Mathematisches Forschungsinstitut Oberwolfach

Report No. 25/2016

DOI: 10.4171/OWR/2016/25

Factorization Algebras and Functorial Field Theories

Organised by
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8 May – 14 May 2016

Abstract. Factorization algebras are a new mathematical approach to quantum field theory. They are related to functorial field theories, another approach to quantum field theory. Factorization algebras also figure in current research in manifold topology, homotopy theory and algebraic geometry. The workshop brought together researchers from many different fields to understand and deepen these connections.

Mathematics Subject Classification (2010): 14xx, 55xx, 57xx, 81xx, 53xx, 52xx.

Introduction by the Organisers

The workshop *Factorization Algebras and Functorial Field Theories*, organized by Owen Gwilliam (Bonn), Stephan Stolz (Notre Dame), Peter Teichner (Bonn), and Mahmoud Zeinalian (New York) had 50 diverse international participants.

In recent years, the interplay between topology and theoretical physics — in particular quantum field theory — has played a significant role in the work of many researchers. This workshop brought together people from several fields so that they could exchange their results and perspectives.

In the setting of physics, a d-dimensional quantum field theory (QFT) is typically obtained by quantization of a classical field theory. The data of a classical field theory is a space of fields $\mathcal{F}$, usually the sections of a bundle over a "space-time" manifold Σ, and an action functional S on $\mathcal{F}$. Only the extrema of the action S are physically allowed states, and they are solutions to the Euler-Lagrange equations determined by S. By contrast, in the quantum theory, all fields are relevant and the physically important information is encoded by correlation functions, which are obtained by integrating functions on $\mathcal{F}$ (or some suitable subspace) against a probability-type measure governed by S. Unfortunately, this functional integral rarely admits a mathematically rigorous formulation in existing mathematics.

For example, for each simple Lie group G, there is a Chern-Simons theory defined on oriented 3-dimensional manifolds whose space of fields $\mathcal{F}$ is the stack of principal G-bundles with connections. The classical theory studies the stack of flat G-bundles. As another example, given two Riemannian manifolds Σ and X, there is a non-linear σ-model whose fields consist of the space of smooth maps from Σ to X. The classical theory studies the space of harmonic maps. In the last few decades, physicists have developed an array of recipes to calculate the functional integrals that encode the quantization, but the challenge remains to give mathematically rigorous constructions.

In the late 1980s, Atiyah, Segal, and others developed a novel axiomatic approach to QFT, as well as to conformal field theories and topological field theories, that suggested profound connections to topology and geometry. (Historically, mathematics focused on QFT tended to hew closely to analysis and representation theory.) The quantum Chern- Simons theories, often known as Reshetikhin-Turaev-Witten theories, provide a rich class of examples. However, many other field theories arising in physics are difficult to write down in this axiomatic framework. Recently, a relation to factorization algebras (see below) seems to indicate that various physically relevant field theories can be expressed in terms of Atiyah-Segal axioms.

Nonetheless, these axioms have led to interesting work in mathematics, particularly in connection with algebraic topology and higher categories. Originally, a functorial field theory consisted of symmetric monoidal functors from a (geometric) bordism category to a symmetric monoidal category of a linear nature, such as vector spaces with tensor product. Since then, various refinements and structures have been added to make such functors better reflect the formal behavior of a field theory from physics. A high point of research in the last decade was a classification of topological field theories that are fully local — i.e. the bordism categories specify 1-dimensional bordisms between points, 2-dimensional bordisms between the 1-dimensional bordisms, and so on — by Jacob Lurie. (His proof of the Baez-Dolan Cobordism Hypothesis applies to all dimensions. Lurie did the 2-dimensional case with Hopkins, but it was also proved independently by Schommer-Pries.) This result has spurred a lot of recent activity in higher categories and their connections with established topics like quantum groups. Another active direction of research explores a suggestion of Segal that functorial field theories should provide geometric cocycles for certain generalized cohomology theories, notably elliptic cohomology theories. The development and extension of this idea by Stolz,Teichner, and collaborators — particularly in the setting of super-Euclidean field theories — pushes beyond topological field theory (which is the context for the Cobordism Hypothesis).

