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Mini-Workshop: Topological Complexity and Related Topics

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Abstract. Topological complexity is a numerical homotopy invariant of topological spaces, of Lusternik-Schnirelmann type, introduced by Farber and motivated by the motion planning problem from topological robotics. This mini-workshop assembled researchers interested in calculating the topological complexity and its many variants, with the aim of providing a snapshot of the current state of knowledge, and shaping directions of future research.

Mathematics Subject Classification (2010): 55-XX, 68T40.

Introduction by the Organisers

The mini-workshop *Topological Complexity and Related Topics* was attended by 16 participants from 9 different countries. The list of participants was designed to be 'vertically integrated', in the sense that every career stage was represented, from PhD students to professors. The morning speakers presented surveys on some particular aspect or variant of topological complexity, while the afternoon speakers gave shorter and more specialized talks on their current research. The schedule also included ample time for discussion and collaboration.

Topological complexity is a numerical homotopy invariant of topological spaces, closely related to the Lusternik-Schnirelmann category. It was introduced by Michael Farber in the early 2000s as part of his topological study of the motion planning problem from robotics, and has become a very active area of research in applied topology. The computation of topological complexity and its many variants presents several challenging topological problems, each of which may have practical consequences for the design of efficient motion planning algorithms. By now there are several variants of topological complexity (including higher, symmetric, equivariant and rational versions) as well as applications of the ideas to related problems (such as immersions and embeddings of manifolds, or the topological complexity of kinematic maps). These themes were all discussed at the mini-workshop, which concluded with an informal problem session.

The proceedings of the mini-workshop will hopefully be published as an issue of the AMS Contemporary Mathematics series.

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Mini-Workshop: Topological Complexity and Related Topics

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Abstracts

Topological complexity, immersions, and embeddings of real projective spaces

Jesús González

In this talk I survey the work by Farber, Tabachnikov, and Yuzvinsky [5] on the relation between the motion planning problem in real projective spaces and their Euclidean immersion dimension, as well as the work by González and Landweber [6] on the embedding dimension of those manifolds and their Euclidean embedding dimension.

In the early 1970's, Adem, Gitler and James clarified the relationship between the existence of immersions of real projective space P^n in $\mathbb{R}^{n+k}$ and the existence of axial maps of type (n, k), that is, maps of the form

$$P^n \times P^n \to P^{n+k} \tag{1}$$

which are homotopically nontrivial over each axis. Their analysis can be carried over in the case of 2^e-torsion lens spaces: B. J. Sanderson observed in [10] that an axial map as above can be obtained from a given immersion $P^n \subseteq \mathbb{R}^{n+k}$ and that, at least in the metastable range (that is, when $n < 2k$), such an immersion can homotopically be recovered from (1). Sanderson's work relies on results of Hirsch and Haefliger ([7], also considered by James in [9]) and takes advantage of the so-called *twisted normal bundle* associated to an immersion $P^n \subseteq \mathbb{R}^{n+k}$. The success of the technique depends on the fact that the canonical real line bundle over P^n has multiplicative order 2. However, this is precisely the main drawback in a first attempt to generalize the ideas for higher 2^e-torsion lens spaces. Indeed, on the one hand, Hirsch's basic result on immersing manifolds [8] implies that the codimension in an optimal immersion for $L^{2n+1}(2^e)$ —the $(2n + 1)$-dimensional 2^e-torsion lens space— agrees with the geometric dimension of $-(n+1)\xi_{n,e}$, where $\xi_{n,e}$ is the realification of the canonical complex line bundle over $L^{2n+1}(2^e)$; but on the other hand, $\xi_{n,e}$ is not even a unit in $\mathrm{KO}(L^{2n+1}(2^e))$. The situation can be straighten by following a path, first suggested in [1] by Adem, Gitler and James, which naturally leads to the concept of generalized e-axial maps.

Theorem. If $L^{2n+1}(2^e)$ immerses in $\mathbb{R}^{2n+k+1}$, then there is an *e-axial* map $\alpha\colon S^{2n+1} \times S^{2n+1} \to S^{2n+k+1}$, i.e. a map satisfying $\alpha(-x, y) = -\alpha(x, y)$ as well as $\alpha(\omega x, y) = \alpha(x, \omega y)$ for any 2^e root of unity. The converse holds except perhaps for $n = 2$, 3 or 5.

On the other hand, while the topological complexity of a number of mechanical systems (robot arms, rigid body motion and particles moving with/without obstacles) as well as other related spaces (products and wedges of a given sphere, compact orientable surfaces) —all of them calculated in [2, 3]— is described by rather simple formulas, it turns out to be amazingly difficult to compute (as a function of n) the topological complexity of the naive system formed by the n-dimensional rotations of a line fixed at a base point by a revolving joint —that is, the topological complexity of P^n. Farber, Tabachnikov, and Yuzvinsky proved in [5]

$$\mathrm{TC}(P^n) = I_n \tag{2}$$

(equality is off by one unit in the three special cases where P^n is parallelizable), where I_n denotes the dimension of the smallest Euclidean space where P^n admits an immersion. In extending the ideas behind (2), from projective spaces to 2^e-torsion lens spaces, one finds that the numeric value of $\mathrm{TC}(L^{2n+1}(2^e))$ is determined by the existence of $\mathbb{Z}/2^e$-biequivariant maps $\beta\colon S^{2n+1} \times S^{2n+1} \to S^{2m+1}$, that is maps which satisfy $\beta(\omega x, y) = \beta(x, \omega y) = \omega\beta(x, y)$ for any 2^e root of unit ω.

The obvious relation between e-axial maps and $\mathbb{Z}/2^e$-biequivariant maps suggests to approach the study of the immersion problem for projective spaces by understanding the subtleties in the determination of the topological complexity of 2^e-torsion lens paces as the parameter e decreases. This idea is particularly appealing because it is possible to prove that $4n \leq \mathrm{TC}(L^{2n+1}(2^e)) \leq 4n + 1$ as long as e is larger than the number of ones in the binary expansion of n. In other words, the start of the *lens-space approach* to I_{2n+1} is reasonably well-understood.

In the talk we explain a away to materialize this approach by substituying the parameter e (coming from the torsion of the lens spaces) by the parameter s coming from the *higher* topological complexity TC_s of real projective spaces.

In the final part of the talk I review the required considerations connecting the Euclidean embedding dimension of real projective spaces with Farber-Grant's concept of symmetric topological complexity TC^s introduced in [4].

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Algorithmic approach to topological complexity

Aleksandra Franc
(joint work with Neža Mramor)

Our goal is to find an algorithm that computes the topological complexity of a configuration space X as defined in [2]:

Definition 1. *Topological complexity* $\mathrm{TC}(X)$ is the minimal k such that there exists an open cover

$$U_1 \cup \ldots \cup U_k = X \times X$$

with the property that for all $i = 1, \ldots, k$ the map $\pi\colon X^I \to X \times X$, $\alpha \mapsto (\alpha(0), \alpha(1))$, admits a continuous section $s_i\colon U_i \to PX$ over U_i.

While there are plenty of reasons to believe that reaching this goal is, in fact, not possible in general, we hope to at least get good upper bounds for reasonably large families of spaces, as well as explicit covers that in turn correspond to explicit motion planners. We achieve this by running the algorithm repeatedly and keeping track of the minimal output this produces.

To construct these covers we will use the methods of discrete Morse theory developed in [3]. Instead of checking for existence of the section to the path-space X^I we will use the following alternative characterization of TC from [6], which instead deals with vertical homotopies (i.e. homotopies which are constant when composed with the projection $\mathrm{pr}_1\colon X \times X \to X$ to the first component):

Theorem 2. The topological complexity $\mathrm{TC}(X)$ of X is equal to the least integer n for which there exists an open cover $\{U_1, U_2, \ldots, U_n\}$ of $X \times X$ such that each U_i is compressible to the diagonal $\Delta_X = \{(x, x) \mid x \in X\}$ via a vertical homotopy.

The monoidal topological complexity $\mathrm{TC}^M(X)$ of X is equal to the least integer n for which there exists an open cover $\{U_1, U_2, \ldots, U_n\}$ of $X \times X$ such that each U_i contains the diagonal Δ_X and is compressible to the diagonal via a vertical homotopy which is stationary on Δ_X.

This is where the first potential threat to optimality comes from. It is not currently known whether $\mathrm{TC}(X) = \mathrm{TC}^M(X)$ although $\mathrm{TC}^M(X) \le \mathrm{TC}(X) + 1$, so the disparity is not overly large. Our algorithm produces sets which contain a neighbourhood of Δ_X and contract to it via a vertical homotopy, stationary on Δ_X, so $\mathrm{TC}^M(X)$ is a lower bound for the numbers of elements in the covers obtained by our approach.

We assume for now that X is a finite simplicial complex, although this approach would work for more general spaces as well, such as regular CW complexes. The product $X \times X$ is a prodsimplicial complex with cells consisting of all products of the form $\sigma \times \tau$ for all simplices σ, τ of X.

Some of the problems arising from these assumptions will be similar to those encountered in [1]. Elements of the cover will necessarily be unions of cells collapsible (and not merely contractible) to a subcomplex and an optimal cover satisfying these additional restrictions will in general contain more than $\mathrm{TC}(X)$ or $\mathrm{TC}^M(X)$ sets. See Example 5.

The main tools for constructing vertical contractions will be discrete gradient vector fields on X as defined in [4]:

Definition 3. A *discrete vector field* V on X is a collection of pairs $\{\alpha < \beta\}$ of simplices of X with $\dim(\alpha) = \dim(\beta) - 1$, such that each simplex is in at most one pair of V. All simplices not contained in any of the pairs are called *critical.*

We introduce the notation V_σ for a vector field on X which is critical on σ (but may have other critical simplices as well). Furthermore, given such V_σ let V_σ^2 be the collection of pairs $\{\sigma \times \alpha, \sigma \times \beta\}$ for $\{\alpha < \beta\} \in V_\sigma$. In other words, V_σ^2 is a discrete gradient vector field on $\sigma \times X$. Vector fields V_σ^2 help us define vertical collapses of certain subsets of $X \times X$ to the *prodsimplicial diagonal* Δ (the union of products $\sigma \times \sigma$ over all simplices σ of X).

Definition 4. A *vertical discrete gradient vector field* (VDGVF) on the fiberwise space $\mathrm{pr}_1 \colon X \times X \to X$ is a family

$$\{V_\sigma^2 \mid \sigma \text{ a simplex in } X\}$$

of discrete gradient vector fields.

