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Mini-Workshop: Arrangements of Subvarieties, and their Applications in Algebraic Geometry

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Abstract. While arrangements of hyperplanes have been studied in algebra, combinatorics and geometry for a long time, recent discoveries suggest that they (and more generally arrangements of nonlinear subvarieties) play an even more fundamental role in major problems in algebraic geometry than has yet been understood. The workshop brought into contact experts from commutative algebra and algebraic geometry working on these problems – it provided opportunities to get updated on the latest developments through talks of the participants, but also reserved time for working groups in which participants brainstormed ideas and insights in the context of high-intensity discussions aimed at initiating immediate progress on proposed problems, thereby setting the stage for on-going collaborations after the workshop.

Mathematics Subject Classification (2010): 14C20, 13F20, 14N20, 32S22, 52C35.

Introduction by the Organisers

The theme of the workshop

Hyperplane arrangements have long been objects of attention in algebra, combinatorics and geometry, but recent discoveries suggest they (and more generally arrangements of nonlinear subvarieties) play an even more fundamental role in major problems in algebraic geometry than has yet been understood. They arise both in showing the containment theorem of Ein-Lazarsfeld-Smith and Hochster-Huneke [11, 15] is optimal [5], and they underlie the only known counterexamples to attempts to improve on that theorem [9, 10, 1]. Some of the same arrangements that give containment counterexamples also mysteriously have arisen in work on the Bounded Negativity Conjecture [4]. Others, the so-called star configurations [13], are extremal for a conjecture of Chudnovsky which gives putative bounds on Waldschmidt constants. For all of these problems, a fundamental issue is to understand the Hilbert function of ideals of arrangements of possibly fat linear subspaces, or equivalently of the restriction maps discussed below in this report. Issues of interest include better understanding of why arrangements of linear subspaces arise in these problems, development of new techniques for identifying, studying and classifying specific arrangements relevant to each problem, and exploring the behavior and possible applicability of nonlinear arrangements.

As a critical element for promoting the desired progress, the workshop brought into contact the largely disjoint groups of experts on these problems. It established a framework both for disseminating the latest research and for stimulating collaborations. This was achieved by providing speaking opportunities for the participants to share their expertise and knowledge of new developments, while also reserving time for working groups in which participants brainstorm ideas and insights in the context of high-intensity discussions aimed at initiating immediate progress on the proposed problems, thereby setting the stage for on-going collaborations after the workshop.

The format of the workshop: talks and working groups

In the mornings, the participants had the opportunity to give and hear talks in order to get updated on current work on arrangements of subvarieties, and in order to exchange methods and ideas between the more commutative algebraic oriented part of the group and the more algebro-geometric oriented part. In the afternoons we reserved time for research activities in informal groups, but we also organized working groups on specific topics. For the latter, we had proposed – well ahead of the workshop – a set of research problems and made it available to the participants. In a common meeting on the first day, three working groups were formed on problems selected from our list:

(1) Postulation of lines and a fat flat. It has been proved by Hartshorne and Hirschowitz in [14] that general lines in $\mathbb{P}^n$ impose the expected number of conditions on linear series passing through them. This result has recently attracted some attention and has been extended to general lines and one fat point (there is a final list of exceptions) in [6] and general lines and double points (again with a final list of exceptions) in [2]. Along these lines we studied the following problem.

> *Problem.* Let $X \subset \mathbb{P}^{2n+1}$ be a subscheme consisting of $s \geq 1$ generic reduced linear subspaces of dimension n and a single generic linear subspace of dimension $n-1$ and multiplicity $m \geq 1$. Let I_X be the homogeneous ideal of X and let $(I_X)_t$ denote the vector space of elements of degree t in I_X. Verify if the only cases where the Hilbert function of I_X differs from the Hilbert polynomial of I_X occur for $m = t$ and $2 \leq s \leq t$.

(2) Resurgence for extremal line configurations. Let I be a nontrivial homogeneous ideal in the ring of polynomials $\mathbb{C}[x_0, \dots, x_n]$. Ein, Lazarsfeld and Smith [11] showed that if $c \geq 1$ is real, it is enough to take $c = n$ to guarantee that

$$I^{(m)} \subseteq I^r \text{ for all } m \geq rc, \tag{1}$$

where $I^{(m)}$ denotes the mth symbolic power of the ideal I. An analogous result in arbitrary characteristic was obtained shortly thereafter by Hochster and Huneke [15]. No ideal I is known such that the least c that works in (1) is n.

Bocci and Harbourne introduced in [5] the *resurgence* which provides in effect for a fixed ideal I the optimal constant c in (1). Quite expectedly, this invariant (and its cousins e.g. the asymptotic resurgence) are very hard to compute. In the recent paper [10] they are computed for some configurations of lines.

> *Problem.* Compute the resurgences of the Klein and Wiman configurations (see [4, Sections 4.1 and 4.2]).

(3) Independence of conditions imposed on incomplete linear systems. The well known SHGH Conjecture addresses the problem of when a fat point subscheme $X = m_1p_1 + \cdots + m_rp_r \subset \mathbb{P}^2$ supported at general points p_i fails to impose the expected number of conditions on the complete linear system $V = R_t$ of all forms of some degree t in $R = k[\mathbb{P}^2]$, and in principle gives a classification of all such failures. This raises the question of what happens if we only require $V \subseteq R_t$. In this direction, let $Z \subset \mathbb{P}^2$ be a fat point subscheme and let $V = I(Z)_t \subseteq R_t$. We say X fails to impose the expected number of conditions on V if

$$\dim(I(X)_t \cap V) > \max\Big(\dim V - \sum_i \binom{m_i+1}{2}, 0\Big).$$

The SHGH Conjecture addresses the case that $Z = 0$. It also gives a conjectural answer when Z is supported at general points. The case that Z is reduced with $X = m_1p_1$ and $t = m + 1$, building on [8, 12], is addressed in [7], and establishes connections with line arrangements.

> *Problem.* Find interesting cases where one can characterize or classify triples $\Lambda = (t, X, Z)$ for which X fails to impose the expected number of conditions on $[I(Z)]_t$.

This format – with talks on the one hand and working groups on pre-selected topics on the other hand – had already proven extremely successful in a previously organized workshop by the same organizers [3]: It lead to a number of new research collaborations and to a significant number of publications. And again in the present workshop, the format seems to have worked extremely well. Already at the time of this writing, preprints are being prepared with results from the three working groups described above.

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Mini-Workshop: Arrangements of Subvarieties, and their Applications in Algebraic Geometry (Toc)

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Abstracts

Fixed points of endomorphisms on two-dimensional complex tori

Thomas Bauer
(joint work with Thorsten Herrig)

Consider a holomorphic map $f : X \to X$ on a complex manifold (or a morphism on a smooth projective variety). One of the natural questions about f is as to how many fixed points is has. We tackle this question here motivated by the asymptotic perspective which has proven extremly fruitful in various settings in Algebraic Geometry – examples being the study of base loci [ELMNP], the growth of higher cohomology [DKL], syzygies [EL] and Betti numbers [EEL]. The general theme is that one can expect more regular behaviour from an asymptotic point of view. We therefore address here an asymptotic version of the fixed points question:

Given a holomorphic map $f : X \to X$ on a complex manifold, what is the asymptotic behaviour of the fixed-points function

$$n \mapsto \#\mathrm{Fix}(f^n) \qquad (*)$$

where $f^n = f \circ \cdots \circ f$ is the n-th iterate of f?

The growth of the fixed-points function is also of interest in purely analytic contexts (e.g. [SS]). In our project, we aim to understand $(*)$ when f is a holomorphic map on a complex torus. In that setting, there is previous work (from a non-asymptotic perspective) by Birkenhake and Lange [BL], who focused on the classification of fixed-point free automorphisms.

Examples. Consider a complex torus $X = \mathbb{C}^g/\Lambda$, where $\Lambda \subset \mathbb{C}^g$ is a lattice, and for a given integer $m \geq 2$, let f be the multiplication map

$$\begin{array}{rcl} f : X & \to & X \\ x & \mapsto & mx = x + \cdots + x \end{array}$$

Then the fixed points of f are just the $(m-1)$-torsion points, and hence the fixed-points number

$$\#\mathrm{Fix}((m_X)^n) = (m^n - 1)^{2g}$$

grows exponentially with n.

If, on the other hand, we take f to be the involution $x \mapsto -x$, then the fixed-points number alternates between 2^{2g} and 0.

In view of these first examples, one is led to wonder what other behaviour, if any, might occur. For two-dimensional complex tori we have a complete answer (see [BH] for details and proofs):

Theorem 1. *Let X be a two-dimensional complex torus and let $f : X \to X$ be a non-zero endomorphism. Then the fixed-points function $n \mapsto \#\mathrm{Fix}(f^n)$ has one of the following three behaviours:*

(B1) *It grows exponentially in n, i.e., there are real constants $A, B > 1$ and an integer N such that for all $n \geq N$,*

$$A^n \leq \#\mathrm{Fix}(f^n) \leq B^n .$$

(B2) *It is a periodic function.*

(B3) *It is of the form*

$$\#\mathrm{Fix}(f^n) = \begin{cases} 0, & \textit{if } n \equiv 0 \pmod r \\ h(n), & \textit{otherwise} \end{cases}$$

where $r \geq 2$ is an integer and h is an exponentially growing function.

All three behaviours occur already in the projective case, i.e., on abelian surfaces.

Fixed points and eigenvalues. The possible behaviours (B1)–(B3) are in fact governed by the eigenvalues λ_1 and λ_2 of the analytic representation $\rho_{\mathrm{an}}(f) : \mathbb{C}^2 \to \mathbb{C}^2$ of f:

- In Case (B1) both eigenvalues are of absolute value $\neq 1$.
- In Case (B2) the non-zero eigenvalues of f are roots of unity, and they are contained in the set of k-th roots of unity where $k \in \{1, \ldots, 6, 8, 10, 12\}$.
- In Case (B3) one of the eigenvalues is of absolute value > 1 and the other is a root of unity.

The mere fact that the fixed-points behaviour is related to the eigenvalues of f is quite immediate, since one has as a consequence of the Lefschetz fixed-point formula (in any dimension):

$$\#\mathrm{Fix}(f^n) = \Big| \prod_{i=1}^{g} (1 - \lambda_i^n) \Big|^2$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvales of the analytic representation $\rho_{\mathrm{an}}(f) : \mathbb{C}^g \to \mathbb{C}^g$. The issue thus becomes:

What kind of eigenvalues can endomorphisms of complex tori have?

If, for instance, one has $|\lambda_i| > 1$ for all eigenvalues (as it is the case for the multiplication maps considered above), then $\#\mathrm{Fix}(f^n)$ clearly grows exponentially. The essential content of the theorem lies thus in the analysis of what happens in the presence of eigenvalues λ with $|\lambda| \leq 1$. Somewhat surprisingly, it can in fact happen that eigenvalues are arbitrarily small:

Proposition. *For every $\varepsilon > 0$ there exists a 2-dimensional complex torus (and even a simple abelian surface) with an endomorphism having an eigenvalue λ with*

$$|\lambda| < \varepsilon .$$

This is in constrast to the case of elliptic curves, where one always has $|\lambda| \geq 1$ for every endomorphism λ.

It seems that eigenvalues of endomorphisms on complex tori have so far not been studied much – exceptions being work by McMullen [McM, Sect. 4] and Reschke [Res] on entropies of automorphisms.

Fixed points and endomorphisms algebras. While the theorem above allows in principle to determine the fixed-point behaviour of a given endomorphism, it does not tell which of the fixed-point behaviours occur on a *given* X. For abelian surfaces, it is desirable to answer this question in terms of their endomorphism algebra $\mathrm{End}_{\mathbb{Q}}(X) := \mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Our second result provides this answer (see [BH]):

Theorem 2. *Let X be a simple abelian surface. Then the fixed-points function of any non-zero endomorphism $f \in \mathrm{End}(X)$ is either exponential* (B1) *or periodic* (B2)*, but has never behaviour* (B3)*. Specifically, we have:*

(a) *Suppose that X has real multiplication, i.e.,* $\mathrm{End}_{\mathbb{Q}}(X) \simeq \mathbb{Q}(\sqrt{d})$ *for a square-free integer $d > 0$. Then $\#\mathrm{Fix}(f^n)$ is periodic if $f = \pm \mathrm{id}_X$, and it grows exponentially otherwise.*

(b) *Suppose that X has indefinite quaternion multiplication, i.e.,* $\mathrm{End}_{\mathbb{Q}}(X)$ *is of the form $\mathbb{Q}+i\mathbb{Q}+j\mathbb{Q}+ij\mathbb{Q}$, where $i^2 = \alpha \in \mathbb{Q}\backslash\{0\}$, $j^2 = \beta \in \mathbb{Q}\backslash\{0\}$ with $ij = -ji$ and $\alpha > 0$, $\alpha \geq \beta$. Write $f \in \mathrm{End}(X)$ as $f = a+bi+cj+dij$ with $a, b, c, d \in \mathbb{Q}$. Then $\#\mathrm{Fix}(f^n)$ is periodic if $|a + \sqrt{b^2\alpha + c^2\beta - d^2\alpha\beta}| = 1$, and it grows exponentially otherwise.*

(c) *Suppose that X has complex multiplication, and let $\sigma : \mathrm{End}(X) \hookrightarrow \mathbb{C}$ be an embedding. Then f has periodic fixed-point behaviour if $|\sigma(f)| = 1$, and it has exponential fixed-points growth otherwise.*

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An unexpected menagerie

David Cook II
(joint work with Brian Harbourne, Juan Migliore, Uwe Nagel)

Brian Harbourne's talk introduced the idea of unexpected curves through Z, a finite set of points in $\mathbb{P}^2$. In particular, for a general point P, we define

- $m_Z = \min\{j \geq 0 \text{st} h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1)) > 0\}$,
- $t_Z = \min\{j \geq 0 \text{st} h^0(\mathcal{I}_Z(j+1)) - \binom{j+1}{2} > 0\}$,
- $u_Z = \min\{j \geq 0 \text{st} h^1(\mathbb{P}^2, \mathcal{I}_{Z+jP}(j+1)) = 0\}$, and
- we say Z is *generally special* if $m_Z < u_Z$.

