in \mathbb{N}^2_0,\ k \in \mathbb{Z}^2, \end{cases}$$

where $c_{j_1}, c_{j_2}, c_{j_1,j_2}$ are weighted second order difference operators in x_1, x_2 and both x_1, x_2 direction, respectively; see [6] for details. With this notion, we define the Besov (quasi-)norm in (4) as

$$\begin{aligned} \|f\|_{B^r_{p,q}(\mathbb{T}^2)} \quad := \quad & \|f\|_{L_p(\mathbb{T}^2)} + \left(\sum_{j_1=0}^{\infty} 2^{r j_1 q} \sup_{|h_1| \le 2^{-j_1}} \left\| \Delta^m_{h_1,1} f \right\|^q_{L_p(\mathbb{T}^2)} \right)^{1/q} \\ & + \left(\sum_{j_2=0}^{\infty} 2^{r j_2 q} \sup_{|h_2| \le 2^{-j_2}} \left\| \Delta^m_{h_2,2} f \right\|^q_{L_p(\mathbb{T}^2)} \right)^{1/q} \end{aligned}$$

$$+\left(\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}2^{r(j_1+j_2)q}\sup_{|h_i|\le 2^{-j_i},\ i=1,2}\left\|\Delta_{h_1,h_2}^{m,m}f\right\|_{L_p(\mathbb{T}^2)}^q\right)^{1/q}$$

for $m > r > 1/p$ where $\Delta_{h_1,h_2}^m f$ denotes the mth order difference operator applied to f.

Consider now the case that X is a Hammersley point set with $N = 2^n$ number of points. It was proved for the quadrature rule (2) with weights $\alpha_i = 1$ in [6] that the error of the Quasi-Monte-Carlo method for functions in $B_{p,q}^r(\mathbb{T}^2)$ decays as

$$\sup_{\|f\|_{B_{p,q}^r(\mathbb{T}^2)}\le 1}|I(f) - I_N(f)| \lesssim N^{-r}(\log N)^{1-1/q} \tag{6}$$

for $1 \le p, q \le \infty$ and $1/p < r < 2$. Specifically, for $q = 1$, the log-term disappears. Thus, by employing Quasi-Monte-Carlo methods on certain deterministic point sets, the convergence is substantially faster than in the case of the classical Monte-Carlo method. The proof of (6) is based on the fact that the specific form of the Faber basis employed in (5) allows one to estimate the left hand side of (3) in terms of local contributions, i.e.,

$$|I(f) - I_N(f)| = \left|\sum_{j\in\mathbb{N}_{-1}^2}\sum_{k\in D_j} d_{j,k}^2(f)\, V_{j,k}\right|$$

where $V_{j,k} := N^{-1}\sum_{x_i\in X} v_{j,k}(x_i) - \int_{\mathbb{T}^2} v_{j,k}(x)dx, j \in \mathbb{N}_{-1}^2, k \in D_j$.

Employing wavelet characterizations of Besov spaces as in [3], we can extend the estimate (6) to arbitrary space dimensions d and arbitrary order r [2]. We will further investigate quadrature rules on more general point sets, so-called 'digital nets' to which the Hammersley type point sets belong as a special case. Finally, the periodic setting can be replaced by $[0,1]^d$ for which tensor products of the biorthogonal spline-wavelets on $[0,1]$ constructed as in [4, 5] can be employed.

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A Survey of Maximum-norm Estimates for Finite Element Methods

Alan Demlow

Maximum-norm error estimation for finite element methods has been an active area of research since the 1970's. Our goal in this talk is to survey the current state of research in the area, give a brief overview of techniques for proving such estimates, and discuss extension of existing results to isogeometric methods.

We consider the model problem

$$-\Delta u = f \text{ in } \Omega$$

with homogenous Dirichlet boundary conditions. In addition, let $\mathcal{T}_h$ be a triangulation of Ω and $S_h \subset H_0^1(\Omega)$ a standard Lagrange (C^0) finite element space of polynomial degree r, and let $u_h \in S_h$ be the standard finite element approximation to u. Let $h_T = \operatorname{diam} T$, $T \in \mathcal{T}_h$, and $h = \max_{T\in\mathcal{T}_h} h_T$. We assume throughout that $\mathcal{T}_h$ is shape regular, and for the time being that $\mathcal{T}_h$ is quasi-uniform. The basic goal of maximum-norm error analysis is to prove the following optimal-order error estimates:

$$(\ln 1/h)^{-\delta_{1r}}\|u-u_h\|_{L_\infty(\Omega)} + h\|u-u_h\|_{W^1_\infty(\Omega)} \leq Ch^{r+1}|u|_{W^{r+1}_\infty(\Omega)}. \tag{1}$$

Such estimates are contained in a large number of works by authors including Nitsche, Rannacher, Scott, Schatz, Wahlbin, and others; we do not give precise references as our goal is mostly to give an informal overview. It is generally assumed either that Ω is a convex polyhedral domain in $\mathbb{R}^2$ or $\mathbb{R}^3$, or that $\partial\Omega$ is smooth in order to prove such results. Note also that the logarithm factor above is only present when estimating the error in L_∞ (and not in W^1_∞) and only for piecewise linear elements. In this case it is however known to be necessary [7]. In addition to optimal-order error estimates, sharper stability estimates are also of interest and known in many cases:

$$\|u_h\|_{L_\infty(\Omega)} \leq C(\ln 1/h)^{\delta_{1r}}\|u\|_{L_\infty(\Omega)},\ \|u_h\|_{W^1_\infty(\Omega)} \leq C\|u\|_{W^1_\infty(\Omega)}.$$

Proofs of the above error and stability estimates are significantly more involved than corresponding proofs of standard energy or L_2 error estimates. To understand the difficulty, recall that when proving error estimates in L_2 the first step is typically to represent the error by duality. That is, we write $\|u-u_h\|_{L_2(\Omega)} = \sup_{v\in L_2(\Omega)}(u-u_h,v)$. For such v, we then solve $-\Delta z = v$. Along with standard tools (Galerkin orthogonality, energy estimates, etc.), the H^2 regularity result $\|z\|_{H^2(\Omega)} \leq C\|v\|_{L_2(\Omega)}$ is a necessary part of the argument. This pathway does not work as smoothly in L_∞, since L_1 is not dual to L_∞ and the regularity result $\|z\|_{W^2_1(\Omega)} \leq C\|\Delta z\|_{L_1(\Omega)}$ does *not* hold. Two complementary techniques have been used to circumvent these difficulties, one involving manipulation of weighted norms and the other involving error representation using a regularized Green's function combined with local energy error estimates.

The steps needed to prove maximum-norm a priori error estimates are now well-established, and can be applied to Galerkin methods for which the discrete

subspaces satisfy certain basic properties. These include local approximation properties in a reasonable range of norms (L_p and W_p^1 for $1 \le p \le \infty$), local inverse estimates, and a superapproximation property [8] that largely follows from approximation and inverse assumptions. Proof of these properties for variational isogeometric methods appears to be a reasonable simple extension of existing estimates. On the other hand, isogeometric collocation methods have also gained in popularity. Here L_∞ Sobolev norms are a natural setting for error analysis, but the techniques used to prove estimates for Galerkin methods do not immediately apply. Convergence proofs in such cases remain an open question except in one space dimension.

Residual-type a posteriori error estimates also have been proved in a number of works beginning in the mid-1990's [9, 6, 1, 2, 3]. Let $\mathcal{T}_h$ now be shape-regular only. For $T \in \mathcal{T}_h$, define $\eta_\infty(T) = h_T^2 \|f + \Delta u_h\|_{L_\infty(T)} + h_T \|[\![\nabla u_h]\!]\|_{L_\infty(\partial T)}$, where $[\![\nabla u_h]\!]$ is the jump in ∇u_h across the element boundary. Let also $\ell_h = \max_{T\in\mathcal{T}_h} \ln(1 + 1/h_T)$. Then if Ω is a polyhedral domain in $\mathbb{R}^2$ or $\mathbb{R}^3$,

$$\|u - u_h\|_{L_\infty(\Omega)} \le C\ell_h \max_{T\in\mathcal{T}_h} \eta_\infty(T). \tag{2}$$

To prove such estimates, one represents the error using the standard Green's function G^{x_0} for the continuous problem, that is, $(u-u_h)(x_0) = (\nabla(u-u_h), \nabla G^{x_0})$. A series of standard manipulations then reduces proof of error bounds to establishing certain regularity bounds for G^{x_0}.

As in the priori case, it is natural to ask whether the logarithmic factor in (2) is necessary. This question is answered in the affirmative in [3], at least in 2D and when $r = 1$. The proof yields further insight into maximum-norm error behavior. Let $\|\cdot\|_{BMO_{-\Delta}(\Omega)}$ be an operator-adapted BMO (bounded mean oscillation) norm; cf [4] for details. BMO is weaker than L_∞ but stronger than any L_p norm for any $p < \infty$ and so naturally serves as a proxy for L_∞ in many situations in PDE analysis. In addition, in contrast to L_∞ it is a reflexive space with its dual space being the atomic Hardy space $\mathcal{H}^1$. Finally, regularity estimates for elliptic BVPs hold in $\mathcal{H}^1$ but not in L_1 spaces, that is, $\|D^2 z\|_{\mathcal{H}^1(\Omega)} \le C\|\Delta z\|_{\mathcal{H}^1(\Omega)}$.

From a finite element standpoint, it becomes reasonably straightforward to prove error estimates in BMO once these facts are established. A priori BMO convergence estimates similar to (1) were given in [5], but with no log factor present. In [3] we established that for convex polyhedral domains,

$$\|u - u_h\|_{BMO_{-\Delta}(\Omega)} \simeq \max_{T\in\mathcal{T}_h} \eta_\infty(T) + \mathrm{osc}_\infty(\mathcal{T}_h).$$

Here $\mathrm{osc}_\infty(\mathcal{T}_h)$ is the *data oscillation* which measures the deviation of the data f from a broken piecewise polynomial space of degree $r-1$. Taken together, these estimates show that BMO is a more natural norm than L_∞ from an analytical (though not necessarily application) standpoint. In the a posteriori context in particular, we have shown that the error estimator $\max_{T\in\mathcal{T}_h} \eta_\infty(T)$ that we have been using to measure the L_∞ norm of the error in fact is an upper and lower bound (up to nonessential constants and data oscillation) for the BMO norm of the error. Adaptive finite element methods for L_∞ errors thus are actually more

directly controlling BMO errors. In future research it may be advantageous to consider BMO as a tool for understanding finite element error behavior in L_∞-like norms.