Given such a VDGVF we can construct a subset $U_1 \subset X \times X$ as union of descending paths starting at Δ. Then we start constructing U_2 using the same procedure but giving priority to any cells of $X \times X$ not yet covered by U_1. Repeating this as many times as necessary we eventually obtain a cover $\{U_1, U_2, \ldots, U_k\}$ of $X \times X$ with sets that are vertically collapsible to the diagonal Δ. We will denote the minimal number of sets over all such covers by $\mathrm{TC}_a(X)$. Obviously, $\mathrm{TC}(X) \leq \mathrm{TC}_a(X)$ and $\mathrm{TC}^M(X) \leq \mathrm{TC}_a(X)$. Unfortunately, the inequalities can be strict as shown by the following example.

Example 5. If X is any contractible non-collapsible space (such as the dunce hat or Bing's house with two rooms), then $TC(X) = 1$, but $\mathrm{TC}_a(X) \neq 1$ because a collapse of $X \times X$ to Δ (restricted to a fibre over a point) would induce a collapse of X to a point.

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Survey of equivariant notions of topological complexity

Andres Angel

(joint work with Hellen Colman)

In this talk I described several existing notions of topological complexity that encode symmetries.

There are (at least) 5 different notions of topological complexity of a space with a group acting on it.

- (Colman-Grant) Equivariant topological complexity. [1]
- (Lubawski-Marzantowicz) Invariant topological complexity. [2]
- (Dranishnikov) Strongly equivariant topological complexity. [3]
- (Błaszczyk-Kaluba) Effective topological complexity. [4]
- (A.-Colman) Groupoid topological complexity.

Let G be a topological group and X a G-space. We can consider the free path space X^I as a G-space and $X \times X$ as a G-space with the diagonal action. The evaluation map $ev : X^I \to X \times X$ is a G-fibration.

The equivariant topological complexity of Colman and Grant ($TC_G(X)$) is the least integer k such that $X \times X$ may be covered by k G-invariant open sets $\{U_1, \ldots, U_k\}$, on each of which there is a G-equivariant section $U_i \xrightarrow{s_i} X^I$ such that the diagram commutes:

$$\begin{tikzcd} & X^I \arrow[d, "\mathrm{ev}"] \\ U_i \arrow[ur, "s_i"] \arrow[r] & X \times X \end{tikzcd}$$

We have,

$$TC(X) \leq TC_G(X)$$

and

Theorem 1. Let $E \to B$ be a numerable principal G-bundle and X a G-space, then

$$TC(X_G) \leq TC_G(X)TC(B)$$

where $X_G = E \times_G X$ is the total space of the associated bundle over B.

The strongly equivariant topological complexity of Dranishnikov ($TC^*_G(X)$) is the least integer k such that $X \times X$ may be covered by k $G \times G$-invariant open sets $\{U_1, \ldots, U_k\}$, on each of which there is a G-equivariant section $U_i \xrightarrow{s_i} X^I$ such that the diagram commutes:

$$\begin{tikzcd} & X^I \arrow[d, "\mathrm{ev}"] \\ U_i \arrow[ur, "s_i"] \arrow[r] & X \times X \end{tikzcd}$$

We have,

$$TC(X) \leq TC_G(X) \leq TC^*_G(X)$$

and

Theorem 2. Let $E \to B$ be a principal G-bundle (between locally compact ANR spaces) and X a proper G-space, then

$$TC(X_G) \leq TC^*_G(X) + TC(B) - 1$$

where $X_G = E \times_G X$ is the total space of the associated bundle over B.

The invariant topological complexity of Lubawski-Marzantowicz ($TC^G(X)$) is the least integer k such that $X \times X$ may be covered by k $G \times G$-invariant open sets $\{U_1, \ldots, U_k\}$, on each of which there is a $G \times G$-equivariant section $U_i \overset{s_i}{\to} X^I \times_{X/G} X^I$ such that the diagram commutes:

$$\begin{tikzcd} & X^I \times_{X/G} X^I \arrow[d, "\mathrm{ev}"] \\ U_i \arrow[ur, "s_i"] \arrow[r] & X \times X \end{tikzcd}$$

In terms of the equivariant version of the Clapp-Puppe invariant, we have that

$$TC_G(X) =_{\Delta(X)} cat_G(X \times X)$$

and

$$TC^*_G(X) =_{\daleth(X)} cat_{G\times G}(X \times X)$$

where $\daleth(X)$ is the $G \times G$-saturation of the diagonal $\Delta(X) \subseteq X \times X$.

The effective topological complexity of Błaszczyk-Kaluba ($TC^{G,\infty}(X)$) is the minimum of the numbers $TC^{G,n}$ which are the least integer k such that $X \times X$ may be covered by k open sets $\{U_1, \ldots, U_k\}$, on each of which there is a section (not necessarily equivariant) $U_i \overset{s_i}{\to} X^I \times_{X/G} X^I \times_{X/G} \cdots \times_{X/G} X^I =: \mathcal{P}_n(X)$ such that the diagram commutes:

$$\begin{tikzcd} & \mathcal{P}_n(X) \arrow[d, "\pi_n"] \\ U_i \arrow[ur, "s_i"] \arrow[r] & X \times X \end{tikzcd}$$

We have,

$$TC^{G,\infty}(X) \leq \ldots \leq TC^{G,n+1}(X) \leq TC^{G,n}(X) \leq \ldots \leq TC^{G,1}(X) = TC(X)$$

and

$$TC^{G,2}(X) \leq TC^G(X)$$

We also have,

Theorem 3. If G acts freely on X, $TC^{G,n+1}(X) = TC^{G,n}(X)$ for $n \geq 2$.

The groupoid topological complexity of Angel and Colman $TC(\mathcal{G})$ is the least integer k such that $G_0 \times G_0$ may be covered by k $G \times G$-invariant open sets $\{U_1, \ldots, U_k\}$, on each of which there is a generalized section

$$G \times G \ltimes U_i \stackrel{\epsilon}{\leftarrow} \mathcal{K} \stackrel{s}{\rightarrow} P(\mathcal{G})$$

such that the diagram commutes up to natural transformation:

$$\begin{tikzcd} \mathcal{K} \arrow[r, "s"] \arrow[d, "\epsilon"'] & P(\mathcal{G}) \arrow[d, "\mathrm{ev}"] \\ \mathcal{U}_i \arrow[r] & \mathcal{G} \times \mathcal{G} \end{tikzcd}$$

where ϵ is an essential equivalence.

We have that $TC(\mathcal{G})$ is an invariant under Morita equivalence and we can define an invariant for a G-space X by using the translation groupoid $G \ltimes X$ and considering $TC(G \ltimes X)$. For example, we have

Theorem 4. If G acts freely on X, then $TC(G \ltimes X) = TC(X/G)$

and in fact $TC(\mathcal{G})$ is an invariant of groupoid homotopy type.

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On Equivariant Topological Complexity of $\mathbb{Z}/_p$-spheres

Marek Kaluba

(joint work with Zbigniew Błaszczyk)

Consider the space X of all possible configurations of a mechanical system. The motion planning problem is to describe a continuous algorithm ("motion planner") which, given a pair $(x, y) \in X \times X$, outputs a continuous path in X between x and y. In order to measure discontinuity of the process of motion planning, Farber [4] introduced the notion of topological complexity of X as the minimal number of sets $U \subset X \times X$ with planners which are needed to cover $X \times X$.

There are versions of topological complexity aimed at exploiting the presence of a group action: "equivariant topological complexity" (TC_G) defined by Colman and Grant [3] and "invariant topological complexity" (TC^G) introduced by Lubawski and Marzantowicz [6]. The founding ideas of these two invariants are quite different. Roughly speaking, TC_G records the minimal number of domains of continuity which preserve symmetries, while TC^G tries to take advantage of symmetries to ease the effort of motion planning.

We investigate equivariant and invariant topological complexity of spheres endowed with smooth non-free actions of cyclic groups of prime order. Despite different foundations it turns out that these complexities share a common behaviour *in practice*, hence we will use ETC to denote either $TC_{\mathbb{Z}/_p}$ or $TC^{\mathbb{Z}/_p}$. We prove that linear (or semilinear) G-spheres have both invariants either 2 or 3 and calculate exact values for $G = \mathbb{Z}/_p$ in all but two cases. On the other hand, we exhibit examples which show that these invariants can be arbitrarily large in the class of smooth $\mathbb{Z}/_p$-spheres. These results are possible due to

- lower bound: the ordinary TC of the fixed point set:

$$\mathrm{cat}(X^G) \leq TC(X^G) \leq ETC(X). \tag{1}$$

- upper bound: the G-categorical bound:

$$ETC(X) \leq 2\,\mathrm{cat}_G(X) - 1 \tag{2}$$

These bounds apply to both of the invariants with varying assumptions in general. E.g. a sufficient assumption is the non-emptiness of the fixed point set and this is the case for $\mathbb{Z}/_p$-actions on spheres.

1. Linear actions

The G-category of linear spheres can be computed directly by constructing an explicit cover of S^n by two sets (namely the extended northern and southern hemispheres in appropriate linear embedding). Thus the upper bound amounts to 3 (note that the fixed point set of $\mathbb{Z}/_p$-sphere is non-empty) and we have the the following theorem.

Corollary 1. If S^n is a linear G-connected sphere with non-empty fixed point set, then

$$2 \leq ETC(X) \leq 3.$$

When we specialise to $G = \mathbb{Z}/_p$ the exact values can be nailed for equivariant TC:

Proposition 2. Let S^n be a linear $\mathbb{Z}/_p$-sphere such that $(S^n)^{\mathbb{Z}/_p} = S^k$. Then $TC_{\mathbb{Z}/p}(S^n) = 2$ if and only if both n and k are odd.

The proof follows from constructing an explicit cover and prescribing planners with the aid of equivariant vector fields. If n is even we use the property $TC(X) \leq TC_{\mathbb{Z}/_p}(X)$; if k is even the fixed point set bound 1 applies.

For the invariant TC we have the following result.

Proposition 3. Let S^n be a linear $\mathbb{Z}/_p$-sphere with $(S^n)^{\mathbb{Z}/_p} = S^k$ for $0 < k < n$. If either

- $k < n - 2$, or
- $k = n - 2$ and n is even, or
- $k = n - 1$ and n is odd,

then $TC^{\mathbb{Z}/_p}(S^n) = 3$.

The two remaining cases are:

a) $\mathbb{Z}/_2$-action on S^n by reflection along $(n-1)$-dimensional hyperplane
b) $\mathbb{Z}/_p$-action on S^n for odd n, with codimension-2 fixed point set.