Theorem. *Z admits an unexpected curve if and only if $m_Z < t_Z$; in this case, Z has an unexpected curve of degree j if and only if $m_Z < j \leq u_z$.*

Thus being generally special is required for unexpected curves, though not necessary.

Juan Migliore's talk expanded upon this and gave a connection to line arrangements. Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$ corresponding to a product f of linear forms. Associated to $\mathcal{A}$ is its *splitting type* (a, b), $a \leq b$ (if $\mathcal{A}$ is free, these are the *exponents*).

Part (i) of the following theorem was shown by Faenzi and Vallés [2].

Theorem. *Let Z be the set of points dual to $\mathcal{A}$. Then (i) $a = m_Z$ and (ii) $b = u_z + 1$.*

Thus, in particular, Z is generally special if and only if $b - a \geq 2$. Moreover, this allows us to use combinatorics to compute m_Z and u_Z.

Our first family shows that not all arrangements have generally special dual points. Let $\mathcal{A}_d$ be a configuration of d lines in $\mathbb{P}^2$ such that no three lines meet in a point.

Lemma. *The splitting type of $\mathcal{A}_d$ is $\left(\lfloor \frac{d-1}{2} \rfloor, \lceil \frac{d-1}{2} \rceil\right)$. Moreover, $\mathcal{A}_d$ is free if and only if $d \leq 3$.*

Sketch. This follows by induction on d via showing that the derivation bundle is stable and applying the Grauert-Mülich Theorem. □

A set of points in $\mathbb{P}^2$ is in *linear general position* if no three are on a common line. Thus, in particular, its dual set of lines is of the form above.

Theorem. *A set of points in linear general position is never generally special hence has no unexpected curves.*

Our second family provides examples admitting multiple unexpected curves. The t^{th} Fermat (or monomial) arrangement $\mathcal{A}_t$ consist of $3t$ lines ($t \geq 1$) defined by the linear factors of $f_t = (x^t - y^t)(x^t - z^t)(y^t - z^t)$. If $t > 3$, there are t^2 triple points and 3 points where exactly t lines cross.

The following lemma is easy to see and has been previously demonstrated.

Lemma. *If $t > 2$, then $\mathcal{A}_t$ is free with splitting type* $(t+1, 2t-2)$.

Sketch. We show $\mathcal{A}_t \cup \{x, y\}$ is supersolvable, hence free with splitting type $(t+1, 2t)$. Using the Addition-Deletion Theorem, removing x then y yields the desired result. □

Thus the dual set of points to $\mathcal{A}_t$ is generally special if and only if $t \geq 5$.

Theorem. *The dual set of points Z to $\mathcal{A} = \mathcal{A}_t$ admits unexpected curves of degrees $t+2, \ldots, 2t-3$ if $t \geq 5$.*

Sketch. The previous lemma implies $m_Z = t+1$ and $u_Z = 2t-3$. Brian Harbourne previously [3] proved that the $3t$ points of Z impose independent conditions on forms of degree at least $t+1$, i.e., $t_Z > m_Z$. Thus the desired unexpected curves exist. □

Our third family provides examples admitting an *irreducible* unexpected curve. In particular, this family generalizes the B_3 arrangement.

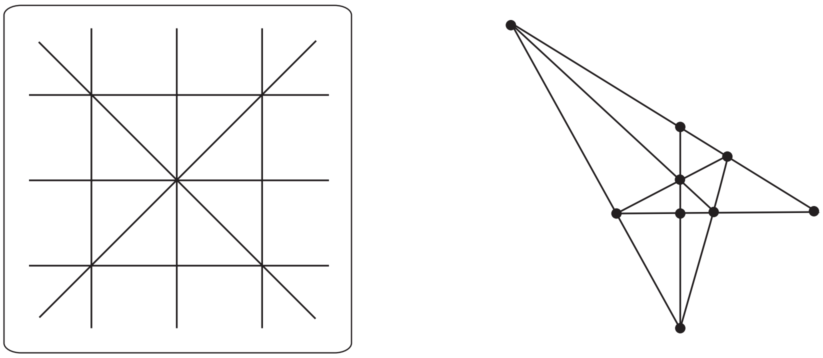

FIGURE 1. The B_3 configuration and its dual set of points. The right-most picture is from [1].

Let $\mathcal{A}_0$ be the arrangement of 5 lines defined by $x = 0, y = 0, z = 0, x - y = 0, x + y = 0$. We then construct $\mathcal{A}_t$ by adding t lines to $\mathcal{A}_0$ similar to how $B_3 = \mathcal{A}_4$ is constructed.

Using an induction argument along with the Addition-Deletion Theorem, it is easy to see the following.

Lemma. *$\mathcal{A}_{4k}$ is free with splitting type* $(2k+1, 2k+3)$.

In particular, $\mathcal{A}_{4k}$ is generally special.

Theorem. *The dual set of points Z to $\mathcal{A} = \mathcal{A}_{4k}$ admits an irreducible unexpected curve of degree $2k+2$.*

Sketch. Juan already showed us that it suffices to demonstrate that removing any line from $\mathcal{A}_{4k}$ reduces the splitting type to $(2k+1, 2k+2)$.

Any line L not defined by $z = 0$, meets the other lines in $2k+2$ points so deleting it yields a free arrangement with splitting type $(2k+1, 2k+2)$ as desired. For the line defined by $z = 0$, it requires a more subtle argument showing that the derivation bundle is semistable and applying the Grauert-Mülich Theorem. □

Our fourth and final example demonstrates that being generally special is necessary but not sufficient to force unexpected curves. This example will be of particular interest for Uwe Nagel's talk.

Let $\mathcal{A}$ be the arrangement defined by the lines $z, x, x+z, x+2z, y, y+z, \ldots, y+12z$. It is not hard to see that $\mathcal{A}$ is supersolvable, hence free, with splitting type $(3, 13)$. Thus the dual set of points Z has $m_Z = 3$ and $u_Z = 12$, hence is generally special. Moreover, we notice that

$$h^0((I_Z)(2+1)) - \binom{2+1}{2} = 0 \text{ and } h^0((I_Z)(3+1)) - \binom{3+1}{2} = 1,$$

thus $t_Z = 3 = m_Z$. Hence Z does not admit unexpected curves despite being generally special.

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Laplace equations, Lefschetz properties and Arrangements, II

Roberta Di Gennaro
(joint work with Giovanna Ilardi, Jean Vallès)

In [2] we give an equivalence between artinian ideals failing Strong Lefschetz Property (SLP) and singular hypersurfaces contained in the apolar of the ideal. In the case of ideals generated by suitable power of linear forms in three variables we give an equivalence between failing SLP and unstability of the derivation bundle of a line arrangement and a conjecture in terms of SLP equivalent to Terao conjecture.

We recall some notation and definitions. K is the complex field, $R = K[x_0, \ldots, x_n$ is the polynomial ring in $n+1$ variables and $R_d = \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^n}(d))$ is the complex vector space of the degree d forms of $\mathbb{P}^n$ and r_d its dimension,$k, t, \delta \geq 1$ are integers. For any vector space V, by V^* we denote the dual vector space of V and by $\ell^\vee$ the dual of ℓ.

Definition 1. *An artinian ideal $I \subset R$ fails the Strong Lefschetz Property (SLP) at range k and degree t by δ if for any linear form ℓ the multiplication map $\times \ell^k : \left(\frac{R}{I}\right)_t \to \left(\frac{R}{I}\right)_{t+k}$, has rank $\min\{H_{R/I}(t), H_{R/I}(t+k)\} - \delta$ (so it has not maximal rank.) For $k=1$ we talk about failing Weak Lefschetz Property.*

Let $I = (F_1, \dots, F_r)$ be an ideal generated by r forms of degree d, we denote by I_d the vector subspace of R_d generated by the $F_1, \dots, F_r$ and $I_{d+i} = R_iF_1 + \cdots + R_iF_r$, for any $i \geq 0$.

Definition 2. *The apolar space of I in degree $d+i$ is*

$$I^{\perp}_{d+i} = \{\delta \in R^*_{d+i} \,|\, \delta(F) = 0, \ \forall F \in I_{d+i}\},$$

where the canonical basis of R^*_d is given by the $r_d = \binom{d+n}{n}$ derivations $\frac{\partial^d}{\partial x_0^{i_0} \dots \partial x_n^{i_n}}$ with $i_0 + \ldots + i_n = d+i$.

Of course one can identify $R_{d+i}/I_{d+i} \simeq (I^{\perp}_{d+i})^*$.

We use here the following two equivalent conditions from the main result in [2].

Theorem 1. [2, Theorem 5.1] *Let $I = (F_1, \dots, F_r) \subset R$ be an artinian ideal generated by r homogeneous polynomials of degree d. Let $i \geq 0, k, \delta \geq 1$ be integers, $N(r,i,k,d) := r(r_i - r_{i-k}) - (r_{d+i} - r_{d+i-k})$, $N^+ = \sup(0, N(r,i,k,d))$ and $N^- = \sup(0, -N(r,i,k,d))$. Assume that there is no syzygy of degree i among the F_j's. The following conditions are equivalent:*

(1) *The ideal I fails the SLP at the range k in degree $d+i-k$.*
(2) *For any $\ell \in R_1$, $\dim_{\mathbb{C}}((I^{\perp}_{d+i})^* \cap \mathrm{H}^0(\mathcal{I}^{d+i-k+1}_{\ell^\vee}(d+i)) = N^- + \delta$, with $\delta \geq 1$.*

When $I = (\ell_1^d, \dots, \ell_r^d)$ where ${\ell_i}^\vee \in \mathbb{P}(R_1^*)$ we link the failing SLP of I and the unstability of the derivation bundle of the line arrangement $\mathcal{A} = \ell_1 \cdots \ell_r$.

Definition 3. *Let $\mathcal{A} = \ell_1 \cdots \ell_r$ be a line arrangement in $\mathbb{P}^2$ and $Z = \{{\ell_1}^\vee, \dots, {\ell_r}^\vee]\}$ be the reduced 0-dimensional scheme in the dual space. The* derivation bundle *$\mathcal{D}(Z)$ is defined by*

$$0 \longrightarrow \mathcal{D}(Z) \longrightarrow \mathscr{O}^3_{\mathbb{P}^2} \xrightarrow{(\partial f)} Jac_Z(d-1) \longrightarrow 0.$$

It is a bundle of rank 2 so on a general line ℓ it splits.
The general splitting type of $\mathcal{A}$ is $a,b)$ if $\mathcal{D}(Z)_{|\ell} = \mathscr{O}_\ell(-a) \oplus \mathscr{O}_\ell(-b)$.
The arrangement $\mathcal{A}$ is free of exponent (a,b) if and only if $\mathcal{D}(Z) = \mathscr{O}_{\mathbb{P}^2}(-a) \oplus \mathscr{O}_{\mathbb{P}^2}(-b)$.

The following two results are our main ingredients.

Lemma 1. [4] *If $I = (\ell_1^d, \dots, \ell_r^d)$ and $Z = \{{\ell_1}^\vee, \dots, \ell_r^\vee$ then $I^{\perp}_d = \mathrm{H}^0(\mathscr{I}_Z(d))$*

From Lemma 1 the singular curves in Theorem 2 are curves through Z: $(I^{\perp}_{d+i})^* \cap \mathrm{H}^0(\mathscr{I}^{d+i-k}_{L^\vee}(d+i)) = \mathrm{H}^0(\mathscr{I}_Z \otimes \mathscr{I}^{d+i-k}_{L^\vee})(d+i)$

Lemma 2. [5, Theorem 4.3] *Let $Z \subset {\mathbb{P}^2}^*$ be a set of $a+b+1$ distinct points with $1 \leq a \leq b$ and ℓ be a general line in $\mathbb{P}^2$. Then the following conditions are equivalent:*

(1) $\mathcal{D}(Z) \otimes \mathscr{O}_\ell = \mathscr{O}_\ell(-a) \oplus \mathscr{O}_\ell(-b)$.
(2) $\mathrm{h}^0((\mathscr{I}_Z \otimes \mathscr{I}^a_{\ell^\vee})(a+1)) \neq 0$ *and* $\mathrm{h}^0((\mathscr{I}_Z \otimes \mathscr{I}^{a-1}_{\ell^\vee})(a)) = 0$.

Lemma 2 suggests us to consider failing SLP at range 2 in degree $d+i-2$, as it is equivalent to $\mathrm{h}^0(\mathscr{I}_Z \otimes \mathscr{I}^{d+i-1}_{L^\vee})(d+i) > N^-$.

Recalling that a rank two vector bundle E on $\mathbb{P}^n, n \geq 2$ is unstable if and only if $E_\ell = \mathscr{O}_l(-a) \oplus \mathscr{O}_\ell(-b)$ on a general line ℓ with $\mid a-b \mid \geq 2$, we deduce the following result.

Proposition 1. [2, Proposition 7.2] *Let* $\ell_1, \dots, \ell_{2d+1} \in R = \mathbb{C}[x,y,z]$ *distinct linear forms and suppose that the ideal* $I = (\ell_1^d, \dots, \ell_{2d+1}^d)$ *is minimally generated; let* $Z = \{\ell_1^\vee, \dots, \ell_{2d+1}^\vee\}$ *be the corresponding set of points in* $\mathbb{P}^{2\vee}$. *Then the ideal* I *fails the* SLP *at the range* 2 *in degree* $d-2$ *if and only if the derivation bundle* $\mathcal{D}(Z)$ *is unstable.*

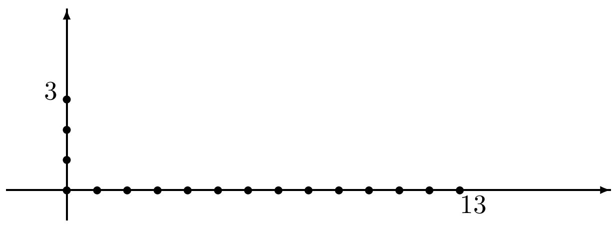
The hypothesis *minimally generated* (i.e. there is no syzygy in degree 0 as Theorem 1 requires) that we implicitly use in the proof is actually missed as missprint in the original statement. Actually, if there is no 0-syzygy our integer N^- is 0 and the existence of a singular suitable curve is enough to the failure of SLP. Moreover, this hypothesis is necessary to the thesys, otherwise the unstability is not enough to get an ideal that fails SLP. This is highlighted in [1] with a nice example.