Recently, there has appeared a new approach to organizing mathematically the data of a quantum field theory, via the notion of factorization algebras. The idea originated in work on conformal field theory by Beilinson and Drinfeld, who sought to find a structure living on an algebraic curve whose local behavior is given by a vertex algebra. Francis, Gaitsgory, and Lurie recognized that an analogous structure appears in manifold topology, where the local structure on an n-dimensional manifold is given by an E_n algebra, i.e. an algebra over the little n-disks operad. They also explicated a relationship with functorial topological field theory. Costello and Gwilliam then formulated a version of factorization algebras well-suited to smooth manifolds and general QFTs. Indeed, they showed that a rigorous version of quantization—using renormalization and Feynman diagrams—naturally produces a factorization algebra of observables living on the space-time. Many important examples of theories have been quantized in this formalism, including topological field theories such as Chern-Simons theory, the B-model, and Rozansky-Witten theory, and non-topological field theories such as Yang-Mills theory and the curved $\beta\gamma$ system. (Since these techniques are a formalization of standard tools in physics, any QFT treated by diagrammatics should produce a factorization algebra of observables.) In the topological cases, there is then a functorial field theory determined by the factorization algebra, by work of Scheimbauer, and hence a direct connection between the action functional description of a field theory and an Atiyah-Segal description.

Our workshop contained three connected lecture series, aimed at explaining the key ideas and techniques involved in the formalism of Costello and Gwilliam. The other lectures covered a broad range of issues related to mathematical approaches to field theory, from other approaches to QFT, to appearances of factorization algebras in homotopy theory and algebraic geometry and convex geometry, to the treatment and application of defects in field theory.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, “US Junior Oberwolfach Fellows”.

Workshop: Factorization Algebras and Functorial Field Theories

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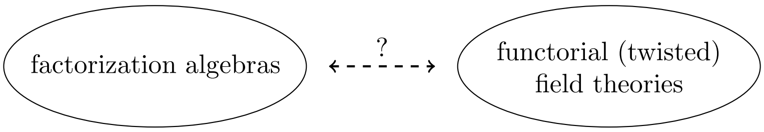
Abstracts

Factorization algebras and (twisted) functorial field theories – the topological case

Claudia Scheimbauer

(joint work with Damien Calaque, Theo Johnson-Freyd)

In recent years, quantum field theories have been studied by mathematicians using, among others, two approaches: *functorial field theories* and their variations following [1, 13, 15] axiomatizing the state space and the partition function; and the more recent approach via *factorization algebras* (cf. [4]) axiomatizing the structure of the observables of perturbative quantum field theories. In this talk, after providing an introduction to both, we will explain how to relate these two concepts in the case of topological field theories, see also [5, 6, 12].



Unravelling the axiomatization by Atiyah, n-dimensional topological field theories (nTFTs) are symmetric monoidal functors out of a suitable category of spacetimes, called *bordisms*, which are n-dimensional topological manifolds, perhaps required to be smooth, or equipped with some tangential structure such as an orientation or a framing. In general, functorial field theories allow for spacetimes endowed with more general types of geometries, such as a conformal or Euclidean structure, encoded as a sheaf $\mathcal{G}$ on the site of n-dimensional manifolds valued in a suitable target category, usually taken to be sets or, to allow for homotopical versions, spaces. Passing to higher categories of cobordisms as defined in joint work with Calaque in [5] leads to (fully) extended field theories which in the topological case describe locality of the field theory: the Cobordism Hypothesis [10] shows that fully extended nTFTs are fully determined by their value at a point.

However, important examples of quantum field theories may not fit into this framework, as can already be seen in Segal's weakly conformal field theories of [13]. This leads to the generalization in form of *twisted field theories* as defined by Stolz-Teichner in [14], which implement the idea that the partition function may not be a number, but rather an element in some line or vector space. A twisted n-dimensional field theory is an (op)lax natural transformation from the trivial field theory (sending everything to the monoidal unit) to an n-dimensional field theory called the *twist*. In the fully extended case, a definition requires a notion of (op)lax natural transformations in the setting of higher categories, as was given in joint work with Johnson-Freyd in [8].

The target of an (extended) nTFT should be taken to be a (delooping of) the category of (dg) vector spaces (n)Vect. An example in the once-extended case is the bicategory of Bimod of algebras, bimodules, and intertwiners. Examples of deloopings of BIMOD as higher categories have been constructed in joint work with Calaque [6] and Johnson-Freyd [8].

Factorization algebras should be thought of as multiplicative versions of cosheaves. As such they encode the structure of the observables of perturbative quantum field theories, as we'll see in the lecture series by Ryan Grady, Si Li, and Brian Williams.

