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The Renormalization Group

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Abstract. The renormalization group was originally introduced as a multi-scale approach to quantum field theory and the theory of critical phenomena, explaining in particular the universality observed e.g. in critical exponents. Since then it has become a hugely important tool in statistical mechanics, condensed matter and high energy physics. More recently, renormalization has also played a decisive role in mathematics as a method of proof, applicable in quantum field theory, differential equations, probability, and other fields. The workshop has focused on new developments along the lines of these two traditions. Besides discussing methodical progress and current applications, we have explored new challenges and problems that may in the future be tackled with the help of the renormalization group.

Mathematics Subject Classification (2010): 81T15, 81T16, 81T17, 82B05, 82C10.

Introduction by the Organisers

The renormalization group has its origins in quantum field theory and statistical mechanics. The idea is to implement scale transformations, possibly combined with other operations such as averaging, on function spaces and operator algebras, and hence on the action functionals and Hamiltonians that define classical or quantum ensembles and dynamics on these spaces. In this way, the transition from a microscopic to a macroscopic scale in systems with many degrees of freedom is formulated in terms of a dynamical system. In the past half-century, this concept has led to a uniform description of diverse physical phenomena, providing a natural explanation why physical phenomena exhibit universal features. In the sequel, the method has entered mathematics as a method of proof that has gradually been applied to solve a great variety of problems in mathematical physics, but also in

probability and partial differential equations. To provide a context, we first sketch the roots and some important developments of the field. (We do not attempt completeness, as this would require too much space.)

Renormalization in quantum field theory (QFT) started out in the 1950s as perturbative renormalization, with the aim of getting well-defined (finite) results at any fixed order in an expansion in a coupling parameter, usually organized in terms of Feynman graphs. It was converted to a mathematical theory in the 1960s and early 1970s, notably by Bogoliubov, Parasiuk, Hepp, and Zimmermann, culminating in an inductive proof of finiteness to all orders of the Green functions of physically important QFTs like quantum electrodynamics. Zimmermann's famous forest formula provides an explicit solution to the recursion of perturbative renormalization, which captures both the analytical and the combinatorial aspects of the procedure in a concise way. In parallel, Epstein and Glaser developed the viewpoint of renormalization as the definition of causal products of distributions. The group of scaling transformations plays a role because the careful mathematical analysis requires introducing a renormalization scale, the variation of which leads to the renormalization group. In the 1970s, 't Hooft and Veltman proved renormalizability of nonabelian gauge theory to first order in a loop expansion, using dimensional renormalization. Breitenlohner and Maison then established the action principle and the BRST identities for this scheme to all orders, and Piguet and Sibold treated supersymmetric theories in the 1980s.

Renormalization in statistical mechanics originated in the idea that at a critical point of a statistical mechanical system, scale invariance should set in. Kadanoff first introduced block-spin transformations in the 1960s. Wilson continued and extended these ideas and developed the renormalization group as a combination of averaging operations and scale transformations. Wilson's RG phenomenology of a flow of effective actions parametrized by length scale shifted the focus from studying individual fixed models to considering a dynamical system on "the space of all theories". This new formulation of renormalization changed the concepts and techniques used in theoretical physics completely, far beyond their original application to critical phenomena. It also allowed to go beyond formal perturbation expansions and to construct models of QFT and statistical mechanics in a mathematical sense.

The renormalization group in the sense of Kadanoff and Wilson was developed into a mathematical theory in the 1980s in pioneering work by Gallavotti and Nicolò, by Gawędzki and Kupiainen, by Feldman, Magnen, Rivasseau and Sénéor, and by Bałaban. Beginning with these works, the method has led to breakthroughs in mathematical results about quantum field theories and models of statistical mechanics. Indeed, virtually all mathematical constructions of quantum field theories in dimensions three or higher involve the RG, and the RG has become a powerful method in the study of systems with infinitely many degrees of freedom in general, and in the construction and analysis of non-Gaussian measures on infinite-dimensional spaces.

These two seemingly different strands of renormalization were joined in the 1970s and 1980s in the work of the above-mentioned people, and by Wegner, Polchinski, and others. In particular, Polchinski used Wilson's flow equation RG to give a proof of perturbative renormalizability that bypasses most of the combinatorial complications of earlier works, and which has since been developed into a powerful tool that has led to many new results, e.g. to a recent proof of perturbative renormalizability of the standard model of elementary particle physics on curved spacetime. Brydges and Kennedy used Polchinski's formulation to shed new light on the forest formula, and to develop new tree expansions that have become standard tools in constructive QFT. Brydges and Yau developed a setup in which the decomposition in large and small fields is avoided.

Much further work has also been done in simplifying perturbative renormalization, extending it to more general situations, and providing new mathematical viewpoints. The conceptually very clear method of Epstein and Glaser was revived by Scharf and collaborators, and then further developed using tools of microlocal analysis by Radzikowski, and then by Brunetti and Fredenhagen and coworkers, and by Hollands and Wald. Kreimer, and then Connes and Kreimer, provided a Hopf-algebraic formulation of the graph operations implementing renormalization in formal perturbation theory. Costello gave another variant of Wilson's effective-action approach to perturbative renormalization.

Last, but not least, starting with the work of Feldman and Trubowitz, as well as Benfatto and Gallavotti, in the early 1990s, the renormalization group has led to a host of new results in mathematical condensed-matter theory, in particular laying rigorous foundations for Luttinger and Fermi liquid theory and numerous new results about spin systems. Moreover, it has placed the proof of Bose-Einstein condensation in the thermodynamic limit within reach.

The 2016 Oberwolfach workshop *The Renormalization Group*, the fourth of its kind at Oberwolfach, organised by M. Disertori (Bonn), M. Salmhofer (Heidelberg) and and W. De Roeck (Leuven), was attended by 43 participants, from universities and research institutes mainly in Europe and North America, with backgrounds ranging from theoretical physics over mathematical physics to pure mathematics.

The talks and discussions covered most of the broad range of topics exposed above. There were talks on novel views and methods of perturbative renormalization (Nguyen, Rejzner), on the operator product expansion (Hollands), on the rigorous construction and analysis of models in classical and quantum statistical mechanics and condensed-matter physics (Feldman, Lohmann, Giuliani, Porta, Pizzo), on probabilistic applications (Brydges, Slade), on specific analysis tools (Buchholz), on applications to nontrivial QFT models motivated by high-energy physics and combinatorics (Abdesselam, Rivasseau), and spectral theory and quantum dynamics (Sigal). Moreover, there were talks about topics that may become interesting for renormalizers (Fröhlich, Gawedzki, Warzel).

The presentations of T. Nguyen and K. Rejzner concerned the Batalin-Vilkovisky approach to the quantization and renormalization of theories with large symmetry groups (e.g. theories with local gauge or diffeomorphism invariance). Rejzner

used the causal perturbation theory approach of Epstein and Glaser in Lorentzian signature, while Nguyen used Costello's version of the effective action approach on Riemannian spaces, with a standard heat kernel regularization, to derive results about the absence of anomalies in nonlinear sigma models.

S. Hollands presented his work on the operator product expansion (OPE), touching a number of different aspects, namely the convergence of the expansion in Euclidian scalar field theory (in every fixed order in perturbation theory), proven by an extension of Polchinski's flow equation method together with Holland and Kopper, the idea of regarding the OPE as the defining feature of a QFT, and a novel construction of the Gross-Neveu model via a new recursion formula for the OPE.

V. Rivasseau talked about renormalization of tensor field theory, which is related to models of quantum gravity, and which poses interesting new features and aspects of renormalization theory.

S. Buchholz, one of the PhD researchers at the workshop, outlined a novel approach to finite-range decompositions of covariances, an analytical tool that has recently helped to simplify cluster expansions needed in renormalization. Among other things, the finite-range decomposition allows to treat non-analytic perturbations, which occur in applications to models for elasticity.

Several talks dealt with applications of the mathematical renormalization group. We had two talks, by J. Feldman and M. Lohmann, on the program of controlling interacting bosons with spontaneous symmetry breaking. This program has as its medium-term goal a proof of Bose-Einstein condensation in the thermodynamic limit, which can be considered one of the most important currently open problems in mathematical physics. G. Slade spoke about $O(n)$ models with a long-range kinetic term, engineered so that these models are slightly below the upper critical dimension. By applying machinery from earlier work with Brydges and Bauerschmidt, the critical point can be studied. I.M. Sigal described some new results on open quantum systems consisting of baths coupled to small systems. In particular, decoherence rates have been exhibited and contrasted with the rate of thermalization in such models. This program relies on the use of the Feshbach map spectral renormalization group. The talk by M. Porta dealt with the universality of the Hall conductance. This universality is very well understood for non-interacting electrons where it is a consequence of topology. Porta and coworkers employed fermionic perturbation theory to extend this to interacting electrons, as well. A. Giuliani presented an application of fermionic perturbation theory to planar dimer models. These models are solvable in the close-packed limit, when the interaction only includes a hard-core repulsion that prevents dimers from overlapping. The new results allow to treat more general dimer interactions. This gives rise to a continuous family of critical models, with varying critical exponents. A. Abdesselam considered field theory on the p-adic numbers, which can be viewed as a model in which the hierarchical approximation to the renormalization group becomes exact. In particular, one of the achievements in this framework is the construction of a theory with anomalous power-law decay of correlations. The

talk by A. Pizzo concerned the ground state of interacting bosons approaching the mean field limit. A lot of work has been done on this problems in the past decade. Pizzo's approach is based on the Feshbach renormalization group and it yields new results on that ground state, making precise the well-known Bogoliubov approximation.

Finally, we had a few interesting talks that were not, at least not explicitly, connected to the renormalization group: D. Brydges outlined an application of the lace expansion to the $n = 2$ component ϕ^4 theory in high dimensions, proving that critical exponents are given by mean field theory. S. Warzel reported on work on planar Ising models. While it has been known already since Onsager that 2-dimensional models often can be analyzed in terms of free fermions, the presented work has significantly extended and generalized this connection. R. Kotecký presented work on the transition from a metastable supercooled gas phase to the stable liquid phase in the Widom-Rowlinson model. The talk of K. Gawedzki focused on periodically driven non-interacting systems, thus bringing us to non-equilibrium physics. Such periodically driven systems can be described by a Floquet operator, which replaces the Bloch Hamiltonian. The topic addressed here was how to define topological invariants for such systems, like, e.g. for topological insulators.

J. Fröhlich gave an overview of the 'Gauge Theory of States of Matter', developed by him and coworkers throughout the 90's. This is a program that classifies relevant effective field theories in condensed matter, thus predicting important phenomena including topological insulators. The talk also touched on more recent works, proposing models for dark matter and dark energy.

The schedule was set up so as to leave enough time for individual discussions and collaboration. The feedback from the participants was positive, suggesting to us that the meeting has been fruitful. The Oberwolfach atmosphere and the perfect organization at the Forschungsinstitut, as well as the friendly and efficient service by the staff were greatly appreciated.

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Workshop: The Renormalization Group

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Abstracts

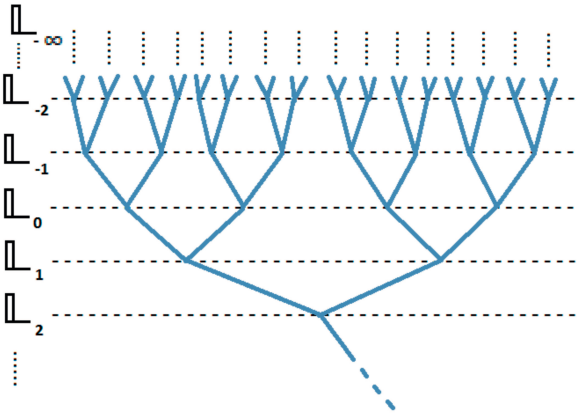
A Field With Anomalous Scaling Dimension via Rigorous Wilson ϵ-expansion, or Some Steps Towards Rigorous 3D CFT

Abdelmalek Abdesselam

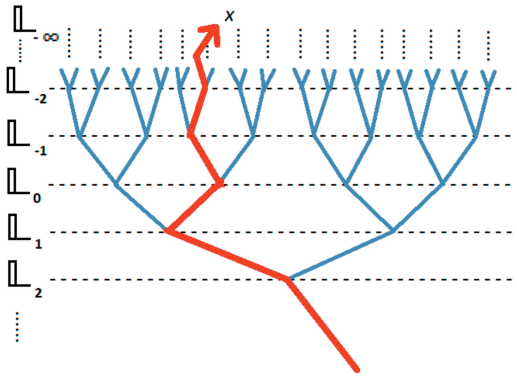
In this extended abstract I will report on the result obtained in [3] which is joint work with Ajay Chandra and Gianluca Guadagni, and which fits in the program outlined in [1] for the rigorous study of conformal field theory (CFT) in three dimensions. The model considered in [3] is p-adic, i.e., a hierarchical toy model for a scalar ϕ^4 model with fractional Laplacian. In the d-dimensional Euclidean setting, the corresponding functional integral has the form $\int \cdots e^{-S(\phi)}\ D\phi$ with the action

$$S(\phi) = \frac{1}{2}\langle \phi, (-\Delta)^{\frac{\alpha}{2}}\phi\rangle_{L^2} + \int_{\mathbb{R}^d} \left\{ g\phi(x)^4 + \mu\phi(x)^2 \right\}\ dx\ .$$

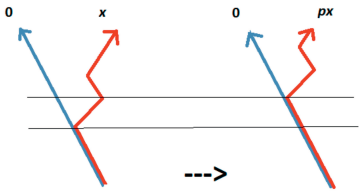
Of particular interest is the case with $d = 3$ and $\alpha = \frac{3+\epsilon}{2}$ with ϵ positive and small which allows a small ϵ expansion which is a version of the famous ϵ-expansion developed by Kenneth Wilson. One has the existence of a nontrivial RG fixed point at distance $\sim \epsilon$ from the trivial or Gaussian fixed point [5]. The scale invariant theory corresponding to this fixed point is believed to be a 3D CFT [6]. I will now introduce the p-adic toy version of this model. Let p be an integer > 1 (in fact a prime number). Let $\mathbb{L}_k$, $k \in \mathbb{Z}$, be the set of boxes $\prod_{i=1}^d \left[a_i p^k, (a_i+1)p^k\right)$ for $a_1, \ldots, a_d \in \mathbb{N}$. The cubes in $\mathbb{L}_k$ form a partition of the octant $[0,\infty)^d$. Then $\mathbb{T} = \cup_{k\in\mathbb{Z}}\mathbb{L}_k$ naturally has the structure of a doubly infinite tree organized in layers or generations $\mathbb{L}_k$:



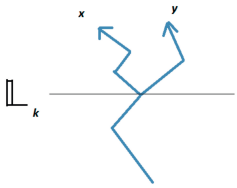
The picture is for $d = 1$, $p = 2$. Now forget about $[0,\infty)^d$ and $\mathbb{R}^d$. Define the substitute for the continuum $\mathbb{Q}_p^d :=$ set of leafs at infinity "$\mathbb{L}_{-\infty}$". More precisely, this is the set of upward paths in the tree.



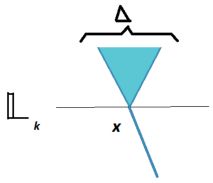
A point $x \in \mathbb{Q}_p^d$ encoded by a sequence $(b_n)_{n\in\mathbb{Z}}$, $b_n \in \{0, 1, \ldots, p-1\}^d$. Let $0 \in \mathbb{Q}_p^d$ correspond to sequence with all digits equal to zero. b_n represents local coordinates of a $\mathbb{L}_{-n-1}$ box inside a $\mathbb{L}_{-n}$ box. Moreover, scaling is defined as follows: if $x = (b_n)_{n\in\mathbb{Z}}$ then $px = (b_{n-1})_{n\in\mathbb{Z}}$, i.e., this is an upward shift.



Likewise $p^{-1}x$ is downward shift and so on for defining $p^k x$, $k \in \mathbb{Z}$. If $x, y \in \mathbb{Q}_p^d$, define their distance as $|x - y| := p^k$ where k is the depth where the bifurcation between the two paths occurs



Also, define $|x| := |x-0|$. Because of the strange notation $|px| = p^{-1}|x|$. Closed balls Δ of radius p^k correspond bijectively to points $\mathbf{x} \in \mathbb{L}_k$:



The metric space structure on $\mathbb{Q}_p^d$ defines a Borel σ-algebra on which lives a natural analogue of the Lebesgue measure $d^d x$ which gives a volume p^{dk} for a closed ball of radius p^k. To construct it, take the product of uniform probability measures on $(\{0,1,\ldots,p-1\}^d)^{\mathbb{N}}$ for $\overline{B}(0,1)$ and similarly for other balls of radius 1, and then collate.

To any group G of offsprings for a site $\mathbf{z} \in \mathbb{L}_{k+1}$, associate a centered Gaussian vector $(\zeta_{\mathbf{x}})_{\mathbf{x}\in G}$ with $p^d \times p^d$ covariance matrix with $1-p^{-d}$ on diagonal and $-p^{-d}$ everywhere else. These vectors are set to be independent for different groups or layers. Note that $\sum_{\mathbf{x}\in G} \zeta_{\mathbf{x}} = 0$ a.s. One can also define an ancestor function: for $k < k'$, $\mathbf{x} \in \mathbb{L}_k$, let $\mathrm{anc}_{k'}(\mathbf{x})$ be the ancestor in $\mathbb{L}_{k'}$. Likewise, one can define $\mathrm{anc}_{k'}(x)$ for $x \in \mathbb{Q}_p^d$. The massless Gaussian field $\phi(x)$, $x \in \mathbb{Q}_p^d$ with engineering scaling dimension $[\phi]$ is

$$\phi(x) = \sum_{k\in\mathbb{Z}} p^{-k[\phi]} \zeta_{\mathrm{anc}_k(x)} \quad , \quad \langle \phi(x)\phi(y)\rangle = \frac{c}{|x-y|^{2[\phi]}} \ .$$

This is only formal since ϕ is not defined pointwise. One needs a notion of random distributions. $f : \mathbb{Q}_p^d \to \mathbb{R}$ is called smooth if it is locally constant

$$S(\mathbb{Q}_p^d) := \{\text{smooth compactly supported functions}\} \ = \ \cup_{n\in\mathbb{N}} S_{-n,n}(\mathbb{Q}_p^d)$$

where for $t_- \le t_+$, $S_{t_-,t_+}(\mathbb{Q}_p^d)$ is the space of functions which are constant in closed boxes of radius p^{t_-} and support in $\overline{B}(0,p^{t_+})$. The topology is the one generated by the set of all possible seminorms. The space of distributions is $S'(\mathbb{Q}_p^d)$, namely, the topological dual with the strong topology (here equal to the weak-$*$ topology). One has $S(\mathbb{Q}_p^d) \simeq \oplus_{\mathbb{N}}\mathbb{R}$ and thus $S'(\mathbb{Q}_p^d) \simeq \mathbb{R}^{\mathbb{N}}$ with the product topology. Since $S'(\mathbb{Q}_p^d)$ is a Polish space, probability theory on it is very nice! The following classical tools or facts hold in this somewhat exotic-looking context:

- Prokhorov's Theorem,
- Bochner's Theorem,
- Levy's Continuity Theorem,
- Uniform convergence of characteristic functions in a complex neighborhood of the origin implies weak convergence of probability measures (using moments or the Vitali-Porter Theorem).
- The analytic RG and dynamical systems methods introduced in [3] deliver exactly that.
- $S'(\mathbb{Q}_p^d) \times S'(\mathbb{Q}_p^d) \simeq S'(\mathbb{Q}_p^d)$ so the same tools work for joint law of pair of distributional random fields, e.g., $(\phi, N[\phi^2])$.