Example. *Let* $\mathcal{A} = xyz(x+z)(x+2z)\prod_{j=1}^{12}(y+jz) =$ *be the free arrangement of splitting type* $(3,13)$. *Of course it is unstable, but the ideal* $I = (x^8, y^8, z^8, (x+z)^8, (x+2z)^8, (y+jz)^8 | 1 \leq j \leq 12)$ *has the SLP at range* 2.

Actually $H_{R/I}(8) = 33 = 45-(17-5)$ that means that there are 5 independent 0-syzygies. As the difference between Hilbert functions is $H_{R/I}(8) - H_{R/I}(6) = -5$ we get the idea to generalize our Theorem 1 when there are syzygies in degree i and not in degree $i-k$ just by replacing the integer $N(r,i,k,d)$ defined before with $N(r,i,k,d) = H_{R/I}(d+i-k) - H_{R/I}(d+i)$: in the example we should get $N = -5$ so $N^- = 5$ and the existence of singular curves is expected (in the sense of [1]) and is not enough to the failure of SLP.

Moreover, the existence of the syzygies in the example can be geometrically nicely deduced: from Veronese projection v_d aligned points $\ell_1^\vee, \dots, \ell_r^\vee \in L$ go in points $\ell_1^d, \dots, \ell_r^d \in v_d(L) = C_d$, where $C_d \subset \mathbb{P}^d$ is the rational normal curve of degree d; so $d+1$ of them are linearly independent, meanwhile $d+2$ are dependent; here, for $d = 8$ we can have at most 9 aligned points corresponding to $l_1^8, \dots, l_9^8$ without 0-syzygies, but we have 14 aligned point so any other aligned point gives a syzygy (14-9=5 independent 0-syzygies).

Thanks to this example we are improving our results by deducing minimality of generators of ideals from the alignement of points.



Finally, it is an open and interesting problem if Terao conjecture holds.

Definition 4. *Let $\mathcal{A}$ be a line arrangement, then the combinatorics of $\mathcal{A}$ is the intersection lattice with reverse order.*

Conjecture. *(Terao) Let $\mathcal{A}$ be a free arrangement and $\mathcal{A}'$ an arrangement with the same combinatorics of $\mathcal{A}$ tehn $\mathcal{A}'$ is free, too (i.e. is the freeness is a combinatorial property).*

This conjecture for line arrangements is true up to 12 lines ([5]). In [2], as we relate freeness and unstability with failing SLP, we give an equivalent conjecture in terms of SLP. It is a calculation from [3] (also [1]) that

Proposition 2. *If $\mathcal{A}$ is free of exponent (a, b) and $\mathcal{A}'$ is not free with the same combinatorics, then the general splitting type of $\mathcal{A}'$ is $(a - s, b + s)$, with $s \geq 1$.*

So, arguing again with Lemma 2 and adding $b - a$ points in general position to the dual set of points Z, Z' and considering the union of previous curves with the $b - a$ lines joining any added point with $\ell^\vee$ we get that starting from a free arrangement $\mathcal{A}$ with exponents (a, b) we can construct a semistable arrangement $\mathcal{B}$ with splitting type (b, b); starting from a non-free arrangement $\mathcal{A}'$ with the same combinatorics of $\mathcal{A}$ we can construct an unstable arrangement $\mathcal{B}'$ with splitting type $(b - s, b + s), s \geq 1$ and the same combinatorics of $\mathcal{B}$. So we deduce that the following conjecture is equivalent to Terao conjecture

Conjecture. *Let $I = ({\ell_1}^b, \ldots, {\ell_{2b+1}}^b)$ be a minimally generated ideal that has SLP at range 2 in degree $b-2$ and $J = ({h_1}^b, \ldots, {h_{2b+1}}^b)$ be a minimally generated such that the corresponding arrangements $\ell_1 \cdots \ell_{2b+1}$ and $h_1 \cdots h_{2b+1}$ have the same combinatorics then J has SLP at range 2 in degree $b - 2$.*

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Gauss maps

Sandra Di Rocco
(joint work with Kelly Jabbusch, Anders Lundman)

Let $X \subset \mathbb{P}^N$ be an n-dimensional irreducible, nondegenerate projective complex variety. The (classical) Gauss map is the rational morphism $\gamma : X \dashrightarrow Gr(n, N)$ that assigns to a smooth point x the projective tangent space of X at x, $\gamma(x) = \mathbb{T}_{X,x} \cong \mathbb{P}^n$. It is known that the general fiber of γ is a linear subspace of $\mathbb{P}^N$, and that the morphism is finite and birational if X is smooth unless X is all of $\mathbb{P}^N$, [8, 5, 4].

In [8], Zak defines a generalization of the above definition as follows.

For $n \leq m \leq N-1$, let $Gr(m, N)$ be the Grassmanian variety of m-dimensional linear subspaces in $\mathbb{P}^N$, and define $\mathcal{P}_m = \overline{\{(x, \alpha) \in X_{sm} \times Gr(m, N) | \mathbb{T}_{X,x} \subseteq L_\alpha\}}$, where L_α is the linear subspace corresponding to $\alpha \in Gr(m, N)$ and the bar denotes the Zariski closure in $X \times Gr(m, N)$.

The **m-th Gauss map** is the projection $\gamma_m : \mathcal{P}_m \to Gr(m, N)$. When $m = n$ we recover the classical Gauss map, $\gamma_n = \gamma$. These generalized Gauss maps still enjoy the property that a general fiber is a linear subspace, [8, 2.3 (c)]. Moreover a general fiber is always finite if X is smooth and $n \leq m \leq N - n + 1$, [8, 2.3 (b)].

Consider now a different generalization of the Gauss map where, instead of higher dimensional linear spaces tangent at a point, we use linear spaces tangent to higher order, namely the osculating spaces. The osculating space of order k of X at a smooth point $x \in X_{sm}, Osc^k_x$, is a linear subspace of $\mathbb{P}^N$ of dimension t, where $n \leq t \leq \binom{n+k}{n}$. The maximal dimension is obtained generically, we denote such dimension with d_k. We call the rational map

$$\gamma^k : X \dashrightarrow Gr(d_k - 1, N)$$

that assigns to a general smooth point x the k-th osculating space of X at x, the **Gauss map of order k.** Notice that when $k = 1$, we recover the classical Gauss map. This definition was originally introduced in [1] and later studied in [7] under the name of associated maps. Higher order Gauss maps have subsequently been studied in connection to higher fundamental forms in [6] and [2], while Gauss maps of order 2 have been investigated in [3].

We generalize the classical result by proving that the higher order Gauss maps are also finite with the only exceptions being Veronese embeddings:

Assume that X is a nonsingular complex variety and that the embedding $i : X \hookrightarrow \mathbb{P}^N$ is such that the osculating dimension at every point is $\binom{n+k}{n}$. Then the Gauss maps γ^s are finite for all $s \leq k$, unless $X = \mathbb{P}^n$ is embedded by the Veronese embedding of order s.

The assumption that the osculating dimension is maximal at every point is not something that we can relax. In fact we can consider the Del Pezzo surface of degree 5, given by the blow up of $\mathbb{P}^2$ in 4 points in general position embedded by the anticanonical bundle $-K_X$. The 2-osculating dimension is maximal at every point outside the 4 exceptional divisors $E_1, \ldots, E_4$ and thus the Gauss map of second order is the rational map $\gamma^2 : X \dashrightarrow Gr(5,5) = \mathrm{pt}$, contracting $X \setminus (E_1 \cup \ldots \cup E_4)$ to a point and is in particular not generically finite.

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Line arrangements and SHGH

Brian Harbourne

(joint work with David Cook II, Juan Migliore, Uwe Nagel)

This work [1] was inspired by a paper of Roberta Di Gennaro, Giovanna Ilardi and Jean Valles [2]. This talk is first in a sequence of four talks on [1] by the authors.

Conceptual Question: When do general fat points in the plane fail to impose independent conditions on forms of degree t?

Goal: Classify all triples $\Lambda = (t, X = \sum_i m_i p_i, V)$ where

$$V \subseteq (k[\mathbf{P}^2])_t,$$

$$p_i \in \mathbf{P}^2 \text{ are general, and}$$

$$I(X) = \cap_i (I(p_i)^{m_i}),$$

such that

(*) $$\dim[I(X) \cap V]_t > \max(0, \dim V - \textstyle\sum_i \binom{m_i+1}{2})$$

Example 1. $V = (k[\mathbf{P}^2])_t$: Here the SHGH Conjecture gives a conjectural value for $\dim(I(X)_t)$ for all X and t and hence in principle it gives a classification of all $\Lambda = (t, X = \sum_i m_i p_i, V = (k[\mathbf{P}^2])_t)$.

Example 2. $V = (Z)_t$ for some $Z = \sum a_j q_j$, $q_j \in \mathbf{P}^2$:

Case (a): $Z = 0 \Rightarrow$ same as Example 1.

Case (b): q_j general: SHGH converts this case to a solvable numerical problem.

Case (c): $Z = \sum_j q_j$ reduced, $X = mp$ general, $t = m + 1$:

Theorem (CHMN): (*) holds for $\Lambda = (t, X, V) = (m+1, mp, I(Z)_{m+1})$ for some m and general $p \in \mathbf{P}^2$ if and only if $m_Z < t_Z$, in which case $m_Z \le m < u_Z$, where:
m_Z is the least d such that $I(dp+Z)_{d+1} \neq 0$;
t_Z is the least d such that $[I(dp+Z)]_{d+1}$ is expected to be nonzero
(i.e., such that $\dim[I(Z)]_{d+1} > \binom{d+1}{2}$); and
u_Z is the least d such that $dp+Z$ imposes independent conditions on $[k[\mathbf{P}^2]_{d+1}]$
(i.e., such that $\dim[I(dp+Z)]_{d+1} = \binom{d+3}{2} - \binom{d+1}{2}$).

Note: t_Z is typically easy to compute. Although in principle m_Z and u_Z can also be computed using software, the obvious way of doing the calculation is often too big to successfully complete. But there is an alternate approach based on work of Faenzi and Valles [3], and Di Gennaro, Ilardi and Valles [2] relating m_Z to the arrangement of lines dual to Z.

Given $Z = q_1 + \cdots + q_s$; we have the linear forms L_j dual to the points q_j and hence $F = L_1 \cdots L_s$ and the sheafification $\tilde{J}$ of the Jacobian ideal $J = (F_x, F_y, F_z)$ and the exact sequence

$$0 \to \mathcal{D} \to \mathcal{O}^3_{\mathbf{P}^2} \to \tilde{J}(s-1) \to 0.$$

Then (over $\mathbf{C}$) the restriction $\mathcal{D}|_L$ of $\mathcal{D}$ to a general line L splits as $\mathcal{O}_{\mathbb{P}^1}(-m_Z) \oplus \mathcal{O}_{\mathbb{P}^1}(-u_Z - 1)$.

Simplest Example of the theorem: Here Z is the 7 points of the Fano plane and $X = 2p$ for a general point p of the plane. The space of cubic forms vanishing on Z had dimension 3. There should be no cubic curve singular at a general point p containing Z. But in fact there is an unexpected cubic curve C_p containing Z and singular at p for every point p. We have many other examples of Z with such unexpected curves.

Question: Is it possible to be more specific about the point sets Z that arise this way?

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Brill-noether problems for vector bundles

Jack Huizenga
(joint work with Izzet Coskun)

Let (X, H) be a smoth polarized surface. We write $M_H(\mathbf{v})$ for the moduli space of H-Gieseker semistable sheaves on X with Chern character $\mathbf{v}$. The spaces $M_H(\mathbf{v})$ are often well-behaved, and for example irreducible. The main question we consider is the following.

Problem (The weak Brill-Noether problem)**.** *Let $\mathcal{E} \in M_H(\mathbf{v})$ be a general sheaf. What is the cohomology of $\mathcal{E}$? In particular, when does $\mathcal{E}$ only have nonzero cohomology in at most one degree? If $\chi(\mathbf{v}) = 0$, does $\mathcal{E}$ have no nonzero cohomology?*

Questions of this sort are important in the study of the birational geometry of Hilbert schemes of points and moduli spaces of sheaves. Most interesting constructions of effective divisors on these spaces stem from the existence of an effective *theta divisor* Θ on a moduli space $M_H(\mathbf{v})$ with $\chi(\mathbf{v}) = 0$. If the general $\mathcal{E} \in M_H(\mathbf{v})$ has no cohomology and every $\mathcal{E}$ has $H^2(\mathcal{E}) = 0$, then the locus of sheaves with a section is an interesting effective divisor Θ.

This problem also naturally generalizes questions about the cohomology of line bundles on surfaces. For example, the Segre-Harbourne-Gimigliano-Hirschowitz conjecture [6, 4, 2, 5] claims to compute the cohomology of line bundles on general blowups X of $\mathbb{P}^2$. A line bundle L on X will have unexpected cohomology precisely if there is a (-1)-curve C on X such that $C.L < -1$.

The problem in question has been solved in the case of $\mathbb{P}^2$ by Göttsche and Hirschowitz.

Theorem (Göttsche-Hirschowitz [3])**.** *If $\mathbf{v}$ is a Chern character on $\mathbb{P}^2$ with rank $r \geq 2$, then the general sheaf $\mathcal{E} \in M(\mathbf{v})$ has nonzero cohomology in at most one degree.*

Furthermore, if $\chi(\mathbf{v}) = 0$ and $c_1(\mathbf{v}).H \geq 0$, then the general sheaf $\mathcal{E} \in M(\mathbf{v})$ has no cohomology and admits a resolution of the form

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus a} \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus b} \to \mathcal{E} \to 0$$

Observe that any sheaf $\mathcal{E}$ with a resolution as above automatically has no cohomology since the line bundles $\mathcal{O}_{\mathbb{P}^2}(-2)$ and $\mathcal{O}_{\mathbb{P}^2}(-1)$ have no cohomology. Thus, one way to prove cohomology vanishing results is to find nice resolutions that exhibit the vanishing. Note also that the hypothesis $r \geq 2$ in the theorem is necessary, since every sheaf in the moduli space containing sheaves of the form $I_p(-3)$ for $p \in \mathbb{P}^2$ has nonzero first and second cohomology.