Definition. *Let $\mathcal{G}$ be a geometry on n-dimensional manifolds. A $\mathcal{G}$-*factorization algebra *is a symmetric monoidal functor*

$$\mathcal{F} : \mathrm{MFLD}^{\mathcal{G},\amalg} \longrightarrow \mathcal{S}^{\otimes}$$

satisfying descent for Weiss covers. Here, $\mathrm{MFLD}^{\mathcal{G},\amalg}$ is a category of $\mathcal{G}$-manifolds and $\mathcal{G}$-isometric embeddings.

If the target $\mathcal{S}$ naturally is a homotopical category, i.e. a symmetric monoidal $(\infty,1)$-category, this definition should be modified to this setting. When considering a fixed n-dimensional manifold M, a factorization algebra on M does not see any geometry:

Definition. *A* factorization algebra $\mathcal{F}$ on M *is an algebra over the colored operad with open sets in M as colors and*

$$\mathcal{P}reFact_M(U_1,\dots,U_n;V) = \begin{cases} \{*\} & \text{if } U_1 \amalg \dots \amalg U_n \subseteq V; \\ \emptyset & \text{otherwise,} \end{cases}$$

satisfying multiplicativity, i.e. $\mathcal{F}(U)\otimes\mathcal{F}(V) \xrightarrow{\simeq} \mathcal{F}(U\amalg V)$, and descent for Weiss covers.

We will see several examples and variations appearing throughout the talks this week: conformal nets (Henriques), structures appearing in algebraic quantum field theory (Rejzner), algebro-geometric versions (Cliff), and several topological examples (Kapranov, Knudsen). Topological factorization algebras are obtained from *factorization homology* [9, 2, 3, 7, 11]: the are defined locally and "glued together" using the tangential structure of a manifold. The local data needed for this procedure in the framed case is that of an *E_n-algebra in $\mathcal{S}$*.

Factorization homology is the key ingredient in relating topological factorization algebras and functorial topological field theories. The target of the latter will be a symmetric monoidal Morita-$(\infty, n+1)$-category $\mathrm{ALG}_n^{ptd}(\mathcal{S})$. Its objects are E_n-algebras, morphisms from A to B are E_{n-1}-algebras which are (A,B)-bimodules, 2-morphisms are bimodules of bimodules, ... and n-morphisms are *pointed* bimodules of ... of bimodules. Its $(n+1)$-morphisms are intertwiners, but the non-invertible ones cannot be seen by the n-dimensional theory. This $(\infty, n+1)$-category can be built using factorization algebras on $\mathbb{R}^n$ which satisfy certain constructibility conditions to encode the objects and k-morphisms for $1 \leq k \leq n$. We restrict to explaining the framed case. The main theorem of my thesis [12], of which I will outline the proof in the talk, is the following:

Theorem. *(Calaque-S. [6]) Let $\mathcal{S}$ be a symmetric monoidal $(\infty,1)$-category which is $\otimes$-sifted-cocomplete. Given any object in $\mathrm{Alg}_n^{ptd}(\mathcal{S})$, i.e. an E_n-algebra A in $\mathcal{S}$, the assignment sending a point to A extends to a fully extended framed n-dimensional topological field theory*

$$\mathcal{FH}_n(A) : \mathrm{Bord}_n^{fr} \longrightarrow \mathrm{Alg}_n^{ptd}(\mathcal{S}).$$

Moreover, any fully extended framed nTFT with target $\mathrm{Alg}_n^{ptd}(\mathcal{S})$ arises this way.

The target is not a delooping of Vect since its bimodules are pointed. However, forgetting the pointings yields a forgetful functor of $(\infty, n+1)$-categories $\mathcal{U}$: $\mathrm{Alg}_n^{ptd}(\mathcal{S}) \to \mathrm{Alg}_n(\mathcal{S})$ and a twisted framed nTFT with twist $T = \mathcal{U} \circ \mathcal{FH}_n(A)$,

$$\begin{tikzcd} \mathrm{Bord}_n^{fr} \arrow[r, bend left, "1"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{Alg}_n(\mathcal{S}). \end{tikzcd}$$

Corollary. *Let $\mathcal{S}$ be a symmetric monoidal $(\infty,1)$-category which is $\otimes$-sifted-cocomplete. Equivalent data are*

(1) *topological factorization algebras with target $\mathcal{S}$*
(2) *twisted framed nTFTs with target $\mathrm{Alg}_n(\mathcal{S})$.*