From now on we set $d = 3$, $[\phi] = \frac{3-\epsilon}{4}$, and $L = p^l$ for the RG step ratio. The UV cut-off is $r \in \mathbb{Z}$, $r \to -\infty$. It eliminates length scales less than L^r. The IR cut-off is $s \in \mathbb{Z}$, $s \to \infty$. It encloses the sytem in a volume of linear size L^s. The cut-off Gaussian measure μ_{C_r} is the law of

$$\phi_r(x) = \sum_{k=lr}^{\infty} p^{-k[\phi]} \zeta_{\mathrm{anc}_k(x)} \ .$$

Sample paths are functions that are locally constant at scale L^r. These Gaussian measures are scaled copies of each other. If the law of $\phi(\cdot)$ is μ_{C_0}, then the law of $L^{-r[\phi]}\phi(L^r\cdot)$ is μ_{C_r}. Introduce fixed parameters g, μ and cut-off dependent bare couplings $g_r = L^{-(3-4[\phi])r} g$ and $\mu_r = L^{-(3-2[\phi])r}\mu$. Let $\Lambda_s = \overline{B}(0, L^s)$ define the confining volume or IR cut-off. Let

$$V_{r,s}(\phi) = \int_{\Lambda_s} \{ g_r : \phi^4 :_{C_r} (x) + \mu_r : \phi^2 :_{C_r} (x) \} d^3x$$

and define the probability measure

$$d\nu_{r,s}(\phi) = \frac{1}{\mathcal{Z}_{r,s}} e^{-V_{r,s}(\phi)} d\mu_{C_r}(\phi) \ .$$

Let $\phi_{r,s}$ be a random variable in $S'(\mathbb{Q}_p^3)$ sampled according to $\nu_{r,s}$ and define the square field $N_r[\phi^2_{r,s}]$ which is the deterministic $S'(\mathbb{Q}_p^3)$-valued function of $\phi_{r,s}$ given by

$$N_r[\phi^2_{r,s}](j) = Z_2^r \int_{\mathbb{Q}_p^3} \{ Y_2 : \phi^2_{r,s} :_{C_r} (x) - Y_0 L^{-2r[\phi]} \} \ j(x) \ d^3x$$

with Z_2, Y_0, Y_2 parameters to be adjusted. Our main result concerns the limit law of the pair $(\phi_{r,s}, N_r[\phi^2_{r,s}])$ in $S'(\mathbb{Q}_p^3) \times S'(\mathbb{Q}_p^3)$ when $r \to -\infty$, $s \to \infty$ regardless of the order of limits. We will need the approximate fixed point coupling

$$\bar{g}_* = \frac{p^\epsilon - 1}{36 L^\epsilon (1 - p^{-3})} \ .$$

Theorem 1. [3] $\exists \rho$, $\exists L_0$, $\forall L \geq L_0$, $\exists \epsilon_0 > 0$, $\forall \epsilon (0, \epsilon_0]$, $\exists [\phi^2] > 2[\phi]$, $\exists$ *functions* $\mu(g)$, $Y_0(g)$, $Y_2(g)$ *on the interval* $(\bar{g}_* - \rho\epsilon^{\frac{3}{2}}, \bar{g}_* + \rho\epsilon^{\frac{3}{2}})$ *such that if one sets* $\mu = \mu(g)$, $Y_0 = Y_0(g)$, $Y_2 = Y_2(g)$ *and* $Z_2 = L^{-([\phi^2]-2[\phi])}$ *then the law of* $(\phi_{r,s}, N_r[\phi^2_{r,s}])$ *converges weakly and in the sense of moments to that of a pair* $(\phi, N[\phi^2])$ *such that:*

1. $\forall k \in \mathbb{Z}$, $(L^{-k[\phi]}\phi(L^k\cdot), L^{-k[\phi^2]} N[\phi^2](L^k\cdot)) \stackrel{d}{=} (\phi, N[\phi^2])$.
2. $\langle \phi(\mathbf{1}_{\mathbb{Z}_p^3}), \phi(\mathbf{1}_{\mathbb{Z}_p^3}), \phi(\mathbf{1}_{\mathbb{Z}_p^3}), \phi(\mathbf{1}_{\mathbb{Z}_p^3}) \rangle^{\mathrm{T}} < 0$ *i.e., ϕ is non-Gaussian. Here $\mathbf{1}_{\mathbb{Z}_p^3}$ is the indicator function of $\overline{B}(0,1)$.*
3. $\langle N[\phi^2](\mathbf{1}_{\mathbb{Z}_p^3}), N[\phi^2](\mathbf{1}_{\mathbb{Z}_p^3}) \rangle^{\mathrm{T}} = 1$.

Mixed correlations satisfy in the sense of distributions

$$\begin{aligned} &\langle \phi(L^{-k}x_1) \cdots \phi(L^{-k}x_n) N[\phi^2](L^{-k}y_1) \cdots N[\phi^2](L^{-k}y_m) \rangle \\ &= L^{-(n[\phi]+m[\phi^2])k} \langle \phi(x_1) \cdots \phi(x_n) N[\phi^2](y_1) \cdots N[\phi^2](y_m) \rangle \ . \end{aligned}$$

For the p-adic model we also proved $[\phi^2] - 2[\phi] = \frac{1}{3}\epsilon + o(\epsilon)$ as expected in the Euclidean model. This, $0.33\ldots$, is not too far, if one sets $\epsilon = 1$, from the 3D short-range Ising model for which recent progress in conformal bootstrap gives $[\phi^2] - 2[\phi] = 0.3763\ldots$ (Simmons-Duffin 2015). The law $\nu_{\phi\times\phi^2}$ of $(\phi, N[\phi^2])$ is independent of g.

Theorem 2. [4] *$\nu_{\phi\times\phi^2}$ is fully scale invariant, i.e., invariant under the action of $p^{\mathbb{Z}}$ instead of just $L^{\mathbb{Z}}$. Moreover, $\mu(g)$ and $[\phi^2]$ are independent of the RG step L.*

The two-point functions are given as distributions by

$$\langle \phi(x)\phi(y)\rangle = \frac{c_1}{|x-y|^{2[\phi]}} \quad , \quad \langle N[\phi^2](x)\ N[\phi^2](y)\rangle = \frac{c_2}{|x-y|^{2[\phi^2]}} \ .$$

Note that $3 - 2[\phi^2] = 3 - \frac{1}{3}\epsilon + o(\epsilon)$ so the correlations are still $L^{1,\mathrm{loc}}$!

Theorem 3. *(A.A., May 2015) Let ψ_i denote ϕ or $N[\phi^2]$. Then for every mixed correlation $\exists$ smooth function $\langle \psi_1(z_1)\cdots\psi_n(z_n)\rangle$ on $(\mathbb{Q}_p^3)^n\backslash\mathrm{Diag}$ which is locally integrable (even on the big diagonal* Diag*) such that*

$$\langle \psi_1(f_1)\cdots\psi_n(f_n)\rangle = \int_{(\mathbb{Q}_p^3)^n\backslash\mathrm{Diag}} \langle \psi_1(z_1)\cdots\psi_n(z_n)\rangle f_1(z_1)\cdots f_n(z_n)\ d^3z_1\cdots d^3z_n$$

for all test functions $f_1, \ldots, f_n \in S(\mathbb{Q}_p^3)$.

This local integrability is the starting point of the investigations in [2] which allow one to define the square field $N[\phi^2]$ as a deterministic Borel-measurable local functional of the elementary field ϕ. As far as I know, the result in [3], albeit on a hierarchical model, is the first mathematical justification of Wilson's ϵ-expansion, in the sense that it is the first construction of a field with anomalous power law decay of correlations due to an nontrivial isolated RG fixed point.

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The lace expansion for the lattice $|\phi|^4$ model

David Brydges
(joint work with Tyler Helmuth, Mark Holmes)

We prove that the $|\phi|^4$ model with $n = 0, 1, 2$ component field ϕ on the lattice $\mathbb{Z}^d$ has a lace expansion that converges at the critical point for sufficiently weak coupling in dimensions $d \geq 5$. Here the $n = 0$ terminology only means that the lattice Edwards model for weakly self-avoiding walk is included. These models are defined below. This lace expansion generalises to allow short range ferromagnetic couplings, but here we state formulas only for the nearest neighbour model. The new result in our work is the construction of a lace expansion for the $n = 2$ component model. We are unable to extend our result to $n > 2$ because our argument requires a form of the Griffiths-Hurst-Sherman correlation inequalities for ferromagnetic lattice models, which are not known for $n > 2$.

We were inspired by recent work of Akira Sakai. He has constructed convergent lace expansions for ferromagnetic Ising models [6] and the one component lattice ϕ^4 model [7] in sufficiently high dimensions. His construction for the ϕ^4 model relies on the Griffiths-Simon approximation of the ϕ^4 model by the Ising model. He proves that, if the strength $g \geq 0$ of nonlinearity is sufficiently small, then the critical ϕ^4 two-point function $\langle \phi_0 \phi_x \rangle$ is asymptotically $|x|^{2-d}$ times a model-dependent constant. His results hold for interactions that are ferromagnetic and short-range. This sharpens and extends to short range ferromagnetic models the infrared bounds proved in [4] for the more special class of reflection positive models (nearest neighbour interactions).

The lace expansion has been primarily a tool for proving that models in high dimensions are governed by mean field theory. We have not yet checked details for our models, but the standard consequences of a convergent lace expansion include the determination of the leading and sub-leading power law decay of the critical two-point function, as in, for example, self-avoiding walk in high dimensions [5]. In this case, the sub-leading decay implies that the critical two-point function of the scaling limit of critical self-avoiding walk is Euclidean invariant. While it is expected that many lattice models have euclidean invariant critical two-point functions it is usually difficult to prove that the scaling limit of a lattice correlation is euclidean invariant.

The lace expansion first appeared in [1]. Subsequently other authors extended it to a method for proving mean field behaviour in high dimensions for many models including self-avoiding walk, percolation, oriented percolation and lattice self-avoiding trees. For some percolation models it has even been possible to study the incipient infinite cluster by lace expansions. These and other applications are described in [8].

The lace expansion, when it exists, is a formula for the two-point function $G = G(a, b)$ of a statistical mechanical model on a lattice which we assume is $\mathbb{Z}^d$. The arguments a, b are the two points in $\mathbb{Z}^d$. For translation invariant models $G(a, b) = G(0, b - a)$. In the models we consider, G is positive and depends on

a real parameter ν with a critical value ν_c such that $G(0,x)$ decays integrably iff $\nu > \nu_c$. Therefore for $\nu > \nu_c$ the Fourier transform $\hat{G} = \hat{G}(k)$ of $G(0,x)$ exists. We show that there is a function $\hat{\Pi} = \hat{\Pi}(k)$ called the *self-energy* such that

$$\hat{G} = (-\hat{\Delta} + \hat{\Pi})^{-1}, \tag{1}$$

where $-\hat{\Delta} = -\hat{\Delta}(k)$ is the Fourier transform $\sum_{x\sim 0}(1 - e^{ik\cdot x})$ of the discrete Laplacian. The lace expansion is a series

$$\Pi = \sum_{i\geq 1} \Pi^{(i)}$$

for Π such that

(1) $\Pi^{(i)}(a,b)$ is $O(g^i)$ where $g > 0$ is a coupling constant in the model: see definitions below.
(2) $\Pi^{(i)}(a,b)$ is bounded in absolute value by an edge irreducible Feynman diagram whose edges represent factors of G.
(3) If $G(a,b) = O(|a-b|^{2-d})$, then $\Pi^{(i)}(a,b) = O(|a-b|^{3(2-d)+\epsilon})$.

Property (3) is a consequence of the edge irreducibility in Property (2); the Feynman diagram has three independent paths joining a to b. Let $d \geq 5$ and let g be sufficiently small and positive. Then, by a bootstrap argument like the one that follows Lemma 2.1 in [10], the lace expansion Properties (2,3) together with equation (1) imply that the infrared bound $G(0,x) = O(|x|^{2-d})$ holds uniformly for $\nu > \nu_c$. In the bootstrap argument the hypothesis $d > 4$ ensures that the second moment $\sum_b |\Pi^{(i)}(0,x)x^2|$ exists. This infrared bound extends to $\nu = \nu_c$ for the $|\phi|^4$ model with $n = 0, 1, 2$ components because these models have the second order phase transition property that $\sum_x G(0,x)$ diverges to infinity as $\nu \downarrow \nu_c$.

The starting point for our lace expansion is continuous time simple random walk $s \mapsto X_s$ on a finite subset Λ of $\mathbb{Z}^d$. The walk is killed at the boundary of Λ and it is generated by the discrete Laplacian Δ. For $\ell \geq 0$ let

$$\tau_x = \int_{[0,\ell]} \mathbb{1}_{\{X_u = x\}} du, \qquad \tau = (\tau_x)_{x\in\Lambda}$$

be the random time X spends at a site $x \in \Lambda$ up to time ℓ. Given a deterministic function $Z : [0,\infty)^\Lambda \to (0,\infty)$, $\tau \mapsto Z_\tau$ we define the *Green's* function

$$G(a,b) = \int_{[0,\infty)} d\ell \ \mathbb{E}_a \left[\frac{Z_\tau}{Z_0} \mathbb{1}_{\{X_\ell = b\}} \right], \tag{2}$$

where a, b are sites in Λ and $\mathbb{E}_a$ is the expectation for X with $X_0 = a$. Thus the ratio of Z's specifies a self-interaction of the walk X. For example when $Z_\tau = Z_0$ this ratio is one and $G(a,b) = (-\Delta)^{-1}(a,b)$ is the Green's function for the simple random walk X. If instead we choose, for parameters $g > 0$ and $\nu \in \mathbb{R}$,

$$Z_\tau = e^{-\sum(g\tau_x^2 + \nu\tau_x)} = e^{-\iint_{[0,\ell]^2} g\mathbb{1}_{\{X_s = X_t\}} ds dt} e^{-\nu\ell},$$

then, in the infinite volume limit $\Lambda \uparrow \mathbb{Z}^d$, we obtain the *lattice Edwards model* of weakly self-avoiding walk. This model is conventionally called the $n = 0$ component $|\phi|^4$ model for good reasons that are not relevant to the present discussion.

The n component $|\phi|^4$ lattice model. For any integer $n \geq 1$ and for $x \in \mathbb{Z}^d$ let $x \mapsto \phi_x \in \mathbb{R}^n$ be a function on $\mathbb{Z}^d$ that vanishes for x not in Λ. Let $d\phi$ be Lebesgue measure on the space $\mathbb{R}^{n\Lambda}$ of all such functions. In terms of the set E of nearest neighbour edges in $\mathbb{Z}^d$ define the Dirichlet form

$$(\nabla\phi, \nabla\phi) = \sum_{xy\in E} (\phi_x - \phi_y)^2,$$

and the ϕ^4 interaction

$$V(\phi_x^2) = g(\phi_x^2)^2 + \nu\phi_x^2.$$

The partition function for the ϕ^4 model is

$$Z_0 = \int e^{-\frac{1}{2}(\nabla\phi,\nabla\phi)} e^{-\sum_x V(\phi_x^2)}\, d\phi$$

and the finite volume two-point function for the first component ϕ_x^1 of the vector ϕ_x is

$$\langle \phi_a^1 \phi_b^1 \rangle = \frac{1}{Z} \int \phi_a^1 \phi_b^1 \; e^{-\frac{1}{2}(\nabla\phi,\nabla\phi)} \; e^{-\sum_x V(\phi_x^2)}\, d\phi.$$

Define a deterministic function $\tau \mapsto Z_\tau$ by

$$Z_\tau = \int e^{-\frac{1}{2}(\nabla\phi,\nabla\phi)} e^{-\sum_x V(\phi_x^2 + 2\tau_x)}\, d\phi.$$

With this choice, by the random walk representation [9], [2], [3],

$$G(a,b) \equiv \int_{[0,\infty)} d\ell \; \mathbb{E}_a \left[\frac{Z_\tau}{Z_0} \mathbb{1}_{\{X_\ell = b\}} \right] = \langle \phi_a^1 \phi_b^1 \rangle.$$

To summarise: the infinite volume limit of the class of functions $G(a,b)$ defined by (2) includes the $n = 0, 1, \ldots$ component $|\phi|^4$ models. We define a candidate expansion for all models of the form (2) where $\tau \to Z_\tau$ is sufficiently regular, but are only able to prove that this candidate expansion has Property (2) listed above when $n = 0, 1, 2$.

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Finite range decompositions with improved regularity

Simon Buchholz

We consider random fields $\varphi : \Lambda \to \mathbb{R}^m$ on a discrete lattice $\Lambda \subset \mathbb{Z}^d$. The energy of a field configuration is of gradient type (massless) and given by the Hamiltonian

$$\mathcal{H}(\varphi) = \sum_{x\in\Lambda}\sum_{i=1}^{d} \frac{1}{2}|\nabla_i\varphi(x)|^2 + V(\nabla_i\varphi(x))$$

where $\nabla_i\varphi(x) = \varphi(x+e_i) - \varphi(x)$ denotes the discrete gradient and the potential V is a perturbation of the quadratic potential. This type of model is called continuous Ising model and is well studied in mathematics and physics. In the case of scalar fields ($m = 1$) it is an effective model for an interface between two phases embedded in $\mathbb{R}^{d+1}$ where the height at point $x \in \mathbb{R}^d$ is given by $\varphi(x) \in \mathbb{R}$, i.e., the discrete interface is parametrised by the set $\{(x, \varphi(x)) \in \mathbb{R}^{d+1} | x \in \Lambda\}$. When $m = d$ this corresponds to a model for a d-dimensional crystal with a nearest neighbour interaction where $\varphi(x)$ denotes the displacement of the atom at position x. In equilibrium the behaviour of this system is given by the Gibbs measure $Z^{-1}e^{-\beta\mathcal{H}}\,\mathrm{d}\lambda$. The analysis of this model is subtle because gradient Gaussian fields exhibit long-ranged correlations which decay critically with $|x-y|^{-d}$. If the entire potential is strictly convex the model is well understood [13],[15], [12], [14]. In the non-convex case there are some results in the high temperature regime [11] and [10] but the low temperature case is mostly open.

