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Topological Recursion and TQFTs

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Abstract. The topological recursion is an ubiquitous structure in enumerative geometry of surfaces and topological quantum field theories. Since its invention in the context of matrix models, it has been found or conjectured to compute intersection numbers in the moduli space of curves, topological string amplitudes, asymptotics of knot invariants, and more generally semiclassical expansion in topological quantum field theories. This workshop brought together mathematicians and theoretical physicists with various background to understand better the underlying geometry, learn about recent advances (notably on quantisation of spectral curves, topological strings and quantum gauge theories, and geometry of moduli spaces) and discuss the hot topics in the area.

Mathematics Subject Classification (2010): 4Nxx, 14H60, 14H70, 37K10, 51PXX, 53Zxx, 81Qxx, 81Txx.

Introduction by the Organisers

At the core of the topological recursion procedure lies the notion of spectral curve: It is a Lagrangian curve $\mathcal{C}$ in $\mathbb{C} \times \mathbb{C}$, a one-form $\omega_{0,1}$ on $\mathcal{C}$ which is the restriction of an antiderivative of the symplectic form on $\mathbb{C} \times \mathbb{C}$, and a bi-form $\omega_{0,2}$ on $\mathcal{C}^2$ allowing a form-cycle duality on $\mathcal{C}$. Out of $\omega_{0,1}$ and $\omega_{0,2}$, it defines a sequence of n-forms $\omega_{g,n}$ on $\mathcal{C}^n$ (the correlators) by a recursion on $2g-2+n > 0$, and a sequence of numbers $\omega_{g,0}$ (the free energies). This definition is made to solve a set of loop equations, which are closely related to the Virasoro constraints. Already for simple examples of spectral curves, the $\omega_{g,n}$'s encode interesting geometric information, e.g. the intersection number of ψ classes on the moduli space $\mathcal{M}_{g,n}$ of genus g Riemann surfaces with n punctures, or numbers of coverings of the sphere by genus g surfaces (simple Hurwitz numbers). In general, the $(\omega_{g,n})_{g,n}$ have many interesting properties: Seiberg-Witten like relations for the variations of initial data, symplectic invariance (change of antiderivative $\omega_{0,1}$), modular properties/holomorphic anomaly in relation with deformations of $\omega_{0,2}$, explicit representation of $\omega_{g,n}$ in terms of integrals of tautological classes on $\overline{\mathcal{M}}_{g,n}$, etc.

This definition is strikingly universal, and has found a broad range of applications in the last 10 years, that motivated our workshop. It appears for instance in large size asymptotic expansion in Hermitean matrix models, $\hbar \to 0$ asymptotic expansion in integrable systems, and in enumerative geometry of surfaces. The latter is ubiquitous in mathematical physics, and includes random maps ($2d$ quantum gravity), invariants of 3-manifolds (Kontsevich integral, Chern-Simons theory, for example), topological string theory, gauge theories, etc. This commonality comes from the (sometimes unprecise) observation that, behind all those problems, there exist Feynman diagrams embedded on surfaces. The general properties of the topological recursion often have interesting interpretations in the problem where it is applied: construction of partition function which are automorphic forms, symplectic invariance seen as framing independence of the closed string sector, ELSV type formulae for enumerative problems, and many more.

The developments of the theory of the topological recursion and its relations to many problems of topological quantum field theory have formed the main topic of the workshop. It aimed at a better abstract understanding of the underlying geometry.

This motivated the presence of many specialists of quantum field theories from mathematics and theoretical physics who are not always working with topological recursion, but are handling problems of related interest: topological strings (Alim, Kashaev, Klemm), gauge theories (Dimofte, Hollands, Scheidegger, Teschner), deformation quantization (Petit), geometry of moduli spaces (Andersen, Mulase). Two other important round of topics which exhibited recent advances were especially discussed: quantum curves (Bouchard, Belliard, Petit, Sulkowski, Mulase) and Frobenius manifolds & cohomological field theories (Dunin-Barkowski, Do, Orantin, Milanov), with applications to enumerative geometry and integrable systems. The workshop was an occasion to diffuse those ideas to a broader community.

The workshop counted 27 participants (including the organizers), from all over Europe, Canada, the US, Australia and Japan. They were a balanced mix of established senior scientists and younger researchers, as well as 3 postdocs and Ph.D. students (Belliard, Dunin-Barkowski, Zenkevich). In order to boost scientific exchange, a problem session was arranged to probe still not completely shaped ideas. Moreover, a special, rather informal, evening session joint with the parallel workshop on *Hochschild Cohomology in Algebra, Geometry, and Topology* was organised with three short talks from each side. It was a very good idea to have this other mini-workshop at the same time as ours as many scientific discussions (e.g. on Hochschild cohomology and higher categories) relevant for the topic of our workshop, have resulted from it.

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Workshop: Topological Recursion and TQFTs

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Abstracts

Geometric hints of non-perturbative topological strings

Murad Alim
(joint work with S.T. Yau and J. Zhou)

This work appeared in Ref. [1]. We are interested in the topological string partition function defined by:

$$\mathcal{Z}(\lambda,t)=\exp\left(\sum_{2g-2+n>0}\frac{\lambda^{2g-2}}{n!}\mathcal{F}^g_{i_1\ldots i_n}(t)x_{i_1}\ldots x_{i_n}\right), \tag{1}$$

and where $\mathcal{M}$ denotes the moduli space of a Calabi-Yau threefold on either side of mirror symmetry and t denotes the local coordinates of a point in $\mathcal{M}$. $\mathcal{Z}$ is a formal power series in λ, which is in fact asymptotic. This is a phenomenon which also occurs in ODEs, hinting at missing, exponentially suppressed parts of the solution, see Ref. [2]. In the following, we will derive a differential equation in λ for a universal piece of $\mathcal{Z}$. This differential equation can be transformed into an Airy equation, which has two independent solutions. One solution gives a universal all genus result, the other solution is non-perturbative in λ.

1. Special Kähler geometry

Let $\mathcal{M}$ be the moduli space of a Calabi-Yau threefold, this can be the moduli space of complexified Kähler forms of a CY X or the moduli space of complex structures of the mirror $\check{X}$. We will use local coordinates $x^i\,, i=1,\ldots,n=\dim(\mathcal{M})$ and denote $\partial_i=\partial/\partial x^i\,,\partial_{\bar{\jmath}}=\partial/\partial\bar{x}^{\jmath}$. e^{-K} is a metric on $\mathcal{L}$ with connection K_i, it provides a Kähler form for a Kähler metric on $\mathcal{M}$, whose components and Levi-Civita connection are given by:

$$G_{i\bar{\jmath}}:=\partial_i\partial_{\bar{\jmath}}K\,,\quad \Gamma^k_{ij}=G^{k\bar{k}}\partial_i G_{j\bar{k}}\,. \tag{2}$$

We further introduce the holomorphic Yukawa couplings or threepoint functions

$$C_{ijk}\in\Gamma\left(\mathcal{L}^2\otimes\mathrm{Sym}^3T^*\mathcal{M}\right), \tag{3}$$

which satisfy

$$\partial_{\bar{\imath}}C_{ijk}=0\,,\quad D_iC_{jkl}=D_jC_{ikl}, \tag{4}$$

the curvature is then expressed as:

$$-R_{i\bar{\imath}\ j}^{\ \ l}=[\bar{\partial}_{\bar{\imath}},D_i]^l{}_j=\bar{\partial}_{\bar{\imath}}\Gamma^l_{ij}=\delta^l_iG_{j\bar{\imath}}+\delta^l_jG_{i\bar{\imath}}-C_{ijk}\overline{C}^{kl}_{\bar{\imath}}, \tag{5}$$

where D_i denotes the covariant derivative and:

$$\overline{C}^{jk}_{\bar{\imath}}:=e^{2K}G^{k\bar{k}}G^{j\bar{\jmath}}\overline{C}_{\bar{\imath}\bar{k}\bar{\jmath}}. \tag{6}$$

Definition 1.1. *A hermitian metric $G_{i\bar{\jmath}}$ is* special Kähler *if*[1]

[1] This definition is following Ref. [3], earlier equivalent definitions were given in Ref. [4]

- *It is a Kähler metric such that the corresponding Kähler form is 2π times the first Chern class of a line bundle $\mathcal{L}$ as given above.*
- *There is a holomorphic symmetric tensor with values in $\mathcal{L}^2$ as given in Eq. (3) satisfying Eqs. (4) such that the curvature satisfies Eq. (5).*

We also introduce the objects S^{ij}, S^i, S, which are sections of $\mathcal{L}^{-2} \otimes \mathrm{Sym}^m T\mathcal{M}$ with $m = 2, 1, 0$, respectively, and give local potentials for the non-holomorphic Yukawa couplings:

$$\partial_{\bar{\imath}} S^{ij} = \overline{C}_{\bar{\imath}}^{ij}, \qquad \partial_{\bar{\imath}} S^j = G_{i\bar{\imath}} S^{ij}, \qquad \partial_{\bar{\imath}} S = G_{i\bar{\imath}} S^i. \tag{7}$$

2. Polynomial structure of topological strings

2.1. **Holomorphic anomaly equations.** The topological string amplitudes at genus g with n insertions $\mathcal{F}^g_{i_1 \ldots i_n}$ are defined in Ref. [3] are sections of the line bundles $\mathcal{L}^{2-2g}$ over $\mathcal{M}$. These are only non-vanishing for $(2g-2+n) > 0$. They are related recursively in n by

$$D_i \mathcal{F}^g_{i_1 \cdots i_n} = \mathcal{F}^g_{i i_1 \cdots i_n}, \tag{8}$$

as well as in g by the holomorphic anomaly equation for $g = 1$ [5]

$$\bar{\partial}_{\bar{\imath}} \mathcal{F}^1_j = \frac{1}{2} C_{jkl} \overline{C}^{kl}_{\bar{\imath}} + (1 - \frac{\chi}{24}) G_{j\bar{\imath}}, \tag{9}$$

where χ is the Euler character of the CY threefold. As well as for $g > 2$ [3]:

$$\bar{\partial}_{\bar{\imath}} \mathcal{F}^g = \frac{1}{2} \overline{C}^{jk}_{\bar{\imath}} \left(\sum_{r=1}^{g-1} D_j \mathcal{F}^r D_k \mathcal{F}^{g-r} + D_j D_k \mathcal{F}^{g-1} \right). \tag{10}$$

2.2. **Feynman diagram solution and polynomial structure.** BCOV showed that the higher genus amplitudes can be cast in terms of Feynman diagrams with propagators S^{ij}, S^i, S and vertices $\mathcal{F}^g_{i_1 \ldots i_n}$. To obtain that solution, the $\partial_{\bar{\imath}}$ derivatives are integrated on both sides of Eq. (10) using

$$\partial_{\bar{\imath}} S^{ij} = \overline{C}^{ij}_{\bar{\imath}},$$

and the integrated special geometry relation which can be obtained from Eq. 5.

$$\Gamma^l_{ij} = \delta^l_i K_j + \delta^l_j K_i - C_{ijk} S^{kl} + s^l_{ij}, \tag{11}$$

where s^l_{ij} are holomorphic functions.

In Ref. [6] it was shown that the topological string amplitudes can be expressed as polynomials in finitely many generators of differential ring of multi-derivatives of the connections of special geometry. This construction was generalized in Ref. [7] for any CY manifold. It was shown there that $\mathcal{F}^g_{i_1, \ldots, i_n}$ is a polynomial of degree $3g - 3 + n$ in the generators S^{ij}, S^i, S, K_i where degrees $1, 2, 3, 1$ were assigned to these generators respectively. The purely holomorphic part of the construction as well as the coefficients of the monomials would be rational functions in the algebraic moduli.

3. Universal all genus structure

In the polynomial expression of $\mathcal{F}^g$ we consider the highest degree term in the generator S^{zz} which is a monomial of the form: $f(z)(S^{zz})^{3g-3}$, where $f(z)$ is a rational function of the modulus z. From the Feynman diagram rules of Ref. [3] and the polynomial structure we know that it is of the form: $f(z) = a_g\, C_{zzz}^{2g-2}, a_g \in \mathbb{Q}$. We introduce the topological string partition function, which only captures these terms in the following.

$$\mathcal{Z}_{top,s} = \exp(\sum_{g=2}^{\infty} a_g \lambda_s^{2g-2}) \ , \tag{12}$$

where we defined the rescaled coupling $\lambda_s^2 = \lambda^2\, C_{zzz}^2 (S^{zz})^3$.

Proposition 1. *$\mathcal{Z}_{top,s}(\lambda_s)$ satisfies the following differential equation* [1] *:*

$$\left(\left(\theta_{\frac{1}{3\lambda_s^2}}\right)^2 - \left(\left(\frac{1}{3\lambda_s^2}\right)^2 + \frac{1}{9}\right)\right) \lambda_s e^{1/3\lambda_s^2} \mathcal{Z}_{top,s} = 0\,, \quad \theta_{\frac{1}{3\lambda_s^2}} := \frac{1}{3\lambda_s^2} \frac{\partial}{\partial\left(\frac{1}{3\lambda_s^2}\right)}\,. \tag{13}$$

This is the modified Bessel differential equation in terms of the variable $\frac{1}{3\lambda_s^2}$ and the general solution in terms of the modified Bessel functions $I_{1/3}, K_{1/3}$ is given by:

$$\mathcal{Z}_{top,s} = \frac{e^{-\frac{1}{3\lambda_s^2}}}{\lambda_s} \left(I_{\frac{1}{3}}\left(\frac{1}{3\lambda_s^2}\right) + \zeta K_{\frac{1}{3}}\left(\frac{1}{3\lambda_s^2}\right)\right)\,. \tag{14}$$

The first term in this solution gives the universal all genus perturbative piece of the partition function. The second part does not contribute to the perturbative expansion around $\lambda_s = 0$, the parameter ζ is therefore free and should be determined from further insights on the non-perturbative structure of topological string theory.

We can further make the change of variables:

$$z = (2\lambda_s^2)^{-2/3}, v = 2^{-1/3} e^{1/3\lambda_s^2} \lambda_s^{1/3} \mathcal{Z}_{top,s}.$$

Eq. (13) then becomes the Airy equation:

$$\left(\partial_z^2 - z\right) v(z) = 0\,. \tag{15}$$

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Geometric quantisation of moduli space, conformal field theory and modular functors

Jørgen Ellegaard Andersen

1. TQFT background

Since their introduction by Atiyah [Ati], Segal [Seg] and Witten [Wit], topological quantum field theories (TQFT's) have been studied intensely using a wide range of techniques. The first construction in 2 + 1 dimensions was given by Reshetikhin and Turaev [RT1, RT2, Tur] using representation theory of quantum groups at roots of unity to construct link invariants and in turn derive invariants of 3-manifolds through surgery and Kirby calculus. Shortly thereafter, a combinatorial construction was given by Blanchet, Habegger, Masbaum and Vogel [BHMV1, BHMV2] in the language of skein theory.

2. Geometric quantization of moduli spaces

A geometric realization was proposed by Witten [Wit], suggesting the use of quantum Chern-Simons theory or conformal field theory to construct the 2-dimensional part. The gauge theoretic approach was studied independently by Axelrod, Della Pietra and Witten [ADPW] and Hitchin [Hit], proving that the quantum spaces arising from geometric quantization of Chern-Simons theory for compact gauge group are indeed independent of the conformal structure on the surface, in the sense that they are identified by parallel transport of a projectively flat connection over the Teichmüller space of the surface. These constructions have been expressed and generalized in purely differential geometric terms in [And5] and [AGL].

3. CFT and Modular Functors

The other construction proposed by Witten, through conformal field theory, was provided by Tsuchiya, Ueno and Yamada [TUY], and further by the work of Andersen and Ueno [AU1, AU2, AU3, AU4].

In fact, in [AU1] we described how to reconstruct the rank one ghost theory first introduced by Kawamoto, Namikawa, Tsuchiya and Yamada in [KNTY] from the the point of view of [TUY]. In [AU2] we described how one combines the work of Tsuchiya, Ueno and Yamada [TUY] with [AU1] to construct the vacua modular functor for each simple Lie algebra and a positive integer K called the level. Let us here denote the theory we constructed for the Lie algebra $\mathfrak{sl}(N)$ and level K by $\mathcal{V}^{\dagger}_{N,K}$.

We recall that a modular functor in the sense of Kevin Walker (see [AU2] for the precise axioms used) is a functor from a certain category of extended labeled marked surfaces (see again [AU2] for details) to the category of finite dimensional vector spaces. Note that we do not consider the duality axiom as part of the definition of a modular functor. We consider the duality axioms as extra data. For modular functors which satisfies the duality axiom, we say that it is a modular functor with duality.

In [Bla] Blanchet constructed a modular tensor category which we will here denote $H_K^{\mathrm{SU}(N)}$. It is constructed using skein theory and one can build a modular functor and a TQFT from this category following either the method of [BHMV2] or [Tur]. We denote the resulting modular functor $\mathcal{V}_K^{SU(N)}$. It is easy to check that the two modular functors $\mathcal{V}^\dagger_{N,K}$ and $\mathcal{V}_K^{SU(N)}$ have the same label set $\Gamma_{N,K}$. The main theorem of [AU4] reads as follows.

Theorem 1. *There is an isomorphism of modular functors*

$$I_{N,K} : \mathcal{V}_K^{SU(N)} \to \mathcal{V}^\dagger_{N,K},$$

i.e. for each extended labeled marked surface $(\mathbf{\Sigma}, \lambda)$ *we have an isomorphism of complex vector spaces*

$$I_{N,K}(\mathbf{\Sigma}, \lambda) : \mathcal{V}_K^{SU(N)}(\mathbf{\Sigma}, \lambda) \to \mathcal{V}^\dagger_{N,K}(\mathbf{\Sigma}, \lambda),$$

which is compatible with all the structures of a modular functor.

4. Applications

We have the following geometric application of our construction.

Theorem 2. *The connections constructed in the bundle of vacua for any holomorphic family of labeled marked curves given in* [TUY] *preserves projectively a unitary structure which is projectively compatible with morphism of such families.*

This Theorem is an immediate corollary of our main Theorem 1. By definition $\mathcal{V}^\dagger_{N,K}(\mathbf{\Sigma}, \lambda)$ is the covariant constant sections of the bundle of vacua twisted by a fractional power of a certain ghost theory over Teichmüller space as described in [AU2]. Using the isomorphism $I_{N,K}$ from our main Theorem 1, we transfer the unitary structure on $\mathcal{V}_K^{SU(N)}(\mathbf{\Sigma}, \lambda)$ to the bundle of vacua over Teichmüller space. Here we have used the preferred section of the ghost theory, to transfer the unitary structure to the bundle of vacua (see [AU2]). Since the unitary structure on $\mathcal{V}_K^{SU(N)}(\mathbf{\Sigma}, \lambda)$ is invariant under the extended mapping class group, the induced unitary structure on the bundle of vacua will be projectively invariant under the action of the mapping class group. But since the bundle of vacua for any holomorphic family naturally is isomorphic to the pull back of the bundle of vacua over Teichmüller space, we get the stated theorem. As a further application we get that

Theorem 3. *The Hitchin connections constructed in the bundle $\mathcal{H}^{(K)}$ over Teichmüller space, whose fiber over an algebraic curve (representing a point in Teichmüller space) is the geometric quantization at level K of the moduli space of semi-stable bundles of rank N and trivial determinant over the curve, projectively preserves a unitary structure which is projectively preserved by the mapping class group.*

This is an immediate corollary of Theorem 2 and then the theorem by Laszlo in [Las], which provides a projective isomorphism of the bundle $\mathcal{H}^{(K)}$ with its Hitchin connection [Hit] and then the bundle of vacua with the TUY-connections over Teichmüller space. We also get the following corollary

Corollary 4.1. *The projective monodromy of the Hitchin connection contains elements of infinite order for N and $k \notin \{1, 2, 4, 8\}$.*

Masbaum proved the corresponding result for the Witten-Reshetikhin-Turaev theory for $N = 2$ in [Mas]. This theorem is therefore an immediate corollary of Theorem 1 and the results of [AMU], which shows that Masbaum's arguments for $N = 2$ implies the statement for any N for the Witten-Reshetikhin-Turaev Theory. Prior to this, Funar proved, in the case of $N = 2$, that the monodromy group was infinite for the same levels for the Witten-Reshetikhin-Turaev theory in [Fur]. For a purely algebraic geometric proof of this result for $N = 2$, see [LPS].