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Optimality and Approximation: A Quantitative Assessment of the Approximation Properties of Spline, Polynomial, and Fourier Bases

John A. Evans

(joint work with Ivo Babuška, Yuri Bazilevs, Joseph Benzaken, Jesse Chan, and Thomas J.R. Hughes)

A fundamental question that arises in numerical approximation is how "optimal" a given space of approximating functions is. This question involves not only the approximation rate for a given space but also the constant of approximation. To make this discussion more precise, consider the following common *a priori* estimate arising in the theory of finite elements:

$$\|u - \pi_n u\|_{H^k(\Omega)} \leq C n^{(l-k)/d} \|u\|_{H^l(\Omega)}$$

where $u \in H^s(\Omega)$ denotes some function one wishes to approximate and:

$$l = \min(p+1, s)$$

Moreover, $\pi_n : H^s(\Omega) \to X_n$ denotes some projection operator onto a space of approximating functions with n degrees of freedom, p denotes the polynomial degree of this space, and d denotes the spatial dimension. A space of approximating functions is said to be quasi-optimal if the approximation constant C is independent of n, but the question of optimality also involves its actual quantitative value.

Unfortunately, the value for C is typically unknown and it depends upon, for example, both the polynomial degree and inherent smoothness of the underlying approximating functions.

In this talk, I introduce a quantitative framework for assessing the optimality properties of a given space of approximating functions. This framework hinges upon the concepts of distance, deviation, and n-width. Let X be a normed linear space, x be a member of X, and X_n be a subset of X. We define the distance of X_n from member x as:

$$E(x, X_n; X) = \inf_{y_n \in X_n} \|x - y_n\|_X$$

and the deviation of X_n from set A as:

$$E(A, X_n; X) = \sup_{x \in A} \inf_{y_n \in X_n} \|x - y_n\|_X$$

Within the context of approximation theory, X_n is interpreted as the space of approximating functions while A is interpreted as the space of candidate functions to be approximated. The question of optimality then is: how close is the deviation of X_n from set A from the optimal space of approximating functions? To answer this question, the concept of an n-width was introduced. With the notation established above, we define the n-width as:

$$\begin{aligned} d_n(A; X) &= \inf \{E(A, X_n; X) : X_n \subset X, \dim(X_n) = n\} \\ &= \inf_{\substack{X_n \subset X \\ \dim(X_n) = n}} \sup_{x \in A} \inf_{y_n \in X_n} \|x - y_n\|_X \end{aligned}$$

Then, a space of approximating functions is optimal if its deviation is equal to the n-width, and to assess its optimality, we simply compare its deviation to the n-width using the optimality ratio:

$$\Lambda(A, X_n; X) = \frac{E(A, X_n; X)}{d_n(A; X)}$$

If the optimality ratio is $\Lambda(A, X_n; X) = 1$, then the space of approximating functions is optimal, and if it is much greater than one, it is far from optimal. While the concept of n-width is quite useful in defining a quantitative notion of optimality, it is useless in practice unless it can be computed. Fortunately, in the Hilbert space setting, both the deviation and n-width can be computed using variational eigenproblems. Colleagues and I introduced an isogeometric framework for solving said variational eigenproblems in the one-dimensional setting in 2009 [1], and we have recently extended this framework to the multi-dimensional setting using a combination of Cholesky factorization, multigrid-preconditioned conjugate gradient methods, and Lanczos iteration. This framework allows one to compute the constant of approximation exactly for a given approximation space. In this talk, I review both this quantitative assessment framework as well as corresponding numerical results for splines, polynomial, and Fourier bases.

I finish this talk by trying to answer the following questions using my new quantitative assessment framework for optimality.

(1) How accurate and robust are splines of maximal continuity in numerical approximation?
(2) How accurate and robust are splines of lowest continuity (i.e., classical finite elements) in numerical approximation?
(3) How do classical finite elements and splines compare?
(4) How do the answers to all of the above depend upon dimensionality?
(5) How accurate and robust are divergence-conforming and curl-conforming splines in numerical approximation?
(6) Which is better for multi-dimensional approximation: tensor-product polynomials or standard isotropic polynomials?
(7) Which is better for multi-dimensional approximation: tensor-product Fourier bases, Fourier bases "complete" to a given wavenumber, or the minimal (sparsest) Fourier basis required for exponential convergence?
(8) Are there different situations in which certain polynomial or Fourier bases may be preferred, and can we quantify said situations?

The answers to these questions certainly have important consequences regarding the effectiveness of isogeometric analysis as a numerical solution procedure for partial differential equations [2, 3, 4, 5], but the answers to the above questions also have important consequences regarding the effectiveness of sparse approximations for multi-dimensional problems which are relevant in parameter space exploration and uncertainty quantification [6].

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On Fast Boundary Element Methods for Parametric Surfaces

Helmut Harbrecht

In many situations, practical problems arising from science and engineering can be formulated in terms of differential equations for an unknown function. If a Green's function of the underlying differential operator is known, they may be reformulated by means of a boundary integral equation

$$(\mathcal{A}u)(\mathbf{x}) = \int_\Gamma G(\mathbf{x},\mathbf{y})u(\mathbf{y})\,\mathrm{d}\,o_{\mathbf{x}} = f(\mathbf{x}), \quad \mathbf{x}\in\Gamma.$$

The Green's function $G(\cdot,\cdot)$ is, for instance, known in case of the Laplace equation, the Helmholtz equation, and the heat equation. The major advantage of considering boundary integral equations is the reduction of the problem's dimensionality, which especially gives the possibility for easily treating also exterior boundary value problems.

In general, boundary integral equations are solved by the boundary element method. However, due to the non-locality of the integral operator $\mathcal{A}$, one usually ends up with large and densely populated system matrices. Thus, the numerical solution of such problems is rather challenging. Different approaches have been proposed to overcome this obstruction, such as the fast multipole method, adaptive cross approximation, or wavelet matrix compression.

This talk intends to give an overview on fast boundary element methods which are tailored to the context of parametric surfaces. This means, the surface Γ is subdivided into several smooth *patches*

$$\Gamma = \bigcup_{i=1}^{M} \Gamma_i$$

such that the intersection $\Gamma_i\cap\Gamma_{i'}$ consists at most of a common vertex or a common edge for $i\neq i'$. Moreover, for each patch, there exists a smooth diffeomorphism

$$\boldsymbol{\gamma}_i : \square \to \Gamma_i \quad \text{with} \quad \Gamma_i = \boldsymbol{\gamma}_i(\square) \quad \text{for} \quad i = 1,2,\ldots,M,$$

where $\square := [0,1]^2$ denotes the unit square. This surface representation is in contrast to the common approximation of surfaces by flat panels. Nonetheless, parametric surface representations are easily accessible from Computer Aided Design (CAD) and are the topic of recent studies in isogeometric analysis.

We address different issues of fast boundary element methods for the solution of boundary integral equations on parametric surfaces as considered in [1]–[5]. This includes storage requirements, adaptivity, and higher-order ansatz functions. In particular, it turns out that the additional information, which is imposed by the parametric surface representation, enables numerous simplifications and optimizations of the underlying data structures and algorithms. Several numerical examples are provided in order to quantify and qualify the proposed methods.

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Isogeometric Analysis: Overview

Thomas J.R. Hughes

I began the workshop on the mini-workshop Mathematical Foundations of Isogeometric Analysis with an overview of Isogeometric Analysis (IGA). I mentioned that the field is very young, 10 years old [1], but has grown to the point where it is impossible to even briefly mention all the topics that have been pursued to date. On Thomson Reuters Web of Science, there were zero citations to the topic 10 years ago, but in 2015 alone there were over 4,500 citations! I started by comparing the growth of the subject to that of Finite Element Analysis (FEA) in its early years. Needless to say, it is growing much, much faster than FEA. Then I briefly mentioned the original motivation behind the concept and the basic analytical tools coming from Computer Aided Geometric Design. I discussed their approximation properties from the perspectives of functional analysis and spectral analysis. I highlighted the important role of new developments in numerical quadrature, the renaissance in shell structural analysis, and collocation methods facilitated by IGA. Then I presented an application to modeling and analysis of a patient-specific heart valve, and the enormous superiority over an optimized traditional FEA approach. I spent some time reviewing design-through-analysis procedures, specifically T-splines, immersed methods, hierarchically refined B-splines, and IGA boundary element methods. I finished with a few applications to fluid-structure interaction of bioprosthetic heart valves, and the original phase-field theory, that is, the Navier-Stokes-Korteweg equations, which have already been utilized to simulate boiling phenomena, and have the potential to represent cavitation. I summarized the significant opportunities of further mathematical and engineering research in IGA and presented a list of 11 problems that I would like to see solved. Here they are:

- Eliminate outliers
- Fix extraordinary points
- Algorithms to automatically generate unstructured surface splines
- Algorithms to automatically generate unstructured volumetric splines

- Algorithms to determine optimal quadrature rules for unstructured spline spaces
- Algorithms to automatically construct untrimmed, smooth spaces from trimmed NURBS
- inf-sup stable spline spaces for incompressible problems (i.e., mappings, not vector fields)
- Approximation results for immersed (i.e., trimmed) spline formulations
- Prove gamma convergence for fourth-order phase-field theory
- Construct spline spaces that solve the membrane-bending locking problem
- A general mathematical theory of collocation

Obviously, these problems are vaguely defined and they need to be made more specific, but that is a significant part of all the problems, to define them precisely.