2. Arbitrary smooth actions

Arbitrary smooth $\mathbb{Z}/_p$-spheres tell a different story. A general $\mathbb{Z}/_p$-smooth action would have a $\mathbb{Z}/_p$-homology sphere as its fixed point set, and non-trivial homology spheres have TC at least 4. Thus ETC of a *generic* smooth $\mathbb{Z}/_p$-sphere is by the lower bound (1) at least 4. This is somehow expected as the G-category is supposed to measure the complexity of an action, hence we should not have a cheap (or directly computable) replacement for the upper bound (2). Following this intuition we prove the following theorem.

Theorem 4. There exist smooth $\mathbb{Z}/_p$-actions on S^n such that $ETC(S^n) \geq n-2$.

The main two ingredients in the proof are the lower bound (1) and realisation of homology spheres as the fixed point sets of actions on spheres. We will construct a smooth $\mathbb{Z}/_p$ action on S^n with codimension-2 fixed point set F such that $TC(F) = n-2$. The following proposition will allow us to push the lower bound arbitrarily high.

Proposition 5. There exist k-essential smooth homology k-spheres.

Using results of [1] we construct a $B\pi$ with homology of an k-sphere. The Quillen's $+$-construction assures that the space is k-essential, and surgery classification of smooth homology spheres provided by [5] allows us to find a smooth homology sphere Σ^k k-equivalent to $B\pi$.

Proof of Theorem 4 (a sketch). By results of [7] k-essential manifolds have TC equal $k+1$. Thus to prove the theorem it is enough to realise Σ as the fixed point set of smooth $\mathbb{Z}/_p$-action on S^{k+2}. The boundary of $X \times D(\varrho)$ is such a sphere, where X is a contractible manifold bounded by Σ and ϱ is any $\mathbb{Z}/_p$-representation fixing only origin. □

The full account of results and proofs can be found in [2].

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Yet another approach to topological complexity of robots with symmetric configuration spaces

Zbigniew Błaszczyk
(joint work with Marek Kaluba)

The talk is aimed at introducing a variant of topological complexity, the "effective TC", suitable for investigating mechanical systems whose configuration spaces exhibit symmetries. The idea of weaving symmetries into the concept of TC is not new and has been pursued previously by Colman and Grant [2], Dranishnikov [3], and Lubawski and Marzantowicz [4], who introduced, respectively, "equivariant", "strongly equivariant" and "invariant" topological complexities. However, all three invariants have a common lower bound in $TC(X^G)$, where X^G is the fixed point set of X, and, as a result, they can be arbitrarily larger than $TC(X)$. Effective TC never exceeds classical TC, and often is actually smaller, thus quantifying the idea that symmetries present in configurations spaces can be used to ease the effort of motion planning. For example, a sphere equipped with an involution which flips the two hemispheres has effective TC equal to 1.

To elaborate a little, let $k \geq 1$ be an integer and G a topological group. Given a G-space X, write

$$\mathcal{P}_k(X) = \big\{(\gamma_1, \dots, \gamma_k) \in (PX)^k \mid G\gamma_i(1) = G\gamma_{i+1}(0) \text{ for } 1 \leq i \leq k-1\big\}$$

and define a map $\pi_k\colon \mathcal{P}_k(X) \to X \times X$ by setting

$$\pi_k(\gamma_1, \dots, \gamma_k) = \big(\gamma_1(0), \gamma_k(1)\big) \text{ for } (\gamma_1, \dots, \gamma_k) \in \mathcal{P}_k(X).$$

It is not difficult to see that π_k is a fibration. A (G,k)-*motion planner* on an open subset $U \subseteq X \times X$ is a section of π_k over U, i.e. a map $s\colon U \to \mathcal{P}_k(X)$ such that $\pi_k \circ s = \mathrm{id}_U$. Denote by $TC^{G,k}(X)$ the least integer $\ell \geq 1$ such that there exists an open cover of $X \times X$ by ℓ sets which admit (G,k)-motion planners.

By design, the following hold:

(1) $TC^{G,k+1}(X) \leq TC^{G,k}(X)$,
(2) $TC^{G,k}(X) \leq TC^{H,k}(X)$, where $H \subseteq G$ is any subgroup.

It is also true, although less apparent, that $TC^{G,k}$ is a G-homotopy invariant. The bottom line is that for any G-space X, $\big(TC^{G,k}(X)\big)_{k=1}^{\infty}$ is a decreasing sequence of G-homotopy invariants, and since the sequence is bounded from below by 1, it stabilises at some point. Set

$$TC^{G,\infty}(X) = \lim_{k\to\infty} TC^{G,k}(X).$$

This is the *effective topological complexity* of X. A selection of its properties will be discussed during the talk. In particular, we will see that it enjoys a lower bound in terms of nilpotency of the kernel of the cup product homorphism corresponding to the orbit space X/G, provided that G is a finite group.

Motion planning in the context of effective TC can be interpreted as follows. A path output by a (G, ∞)-motion planner is typically no longer continuous, but its discontinuities are of prescribed nature — they are parametrised by symmetries. Whenever a robot follows such a path and runs into a point of discontinuity, it re-interprets its position accordingly within a batch of symmetric positions, and then resumes normal movement.

The talk is based on [1].

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Survey talk on rational topological complexity

J.G. Carrasquel-Vera

This series of talks aims to give an elementary introduction to how rational homotopy methods can be applied to study topological complexity.

More explicitly, the topological complexity of a space is usually computed through the use of bounds. The most important and used lower bound for it is the *zero-divisor cup lenght* of the space. If we work over $\mathbb{Q}$, we can improve this lower bound by taking invariants that come from Sullivan's *minimal model* of the space.

The first part of the talk is an introduction to Sullivan's rational homotopy theory[15]. Here we explain the rationalisation functor and give an explicit description of the category of *commutative differential graded algebras.* We introduce the minimal model of a space and show how rational homotopical information on the space can be obtained through its minimal model[8]. As examples we build minimal models of spheres and introduce the concept of formality. To finish, we give an example of a space which is not formal.

The second part of the talk is focused on *rational Lusternik-Schnirelmann category.* This part serves as a smooth transition to our main objective. This is because LS category is a simpler case of sectional category than topological complexity. We start by briefly describing the Ganea characterisation of LS category and using it to define rational invariants related to the LS category of X such as the *rational Toomer invariant*, *module LS category* and *rational LS category.* We then state the important theorem of Félix and Halperin [6] that gives a way to compute these invariants through minimal models. We compute some examples showing that some of these invariants do not coincide and state a theorem of Hess saying that module category and rational category coincide[11]. To finish this part, some theorems on category of products and Poincaré duality complexes of Félix-Haleperin-Lemaire are exposed[7], also the mapping theorem is explained.

The third part is basically a generalisation to topological complexity of previous part. We start with the mapping theorem for topological complexity due to Grant-Lupton-Oprea[10]. Then we introduce analogous invariants for topological complexity through the Ganea characterisation. We also explain explicity semi-free models for this construction given in [9]. We state a generalisation of Félix-Halperin's theorem for topological complexity[1] and explain how it gives a proof to the Jessup-Murillo-Parent conjecture[12]. We then show how, for the sake of computation, some of the hypotheses on this theorem can be relaxed. Several examples are then explained and generalisations of Félix-Halperin-Lemaire theorems are given[4, 5]. To finish, we talk about a possible generalisation of Hess' theorem and give a cautionary example of Stanley[14].

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Topological Complexity and Invariant Topological Complexity

Marzieh Bayeh
(joint work with Soumen Sarkar)

In the first part of the talk we compute the topological complexity of some locally standard torus manifolds.

Topological complexity of the configuration space of a mechanical system was introduced by M. Farber [6] to estimate the complexity of a motion planning algorithm. For a topological space X the topological complexity of X, denoted by $TC(X)$, is the least number of open subsets that form a covering for $X \times X$ in which each open subset admits a section to the following fibration

$$\pi : PX \to X \times X.$$

In fact $TC(X)$ is the Schwarz genus [9] of the map $\pi : PX \to X \times X$.

Davis and Januszkiewicz [5] introduced the topological counterpart of complex projective variety; these are called quasitoric manifolds in [1]. Generalizing this idea, Masuda [8] introduced *torus manifolds*.

A torus manifold is a $2n$-dimensional closed connected orientable smooth manifold M with an effective smooth action of an n-dimensional torus $\mathbb{T}^n = (\mathbb{S}^1)^n$.

Let P be an n-dimensional nice manifold with corners (see [4] for the definition of a nice manifold with corners). A closed, connected, oriented, and smooth $2n$-dimensional $\mathbb{T}^n$-manifold M is called a *locally standard torus manifold* over P if the following conditions are satisfied:

(1) the $\mathbb{T}^n$-action is locally standard, meaning, for every point $y \in M$ there exist
  (a) a $\mathbb{T}^n$-invariant open neighborhood U_y of y in M;
  (b) a $\mathbb{T}^n$-invariant open subset V in $\mathbb{C}^n$;
  (c) a diffeomorphism $\psi : U_y \to V$;
  (d) an isomorphism $\delta_y : \mathbb{T}^n \to \mathbb{T}^n$;
  such that for all $(t, x) \in \mathbb{T}^n \times U_y$,
$$\psi(tx) = \delta_y(t)\psi(x);$$
(2) $\partial P \neq \emptyset$, where ∂P is the boundary of P;
(3) there is a projection map $\mathfrak{q} : M \to P$, constant on orbits, which maps every l-dimensional orbit to a point in the interior of an l-dimensional face of P.

In the case that P is a simple polytope, M is called a *quasitoric manifold*.

We compute the topological complexity of locally standard torus manifold over a nice manifold with corners P, while ∂P contains the boundary of a simple polytope.

Theorem 1. [3, Theorem 5.7] Let M be a $2n$-dimensional locally standard torus manifold over a nice manifold with corners P, such that ∂P contains the boundary of a simple polytope.Then

$$TC(M) \geq 2n + 1.$$

For the case that the orbit space is simply connected, we show the following theorem.

Theorem 2. [3, Theorem 5.8] Let M be a $2n$-dimensional locally standard torus manifold with a simply connected orbit space P. If a connected component of ∂P is the boundary of a simple polytope, then

$$TC(M) = 2n + 1.$$

Corollary 3. [3, Corollary 5.9] Let M be a $2n$-dimensional quasitoric manifold over a simple polytope P. Then

$$TC(M) = 2n + 1.$$

In [2], it is shown that if M_1 and M_2 are two $2n$-dimensional quasitoric manifolds, then the equivariant connected sum of M_1 and M_2, denoted by $M_1 \# M_2$, is simply connected for all n and k except $k = n = 1, 2$. Therefore we have the following results.