A further corollary is that the projective space of projective flat sections of the bundle $\mathcal{H}^{(K)}$ over Teichmüller space of any closed surface Σ is isomorphic to the space $\mathbb{P}\mathcal{V}_K^{SU(N)}(\Sigma)$. Therefore we are allowed to use the geometric quantization of the moduli space of flat $SU(N)$-connections to study the Witten-Reshetikhin-Turaev TQFT's.

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Loop equations from differential systems

Raphaël Belliard
(joint work with Bertrand Eynard, Olivier Marchal)

To any differential system $d\Psi = \Phi\Psi$ where Ψ belongs to a Lie group (a fiber of a principal bundle) and Φ is a Lie algebra g valued 1-form on a Riemann surface Σ, is associated an infinite sequence of "correlators" W_n that are symmetric n-forms on Σ_n. We show that these correlators always satisfy "loop equations", the same equations satisfied by correlation functions in random matrix models, or the same equations as Virasoro or W-algebra constraints in CFT. In particular, under certain assumptions called the "Topological Type" property, they can be computed perturbatively using the topological recursion of Eynard and Orantin.

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Quantum curves and topological recursion

Vincent Bouchard

(joint work with Bertrand Eynard)

The Eynard-Orantin topological recursion is now understood as being a rather universal formalism that reconstructs generating functions for various enumerative invariants from the data of a spectral curve. While it originated in the context of matrix models, it has now been shown to be closely related to other fundamental structures in enumerative geometry, such as Virasoro constraints, Frobenius structures, Givental formalism and cohomological field theories. This explains, in part, why the topological recursion appears in so many different algebro-geometric context.

Another connection between the topological recursion and fundamental mathematical structures has been studied in recent years. With the intuition coming from determinental formulae in matrix models, it has been conjectured that the topological recursion reconstructs the WKB asymptotic solution of Schrödinger-like ordinary differential equations, known as *quantum curves.* More precisely, the claim is that there exists a Schrödinger-like ordinary differential operator, which is a *quantization* of the original spectral curve (which is why it is called a quantum curve), and whose WKB asymptotic solution is reconstructed by the topological recursion applied to this spectral curve.

The starting point of the topological recursion is a spectral curve. Here a spectral curve will mean a triple (Σ, x, y) where Σ is a Torelli marked compact Riemann surface and x and y are meromorphic functions on Σ, such that the zeroes of dx do not coincide with the zeroes of dy. Then x and y must satisfy an irreducible polynomial equation

$$P(x, y) = 0. \tag{1}$$

We say that a spectral curve is *admissible* if its Newton polygon has no interior point, and it is smooth as an affine curve. In particular, admissible curves all have genus zero.

Out of this spectral data, the topological recursion produces an infinite tower of meromorphic differentials $W_{g,n}(z_1, \ldots, z_n)$ on Σ^n. Then we construct a "wave-function" as

$$\Psi(z) = \exp\left(\frac{1}{\hbar} \sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{2g+n-1}}{n!} \int_a^z \cdots \int_a^z W_{g,n}(z_1, \ldots, z_n) \right), \tag{2}$$

where $a \in \Sigma$ is a choice of base point for integration of the meromorphic differentials. We generally take a to be a pole of x, but we can generalize the definition above for arbitrary integration divisors.

Then the question is whether there exists a *quantum curve*, that is, a quantization $\hat{P}(\hat{x}, \hat{y}; \hbar)$ of the spectral curve $P(x, y) = 0$ that kills the wave-function:

$$\hat{P}(\hat{x}, \hat{y}; \hbar)\Psi = 0. \tag{3}$$

In other words, the question is whether the asymptotic series in $\hbar$ given by (2) reconstructs the WKB expansion of (3). Note that by quantization of $P(x, y) = 0$, we mean that the $\mathcal{O}(\hbar^0)$ of the asymptotic expansion of (3) gives

$$P(x, S_0') = 0, \tag{4}$$

where $S_0' = \frac{d}{dx} \int_a^z W_{0,1}(z) = y(z)$.

In [1] we prove that there exists such a quantum curve for all admissible spectral curves. Further, the quantum curve has at most a finite number of $\hbar$ corrections. We also reconstruct the quantum curve explicitly from the topological recursion for arbitrary admissible spectral curves. This includes all of the genus zero quantum curves that have already been studied in the literature (to our knowledge), and many more. It includes many quantum curves of rank greater than two.

We also find an explicit dependence between the form of the quantum curve and the choice of integration divisor to reconstruct the asymptotic expansion (2) from the meromorphic differentials produced by the topological recursion. Different choices of integration divisors, for the same spectral curve, give rise to different quantum curves that are all quantizations of the original spectral curve; they generally differ by some choice of ordering in the quantization.

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Quantum modularity and Chern-Simons theory

Tudor Dimofte

(joint work with Stavros Garoufalidis, Don Zagier)

This talk was based on joint work [1] with Stavros Garoufalidis, inspired by Zagier's Quantum Modularity Conjecture [4], and many numerical results of Garoufalidis and Zagier.

Loosely speaking, Zagier's Quantum Modularity Conjecture (QMC) describes the asymptotics of colored Jones polynomials "near roots of unity," and predicts a relation (quantum modularity) among different roots of unity. More explicitly, recall that the colored Jones polynomials are a sequence of Laurent polynomials

$$J_{K,N}(q) \in \mathbb{Z}[q^{\pm 1}], \qquad N \in \mathbb{N} \tag{1}$$

assigned to any knot $K \subset S^3$. For any rational number a/c with a, c coprime and $c > 0$, one defines

$$J_K(a/c) := J_{K,c}(e^{2\pi i \frac{a}{c}}). \tag{2}$$

As $\alpha \to a/c$ through the rationals, the values $J_K(\alpha)$ tend to infinity in a very particular way. The conjecture is that for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ there is an asymptotic expansion

$$J_K\Big(\frac{aX+b}{cX+d}\Big) \sim (cX+d)^{\frac{3}{2}} J_K(X) \cdot e^{V_K/(c\hbar)} \phi_{K,a/c}(\hbar) \tag{3}$$

as $X \to \infty$ through rationals with bounded denominator, where V_K is the hyperbolic volume of the knot complement $S^3 \backslash K$, $\hbar := 2\pi i/(cX+d)$, and $\phi_{K,a/c}(\hbar) \in \mathbb{C}[\![\hbar]\!]$ is a formal power series whose coefficients depend only on K and a/c. Moreover, it is conjectured that the coefficients of $\phi_{K,a/c}(\hbar)$ are algebraic numbers of a certain type; in particular, $(\phi_{K,a/c}(0))^{2c}$ and all the coefficients of the unital series $\phi_{K,a/c}(\hbar)/\phi_{K,a/c}(0)$ are meant to lie in the trace field $F_K(e^{2\pi i \frac{a}{c}})$ of K with a root of unity adjoined.

This conjecture may be viewed as a generalization of the famous Volume Conjecture of Kashaev and Murakami-Murakami, and its extension by Gukov and Garoufalidis. The Volume Conjecture concerns the asymptotics of colored Jones polynomials at $q=1$, and follows by taking $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $X \in \mathbb{N}$ in (3).

It was argued by Gukov [2] that the Volume Conjecture may be motivated by a semi-classical expansion of complex Chern-Simons theory with gauge group $SL(2,\mathbb{C})$, specifically of the $SL(2,\mathbb{C})$ Chern-Simons partition function on a knot complement $S^3\backslash K$ together with a boundary condition enforcing parabolic holonomy at the knot. One of our goals was to similarly relate the QMC to asymptotics in $SL(2,\mathbb{C})$ Chern-Simons theory.

Complex Chern-Simons theory [1] on an oriented 3-manifold M has an action

$$S_{k,s}[M;\mathcal{A},\bar{\mathcal{A}}] = \frac{k+is}{8\pi}\int_M \mathrm{Tr}\big(\mathcal{A}d\mathcal{A} + \tfrac{2}{3}\mathcal{A}^3\big) + \frac{k-is}{8\pi}\int_M \mathrm{Tr}\big(\bar{\mathcal{A}}d\bar{\mathcal{A}} + \tfrac{2}{3}\bar{\mathcal{A}}^3\big)\,, \tag{4}$$

where $\mathcal{A}$ is an $SL(2,\mathbb{C})$ (say) connection on M, represented locally as a Lie-algebra-valued one-form; $\bar{\mathcal{A}}$ is its conjugate; and k,s are coupling constants or "levels." In order for the (schematic) path integral $\mathcal{Z}_{k,s}[M] = \int D\mathcal{A}\, D\bar{\mathcal{A}}\, e^{iS_{k,s}[\mathcal{A},\bar{\mathcal{A}}]}$ to be well defined, k must be restricted to integer values. This is analogous to the familiar quantization of the level in Chern-Simons theory with compact gauge group. Unlike the compact case, there is a second level s that may take arbitrary real values, or even complex values if one does not require unitarity.

We propose that in the singular limit $s \to -ik$ (with k fixed), the $SL(2,\mathbb{C})$ Chern-Simons partition function on a hyperbolic knot complement $M = S^3\backslash K$ has an asymptotic expansion

$$\mathcal{Z}_{k,s}[M] \sim \big(\tfrac{2\pi i}{\hbar}\big)^{3/2} e^{V_K/k\hbar} \phi_{K,1/k}(\hbar)\,, \tag{5}$$

of the same form as (3), with exactly the same formal series $\phi_{K,1/k}(\hbar)$, where now $\hbar = \frac{k-is}{k+is}$. We identify the subleading term $\phi_{K,1/k}(0)$ as a twisted Reidemeister-Ray-Singer torsion of $S^3\backslash K$. In addition, the complex Chern-Simons partition

[1] Many fundamental properties of this theory were introduced by Witten in [3].

function on a hyperbolic knot complement has an explicit computation (mathematically: a proposed definition) via a state-integral model, developed by Dimofte and Andersen-Kashaev. The state-integral model depends on a topological ideal triangulation of M, and allows an easy and explicit evaluation of the series $\phi_{K,1/k}(\hbar)$ in (5) in terms of the combinatorics of the ideal triangulation. We prove that for any M (and any choice of ideal triangulation), the series $\phi_{K,1/k}(\hbar)$ evaluated via the state-integral model has all the desired number-theoretic properties as the asymptotics in the QMC; namely $(\phi_{K,1/k}(0))^{2k}$ and all the coefficients of $\phi_{K,1/k}(\hbar)/\phi_{K,1/k}(0)$ belong to the number field $F_K(e^{\frac{2\pi i}{k}})$.

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Topological recursion for irregular spectral curves (A tale of two tau-functions)

Norman Do

(joint work with Paul Norbury)

Counting dessins d'enfant with topological recursion. Consider the number $B_{g,n}(\mu_1, \ldots, \mu_n)$ of genus g dessins d'enfant (in other words, bicoloured maps) with n labelled faces of perimeters $2\mu_1, \ldots, 2\mu_n$. The work of Kazarian and Zograf [9] and separate work of the authors [4] assert that this enumeration is governed by the topological recursion of Chekhov, Eynard and Orantin [3, 8]. More precisely, the numbers $B_{g,n}(\mu_1, \ldots, \mu_n)$ appear in the expansion coefficients of the correlation differentials $\omega_{g,n}$ obtained from applying the topological recursion to the rational spectral curve given by $x(z) = z + \frac{1}{z} + 2$ and $y(z) = \frac{z}{1+z}$. The genesis of our current work was the simple observation that this spectral curve possesses two branch points (in other words, zeros of $\mathrm{d}x$) given by $z = +1$, at which y is regular, and $z = -1$, at which y has a simple pole.

Regular spectral curves. We say that a spectral curve (C, x, y) — where x and y are non-zero meromorphic functions on the compact Riemann surface C — is *regular* if the zeroes of $\mathrm{d}x$ are simple and do not coincide with the zeroes of $\mathrm{d}y$ *nor the poles of* y. Note that this last restriction has largely been ignored in the previous literature on topological recursion.

At its branch points, a regular spectral curve looks locally like the *Airy spectral curve*, which is given by $x(z) = \frac{1}{2}z^2$ and $y(z) = z$. This universal local structure of regular spectral curves lifts to a universal local structure of the associated correlation differentials produced by the topological recursion, thus endowing the Airy spectral curve with central importance. The following are results concerning its correlation differentials, partition function and wave function.

- *Local behaviour of correlation differentials.* For $2g-2+n>0$, the correlation differentials of a regular spectral curve, up to leading order and a linear change of coordinates, satisfy

$$\omega_{g,n} \sim \omega_{g,n}^{\mathrm{Airy}} = \sum_{a_1,\ldots,a_n} \left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \right) \prod_{i=1}^{n} \frac{(2a_i+1)!!}{z_i^{2a_i+2}} \, \mathrm{d}z_i.$$

  The integral here is the evaluation of a product of psi-classes $\psi_1, \ldots, \psi_n \in H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ on the fundamental class of the moduli space of curves $\overline{\mathcal{M}}_{g,n}$.
- *Integrability.* The Airy partition function is a KdV tau-function.

$$Z_A(\mathbf{t};\hbar) = \exp\left[\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{2g-2+n}}{n!} \sum_{a_1,\ldots,a_n} \left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \right) \prod_{i=1}^{n} t_{a_i} \right]$$

- *Quantum curve.* The wave function $\psi(z,\hbar) = e^{\frac{z^3}{3\hbar}} z^{-\frac{1}{2}} Z_A(t_i = \frac{(2i-1)!!}{z^{2i-1}}; \hbar)$ satisfies the Airy equation $\left(\frac{1}{2}\hat{y}^2 - \hat{x}\right)\psi(z,\hbar) = 0$, where we use the operators $\hat{x} = x = \frac{1}{2}z^2$ and $\hat{y} = \hbar \frac{\partial}{\partial x} = \frac{\hbar}{z}\frac{\partial}{\partial z}$.

Irregular spectral curves. A simple branch point of a spectral curve can be described locally by $x(z) = z^2$ and $y(z) = z^m$ for some non-zero integer m.

- $m > 1$: The correlation differentials lose the key property of symmetry.
- $m = 1$: The correlation differentials are locally governed by the Airy spectral curve, as discussed above.
- $m = -1$: The correlation differentials are locally governed by the Bessel spectral curve, as discussed below.
- $m < -1$: We obtain $\omega_{g,n} = 0$, so there is no interesting local behaviour.

This classification suggests that the only irregular branch points one should consider are those where y has a simple pole. It would be interesting to extend the classification above to include higher order branch points, by using the so-called *global topological recursion* of Bouchard and Eynard [2].

The Bessel spectral curve. Define the *Bessel spectral curve* by $x(z) = \frac{1}{2}z^2$ and $y(z) = \frac{1}{z}$. The associated correlation differentials can be expanded to obtain

$$\omega_{g,n}(z_1,\ldots,z_n) = \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} U_{g,n}(\mu_1,\ldots,\mu_n) \prod_{i=1}^{n} \frac{\mu_i}{z_i^{\mu_i+1}} \, \mathrm{d}z_i.$$

Theorem 1 (Do–Norbury [5]). *The numbers $U_{g,n}(\mu_1,\ldots,\mu_n)$ satisfy the following recursion, where $S=\{2,3,\ldots,n\}$.*

$$\mu_1 U_{g,n}(\mu_1,\boldsymbol{\mu}_S)=\sum_{k=2}^{n}(\mu_1+\mu_k-1)\,U_{g,n-1}(\mu_1+\mu_k-1,\boldsymbol{\mu}_{S\setminus\{k\}})$$
$$+\frac{1}{2}\sum_{i+j=\mu_1-1} ij\left[U_{g-1,n+1}(i,j,\boldsymbol{\mu}_S)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S}}U_{g_1,|I|+1}(i,\boldsymbol{\mu}_I)\,U_{g_2,|J|+1}(j,\boldsymbol{\mu}_J)\right].$$

A simple corollary of this theorem is the fact that $U_{g,n}(\mu_1,\ldots,\mu_n)\neq 0$ if and only if $\mu_1,\ldots,\mu_n$ are positive odd integers whose sum is $2g-2+n$. Don Zagier has computed these numbers up to $g=26$ and observed striking properties.

Theorem 2 (Do–Norbury [5]).

- Integrability. *The Bessel partition function is a KdV tau-function with times $t_k=(2k-3)!!\,p_{2k-1}$.*

  $$Z_B(\mathbf{p};\hbar)=\exp\left[\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{2g-2+n}}{n!}\sum_{\mu_1,\ldots,\mu_n}U_{g,n}(\mu_1,\ldots,\mu_n)\prod_{i=1}^{n}p_{\mu_i}\right]$$

- Virasoro constraints. *The partition function $Z_B(\mathbf{p};\hbar)$ is annihilated by operators $L_0,L_1,L_2,\ldots$ satisfying the Virasoro commutation relation.*
- WDVV equation. *We have $\frac{\partial Z_B}{\partial\hbar}=MZ_B$ and $Z_B=\exp(\hbar M)1$, where*

  $$M=\frac{1}{8}\,p_1+\sum_{i,j\ \mathit{odd}}\left[\frac{1}{2}ij\,p_{i+j+1}\frac{\partial^2}{\partial p_i\partial p_j}+(i+j-1)\,p_ip_j\frac{\partial}{\partial p_{i+j-1}}\right].$$

- Quantum curve. *The wave function $\psi(z,\hbar)=e^{\frac{z}{\hbar}}z^{-\frac{1}{2}}Z_B(p_i=\frac{1}{z^i};\hbar)$ satisfies the Bessel equation $\left(\hat{y}\hat{x}\hat{y}-\frac{1}{2}\right)\psi(z,\hbar)=0$.*

Matrix models and geometry. The tau-function Z_B has appeared previously in the literature as the Brézin–Gross–Witten partition function [1].

Theorem 3 (Do–Norbury [5]). *Up to rescaling, the partition function Z_B equals*

$$Z_{BGW}=\int_{U(N)}\mathrm{d}U\,e^{\mathrm{Tr}\,(M^\dagger U+MU^\dagger)}=\int_{H(N)}\mathrm{d}H\,e^{\mathrm{Tr}\,(-H^{-1}+MH-N\log H)},$$

with the time-variables $t_k=\frac{1}{k}\mathrm{Tr}\,(MM^\dagger)^k$ and $t_k=\frac{1}{k}\mathrm{Tr}\,M^{-k}$, respectively.

