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Mini-Workshop: PBW Structures in Representation Theory

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Abstract. The PBW structures play a very important role in the Lie theory and in the theory of algebraic groups. The importance is due to the huge number of possible applications. The main goal of the workshop was to bring together experts and young researchers working in the certain areas in which PBW structures naturally appear. The interaction between the participants allowed to find new viewpoints on the classical mathematical structures and to launch the study of new directions in geometric, algebraic and combinatorial Lie theory.

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Introduction by the Organisers

The mini-workshop *PBW Structures in Representation Theory*, organised by Evgeny Feigin (Moscow), Ghislain Fourier (Glasgow) and Martina Lanini (Edinburgh) took place February 28th–March 5, 2016. It was attended by 17 participants, including a consistent number of young researchers, from France, Germany, Italy, Russia, Spain, the UK and the US.

The first two days of the mini-workshop were dedicated to four two-hour introductory lecture series, whose aim was to provide all participants with some common background. They recalled the state of art, explained open and interesting conjectures, and gave possible directions for future projects and interactions. Within the week, we also had 7 research talks, most of them given by the younger participants. All presentations were quite informal, with a very active participation of the audience, strong interactions with the speaker, and several questions and answers during the talks, which served as a starting point for the discussion sessions. We have the evidence that the workshop was properly sized and quite focused as in all talks, almost all participants were directly involved in the interactions. We have had daily discussion sessions which played a central role in the miniworkshop. These informal sessions were of two types: either a participant was explaining a problem he/she is working on, or a group of participants were working together on an existing or new project.

The topics discussed within the workshop were centred around the PBW (Poincaré-Birkhoff-Witt) theorem and its appearances and applications in different areas of mathematics, such as representation theory, algebraic geometry, and combinatorics. The central idea is to consider an algebraic object, e.g. a simple complex Lie algebra, and apply the PBW Theorem to produce a filtration on it. This gives a machinery to generate new mathematical objects starting from a given one by applying certain algebra or group of operators, it provides a powerful link between Lie theory and commutative algebra. Such a strategy has been successfully applied in (algebraic, geometric and combinatorial) Lie theory and proved to be very powerful both in the theoretical questions and in applications.

The goal of the mini-workshop was to further study this sort of phenomena and, in particular, investigate applications in new fields by bringing together researchers with different backgrounds. Indeed, the range of topics discussed within the workshop was broad: Newton-Okounkov bodies, toric degenerations, quiver Grassmannians, affine and finite-dimensional Grassmannians and flag varieties, representation theory of Kac-Moody Lie algebra, theory of Macdonald polynomials (symmetric and nonsymmetric). The interdisciplinary of the event was further fostered by interesting exchanges with several researchers working on toric degenerations and Newton-Okounkov bodies, who were attending a parallel MFO mini-workshop ("Arrangements of Subvarieties, and their Applications in Algebraic Geometry").
Several new projects were initiated because of the workshop. The exchanges between experts in representation theory and experts of Newton-Okounkov bodies led to very promising new collaborations. The concentration of researchers coming from different areas (cluster algebras, Lie algebras, tropical geometry) but working on the common ground of toric degenerations has given a much needed impact on the project of relating all these various points of view. Further, some of the world leading researchers in the representation theory of current algebras were part of the workshop, and helped to investigate further on the relation between this representation theory, the PBW filtration and Macdonald polynomials.

These were just three of several discussed topics and initiated collaborations, we are looking forward to seeing the results emerging from this workshop in the near future.

Summarizing, the MFO mini-workshop *PBW Structures in Representation Theory* was very successful and succeeded all high expectations of the organisers.

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Mini-Workshop: PBW Structures in Representation Theory

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Abstracts

PBW filtration for quantum groups

Teodor Backhaus
(joint work with Xin Fang, Ghislain Fourier)

Let $\mathfrak{g}$ be a simple finite dimensional complex Lie algebra with a triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. Let $U(\mathfrak{n}^-)$ be the corresponding universal enveloping algebra. By setting the degree of each $x \in \mathfrak{n}^- \setminus \{0\}$ to 1 we obtain a $\mathbb{N}$-filtration on $U(\mathfrak{n}^-)$:

$$U(\mathfrak{n}^-)_s := \text{span}\{x_1 x_2 \dots x_l \mid x_i \in \mathfrak{n}^-,\ l \leq s\}.$$

The PBW theorem implies that the associated graded algebra gr $U(\mathfrak{n}^-)$ is isomorphic to the symmetric algebra $S(\mathfrak{n}^-)$. Denote by $V(\lambda)$, $\lambda \in P^+$ a dominant integral weight, the simple $\mathfrak{g}$-module of highest weight λ and let v_λ be a highest weight vector. Then the filtration above induces the PBW filtration on $V(\lambda)$ by $V(\lambda)_s := U(\mathfrak{n}^-)_s.v_\lambda$. We obtain that gr $V(\lambda)$ is a cyclic $S(\mathfrak{n}^-)$-module with cyclic generator v_λ. Denote by $I(\lambda)$ the kernel of the surjection $S(\mathfrak{n}^-) \to S(\mathfrak{n}^-)v_\lambda$, note that $I(\lambda)$ is not a monomial ideal in general.

In the last years monomial bases of gr $V(\lambda)$, for arbitrary $\lambda \in P^+$, have been provided in type $\mathtt{A_n}, \mathtt{C_n}, \mathtt{B_3}, \mathtt{D_4}, \mathtt{G_2}$, see [1] for type $\mathtt{A_n}$.

Let $U_q(\mathfrak{g})$ be the quantum group over $\mathbb{C}(q)$ associated with $\mathfrak{g}$ with triangular decomposition $U_q(\mathfrak{g}) \cong U_q(\mathfrak{n}^+) \otimes U_q^0 \otimes U_q(\mathfrak{n}^-)$. We shall use the quantum PBW bases of $U_q(\mathfrak{n}^-)$ provided by Lusztig [3] which is constructed as follows.

Let $\underline{w}_0 = s_{i_1} \dots s_{i_N}$ be a reduced decomposition of the longest Weyl group element. We associate a sequence of elements $F_{\beta_1}, \dots, F_{\beta_N} \in U_q(\mathfrak{n}^-)$, where $\{\beta_1, \dots, \beta_N\}$ is the set of positive roots and F_{β_i} is a quantum PBW root vector of weight $-\beta_i$. Then Lusztig has shown that ordered monomials in $F_{\beta_1}, \dots, F_{\beta_N}$ form a basis of $U_q(\mathfrak{n}^-)$.

The commutation relations of those vectors in $U_q(\mathfrak{n}^-)$ are given by the following Levendorskii-Soibelman (L-S for short) formula: for any $i < j$:

$$F_{\beta_j} F_{\beta_i} - q^{-(\beta_i, \beta_j)} F_{\beta_i} F_{\beta_j} = \sum_{n_{i+1}, \dots, n_{j-1} \geq 0} c(n_{i+1}, \dots, n_{j-1}) F_{\beta_{i+1}}^{n_{i+1}} \dots F_{\beta_{j-1}}^{n_{j-1}}.$$

These commutation relations depend heavily on the choice of reduced decomposition $\underline{w}_0$. For a given reduced decomposition $\underline{w}_0$, we seek for degree functions on the set of positive roots

$$\mathbf{d} : \Delta_+ \longrightarrow \mathbb{N}$$

such that letting $\deg(F_\beta) = \mathbf{d}(\beta)$ for $\beta \in \Delta_+$ defines a filtered algebra structure on $U_q(\mathfrak{n}^-)$ by

$$U_q(\mathfrak{n}^-)_s^{\mathbf{d}} := \text{span}\{F_{\beta_1}^{c_1} \cdots F_{\beta_N}^{c_N} \mid c_1 \mathbf{d}(\beta_1) + \cdots + c_N \mathbf{d}(\beta_N) \leq s\}$$

and the associated graded algebra is a skew-polynomial algebra. Inspired by the L-S formula, we define for any reduced decomposition $\underline{w}_0$ the *quantum degree cone*

$$\mathcal{D}^q_{\underline{w}_0} := \{(d_\beta) \in \mathbb{R}^N_+ \mid \forall\, i < j :\ d_{\beta_i} + d_{\beta_j} > \sum_{k=i+1}^{j-1} n_k d_{\beta_k} \text{ if } c(n_{i+1}, \dots, n_{j-1}) \neq 0\}.$$

Theorem A.
Let $\underline{w}_0$ be a reduced decomposition. Then

(1) the set $\mathcal{D}^q_{\underline{w}_0}$ is a non-empty, open polyhedral cone;
(2) a degree function $\mathbf{d} : \Delta^+ \longrightarrow \mathbb{N}$ satisfies $\mathrm{gr}^{\mathbf{d}}\, U_q(\mathfrak{n}^-) \cong S_q(\mathfrak{n}^-)$ if and only if $\mathbf{d} \in \mathcal{D}^q_{\underline{w}_0} \cap \mathbb{Z}^N$.
(3) for any simple finite dimensional Lie algebra $\mathfrak{g}$ of rank 3 or greater we have, here $R(w_0)$ is the set of all reduced decompositions of w_0:

$$\bigcap_{\underline{w}_0 \in R(w_0)} \mathcal{D}^q_{\underline{w}_0} = \emptyset.$$

We turn from the quantum situation to the classical one. Consider the *classical degree cone*:

$$\mathcal{D}_{\underline{w}_0} := \{(d_\beta) \in \mathbb{R}^N_+ \mid\ \forall\, \alpha, \beta, \gamma \in \Delta_+ : \alpha + \beta = \gamma \implies\ d_\alpha + d_\beta > d_\gamma\}.$$

It is immediate that $\mathbf{d} \in \mathcal{D} \cap \mathbb{Z}^N$ induces a $\mathbb{N}$-filtration on $U(\mathfrak{n}^-)$ such that $\mathrm{gr}^{\mathbf{d}} U(\mathfrak{n}^-) \cong S(\mathfrak{n}^-)$. As before we obtain an induced filtration on $V(\lambda)$ and $\mathrm{gr}^{\mathbf{d}} V(\lambda)$ is cyclic $S(\mathfrak{n}^-)$-module, where we denote the defining ideal by $I^{\mathbf{d}}(\lambda)$. We define

$$S_{\mathrm{gm}} := \{\mathbf{d} \in \mathcal{D} \cap \mathbb{Z}^N \mid I^{\mathbf{d}}(\lambda) \text{ is a monomial ideal } \forall\, \lambda \in P^+\},$$

the set of all *global monomial degrees*.

As the other main result, all monomial bases appearing in the context of PBW filtration in the literature can be actually obtained through a global monomial degree.

Theorem B.
Let $\mathfrak{g}$ be a simple Lie algebra of type $\mathtt{A_n}, \mathtt{C_n}, \mathtt{B_3}, \mathtt{D_4}, \mathtt{G_2}$. Then $S_{\mathrm{gm}} \neq \emptyset$.

We provide a global monomial degree in each case (for the $\mathtt{A_n}$-case this has been done already in [2]). Based on the evidence of several further examples, we conjecture:

Conjecture.

(1) $S_{\mathrm{gm}} \neq \emptyset$ for any simple finite-dimensional Lie algebra $\mathfrak{g}$.
(2) For any simply-laced simple Lie algebra $\mathfrak{g}$, $S_{\mathrm{gm}} \cap \mathcal{D}^q_{\underline{w}_0}$ is non-empty for some $\underline{w}_0 \in R(w_0)$.

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Birational sequences and Demazure characters

Lara Bossinger

Birational sequences are defined by Fang, Fourier and Littelmann in [1] to obtain toric degenerations of flag varieties using Newton–Okounkov bodies and essential monoids. The aim of this project is to give a class of examples with particularly nice properties that are well-adapted to computations.