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ANTS: Basis Functions for Partially Nested Spline Refinement

Bert Jüttler

(joint work with Nora Engleitner and Urška Zore)

Truncated hierarchical B-splines, which were introduced in [3] based on earlier results concerning hierarchical spline refinement [1, 5], have been established as one of the approaches to perform local refinement in geometric modeling and isogeometric analysis [2]. Besides forming a non-negative partition of unity, these basis functions possess good properties regarding stability [4], algebraic completeness [6], and approximation power [7].

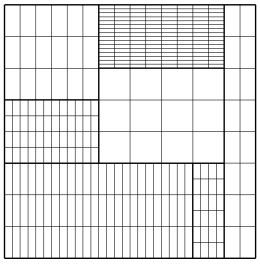

Figure 1. A mesh obtained by partially nested hierarchical spline refinement.

The established construction of truncated hierarchical B-splines starts from a given sequence of nested spline spaces, and hence it is not possible to pursue independent refinement strategies in different parts of a model. In order to overcome

this limitation, we generalize hierarchical B-splines to sequences of *partially* nested spline spaces. This allows us to define basis functions for splines defined on more general meshes, see Fig. 1 for an example.

We identify assumptions that enable us to define a hierarchical spline basis, to establish a truncation mechanism, and to derive a completeness result. The resulting basis functions are denoted as ANTS, which is the acronym of "ANTS are Not T-Splines". The application potential of the proposed generalization is demonstrated by first result on least-squares approximation.

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Adaptive Approximations of Parametric Parabolic PDE-Constrained Control Problems

Angela Kunoth

(joint work with Max Gunzburger and with Christoph Schwab)

Optimization problems constrained by linear PDEs (partial differential equations) are challenging from a computational point of view: one needs to solve a *system of PDEs* coupled globally in space, and, in addition, *globally* in time if the underlying PDE is time-dependent. This global coupling is an unavoidable feature of such control problems as specified next: typically an adjoint PDE comes into place.

PDE-constrained control problems. Let Y, U be Hilbert spaces over $\mathbb{R}$ which shall host the *state* y of a system and a *control* by which the state can be influenced. Let $J : Y \times U \to \mathbb{R}$ be a twice differentiable functional, and let $K : Y \times U \to Y'$ be a (in y, u Fréchet-) differentiable function where Y' denotes the topological dual of Y. Consider the constrained minimization problem

$$\inf_{(y,u)\in Y\times U} J(y,u) \quad \text{subject to } K(y,u) = 0. \tag{1}$$

For the constraints $K(y, u) = 0$ (the PDE), we assume that there exists a unique solution $y \in Y$ when $u \in U$ is given. A typical way to solve (1) is to compute

the zeroes of the first order Fréchet derivatives of the Lagrangian functional which is defined by introducing the *co-*(or *adjoint) state* p by which the constraints are appended to the functional J, i.e., $L(y,u,p) := J(y,u) + \langle K(y,u),p\rangle_{Y'\times Y}$ with $L : Y \times U \times Y \to \mathbb{R}$. Denoting by $L_z(y,u,p) := \frac{\partial}{\partial z}L(y,u,p)$ and $L_{zz}(y,u,p) := \frac{\partial^2}{\partial z^2}L(y,u,p)$ the first and second variation, of L with respect to $z = y,u,p$, and assuming that J is quadratic in both y,u and that K is linear in y,u, the necessary conditions for optimality yield the linear system of equations

$$(2)\quad \begin{pmatrix} L_{yy} & L_{yu} & K_y^* \\ L_{uy} & L_{uu} & K_u^* \\ K_y & K_u & 0 \end{pmatrix} \begin{pmatrix} y \\ u \\ p \end{pmatrix} = g \Longleftrightarrow: \begin{pmatrix} \mathcal{A} & \mathcal{B}^* \\ \mathcal{B} & 0 \end{pmatrix} \begin{pmatrix} (y,u)^\top \\ p \end{pmatrix} = g \Longleftrightarrow: \; G\,q = g$$

with some right hand side g and C^* the dual of C. The Hessian of L or the Karush-Kuhn-Tucker (KKT) operator G has for such *linear-quadratic problems* constant entries, and the necessary conditions are also sufficient. Moreover, if J or K do not contain products yu, one has $L_{yu} = L_{uy} = 0$ so that $\mathcal{A}$ is a block diagonal operator. Typically, the quadratic functional (1) contains inner products so that the resulting Riesz operators L_{yy}, L_{uu} are symmetric which implies that $\mathcal{A}$ and, thus, G is *symmetric*. In all the cases we consider, the operators $\mathcal{A} : \mathcal{V} \to \mathcal{V}, \mathcal{B} : \mathcal{V} \to \mathcal{Q}'$ (for some Hilbert spaces) $\mathcal{V}$, $\mathcal{Q}$ are continuous; $\operatorname{Im}\mathcal{B} = \mathcal{Q}'$; and $\mathcal{A}$ is invertible on $\operatorname{Ker}\mathcal{B}$ so that the saddle point problem (2) has for $g \in \mathcal{V}' \times \mathcal{Q}'$ a unique solution $q \in \mathcal{V} \times \mathcal{Q}$ by the Brezzi-Fortin theory. Thus, we can consider constrained linear-quadratic minimization problems (1) as *symmetric saddle point problems* (2) with a boundedly invertible linear mapping $G : \mathcal{V} \times \mathcal{Q} \to \mathcal{V} \times \mathcal{Q}'$ where $\mathcal{V} := Y \times U$ and $\mathcal{V} := Q$. Some standard examples from [7] to which this scenario applies are the following.

Dirichlet problem with distributed control. Consider the standard weak formulation of a second order elliptic PDE with homogeneous Dirichlet boundary conditions. Choosing $Y := H_0^1(\Omega)$ and $U := Y'$, we consider for given $f \in Y'$ the linear operator equation

$$(3)\qquad K(y,u) := Ay - f - u = 0$$

and the quadratic objective functional

$$(4)\qquad J(y,u) := \frac{1}{2}\|y - y_*\|_Y^2 + \frac{\omega}{2}\|u\|_{Y'}^2$$

for a given target state $y_* \in Y$ and any fixed weight parameter $\omega > 0$. We assume that $A : Y \to Y'$ is a linear (not necessarily symmetric) boundedly invertible operator. The norms in (4) can more generally be norms on Hilbert spaces as long as the constrained optimization system (1) possesses a unique solution.

Denote by $R : Y \to Y'$ the Riesz operator defined by the inner product $(\cdot,\cdot)_Y$ inducing $\|\cdot\|_Y$, $\langle v, Rw\rangle_{Y\times Y'} := (v,w)_Y$, $v,w \in Y$. Since $(\cdot,\cdot)_Y$ is symmetric, R is also. Consequently, the Lagrangian is

$$(5)\quad L(y,u,p) = \frac{1}{2}\langle y-y_*, R(y-y_*)\rangle_{Y\times Y'} + \frac{\omega}{2}\langle u, R^{-1}u\rangle_{Y\times Y'} + \langle Ay-f-u, p\rangle_{Y'\times Y}$$

implying that the system (2) becomes

$$G \begin{pmatrix} y \\ u \\ p \end{pmatrix} := \begin{pmatrix} R & 0 & A^* \\ 0 & \omega R^{-1} & -I \\ A & -I & 0 \end{pmatrix} \begin{pmatrix} y \\ u \\ p \end{pmatrix} = \begin{pmatrix} Ry_* \\ 0 \\ f \end{pmatrix}, \tag{6}$$

i.e., $\mathcal{A} = \mathrm{diag}(R, \omega R^{-1})$ and $\mathcal{B} = (A, -I)$. The system matrix G defined in (6) is symmetric since R is. Moreover, $\mathcal{A}$ is positive definite and $\mathcal{B}$ has full rank since, by assumption, the PDE constraints (3) have a unique solution for given u. Thus, the resulting saddle point operator G is symmetric and boundedly invertible.

Parabolic PDE with distributed control. The constraint $K(y, u) = 0$ in (1) is here a linear parabolic evolution PDE in a variation of the full space-time weak formulation from [10]. The parabolic operator equation is formulated such that the resulting operator B is boundedly invertible from $\mathcal{X} := L^2(I) \otimes Y$ to $\mathcal{Y}' := ((L^2(I) \otimes Y) \cap (H^1_T(I) \otimes Y'))'$ where $H^1_T(I)$ is the closure of the functions in $H^1(I)$ which vanish at end time T and $I := (0, T)$ denotes the time interval. The constraints are of the form (3) with the parabolic evolution operator $B = \partial_t + A$ in full weak space-time form in place of A, see [6] for details. Choosing the objective function then as in (4) with the obvious changes for the norms, i.e., using the norms for $\mathcal{X}$, $\mathcal{Y}$, we arrive at a system very similar to (6) with symmetric $\mathcal{A} = \mathrm{diag}(R_1, \omega R_2)$ with the respectively defined Riesz operators. Finally, the resulting operator G is a boundedly invertible mapping from $\mathcal{Z} := \mathcal{X} \times \mathcal{Y}' \times \mathcal{X}$ onto $\mathcal{Z}'$.

Adaptive wavelet methods for the parabolic PDE-constrained control problem. In view of the fully in *space and time* coupled system (2), conventional time-stepping methods require an enormous storage. In contrast, adaptive methods in both space and time which aim at distributing the available degrees of freedom in an a-posteriori-fashion to capture singularities are most promising. Employing wavelet schemes for full weak space-time formulations of the parabolic PDEs, we can prove convergence and optimal complexity for control problems constrained by a linear parabolic PDE [6], generalizing the ideas from [4] for control problems constrained by an elliptic PDEs.

Parametric PDE-constrained control problems. Yet another level of challenge are control problems constrained by evolution PDEs involving stochastic or countably many infinite parametric coefficients: for each instance of the parameters, this requires the solution of the complete control problem (2).

Our method of attack is based on the following new theoretical paradigm developed for elliptic PDEs in [2, 3]. It is first shown for control problems constrained by evolution PDEs, formulated in full weak space-time form as in [10], that state, costate and control are *analytic* as functions depending on these parameters. We establish that these functions allow expansions in terms of sparse tensorized generalized polynomial chaos (gpc) bases. Their sparsity is quantified in terms of p-summability of the coefficient sequences for some $0 < p \leq 1$. Resulting a-priori estimates establish the existence of an index set for *simultaneous* approximations of state, co-state and control for which the gpc approximations attain rates of best N-term approximation. This entails corresponding *sparse realizations* in terms of

deterministic adaptive Galerkin approximations of state, co-state and control on the entire, possibly infinite-dimensional parameter space, see [7]. We specify in [8] how to realize these Galerkin approximations by the techniques in [9] and the realizations in [5] for a single PDE.