We observe that the latter matrix integral is an example of a *generalised Kontsevich model*, which in other cases stores Witten's r-spin intersection numbers. Furthermore, these intersection numbers are known to be governed by the topological recursion applied to the spectral curve given by $x(z)=z^r$ and $y(z)=z$ [6].

The previous discussion leads us inexorably to the following conjecture.

The Bessel spectral curve stores (-2)-spin intersection numbers!

This conjecture should be taken with a grain of salt, given that Witten's r-spin intersection numbers are usually defined only for $r\geq 2$. Nevertheless, this statement may help to reveal the underlying geometry of the Bessel spectral curve.

The moral of our story is that spectral curves with simple branch points have local behaviour that is governed not only by the Airy spectral curve, but also by its Bessel counterpart. Furthermore, as evidenced by the enumeration of dessins d'enfant, such behaviour appears "in nature". It is thus natural to expect that the relation between Givental decomposition and topological recursion [7] generalises to irregular spectral curves.

> *Givental decomposition can be extended by associating Z_A to regular branch points and Z_B to irregular branch points.*

Evidence for this conjecture lies in the work of Alexandrov, Mironov and Morozov, who observe such decomposition formulas in the realm of matrix models [1].

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From cellular graphs to Topological Quantum Field Theories

Olivia Dumitrescu
(joint work with Motohico Mulase)

1. Introduction

We formulate $2D$ Topological Quantum Field Theory (TQFT) in terms of Edge-Contraction Axioms based on cell graphs. This set of rules based on edge contraction also represent the key structure of topological recursion. There is also a direct relation between a Frobenius algebra and Gromov-Witten theory when A is given by the quantum cohomology of a target space. In the case that the target space is 0-dimensional, the TQFT captures the whole Gromov-Witten theory, using the topological recursion with values in A. For example the case $A = \mathbb{Q}$ is related to the Gromov-Witten theory of a point, while the case $A = Z\mathbb{C}[G]$, i.e. the Frobenius algebra is the center of the group algebra of a finite group, is related to the Gromov-Witten theory of the classifying space BG.

2. Edge Contraction Axioms

2.1. **Background and notations.** We introduce the notations we will use in this section.

Let A be a Frobenius algebra, $\eta : A \times A \to K$ its Frobenius form and $\cdot$ denote the multpiplication structure of A. If $\{e_1, \ldots, e_n\}$ is a basis of A, let $\eta_{ab} := \eta(e_a, e_b)$ and its inverse matrix $[\eta^{ab}] := [\eta_{ab}]^{-1}$.

Properties 2.1. If A is a Frobenius algebra then the following statements hold

(1) There exists a unique counit $\epsilon : A \to K$ with $\epsilon(v) := \eta(1, v)$ for every $v \in A$.
(2) There exist an isomorphism $\lambda : A \to A^*$, $\lambda(u)(v) := \eta(u, v)$ for every $u, v \in A$.
(3) There exist a unique commultiplication $\delta : A \to A \otimes A$ that makes the following diagram commutative

$$\begin{array}{ccc} A & \xrightarrow{\ \delta\ } & A \otimes A \\ \lambda \downarrow & & \downarrow \lambda\otimes\lambda \\ A^* & \xrightarrow[m^*]{} & A^* \otimes A^*. \end{array}$$

In terms of basis elements the comultiplication is given by

$$\delta(v) = \sum_{i,j,a,b} \eta\big(v, e_i \cdot e_j\big)\eta^{ia}\eta^{jb} e_a \otimes e_b. \tag{2.1}$$

The axiomatic formulation of TQFT was introduced by Atiyah and Segal [1, 5] in 1980s. The sewing axiom of Atiyah endows the vector space a Frobenius algebra structure, A.

A 2D TQFT is a functor, Z, from the cobordism category of oriented surfaces to the tensor category of finite-dimensional vector spaces over a field satisfying a set of axioms. If the all boundaries of the cobordism, $\Sigma_{g,n}$, have the induced orientation we denote

$$\Omega_{g,n} := Z(\Sigma_{g,n}) : A^{\otimes n} \longrightarrow K.$$

Cohomological field theories were indroduced by Kontsevich and Manin in [4].

Remark 2.2. If $2g - 2 + n > 0$ an assignment $\Omega_{g,n} : A^{\otimes n} \longrightarrow K$ is a 2D TQFT if is the degree zero part of a CohFT. We refer to [2, Section 3] for more details.

We denote by $\Gamma_{g,n}$ the set of connected cell graphs of type (g, n) with labeled vertices.

Remark 2.3. Definition 2.4 will introduce a set of Edge Contraction Axioms on ribbon graphs. We emphasize that the edge contraction of Figure 2.1 corresponds to the multiplication structure m of the Frobenius algebra (see Axiom (2.3)) while the loop contraction of Figure 2.2, corresponds to the commultiplication δ (see Axioms (2.4) and (2.6)).

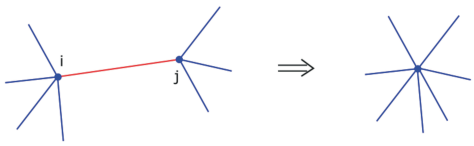

FIGURE 2.1

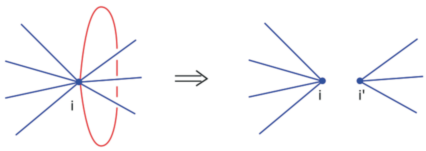

FIGURE 2.2

Definition 2.4. [2, Definition 4.4] Define the **edge-contraction axioms** as aset of rules for $\Omega(\gamma) : A^{\otimes n} \longrightarrow K$, a multilinear map assigned to each cell graph $\gamma \in \Gamma_{g,n}$. Assigning $v_i \in A$ to the i-th vertex of γ one can view $\Omega(\gamma)$ an n-variable function $\Omega(\gamma)(v_1, \ldots, v_n)$.

- **ECA 0**: For the cell graph consisting of only one vertex without any edge, $\gamma_0 \in \Gamma_{0,1}$, define

$$\Omega(\gamma_0)(v) = \epsilon(v), \qquad v \in A. \tag{2.2}$$

- **ECA 1**: Suppose there is an edge E connecting the i-th vertex and the j-th vertex for $i < j$ in $\gamma \in \Gamma_{g,n}$. Let $\gamma' \in \Gamma_{g,n-1}$ denote the cell graph obtained by contracting E as in Fig 2.1. Then

$$\Omega(\gamma)(v_1, \ldots, v_n) = \Omega(\gamma')(v_1, \ldots, v_{i-1}, v_i v_j, v_{i+1}, \ldots, \widehat{v_j}, \ldots, v_n), \tag{2.3}$$

  Here $\widehat{v_j}$ means we omit the j-th variable v_j at the j-th vertex of γ.
- **ECA 2**: Suppose there is a loop L attached at the i-th vertex of $\gamma \in \Gamma_{g,n}$. Let γ' denote the possibly disconnected graph obtained by contracting L and separating the vertex to two distinct vertices labeled by i and i' as in Fig 2.2. We assign an ordering $i - 1 < i < i' < i + 1$.

  If γ' is connected, then it is in $\Gamma_{g-1,n+1}$. We call L a *loop of a handle.* We then impose

$$\Omega(\gamma)(v_1, \ldots, v_n) = \Omega(\gamma')(v_1, \ldots, v_{i-1}, \delta(v_i), v_{i+1}, \ldots, v_n), \tag{2.4}$$

where the outcome of the comultiplication $\delta(v_i)$ is placed in the i-th and i'-th slots.

If γ' is disconnected, then write $\gamma' = (\gamma_1, \gamma_2) \in \Gamma_{g_1,|I|+1} \times \Gamma_{g_2,|J|+1}$, where

$$\begin{cases} g = g_1 + g_2 \\ I \sqcup J = \{1, \dots, \widehat{i}, \dots, n\} \end{cases} . \tag{2.5}$$

In this case L is a *separating loop.* Here, vertices labeled by I belong to the connected component of genus g_1, and those labeled by J on the other component. Let (I_-, i, I_+) (reps. (J_-, i, J_+)) be reordering of $I \sqcup \{i\}$ (resp. $J \sqcup \{i\}$) in the increasing order. We impose

$$\Omega(\gamma)(v_1, \dots, v_n) = \sum_{a,b,k,\ell} \eta(v_i, e_k e_\ell)\eta^{ka}\eta^{\ell b}\Omega(\gamma_1)(v_{I_-}, e_a, v_{I_+})\Omega(\gamma_2)(v_{J_-}, e_b, v_{J_+}). \tag{2.6}$$

Definition 2.5. Denote by $\mathbf{e}$ the Euler element of a Frobenius algebra obtained by applying the multiplication m composed with the comultiplication at the unit element $\mathbf{e} := (m \circ \delta)(1)$.

Extending the set of axioms of 2D TQFT of Remark (2.2) to the unstable cases we prove the following equivalence see [2][Theorem 3.8, Corollary 4.8].

Theorem 2.6. *Let A be a Frobenius algebra. The axioms of 2D TQFT in Remark* (2.2) *and Definition* (2.4) *are equivalent. More precisely,*

(1) $\Omega_{g,n}(v_1, \dots, v_n) = \epsilon(v_1 \cdot \dots \cdot v_n \cdot \mathbf{e}^g)$, *for every* $v_i \in A$.
(2) $\Omega(\gamma)(v_1, \dots, v_n) = \epsilon(v_1 \cdot \dots \cdot v_n \cdot \mathbf{e}^g)$, *for every* $v_i \in A$ *and any cell graph* $\gamma \in \Gamma_{g,n}$.

2.2. **Application: 2D TQFT-twisted Catalan numbers.** Let $C_{g,n}(\mu_1, ..., \mu_n)$ denote the number of **arrowed cell graphs** of labeled vertices of degrees $(\mu_1, ..., \mu_n)$, see [3]. Using edge contraction axioms (2.4) and Theorem 2.6 one can extend the recusion relation of Catalan numbers of [3] by twisting by any 2D TQFT $\Omega_{g,n}$

Proposition 2.7. *The 2D TQFT-twisted Catalan numbers satisfy the following:*

$$\begin{aligned} &C_{g,n}(\mu_1, \dots, \mu_n) \cdot \Omega_{g,n}(v_1, \dots, v_n) \\ &= \sum_{j=2}^{n} \mu_j C_{g,n-1}(\mu_1 + \mu_j - 2, \mu_2, \dots, \widehat{\mu_j}, \dots, \mu_n) \cdot \Omega_{g,n-1}(v_1 v_j, v_2, \dots, \widehat{v_j}, \dots, v_n) \\ &+ \sum_{\alpha+\beta=\mu_1-2} C_{g-1,n+1}(\alpha, \beta, \mu_2, \dots, \mu_n) \cdot \Omega_{g-1,n+2}\big(\delta(v_1), v_2, \dots, v_n\big) \\ &+ \sum_{\alpha+\beta=\mu_1-2} \sum_{\substack{g_1+g_2=g \\ I \sqcup J=\{2,\dots,n\}}} \sum_{a,b,k,\ell} \eta(v_1, e_k e_\ell)\eta^{ka}\eta^{\ell b} \\ &\quad \times \big(C_{g_1,|I|+1}(\alpha, \mu_I) \cdot \Omega_{g_1,|I|+1}(e_a, v_I)\big) \big(C_{g_2,|J|+1}(\beta, \mu_J) \cdot \Omega_{g_2,|J|+1}(e_b, v_J)\big) . \end{aligned} \tag{2.7}$$

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Dubrovin's superpotential as a global spectral curve

Petr Dunin-Barkowski

(joint work with Paul Norbury, Nicolas Orantin, Alexandr Popolitov, Sergey Shadrin)

Paper [4] gave a construction of a local spectral curve for a given (local) semisimple Frobenius manifold which allows to reproduce the corresponding semisimple cohomological field theory (CohFT) with the help of spectral curve topological recursion [6, 1]. Natural question arises: can the local spectral curves produced with that procedure be glued into global curves? Here we answer this question affirmatively, at least for the cases satisfying certain assumptions. We construct these global curves with the help of *Dubrovin's superpotential* [2].

Dubrovin's superpotential. Let $F(t^1, \dots, t^n)$ be the prepotential of a given Frobenius manifold of conformal dimension d with flat coordinates $t^1, \dots, t^n$, η its metric, and $E = \sum_{\alpha=1}^{n}((1-q_\alpha)t^\alpha + r_\alpha)\partial_{t^\alpha}$ the Euler field.

Define $\eta'(\partial', \partial'') := \eta(\partial' \cdot \partial'', E)$ to be the second metric, and let $\tilde{\nabla}$ be the extended connection. Let u^i be canonical coordinates in a semisimple neighborhood on the Frobenius manifold, and let Ψ be the transition matrix from the flat frame to the normalized canonical frame. Define $V_i := \partial_{u_i}\Psi \cdot \Psi^{-1}$ and $V := \sum_i u_i V_i$.

A *superpotential* [2] is a function $\lambda(p, u_1, \dots, u_n)$ of a variable $p \in \mathcal{D}$ in some domain $\mathcal{D}$ that depends on points $(u_1, \dots, u_n) \in B_0 \subset B^{ss}$ in a ball in the semisimple part of the Frobenius manifold, which satisfies certain properties which allow to reconstruct the structure of the Frobenius manifold out of it, such as the following property

$$\eta(\partial' \cdot \partial'', \partial''') = -\sum_{i=1}^{n} \mathrm{Res}_{p\to p_i} \frac{\partial'(\lambda dp)\partial''(\lambda dp)\partial'''(\lambda dp)}{dp\, d_p\lambda}.$$

Consider the extended Gauss-Manin system [2]:

$$d\phi = -(U-\lambda)^{-1} d(U-\lambda)\left(\frac{1}{2} + V\right)\phi + d\Psi \cdot \Psi^{-1}\phi.$$

Here $d = d_\lambda + d_u$ is the total de Rham differential; $U = \mathrm{diag}(u_1, ..., u_n)$. This equation has poles at $\lambda = u_1, \dots, u_n$ on the λ-plane, so we choose parallel cuts $L_1, \dots, L_n$ from the points u_i to infinity (we assume that $u_j \notin L_i$ for $i \neq j$). On $\mathbb{C} \setminus \cup_{i=1}^n L_i$ we choose branches of functions $\sqrt{u_i - \lambda}$, $i = 1, \dots, n$. We denote by R_i the monodromy of the space of solutions of the above equation corresponding to following a small loop around u_i.

Dubrovin proves [2] that there exist a unique system of solutions $\phi^{(1)}, \dots, \phi^{(n)}$ to the extended Gauss-Manin system satisfying the following properties:

$$\phi_j^{(j)} = \frac{1}{\sqrt{u_j - \lambda}} + O(\sqrt{u_j - \lambda}) \text{ for } \lambda \to u_j, \qquad j = 1, \dots, n;$$
$$\phi_a^{(j)} = \sqrt{u_j - \lambda} \cdot O(1) \text{ for } \lambda \to u_j, \qquad a \neq j;\ a, j = 1, \dots, n;$$
$$R_j \phi^{(i)} = \phi^{(i)} - 2G^{ij} \phi^{(j)}, \qquad i, j = 1, \dots, n;$$

where $G^{ij} := (\phi^{(i)})^T (U - \lambda) \phi^{(j)}$ is a bilinear form that doesn't depend on λ and $u_1, \dots, u_n$.

Define $\phi := \sum_{i,j=1}^n G_{ij} \phi^{(j)}$. Consider the function $p = p(\lambda, u)$ given by the formula

$$p(\lambda, u) := \frac{\sqrt{2}}{1 - d} \phi^T (U - \lambda) \Psi \mathbb{1}.$$

This function is analytic in $\mathbb{C} \setminus \cup_{i=1}^n L_i$, with a regular singularity at infinity, and its local behavior for $\lambda \to u_i$ is given by

$$p(\lambda, u) = p(u_i, u) + \Psi_{i,\mathbb{1}} \sqrt{2(u_i - \lambda)} + O(u_i - \lambda), \qquad i = 1, \dots, n.$$

The 1-form $d_\lambda p$ has at most a finite number of zeros. We denote them by $r_1, \dots, r_N$ and we assume that they do not belong to the cuts L_i, $i = 1, \dots, n$. Let D be the image of $\mathbb{C} \setminus \cup_{i=1}^n L_i$ under the map $p(\lambda, u)$. This domain has a boundary given by the unfolding of the cuts L_i, $i = 1, \dots, n$. The inverse function $\lambda = \lambda(p, u)$ is a multivalued function on D. Consider the points $p(r_c, u)$, $c = 1, \dots, N$. We glue a finite number of copies of D along the cuts from the points $p(r_c, u)$ to infinity, $c = 1, \dots, N$. In this way we obtain a domain $\hat{D}$, where the function λ is single-valued. We analytically continue the function λ on $\hat{D}$ beyond the boundary. This procedure is not unique; for instance, we can glue several copies of $\hat{D}$ along the boundaries that are the images of the same cuts on the λ-plane. In any case, we can perform this construction uniformly over a small ball in the space of parameters $u_1, \dots, u_n$. This way we obtain a (not necessarily compact) Riemann surface $\mathcal{D}$, with a function $\lambda = \lambda(\tilde{p}, u) \colon \mathcal{D} \to \mathbb{C}$ (by $\tilde{p}$ we denote some local coordinate on $\mathcal{D}$). Dubrovin proves in [2] that the family of functions $\lambda(\tilde{p}, u)$ defined this way is a superpotential of the Frobenius manifold which was the input of this construction.

Topological recursion. For a spectral curve (Σ, x, y, B) we define a sequence of symmetric n-forms $\omega_{g,k}(z_1, \dots, z_k)$ on $\Sigma^{\times k}$ according to the spectral curve topological recursion [6, 1].

A local version of the recursion was defined in [5] as follows. Consider some small neighborhoods $U_i \subset \Sigma$ of the points c_i. If we look at just the restrictions of $\omega_{g,k}$ to the products of these disks, $U_{i_1} \times \cdots \times U_{i_k}$, we can still proceed by topological recursion, using as an input the restrictions of $\omega_{0,1}$ to U_i, $i = 1, \ldots, n$, and $\omega_{0,2}$ to $U_i \times U_j$, $i, j = 1, \ldots, n$.