Remark. *The implication (1) $\Longrightarrow$ (2) in the corollary is the fully extended topological case of a theorem by Dwyer-Stolz-Teichner, which shows that $\mathcal{G}$-factorization algebras valued in the category of chain complexes $\mathcal{S} = \mathrm{Ch}$ lead to twisted $\mathcal{G}$-field theories with target the bicategory of dg categories, bimodule categories, and intertwiners. We expect this construction to fully extend using similar methods.*

References

[1] Michael Atiyah. Topological quantum field theories. *Inst. Hautes Études Sci. Publ. Math.*, (68):175–186 (1989), 1988.
[2] David Ayala and John Francis. Factorization homology of topological manifolds. *J. Topol.*, 8(4):1045–1084, 2015.
[3] David Ayala, John Francis, and Hiro L. Tanaka. Factorization homology of stratified spaces. arXiv:1409.0848
[4] Kevin Costello and Owen Gwilliam. Factorization algebras in perturbative quantum field theory.
[5] Damien Calaque and Claudia Scheimbauer. A note on the (∞,n)-category of cobordisms. 2015. arXiv:1509.08906, based on the second author's PhD thesis.
[6] Damien Calaque and Claudia Scheimbauer. Factorization homology as a fully extended topological field theory. 2016. In preparation, based on the second author's PhD thesis.
[7] Grégory Ginot, Thomas Tradler, and Mahmoud Zeinalian. Higher Hochschild homology, topological chiral homology and factorization algebras. *Comm. Math. Phys.*, 326(3):635–686, 2014.
[8] T. Johnson-Freyd and C. Scheimbauer. (Op)lax natural transformations, relative field theories, and the "even higher" Morita category of E_d-algebras. 2015. arXiv:1502.06526.
[9] Jacob Lurie. Higher algebra.
[10] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009.
[11] Scott Morrison and Kevin Walker. Blob homology. *Geom. Topol.*, 16(3):1481–1607, 2012.

[12] Claudia Scheimbauer. *Factorization Homology as a Fully Extended Topological Field Theory.* PhD thesis, ETH Zürich, 2014.

[13] Graeme Segal. The definition of conformal field theory. In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 421–577. Cambridge Univ. Press, Cambridge, 2004.

[14] Stephan Stolz and Peter Teichner. Supersymmetric field theories and generalized cohomology. In *Mathematical foundations of quantum field theory and perturbative string theory*, volume 83 of *Proc. Sympos. Pure Math.*, pages 279–340. Amer. Math. Soc., Providence, RI, 2011.

[15] Edward Witten. Supersymmetry and Morse theory. *J. Differential Geom.*, 17(4):661–692 (1983), 1982.

Conformal nets are factorization algebras

André Henriques

In this note, we prove that conformal nets of finite index [1, Def. 1.1 & Def. 3.1] form an instance of the notion of a factorization algebra.

Let Man^n be the category whose objects are n-dimensional manifolds and whose morphisms are embeddings. We equip it with the symmetric monoidal structure given by disjoint union. An open cover $\{U_i \subset M\}$ of a manifold M is a Weiss cover if for every finite subset $S \subset M$, there exists an index i such that $S \subset U_i$ [3, Chapt. 6]. Let $\mathcal{C}$ be a symmetric monoidal category.

Definition ([3, Chapt. 6])**.** *A $\mathcal{C}$-valued factorization algebra is a symmetric monoidal functor $F : \mathsf{Man}^n \to \mathcal{C}$ which is a co-sheaf with respect to Weiss covers.*

Here, being a co-sheaf means that for every Weiss cover $\{U_i \subset M\}$, the natural map

$$\operatorname{colim}\left(\begin{array}{ccc} F(U_1\cap U_2) & & F(U_1) \\ F(U_1\cap U_3) & & F(U_2) \\ F(U_2\cap U_3) & & F(U_3) \\ F(U_1\cap U_4) & & F(U_4) \\ F(U_2\cap U_4) & & \vdots \\ \vdots & & \end{array}\right) \longrightarrow F(M)$$

is an isomorphism. For later notational convenience, we abbreviate the left hand side as $\operatorname{colim}(\{F(U_i \cap U_j)\} \rightrightarrows \{F(U_i)\})$.

A *multi-interval* is an oriented 1-manifold which is diffeomorphic to a finite disjoint union of copies of $[0, 1]$. Let INT be the category whose objects are multi-intervals, and whose morphisms are orientation preserving embeddings. By the split property ([1, Def. 1.1]), a conformal net can be viewed as a symmetric monoidal functor $\mathcal{A} : \mathsf{INT} \to \mathsf{VN}$, where VN is the category of von Neumann algebras equipped with the symmetric monoidal stucture given by spatial tensor product.