Corollary 4. [3, Corollary 5.10] Let M_1 and M_2 be two quasitoric manifolds. Then for any k and n except $k = n = 1, 2$, we have

$$TC(M_1 \#_{\mathbb{T}^k} M_2) = 2n + 1.$$

Corollary 5. [3, Corollary 5.11] Let M be a 4-dimensional locally standard torus manifold over P, such that a connected component of ∂P is the boundary of a polygon. Then

$$TC(M) = 5.$$

In the second part of the talk we examine the cases in which the invariant topological complexity is infinite.

Invariant topological complexity was introduced by W. Lubawski and W. Marzantowicz [7] as a generalization of topological complexity for G-spaces. Let X be a G-space, and $\urcorner(X)$ be the saturation of $\Delta(X)$ with respect to the $G \times G$-action,

$$\urcorner(X) = (G \times G) \cdot \Delta(X) \subset X \times X.$$

Then the invariant topological complexity of X is defined to be,

$$TC^G(X) =_{\urcorner(X)} cat_{G\times G}(X \times X).$$

In [3], we define the concept of orbit class, which is the equivalent class of an orbit, considering the equivalence relation as being G-homotopic. Then we show that if a G-space X has more than one minimal orbit class, then $\urcorner(X)$ does not intersect all minimal orbit classes of the $(G \times G)$-space $X \times X$. Thus we have the following theorem.

Theorem 6. [3, Theorem 4.7] If X has more than one minimal orbit class, then

$$TC^G(X) = \infty.$$

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Remarks on the topological complexity of a map

Petar Pavešić

The concept of the topological complexity of a space X was introduced by M. Farber in [1] as a measure of inherent discontinuities that arise in motion planning algorithms in robotics. The topological complexity of a map $f\colon X \to Y$ is an important generalization suggested by A. Dranishnikov and it can be used to model a much wider range of problems that arise in robotics. One may consult [3] for an introduction geared toward applications.

Let X be a path-connected space and let $f\colon X \to Y$ be a surjective continuous map. We denote by PX the space of all maps from the unit interval to X and by $p\colon PX \to X \times Y$ the map $p(\alpha) := (\alpha(0), f(\alpha(1)))$. A *motion plan* for f consists of a subspace $Q \subseteq X \times Y$ which is a domain of a local section $s\colon Q \to PX$ such that $ps = 1_Q$. For reasons that will be explained later we require that the motion plan domains Q are Euclidean Neighbourhood Retracts (ENR). The *topological complexity* $TC(f)$ of f is defined to be the minimal number of motion plans needed to cover entire $X \times Y$.

Examples

(1) The topological complexity of the identity map on X clearly coincides with the topological complexity of the space X, as defined by Farber.

(2) If X is the configuration space of a system with several linked or free parts (e.g. a humanoid robot or a group of vehicles on a road) one is often interested in a movement of one particular component. Then the motion planning problem can be modelled as the motion planning for the projection map from X to the configuration space of that part that need to be considered.
(3) Manipulation of a robot arm requires control of the movement of its joints that results in a movement of the arm in the cartesian space. The correspondence is given by the forward kinematic map $f\colon X \to Y$ of the arm, where $X \subseteq (S^1)^n$ is a subset of the joint space, and $Y \subset \mathbb{R}^3 \times SO(3)$ is a subspace of the set of poses (i.e. positions and orientations) of the arm actuator. As before, one may also restrict the attention to the spatial position of the actuator or to its orientation.
(4) In many cases different configurations of a mechanism can be functionally equivalent, e.g. when different positions of a robot arm have the same grip. This can be interpreted as a quotient map with respect to a group action $X \to X/G$ and the topological complexity of the quotient map is a version of the equivariant topological complexity.
(5) Similar setting applies to problems in robotics dynamics: if we denote by CX the space of smooth paths in X that has curvature bounded by some constant, then the minimal number of local sections of the evaluation map $ev_{0,1}\colon CX \to X \times X$ measures the ability to navigate a machine when only certain curvatures of path are allowed (like in the car-parking problem). Although the formulation of the problem resembles that of the topological complexity of a single space, the solution methods are closer to the complexity of a map because the map $ev_{0,1}$ is not a fibration.

There are two basic situations when the computation of $TC(f)$ can be easily related to the complexity of X and Y (cf. [3]):

If $f\colon X \to Y$ admits a global section, then $TC(X) \geq TC(f) \geq TC(Y)$.

If $f\colon X \to Y$ is a fibration, then $TC(Y) \geq TC(f)$.

However, note that the forward kinematic maps that appear in applications to robotics are in general not fibrations and they do not admit global inverse kinematic maps.

An important fact that allows much flexibility in the computation of the topological complexity of a space is that the value of $TC(X)$ is independent of the choice of open, closed or ENR sets as domains of its motion plans that cover $X \times X$. The situation is less favourable when one deals with $TC(f)$. In fact, a motion plan in some neighbourhood of a point (x, y) implies the existence of a local section of f in some neighbourhood of y. If f is not locally sectionable around some $y \in Y$, then (x, y) cannot be internal point of any motion plan domain. A similar argument shows that we cannot require $X \times Y$ to be covered only by motion plans with closed domains.

If $TC(f) > 1$ then there exists a point $(x, y) \in X \times Y$ such that every neighbourhood of that point intersects several motion plan domain, so the corresponding motion planer is forced to choose among several possible motion plans. This property is called *instability* of the motion planner. Farber [2] showed that for every motion planer in a space X there is always a point $(x, x') \in X \times X$ such that every neighbourhood of that point intersects at least $TC(X)$ motion plan domains. A similar result can be proved for $TC(f)$:

Theorem. Given any partition of $X \times Y$ into ENRs that admit local motion plans there exists a point $(x, y) \in X \times Y$ such that every neighbourhood of it intersects at least $TC(f)$ motion plan domains.

The following useful formula relates the complexity of a product of two maps to the complexity of its factors.

Theorem. Given maps $f_1 \colon X_1 \to Y_1$ and $f_2 \colon X_2 \to Y_2$ the topological complexity of the product map $f_1 \times f_2 \colon X_1 \times X_2 \to Y_1 \times Y_2$ satisfies

$$TC(f_1 \times f_2) < TC(f_1) + TC(f_2).$$

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Higher TC of some polyhedral product spaces and its asymptotic behavior in random models

Bárbara Gutiérrez

(joint work with Jesús González, Hugo Mas and Sergey Yuzvinsky)

This work is divided in two parts. Both parts are about a homotopy invariant called *Higher (or sequential) topological complexity*. The first part is related to this invariant of a family of polyhedral product spaces and the second part is about the asymptotic behavior of this invariant of a specific random model. Let's start by defining what the higher topological complexity is, and after that, some general facts about this invariant.

For a positive integer $s \in \mathbb{N}$, the s-th (higher or sequential) topological complexity of a path connected space X, $\mathrm{TC}_s(X)$, is defined in [8] as the reduced Schwarz genus of the fibration $e_s = e_s^X : X^{J_s} \to X^s$ given by $e_s(f) = (f_1(1), \ldots, f_s(1))$. Here J_s denotes the wedge of s copies of the closed interval $[0, 1]$, in all of which $0 \in [0, 1]$ is the base point, and we think of an element f in the function space X^{J_s} as an s-tuple $f = (f_1, \ldots, f_s)$ of paths in X all of which start at a common point. Thus, $\mathrm{TC}_s(X) + 1$ is the smallest cardinality of open covers $\{U_i\}_i$ of X^s so that, on each U_i, e_s admits a section σ_i. In such a cover, U_i is called a *local* *domain*, the corresponding section σ_i is called a *local rule*, and the resulting family of pairs $\{(U_i, \sigma_i)\}$ is called a *motion planner*. The latter is said to be *optimal* if it has $\mathrm{TC}_s(X) + 1$ local domains.

For practical purposes, the openness condition on local domains can be replaced (without altering the resulting numeric value of $\mathrm{TC}_S(X)$) by the requirement that local domains to be pairwise disjoint Euclidean neighborhood retracts (ENR).

Since e_s is the standard fibrational substitute of the diagonal inclusion $d_s = d_s^X : X \hookrightarrow X^s$, $\mathrm{TC}_s(X)$ coincides with the reduced Schwarz genus of d_s. This suggests part (a) in the following definition, where we allow cohomology with local coefficients:

Let X be a connected space and R be a commutative ring.

(a) Given a positive integer s, we denote by $\mathsf{zcl}_s\,(H^*(X;R))$ the cup-length of elements in the kernel of the map induced by d_s in cohomology. Explicitly, $\mathsf{zcl}_s\,(H^*(X;R))$ is the largest integer m for which there exist cohomology classes $u_i \in H^*(X^s, A_i)$, where X^s is the s-th Cartesian power of X and each A_i is a system of local coefficients, such that $d_s^*(u_i) = 0$ for $i = 1, \ldots, m$ and $0 \neq u_1 \otimes \cdots \otimes u_m \in H^*(X^s,\, A_1 \otimes \cdots \otimes A_m)$.

(b) The homotopy dimension of X, $\mathrm{hdim}(X)$, is the smallest dimension of CW complexes having the homotopy type of X. The connectivity of X, $\mathrm{conn}(X)$, is the largest integer c such that X has trivial homotopy groups in dimensions at most c. We set $\mathrm{conn}(X) = \infty$ when no such c exists.

One has that, For a path connected space X,

$$\mathsf{zcl}_s\,(H^*(X;R)) \leq \mathrm{TC}_s(X) \leq \frac{s\ \mathrm{hdim}(X)}{\mathrm{conn}(X)+1}.$$

In particular for every path connected X,

$$\mathrm{TC}_s(X) \leq s\,\mathrm{hdim}(X).$$

For a proof see [1, Theorem 3.9] or, more generally, [9, Theorems 4 and 5].

Part I

The spaces we work with arise as follows. For a positive integer k_i consider the minimal cellular structure on the k_i-dimensional sphere $S^{k_i} = e^0 \cup e^{k_i}$. Here e^0 is the base point, which is simply denoted by e. Take the product (also minimal) cell decomposition in

$$\mathbb{S}(k_1, \ldots, k_n) := S^{k_1} \times \cdots \times S^{k_n} = \bigsqcup_J e_J$$

whose cells e_J, indexed by subsets $J \subseteq [n] = \{1, \ldots, n\}$, are defined as $e_J = \prod_{i=1}^n e^{d_i}$ where $d_i = 0$ if $i \notin J$ and $d_i = k_i$ if $i \in J$. Explicitly,

$$e_J = \left\{(x_1, \ldots, x_n) \in \mathbb{S}(k_1, \ldots, k_n) \mid x_i = e^0 \text{ if and only if } i \notin J\right\}.$$

It is well known that the lower bound given by zcl_s is optimal for $\mathbb{S}(k_1, \ldots, k_n)$; our Theroem below asserts that the same phenomenon holds for subcomplexes.