We choose the local coordinates w_i in the domains U_i such that $x|_{U_i} = -w_i^2/2 + x(c_i)$, $i = 1, \ldots, n$. Spectral curve topological recursion produces a CohFT when:

$$R^{-1}(\zeta^{-1})_i^j = -\frac{1}{\sqrt{2\pi\zeta}} \int_{-\infty}^{\infty} \left. \frac{B(w_i, w_j)}{dw_i} \right|_{w_i=0} \cdot e^{(x(w_j) - x(c_j))\zeta}; \tag{1.1}$$

$$\sum_{k=1}^{n} (R^{-1}(\zeta^{-1}))_k^i \Delta_k^{-\frac{1}{2}} = \frac{\sqrt{\zeta}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy(w_i) \cdot e^{(x(w_i) - x(c_i))\zeta}. \tag{1.2}$$

The matrix $R^{-1}(\zeta^{-1})_i^j)$ determines for us a semi-simple CohFT $\{\alpha_{g,k}\}$ with an n-dimensional space of primary field $V := \langle e_1, \ldots, e_n \rangle$. The differentials $\omega_{g,k}$ can be written in terms of the auxiliary functions

$$\xi^i(z) := \int^z \left. \frac{B(w_i, \bullet)}{dw_i} \right|_{w_i=0}$$

as

$$\omega_{g,k} = \sum_{\substack{i_1, \ldots, i_k \\ d_1, \ldots, d_k}} \int_{\overline{\mathcal{M}}_{g,k}} \alpha_{g,k}(e_{i_1}, \ldots, e_{i_k}) \prod_{j=1}^{k} \psi_j^{d_j} d\left(\left(\frac{d}{dx} \right)^{d_j} \xi^{i_j} \right). \tag{1.3}$$

Superpotential as a global spectral curve. Consider Dubrovin's construction of a superpotential on the Riemann surface $\mathcal{D}$ described above. It is associated to a Frobenius manifold with given constants $\Delta_i^{-\frac{1}{2}}$ and the matrix $R^{-1}(\zeta^{-1})_i^j$ at the point with canonical coordinates $u_1, \ldots, u_n$. Consider the points $c_i = p(u_i, u) \in \mathcal{D}$. These points are the critical points of the function $x := \lambda$.

Theorem 1. *Equation* (1.2) *is satisfied for the matrix $R^{-1}(\zeta^{-1})_i^j$ corresponding to the given Frobenius manifold and the spectral curve data $\Sigma = \mathcal{D}$, $x := \lambda$, $y := p$.*

Recall that x defines a local involution σ_i near each zero c_i of dx, $i = 1, \ldots, n$. For a given data of a spectral curve (Σ, x, y, B) (maybe, local) Equations (1.1) and (1.2) imply a certain relation for x, y, and B:

Theorem 2 (Compatibility test)**.** *Equations* (1.1)*, and* (1.2) *are compatible (as equations for the unknown variables forming R^{-1}) if and only if the 1-form*

$$H(z) := d\left(\frac{dy}{dx}(z) \right) + \sum_{i=1}^{n} \mathrm{Res}_{z'=c_i} \frac{dy}{dx}(z') B(z, z'). \tag{1.4}$$

is invariant under each local involution σ_i, $i = 1, \ldots, n$.

Our main statement is then as follows:

Theorem 3. *Given a conformal Frobenius manifold, construct a superpotential $p(\lambda; u)$ which defines the Riemann surface $\mathcal{D}$ according to Dubrovin's construction of the superpotential. Assume the following:*

- *$\mathcal{D}$ is a compact curve of genus g;*
- *there is exactly one critical point in each singular fiber of $\lambda : \mathcal{D} \to \mathbb{C}$.*

Fix a symplectic basis $(\mathcal{A}_i, \mathcal{B}_i)_{i=1}^g$ of $H_1(\mathcal{D}, \mathbb{Z})$ and define $B(p_1, p_2)$ as the only Bergman kernel on $\mathcal{D}$ normalized by

$$\forall i = 1, \dots, g\,, \oint_{p_1 \in \mathcal{A}_i} B(p_1, p_2) = 0. \tag{1.5}$$

Further assume that:

- *the pair $(p, B(p_1, p_2))$ passes the compatibility test (as in Theorem 2).*

Then the total ancestor potential of the CohFT associated to the Frobenius manifold is related by Equation (1.3) to the differentials obtained through spectral curve topological recursion on the Riemann surface $\mathcal{D}$ with $x = \lambda$, $y = p$ and $B(p_1, p_2)$.

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Twisted superpotential of the T_3 theory

Lotte Hollands

(joint work with Andy Neitzke)

Fix a compact Riemann surface C with punctures $\{z_1, \dots, z_m\}$ and a compact Lie group $G = SU(N)$. Then we can define the space

$$B = \{(\varphi_2, \varphi_3, \dots, \varphi_N) \,|\, \varphi_k \in H^0(C, K_C^{\otimes k}((k-1)\sum_{i=1}^m z_i)\}$$

of N-tuples of meromorphic k-differentials on C. It is the Coulomb branch of a $\mathcal{N} = 2$ supersymmetric quantum field theory $T_G[C]$ in four dimensions.

One particularly interesting example is $C = \mathbb{P}^1_{0,1,\infty}$ with $N = 3$. The corresponding quantum field theory T_3 is the conformal E_6 Minahan-Nemeschansky theory. Its Coulomb branch is parametrized by the differentials

$$\varphi_2(z) = 0$$

$$\varphi_3(z) = \frac{u}{z^2(z-1)^2}(dz)^{\otimes 3},$$

where u is the so-called Coulomb parameter. The Seiberg-Witten curve $\Sigma \in T^*C$ is the spectral curve defined by the equation

$$w^3 + \varphi_2(z)w + \varphi_3(z) = 0.$$

Suppose that we put $T_G[C]$ in the so-called partial Ω-background $\mathbb{R}^2_\epsilon \times \mathbb{R}^2$ with complex parameter ϵ. This implies that we effectively compactify the theory to two dimensions. In this talk we proposed how to compute the effective two-dimensional twisted superpotential $\widetilde{W}(\epsilon)$ as a generating function of $G_\mathbb{C}$-opers on C. This proposal generalizes and unifies work by Nekrasov, Rosly and Shatashvili [2] and by Kashani-Poor and Troost [3].

As a first step, we suggested a concrete parametrization of the family of opers on C associated with the data B (generalizing a conjecture by Gaiotto [4], see also [5] when $m = 0$). For instance, the family of opers for the T_3 theory is parametrized by the rank 3 differential equation

$$\left(\partial_z^3 + \frac{1 - z + z^2}{z^2(z-1)^2}\partial_z + \frac{1 - (u + \frac{5}{2})z + (u + \frac{3}{2})z^2 + z^3}{z^3(z-1)^3}\right)\psi(z) = 0.$$

If we introduce the complex parameter ϵ through sending $u \mapsto u/\epsilon^3$, we recover the spectral curve Σ in the semi-classical limit in ϵ. The ϵ-corrections are determined by the projective structure corresponding to the uniformization of $\mathbb{P}^1_{0,1,\infty}$.

Second, we proposed that the twisted superpotential $\widetilde{W}(\epsilon)$ is computed through the WKB analysis of this family of opers. We introduce

$$\psi(z) = \exp\left(\int^z S(z,\epsilon)dz\right)$$

and find a formal (series) solution in ϵ of the form

$$S(z) = \frac{S_{-1}(z)}{\epsilon} + S_0(z) + \epsilon S_1(z) + \ldots.$$

Note that $S_{-1}(z)$ is the Seiberg-Witten differential, i.e. the tautological 1-form on T^*C restricted to Σ. This motivates us to define the *quantum periods*

$$\mathcal{X}_\gamma = \oint_\gamma S(z,\epsilon)dz$$

for any 1-cycle γ on the compactified Seiberg-Witten curve $\overline{\Sigma}$. Given a choice of A- and B-cycles on Σ, the twisted superpotential $\widetilde{W}(\epsilon)$ is then the solution to the coupled set of equations

$$\mathcal{X}_B = \frac{\partial \widetilde{W}(\epsilon)}{\partial \mathcal{X}_A}.$$

For the T_3 theory this leads to

$$\widetilde{W}(\epsilon) = \frac{e^{\frac{4\pi i}{3}}a^2}{2} + 2\pi i \log(a) + \ldots.$$

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Deforming irregular singular points

Jacques Hurtubise
(joint work with Christiane Rousseau)

The geometric data classifying an irregular singularity of a (vector valued) o.d.e, is well known: a formal normal form, and Stokes data, consisting in the generic case basically of a family of $n-1$ upper triangular matrices and $n-1$ lower triangular matrices, the Stokes data, as well as a formal normal form at the singularity, Likewise, generically, for regular singularities one has the monodromy data , as well as formal normal forms at the singularity. These data, while similar, are not the same, and one would like some picure which holds relatively uniformaly in the family. More precisely, let us consider, on a neighbourhood of the origin in $\mathbb{C}$,

$$y' = \frac{A(\epsilon, x)}{p_\epsilon(x)} \cdot y. \tag{1.1}$$

where

$$p_\epsilon(x) = x^{k+1} + \epsilon_{k-1}x^{k-1} + \cdots + \epsilon_1 x + \epsilon_0, \tag{1.2}$$

is a polynomial, and ϵ_i are small complex parameters, with $\epsilon = -0$ corresponding to an irregular singarity. The $n \times n$ matrix $A(\epsilon, x)$ is holomorphic in x, ϵ, on a neighbourhood of the origin. Its leading order term A_0 is supposed to be diagonalisable, with distinct eigenvalues, which, changing variables if necessary, we can assume have distinct real part. One introduces an extra variable t, and rewrites the equation as

$$\dot{y} = A(\epsilon, x) \cdot y \tag{1.3}$$

$$\dot{x} = p_\epsilon(x) \tag{1.4}$$

Formal Normal form
Via a formal gauge transformation, one can gauge the vector equation so as to have $A(\epsilon, x)$ a polynomial of diagonal matrices of order $k-1$ in x, with coefficients holomorphic in ϵ. This transformation does not necessarily converge.

The scalar equation.
If one considers the first the scalar equation (1.4), for $\epsilon = 0$, one finds that the flows $Im(t) =$ constant give in the x-plane $2(k-1)$ separatrices emerging from

infinity and converging to the origin; these realize the boundaries of the Stokes sectors. If one now considers the flows for more general ϵ, one can appeal to results of Douady and Santenac. The separatrices emerging from infinity persist. We impose the genericity conditions of

$$\sum_{\ell\in I_1}\frac{1}{p'_\epsilon(x_\ell)}\notin i\mathbb{R}. \tag{1.5}$$

where x_ℓ is a root of $p_\epsilon(x_\ell)$ and I_1 is a subset of $\{1,..,n\}$. This divides the parameter space into domains S_Γ. Within S_Γ, the phase portrait has the separatrices converging to the singular points x_ℓ, which become naturally linked onto a tree, defining the domain and, along with the separatrices, dividing the sphere into $2(k-1)$ open sectors $\Omega_{j,\pm}(\epsilon)$. These generalize the Stokes sectors.

The vector equation
If one then looks at the vector equation 1.3, and considers the behaviour along a fixed line $Im(t)$ = constant, one sees that the the asymptotics of the equation as one goes to $Re(t) = +\infty$ give a flag E_i^+ in the space of solutions; likewise one has a flag E_i^- associated to $Re(t) = -\infty$. These flags are transverse, and if one chooses a point in each sector $\Omega_{j,\pm}(\epsilon)$, its two associated flags give by their intersection a basis of the solutions, defined up to the action of a torus. Adjacent sectors share one of their flags, and relating their bases gives the generalized Stokes matrices, alternately upper and lower triangular, depending holomorphically on ϵ, continuous at the boundary of the domains S_Γ. One has:

Theorem 1. *Fixing a formal normal form, the generalized Stokes matrices determine the differential equation in the family*

One can use the generalised Stokes matrices to reobtain the monodromy; for a given path, one multiplies the matrices associated to the boundaries of the domains that the path crosses.

Changing domains
As one goes from a sector S_Γ to a sector $S_{\Gamma'}$, the phase diagram of the scalar equation bifurcates. One can ask what the conditions of compatibility of Stokes factors are; it turns out that the only requirement is that they define the same monodromy representation on the generic locus.

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The quantum dilogarithm and operators from topological strings

Rinat Kashaev
(joint work with Marcos Mariño, Szabolcs Zakany)

The partition function of closed topological strings propagating on a Calabi–Yau manifold X is known to generate the Gromov–Witten invariants of X, thus providing a purely geometrical application of topological string theory. Taking into account the fact that string partition functions are usually defined only perturbatively in terms of genus expansion, the recent conjecture of Grassi–Hatsuda–Mariño [1] opens a new perspective by suggesting that topological strings on a toric Calabi–Yau threefold X are captured non-perturbatively through the spectral theory of a positive trace class operator ρ obtained by quantization of the mirror curve of X.

One way of formulating the spectral theory/mirror symmetry correspondence of [1] is by considering the "fermionic" traces $Z(N, \hbar)$ of the operator ρ defined as the expansion coefficients of the spectral or Fredholm determinant

$$\det(1 + \kappa\rho) = 1 + \sum_{N=1}^{\infty} Z(N, \hbar)\kappa^N, \tag{1.1}$$

where the series converges on the entire complex plane of the variable κ due to the trace class property of ρ. It turns out that in the 't Hooft limit $N \to \infty$, $\hbar \to \infty$ with fixed ratio $\lambda = \frac{N}{\hbar}$, one expects the asymptotic expansion

$$\log Z(\lambda\hbar, \hbar) \sim \sum_{g=0}^{\infty} \mathcal{F}_g(\lambda)\hbar^{2-2g}$$

where $\mathcal{F}_g(\lambda)$ are the genus g free energies of the standard topological strings in the so-called conifold frame. The 't Hooft parameter λ is a flat coordinate for the Calabi–Yau moduli space given by the vanishing period at the conifold point (assuming that the mirror curve has genus one). In this way, the weakly coupled topological strings emerge in a limit where the quantum mechanical problem is strongly coupled (since $\hbar \to \infty$). On the other hand, the double-scaling limit $N \to \infty$, $\hbar \to 0$ with fixed product $N\hbar$ corresponds to the WKB expansion of the quantum-mechanical problem, and it is expected to be captured by the so-called Nekrasov–Shatashvili limit of the refined topological strings. Thus, the Grassi–Hatsuda–Mariño conjecture implies that the fermionic traces $Z(N, \hbar)$ of the trace class operator ρ provide a non-perturbative definition of the topological string partition function. In the case of some simple toric Calabi–Yau threefolds, the associated trace class operators can be explicitly expressed in terms of Faddeev's quantum dilogarithm [3, 4], and that allows us to express the fermionic traces as concrete finite dimensional integrals. Let us briefly review the derivation of these results.

Recall that Faddeev's quantum dilogarithm, defined by the formula

$$\Phi_{\mathsf{b}}(x) = \exp\left(\int_{\mathbb{R}+\mathrm{i}\epsilon} \frac{e^{-2\mathrm{i}xz}}{4\sinh(z\mathsf{b})\sinh(z\mathsf{b}^{-1})}\frac{\mathrm{d}z}{z}\right), \tag{1.2}$$

is a meromorphic function of complex variable x, satisfies the functional equations

$$\frac{\Phi_{\mathsf{b}}(x - \mathrm{i}s)}{\Phi_{\mathsf{b}}(x + \mathrm{i}s)} = 1 + e^{4\pi s x}, \quad s \in \left\{\frac{\mathsf{b}}{2}, \frac{1}{2\mathsf{b}}\right\}, \tag{1.3}$$

and, under complex conjugation, possesses the unitarity property

$$\overline{\Phi_{\mathsf{b}}(x)} = \frac{1}{\Phi_{\mathsf{b}}(\bar{x})}, \tag{1.4}$$

where we assume that the parameter b is chosen to be a positive real number.

Three-term operators. Consider a generic three-term quantum mechanical operator

$$\mathsf{O}_{m,n} = e^{\mathsf{x}} + e^{\mathsf{y}} + e^{-m\mathsf{x}-n\mathsf{y}}, \quad m, n \in \mathbb{R}_{>0}, \tag{1.5}$$

where x and y are self-adjoint operators satisfying the Heisenberg commutation relations

$$\mathsf{xy} - \mathsf{yx} = \mathrm{i}\hbar. \tag{1.6}$$

A systematic way of deriving the operator $\mathsf{O}_{m,n}$ from a toric Calabi–Yau geometry is given in [2]. In particular, the case $m = n = 1$ is known to be related with the quantization of the mirror curve to the local $\mathbb{P}^2$ geometry.

By using (1.3) with $s = \mathsf{b}/2$, we notice that

$$e^{\mathsf{x}} + e^{\mathsf{y}} = e^{\mathsf{x}/2}\left(1 + e^{\mathsf{y}-\mathsf{x}}\right)e^{\mathsf{x}/2} = e^{\mathsf{x}/2}\frac{\Phi_{\mathsf{b}}\left(\frac{\mathsf{y}-\mathsf{x}}{2\pi\mathsf{b}} - \mathrm{i}\frac{\mathsf{b}}{2}\right)}{\Phi_{\mathsf{b}}\left(\frac{\mathsf{y}-\mathsf{x}}{2\pi\mathsf{b}} + \mathrm{i}\frac{\mathsf{b}}{2}\right)}e^{\mathsf{x}/2} = (\mathsf{a}\mathsf{a}^*)^{-1}, \tag{1.7}$$

where

$$\mathsf{a} \equiv e^{-\mathsf{x}/2}\Phi_{\mathsf{b}}\left(\frac{\mathsf{y}-\mathsf{x}}{2\pi\mathsf{b}} + \mathrm{i}\frac{\mathsf{b}}{2}\right), \tag{1.8}$$

so that

$$\mathsf{a}^*\mathsf{O}_{m,n}\mathsf{a} = 1 + \mathsf{a}^* e^{-m\mathsf{x}-n\mathsf{y}}\mathsf{a} = 1 + \mathsf{c}^*\mathsf{c}, \tag{1.9}$$

where

$$\mathsf{c} \equiv e^{-((1+m)\mathsf{x}+n\mathsf{y})/2}\Phi_{\mathsf{b}}\left(\frac{\mathsf{y}-\mathsf{x}}{2\pi\mathsf{b}} + \mathrm{i}\frac{\mathsf{b}}{2}\right). \tag{1.10}$$

Let b be such that

$$\frac{\mathsf{b}}{2} = (1 + m + n)\frac{\hbar}{4\pi\mathsf{b}} \quad \Leftrightarrow \quad \hbar = \frac{2\pi\mathsf{b}^2}{1 + m + n}. \tag{1.11}$$

Then, the operator c in (1.10) can equivalently be rewritten in the form

$$\mathsf{c} = \Phi_{\mathsf{b}}\left(\frac{\mathsf{y}-\mathsf{x}}{2\pi\mathsf{b}}\right)e^{-((m+1)\mathsf{x}+n\mathsf{y})/2}, \tag{1.12}$$

which implies that

$$\mathsf{c}^*\mathsf{c} = e^{-(m+1)\mathsf{x}-n\mathsf{y}}, \tag{1.13}$$

and from (1.9) we immediately conclude that the operator $\mathsf{O}_{m,n}$ is explicitly invertible with the inverse

$$\rho_{m,n} \equiv \mathsf{O}_{m,n}^{-1} = \mathsf{a}\left(1 + e^{-(m+1)\mathsf{x}-n\mathsf{y}}\right)^{-1}\mathsf{a}^*. \tag{1.14}$$