We introduce the following slight modification of the notion of a Weiss cover:

Definition. *Let X be a topological space. A family of closed subsets $\{V_i \subset X\}$ is a Weiss c-cover if the interiors of the V_i form a Weiss cover of X.*

The following is what we mean by "conformal nets are factorization algebras":

Theorem 1. *Let $\mathcal{A} : \mathsf{INT} \to \mathsf{VN}$ be a conformal net of finite μ-index. Then $\mathcal{A}$ is a co-sheaf with respect to Weiss c-covers.*

Proof. Let $\{I_i \subset I\}$ be a Weiss c-cover. We first note that, by the strong additivity property of conformal nets (see [1, Def. 1.1]), the map

$$q : \operatorname{colim}\big(\{\mathcal{A}(I_i \cap I_j)\} \rightrightarrows \{\mathcal{A}(I_i)\}\big) \;\to\; \mathcal{A}(I)$$

has dense image. It is therefore surjective, as any morphism of von Neumann algebras whose image is dense is automatically surjective. To show that q is injective, pick a faithful representation

$$\pi : \operatorname{colim}\big(\{\mathcal{A}(I_i \cap I_j)\} \rightrightarrows \{\mathcal{A}(I_i)\}\big) \to \mathbf{B}(H)$$

and let $\rho_i := \pi|_{\mathcal{A}(I_i)}$. By Lemma 1, this extends to an action $\rho : \mathcal{A}(I) \to \mathbf{B}(H)$. As π is injective and $\pi = \rho \circ q$, the map q is also injective. $\square$

Lemma 1. *Let H be a Hilbert space equipped with actions $\rho_i : \mathcal{A}(I_i) \to \mathbf{B}(H)$ satisfying $\rho_i|_{\mathcal{A}(I_i \cap I_j)} = \rho_j|_{\mathcal{A}(I_i \cap I_j)}$. Then those maps extend to an action of $\mathcal{A}(I)$.*

Proof. We only treat here the case when I is connected, as it contains the most important idea. The proof of [1, Lem. 1.9] can be adapted word-for-word to show that for every interval $J \subsetneq I$, the actions of $\mathcal{A}(I_i \cap J)$ extend (uniquely) to an action of $\mathcal{A}(J)$. We may therefore assume, without loss of generality, that $I = [0,5]$, and that that the Weiss c-cover contains $[0,2] \cup [3,5]$ and $[1,4]$ as elements. Recall that $L^2(-)$ is the unit for the operation $\boxtimes$ of Connes fusion. We have

$$H \;\cong\; L^2\mathcal{A}([1,4]) \boxtimes_{\mathcal{A}([1,4])} H,$$

both as $\mathcal{A}([1,4])$-modules and as $\mathcal{A}([0,2] \cup [3,5])$-modules. By [2, Cor. 2.9], the vacuum sector $L^2\mathcal{A}([1,4])$ is isomorphic to

$$L^2\mathcal{A}([2,3]) \boxtimes_{\mathcal{A}([2,3]\cup_{\{2,3\}}[2,3])} \Big(L^2\mathcal{A}([1,4]) \boxtimes_{\mathcal{A}([1,2]\cup[3,4])} L^2\mathcal{A}([1,4])\Big)$$

as an $\mathcal{A}([1,4])$-$\mathcal{A}([1,4])$-bimodule, where $\mathcal{A}([2,3]\cup_{\{2,3\}}[2,3])$ is as in [2, Prop. 1.25]. Combining the above two facts, one gets

$$\begin{aligned}
H &\cong L^2\mathcal{A}([1,4]) \boxtimes_{\mathcal{A}([1,4])} H \\
&\cong \Big(L^2\mathcal{A}([2,3]) \boxtimes_{\mathcal{A}([2,3]\cup_{\{2,3\}}[2,3])} \big(L^2\mathcal{A}([1,4]) \boxtimes_{\mathcal{A}([1,2]\cup[3,4])} L^2\mathcal{A}([1,4])\big)\Big) \boxtimes_{\mathcal{A}([1,4])} H \\
&\cong L^2\mathcal{A}([2,3]) \boxtimes_{\mathcal{A}([2,3]\cup_{\{2,3\}}[2,3])} \Big(L^2\mathcal{A}([1,4]) \boxtimes_{\mathcal{A}([1,2]\cup[3,4])} L^2\mathcal{A}([1,4]) \boxtimes_{\mathcal{A}([1,4])} H\Big) \\
&\cong L^2\mathcal{A}([2,3]) \boxtimes_{\mathcal{A}([2,3]\cup_{\{2,3\}}[2,3])} \Big(L^2\mathcal{A}([1,4]) \boxtimes_{\mathcal{A}([1,2]\cup[3,4])} H\Big)
\end{aligned}$$