In a recent work [2] strict convexity of the surface tension as a function of the tilt is shown for small temperatures and small tilts. This work is uses a renormalization group approach in the spirit of Brydges, Yau and Slade [6], [7], [4]. The renormalisation group technique is a multi-scale analysis based on a finite-range decomposition of a Gaussian field $\xi = \sum_{k=1}^{N+1} \xi_k$ into independent Gaussian fields which have increasing ranges. A suitable decomposition for elliptic, discrete difference operators $\nabla^* A\nabla$ has been constructed in [1] based on the decompositions in [8] and [5] and with a new approach in [3]. Successive integrations with respect to the fields ξ_k allows to gain control on the behaviour of functionals $F(\xi) = F(\sum_{k=1}^{N+1} \xi_k)$.

One difficulty in [2] is a loss of regularity of the renormalisation map that integrates out the Gaussian field ξ_k, i.e., the map that acts on a functional F via $F \to R_k^{\mathcal{C}_k} F = \mathbb{E}_{\mathcal{C}_k} F(\cdot + \xi_k)$ where the field ξ_k has covariance $\mathcal{C}_k$. More precisely the derivatives of $R_k^{\mathcal{C}_k} F$ with respect to the covariance $\mathcal{C}_k$ can a-priori only be estimated by higher order norms of F. The estimates of these derivatives are needed because one has to extract the correct renormalised Gaussian part $(\nabla\varphi, A\nabla\varphi)$ such

that the renormalisation group iteration converges to the Gaussian fixed point via an implicit function theorem. The loss of regularity at this point causes several technical difficulties, in particular the work [2] relies on chain rules and implicit function theorems with loss of regularity. The goal of this work is to explain how this loss of regularity can be avoided when the finite range decompositions is suitably modified.

The loss of regularity may appear surprising because in principle the heat semigroup is smoothing. However, the covariance $\mathcal{C}_k$ decay very fast in k which means they are less and less smoothing (think about the δ-distribution which is the Gaussian measure with vanishing covariance). Therefore a lower bound on the covariances $\mathcal{C}_k$ is needed to control their smoothing behaviour. In [9] we construct a new finite range decomposition based on the decomposition in [1] which in addition satisfies strong lower bounds. The key idea is that lower bounds can be obtained from linear combinations of the original decomposition. In a second step we construct again using linear combinations another decomposition that is very regular with respect to the parameter A. With this new decomposition and a localisation argument that exploits the structure of the functionals in the renormalisation group approach the loss of regularity can be avoided.

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Complex Bosonic Many-body Models

Joel Feldman

(joint work with Tadeusz Balaban, Horst Knörrer, Eugene Trubowitz)

It is our long term goal to rigorously demonstrate symmetry breaking in a gas of bosons hopping on a three dimensional lattice.

This program was initiated in [1, 2], where we expressed the positive temperature partition function and thermodynamic correlation functions in a periodic box (a discrete three–dimensional torus) as 'temporal' ultraviolet limits of four–dimensional (coherent state) lattice functional integrals. By a lattice functional integral we mean an integral with one (in this case complex) integration variable for each point of the lattice. By a 'temporal' ultraviolet limit, we mean a limit in which the lattice spacing in the inverse temperature direction (imaginary time direction) is sent to zero while the lattice spacing in the three spatial directions is held fixed.

In [3][1], by a complete large field/small field renormalization group analysis, we expressed the temporal ultraviolet limit for the partition function, still in a periodic box, as a four–dimensional lattice functional integral with the lattice spacing in all four directions being of the order one, preparing the way for an infrared renormalization group analysis of the thermodynamic limit.

The next stage of the program, which we have just completed, is contained in [5, 6] and a number of supporting papers cited therein. In that stage we initiated the infrared analysis by tracking, in the small field region, the evolution of the effective interaction generated by the iteration of a renormalization group map that is taylored to a parabolic covariance: in each renormalization group step the spatial lattice directions expand by a factor $L > 1$, the inverse temperature direction expands by a factor L^2 and the running chemical potential grows by a factor of about L^2, while the running coupling constant decreases by a factor of about L^{-1}. Consequently, the effective potential, initially close to a paraboloid, develops into a Mexican hat with a moderately large radius and a moderately deep circular well of minima. [5, 6] ends after a finite number (of the order of the magnitude of the logarithm of the coupling constant) of steps once the chemical potential, which initially was of the order of the coupling constant, has grown to a small 'ϵ' power of the coupling constant. Then the 'Mexican hatness' of the effective potential is strong enough that we can no longer base our analysis on expansions about zero field.

[1] See [4] for a more pedagogical introduction.

In the next stage of the construction, we plan to continue the parabolic evolution in the small field regime, but expanding around fields concentrated at the bottom of the (Mexican hat shaped) potential well rather around zero (much as is done in the Bogoliubov Ansatz) and track it through an additional finite number of steps until the running chemical potential is sufficiently larger than one. At that point we will turn to a renormalization group map with a scaling taylored to an elliptic covariance, that expands both the temporal (inverse temperature) and spatial lattice directions by the same factor L. It is expected that the elliptic evolution can be controlled through infinitely many steps, all the way to the symmetry broken fixed point. The system is superrenormalizable in the entire parabolic regime because the running coupling constant is geometrically decreasing. However in the elliptic regime, the system is only strictly renormalizable.

The final stage(s) of the program concern the control of the large field contributions in both the parabolic and elliptic regimes.

The implementation of the (parabolic) renormalization group in [5, 6] proceeds much as in [3], except that we are restricting our attention to the small field regime and

- we use $1 + 3$ dimensional block spin averages. In [3], we had used decimation, which was suited to the effectively one dimensional problem of evaluating the temporal ultraviolet limit.
- The stationary phase calculation that controls the oscillations is similar, but technically more elaborate.
- The critical fields and background fields are now solutions to (weakly) nonlinear systems of parabolic equations.
- The Stokes' argument that allows us to shift the multi dimensional integration contour to the 'reals' and
- the evaluation of the fluctuation integrals is similar.
- However, there is an important new feature: the chemical potential has to be renormalized.

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Physical Implications of the Chiral Anomaly – from Condensed Matter Physics to Cosmology

Jürg Fröhlich

This lecture contains a survey of some special aspects of what I have called the *"Gauge Theory of States of Matter"*, an approach towards a general classification of states of matter based on analyzing effective actions (= generating functionals of current Green functions). Effective action functionals can be used to describe the response of states of macroscopically large systems of quantum-mechanical matter to turning on various (real or virtual) external gauge fields. Exploiting cluster properties of current Green functions, power counting (i.e., dimensional analysis, using that conserved currents and the gauge fields they are coupled to do not have anomalous dimensions), and gauge invariance, in particular anomaly cancellation between bulk degrees of freedom and degrees of freedom localized near the edge of such systems ("holography"), one is able to determine the general form of effective action functionals of systems of condensed matter. This then yields a general picture of possible phases of condensed matter and, in particular, of bulk- and edge degrees of freedom and their properties.

Organization of the lecture:
Starting with an analysis of chiral edge currents in 2D electron gases exhibiting the quantum Hall effect, I discuss the role of anomalous chiral edge currents and of anomaly inflow in 2D insulators with explicitly broken parity and time-reversal (2D incompressible electron gases in a homogeneous external magnetic field, see [1] – [5], [6][1]) and in time-reversal invariant 2D topological insulators with strong spin-orbit interactions exhibiting edge spin-currents – as originally described in [6]; see also [7]. I derive the topological Chern-Simons theories that yield the correct response equations for the 2D bulk of such materials. A short review of "abelian Hall (electron) fluids" in terms of odd-integral lattices is offered, see [3, 4, 6] for details, and an indication as to the nature of quasi-particle degrees of freedom (2-component "Dirac fermions") in time-reversal invariant 2D topological insulators is given. (The theory of "non-abelian Hall fluids" is developed in [5].)

After an excursion into the theory of 3D topological insulators, including "axionic insulators" (see [7]), I end with a brief discussion of a model of Dark Matter and Dark Energy involving an axion coupled to the instanton (Pontryagin) density of a gauge field. This is based on work in progress with S. Alexander and R. Brandenberger. Somewhat related ideas have previously enabled my collaborators and me to describe a mechanism that may give rise to the growth of tiny, but very

[1] I do not attempt to provide a survey of the relevant literature here, but give some hints to papers I have been involved in writing that have been relevant in preparing my lecture – I apologize for this shortcoming!

homogeneous magnetic fields in the Universe extending over intergalactic scales; see [9, 10]. The growth of magnetic fields is triggered by an instability exhibited by axion electrodynamics. A related mechanism may guarantee that the "slow-roll condition" is valid in our model of Dark Energy and Dark Matter. The ideas in [9, 10] grew out of an attempt to identify higher-dimensional cousins of the quantum Hall effect.

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Topological invariants for time reversal symmetric crystals

Krzysztof Gawędzki

The simplest models of topological crystals are described by self-adjoint Hamiltonians acting in the Hilbert space of vector-valued functions on a crystalline d-dimensional lattice and commuting with the lattice translations. In the Bloch picture that diagonalizes the translations by a discrete Fourier transform, such Hamiltonians are described by smooth families $H(k)$ of $N \times N$ Hermitian matrices that are parameterized by k belonging to the Brillouin torus $\mathbf{T}^d = \mathbf{R}^d/(2\pi\mathbf{Z}^d)$. We shall be interested in the cases where $d = 2, 3$. The system of Bloch Hamiltonians

$H(k)$ describes an insulator if the Fermi energy ϵ_F separating the energy eigenstates that are occupied at zero temperature from the unoccupied states of higher energy is in the spectral gap of all $H(k)$. In such a case, the eigenstates of $H(k)$ with energies lower than ϵ_F form a smooth vector bundle $\mathcal{E}$ over $\mathbf{T}^d$ whose 1^{st} Chern numbers describe the simplest type of topological invariants for insulators [1]. In the presence of time reversal symmetry, the Bloch Hamiltonians satisfy the relation

$$H(k) = \theta H(-k)\theta^{-1}$$

for an antiunitary operator $\theta : \mathbf{C}^N \to \mathbf{C}^N$ that squares to $-I$ realizing the time reversal. In this case, the eigenstates of $H(k)$ and $H(-k)$ come in "Kramers pairs" related by θ, having the same energy. This property implies that the 1^{st} Chern number(s) of the bundle $\mathcal{E}$ vanish and, consequently, that $\mathcal{E}$ is a trivializable vector bundle (of even rank). Let $(e_i(k))_{i=1}^{2m}$ be a trivialization of $\mathcal{E}$ composed of vectors orthonormal for eack k. In [2], it was realized that for $d = 2$ there exists an obstruction $KM \in \mathbf{Z}_2$ to choose a trivialization of $\mathcal{E}$ that is composed of Kramers pairs, i.e. such that

$$e_{2i}(-k) = \theta e_{2i-1}(k).$$

The obstruction was defined in [3] by the multiplicative formula

$$(-1)^{KM} = \prod_{k=-k\in\mathbf{T}^2} \frac{\sqrt{\det w(k)}}{\operatorname{pf} w(k)}$$

where $w(k)$ is the "sewing matrix" with the entries $w_{ij}(k) = \langle e_i(-k)|\theta e_j(k)\rangle$ and the square root $\sqrt{\det w(k)}$ is chosen smoothly over T^d whereas the Pfaffians $\operatorname{pf} w(k)$ are defined only at $k = -k$ where $w(k)$ is antisymmetric.

I the talk, I discuss new expressions for the KM obstruction relating the latter to the Wess-Zumino (WZ) action functional $S_{WZ}(\phi)$ of maps $\phi : \mathbf{T}^2 \to U(n)$ that is defined modulo 2π by the prescription [4]

$$S_{WZ}(\phi) = \int_{\mathcal{B}} \widetilde{\phi}^*\chi$$

for $\widetilde{\phi}$ extending ϕ to an oriented 3-manifold $\mathcal{B}$ with $\partial\mathcal{B} = \mathbf{T}^2$ and for χ being the closed bi-invariant 3-form on the group $U(N)$ with 3-periods in $2\pi\mathbf{Z}$. First, one has

$$(-1)^{KM} = \exp[\sqrt{-1}\, S_{WZ}(w)].$$

Second, if $P(k)$ is the projector of the eigensubspace of $H(k)$ with energies $< \epsilon_F$ then

$$(-1)^{KM} = \exp[\tfrac{1}{2}\sqrt{-1}\, S_{WZ}(I - 2P)]$$

where, in the latter case, the extension $\widetilde{\phi}$ of $\phi = I - 2P$ has to be judiciously chosen to assure that $S_{WZ}(I - 2P)$ is defined modulo 4π. Both formulae may be proven using the techniques of bundle gerbes [5] providing local expressions for the WZ action $S_{wZ}(\phi)$ that enable the localization of the right hand sides at the contributions from $k = -k$.

The second formula suggests how to define a dynamical $\mathbf{Z}_2$-valued indices generalizing the Fu-Kane-Mele invariants to the case of two-dimensional crystalline Floquet systems periodically dependent on time that gives rise to Bloch Hamiltonians

$$H(k,t) = H(k,t+T) = \theta H(-k,-t)\theta^{-1}.$$

Such generalization was obtained in [6, 7]. It extended the proposal of [8] to time reversal symmetric Floquet systems.

Finally, I discuss in the talk the generalization of the above results to three-dimensional crystalline systems with time reversal symmetry, both static and with periodic time dependence [9, 10].

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Height fluctuations and universality relations in interacting dimer models

Alessandro Giuliani

In this talk I discuss height fluctuations and convergence of the height function to the massless Gaussian free field, for a class of interacting dimer models on the two-dimensional square lattice. I report recent work in collaboration with V. Mastropietro and F. Toninelli [5, 6, 7].

Two-dimensional (2D) dimer models have a long history, interconnected with the one of the 2D Ising model. They are used as highly simplified models either of liquids of anisotropic molecules, or of discrete random surfaces. The correspondence between these two different descriptions is mediated by the well-known correspondence between close-packed dimer configurations and the height function (briefly reviewed below). As in the case of the Ising model, they are also studied as

a playground for the theory of the critical point, for understanding the existence and nature of the scaling limit, and for investigating its conformal covariance.

I restrict the attention to the case of close-packing: in other words, the allowed dimer configurations that I consider are those where every vertex of the graph is covered exactly once by a dimer (a dimer being a rigid rod connecting two neighboring vertices), see fig.1. For definiteness, I restrict the discussion to the square lattice, even though other regular *bipartite* lattices can be easily treated by the same methods. The bipartite condition is important: we shall refer to the sites of the two sublattices as to the *even* and *odd* sites.

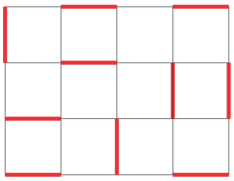

FIGURE 1. An example of an admissible dimer configuration on a finite portion of the 2D square lattice.

I distinguish between the standard, *non-interacting*, dimer model, and the so-called *interacting* model. The non-interacting model consists of the statistical mechanics model, where every admissible dimer configuration D is weighted evenly. On the contrary, the interacting case is characterized by a non-trivial Gibbs weight $e^{-\lambda W_D}$ on the space of allowed dimer configurations. The energy function W_D is a sum of local energies labelled by the plaquettes P of the square lattice, $W_D = \sum_P w_P(D)$, and $w_P(D)$ is an energy depending on the relative orientations of the dimers in the vicinity of P. We require W_D to be invariant under the basic symmetries of the square lattice (reflections and discrete rotations). To be definite, one can think $w_P(D)$ to be equal to -1, if P is occupied by two parallel dimers in D, and zero otherwise, see fig.2. In this case, we shall write $w_P(D) = -N_P(D)$. The interacting model has been introduced as the classical limit of quantum dimer models, which attracted a lot of attention in the last years, as (over-simplified) toy models for high temperature superconductors [14].

With every dimer configuration, we associate a height function, defined on the faces of the lattice, which is unique *up to an overall additive constant*. More precisely, the height difference between two faces f and f' is defined as:

$$h(f') - h(f) = \sum_{b \in C_{f \to f'}} \sigma_b (1_b - 1/4)$$

where: $C_{f \to f'}$ is a lattice path (connecting neighboring faces) going from f to f'; σ_b is a sign, equal to $+1$ or -1, depending on whether b is crossed by $C_{f \to f'}$ with an even site on the right or left, respectively; $!_b$ is the characteristic function of having

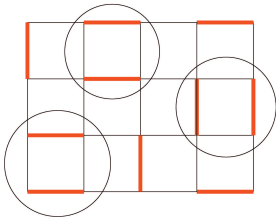

FIGURE 2. A dimer configuration D covering a finite portion of the square lattice, and the plaquettes such that $N_P(D) = 1$.

a dimer on b, i.e., 1_b is equal to 1, if b is occupied by a dimer, and zero otherwise. Note that the definition is well-posed, i.e., the right side is independent of the choice of the lattice path. In other words, the sum in the right side, evaluated along any *closed* path, is zero: this is a simple consequence of the close-packing condition, as the reader can easily check (it is enough to prove it for an elementary closed path encircling a vertex).

At close-packing, non-interacting dimers are exactly solvable [10]: the partition function can be written as the Pfaffian of a complex adjacency matrix, known as the Kasteleyn's matrix, and the multi-point dimer correlations (i.e., the joint probabilities of finding n dimers in prescribed locations) satisfy a suitable *fermionic* Wick rule; more precisely, the n-point function can be written as the Pfaffian of a suitable minor of the inverse Kasteleyn matrix. The explicit expression of these correlations shows that their connected part decays to zero *polynomially* at large distances: this algebraic decay of correlations is referred to by saying that the close-packed model is *critical*.

Thanks to the closed formula for the multipoint dimer correlations, one can also compute the variance of the height difference, as well as all its higher cumulants: the computation shows that $\langle (h(f) - h(f'))^2 \rangle_0 = \frac{1}{\pi^2} \log |f - f'| + O(1)$, as $|f - f'| \to \infty$ (here $\langle \cdot \rangle_0$ is the non-interacting dimer measure, in the thermodynamic limit), while the higher order cumulants are uniformly bounded as $|f - f'| \to \infty$. Recalling that a random variable is zero if and only if all the cumulant of order 3 or higher are zero, we conclude that the height difference for non-interacting dimers is asymptotically gaussian, in the limit of large distances. The logarithmic growth of the variance reveals the underlying massless Gaussian free field (GFF) behavior of the height function. Convergence of the height function to the massless GFF and conformal covariance of the scaling limit has been proved in [3, 11, 12, 13].