For the rest of the abstract, let $X = \mathbb{F}_e$ be the Hirzebruch surface

$$\mathbb{F}_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(e)).$$

The Picard group is generated by classes E and F where $E^2 = -e$, $E \cdot F = 1$, and $F^2 = 0$. We have $K_X = -2E - (e+2)F$. First consider the cohomology of a line bundle $\mathcal{O}_X(aE + bF)$. By Serre duality, we may as well assume $a \geq -1$. It is then straightforward to show that $\mathcal{O}_X(aE+bF)$ has cohomology in at most one degree if and only if

$$(aE+bF).E \geq -1.$$

Our main theorem generalizes the Gottsche-Hirschowitz theorem to the Hirzebruch surface X. We write

$$\nu(\mathcal{E}) = \frac{c_1(\mathcal{E})}{r(\mathcal{E})} = aE + bF,$$

where $a, b \in \mathbb{Q}$. Notice that for a line bundle, $\nu(\mathcal{O}_X(aE+bF)) = aE+bF$.

Theorem (Coskun-H. [1])**.** *Let (X,H) be a Hirzebruch surface with an ample polarization H. Suppose $\mathbf{v}$ is the Chern character of an H-semistable sheaf $\mathcal{E}$, and suppose $\chi(\mathbf{v}) = 0$ and $a \geq -1$, where $\nu(\mathbf{v}) = aE+bF$. If*

$$\nu(\mathbf{v}).E \geq -1,$$

then the general $\mathcal{E} \in M_H(\mathbf{v})$ has no cohomology. Furthermore, in this case $\mathcal{E}$ admits a resolution of the form

$$0 \to \mathcal{O}_X(-E-(e+1)F)^a \to \mathcal{O}_X(-E-eF)^b \oplus \mathcal{O}_X(-F)^c \to \mathcal{E} \to 0.$$

Note that the condition $\nu(\mathbf{v}).E \geq -1$ can be rephrased as $\chi(\mathcal{O}_X(E), \mathcal{E}) \leq 0$. Thus the natural expectation is that if $\nu(\mathbf{v}).E < -1$ then there is a nonzero map $\mathcal{O}_X(E) \to \mathcal{E}$. Composing with the map $\mathcal{O}_X \to \mathcal{O}_X(E)$ then produces a section of $\mathcal{E}$. Making this rigorous proves one direction of the theorem. In the other direction, one shows a general $\mathcal{E}$ has a resolution as above, and thus that the cohomology vanishing holds. When the general sheaf has unexpected cohomology, it can be attributed to the bad curve E, just as in the case of line bundles.

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Laplace equations, Lefschetz properties and arrangements, I

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(joint work with Roberta Di Gennaro, Jean Vallès)

The classical problem of varieties satisfying Laplace equations is brought back to the modern theory of artinian ideals failing Lefschetz properties and both lead to line arrangements.

Let us recall some basic definition.
Let $X \subset \mathbb{P}^N$ be a projective n-dimensional complex variety.

Definition 1. *For $m \geq 1$, the m-th osculating space to X at a general point P, $T_P^m(X)$, is the subspace of $\mathbb{P}^N$ spanned by P and by all the derivative points of degree less than or equal to m of a local parametrization of X, evaluated at P. Of course, for $m = 1$ we get the tangent space $T_P(X)$.*

We remark that the expected dimension of the m-th osculating space is

$$\text{expdim}T_P^m(X) = \inf(\binom{n+m}{n} - 1, N).$$

Definition 2. *A n-dimensional variety $X \subset \mathbb{P}^N$ satisfies δ independent Laplace equations of order m if the m-th osculating space at a general point has*

$$\dim T_P^m(X) = \textit{expdim}T_P^m(X) - \delta.$$

If $N < \binom{n+m}{n} - 1$, then there are always $\binom{n+m}{n} - 1 - N$ relations between the partial derivatives. We call these relations "trivial" Laplace equations of order m.

Among the varieties satisfying non trivial Laplace equations, we are interested in rational varieties which are suitable projections of a Veronese variety.
For any vector space V, $V^* = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ will be the dual space. Let $R_d = \text{H}^0(\mathscr{O}_{\mathbb{P}^n}(d))$ be the complex vector space of the degree d forms of $\mathbb{P}^n$ and r_d its dimension. Let $v_t : [L] \in \mathbb{P}(R_1^*) \hookrightarrow [L^t] \in \mathbb{P}(R_t^*)$ be the t-uple Veronese embedding and the image $v_t(\mathbb{P}^n)$ is the Veronese n-fold of order t.
Let $I = (F_1, \dots, F_r)$ be an ideal generated by r forms of degree d and I_d be the vector subspace of R_d generated by the $F_1, \dots, F_r$ and $I_{d+i} = R_iF_1 + \cdots + R_iF_r$, for any $i \geq 0$.

Definition 3. *The apolar space of I in degree $d + i$ is*

$$I_{d+i}^{\perp} = \{\delta \in R_{d+i}^* \,|\, \delta(F) = 0, \ \forall F \in I_{d+i}\},$$

where the canonical basis of R_d^* is given by the $r_d = \binom{d+n}{n}$ derivations $\frac{\partial^d}{\partial x_0^{i_0} \dots \partial x_n^{i_n}}$ with $i_0 + \ldots + i_n = d + i$.

There is the exact sequence of vector spaces

$$0 \longrightarrow I_{d+i}^{\perp} \longrightarrow R_{d+i}^* \longrightarrow I_{d+i}^* \longrightarrow 0$$

and the corresponding projection map

$$\pi_{I_{d+i}} : \mathbb{P}(R_{d+i}^*) \setminus \mathbb{P}(I_{d+i}^*) \to \mathbb{P}(I_{d+i}^{\perp})$$

Of course one can identify $R_{d+i}/I_{d+i} \simeq (I^{\perp}_{d+i})^*$ and write the decomposition $R_{d+i} = I_{d+i} \oplus (I^{\perp}_{d+i})^*$. If I is an homogeneous artinian ideal, we consider $X = \pi_{I_t}(v_t(\mathbb{P}^n))$.

Remark. *The toric case is the easiest: when I_d is generated by r monomials of degree d, $(I^{\perp}_d)^*$ is generated by the remaining $r_d - r$ monomials.*

The main example is the classical Togliatti surface $\pi_{I_3}(v_3(\mathbb{P}^2))$ with center $I_3 = (x^3, y^3, z^3, xyz)$. This surface verifies exactly one equation of second order [4].
This example has been generalized with the notion of *Togliatti systems* (—, 2006) i.e. ideals such that by projection from suitable I_t we obtain varieties verifying Laplace equations, and recently Mezzetti, Michalek, Mirò-Roig and Ottaviani resume the study of this topic.

Definition 4. *An artinian ideal I is said to fail the Weak Lefschetz Property (WLP) in degree t if for any linear form L there exist t such that the multiplication map $\times L : \left(\frac{R}{I}\right)_t \to \left(\frac{R}{I}\right)_{t+1}$, has not maximal rank.*

For example the ideal $I = (x^3, y^3, z^3, xyz)$ fails the WLP in degree 2. This is no accident, in fact Mezzetti,Miró-Roig, Ottaviani proved the equivalence between rational varieties satisfying Laplace equations of order $d-1$ and artinian ideals failing WLP in degree $d-1$:

Theorem 1. [3] *Let $I \subset R$ be an artinian ideal generated by $r \leq \binom{d+n-1}{n-1}$ forms $F_1, \ldots, F_r$ of degree d. Then the following conditions are equivalent:*

1. *The ideal I fails the WLP in degree $d-1$,*
2. *The homogeneous forms $F_1, \ldots, F_r$ become linearly dependent on a general hyperplane of $\mathbb{P}^n$,*
3. *The n-dimensional variety $\pi_{I_d}(v_d(\mathbb{P}^n))$ satisfies at least one Laplace equation of order $d-1$.*

The hypothesis $r \leq \binom{d+n-1}{n-1}$ ensures that there is no trivial equation.
We generalize this result to ideals failing strong Lefschetz property.

Definition 5. *The artinian ideal I is said to fail the Strong Lefschetz Property (SLP) at range k and degree t by δ if for any linear form L the multiplication map $\times L^k : \left(\frac{R}{I}\right)_t \to \left(\frac{R}{I}\right)_{t+k}$, has rank $\min\{H_{R/I}(t), H_{R/I}(t+k)\} - \delta$ (so it has not maximal rank.)*

For $k = 1$ we recover failing WLP.
Let $i \geq 0$, $k \geq 1$; $N(r, i, k, d) := r(r_i - r_{i-k}) - (r_{d+i} - r_{d+i-k})$, $N^+ = \sup(0, N(r, i, k, d))$ and $N^- = \sup(0, -N(r, i, k, d))$.

Theorem 2. [2, Theorem 5.1] *Let $I = (F_1, \ldots, F_r) \subset R$ be an artinian ideal generated by r homogeneous polynomials of degree d. Let i, k, δ be integers such that $i \geq 0$, $k \geq 1$. Assume that there is no syzygy of degree i among the F_j's. The following conditions are equivalent:*

(1) *The ideal I fails the SLP at the range k in degree $d+i-k$.*
(2) *There exist $N^+ + \delta$, with $\delta \geq 1$, independent vectors $(G_{1j},\dots, G_{rj})_{j=1,\dots,N^++\delta} \in R_i^{\oplus r}$ and $N^+ + \delta$ forms $G_j \in R_{d+i-k}$ such that $G_{1j}F_1 + \ldots + G_{rj}F_r = L^k G_j$ for a general linear form L of $\mathbb{P}^n$.*
(3) *The n-dimensional variety $\pi_{I_{d+i}}(v_{d+i}(\mathbb{P}^n))$ satisfies $\delta \geq 1$ Laplace equations of order $d+i-k$.*
(4) *For any $L \in R_1$, $\dim_{\mathbb{C}}((I^{\perp}_{d+i})^* \cap \mathrm{H}^0(\mathcal{I}_{L^\vee}^{d+i-k+1}(d+i)) = N^- + \delta$, with $\delta \geq 1$.*

Remark. Another interesting case in which we know the apolar of the ideal I is when $I = (L_1^d, \dots, L_r^d)$ where $[L_i] \in \mathbb{P}(R_1^*)$; in this case $(I_d^{\perp})^*$ is generated by degree d polynomials that vanish at the points $[L_i^\vee] \in \mathbb{P}(R_1)$. This allows us to link this theory to line arrangements:
Let $\mathcal{A} = L_1 \cdots L_r$ be a line arrangement in $\mathbb{P}^2$ and $Z = \{[L_1], \dots, [L_r]\}$ be the reduced scheme of points in the dual space. If $I = (L_1^d, \dots, L_r^d)$ then $I_d^{\perp} = \mathrm{H}^0(\mathcal{J}_Z(d))$ and the singular curves in Theorem 2 (4) are curves through Z: $(I^{\perp}_{d+i})^* \cap \mathrm{H}^0(\mathscr{I}_{L^\vee}^{d+i-k+1}(d+i)) = \mathrm{H}^0(\mathscr{I}_Z \otimes \mathcal{J}_{L^\vee}^{d+i-k+1})(d+i)$

Since in this case, but also for other interesting questions, the ideals are generated by powers of linear forms, we note that in this situation the hypothesis on the global syzygy in Theorem 2 is not so restrictive.

Lemma 1. *Let I be the ideal $(L_1^d, \dots, L_r^d)$ where the L_j are linear forms and $r < r_d$. Then, if $i > 0$ there is no syzygy in degree i if and only if $rr_i \leq r_{d+i}$.*

When we are interested in many generators and mainly in linear dependance (i.e. 0-syzygies) (as [1] inspired us) we are proving a generalization of Theorem 2 that we announce here.
As we note that if there are no syzygy in degree i then $N(r,i,k,d) = H_{R/I}(d+i-k) - H_{R/I}(d+i)$, we redefine this number when there are syzygies of degree i and there is no syzygy of degree $i-k$.

Definition 6. *Let $I = (F_1, \dots, F_r) \subset R$ be an artinian ideal generated by r forms of degree d such that there are s syzygies linearly independent of degree i among the F_j's and there is no syzygy of degree $i-k$. We denote*

$$N(r,i,k,d) := r(r_i - r_{i-k}) - s - (r_{d+i} - r_{d+i-k})$$

$$N^+ = \sup(0, N(r,i,k,d)) \text{ and } N^- = \sup(0, -N(r,i,k,d)).$$

Note that again $N(r,i,k,d) = H_{R/I}(d+i-k) - H_{R/I}(d+i)$.

Theorem 3. *Let $I = (F_1, \dots, F_r) \subset R$ be an artinian ideal generated by r forms of degree d. Let $i \geq 0, k, \delta \geq 1$ and assume that there is no syzygy of degree $i-k$ among the F_j's. The following conditions are equivalent:*

(1) *The ideal I fails the SLP at the range k in degree $d+i-k$ by δ.*
(2) *There exist $N^+ + \delta$, with $\delta \geq 1$, independent vectors $(G_{1j},\dots, G_{rj})_{j=1,\dots,N^++\delta} \in R_i^{\oplus r}$ and $N^+ + \delta$ forms $G_j \in R_{d+i-k}$ such that $G_{1j}F_1 + \ldots + G_{rj}F_r = L^k G_j$ for a general linear form L of $\mathbb{P}^n$.*

(3) *The n-dimensional variety $\pi_{I_{d+i}}(v_{d+i}(\mathbb{P}^n))$ satisfies $\delta \geq 1$ Laplace equations of order $d+i-k$.*
(4) *For any $L \in R_1$, $\dim_{\mathbb{C}}((I^{\perp}_{d+i})^* \cap \mathrm{H}^0(\mathscr{I}_{L^\vee}^{d+i-k+1}(d+i)) = N^- + \delta$, with $\delta \geq 1$.*

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Newton–Okounkov bodies, construction of singular divisors, and higher syzygies on abelian surfaces

Alex Küronya
(joint work with Victor Lozovanu)

Let X be a smooth projective surface over the complex numbers, L an ample line bundle on X. Typically no harm is done if we assume L to be very ample, at the same time, no attempts regarding the minimality of our assumptions are made.