For a complex semisimple simply connected algebraic group G we fix a Borel subgroup B and a maximal torus T. Let U^- denote the unipotent radical of the opposite Borel subgroup and N the number of positive roots for G. A sequence $S = (\beta_N, \cdots, \beta_1)$ of positive roots for G is called *birational*, if the multiplication map $U_{-\beta_N} \times \cdots \times U_{-\beta_1} \to U^-$ is birational. Here $U_{-\beta} := \{\exp(sf_\beta) \mid s \in \mathbb{C}\}$ where $f_\beta \in \mathfrak{g}_{-\beta} - \{0\}$ is a root vector in the Lie algebra corresponding to G of weight $-\beta$. A well-known example for a birational sequence is $S_{w_0} = (\alpha_{i_N}, \cdots, \alpha_{i_1})$ corresponding to $w_0 = s_{i_N} \cdots s_{i_1}$ a reduced expression of the longest element in the Weyl group.

For every positive root β one defines the Demazure operator D_β on the character lattice. For λ a dominant integral weight with $\langle \lambda, \check{\beta} \rangle \geq 0$ we have $D_\beta(e^\lambda) = e^\lambda + e^{\lambda-\beta} \cdots + e^{s_\beta(\lambda)}$. These operators give the characters of Demazure modules $V_w(\lambda)$, where w is an element of the Weyl group, as follows: $D_{\alpha_{i_k}} \cdots D_{\alpha_{i_1}}(e^\lambda) = \text{char}V_{s_{i_k}\cdots s_{i_1}}(\lambda)$ (see [2]). In particular, applying all Demazure operators corresponding to S_{w_0} to e^λ gives the character of the irreducible highest weight module $V(\lambda)$.

Starting with S_{w_0} as initial sequence, we explain the notion of subword sequences. These are sequences of positive roots of length N. They have nice properties and are conjectured to be birational. For example the classical Demazure character formula generalizes to subword sequences. For any $1 \leq k \leq N$ there exists a Weyl group element w such that $D_{\beta_k} \cdots D_{\beta_1}(e^\lambda) = w.\text{char}V_{s_{i_k}\cdots s_{i_1}}(\lambda)$. Here $S = (\beta_N, \cdots, \beta_1)$ is a subword sequence obtained from S_{w_0} as initial sequence. We call this equality the *Demazure porperty.* This result has an elegant proof using Weyl group combinatorics, furthermore the word w is explicitly determined. In particular, applying all Demazure operators corresponding to S to e^λ gives the character of $V(\lambda)$ as in the classical case. No twist is needed. One important Corollary is that subword sequences are dominant, i.e. the multiplication map used to define birational sequences is at least dominant. Further, there exists a generalization of subword sequences, called twisted Demazure sequences. It is conjectured that the set of twisted Demazure sequences is excatly the set of all sequences with Demazure property in type $\mathtt{A}_n$. It is known that twisted Demazure sequnces have the Demazure property in all types.

For type $\mathtt{A}_n$ I present an idea on how to prove birationality of subword sequences. We define three local moves on sequences of positive roots, called $\mathtt{A}_2$-*move*, *braid move* and *orthogonal move*, that induce birational maps between the corresponding products of root subgroups. They are expected to give a procedure of transforming subword sequences to known birational sequences, which is explained using an example in type $\mathtt{A}_3$.

Subword sequences are easy to compute and (when proven to be birational) give a large class of examples of birational sequences. For example, there are 3 subword sequnces corresponding to a fixed reduced expression of w_0 in type $\mathtt{A}_2$, 4 in type $\mathtt{B}_2$, 12 in type $\mathtt{A}_3$, 54 in type $\mathtt{A}_4$ and 60 for the Grassmannian $Gr(2,6)$.

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Smooth quiver Grassmannians of minimal dimension

Giovanni Cerulli Irelli

Let Q be a Dynkin quiver with n vertices and let M be a complex, finite–dimensional representation of Q. Given a dimension vector $\mathbf{e} \in \mathbb{Z}^n$, one defines the quiver Grassmannian $Gr_{\mathbf{e}}(M)$ set theorethically as the set of subrepresentations of M of dimension vector $\mathbf{e}$. The Euler characteristic of quiver Grassmannians plays an important role in the study of cluster algebras associated with Q. Namely, to M one attaches the Laurent polynomial

$$CC_M(x_1,\cdots,x_n) := \sum_{\mathbf{e}\in\mathbf{Z}^n} \chi(Gr_{\mathbf{e}}(M))\mathbf{x}^{B\mathbf{e}+\mathbf{g}_M} \in \mathbf{Z}[x_1^{\pm1},\cdots,x_n^{\pm1}].$$

In this formula, B is the adjacency matrix of Q, $\mathbf{g}_M \in \mathbf{Z}^n$ is the index of M, and χ is the Euler–Poincaré characteristic. The main result of [1] states that the set $\{CC_M|\, M$ indecomposable such that $\mathrm{Ext}^1(M,M)=0\} \cup \{x_1,\cdots,x_n\}$ is precisely the set of cluster variables of the coefficient–free cluster algebra associated with Q (the formula can be slightly modified in order to get coefficients and cluster variables of an arbitrary cluster algebra associated with Q). This important result was achieved by induction on the Auslander–Reiten quiver of Q. The heart of the proof was the following: let

$$0 \to \tau M \to E \to M \to 0$$

be an almost split sequence ending in the non–projective indecomposable M. Then

$$\chi(Gr_{\mathbf{e}}(E)) = \chi(Gr_{\mathbf{e}}(M\oplus\tau M)) \text{ for all } \mathbf{e} \neq \mathbf{dim}\, M \tag{1}$$

(For $\mathbf{e} = \mathbf{dim}\, M$, $Gr_{\mathbf{dim}\, M}(E)$ is empty and $Gr_{\mathbf{dim}\, M}(M \oplus \tau M)$ is a reduced point.) Formula (1) was obtained by Caldero and Chapoton by dividing the two quiver Grassmannians into pieces, and then comparing the Euler characteristic of the corresponding pieces on each projective variety. Notice that the pieces did not have much geometric structure.

In a series of articles [3], [4], [5], [6], [7], together with E. Feigin and M. Reineke we developed techniques to study the geometry of quiver Grassmannians associated with representations of Q. By using those techniques, it is quite easy to prove that both $Gr_{\mathbf{e}}(E)$ and $Gr_{\mathbf{e}}(M \oplus \tau M)$ are smooth and irreducible of the same dimension $\langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle$. Here $\langle \cdot, \cdot \rangle$ denotes the Euler form of Q and $\mathbf{d} = \mathbf{dim}\, E = \mathbf{dim}\,(M \oplus \tau M)$. I hence wondered if $Gr_{\mathbf{e}}(E)$ and $Gr_{\mathbf{e}}(M \oplus \tau M)$ share more than just their Euler characteristic. The following surprising result was proved in [2]:

Theorem 1. *The quiver Grassmannians $Gr_{\mathbf{e}}(E)$ and $Gr_{\mathbf{e}}(M \oplus \tau M)$ are diffeomorphic if $\mathbf{e} \neq \mathbf{dim}\, M$. In particular they are the same topological space and hence share the same topological invariants (e.g. Poincaré polynomials, Euler characteristic...).*

Notice that Theorem 1 implies (1) immediately. In principle, one could obtain the proof of theorem 1 by inspection, since the indecomposable Q–representations are well–known. Indeed, this was my first strategy. But then I realized that a much more general result holds true, which follows from well-known facts from differential/algebraic geometry (namely Ehresmann localization theorem and Bialynicky–Birula decomposition theorem). Let hence X be any Q–representation of dimension vector $\mathbf{d}$. It is known (see e.g. [3]) that any non–empty quiver Grassmannian $Gr_{\mathbf{e}}(X)$ associated with X has dimension at least $\langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle$. In case equality holds, we say that the quiver Grassmannian has minimal dimension.

Theorem 2. *Let X and Y be two Q–representations of the same dimension vector $\mathbf{d}$. Given a dimension vector $\mathbf{e}$, if the two quiver Grassmannians $Gr_{\mathbf{e}}(X)$ and $Gr_{\mathbf{e}}(Y)$ are smooth and of minimal dimension, then they are diffeomorphic. In particular, they share the same Poincaré polynomial and the same Euler characteristic.*

As a corollary of Theorem 2 we obtain Theorem 1. Moreover, by induction of the Auslander Reiten quiver of Q, we obatin the following result.

Theorem 3. *Let M be a Q–representation such that $Ext^1_Q(M, M) = 0$ (M is not necessarily indecomposable). Then a non–empty quiver Grassmannian $Gr_{\mathbf{e}}(M)$ has positive Euler characteristic. Moreover the odd homology groups $H_{odd}(Gr_{\mathbf{e}}(M))$ are zero and the even $H_{even}(Gr_{\mathbf{e}}(M))$ are torsion–free.*

The proof of Theorem 3 given in [2] is obtained by induction on the Auslander–Reiten quiver of Q. It can be extended without changes to preprojective and preinjective representations of acyclic quivers.

Conjecture 1. *Let M be as in Theorem 3. I conjecture that the cycle map $\varphi_i : A_i(Gr_{\mathbf{e}}(M)) \to H_{2i}(Gr_{\mathbf{e}}(M))$ is an isomorphism for all i.*

I could prove conjecture 1 if Q is of type A. The proof follows from the fact that in type A, the diffeomorphism of Theorem 1 is actually an isomorphism of projective varieties. This is not the case in type D and E. **Question:** For *any* representation M of a Dynkin quiver Q, is it true that any quiver Grassmannian associated with M admits a cellular decomposition?

A positive answer to this open problem solves Conjecture 1.

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Polytopes arising from mirror plabic graphs

Xin Fang

(joint work with Ghislain Fourier)

For $1 \leq k \leq n$, let $\mathrm{Gr}_{k,n}(\mathbb{K})$ be the Grassmannian variety over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Plabic graphs (plane bicolored graphs) are introduced by Postnikov [Pos] with the aim to parametrize cells in the totally non-negative (TNN) Grassmannians $(\mathrm{Gr}_{k,n}(\mathbb{R}))_{\geq 0}$. These graphs are drawn inside a disk with boundary vertices labelled by $1, 2, \ldots, n$ in a fixed orientation and internal vertices colored by black and white. For a reduced plabic graph $\mathcal{G}$ corresponding to the top cell in the TNN-Grassmannian $(\mathrm{Gr}_{n-k,n}(\mathbb{R}))_{\geq 0}$, Rietsch and Williams [RW] constructed a family of polytopes $\mathrm{NO}^r_{\mathcal{G}}$ for positive integers r as Newton-Okounkov bodies [KK, LM09] associated to the line bundle $r \in \mathbb{Z} \cong \mathrm{Pic}(\mathrm{Gr}_{n-k,n}(\mathbb{C}))$.

When the plabic graph $\mathcal{G} = \mathcal{G}_0$ is properly chosen, it is shown in [RW] that the corresponding Newton-Okounkov body $\mathrm{NO}^r_{\mathcal{G}_0}$ is unimodularly equivalent to the Gelfand-Tsetlin polytope $\mathrm{GT}(r\varpi_{n-k})$ of the finite dimensional irreducible representation $V(r\varpi_{n-k})$ of the Lie algebra $\mathfrak{sl}_n$.

The Newton-Okounkov body is by definition a convex hull of points; but to read off its defining inequalities is a hard problem. In [RW], the authors used mirror symmetry of Grassmannians to obtain these inequalities from the tropicalization of the super-potential on an open set of the mirror Grassmannian arising from the

Landau-Ginzburg model. By applying this symmetry, they give explicit defining inequalities of $\mathrm{NO}^r_{\mathcal{G}_0}$, which can be compared to those of Gelfand-Tsetlin polytopes.

Lattice points in Gelfand-Tsetlin polytopes parametrize bases of finite dimensional irreducible representations of simple classical Lie algebras. When the Lie algebra is $\mathfrak{sl}_n$, motivated by a conjecture of Vinberg, another family of polytopes $\mathrm{FFLV}(\lambda)$ is found by Feigin, Fourier and Littelmann [FeFoL11] whose lattice points parametrize a monomial basis of $V(\lambda)$, where λ is a dominant integral weight of $\mathfrak{sl}_n$.