Let us now consider the interacting case, which has been extensively studied by approximate and numerical methods. In this case, the model is not exactly solvable anymore: the Pfaffian structure breaks down and there are no closed formulas, neither for the thermodynamic, nor for the correlation functions. In [4], by constructive Renormalization Group (RG) mehods, it has been shown that, for weak enough interaction, the system is still in a liquid phase, characterized by polynomial decay of the correleation functions. However, the liquid is *anomalous*,

in the sense that the dimer-dimer correlation is characterized by a modified critical exponent, denoted by $\kappa(\lambda)$, continuously depending on the interaction strength λ and such that $\kappa(0) = 1$. For this reason, one may fear that the logarithmic fluctuations of the height function may be changed into anomalous power law fluctuations, which would be incompatible with a GFF of the interacting height function. Quite remarkably, this is not the case, thanks to systematic cancellations taking place at all orders in renormalized perturbation theory. In short, our main result consists in a rigorous construction of all the multipoint dimer correlations and an asymptotic computation of the height cumulants. We prove that the height variance is renormalized by a non-trivial amplitude, which satisfies a universal relation, relating it to the critical exponent $\kappa(\lambda)$ of the dimer-dimer correlations. More precisely, our main result can be summarized as follows. We denote by $\langle\cdot\rangle_\lambda$ the interacting Gibbs measure in the thermodynamic limit, whose existence is part of our result.

Theorem 1. *There exist: (i) a positive constant λ_0 and a real analytic function $K(\lambda)$ on $|\lambda| < \lambda_0$ satisfying $A(0) = 1$, (ii) positive constants C_n, with $n \geq 2$, and a bounded function $R(x)$ satisfying $|R(x)| \leq C_2$, $\forall x \neq 0$, such that: if $f \neq f'$, then*

$$\langle (h(f) - h(f'))^2 \rangle_\lambda = \frac{K(\lambda)}{\pi^2} \log |f - f'| + R(f - f') \; .$$

Moreover, if $n > 2$, the n-th cumulant of $(h(f) - h(f'))$ is bounded uniformly in $|f - f'|$ as

$$|\underbrace{\langle h(f) - h(f'); \cdots ; h(f) - h(f') \rangle_\lambda}_{n \text{ times}}| \leq C_n \; .$$

As a corollary (of the proof of Theorem 1), we also obtain convergence of the height function to the massless GFF, in the following sense.

Theorem 2. *In order to fix the arbitrary additive constant in the definition of the height, we set $h(f_0) = 0$ at a given face f_0 (playing the role of 'central face'). For every C^∞, compactly supported, test function $\phi : \mathbb{R}^2 \mapsto \mathbb{R}$ satisfying $\int \phi(x) dx = 0$, and $\epsilon > 0$, define*

$$h^\epsilon(\phi) = \epsilon^2 \sum_{\boldsymbol{\eta}} h_{\boldsymbol{\eta}} \phi(\epsilon \boldsymbol{\eta})$$

where the sum runs over the faces of Λ. Then, for every $\alpha \in \mathbb{R}$,

$$\lim_{\epsilon \to 0} \langle e^{i\alpha h^\epsilon(\phi)} \rangle_\lambda = \exp\left(\frac{\alpha^2 K(\lambda)}{4\pi^2} \int \phi(x)\phi(y) \log|x - y| \, dx \, dy \right) .$$

Note that the right side of the last equation is the characteristic function of the massless GFF on the plane X with covariance

$$\langle X(x) X(y) \rangle = G_\lambda(x - y) := -\frac{K(\lambda)}{2\pi^2} \log|x - y|.$$

Its 'stiffness' $K(\lambda)$ satisfies the following remarkable identity:

$$K(\lambda) = \kappa(\lambda)$$

where $\kappa(\lambda)$ is the critical exponent of the dimer-dimer correlation, see [5, Theorem 2]. This is one of the *Haldane relations* adapted to the present context [8, 9, 1, 2].

The proof of these claims is based on a rigorous RG construction of the interacting dimer measure, on a systematic implementation of lattice Ward Identities and on the path independence property of the height function. The RG construction is performed on a fermionic reformulation of the model (which is possible, thanks to the underlying Pfaffian structure of the non-interacting model) and uses multiscale fermionic cluster expansion techniques. For the proof of Theorem 1 and 2, see [5]. The universality relation $K = \kappa$ is proved in [7].

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The operator product expansion

Stefan Hollands

The operator product expansion (OPE) is not only a powerful approximation method for correlation functions, but also conceptually important as a possible way to encode the algebraic content of quantum field theories (QFTs). Informally, it states that a correlation function containing composite operators can be expanded

as [1], [2]

$$\left\langle \prod_{i=1}^{N} \mathcal{O}_{A_i}(x_i)(\text{spectators}) \right\rangle = \sum_C \mathcal{C}^C_{A_1 \dots A_N}(x_1, \dots, x_N) \left\langle \mathcal{O}_C(x_N)(\text{spectators}) \right\rangle .$$

The objects $\mathcal{O}_A(x)$ are the local operators of the theory, which are typically products of derivatives of "basic field(s)" at the point x. The A_i and C a multi-indices labelling the various composite fields, and the "insertion points" $x_i \in \mathbb{R}^d$ have to be pairwise distinct. The OPE coefficients $\mathcal{C}^C_{A_1 \dots A_N}(x_1, \dots, x_N)$ are independent of the "spectators", which for instance can be other composite operators at other points. The convergence properties of the OPE depend on the choice of the spectators. In the early days, it was not in any case believed that the expansion converges at all in the sense of a series, but only as an asymptotic series as $|x_i - x_N| \to 0$ for all i.

Recently, considerable progress in our understanding of the OPE has been made in [4], [5], [6], [7]. One line of research has focussed on the $g\phi^4$-model in Euclidean, $d = 4$-dimensional space. In this model, the composite fields $\mathcal{O}_C$ are the monomials in $\phi, \partial\phi, \partial^2\phi, \dots$, to which a dimension $[C]$ can be assigned using standard "power counting" rules. The correlation functions and OPE coefficients in this model can be defined e.g. by the method of renormalization group flow equations in the form of a renormalized perturbation series in powers of g or $\hbar$ (i.e. "loops"), see [3] for a general exposition of this topic. In [4], [5], sharp bounds on the remainder in the OPE–i.e., the left minus right side of the above equation–were obtained if the sum over "C" is truncated for dimensions $[C] > D$, where D is large, and for the case that the "spectators" are given by

$$\text{spectators} = \prod_{i=1}^{n} \int d^4x \ \phi(x) f_i(x) \ . \tag{1}$$

Consider test functions f_i such that the support of $\hat{f}_1(p_1) \cdots \hat{f}_n(p_n)$ contains only configurations of 4-momenta $p_1, \dots, p_n$ whose magnitude is less than some P, and such that their distance to "exceptional"[1] momentum configurations is at least $\epsilon > 0$. Under these conditions, one has:

Theorem: The remainder of the operator product expansion (for two operators), carried out up to operators of dimension $D = [A_1] + [A_2] + \Delta$, at order $\hbar^l$ (i.e. l loops), is bounded by

$$\begin{aligned} &\Big|\text{remainder}\Big|_{l-\text{loops}} \\ &\leq P^N \sqrt{[A_1]![A_2]!} \left(K\, M\, \sup(1, \frac{P}{M})^{(n+2l+1)} \right)^{[A_1]+[A_2]} \left(\frac{P}{\inf(M, \epsilon)} \right)^{3n} \prod_i \sup|\hat{f}_i| \\ &\times \frac{1}{\sqrt{\Delta!}} \left(K\, M\, |x_1 - x_2|\, \sup(1, \frac{P}{M})^{(n+2l+1)} \right)^{\Delta} \mathcal{P}_{2l+\frac{n}{2}} \left(\log \frac{P}{\inf(M, \epsilon)} \right) , \end{aligned}$$

[1] By an exceptional configuration one means a set $(p_1, ..., p_n)$ such that a strict subsum of momenta vanishes. Singularities at such configurations are common in massless theories.

where K is a constant depending on n, l, and $\mathcal{P}_s$ is a polynomial of degree s whose coefficients depend on n, l. M is a "renormalization scale" in the massless theory and equal to the mass otherwise.

The bound implies that the OPE convergences as $D \to \infty$, and provides an increasingly accurate approximation as $|x_1 - x_2| \to 0$ for large but finite D. In addition, the result gives a rather detailed, and physically plausible, picture of the approximation properties of the OPE also in the following respects:

(1) We see that if P becomes large, then due to the factors of $P^{\Delta(n+2l+1)}$ in the remainder, the OPE converges more slowly as D, hence Δ, goes to infinity. This is physically plausible, because for large P the factorisation phenomenon exhibited by the OPE sets in more slowly if the "state" generated by the smeared spectator fields from the "vacuum" contains more "UV modes".
(2) We see that if the quantity ϵ becomes small, then the bound on the remainder is also larger due to the inverse powers of ϵ. This is also physically plausible, because ϵ characterises how close the momenta in the support of the spectator fields are to becoming "exceptional".

The convergence of the series suggests to take the OPE coefficients as the *defining* structure of the QFT, in much the same way as the structure constants of a finite dimensional algebra define it. In fact, as for ordinary algebras, one can show [7] that there hold powerful *associativity*-type relations among the OPE coefficients. The simplest example of such a relation is

$$\sum_D \mathcal{C}^D_{AB}(x_1, x_2)\mathcal{C}^E_{DC}(x_2, x_3) = \mathcal{C}^D_{ABC}(x_1, x_2, x_3) \ ,$$

where the sums are shown to converge as long as $0 < |x_1 - x_2| < |x_2 - x_3|$.

Based on such relations, we showed in [6], [7] that there holds a differential equation for the OPE coefficients as a function of the coupling g. This differential equation reads as follows in the simplest case. Let $\mathscr{V}$ be the operator representing the interaction of the model, i.e. $\mathscr{V} = -\phi^4$ in the present theory. Then

$$\partial_g \mathcal{C}^C_{AB}(x_1, x_2) = \int_y \Big\{ \mathcal{C}^C_{AB\mathscr{V}}(x_1, x_2, y) - \sum_{[D]\leq C} \mathcal{C}^D_{AB}(x_1, x_2)\mathcal{C}_{\mathscr{V}D}(y, x_2) \\ - \sum_{[D]\leq[A]} \mathcal{C}^D_{\mathscr{V}A}(y, x_1)\mathcal{C}^C_{DB}(x_1, x_2) - \sum_{[D]\leq[B]} \mathcal{C}^D_{\mathscr{V}B}(y, x_2)\mathcal{C}^C_{AD}(x_1, x_2) \Big\}$$

This relation, together with a hierarchy of similar relations with more points $\{x_i\}$ has two remarkable features:

(1) It *only* involves the OPE-coefficients, but no correlation functions.
(2) The integral over the "insertion point", y, is shown to be *absolutely convergent*, and in particular free of any of the seemingly unavoidable UV-divergences in QFT.

The last point is closely related to associativity (). Indeed, if the integration point y gets dangerously close to one of the points x_1, x_2, i.e. the "UV-region", then associativity guarantees that the singular part of the first term in $\{\dots\}$ is precisely cancelled by one of the sums in the second line of $\{\dots\}$. Similarly, if y goes to infinity, the first sum in $\{\dots\}$ cancels precisely the divergent contributions in the "IR region".

The above identities were derived in the context of perturbation theory. However, one may attempt to use them *directly* to construct QFT models non-perturbatively. The $g\phi^4$-theory is not a good candidate in this respect since it is not "asymptotically free". This disadvantage is avoided e.g. in the two-dimensional Gross-Neveu model describing a multiplet of Dirac fermions ψ in two spacetime dimensions. Its classical Lagrange density is $\mathscr{L} = \bar\psi(i\partial\!\!\!/ - m)\psi - g(\bar\psi\psi)^2$, so that the interaction is now $\mathscr{V} = -(\bar\psi\psi)^2$. Our construction proceeds in several steps:

(1) We expand each OPE-coefficient $\mathcal{C}^B_{A_1\dots A_n}(x_1,\dots,x_n;g)$ into an–a priori only formal–power series in g. This is, of course, a realization of the ordinary perturbative expansion. The differential equation () then turns into a recursive formula: From the OPE coefficients at order g^0 (free theory) we can get those at order g^1, and so fourth. Thus, to get started, we only need the OPE coefficients of the underlying free theory, i.e. for the free Lagrangian $\bar\psi(i\partial\!\!\!/ - m)\psi$.

(2) By induction, we get bounds for the OPE coefficients at arbitrary order g^r from bounds on the OPE coefficients in the free theory (order g^0). To get sufficiently sharp bounds in the free theory, in particular bounds that do not produce disastrous r-dependent combinatorial factors upon induction and that reflect sufficiently well the behavior of the coefficients in the points $\{x_i\}$ we use two main ideas: The first idea, which has many antecedents in other works on fermionic theories, is to write the order g^0-coefficients as a Pfaffian of a certain matrix M:

$$(\mathcal{C}_0)^B_{A_1\dots A_n}(x_1,\dots,x_n) = \mathrm{Pf}\, M(\{x_i, A_i\}) \ .$$

If we were to expand out the Pfaffian, we would get a large number of terms leading to disastrous combinatorial factors. To get around this, we use the trick of "Gram bounds" and of "finite range" decompositions pioneered by Brydges, Baurschmidt and coworkers. Since our recursion formula generates products of OPE coefficients organized by trees, we actually need a Pfaffian representation and recursion formulas for those, too.

(3) Combining the previous two steps, the aim is to obtain the following bound for the OPE-coefficients at order g^r:

$$\Big|(\mathcal{C}_r)^B_{A_1\dots A_n}(x_1,\dots,x_n)\Big| \le \frac{(\max\limits_{i\neq j}|x_i-x_j|)^{[B]}}{\prod_j(\min\limits_{i\neq j}|x_i-x_j|)^{[A_j]}}\ K^r \log^r(\frac{L}{\delta}) \ .$$

Here, K is a constant not depending on r, and δ, L are UV- and IR- cutoffs which are introduced into the y-integral () such that $\delta < |x_i - x_j| < L$ for

$i \neq j$. These cutoffs could be taken away, $\delta \to 0, L \to \infty$ at the level of the OPE coefficients $(\mathcal{C}_r)^B_{A_1 \ldots A_n}$ at each order g^r without any problems, since the integral () is absolutely convergent but *not* in the upper bound. There exists an alternative upper bound without cutoffs δ, L, but it contains a bad combinatorial factor in r, which would spoil convergence of the series in g^r.

(4) The bound (3) would establish that the series for $\mathcal{C}^B_{A_1 \ldots A_n}(x_1, \ldots, x_n; g)$ converges as long as, for fixed δ, L, we have $g < [K \log(L/\delta)]^{-1}$ and as long as $\delta < |x_i - x_j| < L$ for $i \neq j$. This is clearly not sufficient to establish the non-perturbative existence of the OPE, as the convergence radius shrinks to zero logarithmically as a function of the cutoffs. In order to deal with this problem, another idea is needed which highlights the flexibility of the OPE. The point is that, in a local quantum field theory, we have the freedom to redefine the fields ("basis change") as $\mathcal{O}_A \to \sum_B Z^B_A \, \mathcal{O}_B$ where Z is an invertible matrix whose entries are non-zero only when $[B] \leq [A]$. The OPE-coefficients transform as tensors under such a change of operator basis. The idea is now to absorb the divergence of the series for the OPE coefficients as $\delta \to 0, L \to \infty$ into such a field-redefinition. Let $\delta = \ell_0 e^{-t}, L = \ell_0 e^{Kt}$, where t is a scaling parameter. It is shown that there exists a field redefinition $Z(g, t)$ such that the OPE coefficients of the redefined operators are equal to the original ones with δ, L replaced by $\delta = \ell_0, L = \ell_0 e^t$ ($\Rightarrow \log(L/\delta) = t$) and g replaced by the running coupling $g(t, g) \sim g/(1 - g\beta_2 K t) \sim 1/(-\beta_2 K t)$ where $\beta_2 < 0$ is the usual 1-loop beta function of the model. So, the running coupling becomes small as $t \to \infty$ (asymptotic freedom), and our series for the OPE coefficient would converge as as the cutoffs are removed.

This program is currently under investigation in collaboration with J Holland, with major steps completed. Financial support of this research via ERC grant QC & C 259562 is gratefully acknowledged.

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Metastability for the Widom-Rowlinson model

Roman Kotecký
(joint work with Frank den Hollander, Sabine Jansen, and Elena Pulvirenti)

The Widom-Rowlinson model is one of the few models of interacting particles in the continuum for which a proof of a "liquid-gas" coexistence at a line of activities $z_c(\beta)$ for large inverse temperatures has been provided. There is a multi-body interaction among particles with the overall energy of a particle configuration proportional to the minus area of the union of balls of equal diameter centred at particle positions. An important feature of the model that facilitates its investigation is the fact that it can be rewritten as a model of two species of particles with mutual hard-core interaction.

In the reported work in progress, we consider a stochastic dynamics of the Widom-Rowlinson model on a two-dimensional torus and discuss a transition from a metastable "supercooled gas phase" to the stable "liquid phase". In particular, we present a proof of Arrhenius law with a nonstandard entropic correction.

We choose the parameters for which the liquid is the stable phase by considering an activity above the coexistence line:

$$z = \kappa z_c(\beta) \text{ with } \kappa \in (1, \infty).$$

Starting now from a low density configuration corresponding to the gas phase, our main object of interest is the random crossover time τ of the transition from a low density gas phase to the stable high density liquid phase. The main claim concerns the *mean crossover time* in the limit $\beta \to \infty$.

In particular, we show that *for every* $\kappa \in (1, \infty)$ *and a fixed sufficiently large torus, asymptotically with* $\beta \to \infty$,

$$\mathbb{E}(\tau) = \exp\Big[\beta \mathcal{U}(\kappa) - \beta^{1/3}\, \mathcal{S}(\kappa) + o(\beta^{1/3})\Big].$$

Here, $\mathcal{U}(\kappa)$ and $\mathcal{S}(\kappa)$ are particular functions related to the *critical droplet* that system needs to create to trigger the crossover.