The section ring

$$R(X,L) = \bigoplus_{m=0}^{\infty} H^0(X, L^{\otimes m})$$

is a fundamental invariants of the polarized variety (X,L), which, in the case of a very ample polarization is simply the homogeneous coordinate ring of X with respect to the embedding arising from L. The algebraic properties of $R(X,L)$ are best studied in the category of modules over the ring

$$S = \mathrm{Sym}^{\bullet} H^0(X,L) \ ,$$

then $R(X,L)$ admits a minimal free graded resolution of the form

$$E_{\bullet} \ : \quad \ldots \to \bigoplus_{r_1} S(-a_{1,j}) \to S \oplus \bigoplus_{r_0} S(-a_{0,j}) \to R(X,L) \to 0 \ .$$

Following Green and Lazarsfeld [8] we ask for the first $p \geq 0$ terms in $E_{\bullet}$ to be as simple as possible; more precisely, we say that L *satisfies property* N_p, if $E_0 = S$, and

$$a_{i,j} = i+1 \quad \text{for all } j \geq 1 \text{ whenever } 1 \leq i \leq p.$$

As an illustration [17],

$$(N_0) \quad \Leftrightarrow \quad \phi_L \text{ defines a projectively normal embedding,}$$

and

$$(N_1) \quad \Leftrightarrow \quad (N_0) \text{ and } I_{X/\mathbb{P}} \text{ is generated by quadrics.}$$

A very large amount of work has gone into the area, with some of the highlights due to Bertram, Green, Ein, Lazarsfeld, and Voising; as a consequence, the case of curves is reasonably well understood. On the other hand, if $\dim X > 1$, then our knowledge is very limited [5, 1].

In the case of abelian varieties, however, the situation appears to be under control. Verifying a conjecture of Lazarsfeld, Pareschi [19] proved that if (X, L) is a polarized abelian variety, then $L^{\otimes p+3}$ will have property (N_p). Observe that the philosophy behing the conjecture was that syzygies on abelian varieties are precisely as on elliptic curves. It turns out however, that in dimension two (and probably in higher dimensions as well) one can obtain a lot more precise, in some sense optimal, statement.

Theorem 1 ([12]). *Let (X, L) be a polarized abelian surface, p a natural number, and assume that $(L^2) \geq 5(p+2)^2$. Then the following are equivalent*

(1) *X does* not *contain a smooth curve C with $(C^2) = 0$ and $1 \leq (L \cdot C) \leq p+2$;*
(2) *L satisfies property (N_p).*

This way, we obtain an equivalent geometric condition for property (N_p) very similar in flavour to Reider's theorem [20] for global generation and very ampleness. Both Reider and Pareschi rely primarily on vector bundle techniques, part of which we will replace by infinitesimal Newton–Okounkov polygons [10, 11] that appear to enable a finer analysis. The proof of the direction (2) ⇒ (1) uses standard methods, notably restricting linear syzygies along the lines of Eisenbud–Green–Hulek–Popescu [4], which is not the topic of this talk.

The strategy of the much more difficult direction (1) ⇒ (2) can be summarized as follows.

(1) On abelian varieties Green and Inamdar [6, 9] reduced verifying property (N_p) to checking the vanishing of the cohomology groups
$$H^i(X^{\times(p+2)}, \times^{p+2} L \otimes I_\Sigma) = 0 \quad \text{for all } i \geq 1,$$
where $\Sigma \subseteq X^{\times(p+2)}$ is a certain union of partial diagonals.
(2) Lazarsfeld–Pareschi–Popa [16] reduced the above vanishing to the construction of divisors with given numerics and given singularities (expressed in terms of multiplier ideals).
(3) We will then use infinitesimal Newton–Okounkov bodies for exhibiting a divisor with the required properties.

The essential contribution of [16] can be formulated as follows.

Theorem 2. [16] *With X, L, and p as above, assume that there exists an effective $\mathbb{Q}$-divisor F_0 such that*

(1) *$F_0 \equiv \frac{1-c}{p+2} L$ for some $0 < c \ll 1$,*
(2) *$\mathcal{J}(X; F_0) = I_{o/X}$, where the latter is the ideal sheaf of the origin.*

Then L has property (N_p).

Lazarsfeld–Pareschi–Popa then use a two-lie genericity argument to conclude

Theorem 3. [16] *Let (X, L) be a polarized abelian variety of dimension n. If*

$$\epsilon(X, L) > (p+2)n \ ,$$

then L satisfies (N_p).

Our argument builds on Theorem 2, but replaces the genericity argument by a somewhat delicate argument using convex geometry when it comes to the construction of singular divisors.

Constructing effective divisors with prescribed singularities is one of the most powerful techniques in higher-dimensional geometry. The method can be used to prove the 'traditional' main theorems of the minimal model program (non-vanishing, base-point freeness, rationality, cone and contraction theorem) along with major positivity results due to Angehrn–Siu and Kollár–Matsusaka.

The power of the method of effective divisors with prescribed singularities comes from the fact that it is a way to extend global sections of adjoint line bundles. At its simplest, the idea goes as follows: let X be a smooth projective variety, $Z \subseteq X$ a subvariety (Z could be a point for instance), L an ample line bundle on X. Then we can consider the restriction map

$$H^0(X, \omega_X \otimes L) \xrightarrow{\mathrm{res}^X_Z} H^0(Z, \omega_X \otimes L|_Z) \ ,$$

ideally we want res^X_Z to be surjective.

Since L is ample, Kodaira vanishing implies that

$$\mathrm{res}^X_Z \text{is surjective} \quad \Leftrightarrow \quad H^1(X, \omega_X \otimes L \otimes I_{Z/X}) = 0 \ .$$

Assuming that Z can be realized as the zero scheme of the multiplier ideal $\mathcal{J}(X; D)$ an effective $\mathbb{Q}$-divisor D on X (and this is a very big 'if', since the problem is extremely difficult), Nadel's vanishing theorem will do the trick.

Theorem 4 (Nadel vanishing)**.** *Let X be a smooth projective variety, L an integral divisor, D an effective $\mathbb{Q}$-divisor on X such that $L - D$ is ample. Then*

$$H^i(X, \omega_X \otimes L \otimes \mathcal{J}(X; D)) = 0 \quad \text{for all } i \geq 1.$$

The difficulty lies in guaranteeing the two conflicting conditions $L - D$ ample and $\mathcal{J}(X; D) = I_{Z/X}$ at the same time. In what follows we will care about the situation when X is a smooth projective surface and Z a point in X. The next statement is our main technical results.

Theorem 5. [12] *Let (X, L) be a smooth polarized surface, p a natural number such that $(L^2) \geq 5(p+2)^2$, let in addition be $x \in X$ be a very general point. Assume that there does* not *exist an irreducible curve $C \subseteq X$ smooth at x with $1 \leq (L \cdot C) \leq p+2$. Write $\pi \colon Y \to X$ for the blowing-up of $x \in X$ with exceptional divisor D, and $B = \frac{1}{p+2}L$.*

The the following hold.

(1) *There exists $z \in E$ such that*

$$\text{length } \Delta_{(E,z)}(\pi^* B) \cap \{2\} \times \mathbb{R}) > 1 \ ;$$

(2) *there exists an effective* $\mathbb{Q}$*-divisor* $D \equiv (1-c)B$ *for some* $0 < c \ll 1$ *such that*

$$\mathcal{J}(X; D)|_{\mathcal{U}} = I_{Z/X}|_{\mathcal{U}}$$

over an open neighbourhood $x \in \mathcal{U} \subseteq X$.

The essential novelty of our method lies in the application (1) $\Rightarrow$ (2). The reason why one obtains much more precise results on abelian surfaces is that for all intents and purposes the origin $o \in X$ behaves like a very general point.

The object $\Delta_{(E,z)}(\pi^* B) \subseteq \mathcal{R}^2$ is an infinitesimal Newton–Okounkov body: with notation as above, a Newton–Okounkov body $\Delta_{Y_\bullet}(\pi^* D)$ on a surface is called *infinitesimal* (over the point $x \in X$), if the curve in the admissible flag is the exceptional divisor of the blow-up of $x \in X$. As it was proven in [14], Newton–Okounkov bodies of complete linear series in dimension two are always polygons, hence so our infinitesimal Newton–Okounkov bodies. For further references on the projective theory of Newton–Okounkov bodies we refer the reader to [15, 14, 10, 13, 11, 12].

As it turns out, infinitesimal Newton–Okounkov bodies are quite an apt tool to detect local positivity. In order to be able to more precise about it, let us define

$$\Delta_t^{-1} = \{(x,y) \mid x, y \geq 0\,,\, x \geq y\}$$

for a positive real number t. With this notation the characterization of local positivity takes the following shape.

Theorem 6. [10] *Let* L *be a big line bundle on a smooth projective variety* X. *Then*

(1) $x \notin \mathbb{B}_+(L) \Leftrightarrow$ *there exists* $t > 0$ *and* $z \in E$ *such that* $\Delta_t^{-1} \subseteq \Delta_{(E,z)}(\pi^* L)$.

(2) *If* L *is ample, then* $\Delta_{\epsilon(\|L\|;x)} \subseteq \Delta_{(E,z)}(\pi^* L) \subseteq \Delta^{-1}_{\mu(L;x)}$ *for all points* $z \in E$.

The natural idea is then to try and use area comparisons between various convex sets to obtain non-trivial inequalities. Indeed, this leads to the construction of interesting divisors.

Theorem 7. [12] *Set* $\Lambda = \{(t,y) \in \mathbb{R}^2 \mid t \geq 2\,,\, y \geq 0\,,\, t \geq 2y\}$, *and let* B *be an ample* $\mathbb{Q}$*-divisor on a smooth projective surface* X, $x \in X$ *an arbitrary point. If*

$$\mathrm{int}\Delta_{(E,z)}(\pi^* B) \cap \Lambda \neq \emptyset \quad \text{for all } z \in E,$$

then there exists an effective $\mathbb{Q}$*-divisor* D *on* X *with* $D \equiv (1-c)B$ *for some* $0 < c \ll 1$ *such that*

$$\mathcal{J}(X; D)|_{\mathcal{U}} = I_{Z/X}|_{\mathcal{U}}$$

over an open neighbourhood $x \in \mathcal{U} \subseteq X$.

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Unexpected curves and line arrangements

Juan Migliore

(joint work with David Cook II, Brian Harbourne, Uwe Nagel)

Inspired by the recent paper [1], we propose a generalization of the problem of finding the number of independent conditions imposed on a linear system of curves in $\mathbb{P}^2$ of given degree by a general collection, X, of fat points. (That is, we require not only that the curves pass through a general set of points, but we also fix the multiplicity with which they pass.) A great deal of work has been done in the case that the linear system is complete, culminating in the famous SHGH Conjecture.

Instead we propose that we start with a smaller linear system, namely those containing a fixed set of points Z (with or without also requiring multiple passage through these points), and again impose the passage through the general set, X, of fat points as above. Now there is an important new feature, namely the geometry of Z itself. In our paper we worked out the first non-trivial case. Specifically, we studied the situation when X consists of only one fat point, and the multiplicity of passage through X is one less than the degree of the curves in the linear system. Our focus was to describe which Z allow X *not* to impose independent conditions, and to describe the curves that arise unexpectedly in this way.

The four authors were all at this workshop, and they presented different aspects of this paper. The current talk focused on introducing the terminology and important invariants that we have introduced, and to state the main results that we have obtained. The paper also gives many new examples, and it gives connections not only to the SHGH conjecture but also to line arrangements, to certain Lefschetz properties, and to the famous Terao Conjecture; these were discussed in talks by the other authors.

We always work over an infinite field k, sometimes of characteristic zero. We will denote X by jP, where P is a general point and j is the desired multiplicity. The linear system of curves of degree j containing Z is the projectivization of $[I_Z]_j$, and the ultimate goal is to see when the dimension of $[I_{Z+jP}]_{j+1}$ is not the expected one. We say that Z has an *unexpected curve* if

$$\dim[I_{Z+jP}]_{j+1} > \max\{\dim[I_Z]_{j+1} - \binom{j+1}{2}, 0\}.$$

This means that jP does not impose the expected number of conditions on $[I_Z]_{j+1}$. Unexpected curves are the main object of study of our paper.

For any zero-dimensional scheme Z in $\mathbb{P}^2$ (although for us, Z is always reduced), we define the *multiplicity index* m_Z (previously introduced in [2]) as

$$m_Z = \min\{j \mid [I_{Z+jP}]_{j+1} \neq 0\}.$$

We say that Z is *generally special* if $Z + m_Z P$ does not impose independent conditions on the complete linear system of curves of degree $m_Z + 1$. We also define

$$t_Z = \min\left\{j \mid \dim[I_Z]_{j+1} - \binom{j+1}{2} > 0\right\}$$

(which depends only on the Hilbert function of Z) and the *specialty index*

$$u_Z = \min\{j \mid Z + jP \text{ imposes independent conditions on curves of degree } j+1\}.$$

It follows from the definition that Z is generally special if and only if $m_Z < u_Z$. If Z has an unexpected curve then it also follows from the definition that Z is generally special. The converse is not true.

One of our main results is the following.

Theorem 1. *Z has an unexpected curve if and only if $m_Z < t_Z$.*

Although Z being generally special does not by itself imply that Z has an unexpected curve, we have the following theorem.

Theorem 2. *Z has an unexpected curve if and only if both of the following two conditions hold:*

(a) *Z is generally special;*
(b) *$h_Z(t_Z) = |Z|$, where h_Z is the Hilbert function of R/I_Z. (Equivalently, $h^1(\mathcal{I}_Z(t_Z)) = 0$.)*

The following theorem summarizes several results in the paper. It shows how the relation between m_Z and u_Z is connected to the number of points of Z. It also shows that "most of the time" there is a unique curve containing Z and vanishing to order m_Z at a general point; an important question is to identify the curves among these that are unexpected. Furthermore, this result shows that $\dim[I_{Z+m_Z P}]_{m_Z+1}$ can only be equal to 1 or 2, and gives some idea of what happens in the latter case.

Theorem 3.

(a) $m_Z < u_Z$ (i.e. Z is generally special) if and only if $|Z| \geq 2m_Z + 3$.
(b) $m_Z = u_Z$ if and only if $|Z| = 2m_Z + 2$.
(c) $m_Z > u_Z$ if and only if $|Z| = 2m_Z + 1$.
(d) $\dim[I_{Z+m_Z P}]_{m_Z+1} = 1$ if and only if either (a) or (b) hold. (In particular, an unexpected curve of minimal degree is always unique.)
(e) $\dim[I_{Z+m_Z P}]_{m_Z+1} = 2$ if and only if (c) holds.

The following result is especially important in giving the connection to arrangements (described in another talk), but is of interest in itself.