In the context of isogeometric analysis, these techniques can be combined with parametric mappings as in [1].

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Multipatch Discontinuous Galerkin Space and Space-time IgA: Error Estimates and Fast Solvers

Ulrich Langer

(joint work with Christoph Hofer, Angelos Mantzaflaris, Stephen Moore, Martin Neumüller, Ioannis Toulopoulos)

1. Multipatch DG IGA Discretizations

In the first part of the presentation, we derive error estimates for multipatch discontinuous Galerkin (dG) Isogeometric Analysis (IgA) approximations

$$a_{dG}(u_h, v_h) = \ell_{dG}(v_h), \quad \forall v_h = \{v_{h,i}\}_{i=1}^N \in V_{dG,h}(\Omega), \tag{1}$$

to heterogeneous elliptic diffusion problems of the form

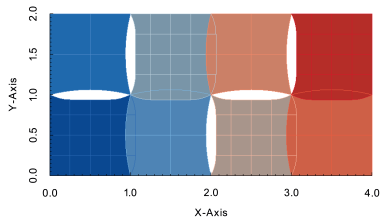
$$-\nabla_\Omega \cdot (\alpha \nabla_\Omega u) = f \text{ in } \Omega, \quad u = g_D \text{ on } \Gamma_D, \quad \alpha \nabla_\Omega u \cdot n = g_N \text{ on } \Gamma_N \tag{2}$$

on open and closed surfaces $\Omega \subset \mathbb{R}^3$ (2d manifolds) and in volumetric computational domains $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. We suppose that $\overline{\Omega} = \cup_{i=1}^N \overline{\Omega}_i$ has a multipatch

representation, where every patch $\Omega_i = F_i(\widehat{\Omega})$ is the image of parameter domain $\widehat{\Omega}$, that is $(0,1)^2$ or $(0,1)^3$, by the NURBS map $F_i : \widehat{\Omega} \to \Omega_i$. Our numerical analysis covers low-regularity solutions, graded meshes, non-matching meshes, and segmentation crimes like non-matching interfaces creating gaps and overlaps, like illustrated in the figure below. The typical discretization error estimate with respect to (wrt) the dG-norm $\|\cdot\|_{dG}$ roughly looks like

$$\|u - u_h\|_{dG} \lesssim \max\{h_j/h_i\}\, h^r + h^{\lambda-0.5}, \tag{3}$$

where u is the weak solution of (2), u_h is the solution of the dG variational scheme (1), $h = \max\{h_i\}$ denotes the maximal mesh size, h_i is the mesh size of the mesh of the patch Ω_i, h^λ denotes the maximum width of the gap with $\lambda \geq 1$, $r = \min\{l - 1 + (d/2) - (d/q), p\}$, and p is the underlying polynomial degree of the NURBS, whereas $l \geq 2$ and $q \in (\max\{1, 2d/(d + 2(l-1))\}, 2]$ are given by the local regularity of the solution u that is supposed to belong to W_q^l at least patch-wise. The table given below shows the convergence rates of the IgA solution u_h to a smooth solution u wrt the dG-norm for $p = 2$ and the domain on the left-hand side (gaps, overlaps, non-matching meshes). The rate is dominated by the term $h^{\lambda-0.5}$ for $\lambda = 1, 2$ and 2.5. If λ is larger than 2.5, e.g., 3, than the rate will not be better than $r = p = 2$. These observations are in perfect agreement with our estimate (3).



Convergence rates for different λ			
h	$\lambda = 1$	$\lambda = 2$	$\lambda = 2.5$
0.03125	0.51	1.52	2.02
0.015625	0.50	1.51	2.01
0.0078125	0.50	1.50	2.00

We refer the reader to our publications [3, 4, 8, 7, 9, 11] for a detailed presentation of the results and the proofs for the different cases mentioned above.

2. Isogeometric Tearing and Interconnecting Solvers

The second part of the presentation is devoted to the construction and analysis of new dual-primal IsogEometric Tearing and Interconnecting (IETI-DP) methods for solving large-scale linear systems $K\underline{u}_h = \underline{f}_h$ of algebraic equations arising from the dG IgA discretization (1) of heterogeneous diffusion problems (2) on multipatch domains with non-matching meshes and interfaces like illustrated in the figure above. In [1], we discuss and analyze the conforming Galerkin multipatch IgA, whereas the discontinuous Galerkin multipatch IgA with non-matching meshes is investigated in [2]. The iteration numbers of a properly scaled Dirichlet preconditioned conjugate gradient IETI iteration behave like $O(1 + \log(H/h))$, where $H/h = \max\{H_i/h_i\}$ and H_i denotes the scaling of the patch (subdomain) Ω_i, $i = 1, \ldots, N$. The numerical results presented in [1] and [2] confirm this theoretical result and show incredible robustness with respect to large jumps in the

diffusion coefficient α across the interfaces, whereas the condition number of the preconditioned IETI system depends on p in a moderate way. Our numerical tests show a logarithmic dependence in 2d and a linear dependence on p in 3d. A new p robust geometrical multigrid method was proposed and rigorously analyzed in [5] based on the approximation results provided in [12].

3. Space-Time IGA for Parabolic Evolution Problems

Finally, we analyze new stable space-time IgA methods of the form:

$$\text{Find } u_h \in V_{0h} : a_h(u_h, v_h) = \ell_h(v_h) \quad \forall v_h \in V_{0h}, \tag{4}$$

for the numerical solution of parabolic diffusion problems like

$$\partial_t u - \Delta u = f \quad \text{in} \quad Q = \{(x,t) \in \mathbb{R}^{d+1} : \ x \in \Omega(t), t \in (0,T)\} \tag{5}$$

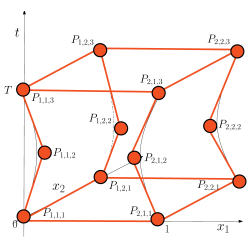
with Dirichlet boundary conditions $u = u_D := 0$ on $\Sigma = \{(x,t) \in \mathbb{R}^{d+1} : \ x \in \partial\Omega(t), t \in (0,T)\}$ and initial conditions $u = u_0 := 0$ on $\overline{\Sigma}_0 = \overline{\Omega(0)} \times \{0\}$ in fixed and moving spatial computational domains Ω resp. $\Omega(t) \subset \mathbb{R}^d$, where

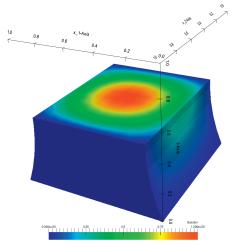
$$\begin{aligned} a_h(u_h, v_h) &:= \int_Q (\partial_t u_h v_h + \theta h \partial_t u_h \partial_t v_h + \nabla_x u_h \cdot \nabla_x v_h - \theta h \partial_t \nabla_x u_h \cdot \nabla_x v_h)\, dxdt \\ &\quad + \theta h \int_{\Sigma_T} \nabla_x u_h \cdot \nabla_x v_h \, ds, \\ \ell_h(v_h) &:= \int_Q f\,[v_h + \theta h \partial_t v_h]\, dxdt, \end{aligned}$$

with some positive θ. The discrete bilinear form $a_h(u_h, v_h)$ is elliptic on the IgA space $V_{0h} = V_{0h}(Q)$ with respect to the discrete energy norm $\|v\|_h = (\|\nabla_x v\|^2_{L_2(Q)} + \theta h \|\partial_t v\|^2_{L_2(Q)} + (1/2)\|v\|^2_{L_2(\Sigma_T)} + \theta h \|\nabla_x v\|^2_{L_2(\Sigma_T)})^{1/2}$. This property together with a corresponding boundedness property, consistency and approximation results for the IgA spaces yields an a priori discretization error estimate of the form

$$\|u - u_h\|_h \le C h^{r-1} \|u\|_{H^r(Q)}, \tag{6}$$

provided that $u \in H^{1,0}_0(Q) \cap H^l(Q)$ with $l \ge 2$, $V_{0h} \subset H^2(Q)$, and θ is sufficiently small, where $r = \min\{l, p+1\}$ and p denotes the underlying polynomial degree of the NURBS used for generating the IgA space $V_{0h}(Q)$. This theoretical convergence result is confirmed by several numerical experiments with low- and high-order IgA spaces, see [10] for more details. The parameter θ can be determined from a constant in an inverse inequality that depends on p and that can be computed by symbolic methods [6]. In the case of fixed spatial domains, θ can be chosen as a fixed positive constant, e.g., 0.1 as in our numerical experiments presented in [10]. The figures below show a moving spatial domain resulting in a curved space-time cylinder Q where the curved lateral faces are described by NURBS. The weights corresponding to the control points $P_{1,1,2}$, $P_{2,1,2}$, $P_{2,2,2}$ and $P_{1,2,2}$ are $1/\sqrt{2}$, whereas the other weights are equal to 1. The corresponding table on the right-hand side provides the discretization error $\|u - u_h\|_h$ and the corresponding convergence rate for the case $p = 4$.





dofs	$\|u - u_h\|_h$	rates
216	6.18145e-03	2.19
512	4.61833e-04	3.74
1728	2.57814e-05	4.16
8000	1.53822e-06	4.07
46656	9.48130e-08	4.02

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B-splines and T-splines

Tom Lyche
(joint work with Cesare Bracco, Carla Manni and with Hendrik Speleers)

Piecewise polynomials, or splines defined over a region in the plane form an indispensable tool in the sciences, with applications ranging from scattered data fitting to finding numerical solutions to partial differential equations using isogeometric methods In this talk we define a generalization of polynomial splines over planar T-meshes known as Tchebycheffian splines and we address the problem of determining their dimension. It is worth pointing out that the extension is non-trivial because the ring structure of algebraic polynomials cannot be used in this general setting. We show that only the bound on the number of (real) roots of the elements of the considered space is required. The results strengthens the structural similarity between algebraic polynomial and general Tchebycheffian spline spaces.