Local $\mathbb{F}_0$. The operator associated to the mirror curve of local $\mathbb{F}_0$ geometry reads

$$\mathsf{O}_{\mathbb{F}_0} = e^{\mathsf{x}} + e^{\mathsf{y}} + e^{-\mathsf{y}} + \zeta e^{-\mathsf{x}}, \tag{1.15}$$

where x and y are self-adjoint and satisfy (1.6). By using (1.7) and (1.8), we have

$$\mathsf{a}^*\mathsf{O}_{\mathbb{F}_0}\mathsf{a} = 1 + \mathsf{g}^*\mathsf{g} + \zeta\mathsf{f}^*\mathsf{f} \tag{1.16}$$

where

$$\mathsf{f} \equiv e^{-\mathsf{x}}\Phi_{\mathsf{b}}\left(\frac{\mathsf{y}-\mathsf{x}}{2\pi\mathsf{b}} + \mathsf{i}\frac{\mathsf{b}}{2}\right), \quad \mathsf{g} \equiv e^{-(\mathsf{x}+\mathsf{y})/2}\Phi_{\mathsf{b}}\left(\frac{\mathsf{y}-\mathsf{x}}{2\pi\mathsf{b}} + \mathsf{i}\frac{\mathsf{b}}{2}\right). \tag{1.17}$$

Choosing b such that $\hbar = \pi\mathsf{b}^2$, we rewrite (1.17)

$$\mathsf{f} = \Phi_{\mathsf{b}}\left(\frac{\mathsf{y}-\mathsf{x}}{2\pi\mathsf{b}}\right)e^{-\mathsf{x}}, \quad \mathsf{g} = \Phi_{\mathsf{b}}\left(\frac{\mathsf{y}-\mathsf{x}}{2\pi\mathsf{b}}\right)e^{-(\mathsf{x}+\mathsf{y})/2} \tag{1.18}$$

so that

$$\begin{aligned} \mathsf{a}^*\mathsf{O}_{\mathbb{F}_0}\mathsf{a} &= 1 + e^{-\mathsf{x}-\mathsf{y}} + \zeta e^{-2\mathsf{x}} = 1 + e^{-(\mathsf{x}+\mathsf{y})/2}\left(1 + \zeta e^{\mathsf{y}-\mathsf{x}}\right)e^{-(\mathsf{x}+\mathsf{y})/2} \\ &= 1 + e^{-(\mathsf{x}+\mathsf{y})/2}\frac{\Phi_{\mathsf{b}}\left(\frac{\mathsf{y}-\mathsf{x}}{2\pi\mathsf{b}} + \mu - \mathsf{i}\frac{\mathsf{b}}{2}\right)}{\Phi_{\mathsf{b}}\left(\frac{\mathsf{y}-\mathsf{x}}{2\pi\mathsf{b}} + \mu + \mathsf{i}\frac{\mathsf{b}}{2}\right)}e^{-(\mathsf{x}+\mathsf{y})/2} \\ &= 1 + \frac{1}{\Phi_{\mathsf{b}}\left(\frac{\mathsf{y}-\mathsf{x}}{2\pi\mathsf{b}} + \mu\right)}e^{-\mathsf{x}-\mathsf{y}}\Phi_{\mathsf{b}}\left(\frac{\mathsf{y}-\mathsf{x}}{2\pi\mathsf{b}} + \mu\right) \\ &= \frac{1}{\Phi_{\mathsf{b}}\left(\frac{\mathsf{y}-\mathsf{x}}{2\pi\mathsf{b}} + \mu\right)}\left(1 + e^{-\mathsf{x}-\mathsf{y}}\right)\Phi_{\mathsf{b}}\left(\frac{\mathsf{y}-\mathsf{x}}{2\pi\mathsf{b}} + \mu\right), \end{aligned} \tag{1.19}$$

where we have used the parameterization $\zeta = e^{2\pi\mathsf{b}\mu}$. Thus, we conclude that the operator $\mathsf{O}_{\mathbb{F}_0}$ is explicitly invertible with the inverse

$$\rho_{\mathbb{F}_0} \equiv \mathsf{O}_{\mathbb{F}_0}^{-1} = \mathsf{u}\left(1 + e^{-\mathsf{x}-\mathsf{y}}\right)^{-1}\mathsf{u}^*, \tag{1.20}$$

where

(1.21) $$\mathsf{u} \equiv e^{-\mathsf{x}/2} \frac{\Phi_{\mathsf{b}}\left(\frac{\mathsf{y}-\mathsf{x}}{2\pi\mathsf{b}} + \mathsf{i}\frac{\mathsf{b}}{2}\right)}{\Phi_{\mathsf{b}}\left(\frac{\mathsf{y}-\mathsf{x}}{2\pi\mathsf{b}} + \mu\right)},$$

and we assume that $\mu \in \mathbb{R}$.

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Elliptically fibered Calabi-Yau manifolds and the ring of weak Jacobi forms

Albrecht Klemm

Abstract. We give evidence that the all genus amplitudes of topological string theory on compact elliptically fibered Calabi-Yau manifolds can be written in terms of meromorphic Jacobi forms whose weight grows linearly and whose index grows quadratically with the base degree. The denominators of these forms have a simple universal form with the property that the poles of the meromorphic form lie only at torsion points. The modular parameter corresponds to the fibre class while the role of the string coupling is played by the elliptic parameter. The numerator is a weak Jacobi form. The fact that the ring of the latter is finitely generated leads to very strong all genus results on these geometries, which check against results from curve counting. This report is based on the results of [13] and [14].

Determining the all genus topological string partition function

(1.1)
$$Z = \exp(F) = \exp\left(\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(\underline{z(\underline{t})})\right) = \exp\left(\sum_{g=0}^{\infty} \sum_{\beta\in H_2(M,\mathbb{Z})} \lambda^{2g-2} r_g^{\beta} \underline{q}^{\beta}\right)$$

on a compact Calabi-Yau manifold M is a benchmark problem with many applications to enumerative geometry and string compactifications.

Here λ is the topological string coupling, t_a are complexified Kähler volumes and $\underline{q}^{\beta} = \exp(2\pi i \sum_a t_a \beta^a)$. The mirror map $\underline{z(\underline{t})}$ can be determined from the solutions of the Picard-Fuchs equations describing the periods of the mirror manifold. At the large volume point $t_a \to \mathrm{i}\infty$ and in the holomorphic limit one can read the genus g Gromov-Witten invariants from the convergent $\underline{q}$ expansion of F_g. In this holomorphic limit $\mathcal{Z} = \lim_{\underline{t}\to\infty} Z$ is equivalently completely determined by integer BPS indices $I_g^\beta \in \mathbb{Z}$ and can be given by a product formula

(1.2)
$$\mathcal{Z}(\underline{t},\lambda) = \prod_\beta \left[\left(\prod_{m=k}^{\infty} (1-y^k q^\beta)^{k n_0^\beta} \right) \prod_{g=1}^{\infty} \prod_{l=0}^{2g-2} (1-y^{g-l-1} q^\beta)^{(-1)^{g+l}\binom{2g-2}{l} I_g^\beta} \right] ,$$

If M is a non-compact toric Calabi-Yau, i.e. the space is typically given by the anticanonical line bundle over a toric base space

$$M_{loc} = \pi : \begin{array}{c} \mathcal{O}(-K_B) \\ \downarrow \\ B \end{array} , \tag{1.3}$$

the problem can be solved in many ways

- By localization w.r.t. to the torus action employing the Atiyah-Bott localization formula [20] .
- Using locally large N-duality of topological string A model to Chern-Simons theory on $\mathbb{S}^3$ [25], which leads to the topological vertex [1].
- Using large N-duality of topological string B-model to a matrix model [3], which lead to the remodeling conjecture [22, 4].
- Using the holomorphic anomaly equation [5], with the gap condition [17] fixing the holomorphic ambiguity completely in the local case [16].

Only the last approach extends to compact Calabi-Yau's, however due to the only partially fixed holomorphic ambiguity, it does not lead to complete results yet.

In this work we like to discuss the simplest compactification of the local models described above by replacing the line bundle by an elliptic fibre bundle

$$M = \pi : \begin{array}{c} \mathcal{E} \\ \downarrow \\ B \end{array} . \tag{1.4}$$

For simplicity we assume that the elliptic fibre has only one section and that it has only fibers of Kodaira type I_1 in codim one in the base.

Generally the holomorphic anomaly can also been viewed as a modular anomaly. In the elliptic manifolds we get an $SL(2,\mathbb{Z})$ acting on the elliptic fibre parameters and expect that the (1.1) is invariant under this action. This turns out to be true and leads with the recursive holomorphic anomaly equation and expression for (1.1) in terms of Jacobi forms.

Indeed we find the following structure of the topological string partition function Z in terms of meromorphic Jacobi forms [13]. Let τ and t_B be the respective Kähler parameters of the elliptic fiber and the base $\mathbb{P}^2$, and put $q = \exp(2\pi i\tau)$ and $Q = \exp(2\pi i t_B)$. We expand $\mathcal{Z}$ in terms of the base degrees d_B as

$$Z(\underline{t},\lambda) = Z_0(\tau,\lambda) + \sum_{\beta\neq 0}^{\infty} Z_\beta(\tau,\lambda) Q^\beta \quad . \tag{1.5}$$

The topological string partition function is characterized by the three properties of $Z_\beta(\tau, g_s)$ for $\beta \neq 0$ [14]:
Property 1: $Z_\beta(\tau, g_s)$ is a *meromorphic Jacobi form* of weight zero $k = 0$ and index

$$m_\beta = \frac{1}{2}\beta \cdot (\beta - c_1(B)), \tag{1.6}$$

where the string coupling g_s is identified up to multiple with the elliptic argument z of the Jacobi form

$$g_s = 2\pi z \ . \tag{1.7}$$

Property 2:

$$Z_\beta(\tau, z) = \frac{1}{\eta^{12\beta\cdot c_1(B)}} \frac{\varphi_\beta(\tau, z)}{\prod_{l=1}^{b_2(B)} \prod_{s=1}^{\beta_l} A(\tau, sz)} \tag{1.8}$$

where $\varphi_\beta(\tau, z)$ is a *weak Jacobi form* of weight

$$w_\beta = 6\beta \cdot c_1(B) - 2 \sum_{l=1}^{b_2(B)} \beta_l \tag{1.9}$$

and index

$$m_\beta = \frac{1}{6} \sum_{l=1}^{b_2(B)} \beta_l(1 + \beta_l)(1 + 2\beta_l) + \frac{1}{2}\beta \cdot (\beta - c_1(B)) \ . \tag{1.10}$$

Property 3: The Castelnuovo bounds that predict the vanishing of BPS indices I_g^β for $g \geq \mathcal{O}(d_e)$ in the classes $\kappa = (d_e, \beta)$ for $\beta = k\tilde\beta$ and $\tilde\beta a$ primitive class, determine together with the genus zero and one results the weak Jacobi form φ_β for all positive $k \in \mathbb{N}$, if

$$m_\beta \leq 0 \ . \tag{1.11}$$

Since the ring of weak Jacobi-Forms are finely generated a finite amount of data suffices to solve the theory for a given base degree to all genus. For example for the so called half K3 $m_b = -\frac{1}{2}b(b - 1)$, where the base degree b is positive, so the theory is completely solved.

After recalling a few properties of Jacobi Forms, we give concrete enumerative results for $B = \mathbb{P}^2$. Jacobi forms $\varphi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ depend on a modular parameter $\tau \in \mathbb{H}$ and an elliptic parameter $z \in \mathbb{C}$. They transform under the modular group [10]
(1.12)

$$\tau \mapsto \tau_\gamma = \frac{a\tau + b}{c\tau + d}, \quad z \mapsto z_\gamma = \frac{z}{c\tau + d} \quad \text{with} \quad \begin{pmatrix} a & b \\ a & c \end{pmatrix} \in \mathrm{SL}(2; \mathbb{Z}) =: \Gamma_0$$

as

$$\varphi\left(\tau_\gamma, z_\gamma\right) = (c\tau + d)^k e^{\frac{2\pi i m c z^2}{c\tau + d}} \varphi(\tau, z) \tag{1.13}$$

and under quasi periodicity in the elliptic parameter as

$$\varphi(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2\tau + 2\lambda z)}\varphi(\tau, z), \quad \forall \quad \lambda, \mu \in \mathbb{Z} \ . \tag{1.14}$$

Here $k \in \mathbb{Z}$ is called the *weight* and $m \in \mathbb{Z}_{>0}$ is called the *index* of the Jacobi form.

The Jacobi forms have a Fourier expansion

$$\varphi(\tau, z) = \sum_{n,r} c(n, r) q^n y^r, \qquad \text{where} \quad q = e^{2\pi i\tau}, \quad y = e^{2\pi i z} \tag{1.15}$$

Because of the translation symmetry one has $c(n, r) =: C(4nm - r^2, r)$, which depends on r only modulo $2m$. For a holomorphic Jacobi form $c(n, r) = 0$ unless $4mn \geq r^2$, for cusp forms $c(n, r) = 0$ unless $4mn > r^2$, while for weak Jacobi forms one has only the condition $c(n, r) = 0$ unless $n \geq 0$.

According to [10] a weak Jacobi form of given index m and even modular weight k is freely generated over the ring of modular forms of level one, i.e. polynomials in $Q = E_4(\tau)$, $R = E_6(\tau)$, $A = \varphi_{0,1}(\tau, z)$, $B = \varphi_{-2,1}(\tau, z)$ [1]

$$J^{weak}_{k,m} = \bigoplus_{j=0}^{m} M_{k+2j}(\Gamma_0) A^j_{-2,1} B^{m-j} \ . \tag{1.16}$$

Enumerative examples for $B = \mathbb{P}^2$: For the case of $d_B = 1$ there only two free coefficients that are are fixed by two genus 0 BPS numbers to be

$$\varphi_1 = -\frac{Q(31Q^3 + 113P^2)}{48}. \tag{1.17}$$

This determines the all genus BPS invariants for base degree 1 by (1.5), (1.8), and the multi-covering formula (1.2).

Up to $g = 6$ and $d_E = 6$ we list them in Table 1.

$g\backslash d_E$	$d_E = 0$	1	2	3	4	5	6
$g = 0$	3	-1080	143370	204071184	21772947555	1076518252152	33381348217290
1	0	-6	2142	-280284	-408993990	-44771454090	-2285308753398
2	0	0	9	-3192	412965	614459160	68590330119
3	0	0	0	-12	4230	-541440	-820457286
4	0	0	0	0	15	-5256	665745
5	0	0	0	0	0	-18	6270
6	0	0	0	0	0	0	21

TABLE 1. Some BPS invariants $n^g_{(d_E,1)}$ for base degree $d_B = 1$ and $g, d_E \leq 6$ as determined by (1.17) for all g, d_E.

For $d_B = 2$ three of 17 coefficients can be fixed by demanding that there is no pole $(2\pi i z)^{-4} = \lambda^{-4}$ in $P_2(\tau, z)$. Note that this pole has to be canceled by the

[1] With E_4, E_4 the standard Eisenstein series and $A = -\frac{\theta_1(\tau,z)^2}{\eta^6(\tau)}$ and $B = 4\sum_{k=2,3,4} \frac{\theta_k(\tau,z)^2}{\theta_k(0,\tau)^2}$.

$(Z_1)^2$ contribution in $P_2(\tau, z)$. This explains the first term in (1.18). the result fixed from three genus zero numbers is

$$
\begin{aligned}
\varphi_2 \quad = \quad & \frac{B^4Q^2\left(31Q^3+113R^2\right)^2}{23887872} + \frac{1}{1146617856}[2507892B^3AQ^7R + 9070872B^3AQ^4R^3 \\
& +2355828B^3AQR^5 + 36469B^2A^2Q^9 + 764613B^2A^2Q^6R^2 - 823017B^2A^2Q^3R^4 \\
& +21935B^2A^2R^6 - 9004644BA^3Q^8R - 30250296BA^3Q^5R^3 - 6530148BA^3Q^2R^5 \\
& +31A^4Q^{10} + 5986623A^4Q^7R^2 + 19960101A^4Q^4R^4 + 4908413A^4QR^6]\ , \qquad (1.18)
\end{aligned}
$$

which predicts the BPS numbers in all genus and fibre classes for $d_B = 2$.

$g \backslash d_E$	$d_E = 0$	1	2	3	4	5	6
$g = 0$	6	2700	-574560	74810520	-49933059660	7772494870800	31128163315047072
1	0	15	-8574	2126358	521856996	1122213103092	879831736511916
2	0	0	-36	20826	-5904756	-47646003780	-80065270602672
3	0	0	0	66	-45729	627574428	3776946955338
4	0	0	0	0	-132	-453960	-95306132778
5	0	0	0	0	0	-5031	1028427996
6	0	0	0	0	0	-18	-771642
7	0	0	0	0	0	0	-7224
8	0	0	0	0	0	0	-24

Table 2. Some BPS invariants for $n^g_{(d_E,2)}$

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Vertex algebras and the topological recursion for A_N singularity

Todor Milanov

Introduction. The main motivation for this work is to derive Bouchard–Eynard type topological recursion for the total descendant potential of a semis-simple Frobenius manifold. Our approach is based on the vertex algebra representation constructed in [2]. We expect that the Bouchard–Eynard topological recursion is equivalent to a generating set of $\mathcal{W}$-constraints in the sense of [2]. In particular, the topological recursion should provide a new method to construct generators of the $\mathcal{W}$-algebra. Here we focus on the case of the Frobenius structure on the space of miniversal unfolding of A_N-singularity.

Frobenius structures. Let $H = \mathbb{C}[x]/f'(x)$ be the local algebra of the A_N-type singularity $f(x) = x^{N+1}/(N+1)$. Let us fix a weighted-homogeneous basis $\{\phi_a(x)\}_{a=1}^N \subset H$ with $\phi_a(x) = x^{N-a}$ and define the so called *Hodge grading* operator

$$\theta : H \to H, \quad \theta(\phi_a) = \Big(-\frac{1}{2} + \frac{a}{h}\Big)\phi_a,$$

where $h := N+1$ is the Coxeter number. We construct a miniversal unfolding $F(s,x) = f(x)+\sum_{a=1}^N s_a\phi_a(x)$, where the deformation parameters $s = (s_1,\ldots,s_N) \in B := \mathbb{C}^N$. It is known that B has a Frobenius structure (see [3, 6, 7]), which allows us to introduce a special *flat* coordinate system $t = (t_1,\ldots,t_N)$ on B. Following Givental (see [5]) we introduce also a set of period vectors $I^{(n)}_\alpha(t,\lambda)$ defined for all $n \in \mathbb{Z}$ and for all reduced homology cycles $\alpha \in \mathfrak{h} := \widetilde{H}_0(f^{-1}(1);\mathbb{C})$, and a *calibration* $S(t,z) = 1 + S_1(t)z^{-1} + \cdots$. Furthermore, we identify $\mathfrak{h} \cong H$, $\alpha \mapsto a$ in such a way that

$$I^{(0)}_\alpha(t,\lambda) = S(t,-\partial_\lambda^{-1})\,\frac{\lambda^{\theta-1/2}}{\Gamma(\theta+1/2)}\,a.$$

Root systems of type A_N. Put

$$\chi_i := \sum_{a=1}^N \eta^{-ia}h^{-a/h}\Gamma\Big(1-\frac{a}{h}\Big)\,\phi_{N+1-a},$$

where $\eta = e^{2\pi\sqrt{-1}/h}$. The set of vanishing cycles is given by

$$\Delta = \{\chi_i - \chi_j \mid 1 \le i \ne j \le N\}$$

and together with the intersection pairing $(\ |\ )$ forms a root system of type A_N. Let us point out that χ_i are the so called 1-point cycles (see [5]) and that the intersection pairing

$$(\chi_i|\chi_j) = -\frac{1}{h} + \delta_{ij}.$$

State–Field correspondence. Let $\widehat{\mathfrak{h}} := \mathfrak{h}[\zeta,\zeta^{-1}] \oplus \mathbb{C}$ be the Heisenberg Lie algebra with Lie bracket

$$[f(\zeta),g(\zeta)] = \mathrm{Res}_{\zeta=0}(f'(\zeta)|g(\zeta))d\zeta.$$

It is convenient to denote $a_{(n)} = a\zeta^n$ for $a \in H$ and $n \in \mathbb{Z}$. Then the above formula is equivalent to

$$[a_{(m)},b_{(n)}] = m(a|b)\delta_{m+n,0}, \quad m,n \in \mathbb{Z}, \quad a,b \in H.$$

The vector space $\mathcal{F} := \mathrm{Sym}(\mathfrak{h}[\zeta^{-1}]\zeta^{-1})$ has a natural structure of a highest-weight $\widehat{\mathfrak{h}}$-module, s.t., $a_{(n)}1 = 0$ for all $a \in H$ and $n \ge 0$. We identify $\mathrm{Sym}(H) \subset \mathcal{F}$ in such a way that $a \mapsto a\zeta^{-1}$ for all $a \in H$.