Theorem 4. *$m_Z + u_Z = |Z| + 2$.*

In the paper we give a careful description of what unexpected curves can look like. In particular, they very often are not irreducible, in which case they have many lines as components. The following result shows that the case where the unexpected curve is irreducible is of special interest, since it avoids the need for (b) in Theorem 2.

Theorem 5. *If Z is generally special and does not consist of a set of points on a line (which is the case $m_Z = 0$), and if $[I_{Z+m_Z P}]_{m_Z+1}$ contains an irreducible curve then Z has an unexpected curve.*

This motivates a criterion for Z having an *irreducible* unexpected curve.

Proposition 6. *Assume that $|Z| \geq 2m_Z + 2$. (Equivalently, assume that $m_Z \leq u_Z$; equivalently, assume that* $\dim[I_{Z+m_Z P}]_{m_Z+1} = 1$*). Then $[I_{Z+m_Z P}]_{m_Z+1}$ contains an irreducible curve if and only if $m_{Z-P} = m_Z$ for all $Q \in Z$.*

An application of this criterion will be described in Cook's talk.

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Unexpected curves, line arrangements, and Lefschetz properties

Uwe Nagel
(joint work with David Cook II, Brian Harbourne, Juan Migliore)

Hilbert functions of fat point schemes have attracted a lot of attention. Indeed, they are related, for example, to interpolation problems, dimensions of secant varieties, and the geometry of rational surfaces. In the recent preprint [1], the focus is a particular instance of this problem, which can be viewed as a first step towards extending the framework of the SGHG-conjecture. This has been described in previous talks by Harbourne, Migliore, and Cook II. Here we want to discuss relations to the study of Lefschetz properties and to the theory of line arrangements.

Let $Z \subset \mathbb{P}^2 = \mathbb{P}^2_K$ be a set of $d > 0$ points, where K is a field of characteristic zero. The *multiplicity index* of Z is

$$m_Z = \min\{j \geq 0 : h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1)) > 0\},$$

where P is a general point of $\mathbb{P}^2$. It is well-defined and satisfies $m_Z \leq \frac{d-1}{2}$.

Identifying the dual projective plane with $\mathbb{P}^2$, each point $P_i \in Z$ corresponds to a line in $\mathbb{P}^2$ defined by a linear form $\ell_i \in R = K[x,y,z]$. Thus, the line arrangement dual to Z is $\mathcal{A} = \mathcal{A}(f)$, where $f = \ell_1 \cdots \ell_d$. The Jacobian ideal of f gives rise to the short exact sequence of graded R-modules

$$0 \to D(f) \to R^3 \to \operatorname{Jac}(f)(d-1) \to 0.$$

The syzygy module $D(f)$ is reflexive. Hence, for a general line $L \subset \mathbb{P}^2$, the line bundles appearing in the restrictions of its sheafification have constant Chern classes, that is, there are integers $a \leq b$ such that

$$\widetilde{D(f)}|_L \cong \mathcal{O}_L(-a) \oplus \mathcal{O}_L(-b).$$

A first connection between Z and $\mathcal{A}$ is given by $m_Z = a$ (see [3, Theorem 4.3]). We note in passing that b has been related to the speciality index u_Z of Z in [1, Corollary 5.3]. One has $b = u_Z + 1$.

If no three points of Z are collinear, it has been shown in [1] that Z does not have an unexpected curve and that the derivation bundle $\widetilde{D(f)}$ is semistable if $d \geq 3$. Moreover, unexpected behavior is related as follows: If Z admits an unexpected curve, then $b - a \geq 2$, and thus $\widetilde{D(f)}$ is not semistable (by the Grauert-Mülich Theorem). The converse is not true.

There is also a connection to a different type of unexpected behavior. Let $A = R/I$ be a graded artininan K-algebra. Then, for a general linear form $\ell \in R$, one often expects that multiplication by a power of ℓ has maximal rank. If $\times\ell^i : [A]_j \to [A]_{i+j}$ has maximal rank for all integers i and j, then one says that A has the *Strong Lefschetz Property.*

Example ([1])**.** Let Z be a set of $d = a + b + 1$ points, where $1 \leq a \leq b - 1$ and b of the points are collinear. Then Z does not admit an unexpected curve, and the dual arrangement $\mathcal{A} = \mathcal{A}(f)$ has splitting type (a, b). Thus, $\widetilde{D(f)}$ is not semistable. Moreover, for each integer j, the multiplication map

$$\times\ell^2 : [R/(\ell_1^{j+1}, \ldots, \ell_d^{j+1})]_{j-1} \to [R/(\ell_1^{j+1}, \ldots, \ell_d^{j+1})]_{j+1}$$

has maximal rank.

The last property is a consequence of the absence of an unexpected curve and the following result.

Theorem ([1])**.** A set Z has an unexpected curve of degree $j + 1$ if and only if

$$\times\ell^2 : [R/(\ell_1^{j+1}, \ldots, \ell_d^{j+1})]_{j-1} \to [R/(\ell_1^{j+1}, \ldots, \ell_d^{j+1})]_{j+1}.$$

does not have maximal rank.

Combined with further results of [1], this has the following consequence:

Corollary ([1])**.** If the splitting type of $\mathcal{A}$ satisfies $b - a \geq 2$ and the unique curve of degree $a + 1$ that contains $Z + aP$ is irreducible, then

$$\times\ell^2 : [R/(\ell_1^{a+1}, \ldots, \ell_d^{a+1})]_{a-1} \to [R/(\ell_1^{a+1}, \ldots, \ell_d^{a+1})]_{a+1}.$$

does not have maximal rank.

Now we relate the above results to Terao's conjecture. A line arrangement $\mathcal{A}(f)$ is said to be *free* if the module $D(f)$ is a free R-module. Terao conjectured that freeness of line arrangements is a combinatorial property, that is, freeness depends only on the intersection lattice induced by the lines of $\mathcal{A}$. Deciding this conjecture is one the main open problems in the theory of hyperplane arrangements.

The following result gives a first connection.

Proposition ([1])**.** Let $\mathcal{A}(f)$ and $\mathcal{A}(g)$ be two line arrangements with the same incidence lattice. Assume $\mathcal{A}(f)$ is free with splitting type (a, b). Then $\mathcal{A}(g)$ has splitting type of $\mathcal{A}(g)$ is $(a - s, b + s)$, where $s \geq 0$.

Moreover, $\mathcal{A}(g)$ is free if and only if $s = 0$.

Corollary ([1])**.** If the splitting type of a line arrangement is a combinatorial property, then Terao's conjecture is true.

We will see below that a weaker condition is already sufficient.

Building on work in [2], the above proposition is used to establish a criterion for Terao's conjecture.

Theorem ([1])**.** The following two conditions are equivalent:

(a) Terao's conjecture is true for line arrangements.

(b) If $\mathcal{A}(f)$ is any free line arrangement with splitting type (a, b), then, for every line arrangement $\mathcal{A}(g)$ with the same incidence lattice as $\mathcal{A}(f)$, the multiplication map

$$[R/J]_{b-2} \xrightarrow{\times \ell^2} [R/J]_b$$

is surjective, where $J = (\ell_1^b, \ldots, \ell_{a+b+1}^b, L_1^b, \ldots, L_{b-a}^b)$ with $g = \ell_1 \cdots \ell_{a+b+1}$ and general linear forms $\ell, L_1, \ldots, L_{b-a} \in R$.

A key ingredient of the proof is the following property of the multiplicity index, established in [1]. If $Q \in \mathbb{P}^2$ is a general point, then

$$m_{Z+Q} = \begin{cases} m_Z + 1 & \text{if } m_Z < \frac{d-1}{2}; \\ m_Z & \text{if } m_Z = \frac{d-1}{2}. \end{cases}$$

Interpreting the above theorem using multiplicity indices, one gets a sufficient condition for Tearo's conjecture.

Corollary ([1])**.** If, for sets of $2k+1$ points, having (maximal) multiplicity index $m_Z = k$ is a combinatorial property, then Terao's conjecture is true for line arrangements.

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Local Negativity, Curve Arragements and Harbourne constants

Piotr Pokora

In recent years there has been growing intrest in questions around the celebrated Bounded Negativity Conjecture [1, 3], which can be formulated as follows.

Conjecture 1 (BNC)**.** *Let X be a smooth complex projective surface. Then there exists an integer $b(X) \in \mathbb{Z}$ such that for every reduced curve $C \subset X$ one has $C^2 \geq -b(X)$.*

There are some surfaces for which the BNC is true, for instance surfaces with the $\mathbb{Q}$-effective anti-canonical divisor, K3 surfaces or Enriques surfaces. However, it is not clear whether the BNC holds for blow ups of the complex projective plane along $s \geq 10$ general points. It is easy to observe that depending on a surface $b(X)$ can be arbitrarily small. In order to observe this phenomenon one needs to consider the blow up of the complex projective plane along $s \gg 0$ points on a line ℓ and then the strict transform of ℓ has the self-intersection equal to $-s+1$. A natural approach to study the BNC on blow-ups of smooth projective surfaces is to define some asymptotic invariants. Here we focus only on a certain variation on the so-called Harbourne constants [2]. By transversal configuration of curves C we mean configurations having only transversal intersection points as singularities and we denote by $\mathrm{Sing}(C)$ the set of all singular points.

Definition 1. *Let X be a smooth projective surface and let $\mathcal{C} = \{C_1, ..., C_k\}$ be a transversal configuration of smooth curves on X. Denote by $C = \sum_{i=1}^{k} C_i$ the associated divisor. The rational number*

$$h(X;\mathcal{C}) = \frac{C^2 - \sum_{P \in \mathrm{Sing}(C)} (\mathrm{mult}_P(C))^2}{\#\mathrm{Sing}(\mathcal{C})}$$

is the Harbourne index of $\mathcal{C}$.

Our first result provides the following bound on Harboure indices for transversal configurations of smooth curves of degree $d \geq 2$ on the complex projective plane.

Theorem 1. *(*[10, Theorem B], [9, Theorem 3.2]*) Let $\mathcal{C} \subset \mathbb{P}^2_{\mathbb{C}}$ be a transversal configuration of $k \geq 4$ smooth curves of degree $d \geq 2$ such that there is no point such that all curves intersect simultaneously. Then*

$$h(\mathbb{P}^2_{\mathbb{C}};\mathcal{C}) \geq -4 - \frac{5}{2}d^2 + \frac{9}{2}d.$$

The proof of the above result is based on Hirzebruch's construction [4] and the Miyaoka-Yau inequality [7].

For a surface X we denote by K_X the canonical divisor and by $e(X)$ its Euler characteristic.

Theorem 2. *(*[6, Theorem 2.2]*) Let X be a smooth complex projective surfaces of non-negative Kodaira dimension and let $\mathcal{C} \subset X$ be a transversal configuration of smooth curves of genus g having n irreducible components. Denote by C the associated divisor to $\mathcal{C}$ and by s the number of all singular points of $\mathcal{C}$. Then*

$$h(X;C) \geq -4 + \frac{K_X^2 - 3e(X) + 2(1-g)n}{s}.$$

We have the following straightforward corollaries.

Corollary 1. *(*[8, Main Theorem 1.4]*) Let X be a smooth hypersurface in $\mathbb{P}^3_{\mathbb{C}}$ of degree $d \geq 4$ containing a configuration $\mathcal{L}$ of $n \geq 2$ lines. Denote by $s \geq 1$ the number of all singular points of $\mathcal{L}$. Then*

$$h(X;\mathcal{L}) \geq -4 + \frac{2n - 2d(d-1)^2}{s}.$$

Corollary 2. *(*[5, Theorem 2.2]*) Let X be a $K3$ or Enriques surface containing a configuration $\mathcal{C}$ of $n \geq 3$ smooth rational curves having only transversal intersection points. Denote by s the number of singular points of $\mathcal{C}$. Then*

$$h(X;\mathcal{C}) \geq -4 + \frac{2n - 72}{s}.$$

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Logarithmic vector fields and curve arrangements in $\mathbb{P}^2$

Hal Schenck

(joint work with Terao, Hiroaki; Yoshinaga, Masahiko)

Let $\mathcal{A} = \bigcup_{i=1}^r C_i \subseteq \mathbb{P}^2_{\mathbb{C}}$ be a collection of plane curves, such that each singular point of $\mathcal{A}$ is quasihomogeneous.

Definition 1. *A hypersurface singularity is quasihomogeneous if there is a holomorphic change of variables so the defining equation is weighted homogeneous.*

In [6], Saito showed that a singularity at $(0,0)$ of a plane curve C is quasihomogeneous iff the Milnor number

$$\mu_{(0,0)}(C) = \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$$

is equal to the Tjurina number of C at $(0,0)$

$$\tau_{(0,0)}(C) = \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f\rangle.$$

Let Y be a divisor on a complex manifold X. In [5], Saito initiated the study of the sheaf of logarithmic vector fields with pole along Y, defined locally as

$$Der(-\log Y)_p = \{\theta \in Der_{\mathbb{C}}(X) | \theta(f) \in \langle f \rangle\},$$

where $f \in \mathcal{O}_{X,p}$ is a local defining equation for Y at p. Saito's work generalized earlier work of Deligne [2], who studied the case for Y a normal crossing divisor.

Because $Der(-\log Y)_p$ is the kernel of the evaluation map $\theta \mapsto \theta(f) \in \mathcal{O}_{Y,p}$, there is a short exact sequence [3]

$$0 \longrightarrow Der(-\log\, Y) \longrightarrow \mathcal{T}_X \longrightarrow J_Y(Y) \longrightarrow 0.$$

In [9], Terao showed that when $\mathcal{A} = \bigcup H_i$ is an arrangement of hyperplanes, then there is a left exact sequence

$$0 \longrightarrow Der(-\log\, \mathcal{A}'(-1)) \longrightarrow Der(-\log\, \mathcal{A}) \longrightarrow Der(-\log\, \mathcal{A}|_H$$

where $\mathcal{A}' = \mathcal{A} \setminus H$. A hyperplane arrangement is called free when $Der(-\log\, \mathcal{A})$ is a direct sum of line bundles; the theorem above gives a way to approach the question inductively. Freeness is a topic of great interest due to Terao's theorem [10] connecting it to topological properties of the complement of the arrangement.