The dimension of polynomial spline spaces on a prescribed T-mesh for a given component-wise degree p and smoothness r has been addressed by several authors using different techniques, and it turns out to be a very challenging problem see for example [1, 3] and references therinn. Lower and upper bounds for the dimension are known, and an explicit expression has been determined in some special cases. In particular, the dimension is known for spline spaces over so-called quasi-cross-cut T-meshes – these are meshes where each edge extends to the boundary – and for spline spaces with $p > 2r$ under some mild conditions on the T-mesh. On the other hand, instability in the dimension can occur if the degree is not large enough with respect to the smoothness.

The dimension problem of spline spaces over T-meshes faces the same difficulties as the dimension problem for polynomial spline spaces of total degree p over triangulations see [4] and references therein. In the latter case, the dimension is known for spline spaces over quasi-cross-cut partitions and for spline spaces with $p \geq 3r + 2$. Instability in the dimension has been illustrated for $p = 2r$. Some similar results are known for spline spaces of total degree p over general rectilinear partitions.

Among the various techniques to tackle the dimension problem, one can use the homological approach proposed in [6], where the techniques for polynomial splines over triangulations has been fine-tuned for polynomial splines over planar T-meshes. To this end, we generalize the techniques and the results presented in [6] for the algebraic polynomial case. More precisely, besides characterizing the Tchebycheffian spline space as a suitable homology space,

- we provide a dimension formula in terms of combinatorial quantities of the T-mesh, the smoothness, the dimensions of the underlying extended Tchebycheff spaces, and homology quantities;
- we derive lower and upper bounds for the dimension (under a specific assumption on the underlying extended Tchebycheff spaces);
- we provide an explicit expression for the (stable) dimension of spline spaces over quasi-cross-cut T-meshes, and of spline spaces with $p \geq 2r + 1$ under

some mild conditions on the T-mesh; the latter conditions are usually satisfied by T-meshes of interest in applications and identify a family of T-meshes larger than the one considered in [6];
- we illustrate that the dimension of Tchebycheffian spline spaces over T-meshes can be unstable, by generalizing the examples given for the polynomial spline case to some non-polynomial Tchebycheff spaces: the case of trigonometric and the case of exponential functions.

We start the talk by giving an introduction to Tchebycheff systems and Tchebycheffian B-splines see [5, 7].

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Spectral Analysis of Matrices from Isogeometric Methods

Carla Manni

(joint work with M. Donatelli, C. Garoni, F. Pelosi, F. Roman, S. Serra-Capizzano, D. Sesana, H. Speleers)

When discretizing a linear PDE by a linear numerical method, the computa- tion of the numerical solution reduces to solving a linear system. The size of this system grows when the discretization parameter n increases, i.e., when we refine the discretization mesh. We are then in the presence of a sequence of linear systems with increasing size. It is usually observed in practice that the corresponding sequence of discretization matrices enjoys an asymptotic spectral distribution. The spectral distribution of a sequence of matrices is a relevant concept. Roughly speaking, if the sequence of matrices $\{A_n\}$ is distributed like the function f then the eigenvalues of A_n behave like a sampling of f over an equispaced grid on the domain of f. In this case, the function f is called the (spectral) symbol of the sequence $\{A_n\}$.

In this talk we consider a general second-order elliptic Partial Differential Equation (PDE),

$$\begin{cases} -\nabla \cdot K\nabla u + \boldsymbol{\alpha} \cdot \nabla u + \gamma u = \mathrm{f}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^d$ is a bounded open domain with Lipschitz boundary, $K : \Omega \to \mathbb{R}^{d\times d}$ is a symmetric positive definite matrix of functions in $L^\infty(\Omega)$, $\boldsymbol{\alpha} : \Omega \to \mathbb{R}^d$ is a vector of functions in $L^\infty(\Omega)$, $\gamma \in L^\infty(\Omega)$, $\gamma \geq 0$ and $\mathrm{f} \in L^2(\Omega)$.

We determine the spectral symbol and we analyze the spectral distributions of the sequences of matrices arising from isogeometric discretizations based on tensor product B-splines of degree $\boldsymbol{p} := (p_1, \ldots, p_d)$ for the problem stated above. We address both Galerkin and collocation methods and, in the spirit of Isogeometric Analysis (IgA), we also consider a geometry map $\mathbf{G} : [0,1]^d \to \overline{\Omega}$, see [2]. The main tool for computing spectral symbols of PDE discretization matrices is the theory of Generalized Locally Toeplitz (GLT) sequences, [10].

Recently, in [5, 6, 7, 8] we have proved the following results:

- a spectral distribution exists and is compactly described by a symbol f;
- the symbol f has a canonical structure incorporating:
  - the approximation technique, identified by a finite set of polynomials in the Fourier variables $\boldsymbol{\theta} := (\theta_1, \ldots, \theta_d) \in [-\pi, \pi]^d$;
  - the geometry, identified by the map $\boldsymbol{G}$ in the parametric variables $\hat{\boldsymbol{x}} := (\hat{x}_1, \ldots, \hat{x}_d)$ defined on the reference domain $[0,1]^d$;
  - the coefficients of the higher-order differential operator of the PDE, namely K, in the physical variables $\boldsymbol{x} := (x_1, \ldots, x_d)$ defined on the physical domain Ω;
- the symbol f is the same in the Galerkin and collocation isogeometric setting, up to a determinant factor $|\det(J_{\boldsymbol{G}})|$, being $J_{\boldsymbol{G}}$ the Jacobian matrix of $\boldsymbol{G}$.

The above analysis has been partially extended to the case of isogeometric discretizations based on tensor product generalized B-splines of degree $\boldsymbol{p}$, see [9].

It is worth to notice that the picture described in the second item above is intrinsic to the approximation of PDEs by any local method, such as Finite Differences (FDs) and Finite Elements (FEs). Actually, the formal structure of the symbol is essentially the same when considering different techniques to approximate the same problem; see [1, 10] and references therein.

Although the formal structure of the symbol is shared by different approximation techniques, some of its analytic features are not so common. For instance, if one of the components of $\boldsymbol{p}$, say p_i, is large, then the symbol f has "numerical zeros" at the points $(\hat{\boldsymbol{x}}, \boldsymbol{\theta}) \in [0,1]^d \times [-\pi, \pi]^d$ where $\theta_i = \pi$. More precisely, if $\theta_i = \pi$, the value $f(\hat{\boldsymbol{x}}, \boldsymbol{\theta})$ tends to 0 exponentially as $p_i \to \infty$. The latter information implies that small eigenvalues are related to high-frequency eigenvectors, and this non-canonical source of ill-conditioning is responsible for the slowdown of all the standard multigrid and preconditioning techniques when one of the p_i grows.

The spectral information provided by the symbol f has been exploited for designing algorithms with convergence speed independent of the fineness parameters and also substantially independent of the approximation parameters $\boldsymbol{p}$; see [3, 4]. Indeed, it was possible to construct iterative solvers of multigrid type which are not only optimal with respect to the mesh-size but also robust with respect to the approximation degree and the dimensionality of the domain.

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Low-Rank Tensor Methods in Isogeometric Analysis

Angelos Mantzaflaris

(joint work with B. Jüttler, B. N. Khoromskij and U. Langer)

Tensor-product splines are among the most popular approaches to construct multivariate representations suitable for isogeometric analysis (IGA). In fact, the multivariate parameterizations in IGA rely almost exclusively on tensor-product representations. In the existing isogeometric technology, multivariate tensor-product spline functions are represented by full coefficient tensors. The number of coefficients, which have to be stored and processed, grows exponentially with respect to the spatial dimension. Even with today's computational power, processing B-spline functions in several dimensions (for instance 3D or even 4D space-time simulations) with a few hundreds of knots per direction (i.e., a few billions of basis functions), is an extremely difficult task. Consequently, in PDE discretizations the size of the system matrix is subject to the same exponential growth.

The field of tensor calculus and its use in numerical simulation is as recent as isogeometric analysis [4, 8, 9]. These methods were introduced in order to overcome the curse of dimensionality, i.e., the exponential complexity with respect to the spatial dimension of the computational domain. In particular, tensor calculus enables the separation of variables of multivariate functions, allowing efficient computations on large scale grids. Recently, the approach has been applied to the decoupling of multivariate polynomial functions [3]. The advantage of tensor

methods has been demonstrated for high-dimensional problems. Efficient computations for a wide class of discrete functions and operators have been made possible. For instance, in applications in biology, the discretization and solution process of multi-dimensional steady-state and dynamical problems is made possible with a logarithmic complexity in the volume size of the computational grid [7].

The very same difficulty of dimensionality is encountered in isogeometric analysis, in particular in the task of matrix assembly. This task is computationally more challenging than in the case of traditional finite element methods. Apart from the rapidly growing matrix size under uniform $h-$refinement, this is also due to factors such as the increased degree and the larger supports of the ansatz functions (tensor-product B-splines), that burden the sparsity pattern and bandwidth of the system matrix. Contrarily to the high-dimensional setting in chemistry or biology, the problems occur already in three dimensions in our setting.

The standard approach to perform the assembly task in IGA consists in using Gaussian quadrature, but this does not give optimal runtimes, due to its high computational cost which is asymptotically bounded by $\mathcal{O}(n^d p^{3d})$. Several approaches have been introduced in order to overcome this problem. Firstly, new approaches reduce the number of quadrature points required per parametric dimension for exact or sufficiently accurate numerical integration [2, 5, 15]. Secondly, discretizations via collocation [14] have been studied, but the theoretical foundations of this mathematical technology are less well understood than in the case of Galerkin projection. In [1], the sum factorization technique is used for IGA, leading to a complexity bound of $O(n^d p^{2d+1})$. An efficient direct solver for certain systems in tensor-product form is used to derive fast IGA preconditioners in [13]. Moreover, this problem has motivated the use of GPU programming [6] for accelerating the assembly process.