Following the construction in [2] we define a *state-field* correspondence $v \mapsto X(v)$, which to every $v \in \mathcal{F}$ associates a *twisted field* $X(v)$. The latter is a differential operator on the set of formal variables

$$\mathbf{t} = \{t_{k,a}\}, \quad 1 \le a \le N, \quad k \ge 0,$$

whose coefficients are Laurent polynomials in $\lambda^{1/h}$. It is defined as follows. First, we set

$$X(a,\lambda) := \Big(\sum_{n\in\mathbb{Z}} I^{(n+1)}_a(0,\lambda)(-z)^n\Big)^{\wedge}, \quad a \in H \tag{1.1}$$

where the quantization rules are

$$(\phi_a z^k)^\wedge = -\hbar^{1/2}\partial_{t_{k,a}}, \quad (\phi_{N+1-a} z^{-k-1})^\wedge = \hbar^{-1/2} t_{k,a},$$

where $\hbar$ is a formal parameter. The definition (1.1) is uniquely extended in such a way that the following Operator Product Expansion axiom holds

$$X(a_{(-n-1)}v, \lambda) = \mathrm{Res}_{\lambda'=\lambda}\, X(a,\lambda')X(v,\lambda)\frac{d\lambda'}{(\lambda'-\lambda)^{n+1}}, \quad a \in H, \quad v \in \mathcal{F}. \tag{1.2}$$

Let us point out that for

$$\gamma_a := h^{-a/h}\Gamma(1-a/h)\,\phi_{h-a}, \quad 1 \le a \le N$$

we have

$$I^{(n)}_{\gamma_a}(0,\lambda) = \frac{\prod_{i=-\infty}^{0}(-a+i\,h)}{\prod_{i=-\infty}^{-n}(-a+i\,h)}\,(h\lambda)^{-n-a/h}\phi_{N+1-a},$$

so the state-field correspondence can be written explicitly for every given $v \in \mathcal{F}$.

Topological recursion. The total descendant potential is a formal series of the type

$$\mathcal{D}(\hbar;\mathbf{t}) = \exp\Big(\sum_{g=0}^{\infty}\hbar^{g-1}\mathcal{F}^{(g)}(\mathbf{t})\Big),$$

where $\mathcal{F}^{(g)}$ are formal power series in $\mathbf{t}$. We refer to [4] for the precise definition. Let us define $\Omega^{(g)}_{j_1,\dots,j_r}$ by the following identity:

$$X(\chi_{j_1}\cdots\chi_{j_r},\lambda)\,\mathcal{D}(\hbar;\mathbf{t}) = \Big(\sum_{g=0}^{\infty}\hbar^{g-r/2}\Omega^{(g)}_{j_1,\dots,j_r}(\lambda;\mathbf{t})\Big)\mathcal{D}(\hbar;\mathbf{t}),$$

where $1 \le j_1 < \cdots < j_r \le h$. Our main result can be stated as follows

$$\frac{\partial\mathcal{F}^{(g)}}{\partial t_{m,a}} = -\,\mathrm{Res}_{\lambda=0}\sum_{i=1}^{h}\sum_{j_1,\dots,j_r}\frac{(I^{(-m-1)}_{\chi_i}(0,\lambda),\phi_a)}{\prod_{s=1}^{r}(I^{(-1)}_{\chi_i-\chi_j}(0,\lambda),1)}\,\Omega^{(g)}_{i,j_1,\dots,j_r}(\lambda;\mathbf{t})d\lambda, \tag{1.3}$$

where the 2nd sum is over all non-empty subsets $\{j_1,\dots,j_r\}$ of $\{1,\dots,i-1,i+1,\dots,h\}$. It is not hard to see that if we give an appropriate weight to each variable $t_{k,i}$, so that the functions $\mathcal{F}^{(g)}$ are homogeneous, then the above identity will give us a recursion that uniquely determines $\mathcal{F}^{(g)}$ for all $g \ge 0$.

Genus-0. Put

$$p_{m,a} = (-a+h)\cdots(-a+(m+1)h)\frac{\partial\mathcal{F}^{(0)}}{\partial t_{m,a}}, \quad 1 \le a \le N, \quad m \ge 0,$$

$$\Phi^{(0)}_a(\lambda,\mathbf{t}) := \sum_{m=0}^{\infty}(x_{m,a}\lambda^m + p_{m,h-a}\lambda^{-m-1}),$$

and define the following numbers

$$C(a_1,\dots,a_r) = \sum_{1\le j_1<\cdots<j_r\le h-1}\frac{\eta^{-j_1a_1}}{1-\eta^{j_1}}\cdots\frac{\eta^{-j_ra_r}}{1-\eta^{j_r}}, \quad 1 \le a_1,\dots,a_r \le N.$$

The genus-0 reduction of the recursion (1.3) can be stated as follows:

$$p_{m,a} =$$
$$- \mathrm{Res}_{\lambda=0} \sum_{a_1,\ldots,a_r=1}^{h-1} C(a_1,\ldots,a_r)\, \Phi^{(0)}_{a_0}(\lambda,\mathbf{t})\Phi^{(0)}_{a_1}(\lambda,\mathbf{t})\cdots\Phi^{(0)}_{a_r}(\lambda,\mathbf{t})\lambda^{m+n+1}\, d\lambda,$$

where the numbers $n \in \mathbb{Z}$ and a_0, $0 \leq a_0 \leq h-1$ are defined by

$$-(a + r + a_1 + \cdots + a_r) = nh + a_0$$

and if $a_0 = 0$ then we set $\Phi^{(0)}_{a_0} = 0$. In particular, if we set $t_{0,a} := t_a$ and $t_{m,a} = 0$ for $m > 0$, then the above identity allows us to compute the primary potential of the Frobenius structure.

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On Gaiotto conjecture

Motohico Mulase

(joint work with Olivia Dumitrescu, Laura Fredrickson, Georgios Kydonakis, Rafe Mazzeo, Andrew Neitzke)

Gaiotto in his seminal paper [1] conjectured surprising construction of Beilinson-Drinfeld opers (globally defined Schrödinger-type equations on a compact Riemann surface) from a particular choice of families of Higgs bundles that are called Hitchin components. We give a purely holomorphic description of the Gaiotto opers, which are constructed as scaling limit of real differentiable objects. Our point of view makes clear that the whole business is deeply related to variation of Hodge structures.

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Topological recursion and cohomological field theories

Nicolas Orantin

1. Introduction

The topological recursion (TR) formalism is a procedure first introduced in the context of random matrix theories for solving a large class of models of statistical physics on a random lattice [4]. Mathematically, these models consists in the enumeration of maps, i.e. in counting polygonalizations of surfaces of given topology. This formalism has been later extended to an abstract setup away from the random matrix formalism [6] and surprisingly proved very efficient for solving many problems of enumerative geometry in a universal way. Among these applications, we can mention the computation of the Weil-Petersson volume of the moduli space of Riemann surfaces, of the Gromov-Witten (GW) invariants of Toric Calabi-Yau threefolds, of GW invariants of $\mathbb{P}^1$ or of simple Hurwitz numbers. However, all these examples were treated case by case and a unifying setup explaining the ubiquity of the TR formalism was still missing.

Such a unifying scheme can actually be found in the context of Frobenius manifold and Cohomological Field Theories (CohFT). In this short note, I review this structure and its connection to the TR formalism as well as some simple examples ranging from Gromov-Witten invariants to modular functors.

2. Cohomological Field Theories

CohFT's are a set of theories axiomatizing the theory of Gromow-Witten invariants [8]. A CohFT is a set of cohomology classes $\Omega := (\Omega_{g,n})_{2g+n-2>0} \in H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes (A^*)^{\otimes n}$ where A is a finite dimensional vector space equipped with a symmetric bilinear form b and A^* is its dual with respect to b. For Ω to define a CohFT, it must satisfy a simple set of axioms.

- For any $2g-2+n>0$, $\Omega_{g,n}$ is invariant under the natural action of the symmetric group S_n.
- There exists a vacuum element $\mathbf{1} \in A$ such that

$$\forall (a_1,a_2) \in A^2\,,\ b(a_1,a_2) = \int_{\overline{\mathcal{M}}_{0,3}} \Omega_{0,3}(a_1 \otimes a_2 \otimes \mathbf{1}).$$

- Under the action of the gluing map $\pi_1 : \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}$, one has

$$\forall (a_1,\dots,a_n) \in A^n\,,\ \pi_1^*\Omega_{g+1,n}(a_1,\dots,a_n) = \sum_{i,j} b(e_i,e_j)\,\Omega_{g,n+2}(e_i,e_j,a_1,\dots,a_n)$$

where $(e_i)_i$ is an arbitrary basis of A.

- Under the action of the gluing map $\pi_2 : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2} \to \overline{\mathcal{M}}_{g,n}$ with $g_1 + g_2 = g$ and $n_1 + n_2 = n$, one has

$$\forall (a_1, \ldots, a_n) \in A^n \,,\ \pi_1^* \Omega_{g,n}(a_1, \ldots, a_n) = \sum_{i,j} b(e_i, e_j)\, \Omega_{g_1,n_1+1}(e_i, a_1, \ldots, a_{n_1}) \otimes \Omega_{g_2,n_2+1}(e_j, a_{n_1+1}, \ldots, a_{n_2}).$$

- Considering the forgetful map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$, one has

$$\forall (a_1, \ldots, a_n) \in A^n \ \ \pi^* \Omega_{g,n}(a_1 \otimes \cdots \otimes a_n) = \Omega_{g,n+1}(a_1 \otimes \cdots \otimes a_n \otimes \mathbf{1}).$$

It should be noted that there exist a natural way for generating families of CohFTs parameterized by a point $u \in U$ such that it equips a manifold U with the structure of a Frobenius manifold as developped by Dubrovin [2].

3. Topological recursion and CohFT

One version of the TR formalism [6] is a procedure which builds by induction a set of symmetric differential forms $(\omega_{g,n})_{g,n} \in H^0 \left(D^n, (K_D(*0))^{\boxtimes n} \right) \otimes (A^*)^{\otimes n}$ in a local neighborhood D of 0 with poles at 0 out of the data of a local spectral curve $\Sigma := (\omega_{0,1}, \omega_{0,2})$ where $\omega_{0,1} \in H^0 (D, K_D) \otimes A^*$ is holomorphic and $\omega_{0,2} \in H^0 \left(D^2, K_D^2(2\Delta)\right) \otimes (A^*)^2$ has double poles on the diagonal Δ.

It was proved in [3] that, under a semi-simplicity assumption, the correlators of a CohFT can be computed by the TR formalism.

Theorem 1. *For any semi-simple CohFT Ω, there exists a local curve $\Sigma_\Omega = (\omega_{0,1}, \omega_{0,2})$ such that the TR formalism produces its correlators. Namely, for all $2g - 2 + n > 0$, one has*

$$\forall (a_1, \ldots a_n) \in A^n \,,$$

$$\omega_{g,n}(a_1, \ldots a_n) = \sum_{\vec{k} \in \mathbb{N}^n} \left[\int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(a_1, \ldots, a_n) \prod_{i=1}^n \Psi_i^{k_i} \right] \prod_{i=1}^n \xi_{k_i}(a_i)$$

where $(\xi_k(a))_{a \in A, k \in \mathbb{N}}$ is a set of differential forms on D computable out of the data of $\omega_{0,2}$.

Even more, the spectral curve Σ_Ω can be computed explicitly.

This theorem is proved by building upon the work of Givental and Teleman. The latter classifies two dimensional topological field theories and CohFT in [10] thus proving a conjecture due to Givental [7] which states how to recover the correlators of a CohFT out of its degree 0 part. This reconstruction procedure expresses the correlators of a CohFT as a sum over stable graphs whose vertices are weighted by the correlators of the degree 0 part of the theory. When the theory is semi-simple, these degree 0 correlators are computable explicitly in terms of intersection of Psi-classes on $\overline{\mathcal{M}}_{g,n}$. It is then possible to identify this sum over stable graph with a representation of the TR correlation functions $\omega_{g,n}$ in terms of stable graphs worked out by Eynard [5].

4. Applications

One of the main applications of CohFTs is the theory of GW invariants. The GW invariants of a manifold X can indeed be written as integrals of some cohomology classes on the moduli space of curves which form a CohFT. This CohFT is semi-simple whenever the quantum cohomology of X is semi-simple.

In this way, the examples of application described in the introduction actually come from a semi-simple CohFT allowing to compute these numbers by the TR formalism.

In [1], another type of example was discussed. It has been motivated by possible applications to low-dimensional topology and conformal field theories. Starting from the data of a modular functor [11], it builds a vector bundle Z over $\mathcal{M}_{g,n}$ which can be extended over its Deligne-Mumford compactification. The total Chern class of Z then defines a CohFT. Its integral over $\overline{\mathcal{M}}_{g,n}$ can thus be computed explicitly by the TR formalism. Among particular cases of modular functors, the bundle Z is the so-called Verlinde bundle [9] describing the space of conformal blocks of a given conformal field theory.

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Quantization of spectral curves and DQ-modules

François Petit

The quantization of spectral curves in terms of quantum curves has recently received a lot of attention due to the close relation between the topological recursion of Eynard-Orantin [2] and quantum curves.

The quantization of spectral curves is a special case of the problem which consists in quantizing a Lagrangian subvariety inside a holomorphic symplectic manifold. The theory of Deformation Quantization (DQ) modules, introduced by Kontsevich in [4] and developed in [3] by Kashiwara and Schapira, provides an adequate framework to study this type of question from the point of view of deformation quantization. In [6], I have studied, from the standpoint of DQ-modules, the problem of quantizing the spectral curve associated to a Higgs bundle. In this situation, the spectral curve is a Lagrangian subvariety inside the cotangent bundle $T^*\Sigma$ of a Riemann surface Σ.

In general (see [1, 5]), a quantum curve quantizing the spectral curve $\mathcal{S} = \{f(x,\xi) = 0\} \subset T^*\Sigma$ associated to a Higgs bundle is understood as an operator P such that

$$P \in R(\mathcal{D}_\Sigma) := \bigoplus_{i\in\mathbb{N}} \hbar^i F_i(\mathcal{D}_\Sigma) \quad \text{and} \quad \lim_{\hbar\to 0} e^{-\frac{x\xi}{\hbar}} P e^{\frac{x\xi}{\hbar}} = f$$

where $\mathcal{D}_\Sigma$ is the sheaf of holomorphic differential operators and $F_i(\mathcal{D}_\Sigma) \subset \mathcal{D}_\Sigma$ is the sheaf of differential operators of order at most i. The operator P give rise to a $R(\mathcal{D}_\Sigma)$-module $\mathcal{M} = R(\mathcal{D}_\Sigma)/R(\mathcal{D}_\Sigma)P$. That is a sheaf on Σ.

Here, we microlocalize the quantization problem and translate it into the framework of DQ-modules. We endow the cotangent bundle $T^*\Sigma$ with its canonical DQ-algebra $\widehat{W}_{T^*\Sigma}(0)$ constructed in [7]. Locally, this DQ-algebra is isomorphic to $(\mathcal{O}_{T^*\Sigma}[[\hbar]], \star)$ where $\star$ is a $\mathbb{C}[[\hbar]]$-bilinear associative law such that

$$\forall f, g \in \mathcal{O}_{T^*\Sigma}, \quad f \star g = \sum_{n\geq 0} \frac{\hbar^n}{n!} \partial_\xi^n f \, \partial_x^n g.$$

In this setting, we are now searching for a sheaf on the cotangent bundle. More precisely, we are looking for a coherent $\widehat{W}_{T^*\Sigma}(0)$-module $\mathcal{M}$ without $\hbar$-torsion supported by $\mathcal{S}$, that is for an holonomic module supported by $\mathcal{S}$. It is possible to translate this problem into a question of cohomology of sheaves via the following lemma

Lemma 1. *Let Σ be a Riemann surface, $\pi : T^*\Sigma \to \Sigma$ the canonical projection and let $\mathcal{L}$ be a line bundle on Σ and $s \neq 0$ be a section of $\pi^*\mathcal{L}$. Assume that $\mathrm{H}^1(T^*\Sigma, \widehat{W}^{\mathcal{L}}_{T^*\Sigma}(0)) = 0$ where $\widehat{W}^{\mathcal{L}}_{T^*\Sigma}(0) := \widehat{W}_{T^*\Sigma}(0) \otimes_{\pi^{-1}\mathcal{O}_\Sigma} \pi^{-1}\mathcal{L}$. Then, there exists a coherent $\widehat{W}_{T^*\Sigma}(0)$-module $\mathcal{M}$ without $\hbar$-torsion supported by the zero locus of s and such that $\mathcal{M}/\hbar\mathcal{M} \simeq \pi^*\mathcal{L}/(s)$ where (s) denotes the $\mathcal{O}_{T^*\Sigma}$-submodule of $\pi^*\mathcal{L}$ generated by s.*

Then, it is possible to give sufficient condition for the vanishing of the cohomology group $\mathrm{H}^1(T^*\Sigma, \widehat{W}^{\mathcal{L}}_{T^*\Sigma}(0))$. In particular, we have the following lemma.

Lemma 2 ([6, lemma 3.10]). *Let Σ be a compact Riemann surface and $\mathcal{L}$ be a line bundle such that* $\mathrm{H}^1(\Sigma, \mathcal{L}) = 0$. *Then,*

$$\mathrm{H}^i(T^*\Sigma, \widehat{W}^{\mathcal{L}}_{T^*\Sigma}(0)) \simeq 0, \text{ for } i > 0.$$

Given a Higss bundle $(\mathcal{E}, \phi)$ of rank r and applying the above lemmas with $\mathcal{L} = (\Omega^1_\Sigma)^{\otimes r}$ and $s = \det(\pi^*\phi - \eta\,\mathrm{id})$ with η the Liouville form of the cotangent bundle, we get the following theorem (see [6, Theorem 3.15]).