For a plabic graph $\mathcal{G}$, its mirror $\mathcal{G}^\vee$ is defined by coloring all white (black) vertices in $\mathcal{G}$ black (white), and reverse the boundary orientation. When the plabic graph $\mathcal{G}$ corresponds to the top cell in $(\mathrm{Gr}_{n-k,n}(\mathbb{R}))_{\geq 0}$, $\mathcal{G}^\vee$ parametrizes the top cell in $(\mathrm{Gr}_{k,n}(\mathbb{R}))_{\geq 0}$. The following theorem answers a question in [FaFoL16]:

Theorem 4 (Fang-Fourier, 2016)**.** *The Newton-Okounkov body $NO^r_{\mathcal{G}_0^\vee}$ is unimodularly equivalent to FFLV($r\varpi_k$).*

Another conjectural way to relate Gelfand-Tsetlin polytopes to FFLV polytopes is via a connection between the corresponding clusters in different cluster algebras. Each reduced plabic graph $\mathcal{G}$ parametrizing the top cell of the TNN-Grassmannian $(\mathrm{Gr}_{n-k,n}(\mathbb{R}))_{\geq 0}$ gives a cluster $\mathcal{C}$ consisting of Plücker coordinates $\Delta_{I_1}, \ldots, \Delta_{I_m}$ where $I_1, \ldots, I_m$ are $(n-k)$-element subsets of $[n] = \{1, 2, \ldots, n\}$ and $\Delta_{I_r}, \ldots, \Delta_{I_m}$ are non-frozen variables in $\mathcal{C}$.

For $I \subset [n]$, let $\overline{I} = [n] \backslash I$ be its complement. Then the set

$$\mathcal{C}' = \{\Delta_{I_1}, \ldots, \Delta_{I_{r-1}}, \Delta_{\overline{I}_r}, \ldots, \Delta_{\overline{I}_m}\}$$

is a cluster for $\mathrm{Gr}_{k,n}(\mathbb{C})$, corresponding to a plabic graph $\overline{\mathcal{G}}$.

Conjecture 2. *The Newton-Okounkov body $NO^r_{\overline{\mathcal{G}}_0}$ is unimodularly equivalent to FFLV($r\varpi_k$).*

This conjecture is verified in the case of $\mathrm{Gr}_{3,6}(\mathbb{C})$.

Recently, in a joint work with G. Fourier and P. Littelmann [FaFoL15], we introduced the notion of birational sequences, giving birational charts by root subgroups of the maximal unipotent subgroup U^- of a reductive group G. Different Newton-Okounkov bodies can be associated to a birational sequence by choosing different total orders, yielding toric degenerations of the corresponding flag varieties. By choosing different birational sequences and total orders, we can not only recover the toric degenerations obtained in [GL, Cal, AB, FeFoL13], but provide a new family of toric degenerations given by Lusztig polytopes, parametrizing the canonical basis by its different leading PBW-term.

Question. Whether all Newton-Okounkov bodies arising from plabic graphs can be obtained via birational sequences of Grassmannians and proper total orders?

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Geometry and combinatorics of Kostka-Shoji polynomials

Michael Finkelberg

(joint work with Andrei Ionov)

1.1. **Kostka polynomials.** Schur functions $s_\lambda,\ \lambda \in \mathcal{P}$, form a $\mathbb{Z}$-basis of the symmetric functions ring Λ. Hall-Littlewood functions $P_\lambda,\ \lambda \in \mathcal{P}$, form a $\mathbb{Z}[t]$-basis of $\Lambda[t]$. We have $P_\lambda|_{t=0} = s_\lambda,\ P_\lambda|_{t=1} = m_\lambda$.

If the number of parts $\ell(\lambda)$ is at most N, then

$$P_\lambda(x_1, \dots, x_N, t) = \sum_{\sigma \in \mathfrak{S}_N / \operatorname{Stab}_\lambda} \sigma \left(x_1^{\lambda_1} \cdots x_N^{\lambda_N} \prod_{\lambda_i > \lambda_j} \frac{x_i - t x_j}{x_i - x_j} \right).$$

The transition matrix consists of Kostka polynomials: $s_\lambda = \sum_{\mu \le \lambda} K_{\lambda\mu}(t) P_\mu$ (dominance order). $K_{\lambda\mu}(t) \in \mathbb{N}[t]$ is a monic polynomial of degree $n(\mu) - n(\lambda)$ where $n(\lambda) = \sum (i-1)\lambda_i$.

1.2. **Theorem.** (G. Lusztig [5]) (1) $t^{-\dim \mathcal{O}_\mu/2} K_{\lambda\mu}(t^{-1}) = \sum_i \dim \mathrm{IC}(\overline{\mathcal{O}}_\lambda)^{-2i}_\mu t^{-i}$ where $\lambda, \mu \in \mathcal{P}(n),\ \mathcal{O}_\mu \subset \overline{\mathcal{O}}_\lambda$ nilpotent orbits of GL_n, and $\mathrm{IC}(\overline{\mathcal{O}}_\lambda) = j_{!*}\underline{\mathbb{C}}_{\mathcal{O}_\lambda}[\dim \mathcal{O}_\lambda]$. Note $\dim \mathcal{O}_\mu = n(n-1) - 2n(\mu)$.

(2) Assume the number of parts of λ, μ is at most N. Then $t^{-\dim \mathrm{Gr}^\mu/2} K_{\lambda\mu}(t^{-1}) = \sum_i \dim \mathrm{IC}(\overline{\mathrm{Gr}}^\lambda)^{-2i}_\mu t^{-i}$ where $\mathrm{Gr}^\mu \subset \overline{\mathrm{Gr}}^\lambda \subset \mathrm{Gr}_{GL_N}$ are $GL_N[[z]]$-orbits in the affine Grassmannian $GL_N((z))/GL_N[[z]]$.

1.3. **Coherent interpretation.** (R. Brylinski [2]) $\Gamma(T^*\mathcal{B}, \mathcal{O}(\mu)) = \Gamma(\mathcal{B}, \mathrm{Sym}^\bullet T\mathcal{B} \otimes \mathcal{O}(\mu)) = \bigoplus_{\lambda\geq\mu} \mathbf{K}^\bullet_{\lambda\mu} \otimes V_\lambda$. Here $\mathcal{B}$ is the flag variety of GL_N, $\mathcal{O}(\mu)$ an ample line bundle, and $\mathbf{K}^\bullet_{\lambda\mu}$ a graded vector space with Poincaré polynomial $K_{\lambda\mu}$.

It follows from (a) $R\Gamma^{>0}(T^*\mathcal{B}, \mathcal{O}(\mu)) = 0$ that follows in turn from the Frobenius splitting of $T^*\mathcal{B}$ [8]. (b) A comparison of the Atiyah-Bott-Lefschetz formula for $\chi(\mathcal{B}, \mathrm{Sym}^\bullet T\mathcal{B} \otimes \mathcal{O}(\mu))$ and a t-analogue of H. Weyl formula [6], [4]:

Let $\mathbb{N}^{N-1}$ be the cone of positive integral linear combinations of the simple roots of GL_N. For $\alpha = (a_1, \dots, a_{N-1})$ set $L_\alpha(t) = \sum p_i t^i$ where p_i is the number of decompositions of α into a sum of i positive roots in $R^+(GL_N)$. Then $K_{\lambda\mu}(t) = \sum_{\sigma\in\mathfrak{S}_N} (-1)^\sigma L_{\sigma(\lambda+\rho)-\rho-\mu}(t)$ Here $\rho = (N, N-1, \dots, 1)$, $\lambda' = \sigma(\lambda+\rho)$, $\mu' = \mu+\rho$, $a_1 = \lambda'_1 - \mu'_1$, $a_2 = \lambda'_1 + \lambda'_2 - \mu'_1 - \mu'_2, \dots$

1.4. **Higher analogues.** $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathcal{P}_r$ a multipartition. Schur functions $s_{\boldsymbol{\lambda}}(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(r)}) = s_{\lambda^{(1)}}(\mathbf{x}^{(1)}) \cdots s_{\lambda^{(r)}}(\mathbf{x}^{(r)}) \in \Lambda^{\otimes r}$ form a $\mathbb{Z}$-basis.

Hall-Littlewood-Shoji functions $P^\pm_{\boldsymbol{\lambda}}$ [9] form a $\mathbb{Q}(t)$-basis of $\Lambda^{\otimes r}(t)$.

The transition matrix consists of Kostka-Shoji functions: $s_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\mu}} K^\pm_{\boldsymbol{\lambda\mu}}(t) P^\pm_{\boldsymbol{\mu}}$. If $r = 2$ then $K^+_{\boldsymbol{\lambda\mu}} = K^-_{\boldsymbol{\lambda\mu}} =: K_{\boldsymbol{\lambda\mu}} \in \mathbb{N}[t]$ vanishes unless $\boldsymbol{\lambda} \geq \boldsymbol{\mu}$ in the sense $\lambda^{(1)}_1 \geq \mu^{(1)}_1,\ \lambda^{(1)}_1 + \lambda^{(2)}_1 \geq \mu^{(1)}_1 + \mu^{(2)}_1,\ \lambda^{(1)}_1 + \lambda^{(2)}_1 + \lambda^{(1)}_2 \geq \mu^{(1)}_1 + \mu^{(2)}_1 + \mu^{(1)}_2, \dots$

For arbitrary r, if the number of parts of all $\lambda^{(i)}, \mu^{(i)}$ is at most N, there are rN such differences, but the last one must vanish: $|\boldsymbol{\lambda}| = |\boldsymbol{\mu}|$ if $\boldsymbol{\lambda} \geq \boldsymbol{\mu}$. So if $\boldsymbol{\lambda} \geq \boldsymbol{\mu}$, then $\boldsymbol{\lambda} - \boldsymbol{\mu} \in \mathbb{N}^{rN-1}$.

We define a "positive root system" $R^+_r \subset \mathbb{N}^{rN-1}$ as the set of intervals of 1's $(0, \dots, 0, 1, \dots, 1, 0, \dots, 0)$ of length $l = 1 \pmod r$, and consider the corresponding Lusztig-Kostant partition function $L^{(r)}_\alpha(t) = \sum p_i t^i$ where p_i is the number of decompositions of $\alpha \in \mathbb{N}^{rN-1}$ into a sum of i positive "roots" in R^+_r.

1.5. **Conjecture.** $K^-_{\boldsymbol{\lambda\mu}}(t) = \sum_{\boldsymbol{\sigma}\in\mathfrak{S}^r_N} (-1)^{\boldsymbol{\sigma}} L^{(r)}_{\boldsymbol{\sigma}(\boldsymbol{\lambda}+\boldsymbol{\rho})-\boldsymbol{\rho}-\boldsymbol{\mu}}(t)$ where $\boldsymbol{\rho} = (\rho, \dots, \rho)$.

It was proved for $r = 2$ by T. Shoji [9] and checked by L. Yanushevich using P. Achar's computer program for $|\boldsymbol{\lambda}| = |\boldsymbol{\mu}| \leq 7$. It implies $K^-_{\boldsymbol{\lambda\mu}}(t) \in \mathbb{Z}[t]$.

1.6. **Coherent geometric interpretation.** We consider the following ordered base $e^{(1)}_1, \dots, e^{(r)}_1, e^{(1)}_2, \dots, e^{(r)}_2, \dots, e^{(1)}_N, \dots, e^{(r)}_N$ of $\mathbb{C}^{rN}$ giving rise to an embedding $GL^r_N \hookrightarrow GL_{rN}$ and an embedding of their upper triangular Borel subgroups $B^r_N \hookrightarrow B_{rN}$. In the strictly upper triangular subalgebra $\mathfrak{n}_{rN} \subset \mathfrak{gl}_{rN}$ consider a vector subspace spanned by the elementary matrices $e_{ij},\ j - i = 1 \pmod r$. This is a submodule of the adjoint representation of B^r_N, and it gives rise to a GL^r_N-equivariant vector bundle T^*_r on the flag variety $\mathcal{B}^r_N = GL^r_N / B^r_N$.