Namely, $\mathcal{U}(\kappa) = \mathcal{U}_\kappa(R_c)$ with $\mathcal{U}_\kappa(R) = \pi R^2 - \kappa\pi(R-2)^2$ being the asymptotic value (with $\beta \to \infty$) of the *bulk free energy* of "liquid" occupying a disk of diameter $R-2$. The function $\mathcal{U}_\kappa$ reaches its maximum $\mathcal{U}(\kappa)$ for $R = R_c = \frac{2\kappa}{\kappa-1}$.

The second term in the exponent corresponds to the *surface free energy* of the critical droplet stemming from fluctuations of its boundary. The function $\mathcal{S}(\kappa)$ has also an explicit expression. Namely, $\mathcal{S}(\kappa) = s\big(R_c(\kappa)\big)$ where $s(R_c(\kappa)) \in (0, \infty)$ is the unique solution of the equation

$$4\sqrt{3\pi C(\kappa)} \int_0^\infty \mathrm{e}^{-s\alpha/2\pi}\, \mathrm{e}^{-C(\kappa)\alpha^3} \sqrt{\alpha}\, d\alpha = 1 \tag{1}$$

with

$$C(\kappa) = \frac{1}{48} R_c(\kappa)^2 (R_c(\kappa) - 2) = \frac{\kappa^2}{6(\kappa-1)^3}. \tag{2}$$

We link the fluctuations of the boundary of the critical droplet to a Brownian bridge conditioned to have integral zero. The proofs rely on large and moderate deviation principles for the volume of the critical droplet, isoperimetric inequalities, and capacity estimates.

On the gas of Bosons

Martin Lohmann

We consider a system of interacting Bosons on the $d = 3$ dimensional lattice $X = \mathbb{Z}^3/L\mathbb{Z}^3$, where L is a large integer that will eventually taken to become infinite in order to study the low energy phenomenology of the system. The Hamiltonian operator defining the system is given by

$$H = \sum_{x,x' \in X} \psi^\dagger(x)\big(- \Delta_{x,x'} - \mu\delta_{x,x'}\big)\psi(x') + \psi^\dagger(x)\psi^\dagger(x')v(x-x')\psi(x)\psi(x'),$$

where $\psi, \psi^\dagger$ are the conventional creation and annihilation operators on Fock space, Δ is the discrete Laplacian, $\mu \in \mathbb{R}$ is the chemical potential, and v is a two body potential which we assume to be exponentially decaying and with uniformly positive Fourier transform. The goal is to study the equilibrium thermodynamics of this system by constructing the correlation functions

$$\varrho_m(\mathbf{x};\mathbf{y}) = \lim_{L\to\infty} \mathcal{Z}^{-1}\mathrm{Tr}\; e^{-\beta H}\prod_{l=1}^{m}\psi^\dagger(x_l)\psi(y_l)$$
$$\mathcal{Z} = \mathrm{Tr}\; e^{-\beta H}$$

at large inverse temperature β. Of interest are the clustering properties of ϱ_m because they are linked to the phase which the system is in. It is conjectured that, for $\mu < \mu_*(\beta)$, the system is in the gaseous phase, for $\mu = \mu_*(\beta)$, it is at a critical point, and for $\mu > \mu_*(\beta)$, it is in the phase of Bose Einstein condensation. At large β, the critical value μ_* is believed to be given approximately by

$$\mu_*(\beta) \approx \hat{v}(0)\int_{\mathbb{R}^3}\frac{\mathrm{d}p}{e^{\beta p^2}-1}.$$

The gaseous phase is characterized by exponential clustering of correlations, such as $\varrho_1(x,y) \sim e^{-m|x-y|}$ or

$$\varrho_2(x_1,x_2,y_1,y_2) - \varrho_1(x_1,y_1)\varrho_2(x_2,y_2) - \varrho_1(x_1,y_2)\varrho_1(x_2,y_1) \sim e^{-m\max_{z,z'\in\{x_1,x_2,y_1,y_2\}}|z-z'|}.$$

At the critical point, the correlations cluster with a power law $|z-z'|^{-1}$. There is no clustering in the Bose Einstein condensation phase: e.g. $\varrho_1(x-y) \sim \rho_0 + \text{const}\,|x-y|^{-2}$, with $\rho_0 > 0$ the condensate density.

Establishing this picture is an important open problem. A well known partial result is the bound $|\varrho_1(x,y)| \leq \text{const}\, e^{-m|x-y|}$ for $\mu < 0$ and pointwise positive v by Ginibre [1]. Scaling limits of the problem in the conjectural Bose Einstein phase have received much attention, see [2] and references therein. In the formalism

used by these methods, it has been hard to explain the mechanism underlying the condensation and fluctuations of the condensate. A convenient language to describe this mechanism heuristically is through formal coherent state functional integrals, which have been applied extensively in theoretical physics.

In a program by Balaban, Feldman, Knörrer and Trubowitz, a rigorous coherent state integral representation, e.g. for $\mathcal{Z}$,

$$\mathcal{Z} = \lim_{p\to\infty} \int \prod_{\substack{x\in X \\ \tau\in\frac{\beta}{p}\mathbb{Z}/\beta\mathbb{Z}}} \frac{\mathrm{d}\phi(x,\tau)^* \wedge \phi(x,\tau)}{2\pi i} e^{-|\phi(x,\tau)|^2} e^{-\phi(x,\tau-\frac{1}{p})^* e^{\frac{\beta}{p}(\Delta-\mu)}\phi(x,\tau)} \chi_p(\phi)$$
$$\times \exp\Big(-\frac{\beta}{p} \sum_{\tau,x,x'} \phi(x,\tau-\tfrac{1}{p})^* \phi(x',\tau-\tfrac{1}{p})^* v(x-x')\phi(x,\tau)\phi(x',\tau)\Big),$$

for a small field characteristic function χ_p, has been derived and subsequently analyzed for small v. See [3]. The temporal ultraviolet limit $p\to\infty$ of this family of oscillatory integrals was constructed uniformly in β, L, up to integrating out the variables $\phi(x,\tau)$ for $\tau \in \theta\mathbb{Z}/\beta\mathbb{Z}$ for fixed, small enough θ. The result is given in the form of a sum over "small field large field decompositions" of X.

The dominant "pure small field" term in this decomposition has since been further investigated with a block spin renormalization group method for $\mu > \mu_*(\beta)$, assuming that additional natural small field conditions can be introduced at will [4]. This is a formidable and laborious challenge, as will be the justification of the new small field approximations. We propose to also study the phase $\mu \le \mu_*(\beta)$ with the block spin renormalization group. The pure small field part of this construction is technically much easier than in the $\mu > \mu_*$ case, and this allows to focus on the large field aspect of the problem, which is technically extremely involved and more complicated than the large field problems encountered in the treatment of the conventional classical unbounded spin systems.

The first result in this direction was given in [5], where the case $\mu < \mu_* - \text{const}\ \hat{v}(0)^{\frac{1}{2}-\varepsilon}$ was investigated with a robust single scale cluster expansion, and exponential clustering of all correlation functions was shown. The corresponding large field problem has a very easy structure, somewhat unnatural in the framework of the program as no stability problem arises. As the critical point μ_* is further approached, block spins have to be introduced and multi scale cluster expansions should be used. If $\mu < \mu_* - \text{const}\ \hat{v}(0)^{1+\varepsilon}$, the problem can be treated by a two scale cluster expansions with very large choice of the scaling parameter $M \sim \hat{v}(0)^{-\frac{1}{4}+\epsilon}, \epsilon < \frac{1}{4}\frac{d-2}{d+2}$, $\hat{v}(0) \le c(\beta)$. The second scale becomes massive after a single mass renormalization. The large field problem is much more subtle in this case, and in particular a careful treatment of the oscillatory nature of the integral seems to be necessary. My analysis of this problem is unfinished at the moment. It would be the first mathematical result on the Bose gas at positive chemical potential.

As $\mu \nearrow \mu_*$, more scales appear in the problem. The perturbation theory of the critical point is known to be superrenormalizable, which gives hope that the analysis of the inductive step in the multi scale induction of this construction can be performed in the same way as in the two scale case described above. A full

treatment of the subcritical phase and the critical point of the Boson system along these lines is the long term goal of the research program outlined here.

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Batalin-Vilkovisky Quantization and the Renormalization Group

Timothy Nguyen

The Batalin-Vilkovisky (BV) approach to perturbation quantization was developed by I. A. Batalin and G. A. Vilkvosky in the early 1980s as a general method of quantizing theories possessing symmetries [1]. Only quite recently however, have systematic and mathematically rigorous formulations of the BV approach appeared, both in Euclidean signature [2] and in Lorenztian signature [5] [3].

In this talk, we give a schematic overview of the approach of K. Costello [2], which combines a rigorous approach to regularization and renormalization together with the cohomological aspects of the BV framework. We then apply this framework to obtain results concerning the perturbative quantization of nonlinear sigma models in two dimensions. In particular, we provide a mathematically rigorous derivation of Friedan's famous result that the one-loop renormalization group flow for the nonlinear sigma model is the Ricci flow on the target:

$$\beta(g_{ij}) = \hbar \mathrm{Ric}_{ij}(g). \tag{1}$$

Of central importance in the BV framework is the *quantum master equation*

$$\{S, S\} + \hbar \Delta S = 0. \tag{2}$$

Here, S is an extended action which consists of a classical action consisting of ordinary fields and then additional terms involving ghosts and antifields that encapsulate the underlying classical symmetries of the system. The bracket $\{\cdot, \cdot\}$ is an odd Poisson bracket coming from the (graded) sympletic manifold structure on the space of all fields (ordinary plus ghost and antifields) and Δ is the BV Laplacian, which is essentially a divergence operator. The problem with (2) is two-fold. First, the operator Δ, being a divergence operator in the infinite-dimensional setting of quantum field theory, needs to be suitably interpreted. Second, since when we quantize a theory, we have to supplement our action S with counterterms in order to render quantities computed via the Feynman diagrammatic expansion finite, we may violate symmetries and hence the equation (2).

To address the first issue, we use a heat kernel regulator. This involves regulating the Green's function for the kinetic operator Δ of the theory (which we assume to be Laplace-type) via

$$\Delta^{-1} \longrightarrow \int_{\varepsilon}^{L} e^{-t\Delta} dt.$$

Here, ε and L are short (ultraviolet) and long (infrared) distance regulators, respectively. Renormalization involves adding ε-dependent local counterterms to the action S in such a way that the $\varepsilon \to 0$ limit of the Feynman diagramattic expansion is well-defined. A priori, since these counterterms are only required to render the theory finite, they need not satisfy any constraints due to symmetry considerations.

The power of the BV approach, however, is that the manner in which the symmetry is violated through renormalization can be expressed in cohomological terms. Namely, there is a scale L quantum master equation (one involving the renormalized interactions of the theory with a fixed infrared cutoff L, which we regard as a scale), which may write schematically as follows:

$$QME_L = O[L],$$

Here, QME_L stands for a regulated version of (2) and $O[L]$, if nonzero, is regarded as an obstruction. Remarkably, it turns out that, order by order in the perturbative parameter $\hbar$, we can perform a change of renormalization scheme that eliminates the obstruction $O[L]$ so long as we have control over the cohomology of of the differential $\{S, \cdot\}$ acting on the space of local functionals. The end result is that we can ensure that symmetries are maintined at the quantum level, i.e., there are no *anomalies*, if we can show that the appropriate cohomology groups induced from $\{S, \cdot\}$ vanish.

In [4], we apply this general framework to the study of nonlinear sigma models. The symmetries we consider are twofold: (i) diffeomorphism covariance for general Riemannian targets; (ii) a transitive group of isometries for homogeneous space targets. By computing the appropriate cohomology groups associated to these symmetries, we show that these symmetries are not anomalous when we quantize (in case (ii) some additional technical assumptions are needed on the target). As an application, by studying a suitable action of scaling on the space of quantizations, we obtain a notion of the renormalization group in the BV formalism. In this way, we are able to derive (1).

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Bose particles in a box. A convergent expansion of the ground state of the Hamiltonian in the mean field limiting regime

Alessandro Pizzo

In this talk we have presented a novel multi-scale technique to study many-body quantum systems where the total number of particles is kept fixed. The method is based on Feshbach map and the scales are represented by occupation numbers of particle states. In the talk, we have mainly considered a *three-modes* (including the zero mode) Bogoliubov Hamiltonian for a sufficiently small ratio between the kinetic energy and the Fourier component of the (positive type) potential corresponding to the two nonzero modes. For such a system, in the talk we have reported on the following results: a) For any space dimension $d \geq 1$ and in the mean field limiting regime (i.e., at fixed box volume $|\Lambda|$ and for a number of particles, N, sufficiently large) this method provides the construction of the ground state and its expansion in terms of the bare operators. b) In the limit $N \to \infty$ the expansion is up to any desired precision. c) In space dimension $d \geq 3$ the method provides similar results for an arbitrarily large (finite) box and a *large but fixed particle density* ρ, i.e., ρ is independent of the size of the box.

In this project we study the Hamiltonian describing a gas of (spinless) nonrelativistic Bose particles that, at zero temperature, are constrained to a $d-$*dimensional* box of side L with $d \geq 1$. The particles interact through a pair potential with a coupling constant proportional to the inverse of the particle density ρ. The rigorous description of this system has many intriguing mathematical aspects not completely clarified yet. In spite of remarkable contributions also in recent years, some important problems are still open to date, in particular in connection to the thermodynamic limit and the exact structure of the ground state vector. In the related papers [Pi1], [Pi2], and [Pi3], the reader can find several references concerning previous results on this topic.

Though the number of particles, N, is fixed we use the formalism of second quantization that turns out to be very convenient to implement the technique. The Hamiltonian corresponding to the pair potential $\phi(x-y)$ and to the coupling constant $\frac{1}{\rho} = \frac{|\Lambda|}{N} =: \lambda > 0$ is

$$H := \int \frac{1}{2m}(\nabla a^*)(\nabla a)(x)dx + \frac{\lambda}{2}\int\int a^*(x)a^*(y)\phi(x-y)a(y)a(x)dxdy \tag{1}$$

where reference to the integration domain $\Lambda := \{x \in \mathbb{R}^d \,|\, |x_i| \leq \frac{L}{2}\,,\ i = 1, 2, \ldots, d\}$ is omitted, periodic boundary conditions are assumed, and dx is Lebesgue measure in d dimensions. Concerning units, we have set $\hbar$ equal to 1. Here, the operators $a^*(x)\,, a(x)$ are the usual operator-valued distributions on the bosonic Fock space

$$\mathcal{F} := \Gamma\left(L^2\left(\Lambda, \mathbb{C}; dx\right)\right)$$

that satisfy the canonical commutation relations

$$[a^{\#}(x), a^{\#}(y)] = 0, \qquad [a(x), a^*(y)] = \delta(x-y)\mathbb{1}_{\mathcal{F}},$$

with $a^{\#} := a$ or a^*. In terms of the field modes they read

$$a(x) = \sum_{\mathbf{j}\in\mathbb{Z}^d} \frac{a_{\mathbf{j}} e^{ik_{\mathbf{j}}\cdot x}}{|\Lambda|^{\frac{1}{2}}}, \quad a^*(x) = \sum_{\mathbf{j}\in\mathbb{Z}^d} \frac{a^*_{\mathbf{j}} e^{-ik_{\mathbf{j}}\cdot x}}{|\Lambda|^{\frac{1}{2}}},$$

where $k_{\mathbf{j}} := \frac{2\pi}{L}\mathbf{j}$, $\mathbf{j} = (j_1, j_2, \ldots, j_d)$, $j_1, j_2, \ldots, j_d \in \mathbb{Z}$, and $|\Lambda| = L^d$, with CCR

$$[a^{\#}_{\mathbf{j}}, a^{\#}_{\mathbf{j}'}] = 0, \qquad [a_{\mathbf{j}}, a^*_{\mathbf{j}'}] = \delta_{\mathbf{j},\mathbf{j}'}\,. \tag{2}$$

The unique (up to a phase) vacuum vector of $\mathcal{F}$ is denoted by Ω ($\|\Omega\| = 1$).

Definition 1. *The pair potential $\phi(x-y)$ is a bounded, real-valued function that is periodic, i.e., $\phi(z) = \phi(z+\mathbf{j}L)$ for $\mathbf{j} \in \mathbb{Z}^d$, and satisfies the following conditions:*

(1) *$\phi(z) = \frac{1}{|\Lambda|}\sum_{\mathbf{j}\in\mathbb{Z}^d} \phi_{\mathbf{j}} e^{ik_{\mathbf{j}}z}$ is an even function, in consequence $\phi_{\mathbf{j}} = \phi_{-\mathbf{j}}$.*

(2) *$\phi(z)$ is of positive type, i.e., the Fourier components $\phi_{\mathbf{j}}$ are nonnegative.*

(3) *The pair interaction has a fixed but arbitrarily large ultraviolet cutoff (i.e., the nonzero Fourier components $\phi_{\mathbf{j}}$ form a finite set) with the requirements below to be satisfied:*

3.1) (Strong Interaction Potential Assumption) The ratio $\epsilon_{\mathbf{j}}$ between the kinetic energy of the modes $\pm\mathbf{j} \neq \mathbf{0} = (0,\ldots,0)$ and the corresponding Fourier component $\phi_{\mathbf{j}} (\neq 0)$ of the potential is sufficiently small.

3.2) For all nonzero $\phi_{\mathbf{j}}$ and some $1 > \mu > 0$, $\theta > 0$

$$\frac{\phi_{\mathbf{j}}}{\Delta_0}\frac{N^\mu}{N(N-N^\mu)} < \frac{1}{2} \quad , \quad \frac{1}{N^\mu} \leq \mathcal{O}((\sqrt{\epsilon_{\mathbf{j}}})^{1+\theta})\,, \tag{3}$$

where $\Delta_0 = \min\left\{k^2_{\mathbf{j}} \,|\, \mathbf{j} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}\right\}$ and N is the number of particles in the box.

The operator H is meant to be restricted to the subspace $\mathcal{F}^N$ with exactly N particles.

Motivations and features of the strategy

We know that, at fixed volume $|\Lambda|$, the expectation value of the number operator[1] $\sum_{\mathbf{j}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} a^*_{\mathbf{j}} a_{\mathbf{j}}$ in the ground state of the Hamiltonian (1) remains bounded in the mean field limit (i.e., $\lambda = \frac{|\Lambda|}{N}$ and $N \to \infty$); see the papers by Seiringer and by Lewin *et al.* quoted in [Pi1]. Starting from this fact, one might think of a multi-scale procedure leading to an effective Hamiltonian for spectral values in a neighborhood of the ground state energy. An obvious candidate for such an effective Hamiltonian is (a multiple of) the orthogonal projection onto the state where all the particles are in the zero mode.