Using the isomorphism $H \cong L^2\mathcal{A}([0,2] \cup [3,5]) \boxtimes_{\mathcal{A}([0,2]\cup[3,5])} H$, and the existence of a (non-canonical) isomorphism of $\mathcal{A}([0,5])$-$\mathcal{A}([0,5])$-bimodules

$$\begin{aligned}
&L^2\mathcal{A}([1,4]) \boxtimes_{\mathcal{A}([1,2]\cup[3,4])} L^2\mathcal{A}([0,2] \cup [3,5]) \\
&\qquad \cong L^2\mathcal{A}([0,2]) \boxtimes_{\mathcal{A}([1,2])^{\mathrm{op}}} L^2\mathcal{A}([1,4]) \boxtimes_{\mathcal{A}([3,4])} L^2\mathcal{A}([3,5]) \;\cong\; L^2\mathcal{A}([0,5])
\end{aligned}$$

(see [1, Cor. 1.33] and [2, Lem. A.4]), we get the following sequence of isomorphisms of $\mathcal{A}([1,4])$- and $\mathcal{A}([0,2]\cup[3,5])$-modules:

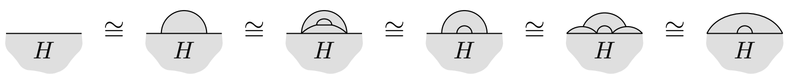
$$\begin{aligned} H &\cong L^2\mathcal{A}([2,3]) \boxtimes_{\mathcal{A}([2,3]\cup_{\{2,3\}}[2,3])} \Big(L^2\mathcal{A}([1,4]) \boxtimes_{\mathcal{A}([1,2]\cup[3,4])} H\Big) \\ &\cong L^2\mathcal{A}([2,3]) \boxtimes_{\mathcal{A}([2,3]\cup_{\{2,3\}}[2,3])} \Big(L^2\mathcal{A}([1,4]) \boxtimes_{\mathcal{A}([1,2]\cup[3,4])} \\ &\qquad\qquad L^2\mathcal{A}([0,2]\cup[3,5]) \boxtimes_{\mathcal{A}([0,2]\cup[3,5])} H\Big) \\ &\cong L^2\mathcal{A}([2,3]) \boxtimes_{\mathcal{A}([2,3]\cup_{\{2,3\}}[2,3])} \Big(L^2\mathcal{A}([0,5]) \boxtimes_{\mathcal{A}([0,2]\cup[3,5])} H\Big) \end{aligned}$$

To finish the proof, one notes that the actions of $\mathcal{A}([1,4])$ and $\mathcal{A}([0,2]\cup[3,5])$ on

$$L^2\mathcal{A}([2,3]) \boxtimes_{\mathcal{A}([2,3]\cup_{\{2,3\}}[2,3])} \Big(L^2\mathcal{A}([0,5]) \boxtimes_{\mathcal{A}([0,2]\cup[3,5])} H\Big)$$

extend to an action of $\mathcal{A}([0,5])$, as they are both acting on $L^2\mathcal{A}([0,5])$. The same property therefore holds for H. $\square$

We finish this note by a graphical rendering of the argument in the proof of Lemma 1:



References

[1] A. Bartels, C. Douglas and A. Henriques. Conformal nets I: Coordinate-free nets. *Int. Math. Res. Not. IMRN* no. 13 (2015), pages 4975–5052.

[2] A. Bartels, C. Douglas and A. Henriques. Conformal nets II: Conformal blocks. *arXiv:1409.8672* (2014).

[3] K. Costello and O. Gwilliam Factorization Algebras in Quantum Field Theory. to be published by Cambridge University Press.

Lecture Series: Observables in the effective BV-formalism; Talk 1: Effective quantum field theory

Ryan Grady

In this talk we describe Costello's mathematical formulation of the low-energy effective field theory approach to perturbative quantum field theory (QFT). Physically, this approach was developed by Kadanoff, Polchinski, Wilson, and others. A key theorem of Costello is a bijection between local functionals on fields and (effective) pertubative QFTs.