Note that $T^*_r \mathcal{B}^r_N$ is a particular case of Lusztig's iterated convolution diagram $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}}$ for the cyclic quiver $\widetilde{A}_{r-1}$ [7]

1.7. **Corollary of the Conjecture.** $\Gamma(T^*_r \mathcal{B}^r_N, \mathcal{O}(\boldsymbol{\mu})) = \Gamma(\mathcal{B}^r_N, \mathrm{Sym}^\bullet T_r \mathcal{B}^r_N \otimes \mathcal{O}(\boldsymbol{\mu})) = \bigoplus_{\boldsymbol{\lambda}\geq\boldsymbol{\mu}} \mathbf{K}^{-;\bullet}_{\boldsymbol{\lambda\mu}} \otimes V_{\boldsymbol{\lambda}}$ as a graded GL^r_N-module.

It follows from the Atiyah-Bott-Lefschetz formula for the equivariant Euler characteristic and the higher cohomology vanishing because of Frobenius splitting of $T^*_r \mathcal{B}^r_N$ (A. Ionov, a variation of [8]).

1.8. **Constructible geometric interpretation.** From now on $r = 2$.

Let $V = \mathbb{C}^n$. Then the orbits of the diagonal action of GL_n on $\mathcal{N}ilp_n \times V$ are naturally numbered by the bipartitions $\mathcal{P}_2(n)$, and the adjacency order on orbits corresponds to the above order on $\mathcal{P}_2(n)$ [1], [10]. Moreover, $\operatorname{codim}_{\mathcal{N}ilp_n \times V} \mathcal{O}_{\boldsymbol{\lambda}} = a(\boldsymbol{\lambda}) = 2n(\lambda^{(1)}) + 2n(\lambda^{(2)}) + |\lambda^{(2)}|$, and $t^{-\dim \mathcal{O}_{\mu}} K_{\boldsymbol{\lambda\mu}}(t^{-1}) = \sum_i \dim \mathrm{IC}(\overline{\mathcal{O}}_{\boldsymbol{\lambda}})^{-i}_{\boldsymbol{\mu}} t^{-i}$ [1]. Furthermore, $\deg K_{\boldsymbol{\lambda\mu}}(t) = a(\boldsymbol{\mu}) - a(\boldsymbol{\lambda})$, and all the powers of t in $K_{\boldsymbol{\lambda\mu}}$ are of the same parity.

1.9. **Corollary.** The orbits of the diagonal action of $GL_N[[z]]$ on $\mathrm{Gr}_{GL_N} \times (\mathbb{C}^N((z)) \setminus 0)$ are naturally numbered by generalized bipartitions $\mathcal{P}'_2 := \{(\lambda^{(1)}_1 \geq \ldots \geq \lambda^{(1)}_N,\ \lambda^{(2)}_1 \geq \ldots \geq \lambda^{(2)}_N)\} \subset \mathbb{Z}^{2N}$ with at most N parts. An orbit $\mathbb{O}_{\boldsymbol{\lambda}}$ contains a point $(L = z^{-\lambda^{(1)}_1 - \lambda^{(2)}_1}\mathbb{C}[[z]]e_1 \oplus \ldots \oplus z^{-\lambda^{(1)}_N - \lambda^{(2)}_N}\mathbb{C}[[z]]e_N,\ v = \sum z^{-\lambda^{(1)}_i} e_i)$.

$t^{-\dim \mathbb{O}_{\mu}} K_{\boldsymbol{\lambda\mu}}(t^{-1}) = \sum_i \dim \mathrm{IC}(\mathbb{O}_{\boldsymbol{\lambda}})^{-i}_{\boldsymbol{\mu}} t^{-i}$ (one can shift both $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ by $(M \geq \ldots \geq M,\ M \geq \ldots \geq M)$, $M \gg 0$, to make sure both $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are real bipartitions) [3].

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Functions on Newton–Okounkov bodies

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(joint work with Sébastien Boucksom, Victor Lozovanu, Catriona Maclean,Tomasz Szemberg)

For the duration of the talk let X be a complex projective variety of dimension n, D a Cartier divisor or line bundle on X, $Y_\bullet$ an admissible flag on X giving rise to a rank n valuation $\nu_{Y_\bullet}$ of $\mathbb{C}(X)$. No harm is done if one assumes that X is a smooth surface and D an ample line bundle on X.

The idea of Newton–Okounkov bodies is to associate a compact convex set $\Delta_{Y_\bullet}(D) \subseteq \mathbb{R}^n$, which has nonempty interior whenever D is big. The Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ can be regarded as a geometic version of $\Gamma_{Y_\bullet}(D)$, the valuation semigroup associated to D and $\nu_{Y_\bullet}$. Unlike the latter, Newton–Okounkov bodies enjoy good formal properties: the scale well, respect numerical equivalence of divisors, even descend continuously to the cone of big divisor classes in $N^1(X)_{\mathbb{R}}$.

On the other hand, the convex body $\Delta_{Y_\bullet}(D)$ depends a lot on the choice of $Y_\bullet$, in particular, there is no combinatorial invariance, and there is in general no way to reconstruct X from the collection of its Newton–Okounkov bodies. An example to keep in mind is the case of curves: if X is smooth of genus g, D an effective divisor of degree d, then $\Delta_{Y_\bullet}(D) = [0, d]$ independently of X.

Our quest is to seek invariants of $\Delta_{Y_\bullet}(D)$ that remain unchanged under the change of the flag $Y_\bullet$. It is classically known [4] that the volume of $\Delta_{Y_\bullet}(D)$ is such:

$$\mathrm{vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(D)) = \lim_{m\to\infty} \frac{h^0(\mathcal{O}_X(mD)}{m^n} = \mathrm{vol}_X(D) ,$$

where the manifestly $Y_\bullet$-independent right-hand side is called the volume of D. Note that the left-hand side is the integral of a constant function over $\Delta_{Y_\bullet}(D)$, hence it seems like a good idea to try to find interesting functions on Newton–Okounkov bodies that can potentially lead to new invariants via integration or otherwise.

The important principle coming from complex analysis (first formalized in this context by Boucksom–Chen [1]) is that multiplicative filtrations on the section ring $R(X, D) = H^0(X, \mathcal{O}_X(mD))$ give rise to concave functions on Newton–Okounkov bodies. An important example of such a filtration is the one induced by the order of vanishing of global sections of D along a smooth subvariety $Z \subseteq X$. Already the case of a point is very interesting. Complex geometry hosts various other sources of multiplicative filtrations: test configurations and certain families of multiplier ideals also give rise to such.

With notation as above, we write $\phi_Z\colon \Delta_{Y_\bullet}(D) \to \mathbb{R}$ for the function arising from the order of vanishing filtration on $R(X, D)$. These functions are extremely difficult to compute in general, it was Donaldson who worked them out for toric surfaces in the context of test configurations. The essential statement is that in the toric setting, the subgraph of ϕ_Z is a rational polytope, or, equivalently, ϕ is a piecewise affine linear function with rational coefficients with respect to a rational polyhedral decomposition of $\Delta_{Y_\bullet}(D)$. The following result is a generalisation of Donaldson's observation.

Theorem 5 (Küronya, Lozovanu, Maclean (unpublished)). *Let X be a smooth projective variety, $Y_\bullet$ an admissible flag on X, D a big divisor on X, $Z \subseteq X$ a smooth subvariety. Then the subgraph of the function ϕ_Z is the Newton–Okounkov body of a big divisor on an $n+1$-dimensional variety.*

For representation-theoretic varieties there is of course hope that one can determine the function ϕ_Z explicitly.

A few words about the formal properties of the functions ϕ_Z.

Theorem 6 ([2]). *With notation as above,*

(1) *if* $\Delta_{Y_\bullet}(D)$ *is a polytope, then* ϕ_Z *is continuous on the whole of* $\Delta_{Y_\bullet}(D)$, *otherwise there exist examples where* ϕ_Z *is not continous along the boundary;*
(2) ϕ_Z *is homogeneous of degree one;*
(3) *if* $D \equiv D'$, *then* $\phi_Z \colon \Delta_{Y_\bullet}(D) \to \mathbb{R}$ *equals* $\phi_Z \colon \Delta_{Y_\bullet}(D) \to \mathbb{R}$.

Finally, here is a result concerning invariants of Newton–Okounkov bodies in terms of the functions ϕ_Z.

Theorem 7. *With notation as above,*

(1) $I(D;Z) \stackrel{\text{def}}{=} \frac{1}{\mathrm{vol}_X(D)} \int_{\Delta_{Y_\bullet}(D)} \phi_Z$ *is independent of* $Y_\bullet$ *(see* [1]*);*
(2) $\max_{\Delta_{Y_\bullet}(D)}$ *is independent of* $Y_\bullet$ *(see* [3]*).*

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Toric degenerations of flag varieties: from algebras with straightening laws to Newton-Okounkov bodies and cluster varieties

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(joint work with Xin Fang, Ghislain Fourier)

One of the beautiful and astonishing properties of the theory of toric varieties is the powerful dictionary translating algebraic geometry properties into combinatorial properties in terms of lattices, cones and polytopes, and vice versa. It is hence tempting to extend this powerful machinery to a larger class of varieties by using flat degenerations of the given variety into a toric variety. The aim of the series of talks was to give an overview starting with ideas of Hodge in the 1940's up to recent developments connecting methods from the theory of Newton-Okounkov bodies with ideas coming from the theory of cluster varieties.

Algebras with straightening laws. It seems that the first flat degeneration of a Graßmann variety into a union of toric varieties has been constructed by Hodge [15]. The ideas of Hodge have been generalized by De Concini, Eisenbud and Procesi in the framework of Hodge algebras [6] (also called algebras with straightening laws). For generalized flag varieties, similar results have been obtained by Chirivì [5] via upgrading the Hodge algebra to Lakshmibai-Seshadri algebras.

The precise description of an algebra with straightening laws is somewhat technical, we try to explain what happens in the special case of Graßmann varieties. Let $I_{d,n}$ be the set of all strictly increasing sequences of length d between 1 and n. Let "$\geq$" be the usual partial order on $I_{d,n}$: $\underline{\mathbf{i}} \geq \underline{\mathbf{j}} \Leftrightarrow i_t \geq j_t$ for all $t = 1, \ldots, d$. A monomial $\mathbf{p}_{\underline{\mathbf{i}}^1} \cdots \mathbf{p}_{\underline{\mathbf{i}}^r}$ of Plücker coordinates is called standard in $\mathbb{C}[\Lambda^d \mathbb{C}^n]$ iff $\underline{\mathbf{i}}^1 \geq \ldots \geq \underline{\mathbf{i}}^r$. The second fundamental theorem in invariant theory describes the relations among the Plücker coordinates $\mathbf{p}_{\underline{\mathbf{i}}}$ considered as functions on the affine cone over the Graßmann variety $\widehat{Gr}_d(\mathbb{C}^n)$. Refine the partial order "$\geq$" to a total order "$\succ$" and denote with the same symbol the induced lexicographic order on the polynomial ring $\mathbb{C}[\Lambda^d \mathbb{C}^n]$, which is a monomial order. One can show: if $\underline{\mathbf{i}}, \underline{\mathbf{j}}$ are not comparable, then

$$(1) \qquad \mathbf{p}_{\underline{\mathbf{i}}}\mathbf{p}_{\underline{\mathbf{j}}} = \sum_{m_k \succ \mathbf{p}_{\underline{\mathbf{i}}}\mathbf{p}_{\underline{\mathbf{j}}}} a_k m_k \mod I(Gr_d(\mathbb{C}^n)),$$

where the m_k are standard monomials of degree two. Such an algorithm expressing non standard monomials as a linear combination of standard monomials is called a *straightening law*. An algebra with such properties: a basis consisting of a special class of monomials, the *standard monomials*, together with relations expressing non standard monomials as a linear combination of (larger) standard monomials, this is roughly what is called an *algebra with straightening laws*. There exists a flat degeneration of this algebra into a *discrete algebra*, i.e. the relation (1) is turned into the relation $\mathbf{p}_{\underline{\mathbf{i}}}\mathbf{p}_{\underline{\mathbf{j}}} = 0$ whenever $\underline{\mathbf{i}}, \underline{\mathbf{j}}$ are not comparable. The corresponding variety is hence a union of toric varieties, and many geometric properties of the original variety (Graßmann variety, Schubert variety) can be read off the combinatorics of the partially ordered set $I_{d,n}$.