Theorem 3. *Let Σ be a compact Riemann surface of genus $g \geq 2$, $(\mathcal{E}, \phi)$ a Higgs bundle of rank $r \geq 2$ on Σ and $\widehat{W}_{T^*\Sigma}(0)$ the canonical quantization of the cotangent bundle $T^*\Sigma$. Then, there exist an holonomic $\widehat{W}_{T^*\Sigma}(0)$-module $\mathcal{M}$ supported by the spectral curve $\mathcal{S}$ of $(\mathcal{E}, \phi)$. Moreover, if this spectral curve is smooth, $\mathcal{M}$ is simple along $\mathcal{S}$.*

The simplicity of $\mathcal{M}$ along $\mathcal{S}$ implies that any other simple DQ-module quantizing $\mathcal{S}$ is locally isomorphic to $\mathcal{M}$.

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Quantisations of some spectral curves

Jörg Teschner

1. Geometric structures associated to spectral curves

There is growing evidence that various problems of mathematical physics are deeply related to certain (quantum) deformations of geometric objects called spectral curves. The list of such problems includes the instanton calculus in $N = 2$, $d = 4$ supersymmetric field theories as well as topological string theory.

Let C be an n-punctured Riemann surface of genus g, and $t = t(u)(du)^2$ be a quadratic differential on C. Out of these ingredients one can define the spectral curve Σ to be the curve in T^*C defined by the equation

$$\Sigma = \{(u, v) \in T^*C; v^2 = t(u)\}. \tag{1.1}$$

The deformation space of such curves can be described as the space $\mathcal{B} \simeq H^0(C, K_C^2)$ fibered over $\mathcal{T}(C)$, the Teichmüller space of deformations of the complex structures on C. This space is isomorphic to the cotangent bundle $T^*\mathcal{T}(C)$, endowing it with a natural symplectic structure.

The space $\mathcal{B}$ carries a geometric structure called special geometry which may be described in terms of the periods $a_r = \int_{\alpha^r} \lambda$, $a_r^{\mathrm{D}} = \int_{\alpha_r^{\mathrm{D}}} \lambda$, $r = 1, \dots, 3g-3+n$, of the canonical differential λ on Σ satisfying $\lambda^2 = t$, with $(\alpha_r, \alpha_r^{\mathrm{D}})$ being a canonical basis for the subspace $H_1^{\mathrm{odd}}(\Sigma, \mathbb{Z})$ of the first homology which is odd under the exchange of sheets of Σ. For given complex coordinates τ_r on $\mathcal{T}(C)$ one may find canonically conjugate coordinates H_r on the cotangent fibers isomorphic to $H^0(C, K_C^2)$ such that (τ_r, H_r) form a system of Darboux coordinates for $T^*\mathcal{T}(C)$. The prepotential $\mathcal{F}(a, \tau)$ satisfying $a_r^{\mathrm{D}} = \int_{\alpha_r^{\mathrm{D}}} \lambda = \partial_{a_r} \mathcal{F}(a, \tau)$ is the generating function for the change of coordinates for $T^*\mathcal{T}(C)$ from (a_r, a_r^{D}) to (τ_r, H_r) [1, Section 7.3].

The special geometry of the space $\mathcal{B}$ canonically defines a complex integrable model, a torus fibration over $\mathcal{B}$ with fibers having complex structure characterised by $\tau_{rs} = \partial_{a_r} \partial_{a_s} \mathcal{F}(a, \tau)$. The resulting integrable model is well-known to be isomorphic to the $SU(2)$ Hitchin system.

2. First Quantisation

The quantisation of the Hitchin system motivates one possible quantisation of the spectral curve Σ, inducing a natural deformation of the prepotential $\mathcal{F}(a, \tau)$ into a one-parameter family of functions $\mathcal{Y}(a, \tau; \epsilon_1)$. Formal quantisation of T^*C, $v \to \epsilon_1 \partial_u$, leads to the differential operator

$$\epsilon_1^2 \partial_u^2 + t(u)\,, \tag{2.1}$$

with $\epsilon_1^{-2} t(u)$ transforming as a projective connection on C. Such differential operators define flat $SL(2, \mathbb{C})$-connections gauge-equivalent to the form $\partial_u + \frac{1}{\epsilon_1} \left(\begin{smallmatrix} 0 & -t(u) \\ 1 & 0 \end{smallmatrix}\right)$ called $\mathfrak{sl}_2$-opers. When $C \simeq \mathbb{P}^1 \setminus \{z_1, \dots, z_n\}$, we may represent $t(u)$ as $t(u) = \sum_{r=1}^n \left(\frac{\mathrm{d}_r}{(u-z_r)^2} + \frac{H_r}{u-z_r} \right)$.

Sklyanin has demonstrated in this case by means of his Separation of Variables method that the eigenvalue equations for the quantum Hitchin Hamiltonians reduce to the equation $(\epsilon_1^2 \partial_u^2 + t(u))\chi(u) = 0$. A natural quantisation problem is to find *single-valued* solutions $\chi(u, \bar{u})$ satisfying both $(\epsilon_1^2 \partial_u^2 + t(u))\chi(u, \bar{u}) = 0$ and the complex conjugate of this equation.

Quantizing spectral curves into opers deforms $T^*\mathcal{T}(C)$ into a twisted cotangent bundle $T^*_{\epsilon_1}\mathcal{T}(C)$. Fixing reference opers defines non-canonical isomorphisms to $T^*\mathcal{T}(C)$ with Darboux coordinates (τ_r, H_r). $T^*_{\epsilon_1}\mathcal{T}(C)$ maps to the moduli space $\mathcal{M}_{\mathrm{flat}}(C)$ of flat $SL(2, \mathbb{C})$-connections by the monodromy map. Introducing Darboux-coordinates (λ_r, κ_r) like those used in [2] for $\mathcal{M}_{\mathrm{flat}}(C)$ allows us to define the generating function $\mathcal{Y}(\lambda, \tau; \epsilon_1)$ for the change of Darboux-coordinates from (τ_r, H_r) to (λ_r, κ_r) defined by the monodromy map. $\mathcal{Y}(\lambda, \tau; \epsilon_1)$ satisfies $\kappa_r = \partial_{\lambda_r} \mathcal{Y}(\lambda, \tau; \epsilon_1)$ and $H_r = -\partial_{\tau_r} \mathcal{Y}(\lambda, \tau; \epsilon_1)$. It has been argued somewhat indirectly in [2, 3], and more directly in [4] that the function $\mathcal{Y}(\lambda, \tau; \epsilon_1)$ defined in this way characterises the solutions to the above-mentioned quantisation problem for the quantised Hitchin systems.

The expansion of $\mathcal{Y}(\lambda,\tau;\epsilon_1)$ when $\epsilon_1 \to 0$ can be calculated using the WKB-expansion of the differential equation $(\epsilon_1^2\partial_u^2 + t(u))\chi(u) = 0$. To leading order in ϵ_1 one finds that λ_r and κ_r are proportional to a_r and $a_r^{\rm D}$, respectively. The Darboux-coordinates (λ_r, κ_r) are thereby identified as non-perturbative versions of the so-called "quantum periods" introduced in [5].

3. Second Quantisation

A connected component of the real slice in $\mathcal{M}_{\rm flat}(C)$ characterised by real values of (λ_r, κ_r) is isomorphic to the zero section $\mathcal{T}(C)$ of $T^*\mathcal{T}(C)$. The canonical quantisation of this real slice yields representations of the quantised algebras of functions on both $\mathcal{M}_{\rm flat}(C)$ and $T^*\mathcal{T}(C)$ [6]. The generating function $\mathcal{Y}(\lambda,\tau;\epsilon_1)$ is deformed into a wave-function $\mathcal{Z}(a,\tau;\epsilon_1,\epsilon_2)$ such that $\lim_{\epsilon_2\to 0}\epsilon_2 \log\mathcal{Z}(a,\tau;\epsilon_1,\epsilon_2) = \mathcal{Y}(\lambda,\tau;\epsilon_1)$. The wave-function $\mathcal{Z}(a,\tau;\epsilon_1,\epsilon_2)$ can be identified as chiral partition function for the Virasoro algebra [6].

It is instructive to observe that the wave-functions $\mathcal{Z}(a,\tau;\epsilon_1,\epsilon_2)$ can be characterised in terms of a "second-quantised" version of the equation for the the quantum spectral curve. The canonical symplectic structure on $T^*\mathcal{T}(C)$ naturally leads to a quantisation where the coordinates H_r get replaced by the differential operators $\epsilon_1\epsilon_2\partial_{\tau_r}$. When $C = \mathbb{P}^1 \setminus \{z_1,\dots,z_n\}$ one may represent the derivatives ∂_{τ_r} in terms of the derivatives ∂_{z_r}, naturally leading to the differential operator

$$\mathcal{D} = \epsilon_1^2\partial_u^2 + \sum_{r=1}^{n}\left(\frac{\delta_r}{(u-z_r)^2} + \frac{\epsilon_1\epsilon_2}{u-z_r}\partial_{z_r}\right). \tag{3.1}$$

Solutions $\Psi(u;a,z)$ to the equation $\mathcal{D}\Psi = 0$ are chiral partition functions of conformal blocks of the Virasoro algebra at $c = 1+6(b+b^{-1})^2$, $b^2 = \epsilon_1/\epsilon_2$, with the insertion of a degenerate field at $u \in C$. There exist solutions to $\mathcal{D}\Psi = 0$ essentially uniquely characterised by the property that the traces of monodromies along a maximal set of non-intersecting closed curves on C are given by the functions $2\cosh(2\pi a_r/\epsilon_1)$ of the variables a. It is possible to reconstruct the chiral partition functions $\mathcal{Z}(a,\tau;\epsilon_1,\epsilon_2)$ without degenerate fields from these solutions.

It would be very interesting to reproduce the precise description of $\mathcal{Z}(a,z)$ outlined above using other approaches, such as the topological recursion.

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Hemisphere Partition Function and Gauged Linear Sigma Model

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(joint work with Johanna Knapp, Mauricio Romo)

The gauged linear sigma model (GLSM) was introduced by Witten in [1] in order to explain why the physics of the nonlinear linear sigma model with target space a Calabi–Yau hypersurface X in a projective space is equivalent to the physics of a certain Landau–Ginzburg orbifold. The GLSM smoothly interpolates between the two theories, and the parameter space turns out to be a complex manifold. This complex manifold is diffeomorphic to the (extended) Kähler moduli space of X, and hence to the complex structure moduli space of its mirror $X^\vee$. The complex structure moduli space $\mathcal{M}$ of a Calabi–Yau threefold carries an additional structure known as special Kähler geometry which in particular includes a Kähler potential K and a flat symplectic connection ∇ satisfying a compatibility condition. It is therefore expected that the parameter space of the GLSM also carries an additional structure which in the limiting points reduces to special Kähler geometry. A few years ago, it has been conjectured in [2] that the sphere partition function of the GLSM computes $\exp(-K)$. Shortly afterwards, it has been conjectured in [3] that the hemisphere partition function of the GLSM computes the central charge function. The latter is expected to be a flat section of an appropriate flat connection ∇. Here, we are going to argue that this is, at least locally, indeed the case [4].

In order to define the GLSM we first define the notion of a GLSM datum. This is a quadruple (G, W, ρ_V, R) where G is a compact real Lie group, $\rho_V : G \to \mathrm{GL}(V)$ a complex representation, $W \in \mathrm{Sym}(V^*)$ a holomorphic function, and $R : \mathrm{U}(1) \to \mathrm{GL}(V)$ a complex representation, satisfying the following conditions: W is G–invariant and has weight 2 under R, ρ_V and R commute, moreover $R(\mathrm{e}^{i\pi}) = \rho_V(J)$ for some $J \in G$. W is known as the superpotential of the GLSM.

Let $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{t} = \mathrm{Lie}(T)$, where T is a maximal torus of G. Decomposing $\mathfrak{g} = \mathfrak{s} + \mathfrak{a}$ into a semisimple and an abelian part, we set $t = (t_1, \ldots, t_s) \in \mathfrak{a}^*_{\mathbb{C}}$ and restrict the t_k to $\mathbb{R} + \mathbb{R}/2\pi i\mathbb{Z}$ and write $t_k = \zeta_k - i\theta_k$. Then we consider the symplectic quotients $X_\zeta = \left(\{\mathrm{d}W^{-1}(0)\} \cap \mu^{-1}(\zeta)\right)/G$ where $\mu : V \to \mathfrak{g}^*$ is the moment map associated to ρ_V. We are mostly interested in the case where X_ζ is Calabi–Yau which requires $\rho_V : G \to \mathrm{SL}(V)$.

The moment map equations divide the parameter space, i.e. the set of ζ_i's into chambers called phases. Each phase is characterized by an ideal $I_\zeta \subset \mathrm{Sym}(V^*) \cong$

$\mathbb{C}[\phi_1,\ldots,\phi_m]$ given by $I_\zeta = \{\phi \in V \mid \mu_\phi^{-1}(\zeta) \text{ is ill defined}\}$. The set of t_i's is diffeomorphic to a neighborhood in the Kähler moduli space of X_ζ.

The running example starts with $G = \mathrm{U}(1)$ and $V \cong \mathbb{C}^{N+1}$ with coordinates $(p, x_1, \ldots, x_N)$. Then we choose $W = pG_N$ where G_N is a homogeneous polynomial of degree N in $x_1, \ldots, x_N$ and the weights of ρ_V and R are $(-N, 1, \ldots, 1)$ and $(2, 0, \ldots, 0)$, respectively. In physics these are known as gauge and R charges. It turns out that there are two phases denoted by $\zeta \gg 0$ and $\zeta \ll 0$: $X_{\zeta\gg 0}$ is a Calabi–Yau hypersurface $\{G_N = 0\} \subset \mathbb{P}^{N-1}$ with characterizing ideal $I_{\zeta\gg 0} = (x_1, \ldots, x_N)$. $X_{\zeta\ll 0}$ is a Landau–Ginzburg orbifold, i.e. a quotient stack together with a holomorphic function $\left(\left[\mathbb{C}^N/\mathbb{Z}_N\right], G_N : \left[\mathbb{C}^N/\mathbb{Z}_N\right] \to \mathbb{C}\right)$ with characterizing ideal $I_{\zeta\ll 0} = (p)$.

An important role is played by the loci in the parameter space, where the GLSM is singular. For the sake of exposition with only discuss the case of $G = \mathrm{U}(1)$. The singular points $t \in \mathfrak{a}_\mathbb{C}^*$ are characterized by a holomorphic function $\widetilde{W}_{\text{eff}} : \mathfrak{a}_\mathbb{C} \to \mathbb{C}/2\pi i\mathbb{Z}$ given by $\widetilde{W}_{\text{eff}}(\sigma) = -t(\sigma) - \sum_i Q_i(\sigma)\log Q_i(\sigma) \mod 2\pi i$ where $Q_i \in \mathfrak{t}^*$ denotes the weights of ρ_V. The GLSM datum is singular along its critical locus $\Delta = \{t \in \mathfrak{a}_\mathbb{C}^* \mid \partial_\sigma \widetilde{W}_{\text{eff}}(\sigma) = 0\}$. In our running example $\widetilde{W}_{\text{eff}}(\sigma) = -t\sigma + N\sigma\log(-N) \mod 2\pi i$ and $\Delta = \{t = N\log N + i\pi N + 2\pi i k, k \in \mathbb{Z}\}$.

Finally the GLSM is the study of maps from a Riemann surface Σ into V respecting all the structure on V, i.e. the coordinates ϕ and σ are promoted to sections $\phi \in \Gamma(\Sigma, P_G \times_{\rho_V} V)$ and $\sigma \in \Gamma(\Sigma, P_G \times_{\text{adj}} \mathfrak{g}_\mathbb{C})$, respectively, where P_G is a principal G–bundle on Σ. The case $\Sigma = \mathbb{R}^2$ is rather trivial. If Σ admits additional isometries, they can be used to evaluate the path integral $Z_\Sigma = \int \mathrm{D}\phi\mathrm{D}\sigma \exp(-S[\phi,\sigma])$. For $\Sigma = S^2$ with the round metric this has been done in [2]. If Σ is the hemisphere D^2, i.e. the disk with the round metric, we need to specify in addition boundary conditions $\mathcal{B}$ along ∂D^2 as well as a boundary action S_{bdy}. The hemisphere path integral then reads $Z_{D^2}(\mathcal{B}) = \int \mathrm{D}\phi\mathrm{D}\sigma\mathrm{D}\psi \exp(-S[\phi,\sigma] - S_{\text{bdy}}[\phi,\sigma,\psi])$, where the boundary fields are relaated to $\mathcal{B}$. Before we can state the result of [3], we need to discuss the boundary conditions $\mathcal{B}$ in detail.

Let $S = \mathrm{Sym}(V^*) = \mathbb{C}[\phi_1,\ldots,\phi_m]$. A boundary datum (also called a GLSM brane or a B–brane) is a quadruple $\mathcal{B} = (M, Q, \rho, \mathbf{r}_*)$ consisting of a $\mathbb{Z}_2$–graded free S–module $M = M^0 \oplus M^1$, an odd endomorphism $Q \in \mathrm{End}_S^1(M)$, a complex representation $\rho : G \to \mathrm{GL}(M)$ and a complex representation $\mathbf{r}_* : \mathfrak{u}(1) \to \mathrm{GL}(M)$ satisfying the following conditions: Q is a matrix factorization of W, i.e. $Q^2 = W\mathrm{id}_M$, and has weight 1 with respect to $\mathbf{r}_*$. Moreover, $\mathrm{d}\rho$ and $\mathbf{r}_*$ commute, and satisfy natural compatibility conditions with ρ_V and R.

B–branes form a triangulated category $\mathcal{D} = \mathcal{D}_{(G,W,\rho_V,R)}$ with G– and $U(1)$–equivariant morphisms $\mathrm{H}^p(\mathcal{B}_1,\mathcal{B}_2) = \mathrm{H}^p_D(\mathrm{Hom}_S(M_1,M_2))$, $p = 0,1$, where $D\psi = Q_2\psi - (-1)^{|\psi|}\psi Q_1$. There are certain subcategories of $\mathcal{D}$ associated to the phases parametrized by ζ. Let $T_\zeta \subset \mathcal{D}$ be the full triangulated subcategory generated by those modules that are annihilated by some power of the ideal I_ζ. Then we define the triangulated quotient $\mathcal{D}_\zeta = \mathcal{D}/T_\zeta$. This category is conjectured to be equivalent to an appropriate derived category associated to X_ζ, i.e. $\mathcal{D}_\zeta \cong D^b(X_\zeta)$ [5, 6].

Now, we are in place to state the result of the evaluation of the path integral defining the hemisphere partition function [3]:

$$Z_{D^2}(\mathcal{B},t) = C\int_{\gamma\subset\mathfrak{t}_{\mathbb{C}}} \mathrm{d}^r\sigma \prod_{\alpha\in\Pi^+}\alpha(\sigma)\sinh\pi\alpha(\sigma)\prod_{j=1}^{m}\Gamma\left(iQ_j(\sigma)+\frac{R_j}{2}\right)\mathrm{e}^{it\sigma}f_{\mathcal{B}}(\sigma).$$

Here, $r=\mathrm{rk}(G)$, C is a normalization constant, Π^+ is the set of positive roots associated to $(\mathfrak{g},\mathfrak{t})$ and

$$f_{\mathcal{B}}(\sigma)=\mathrm{tr}_M\left(\mathrm{e}^{i\pi\mathbf{r}_*}\mathrm{e}^{2\pi\sigma}\right).$$

We observe that $Z_{D^2}(\mathcal{B},t)$ depends only on $\mathcal{B}$ through $f_{\mathcal{B}}$. The latter depends only on the class $[\mathcal{B}]\in K_0(\mathcal{D})$ in the Grothendieck group of $\mathcal{D}$ an can be thought of as an equivariant Chern character. Hence, the conjecture is that $Z_{D^2}: K_0(\mathcal{D})\to\mathbb{C}$ is a central charge function for a Bridgeland stability condition on $\mathcal{D}$ [7].