The setting for field theory is an action $\mathcal{S}$ which is a function on a space of fields

$$\mathcal{S}:\mathcal{E}\to\mathbb{C}.$$

Classical field theory studies the critical set of the function $\mathcal{S}$. A sample computation in quantum field theory is computing the expectation of an observable, i.e., another function $\mathcal{O} : \mathcal{E} \to \mathbb{C}$. The expectation of $\mathcal{O}$ is given (at least formally) by a functional integral

$$\langle \mathcal{O} \rangle = \int_{\mathcal{E}} \mathcal{O}(\varphi) e^{-\mathcal{S}(\varphi)/\hbar} D\varphi.$$

This integral is often ill-defined, but, in good cases, it has a well defined expansion in the limit $\hbar \to 0$. If $\mathcal{E}$ is finite dimensional and $D\varphi$ is the Lebesgue measure, then this $\hbar \to 0$ limit concentrates on a neighborhood of the critical set of $\mathcal{S}$ and this procedure is the classical stationary phase approximation.

A key element in the definition of (effective) perturbative quantum field theory is renormalization flow (called renormalization group flow in [1] and sometimes exact renormalization group flow in the physics literature).

Let V be a finite dimensional vector space over $\mathbb{R}$ and Φ a non-degenerate negative definite quadratic form Φ. Define $P \in \mathrm{Sym}^2 V$ to be the inverse to $-\Phi$. Let

$$\mathscr{O}(V) \stackrel{def}{=} \widetilde{\mathrm{Sym}}(V^\vee),$$

so $\mathscr{O}(V)$ is the ring of formal power series in a variable $v \in V$. Denote by $\mathscr{O}^+(V)[[\hbar]] \subset \mathscr{O}(V)[[\hbar]]$ the subspace of functionals which are at least cubic modulo $\hbar$. For a functional $I \in \mathscr{O}(V)[[\hbar]]$, we write

$$I = \sum_{i,k \geq 0} \hbar^i I_{i,k},$$

where $I_{i,k}$ is homogeneous of degree k.

Given a triple (V, P, I) as above, we define the a new functional $W(P, I) \in \mathscr{O}^+(V)[[\hbar]]$ as follows

$$W(P, I) = \sum_\gamma \hbar^{g(\gamma)} \frac{w_\gamma(P, I)}{|\mathrm{Aut}(\gamma)|},$$

where the sum is over connected (stable) graphs γ, and $g(\gamma)$ is the genus of the graph. The *graph weight* $w_\gamma(P, I) \in \mathscr{O}(V)$ is defined by contracting tensors with the components of I placed on the vertices (a vertex of valency k and internal degree i is labeled by $I_{i,k}$) and internal edges are labeled by P. The map

$$W(P, -) : \mathscr{O}^+(V)[[\hbar]] \to \mathscr{O}^+(V)[[\hbar]]$$

is called the *renormalization flow operator*. The diagrammatic expansion appearing in the definition of $W(P, I)$ can also be understand as an asymptotic series in $\hbar$ for an integral on U (assuming we've normalized the measure on U appropriately):

$$W(P, I)(a) = \hbar \log \int_{x \in U} e^{(\Phi(x,x) + 2I(x+a))/2\hbar}.$$

The integral appearing above doesn't alway make sense in infinite dimensions, however contraction of tensors does. Therefore, we can still define $W(P, I)$ in the case that V is replaced by a nuclear Fréchet space $\mathcal{E}$ (e.g., $\mathcal{E}$ is the space of sections of a vector bundle E over a manifold M); we work with strong duals and use the completed projective tensor product. In particular, for any $P \in \mathrm{Sym}^2$ we have the renormalization flow operator

$$W(P,-) : \mathscr{O}^+(\mathcal{E})[[\hbar]] \to \mathscr{O}^+(\mathcal{E})[[\hbar]].$$

The Wilsonian yoga is that we have a collection of effective actions $\{S[\Lambda]\}$ and that they are related by renormalization flow.