These ideas have been generalized by Chirivì [5]. In his setting the Graßmann variety can be replaced by an arbitrary (partial) flag variety or a Schubert variety therein, the partially ordered set $I_{d,n}$ is replaced by a quotient of the Weyl group with the induced Bruhat order, and the standard monomial theory of Hodge is replaced by the standard monomial theory for flag varieties, a program which was initiated by Seshadri and his coauthors, see [20, 21] and completed in [22].

Caldero's approach. A flat toric degeneration with an irreducible special fiber has been given by Gonciulea and Lakshmibai [12] in the case SL_n/B, where B is a Borel subgroup, using standard monomial theory. This has been interpreted geometrically by Kogan and Miller [19] using geometric invariant theory.

A uniform construction of flat toric degenerations (with an irreducible special fiber) for arbitrary semisimple algebraic groups G has been given by Caldero [4]. Fix a maximal unipotent subgroup U of G. For every choice of a reduced decomposition $\underline{w_0}$ of the longest word w_0 in the Weyl group of G, Caldero constructs a flat toric degeneration of the affine variety $G/\!\!/U$.

Let N be the length of $\underline{w_0}$ and denote by Λ the weight lattice. The coordinate ring of $G/\!\!/U$ has as a basis the so called dual canonical basis $\mathbb{B}^*$. For a fixed reduced decomposition $\underline{w_0}$, this basis can be naturally indexed by a semigroup $\Gamma_{\underline{w_0}} \subset \Lambda \times \mathbf{N}^N$, which by [3, 23] is the semigroup of integral points in a polyhedral cone. Caldero's construction relies on the following multiplicative property of the dual canonical basis (compare this also with (1)):

$$b_{\lambda,\mathbf{m}}b_{\mu,\mathbf{n}} = b_{\lambda+\mu,\mathbf{m}+\mathbf{n}} + \sum_{\mathbf{k}>\mathbf{m}+\mathbf{n}} c^{\mathbf{k}}_{(\lambda,\mathbf{m}),(\mu,\mathbf{n})} b_{\lambda+\mu,\mathbf{k}}, \tag{2}$$

where "$\leq$" denotes the lexicographic ordering on $\mathbb{N}^N$. So the multiplication rule for basis elements can be described as: up to elements which are larger with respect to the lexicographic ordering, the product of basis elements is the same as in the monoid $\Gamma_{\underline{w}_0}$, which is the index system of $\mathbb{B}^*$.

From this, Caldero deduces the existence of an increasing filtration of $\mathbb{C}[G/\!\!/U]$ by $T \times T$-submodules, such that the associated graded algebra, $gr\,\mathbb{C}[G/\!\!/U]$, is isomorphic to the algebra of the monoid $\mathbb{C}[\Gamma_{\underline{w}_0}]$. In geometric terms:

Theorem. [4] *The affine variety $G/\!\!/U$ admits a flat degeneration to a normal affine toric variety $X_0 = \operatorname{Spec}\mathbb{C}[\Gamma_{\underline{w}_0}]$ for the torus $T \times \mathbb{T}$, where we put $\mathbb{T} := (\mathbb{C}^*)^N$. Further, the degeneration is compatible with the actions of $T \times T$ on $G/\!\!/U$ (regarding $T \times T$ as a subgroup of $G \times T$), and on X_0 via the homomorphism of tori $T \times T \to T \times \mathbb{T}$, $(t, t') \mapsto (t^{-1}t', \alpha_{i_1}(t), ..., \alpha_{i_N}(t))$.*

These results have been generalized to spherical varieties by Alexeev and Brion [1], see also the papers of Kaveh and Kiritchenko [17, 18] for another approach via the framework of Newton-Okounkov bodies [16].

Birational sequences and Newton-Okounkov bodies. Let G be a complex semisimple algebraic group. Fix a Borel subgroup $B = TU$, where T is a maximal torus und U is the unipotent radical of B. Denote by U^- the unipotent radical of the opposite Borel subgroup. Fix a sequence of positive roots $S = (\beta_1, \ldots, \beta_N)$. We make no special assumption on this sequence, for example there may be repetitions. Let $\mathbb{T}$ be the torus $(\mathbb{C}^*)^N$, we write $\mathtt{t} = (\mathtt{t}_1, \ldots, \mathtt{t}_N)$ for an element of $\mathbb{T}$. The variety Z_S is the affine space $\mathbb{A}^N$ endowed with the following $T \times \mathbb{T}$-action:

$$\forall (t, \mathtt{t}) \in T \times \mathbb{T} : (t, \mathtt{t}) \cdot (z_1, \ldots, z_N) := (\mathtt{t}_1\beta_1(t)^{-1}z_1, \ldots, \mathtt{t}_N\beta_N(t)^{-1}z_N).$$

We call S a *birational sequence* (see [7]) for U^- if the product map π is birational:

$$\pi : Z_S \to U^-, \quad (z_1, \ldots, z_N) \mapsto \exp(z_1 f_{\beta_1}) \cdots \exp(z_N f_{\beta_N}). \tag{3}$$

For each pair $(S, >)$ consisting of a birational sequence and a monomial order on $\mathbb{C}[Z_S] = \mathbb{C}[x_1, \ldots, x_N]$, we attach to $G/\!\!/U$ a monoid $\Gamma = \Gamma(S, >) \subset \Lambda \times \mathbf{N}^N$. As a side effect we get a vector space basis $\mathbb{B}_\Gamma$ of $\mathbb{C}[G/\!\!/U]$, the elements being indexed by Γ. The basis $\mathbb{B}_\Gamma$ has a multiplication rule very similar to that in (2):

$$\xi_{\lambda,\mathbf{p}}\xi_{\mu,\mathbf{q}} = \xi_{\lambda+\mu,\mathbf{p}+\mathbf{q}} + \sum_{\mathbf{r}>\mathbf{p}+\mathbf{q}} c^{\lambda+\mu,\mathbf{r}}_{\lambda,\mathbf{p};\mu,\mathbf{q}} \xi_{\lambda+\mu,\mathbf{r}}. \tag{4}$$

This makes it possible to transfer the methods of Caldero [4], Alexeev and Brion [1] to this more general setting, once one knows that the monoid Γ is finitely generated and saturated.

Theorem. [7] *If the monoid Γ is finitely generated and saturated, then the affine variety $G/\!\!/U$ admits a flat degeneration to a normal affine toric variety $X_0 =$*

Spec $\mathbb{C}[\Gamma]$ *for the torus* $T\times\mathbb{T}$*, where we put* $\mathbb{T}:=(\mathbb{C}^*)^N$*. Further, the degeneration is compatible with the actions of* $T\times T$ *on* $G/\!\!/U$ *(regarding* $T\times T$ *as a subgroup of* $G\times T$*), and on* X_0 *via the homomorphism of tori* $T\times T\to T\times\mathbb{T}$*,* $(t,t')\mapsto (t^{-1}t',\beta_1(t),...,\beta_N(t))$*.*

The construction can be extended to spherical varieties as in [1] and in [17]. There are quite a few examples where it is known, that Γ is finitely generated and saturated [7]. In fact, we conjecture that Γ is always finitely generated:

The reduced decomposition case I: Let $\{\alpha_1,\dots,\alpha_n\}$ be the set of simple roots, fix a reduced decomposition $\underline{w}_0=s_{i_1}\cdots s_{i_N}$ and set $S=(\alpha_{i_1},\dots,\alpha_{i_N})$. Using the associated Bott-Samelson desingularization, one easily shows that S is a birational sequence. In fact, one recovers the case considered by Caldero, Alexeev and Brion, and Kaveh. One has $\Gamma=\Gamma_{\underline{w}_0}$

The reduced decomposition case II: Fix a reduced decomposition $\underline{w}_0=s_{i_1}\cdots s_{i_N}$ as above, and let $S=(\beta_1,\dots,\beta_N)$ be an enumeration of the positive roots associated to the decomposition, i.e., $\beta_k=s_{i_1}\cdots s_{i_{k-1}}(\alpha_k)$ for $k=1,\dots,N$, then S is a birational sequence. One can show that Γ corresponds in this case to the Lusztig parameterization of the canonical basis. In particular, Γ is the monoid of integral points of a polyhedral cone.

Some other PBW-type cases: Fix an enumeration $\Phi^+=\{\beta_1,\dots,\beta_N\}$ of the positive roots such that $\beta_i>\beta_j$ in the usual partial order on roots implies $i<j$. Note that such an enumeration can not come from a reduced decomposition $\underline{w}_0$. Set $S=(\beta_1,\dots,\beta_N)$, then map π in (3) above induces an isomorphism of affine varieties for any chosen enumeration. Motivated by a conjecture of Vinberg, a new class of toric degenerations has been constructed in [8] for $G=SL_n$ and Sp_{2n} (see also [13, 14]), using monomial bases obtained through refining the PBW-filtration on the corresponding universal enveloping algebras [9, 10]. It turns out that the notation of a birational sequence puts the results in a unifying framework. In particular, in the cases $G=SL_n$ and Sp_{2n}, the corresponding monoid Γ is always the monoid of integral points of a polyhedral cone. □

The monoid Γ depends on the choice of the birational sequence S and the choice of a monomial order on $\mathbb{C}[Z_S]$, which suggests a connection with Newton-Okounkov bodies. Indeed, the birational map in (3) induces an isomorphism between the function fields $\mathbb{C}(Z_S)$ and $\mathbb{C}(G/B)$, so one can use the fixed monomial order to define an induced $\mathbb{Z}^N$-valued valuation ν on $\mathbb{C}(G/B)$. With the construction of Γ one gets naturally for every dominant weight λ a subset $es(\lambda)\subset\Gamma$ such that the union $\Gamma(\lambda)=\bigcup_{n\in\mathbb{N}}es(n\lambda)$ is disjoint and inherits the structure of a monoid. Let $Q_\lambda\supseteq B$ be the parabolic subgroup associated to the dominant weight λ. One can show: the Newton-Okounkov body $\Delta_\nu(\lambda)$ associated to the valuation ν and the ample line bundle $\mathcal{L}_\lambda$ on G/Q_λ is the convex closure

$$\Delta_\nu(\lambda)=\operatorname{conv}(\overline{\bigcup_{n\in\mathbb{N}}\{\frac{\mathbf{m}}{n}\mid \mathbf{m}\in es(n\lambda)\}}). \tag{5}$$

Using the same strategy as in [2] or [1] one can show:

Theorem. [7] *If the monoid* Γ *is finitely generated and saturated, then there exists a family of* T*-varieties* $\pi : \mathcal{Y} \to \mathbb{A}^1$*, where* $\mathcal{Y}$ *is a normal variety, such that* π *is projective and flat,* π *is trivial with fiber* G/Q_λ *over the complement of* 0 *in* $\mathbb{A}^1$*, the fiber of* π *at* 0 *is a toric variety* Y_0 *isomorphic to* $\operatorname{Proj}\mathbb{C}[\Gamma(\lambda)]$*, and the moment polytope associated to* Y_0 *is* $\Delta_\nu(\lambda)$*.*

Toric degenerations, Cluster duality and mirror symmetry. An approach towards toric degenerations via cluster varieties has been suggested by Gross, Hacking, Keel and Kontsevich [11]. The authors believe that all the elementary constructions of toric geometry extend to a larger class of varieties, and they prove many such results in the cluster case. To go into all the technical details would blow up the framework of this overview. We restrict ourself here to some special aspects discovered by Rietsch and Williams [24].