Note that, while $\mathbb{S}(k_1,\ldots,k_n)$ can be thought of as the configuration space of a mechanical robot arm whose i-th node moves freely in k_i dimensions, a subcomplex X of $\mathbb{S}(k_1,\ldots,k_n)$ encodes the information of the configuration space that results by imposing restrictions on the possible combinations of simultaneously moving nodes of the robot arm. Our main theorem states:

Theorem. A subcomplex X of $\mathbb{S}(k_1,\ldots,k_n)$ has $\mathrm{TC}_s(X) = \mathsf{zcl}_s(H^*(X;\mathbb{Q}))$ provided all of the k_i have the same parity.

Our methods imply that the Theorem could equally be stated using cohomology with coefficients in any ring of characteristic 0. We give explicit descriptions of $\mathsf{zcl}_s(H^*(X;\mathbb{Q}))$ that generalize those in [2, 11]. In addition, the optimality of this cohomological lower bound will be a direct consequence of the fact that we actually construct an optimal motion planner. Our construction generalizes, in a highly non-trivial way, the one given first by Yuzvinsky ([10]) for $s = 2$ when X is an arrangement complement and then independently by Cohen-Pruidze ([2], as corrected in [7]) in a more general case.

Part II

For a positive integer n and probability parameter p, $0 < p < 1$, consider the Erdös-Rényi model $\mathcal{G}(n,p)$ of random graphs Γ in which each edge of the complete graph on the n vertices $[n] = \{1,2,\ldots,n\}$ is included in Γ with probability p independently of all other edges. In other words, the random variables e_{ij}, $1 \leq i < j \leq n$, defined by

$$e_{i,j}(\Gamma) = \begin{cases} 1, & \text{if } (i,j) \text{ is an edge in } \Gamma; \\ 0, & \text{otherwise,} \end{cases}$$

are independent and have $P(e_{i,j} = 1) = p$. In this context, the clique random variable $C = C_{n,p}$,

$$C(\Gamma) = \max\{r \in \mathbb{N} : \Gamma \text{ admits a complete subgraph with } r \text{ vertices}\},$$

has been the subject of intensive research since the 1970's. Matula provided in [5] numerical evidence suggesting that C has a very peaked density around $2\log_q n$ where $q = 1/p$. Such a property was established in [4] by Grimmett and McDiarmid who proved that, as $n \to \infty$,

$$\frac{C}{\log_q n} \to 2.$$

A much finer result was proved by Matula. Let $\lfloor x \rfloor$ stands for the integral part of the real number x, and we set

$$z = z(n,p) = 2\log_q n - 2\log_q \log_q n + 2\log_q(e/2) + 1$$

where, as above, $q = 1/p$. The Matula result states:

For $0 < p < 1$ and $\epsilon > 0$,

$$\lim_{n\to\infty} \mathrm{Prob}\Big(\lfloor z - \epsilon \rfloor \leq C \leq \lfloor z + \epsilon \rfloor\Big) = 1.$$

It should be stressed that the probability parameter p is fixed throughout the limiting process.

Since z is logarithmic in n, it is conceivable to ask if any random graph admits the existence of arbitrarily many pairwise disjoint asymptotically-largest-possible cliques. A first step in such a direction was taken in [3] in response to the desire of understanding the stochastic properties of the collision-free motion planning of multiple particles on graphs with a large number of vertices. Roughly speaking, Costa and Farber showed that, with probability tending to 1 as n tends to infinity, a random graph in $\mathcal{G}(n,p)$ has a pair of disjoint asymptotically-largest-possible cliques. In our first main result (see below) we show, more generally, that for any fixed positive integer s, and with probability tending to 1 as n tends to infinity, a random graph in $\mathcal{G}(n,p)$ has s pairwise-disjoint such asymptotically-largest-possible cliques.

An s-th multi-clique of size r of a (random) graph $\Gamma \in \mathcal{G}(n,p)$ is an ordered s-tuple $(V_1,\dots,V_s)$ of pairwise disjoint subsets $V_i \subseteq [n]$, each of cardinality r, such that each of the induced subgraphs $\Gamma_{|V_i}$ is complete.
We do not require that each V_i is a clique of Γ (i.e. a complete subgraph of Γ with the maximal possible number of vertices).

We proved that:

Theorem. Fix a positive integer s, a positive real number ϵ, and a probability parameter $p \in (0,1)$. Then, with probability tending to 1 as $n \to \infty$, a random graph in $\mathcal{G}(n,p)$ has an s-th multi-clique of size $\lfloor z-\epsilon \rfloor$.

We use previous result in order to generalize Costa and Farber's result to the sequential motion planning realm in topological robotics. We then use that result to give the following asymptotical description (with ϵ-resolution spikes of at most s units) of the s-th topological complexity of random right angled Artin groups:

Theorem. For a random graph $\Gamma \in \mathcal{G}(n,p)$, let K_Γ stand for the (random) Eilenberg-MacLane space associated to the right angled Artin group defined by Γ. Then, for any positive real constant ϵ, positive integer s, and probability parameter $p \in (0,1)$, the random variable TC_s given by $\mathrm{TC}_s(\Gamma) = \mathrm{TC}_s(K_\Gamma)$ satisfies

$$\lim_{n\to\infty} \mathrm{Prob}\left(s\lfloor z-\epsilon\rfloor \le \mathrm{TC}_s \le s\lfloor z+\epsilon\rfloor\right) = 1.$$

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Topological complexity and Hopf invariants

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(joint work with Mark Grant and Lucile Vandembroucq)

The sectional category of a fibration $p\colon E \to B$ is the least number of open sets needed to cover B, on each of which p admits a continuous local section. This concept, originally studied by A. S. Schwarz [12] under the name *genus*, has found applications in diverse areas. Notable special cases include the Lusternik–Schnirelmann category (for which the standard reference has become the monograph [2] by Cornea, Lupton, Oprea and Tanré) and Farber's topological complexity [3], both of which are homotopy invariants of spaces which arise as the sectional category of associated path fibrations. The LS-category is classical and related to critical point theory, while topological complexity was conceived in the early part of the twenty-first century as part of a topological approach to the motion planning problem in Robotics. It is common to normalise these invariants so that the sectional category of a fibration with section is zero, a convention which we will adopt in this talk.

Most of the existing estimates for sectional category are cohomological in nature and are based on obstruction theory. We aim at producing more refined estimates using methods from unstable homotopy theory. There is an extensive literature on the application of Hopf invariants to LS-category, originating with Berstein and Hilton [1] and including spectacular applications by Iwase [7], Stanley [14], Strom [15] and others (a nice summary can be found in Chapter 6 of [2]). Building on and generalizing the work of these authors, we develop a theory of generalized Hopf invariants in the setting of sectional category. We then apply our theory to give new computations of topological complexity which we believe would not be possible using obstruction-theoretic arguments.

Our first application is to the computation of the topological complexity of two-cell complexes $X = S^p \cup_\alpha e^{q+1}$. The LS-category of such a space X is determined by the Berstein–Hilton–Hopf invariant

$$H(\alpha) \in \pi_q(\Sigma\Omega S^p \wedge \Omega S^p) \cong \pi_q(S^{2p-1} \vee S^{3p-2} \vee \cdots)$$

of the attaching map $\alpha : S^q \to S^p$ [1]. When $p \geq 2$, we have

$$\mathrm{cat}(X) = \begin{cases} 1 & \text{if } H(\alpha) = 0, \\ 2 & \text{if } H(\alpha) \neq 0. \end{cases}$$

In the metastable range $2p-1 \leq q \leq 3p-3$ we may identify $H(\alpha)$ with its projection onto the bottom cell $H_0(\alpha) \in \pi_q(S^{2p-1})$. If $H_0(\alpha) \neq 0$ then $\mathrm{cat}(X) = 2$, which by standard inequalities implies that $2 \leq \mathrm{TC}(X) \leq 4$. In almost all cases, the usual cohomological bounds fail to determine the exact value of $\mathrm{TC}(X)$, for reasons of dimension. Using Hopf invariants, however, we are able to identify many cases with $\mathrm{TC}(X) \leq 3$, as well as many cases with $\mathrm{TC}(X) \geq 3$:

Theorem. Let $X = S^p \cup_\alpha e^{q+1}$, where $\alpha : S^q \to S^p$ is in the metastable range $2p-1 < q \leq 3p-3$ and $H_0(\alpha) \neq 0$. Then:

(1) $\mathrm{TC}(X) \leq 3$ if and only if $(4 + 2(-1)^p)H_0(\alpha) \circledast H_0(\alpha) = 0$.
(2) $\mathrm{TC}(X) \geq 3$ provided $(2 + (-1)^p)H_0(\alpha) \neq 0$.

The condition $(2 + (-1)^p)H_0(\alpha) \neq 0$ holds automatically if p is odd, while the condition $(4 + 2(-1)^p)H_0(\alpha) \circledast H_0(\alpha) = 0$ holds if q is even. Combining these two theorems we get the precise value $\mathrm{TC}(X) = 3$ for large classes of two-cell complexes. We are also able to draw conclusions about $\mathrm{TC}(X)$ outside of the metastable range, under the additional assumption $H(\alpha) = H_0(\alpha)$.

Example. If p is odd, $2p-1 < q \leq 3p-3$, and the join square $H_0(\alpha) \circledast H_0(\alpha)$ is a non-trivial element of odd torsion, then $\mathrm{TC}(X) = 4$.

We also give a full description of $\mathrm{TC}(X)$ for $X = S^p \cup_\alpha e^{2p}$. The proofs are, however, much more elementary than that of the theorem above.

Our second application is to the analogue of Ganea's conjecture for topological complexity. Recall that the product inequality $\mathrm{cat}(X \times Y) \leq \mathrm{cat}(X) + \mathrm{cat}(Y)$ is satisfied by LS-category. Examples of strict inequality were given by Fox [4], involving Moore spaces with torsion at different primes. Ganea asked, in his famous list of problems [5], if we always get equality when one of the spaces involved is a sphere. That is, if X is a finite complex, is it true that

$$\mathrm{cat}(X \times S^k) = \mathrm{cat}(X) + 1 \qquad \text{for all } k \geq 1?$$

A positive answer became known as *Ganea's conjecture.* The conjecture remained open for nearly 30 years, shaping research in the subject. It was shown to hold for simply-connected rational spaces by work of Jessup [9] and Hess [6], and for large classes of manifolds by Singhof [13] and Rudyak [11], until eventually proven to be false in general by Iwase [8, 7]. Iwase's counter-examples are two-cell complexes X outside of the metastable range, whose Berstein–Hilton–Hopf invariants are essential but stably inessential, from which it follows that $\mathrm{cat}(X) = \mathrm{cat}(X \times S^k) = 2$.