The Feshbach map is a very useful tool to construct effective Hamiltonians. We recall that given the (separable) Hilbert space $\mathcal{H}$, the projections $\mathscr{P}$, $\overline{\mathscr{P}}$ ($\mathscr{P} = \mathscr{P}^2$, $\overline{\mathscr{P}} = \overline{\mathscr{P}}^2$) where $\mathscr{P} + \overline{\mathscr{P}} = \mathbb{1}_{\mathcal{H}}$, and a closed operator $K - z\mathbb{1}$ acting on $\mathcal{H}$ (z in

[1] The operator $\sum_{\mathbf{j}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} a^*_{\mathbf{j}} a_{\mathbf{j}}$ counts the number of particles in the nonzero modes states.

a subset of $\mathbb{C}$) the Feshbach map associated with the couple $\mathscr{P}$, $\overline{\mathscr{P}}$ maps $K - z\mathbb{1}$ to the operator $\mathscr{F}(K - z\mathbb{1})$ acting on $\mathscr{P}\mathcal{H}$ where (formally)

$$\mathscr{F}(K - z\mathbb{1}) := \mathscr{P}(K - z\mathbb{1})\mathscr{P} - \mathscr{P}K\overline{\mathscr{P}}\frac{1}{\overline{\mathscr{P}}(K - z\mathbb{1})\overline{\mathscr{P}}}\overline{\mathscr{P}}K\mathscr{P}\,. \tag{4}$$

The Feshbach map is "isospectral", i.e., assuming that $\mathscr{F}(K-z\mathbb{1})$ is a well defined closed operator on $\mathscr{P}\mathcal{H}$ then: 1) $\mathscr{F}(K - z\mathbb{1})$ is bounded invertible if and only if z is in the resolvent set of K; 2) z is an eigenvalue of K if and only if 0 is an eigenvalue of $\mathscr{F}(K-z\mathbb{1})$. Moreover, the map provides an algorithm to reconstruct the eigenspace corresponding to the eigenvalue z from the kernel of the operator $\mathscr{F}(K - z\mathbb{1})$, and their dimensions coincide.

The use of the Feshbach map for the spectral analysis of quantum field theory systems started with the seminal work by V. Bach, J. Fröhlich, and I.M. Sigal, followed by refinements of the technique and variants; see the references in [Pi1]. In those papers, the use of the Feshbach map is in the spirit of the functional integral renormalization group, and the projections ($\mathscr{P}$, $\overline{\mathscr{P}}$) are directly related to energy subspaces of the free Hamiltonian. However, as a mathematical tool the Feshbach map enjoys an enormous flexibility due to the freedom in the choice of the couple of projections $\mathscr{P}$, $\overline{\mathscr{P}}$. The effectiveness of the choice depends on the features of the Hamiltonian.

In the system that we study the total number of particles is conserved under time evolution. The effective Hamiltonian that we want to construct suggests to relate the Feshbach projections ($\mathscr{P}$, $\overline{\mathscr{P}}$) to subspaces of states with definite number of particles in the modes labeled by $\left\{\frac{2\pi}{L}\mathbf{j}\,;\,\mathbf{j}\in\mathbb{Z}^d\right\}$. More precisely, consider the eigenspace of $\sum_{\mathbf{j}=\pm\mathbf{j}_*} a^*_{\mathbf{j}}a_{\mathbf{j}}$ corresponding to the eigenvalue i, i.e., the subspace of states containing i particles in the modes associated with $\pm\frac{2\pi}{L}\mathbf{j}_*$. Observe that the interaction part of the second quantized Hamiltonian in (1) can connect two eigenspaces corresponding to distinct eigenvalues, i and i', only if $i - i' = \pm 1, \pm 2$. The selection rules of the interaction Hamiltonian with respect to the occupation numbers of the particle states associated with the modes $\left\{\frac{2\pi}{L}\mathbf{j}\,;\,\mathbf{j}\in\mathbb{Z}^d\right\}$ suggest to construct a flow of Feshbach maps associated with projections onto such eigenspaces with decreasing eigenvalue i.

In fact, our method works for a potential ϕ with an ultraviolet cut-off and in the *strong interaction potential regime*: by this we mean that the ratio between each nonzero Fourier component of the potential, $\phi_{\mathbf{j}}$, and the corresponding kinetic energy, $k^2_{\mathbf{j}}$, must be sufficiently large. For a (positive definite) potential $\phi \in L^1$ such that $\int \phi(z)dz > 0$, this is precisely the regime that is relevant in the thermodynamic limit because at fixed $\mathbf{j}$ the ratio $\phi_{\mathbf{j}}/(k_{\mathbf{j}})^2$ diverges like L^2, being $k_{\mathbf{j}} := \frac{2\pi}{L}\mathbf{j}$ and L the side of the box.

In this talk, using a new multi-scale technique inspired by the Feshbach map we have shown how to construct the ground state of a three-modes systems described

by the Bogoliubov Hamiltonian

$$(5)\ H^{Bog}_{\mathbf{j}_*} := \sum_{\mathbf{j}\in\mathbb{Z}^d\setminus\{\pm\mathbf{j}_*\}} k^2_{\mathbf{j}} a^*_{\mathbf{j}} a_{\mathbf{j}} + \sum_{\mathbf{j}=\pm\mathbf{j}_*} (k^2_{\mathbf{j}} + \phi_{\mathbf{j}} \frac{a^*_{\mathbf{0}} a_{\mathbf{0}}}{N}) a^*_{\mathbf{j}} a_{\mathbf{j}} + \phi_{\mathbf{j}_*} \frac{a^*_{\mathbf{0}} a^*_{\mathbf{0}} a_{\mathbf{j}_*} a_{-\mathbf{j}_*}}{N}$$

$$+\phi_{\mathbf{j}_*} \frac{a_{\mathbf{0}} a_{\mathbf{0}} a^*_{\mathbf{j}_*} a^*_{-\mathbf{j}_*}}{N}$$

$$(6)\qquad =: \sum_{\mathbf{j}\in\mathbb{Z}^d\setminus\{\pm\mathbf{j}_*\}} k^2_{\mathbf{j}} a^*_{\mathbf{j}} a_{\mathbf{j}} + \hat{H}^{Bog}_{\mathbf{j}_*}\,.$$

Notice, that after implementing the volume integrations in (1) and keeping only the terms that are quadratic in the operators $a_{\mathbf{0}}$, $a^*_{\mathbf{0}}$ one gets the complete (particles number preserving) Bogoliubov Hamiltonian

$$(7)\qquad H^{Bog} = \frac{1}{2} \sum_{\mathbf{j}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \hat{H}^{Bog}_{\mathbf{j}}\,.$$

Consequently, H^{Bog} can be seen as a collection of three-modes systems. In fact, the three-modes system analyzed in [Pi1] represents the main building block in the construction of the ground state of the Bogoliubov Hamiltonian (see (7)) and of the complete Hamiltonian (see (1)) in the mean field limiting regime, provided the potential fulfills Definition 1; see [Pi2] and [Pi3], respectively.

For the three-modes system described by the Hamiltonian $H^{Bog}_{\mathbf{j}_*}$ where $\epsilon_{\mathbf{j}_*} := \frac{k^2_{\mathbf{j}_*}}{\phi_{\mathbf{j}_*}}$ is sufficiently small, we have considered the Feshbach flow associated with the sequence of projections defined below. Without loss of generality, we can suppose that $H^{Bog}_{\mathbf{j}_*}$ is restricted to $\mathcal{F}^N$ with N even. Then, for i even number, $0 \le i \le N-2$, we define $\overline{\mathscr{P}^{(i)}} := Q^{>i+1}_{\mathbf{j}_*}$, $\mathscr{P}^{(i)} := Q^{(i,i+1)}_{\mathbf{j}_*}$ where:

- $Q^{(0,1)}_{\mathbf{j}_*}$:= the projection (in $\mathcal{F}^N$) onto the subspace generated by vectors with $N-0=N$ or $N-1$ particles in the modes $\mathbf{j}_*$ and $-\mathbf{j}_*$, i.e., the operator $a^*_{\mathbf{j}_*} a_{\mathbf{j}_*} + a^*_{-\mathbf{j}_*} a_{-\mathbf{j}_*}$ has eigenvalues N and $N-1$ when restricted to $Q^{(0,1)}_{\mathbf{j}_*}\mathcal{F}^N$,
- $Q^{(>1)}_{\mathbf{j}_*}$:= the projection onto the orthogonal complement of $Q^{(0,1)}_{\mathbf{j}_*}\mathcal{F}^N$ in $\mathcal{F}^N$,

and, iteratively, up to $i = N-2$

- $Q^{(i,i+1)}_{\mathbf{j}_*}$:= the projection onto the subspace of $Q^{(>i-1)}_{\mathbf{j}_*}\mathcal{F}^N$ spanned by the vectors with $N-i$ or $N-i-1$ particles in the modes $\mathbf{j}_*$ and $-\mathbf{j}_*$,
- $Q^{(>i+1)}_{\mathbf{j}_*}$:= the projection onto the orthogonal complement of $Q^{(i,i+1)}_{\mathbf{j}_*} Q^{(>i-1)}_{\mathbf{j}_*}\mathcal{F}^N$ in $Q^{(>i-1)}_{\mathbf{j}_*}\mathcal{F}^N$.

Hence, we can write

$$(8)\qquad Q^{(>i+1)}_{\mathbf{j}_*} + Q^{(i,i+1)}_{\mathbf{j}_*} = Q^{(>i-1)}_{\mathbf{j}_*}\,.$$

As a final couple of Feshbach projections, we employ $\mathscr{P}^{(N)} := |\eta\rangle\langle\eta|$, $\overline{\mathscr{P}^{(N)}} := Q_{\mathbf{j}_*}^{(>N-1)} - |\eta\rangle\langle\eta|$, where $\eta := \frac{1}{\sqrt{N!}} a_{\mathbf{0}}^* \dots a_{\mathbf{0}}^* \Omega$, i.e., η is the state where all the N particles are in the zero-mode state.

The Feshbach flow is implemented starting from $H_{\mathbf{j}_*}^{Bog} - z\mathbb{1}$ and using iteratively the formula in (4) up to the final step $i = N$. The final Feshbach Hamiltonian is the finite rank operator $f(z)|\eta\rangle\langle\eta|$. In the range $(-\infty, z_{max})$ of the spectral parameter z where the flow can be implemented, we show that there exists a unique z_* such that $f(z_*) = 0$. Then, z_* is an eigenvalue (the ground state energy) of the original Hamiltonian $H_{\mathbf{j}_*}^{Bog}$ due to the isospectrality that holds at each step of the Feshbach flow. Feshbach theory provides also an algorithm to reconstruct the eigenvector of the original Hamiltonian $H_{\mathbf{j}_*}^{Bog}$ associated with the eigenvalue z_* from the eigenvector (η) with eigenvalue zero of the final Feshbach Hamiltonian $f(z_*)|\eta\rangle\langle\eta|$.

Statement of the results and role of the assumptions
In the following list of remarks we specify the results that are obtained in [Pi1] for the three-modes Bogoliubov Hamiltonian, and the role of the *Strong interaction potential assumption* together with Condition 3.2 in Definition 1.

(1) For the implementation of the Feshbach map up to the $N-2-th$ step we shall require $\frac{1}{N} \leq \epsilon_{\mathbf{j}_*}^{\nu}$ for some $\nu > \frac{11}{8}$ and $\epsilon_{\mathbf{j}_*}$ sufficiently small. The bound $\frac{1}{N} \leq \epsilon_{\mathbf{j}_*}^{\nu}$ holds in the mean field limiting regime where the box is kept fixed and the number of particles, N, can be arbitrarily large irrespective of the box size. For space dimension $d \geq 3$, at fixed particle density, the bound $\frac{1}{N} \leq \epsilon_{\mathbf{j}_*}^{\nu}$ is fulfilled (for $\nu < \frac{3}{2}$) if the box is sufficiently large. For $d = 1, 2$, if at fixed $\mathbf{j}_*$ and $\phi_{\mathbf{j}_*}$ the box size tends to infinity the particle density ρ must be suitably divergent to ensure the bound $\frac{1}{N} \leq \epsilon_{\mathbf{j}_*}^{\nu}$.

(2) For the last step of the Feshbach flow, Condition 3.2) in Definition 1 is also necessary for the implementability up to values of the spectral parameter z belonging to a neighborhood of the ground state energy of $H_{\mathbf{j}_*}^{Bog}$. This condition is fulfilled for any dimension d in the mean field limiting regime. At fixed particle density and for $d \geq 2$, Condition 3.2) is fulfilled if L is sufficiently large.

(3) The existence of the point z_* such that $f(z_*) = 0$, i.e., the ground state energy of $H_{\mathbf{j}_*}^{Bog}$, is established for any space dimension $d \geq 1$ in the mean field limiting regime.
With regard to a box of arbitrarily large side $L(< \infty)$, the existence of z_* is achieved if $\rho \geq \rho_0 (L/L_0)^{3-d}$ where ρ_0 is sufficiently large and $L_0 = 1$. Hence, for $d \geq 3$ it is enough to require ρ be sufficiently large but independent of L and the result holds for a finite box of arbitrarily large (finite) volume $|\Lambda|$.

(4) In all cases where the existence of z_* is proven we can construct the ground state. We also show that in the mean field limiting regime $|z_* - E_{\mathbf{j}_*}^{Bog}| \leq$

$\mathcal{O}(\frac{1}{N^\beta})$ for any $0 < \beta < 1$ where $E^{Bog}_{\mathbf{j}} := -\Big[k^2_{\mathbf{j}} + \phi_{\mathbf{j}} - \sqrt{(k^2_{\mathbf{j}})^2 + 2\phi_{\mathbf{j}} k^2_{\mathbf{j}}}\Big]$. Furthermore, in space dimension $d = 3$, for any scaling $\rho = \rho_0(\frac{L}{L_0})^\delta$ with $\delta > 0$ the ground state energy of $H^{Bog}_{\mathbf{j}_*}$ tends to $E^{Bog}_{\mathbf{j}_*}$ as $L \to \infty$. This implies that in space dimension $d \geq 4$ at fixed ρ the ground state energy of $H^{Bog}_{\mathbf{j}_*}$ tends to $E^{Bog}_{\mathbf{j}_*}$ in the thermodynamic limit.
In the mean field limit (i.e., fixed box and $N \to \infty$) we provide the expansion of the ground state vector in terms of the bare operators and the vector η up to any desired precision.

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Hall transitions in the Haldane-Hubbard model

Marcello Porta

(joint work with Alessandro Giuliani, Ian Jauslin, Vieri Mastropietro)

1. Introduction

Two-dimensional condensed matter systems display remarkable transport properties. A paradigmatic example is the *integer quantum Hall effect* (IQHE): at low temperatures, thin samples of insulating materials exposed to strong magnetic fields display a *quantized* transverse conductivity. Let σ be the conductivity matrix: $\sigma_{11} = \sigma_{22}$ is the longitudinal conductivity, while $\sigma_{12} = -\sigma_{21}$ is the transverse, or Hall, conductivity. The conductivity matrix describes the linear response of the system when exposed to a weak electric field $E = (E_1, E_2)$: linear response theory predicts that $J = \sigma E + o(E)$, where J is the current generated by the external field. The integer quantum Hall effect is the following experimental observation:

$$\sigma_{11} = \sigma_{22} = 0\,, \qquad \sigma_{12} = -\sigma_{21} \in \frac{e^2}{h}\mathbb{Z}\,. \tag{1}$$

The quantization of σ_{12} is measured with astonishing precision. Moreover, the value of σ_{12} stays constant as long as the Fermi energy of the system is in a band gap, or in a mobility gap. From a theoretical viewpoint, the IQHE is very well understood in the *absence* of interactions, that is for independent particles [12, 2, 3, 1]. The picture is more complex in the presence of many-body interactions. Preexisting arguments (e.g., [4]), and recent theorems [10], ensure quantization under the assumption that the spectrum of the interacting system has a gap above the ground state (*incompressibility*). This assumption is unproven in most of the physically relevant cases.

In a recent work [7] we proved the universality of the conductivity matrix for general fermionic lattice models, in the presence of short-ranged interactions. We do not make any assumption on the interacting spectrum. Instead, we assume that the *noninteracting* theory has a gap, and that the many-body interaction is weak. The result is that the conductivity matrix, as defined from *Kubo formula*, is *independent* of weak many-body interactions. Thus, [7] proves the stability of the IQHE against weak interactions. The proof is based on fermionic cluster expansion techniques, and lattice Ward identities. The result of [7] is not uniform in the gap of the noninteracting theory: the radius of convergence U_0 shrinks to zero as the gap vanishes. In particular, [7] does not allow to study the *transitions* between different Hall phases, which require the spectral gap to close.

2. Hall transitions in the the Haldane-Hubbard model

The Haldane model is a graphene-like model, with a nontrivial topological phase diagram. In the absence of interactions, this model has been introduced by Haldane in [9]. It describes fermions on the honeycomb lattice; the Hamiltonian is

$$\mathcal{H}^{(0)}_{\rm H} = \mathcal{H}^{(0)}_{\rm g} + \mathcal{H}^{(0)}_{M} + \mathcal{H}^{(0)}_{\phi} \ , \tag{2}$$

where: $\mathcal{H}^{(0)}_{\rm g}$ describes the hopping between nearest-neighbours ("non-interacting graphene"); $\mathcal{H}^{(0)}_{M}$ implements a staggered chemical potential $\pm M$ on the two triangular sublattices; $\mathcal{H}^{(0)}_{\phi}$ introduces a next-to-nearest neighbour hopping, with an alternating hopping parameter $t_2 e^{\pm i\phi}$. The phases $\pm\phi$ describe a magnetic field with *zero net flux* through the hexagonal cell. The Haldane model is the simplest example of a topological insulator, and it is the building brick for more complex models (*e.g.*, the Kane-Mele model). It has been experimentally realized in [11].

For a generic choice of parameters, the model is gapped. Interestingly, even though the magnetic flux is zero, the Hall conductivity is nontrivial. Let $m_\pm = M \pm 3\sqrt{3}t_2 \sin\phi$; $|m_\pm|$ are the amplitudes of the gaps splitting the two conical intersection of the energy bands of $\mathcal{H}^{(0)}_{\rm g}$. The Hall conductivity, as given by Kubo formula, is [9]:

$$\sigma^{(0)}_{12} = \frac{e^2}{h}\big[\mathrm{sgn}(m_-) - \mathrm{sgn}(m_+)\big] \ . \tag{3}$$

In [8] we applied rigorous renormalization group methods to study the topological phase diagram of the *interacting* Haldane model, or Haldane-Hubbard model. The Hamiltonian of the model is $\mathcal{H}^{(0)} + U\mathcal{V}$, where $\mathcal{V}$ is the standard on-site Hubbard interaction. In the gapless regime, the scaling properties of the model are the same of $2+1$-dimensional Dirac fermions, with a quartic local interaction. A crucial point is that the interaction is *irrelevant* in the RG sense, as in [5, 6]. Thus, the ground state correlations can be constructed via *convergent* renormalized series, and the scaling exponents are the same of the noninteracting theory. Nevertheless, the interaction has nontrivial effects: it renormalizes the Fermi velocity, the wave function and the effective masses of the quasi-particles.