The authors do not discuss degenerations, but this is implicitly contained in their approach by the results of Anderson [2]. The most beautiful part of their result is to see how cluster algebras and toric charts can help to understand the connection between two very different ways of looking at polytopes. Rietsch and Williams investigate the case of $X = Gr_{n-k}(\mathbb{C}^n)$, the Graßmann variety of $(n-k)$-dimensional subspaces of $\mathbb{C}^n$. They consider coordinate charts on X and the "mirror dual" Graßmann variety $\check{X} := Gr_k\left((\mathbb{C}^n)^*\right)$, respectively. The charts on both sides are obtained from a choice of combinatorial objects, a reduced plabic graph $\mathcal{G}$. The coordinate system on X is given by an injective map

$$\Phi_{\mathcal{G}} : (\mathbb{C}^*)^{\mathcal{P}_{\mathcal{G}}} \to Gr_{n-k}(\mathbb{C}^n) \tag{6}$$

constructed in [26]. Here $\mathcal{P}_{\mathcal{G}}$ is an index set for a certain set of Plücker coordinates read off from the graph $\mathcal{G}$ by a combinatorial rule. The image of the restriction of $\Phi_{\mathcal{G}}$ to $(\mathbb{R}_{>0})^{\mathcal{P}_{\mathcal{G}}}$ is the totally positive Graßmann variety in its Plücker embedding and thus Rietsch and Williams refer to it as a *positive chart.*

To each positive chart $\Phi_{\mathcal{G}}$ and $r > 0$, the authors associate a Newton-Okounkov polytope $NO^r(\mathcal{G})$. They define $NO^r(\mathcal{G})$ as the convex hull of certain integral points which correspond to the vanishing behavior of sections of the line bundle $\mathcal{L}^r$, similar as in (5). Here $\mathcal{L}$ is the ample generator of $\operatorname{Pic} X$.

It is known [25] that the homogeneous coordinate ring of (the affine cone over) $\check{X}$ admits a cluster algebra structure. Each cluster $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ of this cluster algebra gives rise to a toric chart on $\check{X}$

$$\check{X}_{\mathbf{x}} := \{y \in \check{X} \mid x_i(y) \neq 0,\ 1 \leq i \leq m\}. \tag{7}$$

The index set $\mathcal{P}_{\mathcal{G}}$ labels a collection of Plücker coordinates that form an entire cluster (recall that in general there will be many clusters where the variables do not consist entirely of Plücker coordinates). Therefore, from (7), we get a map

$$\Phi_{\mathcal{G}}^{\vee} : (\mathbb{C}^*)^{\mathcal{P}_{\mathcal{G}}} \to \check{X} \tag{8}$$

called *cluster chart* which satisfies $p_\nu(\Phi_{\mathcal{G}}{}^{\vee}((t_\mu)_\mu)) = t_\nu$ for $\nu \in \mathcal{P}_{\mathcal{G}}$ and p_ν the associated Plücker coordinate.

In mirror symmetry the *mirror* of the Graßmann variety X is a Landau-Ginzburg model, which can be described as the pair $(\check{X}^o, W_q)$, where $\check{X}^o$ is the complement of a particular anticanonical divisor in the Langlands dual Graßmann variety $\check{X}$, and W_q is a regular map on $\check{X}^o$, called superpotential. The condition of the tropicalized version of the superpotential $W_{t^r} \circ \Phi_{\mathcal{G}}{}^\vee$ (i.e. the superpotential written in a cluster expansion in terms of the cluster consisting of Plücker coordinates on $\check{X}$ labeled by $\mathcal{P}_{\mathcal{G}}$ and replacing the q-variable by t^r) to have non-negative value gives rise to a set of linear inequalities defining a polytope $Q^r_{\mathcal{G}}$.

This is the important difference between the two constructions of the polytopes. In the second procedure the description of the polytope is given as the intersection of half spaces while in the first procedure the description of the polytope is given by taking a convex hull of a set of integral points. The main result in [24] is:

Theorem. [24] *The two polytopes $NO^r(\mathcal{G})$ and $Q^r_{\mathcal{G}}$ coincide for all reduced plabic graphs $\mathcal{G}$ with trip permutation $\pi_{k,n}$ and all $r > 0$.*

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Generalized Weyl modules

Ievgen Makedonskyi
(joint work with Evgeny Feigin)

The classical local Weyl modules for a simple Lie algebra are labeled by dominant weights. We generalize the definition to the case of arbitrary weights and study the properties of the generalized modules. We prove that the representation theory of the generalized Weyl modules can be described in terms of the alcove paths and the quantum Bruhat graph. We make use of the Orr-Shimozono formula in order to prove that the $t = \infty$ specializations of the nonsymmetric Macdonald polynomials are equal to the characters of the generalized Weyl modules corresponding to the antidominant weights. We also reprove the result of Chari and Ion that the character of the classical Weyl module coincides the $t = 0$ specialization of the nonsymmetric Macdonald polynomial.

We prove that there exists a decomposition of the generalized Weyl module to subfactors of the same type corresponding to smaller weight. Using this procedure we obtain a basis of a generalized Weyl module. This basis can be described in terms of alcove paths. Also we prove that generalized Weyl module have some fusion-product description.

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Specializations of Macdonald polynomials and the PBW filtration

Daniel Orr
(joint work with Mark Shimozono)

For an arbitrary affine root system, we consider the nonsymmetric Macdonald polynomials $E_\lambda(X;q,t)$ with the parameter t specialized to infinity. In [8], we prove a combinatorial formula for these polynomials involving alcove walks and the quantum Bruhat graph (equivalently, Lusztig's periodic order on alcoves [6]). We arrive at this formula via a careful analysis of the Ram-Yip formula for $E_\lambda(X;q,t)$ of [9] under the specialization $t=\infty$. As our formula is manifestly positive, it follows that the $E_\lambda(X;q^{-1},\infty)$ have coefficients in $\mathbb{Z}_{\geq 0}[q]$, settling a conjecture of [2] in the affirmative.

In this talk, we discuss the work [8] as well as several representation-theoretic interpretations of the positivity. The different representation-theoretic interpretations hold at various levels of generality, depending mainly on the affine root system and the defining weight λ, and they involve:

(1) the PBW filtration on level-one affine Demazure modules [1, 2, 3, 4]
(2) a new degree function on quantum Lakshmibai-Seshadri paths [7]
(3) generalized Weyl modules [5]

By a representation-theoretic interpretation, we mean that $E_\lambda(X;q^{-1},\infty)$ should be realized as the graded character of some graded module for the underlying finite-dimensional simple Lie algebra (or of related objects). Many interesting questions remain. For instance, do there exist direct connections between (1-3) above? And, are there geometric interpretations of the positivity of $E_\lambda(X;q^{-1},\infty)$?

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Quiver Grassmannians

Markus Reineke

1. Introduction

The objective of using quiver Grassmannians in the context of the mini workshop is to model degenerate versions of flag varieties in terms of Grassmannians of subrepresentations of representations of Dynkin quivers.

The main advantage of this point of view is that it yields an approach – in terms of the representation theory of quivers – to the following properties of degenererations of flag varieties (see [1, 2, 3, 4, 5, 7, 8]):

- local properties: irreducibility or parametrization of irreducible components; smoothness, normality and the determination of the singular locus; the property of being locally a complete intersection; the construction of desingularizations
- global properties: construction of cell decompositions, computation of Betti numbers in cohomology, counting rational points over finite fields
- dynamic properties: construction of group actions on degenerations of flag varieties, construction of flat families of degenerations.

However, there are also limitations of this quiver approach: first, it is restricted to degenerations of SL_n-flag varieties (see, however, [7]). Second, it is inherent to the quiver approach that no toric degenerations can be constructed in general.

2. Quiver representations and quiver Grassmannians

The most general definition of quiver Grassmannians is the following:

Definition. Let A be a finite dimensional associative $\mathbf{C}$-algebra, let M be a finite-dimensional left A-modules, and let β be a class in the Grothendieck group $K_0(A)$. Then

$$\mathrm{Gr}_\beta(M) = \{U \subset M \, : U \text{ is a subrepresentation in class } \beta\}$$

is naturally endowed with a structure of a projective variety, since it is closed in the ordinary Grassmannian of linear subspaces of M (viewed as a complex vector space).

More specifically, let Q be a finite quiver with set of vertices Q_0 and arrows written $\alpha : i \to j$. On the free abelian group $\mathbf{Z}Q_0$, we have the Euler form $\langle \_, \_ \rangle_Q$ given by

$$\langle \mathbf{d}, \mathbf{e} \rangle_Q = \sum_{i \in Q_0} d_i e_i - \sum_{\alpha : i \to j} d_i e_j$$

for $\mathbf{d} = (d_i)_{i \in Q_0}, \mathbf{e} = (e_i)_{i \in Q_0} \in \mathbf{Z}Q_0$.

A (complex) representation V of Q consists of $\mathbf{C}$-vector spaces V_i for $i \in Q_0$ and $\mathbf{C}$-linear maps $(f_\alpha : V_i \to V_j)_{\alpha : i \to j}$. A morphism between two such representations $V = ((V_i)_i, (f_\alpha)_\alpha)$ and $W = ((W_i)_i, (g_\alpha)_\alpha)$ consists of $\mathbf{C}$-linear maps $\varphi_i : V_i \to W_i$ for $i \in Q_0$, such that $\varphi_j f_\alpha = g_\alpha \varphi_i$ for all $\alpha : i \to j$.

The resulting category $\mathrm{rep}_{\mathbf{C}} Q$ of $\mathbf{C}$-representations of Q is equivalent to the category of left modules over the so-called path algebra $\mathbf{C}Q$ of Q, thus it is an abelian $\mathbf{C}$-linear category.

Since $\mathbf{C}Q$ is a hereditary algebra, we have $\mathrm{Ext}^{\geq 2}(\_, \_) = 0$ in $\mathrm{rep}_{\mathbf{C}} Q$, and

$$\dim \mathrm{Hom}(V, W) - \dim \mathrm{Ext}^1(V, W) = \langle \mathbf{dim} V, \mathbf{dim} W \rangle_Q,$$

where $\mathbf{dim} V = (\dim V_i)_i \in \mathbf{N}Q_0$.

The general definition of quiver Grassmannians above can now be reformulated for the algebra $A = \mathbf{C}Q$ as follows:

Definition. Fixing a representation $V = ((V_i)_i, (f_\alpha)_\alpha)$ of Q and a dimension type $\mathbf{e} \in \mathbf{N}Q_0$, we define

$$\mathrm{Gr}_{\mathbf{e}}(V) = \{(U_i \subset V_i)_{i \in Q_0},\ f_\alpha(U_i) \subset U_j\} \subset \prod_{i \in Q_0} \mathrm{Gr}_{e_i}(V_i).$$

In this generality, quiver Grassmannians are no specific object of study, since the following holds [9]:

Theorem. *Every projective variety X is isomorphic to a quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(V)$, for a quiver with three vertices, a representation V such that $\mathrm{End}(V) \simeq \mathbf{C}$, and dimension type $\mathbf{e} = (1, 1, 1)$.*

Therefore, we restrict in the following to Q being of Dynkin type, that is, the unoriented graph underlying Q is a disjoint union of Dynkin diagrams of type A_n, D_n, E_6, E_7 or E_8. By Gabriel's theorem, it is then known that there are only finitely many isomorphism classes of indecomposable representations, parametrized by the positive roots of the corresponding root system via the map $\mathbf{dim}$. Consequently, there are only finitely many isomorphism classes of representations of fixed dimension type.