The analogous question for topological complexity (which also satisfies the product inequality) asks whether, for any finite complex X and $k \geq 1$, we always have an equality

$$\mathrm{TC}(X \times S^k) = \mathrm{TC}(X) + \mathrm{TC}(S^k) = \begin{cases} \mathrm{TC}(X) + 1 & \text{if } k \text{ odd}, \\ \mathrm{TC}(X) + 2 & \text{if } k \text{ even}. \end{cases} \tag{1}$$

This question was raised by Jessup, Murillo and Parent [10], who proved that equation (1) holds when $k \geq 2$ for any formal, simply-connected rational complex X of finite type. We give a counter-example to (1) for all even k, using Hopf invariant techniques.

Theorem. Let Y be the stunted real projective space $\mathbb{R}P^6/\mathbb{R}P^2$, and let $X = Y \vee Y$. Then for all $k \geq 2$ even,

$$\mathrm{TC}(X) = 4 \quad \text{and} \quad \mathrm{TC}(X \times S^k) = 5.$$

The main idea leading to the above results is based on the fact that the sectional category of a fibration relative to a subspace increases by at most one on attaching a cone, and moreover the section over the cone can be controlled by the triviality of a certain set of generalized Hopf invariants. We then investigate Hopf invariants for cartesian products of fibrations. Using naturality of the exterior join construction, we prove our key result, which states that Hopf invariants of a product can be obtained as joins of Hopf invariants of the factors, composed with a certain topological shuffle map

$$\Phi^{A,B}_{n,m} : J^n(A) \circledast J^m(B) \to J^{n+m+1}(A \times B),$$

constructed from the standard decomposition of the product of simplices $\Delta^n \times \Delta^m$ into simplices Δ^{n+m}. We describe the effect of this map in homology, in terms of algebraic shuffles.

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Topological complexity of configuration spaces and related objects, I

Daniel C. Cohen

Investigation of the collision-free motion of n distinct ordered particles in a topological space X leads one to study the (classical) configuration space

$$F(X,n) = \{(x_1,\dots,x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\},$$

and the topological complexity of this space. For a path-connected topological space Y, and Y^I the space of all continuous paths $\gamma\colon [0,1] \to Y$ (with the compact-open topology), the *topological complexity* of Y is the sectional category (or Schwarz genus) of the fibration $\pi\colon Y^I \to Y \times Y$, $\gamma \mapsto (\gamma(0),\gamma(1))$, $\mathsf{TC}(Y) = \operatorname{secat}(\pi)$. This homotopy invariant, introduced by Farber, provides a topological approach to the motion planning problem from robotics

In this lecture, and the next, we survey results on the topological complexity of configuration spaces $F(X,n)$ in the case where X is an orientable surface, as well as related objects. The general principle is as follows:

The topological complexity is as large as possible, given natural constraits.

Throughout the discussion, we will make use of the following basic tools. For details and other relevant facts, see Farber's survey [3]. Additional references mentioned but not explicitly cited below are listed at the end of the second lecture.

$$\mathsf{TC}(Y) \leq 2\cdot \operatorname{hdim}(X) + 1 \qquad \mathsf{TC}(Y \times Z) \leq \mathsf{TC}(Y) + \mathsf{TC}(Z) - 1$$

$$\mathsf{TC}(Y) > \mathsf{zcl}\, H^*(Y) = \text{cup length}\big[\ker\big(H^*(Y)\otimes H^*(Y) \xrightarrow{\cup} H^*(Y)\big)\big]$$

We call the first two of these the dimension and product inequalities, and use cohomology with $\mathbb{C}$-coefficients (unless stated otherwise) in the context of the third, the zero-divisor cup length. We use the unreduced notion of topological complexity.

The plane: $X = \mathbb{R}^2 = \mathbb{C}$

Theorem 1 (Farber-Yuzvinsky [6])**.** $\mathsf{TC}(F(\mathbb{C},n)) = 2n-2$ for $n \geq 2$

We recall some relevant facts from the theory of hyperplane arrangements.

$$F(\mathbb{C},n) = \{(x_1,\dots,x_n) \in \mathbb{C}^n \mid x_i \neq x_j \text{ if } i \neq j\} \cong M_n \times \mathbb{C}, \text{ where}$$
$$M_n = \{(y_1,\dots,y_{n-1}) \in \mathbb{C}^{n-1} \mid y_i \neq 0\,\forall\, i, y_i - y_j \neq 0\,\forall\, i<j\}$$

M_n is the complement of an essential, central hyperplane arrangement in $\mathbb{C}^{n-1}$.

An *arrangement* $\mathcal{A} = \{H_1,\dots,H_m\}$ in $\mathbb{C}^\ell$ is a finite collection of affine hyperplanes, $H_i = \{f_i = 0\}$, f_i a linear polynomial. $\mathcal{A}$ is *essential* if $\exists\, \ell$ hyperplanes

in $\mathcal{A}$ whose intersection is a point. $\mathcal{A}$ is *central* if $0 \in H_i$ for each $i \iff f_i$ is a linear form for each i. The complement of $\mathcal{A}$ is $M = M(\mathcal{A}) = \mathbb{C}^\ell \smallsetminus \bigcup_{i=1}^m H_i$.
$\mathcal{A}$ central $\Longrightarrow$ restriction of the Hopf bundle $p\colon \mathbb{C}^\ell \smallsetminus \{0\} \to \mathbb{C}P^{\ell-1}$ to M is trivial.

$$\begin{array}{ccccc} \mathbb{C}^* & & & & \mathbb{C}^* \\ \downarrow & & & & \downarrow \\ M & \to & \mathbb{C}^\ell \smallsetminus H_1 & \to & \mathbb{C}^\ell \smallsetminus \{0\} \\ \downarrow & & \downarrow & & \downarrow \\ p(M) & \to & \mathbb{C}P^{\ell-1} \smallsetminus \mathbb{C}P^{\ell-2} & \to & \mathbb{C}P^{\ell-1} \end{array}$$

Consequently, $M \cong p(M) \times \mathbb{C}^*$
$\mathcal{A}$ essential $\Longrightarrow$ $\operatorname{hdim} p(M) = \ell - 1$
$\mathsf{TC}(M) \leq \mathsf{TC}(p(M)) + \mathsf{TC}(\mathbb{C}^*) - 1$
$\mathsf{TC}(M) \leq 2\ell$
(product, dimension inequalities)

In particular, $F(\mathbb{C}, n) \cong M_n \times \mathbb{C} \simeq M_n \cong p(M_n) \times \mathbb{C}^* \implies \mathsf{TC}(F(\mathbb{C}, n) \leq 2n - 2$.
For the reverse inequality, we use the zero-divisor cup length. The cohomology ring $A = H^*(F(\mathbb{C}, n))$ (with $\mathbb{C}$ coefficients) is classically known, thanks to work of Arnold and Cohen: A is generated by degree one classes $\omega_{i,j} = d\log(x_i - x_j) \in A^1$, $i < j$, with relations consequences of $\omega_{i,j}\omega_{i,k} - \omega_{i,j}\omega_{j,k} + \omega_{i,k}\omega_{j,k} = 0$, $i < j < k$.

Proposition 2 ([6])**.** The zero-divisors $\bar{\omega}_{i,j} = 1 \otimes \omega_{i,j} - \omega_{i,j} \otimes 1 \in A^1 \otimes A^1$ satisfy $\bar{\omega}_{1,2} \cdot \bar{\omega}_{1,3} \cdots \bar{\omega}_{1,n} \cdot \bar{\omega}_{2,3} \cdots \bar{\omega}_{2,n} \neq 0$. Consequently, $\mathsf{zcl}\, H^*(F(\mathbb{C}, n)) \geq 2n - 3$.

With the above considerations, this yields $\mathsf{TC}(F(\mathbb{C}, n)) = 2n - 2$. Similarly:

Theorem 3 (F.-Grant-Y. [5])**.** $\mathsf{TC}(F(\mathbb{C} \smallsetminus \{m \text{ points}\}, n)) = \begin{cases} 2n & m = 1 \\ 2n+1 & m \geq 2 \end{cases}$

Remark 4. The topological complexity of the configuration space of points in a higher dimensional Euclidean space is also known:

$$\mathsf{TC}(F(\mathbb{R}^k, n)) = \begin{cases} 2n - 1 & k \geq 3 \text{ odd } [6] \\ 2n - 2 & k \geq 4 \text{ even } [4] \end{cases}$$

Genus zero: $X = S^2$

Theorem 5 (C.-F. [1])**.** $\mathsf{TC}(F(S^2, n)) = \begin{cases} 3 & n = 1, 2 \\ 4 & n = 3 \\ 2n - 2 & n \geq 4 \end{cases}$

$n \leq 2$: $F(S^2, n) \simeq S^2$, and $\mathsf{TC}(S^2) = 3$.
$n = 3$: $F(S^2, 3) \cong \mathrm{PSL}(2, \mathbb{C}) \simeq \mathrm{SO}(3)$, and $\mathsf{TC}(\mathrm{SO}(3)) = \mathrm{cat}(\mathrm{SO}(3)) = 4$, as $\mathrm{SO}(3)$ is a connected Lie group, see [3].
$n \geq 4$: $F(S^2, n) \simeq \mathrm{SO}(3) \times F(S^2 \smallsetminus \{3 \text{ points}\}, n - 3)$. The results of [5] apply to $F(S^2 \smallsetminus \{3 \text{ points}\}, n-3) \cong F(\mathbb{C} \smallsetminus \{2 \text{ points}\}, n-3)$. The product inequality gives $\mathsf{TC}(F(S^2, n)) \leq 2n - 2$. Then one checks that $\mathsf{zcl}\, H^*(F(S^2, n); \mathbb{Z}_2) \geq 2n - 3$.

Genus one: $X = T = S^1 \times S^1$

Theorem 6 (C.-F. [1])**.** $\mathsf{TC}(F(T, n)) = 2n + 1$

$n = 1$: $F(T, 1) = T$, and $\mathsf{TC}(T) = \mathsf{TC}(S^1 \times S^1) = 3 = 2 + 1$.
$n \geq 2$: Since T is a group, we have $F(T, n) \cong T \times F(T \smallsetminus \{1 \text{ point}\}, n - 1)$ via
$\big((u, v), (uz_1, vw_1), \ldots, (uz_{n-1}, vw_{n-1})\big) \hookleftarrow \big((u, v), \big((z_1, w_1), \ldots, (z_{n-1}, w_{n-1})\big)\big)$.