The result of [8] is a rigorous construction of the renormalized transition curves of the Haldane-Hubbard model; see Fig. 1. The curves solve the equations $m_{R,\pm} =$

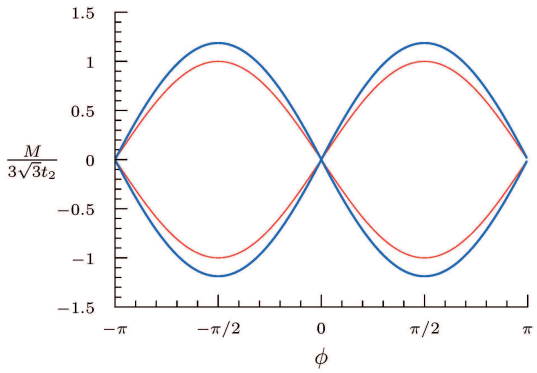

FIGURE 1. In red (inner lines), the noninteracting transition curves. In blue (outer lines), the renormalized transition curves for small $U > 0$ (color online).

0, where $m_{R,\pm} = m_\pm + F_\pm(m_+, m_-; U)$, with $F_\pm = O(U)$ and $F_\pm = 0$ if $m_+ = m_- = 0$. The quantities $|m_{R,\pm}|$ play the role of renormalized masses for the quasi-particles. We prove that, for $U \in (-U_0, U_0)$ with U_0 independent of $m_\pm$, and by suitably choosing μ (renormalized chemical potential):

$$\Big[\lim_{m_{R,\omega}\to 0^+} - \lim_{m_{R,\omega}\to 0^-}\Big]\sigma_{12} = \frac{2e^2}{h}\omega\,, \qquad \omega = \pm\,. \tag{4}$$

That is, the discontinuity of the interacting Hall conductivity at the renormalized critical line is *universal*: it only depends on fundamental constants. Moreover, the longitudinal conductivity *on* the critical line, $\sigma_{11}^{\rm cr}$, is also universal:

$$\sigma_{11}^{\rm cr} = \frac{e^2}{h}\frac{\pi}{4}\,, \tag{5}$$

which is *half* the value obtained for graphene [6]. Finally, in the special points of the phase diagram $m_{R,+} = m_{R,-} = 0$, $\sigma_{11}^{\rm cr}$ is *twice* the value in Eq. (5): each conical intersection in the energy bands contributes with the universal quantity $(e^2/h)(\pi/4)$ to the total longitudinal conductivity. The results (4), (5) follow from a combination of rigorous RG methods and lattice Ward identities.

Our results apply to *clean* interacting systems. It is a very important open questions to understand quantization of charge transport for interacting *disordered* systems: strong disorder is an essential feature to explain the existence of the sharply quantized Hall plateaux.

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Renormalization on Lorentzian manifolds in pAQFT

Kasia Rejzner

1. Causal structure

Let $\mathcal{M} = (M, g)$ be a d-dimensional spacetime, i.e. a smooth d-dimensional manifold with the metric g of signature $(+, -, \ldots, -)$. We assume $\mathcal{M}$ to be oriented, time-oriented and globally hyperbolic (i.e. it admits foliation with Cauchy hypersurfaces). To make this concept clear let me recall a few important definitions in Lorentzian geometry.

Definition 1. *Let $\gamma : \mathbb{R} \supset I \to M$ be a smooth curve in M, for I an interval in $\mathbb{R}$ and let $\dot{\gamma}$ be the vector tangent to the curve. We say that γ is*

- *timelike, if $g(\dot{\gamma}, \dot{\gamma}) > 0$,*
- *spacelike, if $g(\dot{\gamma}, \dot{\gamma}) < 0$,*
- *lighlike (null), if $g(\dot{\gamma}, \dot{\gamma}) = 0$,*
- *causal, if $g(\dot{\gamma}, \dot{\gamma}) \geq 0$.*

The classification of curves defined above is referred to as the *causal structure*.

Definition 2. *Given the global timelike vector field u (the time orientation) on M, a causal curve γ is called future-directed if $g(u,\dot\gamma) > 0$ all along γ. It is past-directed if $g(u,\dot\gamma) < 0$.*

Definition 3. *A causal curve is **future inextendible** if there is no $p \in M$ such that:*

$$\forall U \subset M \text{ open neighborhoods of } p,\ \exists t' \text{ s.t. } \gamma(t) \in U\ \forall t > t' .$$

Definition 4. ***A Cauchy hypersurface** in $\mathcal{M}$ is a smooth subspace of $\mathcal{M}$ such that every inextendible causal curve intersects it exactly once.*

We construct quantum field theory models on globally hyperbolic manifolds using the local geometric data and applying the methods of microlocal analysis. The latter allow us to control propagation of singularities. This way we can construct correlation functions and time-ordered products of fields as well-defined distributions. In general we work with formal power series in $\hbar$ and the coupling constant λ. However, it is believed that convergence in λ can be shown for some simple models, e.g. sin-Gordon in the super-renormalizable regime.

2. Free scalar field

We define the (off-shell) configuration space to be $\mathcal{E} = \mathcal{C}^\infty(M,\mathbb{R})$ and model classical observables as smooth functionals on $\mathcal{E}$. A functional $F \in \mathcal{C}^\infty(\mathcal{E},\mathbb{C})$ is called local if it can be expressed as

$$F(\varphi) = \int_M f(j^k(\varphi))\,,$$

where f is a local, density-valued function on the jet bundle. The space of local functionals is called $\mathcal{F}_{\mathrm{loc}}$. We define the point-wise product of functionals as $F\cdot G(\varphi) \doteq F(\varphi)G(\varphi)$ and call the algebraic completion of $\mathcal{F}_{\mathrm{loc}}$ with respect to this product, the space of multilocal functionals denoted by $\mathcal{F}$. Localization properties of functionals are characterized by the notion of *spacetime support*.

$$\mathrm{supp}(F) \doteq \{x \in M | \forall \mathcal{O} \ni x \text{ open}, \exists \varphi,\psi \in \mathcal{E}(\mathcal{M}) \text{ s.t. } \psi \subset \mathcal{O} \text{ and } F(\varphi+\psi) \neq F(\varphi)\}$$

Regularity properties of a smooth functional are formulated in terms of the WF-sets properties of its derivatives, since $F^{(n)}(\varphi) \in \mathcal{E}'(M^n,\mathbb{C})$. In particular, microcausal functionals, $\mathcal{F}_{\mu\mathrm{c}}$, are those for which

$$\mathrm{WF}(F^{(n)}(\varphi)) \subset \Xi_n\,,$$

where

$$\Xi_n \doteq T^*M^n \setminus \{(x_1,\dots,x_n;k_1,\dots k_n) | (k_1,\dots,k_n) \in (\overline{V}_+^n \cup \overline{V}_-^n)_{(x_1,\dots,x_n)}\}\,,$$

where $\overline{V}_+/\overline{V}_-$ is the future/past closed lightcone. Regular functionals are those for which $\mathrm{WF}(F^{(n)}(\varphi)$ is empty for all $\varphi \in \mathcal{E}$, $n \in \mathbb{N}$.

The equation of motion for the free field reads

$$P\varphi = 0\,, \quad P = -(\Box + m^2)\,.$$

It is crucial that P is a normally hyperbolic operator, so on a globally hyperbolic spacetime it has retarded/advanced Green's functions $\Delta^{\mathrm{R}}/\Delta^{\mathrm{A}}$, respectively. The causal propagator (the commutator function) is defined as $\Delta \doteq \Delta^{\mathrm{R}} - \Delta^{\mathrm{A}}$. The WF set of Δ consists of pairs $(x, k; x', -k')$, such that $(x, k; x', k')$ belongs to the same null geodesic strip (x and x' lie on a null geodesic and k, k' are the cotangent vectors at these points). On globally hyperbolic spacetimes one can split Δ as a sum

$$\frac{i}{2}\Delta = \Delta_+ - H\,, \tag{1}$$

in such a way that

(1) H is symmetric,
(2) Δ_+ is positive, i.e. $\langle \Delta_+, f \otimes \overline{f} \rangle > 0$,
(3) Δ_+ is a distributional bisolution for P and the WF set of Δ_+ is contained in the forward lightcone.

The last condition can be interpreted as the positivity of the energy condition and on Minkowski spacetime would correspond to the split into positive and negative frequency parts in the momentum space. The split (1) is not unique and different choices can be labeled by different symmetric distributions H. Let Had denote the collection of all such distributions. For each of them we can define the algebra $\mathfrak{A}^H(\mathcal{M}) \doteq (\mathcal{F}_{\mu c}[[\hbar]], \star_H)$, where

$$F \star_H G \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}(\varphi), \Delta_+^{\otimes n} G^{(n)}(\varphi) \right\rangle\,.$$

The algebra of Wick-ordered products of tree fields is defined as the family $\mathfrak{A}(\mathcal{M}) = (\mathfrak{A}^H(\mathcal{M}))_{H\in \mathrm{Had}}$, where elements are related by $F_{H'} = \alpha_{H'-H} F_H$, with

$$\alpha_H \doteq e^{\frac{\hbar}{2}\left\langle H, \frac{\delta^2}{\delta\varphi^2}\right\rangle}\,,$$

and the product is

$$(F \star G)_H \doteq F_H \star_H G_H\,.$$

Pick $H \in \mathrm{Had}$. For $F \in \mathcal{F}_{\mathrm{loc}}$ we can define an element of $\mathfrak{A}(\mathcal{M})$, denoted by $:F:$, defined by the requirement that $:F:_H = F$. The subspace of $\mathfrak{A}^H(\mathcal{M})$ generated (as a vector space) by elements arising this way is called $\mathfrak{A}^H_{\mathrm{loc}}(\mathcal{M})$.

3. Interaction

The method to introduce the interaction is motivated by the Dyson formula for the interacting time-evolution operator

$$U_I(t,s) = 1 + \sum_{n=1}^{\infty} \frac{i^n \lambda^n}{n!\hbar^n} \int_{([s,t]\times\mathbb{R}^3)^n} T(:L_I(x_1):\ldots:L_I(x_n):) d^4x_1 \ldots d^4x_n\,.$$

In order to make this formula mathematically rigorous we introduce the formal S-matrix as a formal map on $\mathfrak{A}^H_{\mathrm{loc}}(\mathcal{M})$ defined by

$$\mathcal{S}(\lambda :V:) \doteq e^{\frac{i\lambda}{\hbar} \mathcal{T} :V:} \equiv \sum_{n=0}^{\infty} \left(\frac{i\lambda}{\hbar}\right)^n \frac{1}{n!} \mathcal{T}_n(:V:^{\otimes n}) \,,$$

where $\mathcal{T}_n$ is called the n-fold time-ordered product and it is initially well defined only for arguments with non-coinciding supports and is then extended to arbitrary local functionals by renormalization. For non-coinciding supports we have the explicit formula

$$\mathcal{T}_n(F_1, \ldots, F_n) \doteq m \circ e^{\hbar \sum_{i<j} \left\langle \Delta^{\mathrm{F}}, \frac{\delta}{\delta\varphi_i \delta\varphi_j} \right\rangle} \,,$$

where m is the point-wise multiplication operator and $\Delta^{\mathrm{F}} \doteq \frac{i}{2}\left(\Delta^{\mathrm{A}} + \Delta^{\mathrm{R}}\right) + H$ is the Feynman propagator associated to H. The renormalization problem is now to extend the time-ordered products to arbitrary arguments. Following [EG73], we require these to satisfy the properties:

(1) **Causal factorization:** $\mathcal{T}_n(F_1, ..., F_n) = \mathcal{T}_n(F_1, ..., F_k) \star_H \mathcal{T}_n(F_{k+1}, ..., F_n)$, if the supports of $F_1, ..., F_k$ are not earlier than the supports of $F_{k+1}, ..., F_n$.
(2) $\mathcal{T}_0 = 1$, $\mathcal{T}_1 = \mathrm{id}$
(3) Unitarity, Covariance, etc.

For the complete list of axioms see [BDF09, HW01]. It was shown there, following the original proof of [EG73], that a family of maps $\mathcal{T}_n$ fulfilling the appropriate axioms exists and that the non-uniqueness is governed by the Stückelberg-Petermann renormalization group $\mathcal{R}$, defined as follows. Elements of $\mathcal{R}$ are formal diffeomorphisms Z of $\mathfrak{A}_{\mathrm{loc}}(\mathcal{M})$ such that

(1) $Z(0) = 0$,
(2) $Z^{(1)}(0) = \mathrm{id}$,
(3) $Z = \mathrm{id} + \mathcal{O}(\hbar)$,
(4) $Z(F + G + K) = Z(F + G) + Z(G + K) - Z(G)$, if $\operatorname{supp} F \cap \operatorname{supp} K = \emptyset$.

The main theorem of renormalization states that if $\mathcal{S}$ and $\mathcal{S}'$ are two possible definitions of S-matrices (with time-ordered products satisfying the appropriate axioms), then there exists an element $Z \in \mathcal{R}$ such that

$$\mathcal{S}' = \mathcal{S} \circ Z \,.$$

Also the converse is true, i.e. if $Z \in \mathcal{R}$ and $\mathcal{S}$ is a well defined S-matrix with the properties stated before, then so is $\mathcal{S} \circ Z$.

4. Gauge theories and effective quantum gravity

The framework discussed above can be generalized to treat gauge theories and effective quantum gravity [Hol08, FR12, BFR13]. In both cases, there is a local action of a Lie algebra $\mathfrak{g}$ on the configuration space $\mathcal{E}$ that leaves the space of solutions to the equations of motion invariant. Then one has to replace $\mathcal{E}$ with the graded manifold $\overline{\mathcal{E}} \doteq \mathcal{E} \oplus \mathfrak{g}[1]$ and $\mathcal{F}$ with $\mathcal{O}_{\mathrm{ml}}(\overline{\mathcal{E}})$, the space of multilocal functions on this manifold. Next, we consider the space of multilocal polivector

fields, i.e. the space of functions on $T^*[-1]\overline{\mathcal{E}}$, the odd cotangent bundle of $\overline{\mathcal{E}}$. We denote this space by $\mathcal{BV}$. This is the underlying algebra of the BV complex. It is a bi-graded algebra, where one of the gradings (call it #pg) comes from the underlying manifold $\overline{\mathcal{E}}$, and the other grading, the antifield number #af, is essentially the degree of polivector fields. The grading of the BV complex is given as $\#\mathrm{gh} = \#\mathrm{pg} - \#\mathrm{af}$.

The BV differential is then a sum of two differentials $s = \delta + \gamma$, where the homology of δ characterises the space of functionals on the solutions to the equations of motion and the cohomology of γ characterises the space of invariants under the action of $\mathfrak{g}$. The BV differential can be expanded in powers of fields. The linear $\#\mathrm{af} = 0$ term is denoted by s_0. The gauge invariance of the S-matrix is the condition that

$$s_0(\mathcal{S}(:V:)) = 0\,. \tag{2}$$

This condition is called *quantum master equation* (QME) and can be rewritten as

$$\frac{1}{2}\{S_0 + \lambda V, S_0 + \lambda V\} - i\hbar \triangle (\lambda V) = 0\,, \tag{3}$$

where $\{.,.\}$ is the Schouten bracket on the space of polivector fields, S_0 is the free action, V is the interaction term and $\triangle(\lambda V)$ is the renormalized BV Laplacian, identified with the anomaly term. Solving QME amounts to finding higher order (in $\hbar$) corrections to V such that (3) can be satisfied. This re-definition of the interaction term amounts to acting on V with some $Z \in \mathcal{R}$, so according to the main theorem of renormalization, this is equivalent to redefining the time-ordered products, so that the new S-matrix satisfies the condition (2).

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Constructive Matrix and Tensor Field Theory

Vincent Rivasseau

The tensor track is a program to explore (Euclidean) random fields which are tensors of general rank d. They include as special cases vector (rank 1) and matrix (rank 2) models. Tensor models were introduced as promising candidates for an *ab initio* quantization of gravity. Indeed they are combinatorial objects which do not refer to any background metric, nor even to any background topology. These tensors were initially introduced as symmetric in their indices, a feature which for a long time prevented to investigate rigorously their behavior. In particular in contrast with the famous 't Hooft 1/N expansion for random matrix models, there was until recently no way to probe the large N limit of these symmetric random tensors at rank $d > 2$.

The modern reformulation by R. Gurau and collaborators unlocked the theory by considering un-symmetrized random tensors. Slightly counterintuitively, these objects have in fact a larger symmetry than symmetric tensors, and this larger symmetry allowed to probe their large N limit through 1/N expansions of a new type. Invariants under this symmetry are exactly d-regular edge-colored graphs.

Random tensor models can be further divided into fully invariant models, in which both propagator and interaction are invariant, and tensor field theories (hereafter TFT) in which the interaction is invariant but the propagator is not. This propagator can also incorporate a further gauge invariance to make contact with group field theory, in which case we call the model a tensor group field theory (TGFT).

The first-half of the talk focused on the motivations for random tensors, which come from random geometry and quantum gravity. The tensor model action is known to be a natural discretization of the Einstein-Hilbert action and can be considered as an equilateral form of Regge calculus. The recent associated field theories have added renormalization, asymptotic freedom and exploration of the infrared flow to this older picture.

The second part of the talk focused on another important aspect of the new theories, namely their constructive analysis. In the relatively simple case of models with a quartic interaction, Borel summability of the free energy has been proven for super-renormalizable models on the "stable side" of the coupling constant. The main tool is the so-called multi-scale loop vertex expansion. Although the talk could not enter in that detail, the simpler loop vertex expansion was presented in some detail. It is a constructive technique which bypasses the more traditional cluster expansions and does not require any discretization of space-time by regular lattices, hence it is particularly suited to the study of matrix and tensor models, or of field theories in curved background.

The world of tensor models and tensor field theories is extremely vast, as tensor interactions encode infinitely many triangulations of any piecewise linear manifold with boundaries, and in particular distinguish in four dimensions not only topology, but also smooth structure. They also generalize non-commutative field theories

on Moyal space. Their exploration which has been very active during the last five years, is currently continuing to burgeon in many fascinating directions.