In another attempt to address this problem we developed an interpolation-based approach that approximately transforms the integrands into piecewise polynomials and uses look-up tables to evaluate their integrals [10, 11], with complexity $\mathcal{O}(n^d p^{2d})$. Shortly after, this led us to the use of tensor methods to accelerate the assembly process further [12], notably reducing the complexity to $\mathcal{O}(Rdnp^3)$, where the rank parameter R captures the geometric complexity of the tensor-product patches parameterizing the physical domain. In doing so, we obtained a compact representation of the matrices that occur in IGA as sums of a small number of Kronecker products of auxiliary matrices, which are defined by univariate integrals. This representation, which is based on a low-rank tensor approximation of certain parts of the integrands, makes it possible to achieve a significant speedup of the assembly process without compromising the overall accuracy of the simulation.

The property which allows such low-rank representations in the context of IGA is the presence of the global (patch-wise) geometry map. This map, which describes the shape of the computational domain, is one of the main new features in IGA, compared to the traditional locally parameterized isoparametric finite elements. The tensor-product structure of the map is passed on to the isogeometric

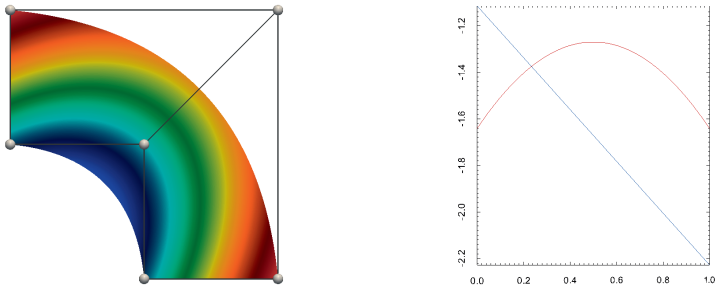

FIGURE 1. A B-spline quarter annulus domain (left) and the "Skeleton functions" of the Jacobian determinant.

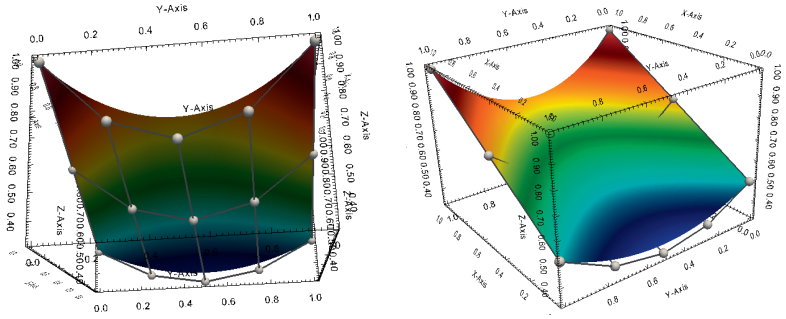

FIGURE 2. Two views of the graph of the Jacobian determinant, plotted over $[0,1]^2$.

discretization space that is derived from it. The involvement of this global mapping in the discretization of PDEs could be regarded as a disadvantage at first glance, as opposed to the purely local nature of classical finite elements. However, using the low-rank representation, we demonstrate that it is possible to benefit significantly from the regularity of its structure. As an example, consider a quarter annulus described by tensor-product B-splines. In Figure 1, the color field on the annulus is the value of the Jacobian determinant. The "skeleton functions" (i.e., the univariate functions) describing the Jacobian determinant is shown on the right side. The linear function is associated with the first direction, while the bubble function describes the determinant along the second parametric direction. In Figure 2 two views of the graph of the Jacobian determinant and its control grid over the parameter domain $[0,1]^2$ are shown. The degree is 2 in the first direction and 4 in the second one. The fact that this determinant is separable as a product of two functions implies that this shape can be treated as efficiently as a simple square domain.

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Isogeometric Analysis on Domains of Arbitrary Topology

Bernard Mourrain

Shapes are not always rectangular. They can have holes, assembled components, ... In this talk, we investigate isogeometric analysis methods on domains with arbitrary topology. This raises the questions of how to describe complex domains and what type of spline spaces we can use to approximate the solutions of Partial Differential Equations on these domains.

We describe two approaches, which represents these domains as collections of faces glued along edges.

The first approach consists in using constant glueing functions between faces that share an edge. We present a type of splines defined over a rectangular mesh with arbitrary topology, which are piecewise polynomial functions of bi-degree

(d, d) and $\mathcal{C}^r$ parameter continuity. We analyze their dimension and exhibit basis functions called Hermite bases for bicubic spline spaces. We investigate their potential applications for solving partial differential equations (PDEs) over a complex physical domain in the framework of Isogeometric analysis. We analyze the property of approximation of these spline spaces for the L^2-norm. Despite the fact that the basis functions are singular at extraordinary vertices, we show that the optimal approximation order and numerical convergence rates are reached by setting a proper parameterization.

The second approach consist in fixing transition maps satisfying some compatibility properties. We show that if the functions and the parametrization are $\mathcal{G}^r$-regular after composition by these transition maps, they induce $\mathcal{C}^r$ functions on the physical domain which can be used for Galerkin-type formulations in the context of Isogeometric Analysis. We analyze the space of geometrically continuous piecewise polynomial functions, or splines, for rectangular and triangular patches, with general rational transition maps. To define these spaces of $\mathcal{G}^1$ spline functions, we introduce the concept of topological surface with gluing data attached to the edges shared by faces. The framework does not require manifold constructions and is general enough to allow non-orientable surfaces. We describe compatibility conditions on the transition maps so that the space of differentiable functions is ample and show that these conditions are necessary and sufficient to construct ample spline spaces. We determine the dimension of the space of $\mathcal{G}^1$ spline functions which are of degree $\leq k$ on triangular pieces and of bi-degree $\leq (k, k)$ on rectangular pieces, for k big enough. A separability property on the edges is involved to obtain the dimension formula. We describe an explicit construction of basis functions attached respectively to vertices, edges and faces; An example of bases of $\mathcal{G}^1$ splines on a simple mesh illustrate the construction.

This talk is based on the two papers [1, 2].

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Linear Finite Elements on Distorted Triangulations

Peter Oswald

The finite element method (FEM) is a workhorse for solving partial differential equations in a variational setting. FEM theory (approximation by piecewise polynomials) is well-established if the underlying domain partitions are well-shaped, e.g., locally quasi-uniform and regular. Nevertheless, applications to problems with boundary and interior layers or more generally to the optimal approximation of functions on higher-dimensional domains with essential support along lower-dimensional sets would benefit from the use of distorted partitions. It is widely

believed that discarding the regularity assumptions is possible only under special circumstances. It was our goal to look into possible extensions of the theory to arbitrary partitions. Using the simplest case of linear C^0 finite element spaces $V_{\mathcal{T}}$ over finite triangulations $\mathcal{T}$ of a bounded polygonal domain $\Omega \subset \mathbb{R}^2$, we have investigated how far certain statements valid for families of shape-regular triangulations can be extended to arbitrary triangulations. The following case studies have been pursued over the years:

Mesh independence of L_∞-bounds for the L_2-orthoprojector. Denote by $P_{\mathcal{T}}$ the L_2-orthoprojector onto $V_{\mathcal{T}}$. As an operator, $P_{\mathcal{T}}$ is well-defined on $L_1(\Omega)$, with range in $C(\Omega)$. Bounds for the norm $\|P_{\mathcal{T}}\|_{L_\infty \to L_\infty}$ are useful in FE analysis, and immediately imply bounds in other L_q-norms ($1 \le q \le \infty$) by duality and interpolation. In [1], we published explicit examples showing that, in contrast to the one-dimensional situation and to shape-regular simplicial partitions in higher dimensions, $\|P_{\mathcal{T}}\|_{L_\infty \to L_\infty}$ may unboundedly grow with the number of triangles $|\mathcal{T}|$ in $\mathcal{T}$. More precisely, we showed that for simple Ω such as a square in $\mathbb{R}^2$

$$\sup_{\mathcal{T}:\,|\mathcal{T}|\le J} \|P_{\mathcal{T}}\|_{L_\infty \to L_\infty} \ge cJ, \qquad J \to \infty, \tag{1}$$

for some absolute constant $c > 0$. This unboundedness result also holds in higher dimensions. It remains an open question if (1) is sharp, i.e., if there exists a matching upper bound of the form $\le CJ$ with an absolute constant $C < \infty$.

On the positive side, in [2] we gave some new upper bounds for $\|P_{\mathcal{T}}\|_{L_\infty \to L_\infty}$ that cover a wide range of practically useful distorted triangulations such as Shishkin and Bakhvalov meshes. Roughly speaking, these upper bounds depend on the maximal valence of the vertices in $\mathcal{T}$ and put restrictions on the possible growth of area in chains of triangles attached to each other. How subtle estimates for $\|P_{\mathcal{T}}\|_{L_\infty \to L_\infty}$ depend on the topology and geometry of $\mathcal{T}$ is demonstrated on the example of triangulations obtained by dividing each cell of a tensor-product partition $\mathcal{P}$ of a rectangle into two triangles. While orthoprojectors onto the bilinear finite element space over the tensor-product partition $\mathcal{P}$ possess a uniform L_∞ bound, the associated $P_{\mathcal{T}}$ may or may not inherit this boundedness, depending on how exactly $\mathcal{T}$ is obtained from $\mathcal{P}$. The positive news is that there is a simple subdivision algorithm applicable to arbitrary such $\mathcal{P}$ and resulting in $\mathcal{T}$ with a uniform bound for $\|P_{\mathcal{T}}\|_{L_\infty \to L_\infty}$. For details and further references, we refer to [2].

Energy-norm convergence of the Galerkin finite element method for Laplace's equation. Consider for simplicity the Poisson problem $-\Delta u = f$ equipped with one of the standard boundary conditions on a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ and right-hand side $f \in L_2(\Omega)$. It is well-known that if the solution u is in $H^2(\Omega)$, and $\mathcal{T}$ satisfies certain restrictions (such as the celebrated maximum angle condition of Babuška and Aziz [3]) then the Galerkin solution $u_{\mathcal{T}} \in V_{\mathcal{T}}$ comes with an optimal H^1 error estimate of the form

$$\|u - u_{\mathcal{T}}\|_{H^1} \le C h_{\mathcal{T}} |u|_{H^2} \tag{2}$$

in terms of the maximal diameter $h_{\mathcal{T}}$ of the triangles in $\mathcal{T}$. In [3], a special family of triangulations of the unit square $\Omega = [0,1]^2$ was used to show that the estimate (2) may not hold if the maximum angle condition is violated. A precise quantitative characterization of this loss of accuracy in the Galerkin method as well as the question of a possible convergence failure remained open.