Recall the classical Fadell-Neuwirth theorem: for X a manifold with $\dim X \geq 2$, and $\ell < n$, the map $F(X,n) \to F(X,\ell)$, $(x_1,\ldots,x_n) \mapsto (x_1,\ldots,x_\ell)$, is a bundle, with fiber $F(X \smallsetminus \{\ell \text{ points}\}, n-\ell)$. These bundles often admit sections. Use this result repeatedly, $F(T \smallsetminus \{1 \text{ point}\}, n-1) \to F(T \smallsetminus \{1 \text{ point}\}, n-2) \to \ldots$ each bundle with fiber homotopy equivalent to a wedge of circles and section, to see that $F(T \smallsetminus \{1 \text{ point}\}, n-1)$ is a $K(G,1)$-space. As G is an iterated semidirect product of free groups, the cohomological and geometric dimensions of G are both equal to $n-1$, $\mathrm{cd}(G) = \mathrm{gd}(G) = n-1$, see [2]. Thus, $\mathrm{hdim}(F(T \smallsetminus \{1 \text{ point}\}, n-1)) = n-1$. Then the dimension and product inequalities yield $\mathsf{TC}(F(T,n)) \leq 2n+1$.

The proof of the theorem is completed by showing that $\mathsf{zcl}\, H^*(F(T,n)) \geq 2n$. The tool here is the Cohen-Taylor/Totaro spectral sequence. For X a closed m-manifold, let $p_i\colon X^n \to X$ and $p_{i,j}\colon X^n \to X^2$ be the obvious projections. The inclusion $F(X,n) \to X^n$ yields a Leray spectral sequence converging to $H^*(F(X,n))$. The initial term is the quotient of the algebra $H^*(X^n) \otimes H^*(F(\mathbb{R}^m,n))$ by the relations $(p_i^*(u) - p_j^*(u)) \otimes \omega_{i,j}$ for $i \neq j$, $u \in H^*(X)$, and $\omega_{i,j}$ the generators of $H^*(F(\mathbb{R}^m,n))$ (from the Arnold/Cohen result noted previously in the case $m=2$). The first nontrivial differential is given by $d(\omega_{i,j}) = p_{i,j}^*(\Delta)$, where $\Delta \in H^m(X \times X)$ is the cohomology class dual to the diagonal.

As shown by Totaro, for X a smooth complex projective variety, this spectral sequence degenerates immediately, d above is the *only* nontrivial differential.

Proposition 7 ([1])**.** For X a smooth complex projective variety, let $H = H^*(X)$, and let I be the ideal in H generated by $\{p_{i,j}^*(\Delta) \mid i < j\}$. Then H/I is a subalgebra of $H^*(F(X,n))$. Thus, $\mathsf{zcl}\, H^*(F(X,n)) \geq \mathsf{zcl}\, H/I$ and $\mathsf{TC}(F(X,n)) \geq \mathsf{zcl}\, H/I + 1$.

In the case $X = T$, these considerations may be used to obtain the needed lower bound on $\mathsf{zcl}\, H^*(F(T,n))$. In this instance, the algebra $A = H/I$ may be described as follows: A is generated by degree one classes x_i, y_i, $1 \leq i \leq n$, with relations $x_i y_i = 0$, $2 \leq i \leq n$, $x_j y_k + x_k y_j$, $2 \leq j < k \leq n$, and their consequences.

Proposition 8 ([1])**.** The zero-divisors $\bar{x}_i = 1 \otimes x_i - x_i \otimes 1$ and $\bar{y}_i = 1 \otimes y_i - y_i \otimes 1$ in $A^1 \otimes A^1$ satisfy $\bar{x}_1 \cdot \bar{y}_1 \cdot \bar{x}_2 \cdot \bar{y}_2 \cdots \bar{x}_n \cdot \bar{y}_n \neq 0$. Consequently, $\mathsf{zcl}\, H^*(F(T,n)) \geq 2n$.

Higher genus: $X = \Sigma_g$, $g \geq 2$

Theorem 9 (C.-F. [1])**.** $\mathsf{TC}(F(\Sigma_g,n)) = 2n+3$

$n=1$: $F(\Sigma_g,1) = \Sigma_g$, and $\mathsf{TC}(\Sigma_g) = 5$.
$n \geq 2$: $F(\Sigma_g,n)$ is a $K(G,1)$, $G =$ pure braid group of Σ_g. Fadell-Neuwirth bundle $F(\Sigma_g,n) \to \Sigma_g$ has a section $\Longrightarrow$ $G \cong \pi_1(F(\Sigma_g \smallsetminus \{1 \text{ point}\}, n-1)) \rtimes \pi_1(\Sigma_g)$ $\Longrightarrow$ $\mathrm{cd}(G) = \mathrm{gd}(G) = n+1$. Thus, $\mathrm{hdim}(F(\Sigma_g,n)) = n+1$ and $\mathsf{TC}(F(\Sigma_g,n)) \leq 2n+3$. The reverse inequality is obtained in a manner analogous to the genus 1 case above.

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Topological complexity of configuration spaces and related objects, II

Daniel C. Cohen

We continue our discussion of the topological complexity of classical configuration spaces and related objects, now focusing primarily on the latter.

Punctured surfaces

Theorem 1 (C.-F. [3]). $\mathsf{TC}(F(\Sigma_g \smallsetminus \{m \text{ points}\}, n)) = 2n+1$ for $g \geq 1$ and $m \geq 1$

$n = 1$: $F(\Sigma_g \smallsetminus \{m \text{ points}\}, 1)$ is a wedge of circles, with topological complexity 3.
$n \geq 2$: Fadell-Neuwirth bundles can be used to show that $F(\Sigma_g \smallsetminus \{m \text{ points}\}, n)$ is a $K(G, 1)$, where G is an iterated semidirect product of free groups with $\mathrm{cd}(G) = \mathrm{gd}(G) = n$. It follows that $\mathsf{TC}(F(\Sigma_g \smallsetminus \{m \text{ points}\}, n)) \leq 2n + 1$.

Establishing the reverse inequality in this context is substantially more involved. As $\Sigma_g \smallsetminus \{m \text{ points}\}$ is *not* a projective variety, Totaro's theorem does not apply directly. Here, the inequality $\mathsf{zcl}\, H^*(F(\Sigma_g \smallsetminus \{m \text{ points}\}, n)) \geq 2n$ is obtained by using mixed Hodge structures (on the cohomology of the quasi-projective variety $F(\Sigma_g, n)$, etc.) in conjunction with Totaro's theorem and its consequences recorded in the previous lecture.

Orbit configuration spaces

Let X be a manifold without boundary, and Γ a (finite) group acting freely on X. The *orbit configuration space* $F_\Gamma(X, n)$ is the space of all ordered n-tuples of points in X which lie in distinct Γ-orbits,

$$F_\Gamma(X, n) = \{(x_1, \ldots, x_n) \mid \Gamma \cdot x_i \cap \Gamma \cdot x_j = \emptyset \text{ if } i \neq j\}.$$

If $\Gamma = \{1\}$ is trivial, $F_{\{1\}}(X, n) = F(X, n)$ is the classical configuration space.

For this discussion, we focus on the case $X = \mathbb{C}^*$, with $\Gamma = \mathbb{Z}_r$ acting by multiplication by $\zeta = \exp(2\pi i/r)$. The associated orbit configuration space is

$$\begin{aligned} F_{\mathbb{Z}_r}(\mathbb{C}^*, n) &= \{(x_1, \dots, x_n) \in (\mathbb{C}^*)^n \mid x_j \neq \zeta^k x_i,\ i \neq j,\ 1 \leq k \leq n\} \\ &= \mathbb{C}^n \smallsetminus \bigcup_{H \in \mathcal{A}_{r,n}} H, \text{ where } \mathcal{A}_{r,n} = \{x_i = 0\}_{1 \leq i \leq n} \cup \{x_j - \zeta^k x_i = 0\}_{\substack{i<j \\ 1 \leq k \leq n}} \end{aligned}$$

The arrangement $\mathcal{A}_{r,n}$ consists of the reflecting hyperplanes of the complex reflection group $G(r,n)$, the full monomial group. For instance, when $r = 2$, this the type B Coxeter group, and $\pi_1(F_{\mathbb{Z}_2}(\mathbb{C}^*, n))$ is the type B pure braid group.

Theorem 2. $\mathsf{TC}(F_{\mathbb{Z}_r}(\mathbb{C}^*, n)) = 2n$

This may be obtained from work of Farber-Yuzvinsky. As shown by Brieskorn (conjectured by Arnold), for any arrangement $\mathcal{A} = \{f_j = 0\}$ with complement M, $H^*(M; \mathbb{Z})$ is torsion free, and is generated by degree one classes $\frac{1}{2\pi i} d \log f_j$ ($\Longrightarrow M$ is $\mathbb{Q}$-formal). The conditions below insure that $\mathsf{zcl}\, H^*(M)$ is as large as possible.

Proposition 3 ([4])**.** Suppose $\mathcal{A}$ is central and essential in $\mathbb{C}^n$. If $\exists H_1, ..., H_{2n-1} \in \mathcal{A}$ with $\{f_1, \dots, f_n\}$ and $\{f_j, f_{n+1}, \dots, f_{2n-1}\}$, $1 \leq j \leq n$, all linearly independent, then $\mathsf{TC}(M) = 2n$.

This result applies to the reflection arrangements $\mathcal{A}_{r,n}$.

Another perspective

For a discrete group G, define $\mathsf{TC}(G) := \mathsf{TC}(Y)$, where Y is a $K(G,1)$-space. It is natural to ask for $\mathsf{TC}(G)$ in terms of algebraic properties of G.

Example 4. Associated to a simple graph Γ on n vertices is a right-angled Artin group G_Γ with generators corresponding to the vertices of Γ, and commutator relators corresponding to the edges. As discussed in the lecture of B. Gutiérrez [6], one has $\mathsf{TC}(G_\Gamma) = z(\Gamma) + 1$, where $z(\Gamma)$ is the maximal number of vertices of Γ covered by two (disjoint) cliques in Γ.