In [7], we generalize results of [8] on lines and conics to prove that if C is an irreducible curve having only quasihomogeneous singularities, such that $C \cap \mathcal{A} \subseteq C_{sm}$ and every singular point of $\mathcal{A} \cup C$ is quasihomogeneous, then there is a short exact sequence relating the $\mathcal{O}_{\mathbb{P}^2}$–module $Der(-\log \mathcal{A})$ of vector fields on $\mathbb{P}^2$ tangent to $\mathcal{A}$ to the module $Der(-\log \mathcal{A} \cup C)$. This gives an inductive tool for studying the splitting of the bundles $Der(-\log \mathcal{A})$ and $Der(-\log \mathcal{A} \cup C)$, depending on the geometry of the divisor $\mathcal{A}|_C$ on C.

Theorem 1. *Let $\mathcal{A} = \bigcup_{i=1}^r C_i$ and $\mathcal{A} \cup C$ be reduced curves in $\mathbb{P}^2$, with C_i and C smooth, such that all singular points of $\mathcal{A}$ and $\mathcal{A} \cup C$ are quasihomogeneous. Then*

$$0 \longrightarrow Der(-\log \mathcal{A})(-n) \longrightarrow Der(-\log \mathcal{A} \cup C) \longrightarrow \mathcal{O}_C(-K_C - R) \longrightarrow 0$$

is exact, where $R = (\mathcal{A} \cap C)_{red}$ is the reduced scheme of $C \cap \mathcal{A}$ and $\deg(C) = n$.

Example 1. *The arrangement A_3 has $Der(-log\, \mathrm{A}_3) \simeq \mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(-3)$* [9].

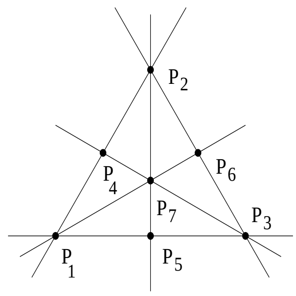

FIGURE 1. The A_3-arrangement

The family of homogeneous cubic polynomials vanishing at the seven singular points of A_3 is parameterized by a $\mathbb{P}^2$; a generic member C of this family is smooth. A computer calculation shows $\mathrm{A}_3 \cup C$ has quasihomogeneous singularities. By Theorem 1, $Der(-log\, \mathrm{A}_3 \cup C)$ is free, isomorphic to $\mathcal{O}_{\mathbb{P}^2}(-4)^2$.

Question 1. *Liao's work* [3] *relates Chern classes of logarithmic vector fields to the Chern-Schwartz-MacPherson class of the complement, and Aluffi* [1] *relates this to the Segre class of the Jacobian scheme. Is it possible to prove a higher dimensional version of Theorem 1 using these methods?*

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On the boundedness of Zariski denominators

David Schmitz

(joint work with Thomas Bauer, Piotr Pokora)

A classical result of Zariski's ([3]) states that for any effective divisor D on a smooth projective surface X there exist effective $\mathbb{Q}$-divisors P and N with $D = P + N$ such that

- P is nef,
- either $N = 0$ or N has negative definite intersection matrix, and
- $PN_i = 0$ for each irreducible component N_i of N.

For geometric use the fact that P and N have rational coefficients is somewhat unpleasant, since in applications one often has to pass to a multiple of D. For instance, for integral mP one finds that the global sections of mD in fact come from mP. It is therefore natural to ask the following

Question 1. *Does there exist a number m depending only on the surface X such that for any effective divisor D on X the Zariski decomposition of mD is integral?*

We observe in [2] that this problem is governed by the possible negative self–intersections of irreducible curves. In fact, we prove that somewhat surprisingly boundedness of the denominators is equivalent to the following long-standing folklore conjecture

Conjecture (Bounded Negativity Conjecture)**.** *Let X be a smooth complex projective surface. Then there exists a number $b(X)$ such that for any irreducible curve C on X the self-intersection C^2 is at least $-b(X)$.*

Note that in positive characteristic this is known to be false (see for example [1, Sect. 2]). We prove the equivalence of bounded negativity and boundedness of Zariski denominators by providing the following bounds.

Theorem 1. *Let X be a smooth projective surface with Picard number ρ. Assume that the self-intersection of irreducible curves on X is bounded from below by $-b$. Then for any denominator d in the Zariski decomposition of an effective divisor we have*

$$d \leq b^{\rho-1}.$$

Theorem 2. *Let X be a smooth projective surface and denote by Δ the discriminant of the Néron-Severi lattice $N^1(X)$. Suppose Zariski denominators of effective divisors on X are bounded from above by d. Then for any irreducible curve C on X,*

$$C^2 \geq -d \cdot d! \cdot |\Delta|.$$

Together with the observation that Bounded Negativity fails in positive characteristic, the above theorem implies that Zariski denominators will be unbounded in those cases, too. In fact it is not hard to produce a sequence of effective divisors whose Zariski denominators tend to infinity from the example mentioned above.

A word about the idea of the proofs: for Theorem 1 we first proves that for an effective divisor D whose negative part has intersection matrix $S \in \mathbb{Z}^{k\times k}$, the denominators of coefficients in the Zariski decomposition divide $\det(S)$. The result then can be deduced from the inequality

$$|\det(S)| \leq \left(\frac{|\operatorname{tr}(S)|}{k}\right)^k$$

together with the fact that the trace of S is the sum of self-intersections of k negative curves, with $k \leq \rho - 1$.

The proof of Theorem 2 amounts to producing for a given curve C with negative self–intersection an effective divisor D such that C^2 is bounded in terms of the Zariski denominator of D. Our approach is to pick an ample divisor A and consider for large k the (effective) divisor $D = A+kC$. It has C as the support of its negative part, with coefficient $k + \frac{(A\cdot C)}{-C^2}$. The challenge then is to prove the following main lemma.

Lemma 1. *If Zariski denominators on X are bounded by d then there exists an ample divisor A such that the greatest common divisor of $(A \cdot C)$ and $-C^2$ divides $d!|\Delta|$.*

Granting the lemma, we see that

$$d \geq \frac{-C^2}{\gcd(AC, C^2)} \geq \frac{-C^2}{d!|\Delta|},$$

which proves the theorem.

Here are two nice consequences of our main theorems.

Corollary 1. *Suppose that on a smooth projective surface X the self-intersections of irreducible curves is at most -1, e.g. X a del Pezzo surface. Then the Zariski decomposition of integral divisors is integral.*

Corollary 2. *Let X be a surface with effective anticanonical divisor and Picard number ρ. Then for any integral divisor D, the Zariski decomposition of $2^{\rho-1}!D$ is integral.*

It is natural to ask whether the bounds given in Theorems 1 and 2 are sharp. This seems not to be the case. For example, so far we have only been able to produce sequences of surfaces such that Zariski denominators grow polynomially in $b(X)$ whereas the theoretical bound predicts growth as $b(X)^{\rho(X)}$.

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Ordinary and symbolic Rees algebras for Fermat ideals of points

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(joint work with Uwe Nagel)

Definition 1. Let $n \geq 3$ be an integer, let k be a field that contains n distinct n^{th} roots of 1. The following ideal of $R = k[x, y, z]$ shall be refer to as a *Fermat* ideal

$$I = (x(y^n - z^n), y(z^n - x^n), z(x^n - y^n))$$

The zero-locus of this ideal is a reduced set of $n^2 + 3$ points in $\mathbb{P}^2$, as shown in [HS, Proposition 2.1]. Specifically, n^2 of these points form the intersection locus of the pencil of curves spanned by $x^n - y^n$ and $x^n - z^n$, while the other 3 are the coordinate points of $\mathbb{P}^2$: $[1 : 0 : 0], [0 : 1 : 0]$ and $[0 : 0 : 1]$. These sets of planar points have gained recent popularity as examples exhibiting the non-containment $I^{(3)} \not\subseteq I^2$ [DST, HS]. This places the Fermat point configurations among the few special known counterexamples to a conjecture posed in [HaHu].

The goal of this note is to describe the ordinary and symbolic Rees algebras for the family of Fermat ideals. These two algebras appear naturally when dealing with the collection of all symbolic and ordinary powers of an ideal.

Definition 2. Let I ba an ideal. The (ordinary) Rees algebra of I is defined as $\mathcal{R}(I) = \bigoplus_{i\geq 0} I^i t^i$ and the symbolic Rees algebra of I is $\mathcal{R}_s(I) = \bigoplus_{i\geq 0} I^{(i)} t^i$.

The importance of the Ress algebra for an ideal I stems from the fact that it gives the coordinate ring of the blow-up of the ambient projective space at the zero locus of I. For this reason, Rees algebras often appear in the literature under the name blow-up algebras.

We first focus on describing the structure of the Rees algebra for Fermat configurations. In general, if $I = (f_1, f_2, \dots, f_s) \subseteq R$ is a homogeneous ideal of a polynomial ring R, a presentation for $\mathcal{R}(I)$ is obtained as follows. Let $S = R[T_1, T_2, \dots, T_s]$ denote a bigraded polynomial ring where the variables of R have degree $(1, 0)$ and the variables T_i have degree $(\deg(f_i), 1)$. The R-algebra epimorphism $R[T_1, T_2, \dots, T_3] \to \mathcal{R}(I)$ sending $T_i \mapsto f_i t$ gives a presentations of $\mathcal{R}(I)$ as a quotient of the bigraded polynomial ring $S = R[T_1, T_2, \dots, T_3]$. Writing L for the kernel of this epimorphism yields:

$$\mathcal{R}(I) = S/L, \text{ where } L = \{F(T_1, T_2, \dots, T_s) \mid F(f_1, f_2, \dots, f_s) = 0\}.$$

The bidegree $(*, 1)$ relations between generators of I are easy to describe and quotienting S by this subideal of L gives rise to the symmetric algebra of I:

$$\operatorname{Sym}(I) = S/L_1, \text{ where } L_1 = \left\{ \sum_{i=1}^{s} b_i T_i \mid \sum_{i=1}^{s} b_i f_i = 0 \right\}.$$

The following useful lemma can be deduced from [Va, Corollary 5.65]:

Lemma 1. *Let I be an ideal defining a reduced set of points in $\mathbb{P}^N$ that can be generated by $N+1$ generators. Then $\mathcal{R}(I) = \operatorname{Sym}(I)$.*

From Lemma 1 we obtain the structure of the Rees algebra for the Fermat ideals.

Corollary 1. *The Rees algebra of the Fermat ideal $I = (x(y^n - z^n), y(z^n - x^n), z(x^n - y^n))$is a complete intersection whose defining ideal is generated by two forms of bidegree $(n+3, 1)$ and $(2n, 1)$:*

$$\mathcal{R}(I) = k[x, y, z, T_1, T_2, T_3]/(x^{n-1}T_1 + y^{n-1}T_2 + z^{n-3}T_3, yzT_1 + xzT_2 + xyT_3).$$

Finally, knowing the minimal free bigraded resolution of the Rees algebra given above, one can deduce the explicit minimal free resolutions and the regularity of all powers of the ideal defining a Fermat configuration of points.

Theorem 3 ([NS])**.** *The minimal free resolution of the ordinary powers of the Fermat ideal is given by*

$$0 \to R(-(r+1)(n+1))^{\binom{r}{2}} \to \begin{array}{c} R(-r(n+1) - n + 1)^{\binom{r+1}{2}} \\ \oplus \\ R(-r(n+1) - 2)^{\binom{r+1}{2}} \end{array} \to R(-r(n+1))^{\binom{r+2}{2}} \to I^r \to 0.$$

The Castelnuovo-Mumford regularity of the ordinary powers of Fermat ideals is

$$\operatorname{reg}(I^r) = \begin{cases} 2n & \text{if } r = 1 \\ r(n+1) + n - 1 & \text{if } r \geq 2. \end{cases}$$

The regularity formula for powers of Fermat ideals obtained above gives an explicit instance of linear growth of the regularity of ordinary powers of an ideal as a function of the power exponent. The fact that $\operatorname{reg}(I^r) = ar+b$ for $r \gg 0$ was proven for all homogeneous ideals in [CHT, Theorem 1.1]. Giving explicit and effective formulas for the regularity of ordinary powers as in Theorem 3, and especially being able to control the constant term, has direct applications in computations of asymptotic invariants, e.g. resurgence [BDHHPSSU].

On the symbolic Rees algebra side, the main problem that arises is the finite generation of the symbolic Rees algebra. Often times symbolic Rees algebras tend to be non-Noetherian. Asking whether a symbolic Rees algebra is finitely generated is equivalent to asking whether there exists a Veronese subalgebra $\bigoplus_{i\geq 0} I^{(mi)}t^{mi}$ of $\mathcal{R}_s(I)$ that is finitely generated. In the case of the Fermat ideals we show that this is indeed the case for the subalgebra of all symbolic powers divisible by three.

Theorem 4 ([NS])**.** *Let* $I = (x(y^n - z^n), y(z^n - x^n), z(x^n - y^n))$ *with* $n \geq 3$*. Then* $I^{(3m)} = \left(I^{(3)}\right)^m$ *and thus* $\mathcal{R}_s(I)$ *is Noetherian.*

In general, whenever the symbolic Rees algebra of an ideal I is Noetherian and $\dim(R/I) = 1$, it can only be shown as in [CHT, Theorem 4.3] that $\operatorname{reg}(I^{(m)})$ is a periodic linear function for m large enough, i.e. there exist integers a_i and b_i such that $\operatorname{reg}(I^{(m)}) = a_i m + b_i$ for $m \equiv i \pmod{s}$ and $m \gg 0$, where s is a numerical invariant depending on a Veronese subalgebra of $\mathcal{R}_s(I)$. We use completely different methods to give an exact formula for Fermat ideals:

Theorem 5 ([NS])**.** *The symbolic powers of the Fermat ideal* I *have their Castelnuovo-Mumford regularity given by* $\operatorname{reg}(I^{(m)}) = m(n+1)$*, for* $m \gg 0$*.*

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Sylvester-Gallai for curves of higher degree

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The classical Sylvester-Gallai theorem asserts that given a set of points $\mathcal{P}$ in the projective plane $\mathbb{P}^2(\mathbb{R})$ exactly one of the two cases holds:

a) either all points are contained in a single line;
b) or there exists a line which contains exactly two points from the set $\mathcal{P}$.

A line satisfying condition b) in the above statement is called an *ordinary line*. It has been long conjectured by proved only recently by Green and Tao [2] that the number of ordinary line for a set $\mathcal{P}$ of s non-collinear points is at least $\lfloor \frac{s}{2} \rfloor$.