Note that while the original path integral is not yet mathematically well–defined, the above contour integral can be well–defined as everything can be expressed in terms of the GLSM datum (G,W,ρ_V,R), the boundary datum $\mathcal{B}$, and $t\in\mathfrak{a}^*_{\mathbb{C}}$. It should therefore be taken as a definition and starting point from the point of view of mathematics. To make it well–defined, it remains to specify the contour $\gamma\subset\mathfrak{t}_{\mathbb{C}}$. For this purpose, we observe that the poles of the integrand all lie on hyperplanes $H_j=\{iQ_j(\sigma)=-\frac{R_j}{2}-k \mid k\in\mathbb{Z}_{\geq 0}\}\subset\mathrm{Im}\,\mathfrak{t}_{\mathbb{C}}$. Let $P=\bigcup_{i=1}^m H_i$ be the union of these polar hyperplanes. We define a contour $\gamma\subset\mathfrak{t}_{\mathbb{C}}$ to be admissible if it is Lagrangian in $\mathfrak{t}_{\mathbb{C}}\setminus P$, if $Z_{D^2}(\mathcal{B},t)$ converges absolutely on γ, and if γ is a continuous deformation of $\mathrm{Re}\,\mathfrak{t}_{\mathbb{C}}$.

At present, admissible contours are only fully understood for $G=\mathrm{U}(1)$. In this case, $f_{\mathcal{B}}$ takes the form

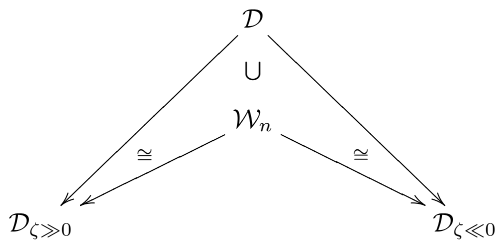
$$f_{\mathcal{B}}=\sum_j\sum_{i=1}^{L_j} n_j^{(i)}\mathrm{e}^{i\pi r_j^{(i)}}\mathrm{e}^{2\pi q_j^{(i)}\sigma}$$

where $r_j^{(i)}$ and $q_j^{(i)}$ are the weights of $\mathbf{r}_*$ and ρ, respectively, and $n_j^{(i)}\in\mathbb{Z}$. An analysis of the asymptotic behaviour of $Z_{D^2}(\mathcal{B},t)$ reveals that any weight q of ρ must satisfy the condition

$$-\frac{N}{2}<\frac{\theta}{2\pi}+q<\frac{N}{2}.$$

This condition has been discovered previously in the study of the subcategories $\mathcal{D}_\zeta$ and is known as the grade restriction rule [6]. To state our result we introduce chambers $T_n=(\pi(N+2n),\pi(N+2n+2))\subset\mathbb{R}$, $n\in\mathbb{Z}$ in the universal covering space of $\mathbb{R}/2\pi\mathbb{Z}\backslash\Delta$. Then we have the following characterization of the contours [4]: Given $\mathcal{B}$ and T_n, there exists an admissible contour γ_t for $\mathcal{B}$ at all points $(\zeta,\theta)\in\mathbb{R}\times T_n$ if and only if all weights of ρ satisfy the grade restriction rule. In particular $\gamma=\mathrm{Re}\,\mathfrak{t}_{\mathbb{C}}$.

The relation to [6] is expressed in the following conjecture: Let $\mathcal{W}_n$ be the full triangulated subcategory of $\mathcal{D}$ of objects $\mathcal{B}$ satisfying the grade restriction rule for $\theta \in T_n$. Then there should exist equivalences such that the following diagram commutes:



Finally, we state an important property of $Z_{D^2}(\mathcal{B})$ in the case of our running example. For this purpose, we introduce the so–called algebraic coordinate of the GLSM: $z = \mathrm{e}^{-i\pi N} N^N \mathrm{e}^{-t}$. We show in [4] that $Z_{D^2}(\mathcal{B})$ viewed as a function of z is a solution to the (generalized) hypergeometric differential equation

$$\left(\theta^{N-1} - z \prod_{j=1}^{N-1} \left(\theta + \tfrac{j}{N}\right)\right) y(z) = 0,$$

where $\theta = z\frac{\mathrm{d}}{\mathrm{d}z}$. In particular, the regular singularities at $z = 0, \infty$ and 1 correspond to the limiting points $t = \infty, t = -\infty$ and the singular set Δ, respectively. This indeed shows that, at least locally, $Z_{D^2}(\mathcal{B})$ is a flat section of an appropriate connection ∇. We emphasize that this is valid in *any* phase of the GLSM. We can, however, give a precise relation to the two phases as follows: Closing the contour to the right and taking residues yields a series expansion $Z_{D^2}^{\zeta\gg 0}(\mathcal{B})$ valid in a neighborhood of $t = \infty$. Similarly, closing the contour to the left yields a series expansion $Z_{D^2}^{\zeta\ll 0}(\mathcal{B})$ valid in a neighborhood of $t = -\infty$. These two series are conjectured to represent the central charge function for Bridgeland stability conditions on $\mathcal{D}_{\zeta\gg 0}$ and $\mathcal{D}_{\zeta\ll 0}$, respectively.

The behaviour of $Z_{D^2}(\mathcal{B})$ in a neighborhood of the singular points $t \in \Delta$ has been studied in [4]. In this case the integrand can be transformed in such a way that the residue theorem can also be applied to give a series expansion near the singular points. Moreover, we give analytic expressions for the analytic continuation of the solutions of the hypergeometric differential equation from $z = 0$ and $z = 1$ [8]. These were previously unknown.

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Quivers of finite mutation type

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(joint work with Anna Felikson, Pavel Tumarkin)

Quivers are directed graphs without loops or two-loops. Quiver mutations play important role in definition of cluster algebra and also appeared independently in form of symmetries of some physical theories (Seiberg duality). Each cluster is equipped by its quiver where transitions between clusters correspond to quiver mutations given by simple combinatorial rules. All quivers obtained one from another by a sequence of quiver mutations form a mutation class (finite or infinite).

We obtain a classification of the cases when the mutation class is finite, and produce certain applications of it.

Quantum curves, matrix models and conformal field theory

Piotr Sułkowski
(joint work with Masahide Manabe)

Quantum curves are actively studied in modern mathematical physics. They appear in various contexts, such as quantization of spectral curves in matrix models, quantization of mirror curves in topological string theory, A-polynomials in knot theory, systems of intersecting branes in string theory, surface operators, in enumerative problems related to moduli spaces of Riemann surfaces, etc. [1, 2, 3]. In all those cases there is a natural classical limit, in which a quantum curve reduces to a complex algebraic curve. It has been claimed that, for a given algebraic curve, a quantum curve can be (more or less) uniquely constructed, and in [1] such a construction, based on the topological recursion [4], has been developed.

In this note, following [3], we show that to a given classical algebraic curve one can in fact assign an infinite family of quantum curves, which are in one-to-one correspondence with, and have the structure of, Virasoro singular vectors. Our construction is based on a matrix model analysis, and in this context a classical algebraic curve should be interpreted as a spectral curve of a matrix model. However this setup can be reformulated in terms of the topological recursion [4], so that a construction of quantum curves can be conducted much more generally. Quantum curves considered previously in literature, and in particular their construction presented in [1], correspond to the simplest non-trivial Virasoro singular vector (at level 2) from the perspective presented here. All other quantum curves that we identify below correspond to singular vectors at levels higher than 2, and we often refer to them as higher level quantum curves.

In more detail, let us consider a classical algebraic curve, defined by a polynomial equation in two complex variables x and y

$$A(x,y)=0. \tag{1.1}$$

A quantum curve associated to this algebraic curve is an operator $\widehat{A}(\hat{x},\hat{y})$ annihilating a wave-function $\widehat{\psi}(x)$

$$\widehat{A}(\hat{x},\hat{y})\widehat{\psi}(x)=0, \tag{1.2}$$

where operators $\hat{x}$ and $\hat{y}$ satisfy a commutation relation $[\hat{y},\hat{x}]=g_s$. In the classical limit $g_s\to 0$, $\widehat{A}(\hat{x},\hat{y})$ should reduce (possibly up to some factor) to $A(x,y)$.

To construct quantum curves we consider a wave-function depending on a parameter α, of the following form

$$\widehat{\psi}_\alpha(x)=\frac{e^{-\frac{2\alpha}{\epsilon_1\epsilon_2}V(x)}}{(2\pi)^N N!}\int_{\mathbb{R}^N}\Delta(z)^{2\beta}\Big(\prod_{a=1}^N(x-z_a)^{-\frac{2\alpha}{\epsilon_2}}\Big)e^{-\frac{\sqrt{\beta}}{\hbar}\sum_{a=1}^N V(z_a)}\prod_{a=1}^N dz_a, \tag{1.3}$$

where $e^{-\frac{2\alpha}{\epsilon_1\epsilon_2}V(x)}/((2\pi)^N N!)$ is certain normalization, $\Delta(z)=\prod_{1\le a<b\le N}(z_a-z_b)$ denotes the Vandermonde determinant, and we also denote

$$\epsilon_1=-\beta^{1/2}g_s,\qquad \epsilon_2=\beta^{-1/2}g_s,\qquad g_s=2\hbar,\qquad b^2=-\beta=\frac{\epsilon_1}{\epsilon_2}. \tag{1.4}$$

For $\alpha=0$ the wave-function (1.3) reduces to the partition function $Z\equiv\widehat{\psi}_{\alpha=0}(x)$ of the standard β-deformed matrix model. For $\beta=1,\frac{1}{2},2$ the above expression is an eigenvalue representation of an integral over hermitian, orthogonal, and symplectic matrices M, and in those cases the term in the bracket in (1.3) is the eigenvalue representation of the determinant $\det(x-M)$, raised to some power parametrized by a parameter α. The exponential term in (1.3) is the eigenvalue representation of $e^{-\frac{\sqrt{\beta}}{\hbar}\mathrm{Tr}V(M)}$, and we assume the potential $V=V(x)$ is a series in times t_n

$$V(x)=\sum_{n=0}^{\infty}t_n x^n. \tag{1.5}$$

The form of the wave-function (1.3) can be shown to arise as the expectation value of $e^{\frac{2\alpha}{g_s}\phi(x)}$, in the β-deformed matrix model realization of the chiral boson field $\phi(x)$, with the background charge $Q=i\big(b+\frac{1}{b}\big)=\frac{\epsilon_1+\epsilon_2}{g_s}$. Our results are therefore a consequence of the conformal invariance of such chiral boson theory in two dimensions, and the associated Virasoro algebra with central charge $c=1-6Q^2$.

We now pose the following question: for which values of parametr α the wave-function satisfies a finite order differential equation in x? We find a beautiful answer to this question: this is so only for discrete values of α of the form

$$\alpha=\alpha_{r,s}=-\frac{r-1}{2}\epsilon_1-\frac{s-1}{2}\epsilon_2,\qquad \text{for } r,s=1,2,3,\dots \tag{1.6}$$

which coincide with degenerate momenta of the chiral boson. We find that for $\alpha = \alpha_{r,s}$ the differential equation for $\widehat{\psi}_\alpha(x)$ has order $n = rs$, and we write it as

$$\widehat{A}_n^\alpha \widehat{\psi}_\alpha(x) = 0, \tag{1.7}$$

where

$$\widehat{A}_n^\alpha = \sum_{p_1+p_2+\ldots+p_k=n} \widehat{c}_{p_1,p_2,\ldots,p_k}(\alpha)\, \widehat{L}_{-p_1}\widehat{L}_{-p_2}\cdots\widehat{L}_{-p_k}. \tag{1.8}$$

The crucial result is that $\widehat{A}_n^\alpha$ has the same structure as an operator that acting on a primary state creates a Virasoro singular vector corresponding to a given value $\alpha_{r,s}$. In particular $\widehat{c}_{p_1,p_2,\ldots,p_k}(\alpha)$ are appropriate constants that appear in expressions for such singular vectors. Furthermore, we find that operators $\widehat{L}_{-p}$ with $p \geq 0$ form the following (seemingly unknown previously) representation of the Virasoro algebra on a space of functions in x and times t_k

(1.9)

$$\widehat{L}_0 = \Delta_\alpha \equiv \frac{\alpha}{g_s}\Big(\frac{\alpha}{g_s} - Q\Big), \qquad \widehat{L}_{-1} = \partial_x,$$

$$\widehat{L}_{-n} = -\frac{1}{\epsilon_1\epsilon_2(n-2)!}\Big(\partial_x^{n-2}\big(V'(x)^2\big) + (\epsilon_1+\epsilon_2)\partial_x^n V(x) + \partial_x^{n-2}\hat{f}(x)\Big), \text{ for } n \geq 2,$$

where $\partial_x^n \widehat{f}(x) \equiv [\partial_x, \partial_x^{n-1}\widehat{f}(x)]$ and

$$\hat{f}(x) = -\epsilon_1\epsilon_2 \sum_{m=0}^{\infty} x^m \partial_{(m)}, \qquad \partial_{(m)} = \sum_{k=m+2}^{\infty} k t_k \frac{\partial}{\partial t_{k-m-2}}. \tag{1.10}$$

As $\widehat{L}_{-n}$ involves derivatives with respect to times (encoded in $\partial_x^{n-2}\hat{f}(x)$), operators (1.8) in general impose partial differential equations in x and t_k. However, in some specific situations these equations can be turned into ordinary differential equations in x. We refer to operators $\widehat{A}_n^\alpha$ in (1.8) as higher level quantum curves.

As an example, imposing a second order differential equation in x for $\widehat{\psi}_\alpha(x)$, we find the following quantum curves at level 2

$$\widehat{A}_2^\alpha \widehat{\psi}_\alpha(x) \equiv \Big(\widehat{L}_{-1}^2 + \frac{4\alpha^2}{\epsilon_1\epsilon_2}\widehat{L}_{-2}\Big)\widehat{\psi}_\alpha(x) = 0, \quad \text{for } \alpha = -\frac{\epsilon_1}{2}, -\frac{\epsilon_2}{2}, \tag{1.11}$$

and for these two values of α the operator $\widehat{A}_2^\alpha$ can be written simply as $\widehat{A}_2^\alpha = \widehat{L}_{-1}^2 + b^{\pm 2}\widehat{L}_{-2}$, which we indeed recognize as usual expressions encoding Virasoro singular vectors at level 2. Explicitly, in terms of representation (1.9) we can write

$$\widehat{A}_2^\alpha = \partial_x^2 - \frac{4\alpha^2}{\epsilon_1^2\epsilon_2^2}V'(x)^2 - \frac{4\alpha^2}{\epsilon_1^2\epsilon_2^2}(\epsilon_1+\epsilon_2)V''(x) - \frac{4\alpha^2}{\epsilon_1^2\epsilon_2^2}\widehat{f}(x). \tag{1.12}$$

Furthermore, rewriting (1.11) as equations for the normalized wave-function $\Psi_\alpha(x) = \widehat{\psi}_\alpha(x)/Z$, where $Z = \widehat{\psi}_{\alpha=0}(x)$, and setting $\beta = 1$, in the classical limit $g_s \to 0$ the above quantum curves reduce to the matrix model spectral curve

$$y^2 - V'(x)^2 - f_{cl}(x) = 0, \tag{1.13}$$

where y is identified with the classical limit of $g_s\partial_x$, and $f_{cl}(x)$ denotes the classical limit of expression that originates from the action of $\widehat{f}(x)$.

For more results, examples, and a discussion of various properties of quantum curves introduced above we refer a reader to [3]. We find it quite intriguing to identify the whole family of quantum curves and their interpretation in terms of Virasoro singular vectors. We are convinced that much more information is encoded in the structure of those quantum curves, and they will turn out to be related to various actively studied problems in mathematical physics.

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Refined topological strings and conformal blocks

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(joint work with Alexei Morozov)

In these notes, based on the paper [1], we report on some new explicit relations between refined topological string amplitudes on toric Calabi-Yau three-folds and q-deformed conformal field theory correlators.

Topological string theory on toric Calabi-Yau three-folds can be effectively solved by the topological vertex technique [2]. The amplitude or partition function in this approach is computed using a Feynman diagram-like prescription, in which the (trivalent) vertices of the three-fold's toric diagram are understood as interactions of particles and the edges play the role of propagators. The analogues of the intermediate particles' momenta are integer partitions (Young diagrams) each associated with an intermediate edge. In this fashion the whole calculation reduces to a certain multiple sum over partitions of very explicit expressions — products of vertices and propagators.

Topological string amplitudes count the number of embeddings of the string worldsheet into the threefold. From a broader string theory perspective these numbers can also be interpreted as the multiplicities of BPS states, e.g. in five-dimensional gauge theory obtained from the compactification of the M-theory on the Calabi-Yau three-fold. Topological string partition function in this case is equal to the trace of a certain operator over the Hilbert space of BPS states in the gauge theory — the quantity called supersymmetric index, or equivariant partition function. The index counts the multiplicities of BPS states, however, it does not distinguish BPS particles spinning in different directions. To circumvent this problem, a more refined version of index is needed, which gives rise to a corresponding (strictly speaking, still hypothetical) refinement of the topological string theory. Calculations for *refined* topological strings entirely rely on the technique similar to topological vertex — which is in this case called the refined topological vertex [3]. For refined vertex calculations an additional decoration of the toric diagram legs, called the *preferred direction,* is needed, which naively breaks the rotational invariance of the diagram and the whole computation. It turns out, however, that the final answer for refined partition function is independent of the preferred direction — an interesting fact, known as the slicing invariance hypothesis.

Quite remarkably, the corresponding index in the gauge theory is also equal to the conformal block of the two-dimensional conformal field theory (CFT) with q-deformed conformal algebra. Conformal block is the holomorphic part of the primary fields correlator, and is the basic, universal and most studied object in CFT. Just as other special functions, it has several different representations. If one inserts the sum over complete basis of intermediate states into the correlator, the conformal block is expressed as an infinite sum over these states. In a different approach, called the Dotsenko-Fateev (DF) integral representation, the block is computed in a particularly simple CFT — the free field theory — and then continued to general values of parameters using a clever trick called dressing, or screening charge insertion.

We give a dictionary, connecting the CFT objects with those in topological strings. In particular, we clarify the relation between the refined topological string amplitudes and multipoint conformal blocks. We also identify two representations of the conformal block mentioned above with two different choices of the preferred direction — one vertical and another horizontal. We give explicit expressions for the matrix elements of the vertex operator for two distinguished choices of the basis in the CFT Hilbert space and relate them to the corresponding amplitudes of refined topological strings. Most importantly, we elucidate on the symmetry between different choices of preferred direction and relate it to a nontrivial linear transformation between different basis sets in CFT. This, essentially, proves the slicing hypothesis for certain classes of toric Calabi-Yau three-folds.

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