Many of the configuration spaces discussed previously are $K(G,1)$-spaces, for surface pure braid groups, for pure braid groups associated to reflection groups... For example, $\pi_1(F(\mathbb{C}, n)) = P_n$ is the Artin pure braid group. From the homotopy exact sequence of the Fadell-Neuwirth bundle $F(\mathbb{C}, m) \to F(\mathbb{C}, m-1)$, with fiber $\mathbb{C} \smallsetminus \{m-1 \text{ points}\}$ and section, we see (inductively) that $F(\mathbb{C}, n)$ is a $K(P_n, 1)$-space, and obtain a split, short exact sequence $1 \to F_{n-1} \to P_n \to P_{n-1} \to 1$, where F_k is the free group on k generators. Thus,

$$P_n = F_{n-1} \rtimes P_{n-1} = F_{n-1} \rtimes (F_{n-2} \rtimes P_{n-2}) = \cdots = F_{n-1} \rtimes (\cdots \rtimes (F_3 \rtimes (F_2 \rtimes F_1)))$$

is an iterated semidirect product of free groups. Further, the action of P_{n-1} on $H_*(F_{n-1}, ; \mathbb{Z})$ (via the Artin representation $P_{n-1} \to \mathrm{Aut}(F_n)$) is trivial.

An *almost-direct product of free groups* is an iterated semidirect product $G = F_{d_n} \rtimes \cdots \rtimes F_{d_1}$ of finitely generated free groups for which F_{d_i} acts trivially on $H_*(F_{d_j}; \mathbb{Z})$ for $i < j$. Thus, P_n is an almost-direct product of free groups.

The pure braid group $P_{r,n} = \pi_1(F_{\mathbb{Z}_r}(\mathbb{C}^*, n))$ associated to the full monomial group $G(r,n)$ also admits this structure. As first shown by Xicoténcatl [7], the map $F_{\mathbb{Z}_r}(\mathbb{C}^*, n) \to F_{\mathbb{Z}_r}(\mathbb{C}^*, n-1)$ defined by forgetting the last coordinate is a bundle, with fiber $\mathbb{C}^* \smallsetminus \{n-1 \text{ orbits}\} = \mathbb{C} \smallsetminus \{r(n-1)+1 \text{ points}\}$. This bundle may be realized as a pullback of the classical configuration space bundle $F(\mathbb{C}, N+1) \to F(\mathbb{C}, N)$ where $N = r(n-1)+1$, see [1]. It follows that this bundle admits a section, and the fundamental group of the base acts trivially on the homology of the fiber. Hence, $P_{r,n}$ is an almost-direct product of free groups.

Theorem 5 (C. [2])**.** *If $G = F_{d_n} \rtimes \cdots \rtimes F_{d_1}$ is an almost-direct product of free groups with $d_j \geq 2$ for each j, and $m \geq 0$, then $\mathsf{TC}(G \times \mathbb{Z}^m) = 2n + m + 1$.*

For an almost-direct product of n free groups G, $\mathrm{cd}(G) = \mathrm{gd}(G) = n$, the homology $H_*(G;\mathbb{Z})$ is torsion free, and the betti numbers are given by $\sum_{k\geq 0} b_k(G)\cdot t^k = (1+d_1t)(1+d_2t)\cdots(1+d_nt)$. Let $N = b_1(G) = d_1+d_2+\cdots+d_n$, and let $\mathfrak{a}\colon G \to \mathbb{Z}^N$ be the abelianization. The induced homomorphism $\mathfrak{a}^2\colon H^2(\mathbb{Z}^N) \to H^2(G)$ in integral cohomology is surjective, denote the kernel by $J = \ker(\mathfrak{a}^2)$, an ideal in the exterior algebra $H^*(\mathbb{Z}^N)$. The integral cohomology ring of G is then given by $H^*(G) \cong H^*(\mathbb{Z}^N)/J$. If $d_j \geq 2$ for each j, one can produce $2n$ zero-divisors in $H^1(G)\otimes H^1(G)$ with nonzero product. These considerations yield $\mathsf{TC}(G) = 2n+1$ for G as in the statement of the theorem. The general case $\mathsf{TC}(G\times\mathbb{Z}^N) = 2n+m+1$ may be obtained from this, the product inequality, and a straightforward analysis of the zero-divisor cup length of $H^*(G \times \mathbb{Z}^m)$.

Several of the results on the topological complexity of discrete groups mentioned above may also be obtained by other (group-theoretic) means.

Theorem 6 (Grant-Lupton-Oprea [5])**.** *If H and K are subgroups of G which satisfy $gHg^{-1} \cap K = \{1\}$ for all $g \in G$, then $\mathsf{TC}(G) \geq \mathrm{cd}(H \times K) + 1$.*

This may be used to recover the topological complexity of the pure braid group, $\mathsf{TC}(P_n) = \mathsf{TC}(F(\mathbb{C}, n)) = 2n-2$. As noted by, for instance, Birman, P_n has a free abelian subgroup $H \cong \mathbb{Z}^{n-1}$, generated in terms of the standard generators $A_{i,j}$ of P_n by $A_{j,j+1}A_{j,j+2}\cdots A_{j,n}$, $1 \leq j \leq n-1$. Let $K < P_n$ be the image of the (right) splitting in the split exact sequence $1 \to F_{n-1} \to P_n \to P_{n-1} \to 1$. The subgroup K consists of pure braids with trivial last strand, and is generated by $A_{i,j}$ with $j < n$. It can be shown geometrically [5], or algebraically, that $gHg^{-1} \cap K = \{1\}$ $\forall\, g \in P_n$. Consequently, $\mathsf{TC}(P_n) \geq \mathrm{cd}(H \times K) + 1 = (n-1)+(n-2)+1 = 2n-2$.

We anticipate that this result may be used to recover the topological complexity of other almost-direct products of free groups, such as the groups $P_{r,n}$.

This result is also used in [5] to find the topological complexity of right-angled Artin groups, and strikingly, to show that $\mathsf{TC}(\mathcal{H}) = 5$ for Higman's acyclic group $\mathcal{H}$.

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Topological Complexity of Non-Generic Arrangement Complements

Nathan Fieldsteel

The goal of this talk is to discuss some progress towards computing the higher topological complexity of central complex arrangement complements.

In what follows, an *arrangement of hyperplanes* is a finite set $\mathcal{A} = \{H_1, \dots, H_n\}$ of codimension 1 linear subspaces of complex affine space $\mathbb{A}^r_{\mathbb{C}}$. The *complement* of $\mathcal{A}$ is the space

$$X_{\mathcal{A}} := \mathbb{A}^r_{\mathbb{C}} \setminus \bigcup_{i=1}^{n} H_i.$$

We will work only with central arrangements, although this is not a serious restriction. The (reduced) s^{th} topological complexity $TC_s(X)$ of a space X is the smallest integer m such that there exists an open cover $\{U_0, \dots U_m\}$ of X^s satisfying that the restriction of the standard path fibration $PX \to X^s$ to each U_i admits a continuous section. The goal of this talk is to discuss progress towards a combinatorial formula for $TC_s(X_{\mathcal{A}})$.

Let $A = H^*(X_{\mathcal{A}}, \mathbb{C})$ and let K be the kernel of the multiplication map

$$A \otimes_{\mathbb{C}} A \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} A \longrightarrow A$$

on the s^{th} tensor power of A. The s^{th} *zero-divisors-cup-length* of A, denoted $zcl_s(A)$, is the largest integer t for which K^t is not the zero ideal. The s^{th} zero-divisors-cup-length of the cohomology of X provides a lower bound for $TC_s(X)$. For all arrangements $\mathcal{A}$ for which $TC_s(X_\mathcal{A})$ is known, this lower bound is equal to $TC_s(X_\mathcal{A})$, and it is conjectured in [5] that this holds for any central arrangement. So we'll start by looking for a combinatorial formula for $zcl_s(A)$.

The cohomology $A = H^*(X_\mathcal{A}, \mathbb{C})$ of the complement of a central arrangement $\mathcal{A}$ is the Orlik-Solomon algebra of the arrangement, and can be presented as the quotient of an exterior algebra by an ideal which is generated by products of linear forms [3]. Specifically, let E be the exterior algebra on $\mathbb{C}e_1 \oplus \ldots \oplus \mathbb{C}e_n$, where the e_i correspond to the hyperplanes H_i. The Orlik-Solomon ideal I of $\mathcal{A}$ is the ideal generated by all products of the form

$$(e_{i_1} - e_{i_2}) \wedge (e_{i_2} - e_{i_3}) \wedge \ldots \wedge (e_{i_{k-1}} - e_{i_k})$$

for any set $\{H_{i_1}, \ldots, H_{i_k}\}$ for which $\operatorname{codim}(H_{i_1} \cap H_{i_2} \cap \ldots \cap H_{i_k}) < k$ in $\mathbb{A}^r_\mathbb{C}$. A is naturally isomorphic to the quotient algebra E/I. We'll associate a monomial ideal I_Δ to the Orlik-Solomon ideal as follows: Let $l_1 \ldots l_t$ be the distinct linear forms appearing among the factors of a chosen set of generators for I, and let E' be the exterior algebra on $\mathbb{C}T_1 \oplus \ldots \oplus \mathbb{C}T_t$. Let $I_\Delta \subset E'$ be the monomial ideal obtained by replacing each l_i by the new variable T_i in the chosen generating set of I. The ideal I_Δ is squarefree and generated by monomials, so it has an associated simplicial complex Δ. Using [1] or [2], we can express $zcl_s(E'/I_\Delta)$ in terms of the simplicial complex Δ.

To recover the Orlik-Solomon algebra A from E'/I_Δ, we re-introduce the exterior variables $e_1, \ldots e_n$, then quotient by the relations $T_i - l_i$. More precisely, let E'' be the exterior algebra on $\mathbb{C}T_1 \oplus \ldots \oplus \mathbb{C}T_t \oplus \mathbb{C}e_1 \oplus \ldots \oplus \mathbb{C}e_n$, and note that I_Δ can be viewed as an ideal in E''. It is not hard to show that

$$zcl_s(E''/I_\Delta) = zcl_s(E'/I_\Delta) + n(s-1).$$

Now if we let L be the ideal in E'' generated by $(T_1 - l_1, \ldots, T_t - l_t)$, we'll have an isomorphism $A \simeq E''/(I_\Delta + L)$. So to compute $zlc_s(A)$ we only need to understand how $zcl_s(E''/I_\Delta)$ changes when we quotient by L.

Using the computational algebra software Macaulay2 (code available on the author's website), we have only ever observed the equality

$$zcl_s(E''/(I_\Delta + L)) = zcl_s(E''/I_\Delta) - t(s-1).$$

Lacking a proof of the above, we can't say anything conclusive. But the empirical data leads us to conjecture that, when A is the Orlik-Solomon algebra of $\mathcal{A}$, the zero-divisors-cup-length of A can be expressed as

$$zcl_s(A) = zcl_s(E'/I_\Delta) + n(s-1) - t(s-1),$$

which depends only on the combinatorics of $\mathcal{A}$, and as discussed above it has been conjectured in [5] that $zcl_s(A) = TC_s(X_\mathcal{A})$.

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