This result admits a geometric interpretation: for a dimension type $\mathbf{d} \in \mathbf{N}Q_0$, consider the affine space $R_{\mathbf{d}} = \bigoplus_{\alpha : i \to j} \mathrm{Hom}(\mathbf{C}^{d_i}, \mathbf{C}^{d_j})$, on which the group $G_{\mathbf{d}} = \prod_{i \in Q_0} \mathrm{GL}_{d_i}(\mathbf{C})$ acts on $R_{\mathbf{d}}$ via base change

$$(g_i)_{i \in Q_0} \cdot (f_\alpha)_\alpha = (g_j f_\alpha g_i^{-1})_{\alpha : i \to j}.$$

By definition, the $G_{\mathbf{d}}$-orbits in $R_{\mathbf{d}}$ correspond naturally to the isomorphism classes of representations of Q of dimension type $\mathbf{d}$. Thus, in case Q is Dynkin, $G_{\mathbf{d}}$ acts on $R_{\mathbf{d}}$ with finitely many orbits for any $\mathbf{d} \in \mathbf{N}Q_0$.

3. Some sample results

To give an idea of the type of results which can be obtained on quiver Grassmannians for Dynkin quivers, and to indicate the interplay of representation theoretic and geometric notions, we list some results from [2, 8]:

Theorem. *Let Q be a Dynkin quiver, V a representation of Q, and $\mathbf{e} \in \mathbf{N}Q_0$ as before.*

(1) *The quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(V)$ is nonempty if and only if, for all indecomposable representations U, we have $\dim \mathrm{Hom}(U, M) \geq \langle \mathbf{dim}U, \mathbf{e} \rangle_Q$.*

(2) *If $\mathrm{Ext}^1(V, V) = 0$, then $\mathrm{Gr}_{\mathbf{e}}(V)$ is smooth of dimension $\langle \mathbf{e}, \mathbf{dim}V - \mathbf{e} \rangle_Q$.*

(3) *The following properties are equivalent:*
   (a) *$\dim \mathrm{Hom}(U, V) \leq \langle \mathbf{dim}U, \mathbf{dim}V - \mathbf{e} \rangle_Q$ for all indecomposable representations U,*
   (b) *There exists a short exact sequence $0 \to P \to V \to I \to 0$, where P is projective of dimension type $\mathbf{e}$ and I is injective.*

   In this case, $\mathrm{Gr}_{\mathbf{e}}(V)$ is an irreducible normal (typically singular) rational locally complete intersection variety of dimension $\langle \mathbf{e}, \mathbf{dim}V - \mathbf{e} \rangle_Q$.

(4) *$\mathrm{Gr}_{\mathbf{e}}(V)$ always carries an action of the algebraic group $\mathrm{Aut}(M)$.*

(5) *$\mathrm{Gr}_{\mathbf{e}}(V)$ is always polynomial count, that is, there exists a scheme $\mathcal{G}\nabla_{\mathbf{e}}(V)$ over $\mathrm{Spec}(\mathbf{Z})$ whose base change to $\mathrm{Spec}(\mathbf{C})$ is isomorphic to $\mathrm{Gr}_{\mathbf{e}}(V)$, and a polynomial $P(x) \in \mathbf{Z}[x]$ with $P(|k|) = |(\mathcal{G}\nabla_{\mathbf{e}}(V))(k)|$ (the set of k-rational points) for almost all finite fields k.*

4. Flat degenerations

We explain a general method to construct flat degenerations between quiver Grassmannians, which is applied in [1, 2] to the construction of flat degenerations of flag varieties.

Inside $R_{\mathbf{d}} \times \prod_{i \in Q_0} \mathrm{Gr}_{e_i}(\mathbf{C}^{d_i})$, we consider the closed subvariety

$$\mathrm{Gr}_{\mathbf{e}}^Q(\mathbf{d}) = \{((f_\alpha)_\alpha, (U_i)_i) \,:\, f_\alpha(U_i) \subset U_j \text{ for all } \alpha : i \to j\}.$$

Then $\mathrm{Gr}_{\mathbf{e}}^Q(\mathbf{d})$ is a $G_{\mathbf{d}}$-homogeneous vector bundle over $\prod_i \mathrm{Gr}_{\mathbf{e}}(\mathbf{C}^{d_i})$ via projection to the second factor, which allows to compute its dimension as

$$\dim \mathrm{Gr}_{\mathbf{e}}^Q(\mathbf{d}) = \dim R_{\mathbf{d}} + \langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle_Q.$$

On the other hand, the projection $\pi : \mathrm{Gr}_{\mathbf{e}}^Q(\mathbf{d}) \to R_{\mathbf{d}}$ to the first factor is projective, and the fibre $\pi^{-1}(V)$ (with its reduced scheme structure) is naturally isomorphic to the variety $\mathrm{Gr}_{\mathbf{e}}(V)$.

Theorem. *Let V_0 be the unique (up to isomorphism) representation of Q of dimension type $\mathbf{d}$ such that $\mathrm{Ext}^1(V_0, V_0) = 0$, and assume that $\mathrm{Gr}_{\mathbf{e}}(V_0)$ is non-empty. If $\dim \mathrm{Gr}_{\mathbf{e}}(V) = \langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle_Q$, then $\mathrm{Gr}_{\mathbf{e}}(V)$ is a flat degeneration of $\mathrm{Gr}_{\mathbf{e}}(V_0)$.*

The proof is easy: the orbit of V_0 in $R_{\mathbf{d}}$ is open, thus π is surjective by assumption. Its generic fibre $\mathrm{Gr}_{\mathbf{e}}(V_0)$ is thus of dimension $\dim \mathrm{Gr}_{\mathbf{e}}^Q(\mathbf{d}) - \dim R_{\mathbf{d}} = \langle \mathbf{e}, \mathbf{d}-\mathbf{e}\rangle_Q$. The restriction of π to the inverse image of the open set U of all points $W \in R_{\mathbf{d}}$ whose orbit closure contains V is thus an equidimensional map between smooth varieties and hence flat. Thus $\pi : \pi^{-1}(U) \to U$ is a flat family with generic fibre $\mathrm{Gr}_{\mathbf{e}}(V_0)$ and special fibre $\mathrm{Gr}_{\mathbf{e}}(V)$.

5. Linear degenerations of SL_{n+1}-flag varieties

We review the main results of [1].

We consider the special case $Q = 1 \to 2 \to \ldots \to n$, $\mathbf{d} = (n+1, \ldots, n+1)$, $\mathbf{e} = (1, 2, \ldots, n)$ of the above. We thus consider the first projection $\pi : Y \to X$ from

$$Y = \{((f_k),(U_i)) \,:\, f_i(U_i) \subset U_{i+1},\, i \le n-1\} \subset \mathrm{End}(\mathbf{C}^{n+1})^{n-1} \times \prod_{i=1}^{n} \mathrm{Gr}_i(\mathbf{C}^{n+1})$$

to $X = \mathrm{End}(\mathbf{C}^{n+1})^{n-1}$, which is equivariant for the base change action of $G = \mathrm{GL}_{n+1}(\mathbf{C}^{n-1}$.

For a subset $I \subset \{1, \ldots, n+1\}$, we denote by $\mathrm{pr}_I : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$ along the basis vectors e_i for $i \in I$, and we define the representations

$$V_0 = (\mathbf{C}^{n+1} \stackrel{\mathrm{id}}{\to} \mathbf{C}^{n+1} \stackrel{\mathrm{id}}{\to} \ldots \stackrel{\mathrm{id}}{\to} \mathbf{C}^{n+1}),$$
$$V_1 = (\mathbf{C}^{n+1} \stackrel{\mathrm{pr}_2}{\to} \mathbf{C}^{n+1} \stackrel{\mathrm{pr}_3}{\to} \ldots \stackrel{\mathrm{pr}_n}{\to} \mathbf{C}^{n+1}),$$
$$V_2 = (\mathbf{C}^{n+1} \stackrel{\mathrm{pr}_{2,3}}{\to} \mathbf{C}^{n+1} \stackrel{\mathrm{pr}_{3,4}}{\to} \ldots \stackrel{\mathrm{pr}_{n,n+1}}{\to} \mathbf{C}^{n+1}).$$

Theorem.

(1) *The generic fibre of π, that is, the fibre of π over V_0, is isomorphic to the* SL_{n+1}*-flag variety,*
(2) *π is flat precisely over the locus of points whose G-orbit closure contains V_2. All fibres over this locus are locally complete intersection varieties admitting a cell decomposition.*
(3) *π is flat with irreducible fibres precisely over the locus of points whose G-orbit closure contains V_1. All fibres over this locus are normal varieties.*
(4) *The number of irreducible components of $\pi^{-1}(V_2)$ equals the n-th Catalan number.*
(5) *The fibre $\pi^{-1}(V_1)$ is isomorphic to the degenerate flag variety of* [6].

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Introduction to Newton–Okounkov bodies

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Inspired by the work of A. Okounkov [13], R. Lazarsfeld and M. Mustaţă [10] and independently K. Kaveh and A. Khovanskii [6] introduced Newton–Okounkov bodies as a tool in the asymptotic theory of linear series on normal varieties, a tool which proved to be very powerful and in recent developments of the theory has gained a central role. An excellent introduction to the subject —not exhaustive due to the rapid development of the theory— can be found in the review [3] by S. Boucksom.

Newton–Okounkov bodies are defined as follows. Let X be a normal projective variety of of dimension n. A flag of irreducible subvarieties

$$\mathbf{Y} = \{X = Y_0 \supset Y_1 \supset \cdots \supset Y_n = \{p\}\}$$

is called *full* and *admissible* if Y_i has codimension i in X and is smooth at the point p. p is called the *center* of the flag. For every non-zero rational function $\phi \in K(X)$, write $\phi_0 = \phi$, and for $i = 1, \dots, n$

$$\nu_i(\phi) = \operatorname{ord}_{Y_i}(\phi_{i-1}) \ , \qquad \phi_i = \left. \frac{\phi_{i-1}}{g_i^{\nu_i(\phi)}} \right|_{Y_i} , \tag{1}$$

where g_i is a local equation of Y_i in Y_{i-1} around p (this makes sense because the flag is admissible). The sequence $\nu_{\mathbf{Y}} = (\nu_1, \nu_2, \dots, \nu_n)$ determines a rank n discrete valuation $K(X)^* \longrightarrow \mathbb{Z}^n_{\text{lex}}$ with center at p [15].

Definition 1. *If X is a normal projective variety, D a big Cartier divisor on it, and $\mathbf{Y}$ an admissible flag, the Newton–Okounkov body of D with respect to $\mathbf{Y}$ is*

$$\Delta_{\mathbf{Y}}(D) = \overline{\left\{ \frac{\nu_{\mathbf{Y}}(\phi)}{k} \,\middle|\, \phi \in H^0(X, \mathcal{O}_X(kD)),\, k \in \mathbb{N} \right\}} \subset \mathbb{R}^n,$$

where $\overline{\{\cdot\}}$ denotes the closure with respect to the usual topology of $\mathbb{R}^2$. Although not obvious from this definition, $\Delta_{\mathbf{Y}}(D)$ is convex and compact, with nonempty interior, i.e., a body (see [6], [10], [3]*).*

Note that a Newton–Okounkov body Δ_ν can be associated to D with respect to any rank n discrete valuation ν, regardless of whether there exists a flag $\mathbf{Y}$ such that $\nu = \nu_{\mathbf{Y}}$ (this point of view was taken in [3]). An application of this approach, useful in the case of toric varieties and Shubert varieties, is that it allows to relax the admissibilty hypothesis on the flag: if $\mathbf{Y}$ is a full flag for which there exists a birational morphism $\tilde{X} \to X$ such that there is a unique admissible flag $\tilde{\mathbf{Y}}$ lifting $\mathbf{Y}$, then $\mathbf{Y}$ also determines a rank n valuation and a Newton–Okounkov body for every big divisor D.