The Sylvester-Gallai theorem has prompted a lot of research activity in various branches of mathematics (e.g. combinatorics, discrete geometry) and has been generalized in many different ways (e.g. colored lines, higher dimensional flats). One of generalizations addresses curves of higher degree. It has been discovered in 1988 by Wiseman and Wilson [3].

Theorem 1 (Wiseman-Wilson)**.** *Let $\mathcal{P} \subset \mathbb{P}^2(\mathbb{R})$ be a finite set of points. Then*

a) *either all points are contained in a conic;*
b) *or there exists a conic which contains exactly* 5 *points from the set $\mathcal{P}$ and it is* determined *by these* 5 *points.*

Definition 1. *We say that a curve C of degree d in $\mathbb{P}^2(\mathbb{R})$ is* determined *by a set of $\frac{d^2+3d}{2}$ points if this is the unique curve of this degree containing these points.*

Our contribution is a new proof of the Wiseman-Wilson Theorem. By using the Cremona transformation we are essentially able to reduce the statement of the Wiseman-Wilson Theorem to the case of lines dealt with by the Sylvester-Gallai Theorem. We refer for details to [1]. Here we turn our attention rather to some open problems in the hope that they sparkle further research on this fascinating topic.

Problem 2. In analogy to the Green-Tao Theorem, it would be interesting to know what is the minimal number of conics determined by subsets of $\mathcal{P}$ in dependance on the cardinality of the set $\mathcal{P}$.

Problem 3. It is well known that the Sylvester-Gallai Theorem fails in the complex projective plane. We were not able to find any counterexamples to the Wiseman-Wilson Theorem in the case of complex conics. It would be interesting to investigate into the question of existence of such counterexamples more systematically. In particular, an attempt to prove the Wiseman-Wilson Theorem in the complex case would require completely new methods.

Problem 4. It is natural to wonder what about the curves of higher degree. Is the following true?

Conjecture 1. *Let $\mathcal{P} \subset \mathbb{P}^2(\mathbb{R})$ be a finite set of points and let d be a positive integer. Then*

a) *either all points in $\mathcal{P}$ are contained in a curve of degree d;*
b) *or there exists a curve of degree d which contains exactly $\frac{d^2+3d}{2}$ points from the set $\mathcal{P}$ and it is* determined *by these $\frac{d^2+3d}{2}$ points.*

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On absolute linear Harbourne constants

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(joint work with Marcin Dumnicki, Daniel Harrer)

Arrangements of lines were introduced to algebraic geometry by Hirzebruch in his papers concerning the geography of surfaces (i.e. construction of surfaces X with prefixed invariants $c_1^2(X)$ and $c_2(X)$), see [1] for an excellent account.

Recently arrangements of lines appeared in the ideas revolving around the Bounded Negativity Conjecture (BNC for short), see [3] and [2] for the background on the Conjecture. In [2] the authors introduced and began to study linear Harbourne constants. Even though the Bounded Negativity fails in positive characteristic, it is clear from Definition 1 that for a fixed d, the linear Harbourne constant $H(d)$ is a finite number. It is natural to wonder about its value or at least some estimates.

A configuration $\mathcal{L}$ is a finite set of mutually distinct lines $\mathcal{L} = \{L_1, \ldots, L_d\}$. Given a configuration $\mathcal{L}$, we define its singular set $\mathcal{P}(\mathcal{L}) = \{P_1, \ldots, P_s\}$ as a set of points where two or more lines intersect. This is the same as the singular locus of the divisor $L_1 + \ldots + L_d$. For a point $P \in \mathcal{P}(\mathcal{L})$, we denote by $m_{\mathcal{L}}(P)$ its *multiplicity*, i.e. the number of lines which pass through P.

Definition 1. *The* linear Harbourne constant of a configuration of lines $\mathcal{L}$ in the projective plane $\mathbb{P}^2(\mathbb{K})$ *is the rational number*

$$H(\mathbb{K}, \mathcal{L}) = \frac{d^2 - \sum_{k=1}^{s} m_{\mathcal{L}}(P_k)^2}{s}. \tag{1}$$

The linear Harbourne constant of d lines over $\mathbb{K}$ *is defined as the minimum*

$$H(\mathbb{K}, d) := \min H(\mathbb{K}, \mathcal{L})$$

taken over all configurations $\mathcal{L}$ of d lines.
Finally the absolute linear Harbourne constant of d lines *is the minimum*

$$H(d) := \min_{\mathbb{K}} H(\mathbb{K}, d)$$

taken over all fields $\mathbb{K}$.

In order to alleviate the notation we define first the set

$$Q = \{q = p^r, \quad p \text{ is prime}, \quad r \in \mathbb{Z}_{>0}\}.$$

For an integer d, we define $q(d)$ as the least number $q \in Q$ satisfying

$$d \le q^2 + q + 1$$

and $r(d)$ as the largest number $r \in Q$ satisfying

$$r^2 + r + 1 \le d.$$

Systematic investigations of absolute linear Harbourne constants $H(d)$ have been initiated in [5]. Results stated there and computer supported experiments have led us to the following conjecture.

Conjecture 1. *For* $d \ge 2$ *let* $q = q(d)$ *and let* $i := q^2 + q + 1 - d$.
If $i \le 2q - 2$, *then*

$$H(d) = h(d)$$

where

$$h(d) = \frac{q^2 + q + 1 - i - \varepsilon_1(i)m_1(i) - \varepsilon_2(i)m_2(i) - t_{q-1}(i)(q-1) - t_q(i)q - t_{q+1}(i)(q+1)}{\varepsilon_1(i) + \varepsilon_2(i) + t_{q-1}(i) + t_q(i) + t_{q+1}(i)},$$

with

$$m_1(i) = q + 1 - i, \qquad m_2(i) = 2q + 1 - i$$

$$\varepsilon_1(i) = \begin{cases} 1 & \text{for } 0 \le i \le q-1 \\ 0 & \text{otherwise} \end{cases}, \qquad \varepsilon_2(i) = \begin{cases} 1 & \text{for } i > q+1 \\ 0 & \text{otherwise} \end{cases},$$

$$t_{q-1}(i) = \begin{cases} qi - q^2 - q & \text{for } i > q+1 \\ 0 & \text{otherwise} \end{cases},$$

$$t_q(i) = \begin{cases} qi & \text{for } i \le q+1 \\ 2q^2 - (i-2)q - 1 & \text{for } i > q+1 \end{cases},$$

$$t_{q+1}(i) = \begin{cases} q^2 + q - iq & \text{for } i \le q+1 \\ 0 & \text{otherwise} \end{cases}.$$

Moreover for $i = 2q - 1$ *we have*

$$H(d) = -\frac{q^3 - q^2 + 2q - 2}{q^2 + q - 1}.$$

Remark 1. We do not know what happens for d such that $d \le (q(d) - 1)^2 + q(d)$. The first such d is $d = 32$ with $q(32) = 7$.

This conjecture has been verified for $d \le 10$ in [5]. The range of the validity of the Conjecture has been extended recently to $d \le 31$, see [4].

Theorem 2. *For $2 \geq d \geq 31$ we have*

d	$H(d)$
2	0
3	-1
4	$-4/3 \approx -1.333$
5	$-3/2 = -1.5$
6	$-12/7 \approx -1.714$
7	-2
8	-2
9	$-9/4 = -2.25$
10	$-29/12 \approx -2.416$
11	$-33/13 \approx -2.538$
12	$-36/13 \approx -2.769$
13	-3
14	$-54/19 \approx -2.842$
15	-3
16	$-16/5 = -3.2$
17	$-67/20 = -3.35$
18	$-24/7 \approx -3.428$
19	$-76/21 \approx -3.619$
20	$-80/21 \approx -3.809$
21	-4
22	$-108/29 \approx -3.724$
23	$-115/30 \approx -3.833$
24	-4
25	$-125/30 \approx -4.166$
26	$-129/30 = 4.3$
27	$-135/31 \approx -4.354$
28	$-140/31 \approx -4.516$
29	$-145/31 \approx -4.677$
30	$-150/31 \approx -4.838$
31	-5

TABLE 1. Values of $H(d)$ for up to 31 lines

Our next main result is the verification of the Conjecture in certain cases.

Theorem 3. *The Conjecture holds for all $d = q(d)^2 + q(d) + 1$.*

Finally we establish certain bounds on absolute linear Harbourne constants.

Theorem 4 (Lower bound on Harbourne constants)**.** *For $d \geq 6$ we have*

$$H(d) \geq -\frac{1}{2}\sqrt{4d-3} + \frac{1}{2}.$$

For $d = q^2 + q + 1$ with $q \in Q$ we have the equality. In this case $H(d) = -q$ is computed by the configuration consisting of all lines in the finite projective plane $\mathbb{P}^2(\mathbb{F}_q)$.

Theorem 5 (Upper bound for Harbourne constants)**.** *For $d \geq 7$ and with $r = r(d)$, we have*

$$H(d) \leq -2\frac{r^4 + r^3 - r - (d-1)^2}{r^4 + 2r^3 - r - d^2 + d - 2}.$$

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On Geography of Surfaces: Results and Techniques

Giancarlo Urzúa
(joint work with Xavier Roulleau, Rodrigo Codorniu)

For a smooth projective surface X over an algebraically closed field $\mathbb{K}$, the coarse moduli invariants are its Chern numbers

$$c_1^2(X) := c_1({\Omega_X^1}^\vee)^2 \quad \text{and} \quad c_2(X) := c_2({\Omega_X^1}^\vee),$$

where Ω_X^1 is the sheaf of differentials. If X is minimal and of general type, then these integers satisfy:
($char = 0$): $c_1^2 > 0$, $c_2 > 0$, $\frac{1}{5}(c_2 - 36) \leq c_1^2 \leq 3c_2$,
($char > 0$): $c_1^2 > 0$, $c_2 \in \mathbb{Z}$, $\frac{1}{5}(c_2 - 36) \leq c_1^2$,
where *char* is the characteristic of $\mathbb{K}$. The number c_2 could be zero or negative in positive characteristic, but those pathological cases have been classified by Shepherd-Barron [5]. Also, it is not uncommon that $c_1^2 > 3c_2$, there are many examples in the literature.

The **problem of geography** [3] is to find a surface (minimal and of general type) representative for each pair (c_1^2, c_2) which satisfies the above conditions. For various reasons, coming mainly from char$=0$, it is interesting to look at **simply connected geography**, this is, the geography problem plus trivial topological (char$=0$), or trivial étale (char>0) fundamental group. It is a hard problem, unsolved in general.

Our approach is via Chern slopes c_1^2/c_2 (in the case of char>0, simply connected surfaces have $c_2 > 0$; see [5]). The possible rational numbers are roughly in $[1/5, 3[$ for char$=0$ (the number 3 is only achieved by ball quotients due to results of Yau and Miyaoka), and in $[1/5, \infty[$ for char>0.

We proved that Chern slopes are dense in $[1, 3]$ for char$=0$ [4] (in collaboration with X. Roulleau), and dense in $[2, \infty[$ for any fixed $\mathbb{K}$ of char>0 [1] (in collaboration with R. Codorniu), even having irregularity $h^1(\mathcal{O}_X) = 0$ (which is not necessarily true for simply connected surfaces in positive characteristic; see [2] for motivation around this issue).

The intention of this talk is to present the techniques involved, in particular connecting **geography of surfaces** with **geography of log surfaces** [6, 7]. We recall that for a smooth projective surface Y and a simple crossings divisor D, the log Chern numbers of (Y, D) are

$$\bar{c}_1^2(Y, D) := c_1\big(\Omega_Y^1(\log D)^\vee\big)^2 \quad \text{and} \quad \bar{c}_2(Y, D) := c_2\big(\Omega_Y^1(\log D)^\vee\big).$$

(Hence for $D = 0$ we recover the Chern numbers of Y.) The relation will be between Chern and log Chern slopes, and for certain arrangements of curves whose log resolution is D. For example, any line arrangement in $\mathbb{P}^2_{\mathbb{K}}$ works.

I will also explain asymptotic minimality for the surfaces constructed from these suitable arrangements of curves. That was used in [1] to prove the density result.

Two interesting questions are:

I. It is known that for nontrivial line arrangements in $\mathbb{P}^2$ we have

$$1 \leq \bar{c}_1^2/\bar{c}_2 \leq 3.$$

Thanks to essentially the work of Hirzebruch, in $\mathbb{P}^2_{\mathbb{C}}$ we have

$$1 \leq \bar{c}_1^2/\bar{c}_2 \leq 8/3.$$

One can prove that $\bar{c}_1^2/\bar{c}_2$ has no accumulation point in $[1, 2[$. On the other hand, the slope $\bar{c}_1^2/\bar{c}_2$ is dense in $[2, 3]$ in positive characteristic, and it is dense in $[2, 2.5]$ over $\mathbb{C}$ (actually over $\mathbb{R}$). Also, line arrangements with $\bar{c}_1^2/\bar{c}_2 > 2.5$ cannot be defined over $\mathbb{R}$, and so they are strictly complex. An intriguing question is: *What are the accumulation points of $\bar{c}_1^2/\bar{c}_2$ in $]5/2, 8/3]$ for these strictly complex arrangements?* The Fermat (or Ceva) arrangements, the dual Hesse arrangement, the Klein arrangement, the Wiman arrangement, and some others belong to this interval, but no accumulation point is know in this range. It may happen that these slopes are isolated in that interval. If so, why? is there a direct relation with the specific fields of definition of the arrangement?

II. Notice that in the previous question, Chern slopes $\bar{c}_1^2/\bar{c}_2$ for line arrangements are discrete in $[1, 2[$. For the suitable arrangements of curves used to connect geography of surfaces with geography of log surfaces, we have been able to produce accumulation points for $\bar{c}_1^2/\bar{c}_2$ in $[1, 3]$ over $\mathbb{C}$, and in $[2, \infty[$ for positive characteristic. The lower potential bound for the slopes $\bar{c}_1^2/\bar{c}_2$ of these arrangements is $1/5$, coming from the Noether's inequality for surfaces. An intriguing question then is: *Is $1/5$ an accumulation point for these arrangements in any characteristic?* We believe it should be true. If so, we would have an uniform treatment for density of Chern slopes of simply connected surfaces of general type, using the techniques explained in this talk.

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