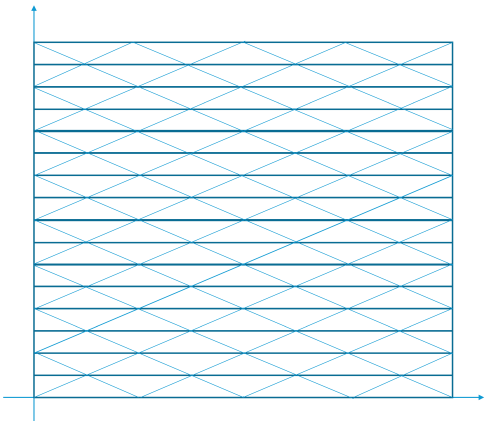

FIGURE 1. Schwarz-Babuška-Aziz triangulation $\mathcal{T}_{4,8}$

In [4], we slightly modified the example from [3], and showed the two-sided estimate

$$\|u - u_{\mathcal{T}_{n,m}}\|_{H^1} \asymp \min(1, m/n^2), \quad m \geq n \to \infty, \tag{3}$$

for a Poisson problem with right-hand side $f(x,y) = 1$ and polynomial solution $u(x,y) = x(1-x)$ on the unit square $[0,1]^2$ (for simplicity, periodic boundary conditions in y-direction and homogeneous Dirichlet conditions at $x = 0, 1$ were assumed). The family of triangulations $\mathcal{T}_{n,m}$ (Fig. 1 shows the case $n = 4$, $m = 8$) was already used in [3] but appeared earlier in work by H. Schwarz (1880). The proof of (3) is elementary, it shows that the Galerkin method converges in energy norm to the (smooth) solution of this Poisson problem for this family of distorted triangulations if $m/n^2 \to 0$, otherwise it does not converge. As strange it may sound, a formal proof of the possible non-convergence of the FEM for a Poisson problem with smooth solution was missing in the literature. V. Kucera (Prague) is currently pursuing the quest for more general necessary and sufficient conditions for convergence of the FEM.

In our talk, we have also briefly mentioned some recent developments in the **convergence theory of multiplicative Schwarz methods**, a modified version of the classical Wiener-Halperin alternating direction method. Roughly speaking, a

spd variational problem on a Hilbert space H of finding $u \in H$ such that

$$a(u, v) = F(v) \qquad \forall\, v \in H \quad (\, F \in H' \,),$$

is iteratively solved using a finite or infinite number of auxiliary spd variational problems with bilinear forms $b_i(\cdot, \cdot)$ on Hilbert spaces H_i, related to the original space by linear operators $R_i : H_i \to H$. The j-th iteration step of the multiplicative Schwarz method consists of choosing a certain index i, computing the restriction of the residual of the current iterate to H_i' using the adjoint $R_i' : H' \to H_i'$, solving the i-th subproblem, and updating the current iterate by this partial inverse using appropriate relaxation parameters α_j and ω_j:

$$u^{j+1} = \alpha_j u^j + \omega_j R_i B_i^{-1} R_i'(F - Au^j).$$

The recent developments concern greedy and random choices for the choice of i in the j-th step [5], the application to solving non-symmetric and overdetermined problems in a least-squares sense (Kaczmarz-type methods, [6, 7]), and the case of infinitely many auxiliary problems [8]. Due to lack of space we refer to the cited papers, and the references therein. We note that our investigations in this direction overlap in many aspects with the huge body of recent research papers on block-iterative solvers for large-scale convex and non-convex optimization. Our specific contributions concern the incorporation of overlapping blocks (redundancy) and the emphasis on the concept of stable space splittings related to preconditioning.

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Design and Analysis for Irregular Quad Layout

Jörg Peters
(joint work with Kęstutis Karciauskas, Thien Nguyen)

Presently the practical construction of smooth parametric surfaces (and other manifolds) in computer aided design is based on glueing together a finite number of surface pieces. Such G^k *constructions* match, after reparameterization (change of variables from one surface piece to the next) derivatives (jets) up to kth order; for example G^2 surfaces have continuous curvatures. [15] provides a survey on the theory of G^k surface construction, including algebraic considerations such as the valence-dependent vertex enclosure problem and how to derive the reparameterization from a G^k surface construction. An alternative to G^k constructions is the use of singular parameterizations. Singular constructions include generalized subdivision surfaces (Catmull-Clark surfaces) that are widely used for visualization and animation (see e.g. [17] for a survey) as well as explicitly singular constructions [13, 1].

In design, the representation of choice for surface pieces are tensor-product splines. These piecewise polynomials follow the outlines of an input mesh consisting of quadrilaterals (*quads*) whose vertices can be interpreted as control points scaling linear combinations of basis functions. Constructions that include mesh *irregularities*, i.e. where three or more than four surface pieces join, have been known since the late 1980s [12, 2, 14, 11]. However, constructions of least polynomial degree and good shape (measured in terms of highlight lines for an obstacle course of input meshes [5]) have only recently been developed: for example explicit recipes – without solving systems of equations – now exist for constructing, high-quality G^2 surfaces of degree bi-6 (or bi-5 when 2×2 pieces are permitted per quad) [7, 6]. These surfaces are linear combinations of B-spline-like functions for irregular quad mesh layout.

In 2014 a formal proof was given that G^k constructions yield C^k finite elements when the same reparameterization is used both for the geometry (surface) and the (analysis) functions [16, 3]. This means that every G^k construction yields a C^k finite element suitable for the iso-parametric IGA framework. [8] is the first account of such *generalized IGA* (gIGA) elements for irregular quad layout in the literature. The paper also compares the gIGA approach to alternatives such as methods of classical finite element analysis and analysis using generalized-subdivision functions.

In gIGA constructions [8, 4, 10, 9]), each point of the quad mesh acts as a B-spline-like control point to determine geometric shape or for IGA computations. The gIGA constructions intentionally do not expose the often asymmetrically-distributed Bézier degrees of freedom that exist after G^1 constraints have been enforced. Rather they harness them to optimize shape. Practice has to show whether this strategy is better than to harvest all possible free Bézier coefficients as degrees of freedom for IGA.

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IGA Collocation, aka "the Ultimate Reduced Quadrature IGA Method": Some Results, Applications, and Open Problems

Alessandro Reali

(joint work with Thomas J.R. Hughes)

Isogeometric analysis (IGA) was first introduced in 2005 [1] with the main aim of bridging Computer Aided Design (CAD) and Finite Element Analysis (FEA). The basic IGA concept, based on the isoparametric paradigm, consisted of adopting the same basis functions (splines) used for geometry representations in CAD systems for the approximation of field variables. The original goal was a cost-saving simplification of the typically expensive mesh generation and refinement processes

required by standard FEA. In addition, thanks to the high-regularity properties of its basis functions, IGA showed a better accuracy per-degree-of-freedom and an enhanced robustness with respect to standard FEA. Such a superior behavior was exploited in a number of applications ranging from solids and structures to fluids and fluid-structure interaction [2]. Moreover, the newly available higher regularity opened also the door to geometrically flexible discretizations of higher-order partial differential equations in primal form. A well-known important issue of IGA is related to the development of efficient integration rules able to reduce the high array formation costs induced by standard Gaussian quadrature, in particular when higher-order approximations are employed. Ad-hoc quadrature rules were proposed by several authors, but the development of a general and effective solution for Galerkin-based IGA methods is still an open problem.

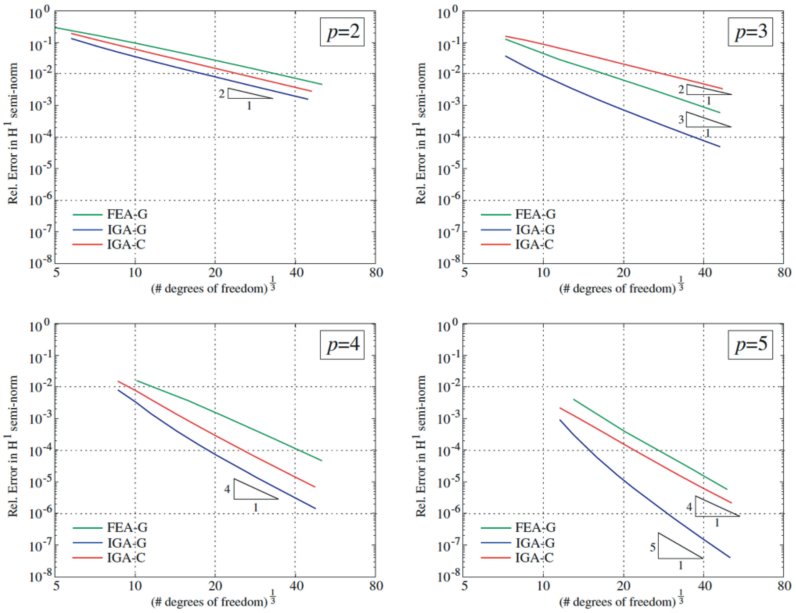

FIGURE 1. 3D elasticity test [4]. Relative error in H^1-seminorm versus number of degrees of freedom per parametric direction for IGA collocation (IGA-C) as well as Galerkin IGA (IGA-G) and FEA (FEA-G).

In an attempt to address the issue above taking full advantage of the special possibilities offered by IGA and in particular by the available higher regularity, isogeometric collocation (IGA-C) schemes have been recently proposed [3]. The main idea of IGA-C consists of the discretization of the governing partial differential equations in strong form, within the isoparametric paradigm, reducing the number of evaluations needed for array formation to only one per degree of

freedom. The aim is to optimize the computational cost still relying on IGA geometrical flexibility and accuracy. In general, IGA-C features look particularly attractive when evaluation and formation costs are dominant, as in the case, e.g., of explicit structural dynamics. Detailed comparisons with both IGA and FEA Galerkin-based approaches were carried out, showing IGA-C advantages in terms of accuracy versus computational cost, in particular for higher-order approximation degrees, as it can be, e.g., seen comparing Figures 1 and 2 [4].