A particularly well-known case of Newton–Okounkov bodies is provided by toric varieties. Indeed, if X is a normal toric projective variety, and the flag $\mathbf{Y}$ is composed by torus-invariant subvarieties, then $\Delta_{\mathbf{Y}}(D)$ is the convex polytope associated to D by the usual correspondence of toric geometry. Several important properties of this correspondence, in particular those describing the asymptotic behavior of D in terms of the convex geometry of the polytope, carry over to the new setting, and allow to connect the asymptotic behavior of D (on an arbitrary normal projective variety X) to the convex geometry of its Newton–Okounkov bodies. In particular, the starting point of the theory of Newton–Okounkov bodies is Okounkov's theorem relating their volume to the volume of the divisor D. Recall that the *volume* of a Cartier divisor D on an irreducible normal projective variety X of dimension r is defined as

$$\operatorname{vol}(D) = \limsup_{m\to\infty} \frac{\dim\left(H^0(X, \mathcal{O}_X(mD))\right)}{m^r/r!} .$$

Theorem 8 (Okounkov)**.** *Let X be an irreducible normal projective variety of dimension r, let D be a big divisor on X, and let ν be a valuation of the field $K(X)$ with value group $\mathbb{Z}^r_{\mathrm{lex}}$. Then*

$$\operatorname{vol}(\Delta_\nu(D)) = \frac{1}{r!}\operatorname{vol}(D),$$

where the volume on the left-hand side denotes the Lebesgue measure in $\mathbb{R}^r$.

In order to understand Newton–Okounkov bodies, one faces the need to understand the underlying valuations and the algebraic constructions stemming from them. Every valuation $\nu\colon K(X)^* \longrightarrow \mathbb{R}^n_{\mathrm{lex}}$ with value group $G = \nu(K(X)^*)$ determines, for each $x \in G$, vector subspaces $\mathcal{P}_{\geq x} = \{f \in K(X)^* \,|\, \nu(f) \geq x\} \cup \{0\}$ and $\mathcal{P}_{>x} = \{f \in K(X)^* \,|\, \nu(f) > x\} \cup \{0\}$. Of these, $R_\nu = \mathcal{P}_{\geq 0}$ is a local ring (usually non-noetherian) with maximal ideal $\mathfrak{M}_\nu = \mathcal{P}_{>0}$. The G-graded algebra of ν is $\mathrm{Gr}_\nu = \bigoplus_{x\in G} \mathcal{P}_{\geq x}/\mathcal{P}_{>x}$.

Rank r valuations of the field of rational functions on an r-dimensional variety (the ones used to define Newton–Okounkov bodies) are "rational" in the sense that $R_\nu/\mathfrak{M}_\nu \cong \mathbb{C}$ and (therefore) each graded piece $\mathcal{P}_{\geq x}/\mathcal{P}_{>x}$ of the graded algebra is a 1-dimensional vector space. There is a natural map $R_\nu \longrightarrow \mathrm{Gr}_\nu$ that sends every nonzero $f \in R_\nu$ with $\nu(f) = x$ to its (nonzero) image in $\mathcal{P}_{\geq x}/\mathcal{P}_{>x}$.

Specializing to sections f in $H^0(X, \mathcal{O}_X(mD))$, one concludes that

$$\dim \frac{H^0(X, \mathcal{O}_X(mD)) \cap \mathcal{P}_{\geq x}}{H^0(X, \mathcal{O}_X(mD)) \cap \mathcal{P}_{>x}} \leq 1 \tag{2}$$

for all $x \in G$, and the set of (x, m) such that the dimension in (2) equals 1 is a subsemigroup Γ of $\mathbb{Z}^{n+1}$ of crucial importance.

Proof of Theorem 8 (sketch). The main steps in Okounkov's proof (also described in detail in [10, Theorem 2.3]) are as follows.

(1) For each $m \geq 0$, let $\Gamma_m = \{x \in \mathbb{Z}^n \,|\, (x, m) \in \Gamma\}$. Then $\#\Gamma_m = \dim H^0(X, \mathcal{O}_X(mD))$. This is an easy consequence of the inequality (2) above.

(2) Let C be the convex cone in $\mathbb{R}^{n+1}$ spanned by Γ. The intersection of C with the hyperplane whose last coordinate is $m = 1$ is exactly $\Delta_\nu(D)$. This follows from the definitions.

(3) Let $\Sigma = C \cap \mathbb{Z}^{n+1}$. If Γ is finitely generated, then there exists (x_0, m_0) such that $(x_0, m_0) + \Sigma \subset \Gamma$. This is a purely semigroup-theoretic statement whose proof can be found, e.g, in [7, §3].

(4) Using the previous step, if Γ is finitely generated, then

$$x_0 + (m - m_0)\Delta_\nu(D) \;\subset\; \Gamma_m \;\subset\; m\Delta_\nu(D).$$

Since both left and right hand sides of these inclusions have $\sim m^d \operatorname{vol}(\Delta_\nu(D))$ elements, it follows that $\lim(\#\Gamma_m)/m^d = \operatorname{vol}(\Delta_\nu(D))$, and by step (1), the claim follows for finitely generated semigroup.

(5) Approximate Γ by finitely generated semigroups, $\Gamma_1 \subset \Gamma_2 \subset \ldots$ such that $\Gamma = \bigcup_n \Gamma_n$, and apply the previous step to each Γ_n. $\square$

Next we list the most important results known to us on Newton–Okounkov bodies at this point, with special attention to those useful in Representation Theory and in the study of Shubert varieties. For simplicity of the statements, we assume that the rational map $X \to \mathbb{P}^N$ induced by D embeds X as a projectively normal variety, referring the reader to the original cited papers for precise statements in the case of general big divisors.

Numerical nature: By [10, Proposition 4.1], $\Delta_{\mathbf{Y}}(D)$ only depends on the numerical equivalence class of D. Reciprocally, by [5] the set of all Newton–Okounkov bodies works as a complete set of numerical invariants of D, in the sense that, if D' is another big Cartier divisor with $\Delta_{\mathbf{Y}}(D) = \Delta_{\mathbf{Y}}(D')$ for all flags $\mathbf{Y}$, then D and D' are numerically equivalent. In the case of surfaces, by [14], even fixing a point $p \in X$, if $\Delta_{\mathbf{Y}}(D) = \Delta_{\mathbf{Y}}(D')$ for all flags $\mathbf{Y}$ centered at p, then the positive parts of the Zariski decomposition of D and D' agree.

Polyhedrality: It is easy to see that if the semigroup Γ above is finitely generated then the Newton–Okounkov body is a rational polytope. However, such finite generation is quite rare, and in dimension $r \geq 3$ bodies need not be polytopes as illustrated in [9]; the key point to construct examples of such behavior is the fact [12] that the volumes of the fibers of the projection to the first coordinate x_1 are "restricted volumes": then, choosing an adequate divisorial part (e.g., equal or containing a suitable abelian surface) one can obtain restricted volumes depending quadratically on x_1. It is worth noting that such behavior can occur even with strong finiteness hypotheses, for instance if X is a Mori dream space. However, on surfaces all Newton–Okounkov bodies are polyhedral, with all vertices but at most two having rational coordinates [9], and on every variety there are choices of flags such that the Newton–Okounkov body is a rational simplex [2].

Toric degenerations: By [1], if the semigroup Γ is finitely generated (so the Newton–Okounkov body is a rational polytope P) then there exists a flat degeneration of X to the toric variety $\operatorname{Proj}\mathbb{C}[\Gamma]$, whose normalization is the toric variety associated to the polytope P. The degeneration can be explicitly described by a Gröbner basis, exploiting the fact that the Γ-graded algebra

$$\bigoplus_{(x,m)\in\Gamma} \frac{H^0(X, \mathcal{O}_X(mD)) \cap \mathcal{P}_{\geq x}}{H^0(X, \mathcal{O}_X(mD)) \cap \mathcal{P}_{>x}}$$

is in fact isomorphic to the semigroup algebra $\mathbb{C}[\Gamma]$ (by the inequality (2) above) and hence finitely generated.

Integrable systems: By [4], if the semigroup Γ is finitely generated and X is smooth, then there is a completely integrable Hamiltonian system on X with moment map $\mu : X \to \mathbb{R}^n$ such that $\mu(X) = \Delta_\nu$. This is constructed by degenerating to the toric variety and "pulling back" the toric moment map.

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Dirac operators and Geometric Invariant Theory in the differentiable setting

Michele Vergne
(joint work with Paul-Emile Paradan)

Let K be a compact Lie group, acting on a compact manifold M. If D is a K-invariant elliptic operator on M, its space $\mathrm{Index}_K(M, D)$ of (virtual) solutions is a virtual representation of K. We wish to understand the space of K-invariant solutions of D as the space $\mathrm{Index}(M_0, D_0)$ of solutions of an elliptic operator D_0 on a "smaller" space M_0.

A familiar setting is when M is a projective manifold, and D the Dolbeaut operator acting on sections of the corresponding line bundle. In this case M_0 is again a projective manifold: the geometric quotient of M defined by Mumford.

Here we consider any Dirac operator on an oriented even dimensional compact K-manifold M. An important example is the case where M is a compact complex manifold, L an holomorphic line bundle, not necessarily ample, and $D = \overline{\partial}_L$ is the Dolbeaut operator with coefficients in L. Then $\mathrm{Index}_K(M, \overline{\partial}_L) = \sum_{i=0}^{\dim M}(-1)^i H^i(M, O(L))$.

If M is a *spin* manifold (so not necessarily complex or even almost complex), we consider the Dirac operator twisted by a line bundle L. More generally, when M is a *spinc* manifold, we consider its determinant line bundle L_2. If the manifold is spin, then $L = L_2^{1/2}$ exists, and for simplicity we assume that we are in this case.

We use a K-invariant Hermitian connection on L. We then can construct a moment map $\Phi : M \to \mathfrak{k}^*$ by Kostant formula $\mathcal{L}(X) = \nabla_X + i\langle\Phi, X\rangle$. Here $X \in \mathfrak{k}$ is in the Lie algebra of K, $\mathcal{L}(X)$ is the infinitesimal action of X on sections of L.

We choose a Cartan subgroup T with Cartan subalgebra $\mathfrak{t}$, a positive root system and let $\rho \in \mathfrak{t}^*$. The element ρ parameterizes the trivial representation of K.

Assume first that the generic stabilizer of the action of K on M is finite. Then, we prove

Theorem

- If ρ is not in the image $\Phi(M)$, $[\mathrm{Index}_K(M, D_L)]^K = 0$.
- If ρ is in the image $\Phi(M)$, we can define an orbifold M_0 and a Dirac operator D_0 on M_0 such that $[\mathrm{Index}_K(M, D_L)]^K = \mathrm{Index}(M_0, D_0)$.

The definition of M_0 and D_0 requires some care: if ρ is a regular value of Φ, then $M_0 = \Phi^{-1}{}^{\text{`}}(\rho)/T$.

When M is complex, and L a line bundle which is not ample, our reduced spaces M_0 are again complex orbifolds, but might be non connected.

When M is a spinc manifold, with an action of S^1, this theorem has been obtained by Cannas da Silva-Karshon-Tolman. The case of toric manifolds and non ample line bundles have been treated in Karshon-Tolman.

In the general case, we prove that the index $\text{Index}_K(M, D)$ is equal to 0, when the semi-simple part $\mathfrak{s}$ of the generic stabilizer $\mathfrak{k}_M$ is not a Levi subalgebra.

Assume that $[\mathfrak{k}_M, \mathfrak{k}_M]$ is a Levi subalgebra $\mathfrak{s}$. In this case, we consider a collection of admissible elements $\rho_a \in \mathfrak{t}^*$ obtained by projection of ρ on certain walls of the Weyl chamber, and we define $M_0 = \cup_a \Phi^{-1}(\rho_a)/K_{\rho_a}$. Our theorem holds $[\text{Index}_K(M, D_L)]^K = \text{Index}(M_0, D_0)$.

This theorem is inspired by the $[Q, R] = 0$ theorem conjectured by Guillemin-Sternberg, and proved by Meinrenken-Sjamaar. However, we work in the spin context and our manifold need not be complex, nor even almost complex.

Some interesting questions remains open. For example, the "Duistermaat-Heckman" measure can be defined independently of the choice of a connection. Can we choose a connection ∇ such that the image of Φ is the support of the Duistermaat-Heckman measure ?

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