On Decoherence, Thermalization and RG

I. M. Sigal
(joint work with Marco Merkli and Gennady Berman)

Decoherence. In this talk, we present an attempt on a conceptual treatment of the theory of quantum decoherence for a class of general, non-solvable, microscopic models.

Earlier results: explicitly solvable (non-demolition) models and non-rigorous results for the 2nd order + Markov approximation (Lindblad evolution).

Some of the literature including experimental works: DiVincenzo, Palma, Suominen, Ekert, Leggett, Chakravarty, Dorsey, Fisher, Garg, Zwerger, Unruh, Zurek, Altepeter, Hadley, Wendelken, Berglund, Kwiat, Ao, Rammer, Berman, Kamenev, Tsifrinovich, Berman, G.P., Bishop, A.R., Borgonovi, F., Dalvit, Duan, Guo, Fedorov, Fedichkin, Paz, Roncaglia, Sorensen, Cirac, Zoller, Utsunomiya, Master, Yamamoto, Schlosshauer.

For a connection to the problem of irreversibility and to the theory of measurement, see recent papers by Jürg Fröhlich et al, which can be found on arXiv.

Model A quantum system interacting with the environment ("reservoir"). The *reservoir* is taken to be an infinitely extended gas of massless excitations (e.g. photons or phonons) at positive temperature and positive density, i.e. in the thermodynamic limit.

This is a classical model which can be found in the book of A. Messiah on Quantum Mechanics or in books on the solid state physics. Relevant references for us are Caldeira - Leggett and Feynman-Vernon. The former used this model to describe the macroscopic tunnelling in superconductors (Josephson junctions). Model. Consider a quantum system interacting with the "reservoir", with the *state space* $\mathcal{H} = \mathcal{H}_{\rm S} \otimes \mathcal{H}_{\rm R}$ and the *Hamiltonian*

$$H = H_{\rm S} \otimes \mathbb{1}_{\rm R} + \mathbb{1}_{\rm S} \otimes H_{\rm R} + \lambda V,$$

where $H_{\rm S}$ and $H_{\rm R}$ are the Hamiltonians of S and R and λV is an *interaction* between them, with a coupling constant $\lambda \in \mathbb{R}$.

In our model, $\mathcal{H}_{\rm R}$ is the *bosonic Fock space*, $\mathcal{F}$, with the *creation/annihilation operators*, $a^*(k)$ and $a(k)$, $H_{\rm R}$ is given by

$$H_{\rm R} = \int_{\mathrm{R}^3} a^*(k)|k|a(k)\mathrm{d}^3k$$

and

$$V = G \otimes \varphi(g).$$

where G is a sa operator on $\mathcal{H}_{\rm S}$, $g^2 \int \left(1+|k|^{-1}\right) \|G_j(k)\|^2 \, dk \ll 1$ and

$$\varphi(g) = \int (a^*(k)g(k) + a(k)\overline{g(k)})d^3k.$$

Here $a^*(k)$ and $a(k)$ are bosonic creation/annihilation operators on $\mathcal{H}_{\rm R}$.
Reduced Density Matrix. The state, R_t, of the *total system* at time t (we think of it as a density matrix though we deal with positive densities) satisfies the von Neumann equation

$$i\partial_t R_t = [H, R_t].$$

The *reduced density matrix* (of the system S) at time t is then formally obtained by tracing out the reservoir degrees of freedom

$$\rho_t = {\rm Tr}_{\rm R}\, R_t.$$

Thermalization and Decoherence. From now on we assume dim $\mathcal{H}_{\rm S} < \infty$.

Jaksic - Pillet, Bach - Fröhlich - IMS: If initially the system is close to a state in which the reservoir is near equilibrium,

$$\rho_t \longrightarrow \rho(\beta,\lambda), \text{ with } \rho(\beta,\lambda) := {\rm Tr}_{\rm R} R(\beta,\lambda),$$

where $R(\beta,\lambda)$ is the *equilibrium state (Gibbs state)* of the *total system* at temperature $T = 1/\beta$. In particular,

$$[\rho_t]_{m,n} \to [\rho(\beta,\lambda)]_{m,n}, \text{ as } t \to \infty. \tag{1}$$

For the *energy basis*, and for $m = n$, this is the *thermalization*, and for $m \neq n$, the *decoherence*.
Thermalization and Decoherence. Generally, $[\rho(\beta,\lambda)]_{m,n} = O(\lambda) \neq 0$, for $m \neq n$. Neglecting it, we arrive at the standard notion of decoherence:

$$[\rho_t]_{m,n} \to 0, \text{ as } t \to \infty, \ \forall m \neq n,$$

in a 'prefered' basis (usually, the energy basis, also called the *computational basis* for a quantum register, i.e. the basis of eigenvectors of the system Hamiltonian $H_{\rm S}$).

In this case, the limiting density matrix is a non-coherent mixture of the pure states, $\sum_n p_n P_n$. where P_n is the orthogonal projection onto the $n-$th eigenfunction of $H_{\rm S}$.
Observables and Decoherence. Let $\{\varphi_j\}_{j\geq 1}$ be an orthonormal basis of $\mathcal{H}_{\rm S}$ diagonalizing $H_{\rm S}$. The matrix elements $[\rho_t]_{m,n} := \langle \varphi_m, \rho_t \varphi_n\rangle$ of the reduced density matrix ρ_t can be written as

$$[\rho_t]_{m,n} = \langle P_{n,m}\rangle_t, \qquad \text{where } \ P_{n,m} = |\varphi_n\rangle\langle\varphi_m|,$$

and $\langle A\rangle_t$ is the average of a *system observable* A,

$$\langle A\rangle_t := {\rm Tr}_{\rm S+R}(R_t(A\otimes \mathbb{1}_{\rm R})) = {\rm Tr}_{\rm S}(\rho_t A).$$

Thus to understand the decoherence and thermalization it suffices to understand behavior of the expectations $\langle A\rangle_t$.

Relaxation of Observables. The above result can be reformulated as: for any system observable A,

$$\langle A\rangle_t \to \langle A\rangle_{\text{thermal}}\,, \quad \text{as} \quad t\to\infty,$$

where

$$\langle A\rangle_{\text{thermal}} = \text{Tr}_{\text{S}}(\rho(\beta,\lambda)A),$$

with

$$\rho(\beta,\lambda) := \text{Tr}_{\text{R}} R(\beta,\lambda).$$

(Recall that $R(\beta,\lambda)$ is the equilibrium state (Gibbs state) of the total system at temperature $T = 1/\beta$.)

Convergence of Observables. Our first main result is the following

Thm 1. $\exists$ *complex energies* ε, lying in the strip $\{z \in \mathbb{C} \mid 0 \le \text{Im}\, z < \frac{\pi}{\beta}\}$, s.t. for any $t \ge 0$

$$\langle A\rangle_t - \langle A\rangle_{\text{thermal}} = \sum_{\varepsilon\neq 0} \text{e}^{\text{i}t\varepsilon} R_\varepsilon(A) + O\left(\lambda^2\, \text{e}^{-\alpha t}\right), \tag{2}$$

where $R_\varepsilon(A)$ are linear functionals of A and (for a special class of couplings)

$$\alpha < \frac{1}{2}\max_\varepsilon\{\text{Im}\,\varepsilon\} + \frac{2\pi}{\beta}.$$

$\text{Im}\,\varepsilon \equiv \text{Im}\,\varepsilon^{(s)}_{m,n} > 0$ encode properties of the *irreversibility* of the reduced dynamics of S (decay of expectations).

Resonances. Now, we formulate our second result:

Thm 2. The complex numbers ε are the *resonances* of a certain explicitly given operator L and 'bifurcate' from the eigenvalues

$$e_{m,n} = e_m - e_n \in \text{spec}(L_S) = \text{spec}(H_{\text{S}}) - \text{spec}(H_{\text{S}})$$

of the system's *Liouville operator* $L_S : A\to[H_{\text{S}}, A]$:

$$\varepsilon \equiv \varepsilon^{(s)}_{m,n} = e_{m,n} + \lambda^2\delta^{(s)}_{m,n} + O(\lambda^4),$$

where $\delta^{(s)}_{m,n}$ are the eigenvalues of a matrix $\Lambda_{m,n}$, called a *level-shift operator*, acting on the eigenspace of L_S corresponding to $e_{m,n}$.

* * * * * * * * *

We define the *thermalization* and *decoherence times* as

$$[\tau_T]^{-1} := \min_m \text{Im}\,\varepsilon_{m,m}(\lambda), \qquad [\tau_D]^{-1} := \min_{m\neq n} \text{Im}\,\varepsilon_{m,n}(\lambda).$$

Two-dimensional systems (qubits). Consider a two-dimensional system (qubit), with state space (of pure states) $\mathcal{H}_{\rm S} = \mathbb{C}^2$, and the system and interaction Hamiltonians are given by

$$H_{\rm S} = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \quad V = \begin{bmatrix} a & c \\ \overline{c} & b \end{bmatrix} \otimes \varphi(g),$$

where $\varphi(g) = \int \varphi(x) g(x)$ is the Bose field operator.

$c = 0$ corresponds to a *non-demolition (energy conserving)* interaction (v *commutes* with the Hamiltonian $H_{\rm S}$ and consequently energy-exchange processes are suppressed).

The property $c \neq 0$ is necessary for *thermalization*.
Decoherence of qubits. Let $\Delta = E_2 - E_1 > 0$ be the energy gap of the qubit. Then

$$[\tau_T]^{-1} = \lambda^2 \pi^2 |c|^2 \xi(\Delta),$$

$$[\tau_D]^{-1} = \frac{1}{2} \lambda^2 \pi^2 \left[|c|^2 \xi(\Delta) + (b-a)^2 \xi(0) \right],$$

modulo $O(\lambda^4)$ and with ($g(k)$ = coupling function)

$$\xi(\eta) = 4 \coth\left(\frac{\beta \eta}{2} \right) |g(\eta)|^2 \eta^2.$$

Exactly solvable models (non-demolition interactions, i.e. $c = 0$): *Palma, Suominen, Ekert, Shao, Ge, Cheng, Mozyrsky, Privman*, and others.
Approach.

- Reformulate the problem in terms of the dynamics

  $$R_t \Longleftrightarrow \Psi_t = e^{iLt} \Psi,$$

  generated by a Liouville operator L, on the *Hilbert space*

  $$\mathcal{H} \otimes \mathcal{H};$$

- Connect the decoherence to the *resonances* of L;
- To compute the resonance energies and life-times use the spectral renormalization group of Bach-Fröhlich-IMS.

The resonances of L are defined as the complex eigenvalues of a non-self-adjoint deformation, $L_\theta, \theta \in \mathbb{C}^+$, of L:

the *resonances* of L = the *eigenvalues* of L_θ.

These complex eigenvalues are independend of the deformation parameter θ, as well as on the method of deformation (the 'vector field').

Critical exponents for long-range $O(n)$ models

Gordon Slade

This talk reports on work which will appear in [11]. It concerns the critical behaviour of $O(n)$ models ($n \geq 0$) on $\mathbb{Z}^d$, with long-range coupling that decays with distance r as $r^{-(d+\alpha)}$, with $\alpha \in (0,2)$.

For $n \geq 1$, we study a long-range n-component $|\varphi|^4$ lattice spin model. This is defined as the infinite-volume limit of a classical continuous unbounded n-component spin field on a discrete d-dimensional torus Λ, as follows. Let $(-\Delta_\Lambda)^{\alpha/2}$ be the fractional Laplacian on Λ. This is a periodic version of the discrete fractional Laplacian on $\mathbb{Z}^d$, whose matrix elements decay as $-(-\Delta_{\mathbb{Z}^d})^{\alpha/2}_{0x} \asymp |x|^{-(d+\alpha)}$ (for large $|x|$). Given $g > 0$ and $\nu \in \mathbb{R}$, we define a function $V : (\mathbb{R}^n)^\Lambda \to \mathbb{R}$ by

$$V(\varphi) = \sum_{x \in \Lambda} \big(\tfrac{1}{4} g |\varphi_x|^4 + \tfrac{1}{2} \nu |\varphi_x|^2 + \tfrac{1}{2} \varphi_x \cdot ((-\Delta_\Lambda)^{\alpha/2} \varphi)_x \big). \tag{1}$$

We are interested in $\nu < 0$, and in particular in the limit $\nu \downarrow \nu_c$, where $\nu_c = \nu_c(g;n)$ is the *critical value* for the appearance of long-range order (divergence of the susceptibility) in the infinite-volume limit $\Lambda \uparrow \mathbb{Z}^d$. The *partition function* is defined by

$$Z_{g,\nu,\Lambda} = \int_{(\mathbb{R}^n)^\Lambda} e^{-V(\varphi)} d\varphi, \tag{2}$$

where $d\varphi$ is the Lebesgue measure on $(\mathbb{R}^n)^\Lambda$. The expectation of a random variable $F : (\mathbb{R}^n)^\Lambda \to \mathbb{R}$ is

$$\langle F \rangle_{g,\nu,\Lambda} = \frac{1}{Z_{g,\nu,\Lambda}} \int_{(\mathbb{R}^n)^\Lambda} F(\varphi) e^{-V(\varphi)} d\varphi. \tag{3}$$

For $n = 0$, we study the weakly self-avoiding walk via an exact representation as a supersymmetric version of the $|\varphi|^4$ model [5]. These models have upper critical dimension $d_c = 2\alpha$ for all $n \geq 0$.

For $d = 1, 2, 3$ and small $\epsilon > 0$, we choose $\alpha = \frac{1}{2}(d + \epsilon)$, so $d = d_c - \epsilon$ is below the upper critical dimension. For small ϵ and small g, we prove existence of and compute the values of the critical exponent γ for the susceptibility ($n \geq 0$) and the critical exponent α_H for the specific heat ($n \geq 1$), to order ϵ. For the susceptibility, $\gamma = 1 + \frac{n+2}{n+8} \frac{\epsilon}{\alpha} + O(\epsilon^2)$, and a similar result is proved for the specific heat. Expansion in ϵ for such long-range models was first carried out in the physics literature in 1972, by Fisher, Ma and Nickel [8].

Our proof adapts and applies a rigorous renormalisation group method developed in previous papers with Bauerschmidt and Brydges for the nearest-neighbour models in the critical dimension $d = 4$. The large-field problem is solved via an application of the main result of [7], and parts of the analysis of [3, 4] are adapted and applied. We use the finite-range decomposition of the fractional Laplacian introduced in [9], and our treatment of the stable manifold theorem is inspired by ideas in [6, 10] (see also [1, 2] for related work).

The proof is based on the construction of a non-Gaussian renormalisation group fixed point. Several aspects of the method used for the nearest-neighbour model in dimension $d = 4$ simplify significantly below the upper critical dimension, and new ideas with potential future application are introduced.

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Random Currents and Fermionic Correlation Functions in Planar Ising models

Simone Warzel

(joint work with Michael Aizenman and Hugo Duminil-Copin)

Our purpose is to present a new perspective on the Pfaffian nature of correlation functions for systems of Ising spins on general planar graphs $G = (V(G), E(G))$, i.e.

$$H_G(\sigma) := -\sum_{(x,y)\in E(G)} J_{x,y}\sigma_x\sigma_y$$

with couplings $(J_{x,y})_{(x,y)\in E(G)}$ of arbitrary sign and strength between neighboring spins $\sigma : V(G) \mapsto \{-1, 1\}$. The corresponding Gibbs equilibrium state at inverse temperature $\beta \geq 0$ is the probability measure such that

$$\langle f\rangle_{G,\beta} := \frac{\sum_{\sigma\in\{\pm1\}^{V(G)}} f(\sigma)e^{-\beta H(\sigma)}}{Z(G,\beta)}$$

for any $f : \{-1,1\}^{V(G)} \longrightarrow \mathbb{C}$. Above, $Z(G,\beta) := \sum_{\sigma\in\{\pm1\}^{V(G)}} e^{-\beta H_G(\sigma)}$ is the partition function. For any such system, at any temperature, the correlation functions for spins located along the boundaries of a finite domain are given by Pfaffians (Pf) of the corresponding boundary-to-boundary two point function, i.e. satisfy a *fermionic* version of Wick's rule:

Theorem 1 ([3])**.** *Fix a planar graph G, arbitrary couplings J, and $\beta \geq 0$. Then, for any cyclicly ordered $2n$-tuple $(x_1,\ldots,x_{2n})$ of sites on the boundary of a fixed face of G, we have*

$$\langle \sigma_{x_1}\cdots\sigma_{x_{2n}}\rangle_{G,\beta} \;=\; \mathrm{Pf}\Big(\big[\langle\sigma_{x_i}\sigma_{x_j}\rangle_{G,\beta}\big]_{1\leq i<j\leq 2n}\Big)\,. \tag{1}$$

The observation that fermionic structures emerge within the classical 2D Ising model was made soon after Onsager's exact solution [18]. The point was initially made by Kaufman [16] and was further elaborated by Schultz-Mattis-Lieb [20] and Kadanoff [13, 14] through analysis involving the transfer matrix, and many works derived since then (cf. [17, 4] and references therein). Hints of this fermionic structure can also be seen in the successful reductions of the 2D Ising model's partition function to Pfaffians, by Hurst and Green [12], Kasteleyn [15] and Fisher [9] (see also [5] and references therein.) Unlike some previous results of this kind, our method of proof does not require the graph to have the shift invariance which is typically assumed in transfer matrix calculations.

Next it is shown in [3] that the fermionic structure of boundary spin functions (1) can be extended to correlation functions of operators suitably defined throughout the bulk. For that, the spins are paired with disorder variables. The resulting order-disorder operators correspond to Kaufman spinors, whose properties were further discussed in [13]. These order-disorder operators were later used to prove the conformal invariance of the model [7].

The fermionic relations are derived in [3] using the random current representation of Ising models [11, 1] and topological arguments involving planarity. Curiously, the derivation of the fermionic rules for planar model employs identities which have previously been used to prove that in high dimensions the scaling limits of critical correlations of ferromagnetic models obey the bosonic Wick rule [1].

The results apply to planar models at all temperatures. However, at that level of generality they do not extend to non-planar interactions, e.g. two dimensional models with pair interactions reaching beyond the range of nearest neighbors. For such extensions, we present a conjecture that planar-type Pfaffian relations emerge at and above the critical points. The conjecture is motivated, within the correlations functions' random current representation, by the generally credible picture of fractality of the models (in the spirit of the extension of the RSW theory to Ising model [8, 6], and the fractality criteria of [2]). Since energy correlation functions are a special case of order-disorder correlations, this approach might also offer an alternative to existing proofs of the emergent fermionic structure of energy correlations of certain non-planar models at criticality [10, 19, 21].

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