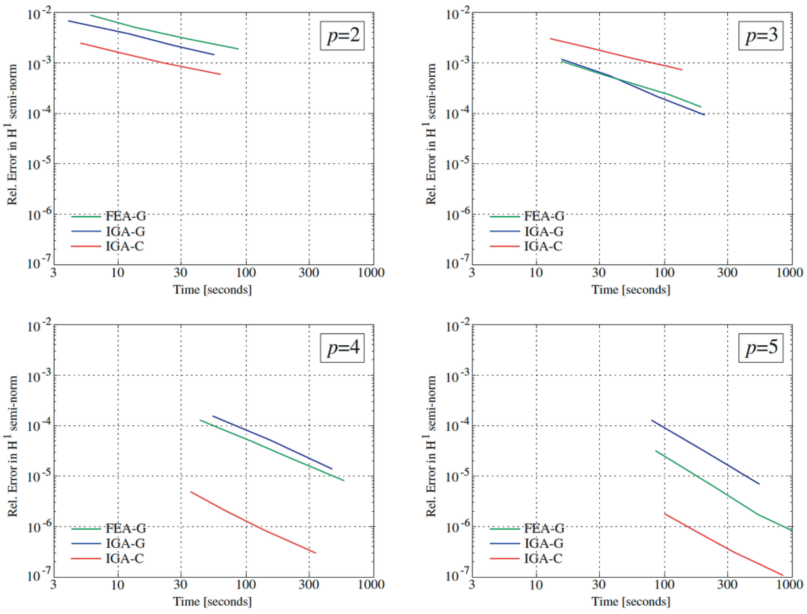

FIGURE 2. 3D elasticity test [4]. Relative error in H^1-seminorm versus computational time for IGA collocation (IGA-C) as well as Galerkin IGA (IGA-G) and FEA (FEA-G).

Since its introduction, many promising significant works on IGA-C were published in different fields, including, among others, phase-field modeling, linear and nonlinear elasticity, contact, as well as several interesting studies in the context of structural elements (see, e.g., [5] and references therein).

In particular, IGA-C allows to reach new frontiers for mixed formulations, where methods that are known to be unstable in the Galerkin framework seem to be stable and very efficient when combined with IGA-C.

Finally, it has been recently shown that IGA-C can be conveniently combined with many different spline spaces, able for example to be locally refinable or possessing other desirable properties not available with classical B-splines or NURBS (like, e.g., Hierarchical NURBS, T-splines, or Generalized B-splines).

All these results naturally propose collocation as one of the most promising research directions in the field of IGA, able to combine simplicity and efficiency with an incredibly high potential. Unfortunately, a sound mathematical understanding of this approach is still missing, as proven by the lack of mathematical theory for the multi-dimensional case, currently representing one of the main open problems in this context.

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Analysis-suitable C^1 Multipatch Isogeometric Spaces

Giancarlo Sangalli
(joint work with Annabelle Collin and Thomas Takacs)

Since higher dimensional spline spaces possess a tensor-product structure, the representation of domains that have a complex geometry is non-trivial. In this paper we focus on multi-patch representations. While the implementation of C^0-continuity over multi-patch domains is well understood (see e.g. [8, 1, 12] for strong and [3] for weak imposition of the C^0 conditions), C^1-continuity is not. Several studies have tackled the problem of constructing function spaces of C^1 or higher order smoothness, such as [4, 6] for spline spaces, [9] for triangulations and [10] comparing both. Nevertheless, the construction of smooth isogeometric spaces with optimal approximation properties on complex geometries is still an open and challenging problem. This is related to the problem of finding parametrizations of smooth surfaces having complex topology, which is a fundamental area of research in the community of Computer Aided Geometric Design (CAGD) over the last decades.

We review two different strategies for constructing smooth multi-patch geometries and corresponding isogeometric spaces. One possibility is to adopt a geometry parametrization which is globally smooth almost everywhere, with the exception of a neighborhood of the extraordinary points (or edges in 3D), see Figure 1 (left). The other possibility is to use geometry parametrizations that are only C^0 at patch interfaces, see Figure 1 (right). The first option includes subdivision surfaces [5] and the T-spline construction in [11] and, while possessing attractive

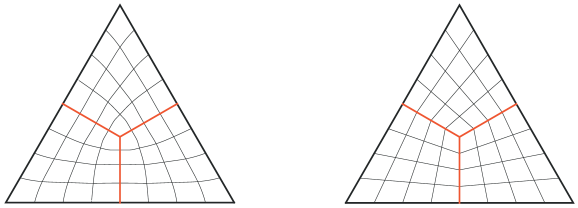

FIGURE 1. Two possible parametrization schemes: C^1 away from extraordinary points (left) and C^0 everywhere (right).

features, they seem to possess optimal approximation properties for some configurations, but in general lack accuracy [10, 6]. In our work we consider the second possibility, corresponding to the right part of Figure 1. The construction of C^1 isogeometric functions over a C^0 parametrization can be interpreted conveniently as geometric continuity G^1 of the graph parametrization. Following this approach, [7] constructed a basis for the shape functions, then analyzed its dimensionality of some configurations and numerically tested the order of convergence on h-refined meshes, obtaining optimal convergence with spline degree 3 or higher and global continuity C^1 on a two-patch bilinear geometry. The authors also show an example of *over-constrained* C^1 isogeometric spaces on a non-bilinear geometry. We refer to the latter situation as C^1 *locking*. Bilinear parametrizations of a multi-patch planar domain have been analyzed in [2], where it was shown that there exists a minimal determining set with local degrees of freedom for the C^1 smooth space if the polynomial degree is high enough (4 if some additional conditions are fulfilled, 5 otherwise).

Our work further develops the underlying theory in the direction of [2, 7]. Note that they give explicit charcaterizations in the form of minimal determining sets [2] or basis constructions [7] whereas we present an implicit characterization of the continuity conditions and derive restrictions on the function spaces. We moreover focus on the impact of refinement schemes as they are employed in isogeometric analysis. Within the isogeometric framework, the concepts of h-refinement, equivalent to knot insertion, as well as k-refinement, simultaneous elevation of the polynomial degree and interior smoothness, are the two main concepts to increase accuracy of the approximating function space. We study these refinement schemes both theoretically and numerically and point out their limitations. We analyze the structure of C^1 isogeometric spaces over AS G^1 two-patch geometries. From that the optimal order of approximation for p-degree isogeometric functions having C^1 continuity across the patch interface and up to C^{p-2} continuity in the interface direction, and infer the C^1 locking for C^{p-1} continuity. Note that in this paper we do not give a detailed proof for the approximation error estimates, which may be the topic of future reserch. We analyze C^1 isogeometric spaces constructed over more general geometries and conclude that h-convergence is suboptimal beyond AS G^1 geometries. The extensions to surface domains and to NURBS are briefly discussed. Numerical tests on two- and multi-patch domains are reported. A key

question that remains to be studied is the flexibility of AS G^1 parametrizations, as well as construction methods for AS G^1 parametrizations on general geometries. We hope this work offers an innovative point of view which leads to a better understanding of this difficult question.

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On the Isogeometric Version of the Scaled Boundary Finite Element Method

Bernd Simeon

(joint work with Sven Klinkel, Lin Chen, Wolfgang Dornisch, Carlo Lovadina)

Recently, a new variant of Isogeometric Analysis has been introduced that discretizes the partial differential equation at hand based on a type of generalized polar coordinates, thus avoiding the construction of tri-variate NURBS parametrizations [1, 2].

While in the Boundary Element Method, the problem is mapped to the boundary using Green's function, this *Scaled Boundary Method* discretizes also the boundary or surface patches of the domain, respectively, but generates then a

formulation on a finite number of rays that emanate from an origin in the computational domain to the control points on the surface. Independent of the spatial dimension, this leads to a 1D boundary value problem on the rays. However, in this method a singularity in the origin of the boundary scaling arises that might affect the performance of numerical methods.

The talk reports on current research for this method class in the context of Isogeometric Analysis. In particular, a conjecture is formulated:
For certain classes of domains and PDEs (which ones?), the Galerkin projection in circumferential and radial direction is equivalent with standard isogeometric discretization for the corresponding polar parametrization.

The figure below illustrates the two concepts. The domain is parametrized via

$$\mathbf{x} = \mathbf{x}_0 + \xi(\mathbf{N}(\eta)\mathbf{X} - \mathbf{x}_0)$$

where $\mathbf{x} \in \Omega \subset \mathbb{R}^2$ is a point in Cartesian coordinates and $\mathbf{x}_0$ the scaling origin or center point. The new coordinates ξ and η are element of $[0,1]^2$, and $\mathbf{N}(\eta)\mathbf{X}$ stands for the boundary representation in terms of B-splines (or NURBS) $\mathbf{N}$ and control points $\mathbf{X}$, both arranged in matrices of appropriate dimensions. On the left, the coordinate transformation generates a 1D problem along the rays (indicated in red) that run from the origin to the control points. On the right, the parametrization directly induces a mesh.

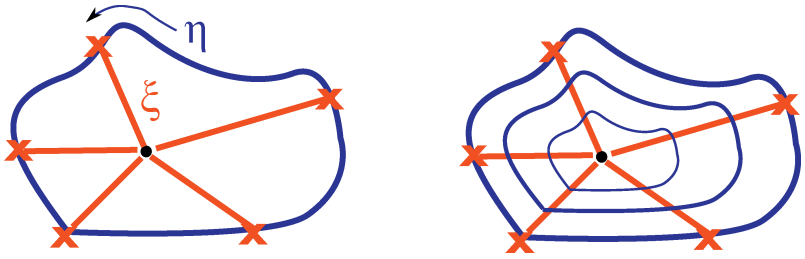

FIGURE 1. Sketch of polar coordinate system and isogeometric mesh

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