Let us discuss this paradigm in the setting of scalar field theory on a compact Riemannian manifold M. In this case, our fields are just the smooth functions $C^\infty(M)$. Let D be the (positive) Laplacian on M and $m \in \mathbb{R}_{>0}$, we assume our effective action has the form

$$S[\Lambda](\phi) = -\frac{1}{2}\langle \phi, (D+m^2)\phi\rangle + I[\Lambda](\phi).$$

The functional $I[\Lambda]$ (which is at least cubic modulo $\hbar$) is called the *effective interaction.* In this picture Λ corresponds to "energy" and let $C^\infty(M)_{[\Lambda',\Lambda)}$ denote the span of functions whose eigenvalues lie between Λ' and Λ. The key requirement is that the effective interactions satisfy the *flow equation*:

$$I[\Lambda'](a) = \hbar \log \int_{\phi \in C^\infty(M)_{[\Lambda',\Lambda)}} e^{(-\langle \phi,(D+m^2)\phi\rangle + 2I[\Lambda](\phi+a))/2\hbar}.$$

If we define a cut off kernel $P_{[\Lambda',\Lambda)}$ (we sum only over certain eigenvalues of the operator $(D+m^2)$), then we can rewrite the flow equation as

$$I[\Lambda'](a) = W(P_{[\Lambda',\Lambda)}, I[\Lambda])(a).$$

For a number of reasons, we actually use a smooth cut-off based on the heat kernel. For $l \in \mathbb{R}_{>0}$, let K_l be the kernel for the operator $e^{-l(D+m^2)}$. Our *propagator* with infrared cut-off L and ultraviolet cut-off ϵ ($\epsilon, L \in [0,\infty]$), is given by

$$P(\epsilon, L) = \int_{l=\epsilon}^{L} K_l dl.$$

The operator $W(P(\epsilon,L),-)$ implements *renormalization flow from length scale ϵ to length scale L.*

Lastly, we call a functional $I \in \mathscr{O}(C^\infty(M))$ *local* if it is given by an integral of some Lagrangian density.

Definition 1. *A perturbative QFT, with fields $C^\infty(M)$ and kinetic action $-\frac{1}{2}\langle\phi,(D+m^2)\phi\rangle$, is given by a set of effective interactions $I[L] \in \mathscr{O}^+(C^\infty(M))[[\hbar]]$ for all $L \in (0,\infty]$, such that*

(1) *The flow equation is satisfied for all $\epsilon, L \in (0,\infty]$:*
$$I[L] = W(P(\epsilon,L), I[\epsilon]).$$
(2) *For each i,k, $I_{i,k}[L]$ has a small L asymptotic expansion by local functionals.*

There is an extension of this definition to vector-bundle valued theories, i.e., where the space of fields is given by the space of sections of a vector bundle over M.

Theorem 1 (Costello). *Fix a renormalization scheme. There is a bijection between the set of perturbative QFTs and the set of local action functions* $I \in \mathscr{O}^+_{loc}(C^\infty(M))[[\hbar]]$.

A renormalization scheme is a way to extract the singular part of certain functions of one variable; we won't belabor this detail. The proof of the theorem above is constructive. Given a local functional I, we can construct a series of counterterms $I^{\mathrm{CT}}(\epsilon)$ which cancel certain ultraviolet divergences, so that the effective interaction is given by

$$I[L] = \lim_{\epsilon \to 0} W(P(\epsilon, L), I - I^{\mathrm{CT}}(\epsilon)).$$

Conversely, if $I[L]$ is a family of effective interactions, then a certain renormalized limit as $L \to 0$ defines a local functional (the naive limit doesn't exist and certain counter terms must be subtracted).

References

[1] K. Costello, *Renormalization and effective field theory*, AMS, 2011.

Lecture Series: Observables in the effective BV-formalism; Talk 2: A rapid introduction to the BV-formalism

Brian Williams

The goal of classical field theory is to describe the critical locus of the action functional. The classical BV-formalism is a description of a critical locus of such an action functional in terms of homological algebra.

Suppose V is a finite dimensional vector space and that $S : V \to \mathbb{C}$ is a quadratic function. The critical locus of S is, by definition

$$\mathrm{Crit}(S) := \{v \in V \mid \mathrm{d}S(v) = 0\}$$

The exterior derivative $\mathrm{d}S$ is a linear function on the space V. That is, we can view it as a linear map

$$\mathrm{d}S : V \to V^\vee \quad v \mapsto (w \mapsto \mathrm{d}S_v(w)). \tag{1}$$

The first step is to interpret (1) as a two-term complex with V in degree zero, $V^\vee$ in degree one, and with differential $\mathrm{d}S$. I.e.

$$V \xrightarrow{\mathrm{d}S} V^\vee[-1].$$

The *classical BV-complex* is the space of algebraic functions on the differential graded vector space above. Explicitly

$$\mathcal{O}\left(V \xrightarrow{\mathrm{d}S} V^\vee[-1]\right) = \left(\mathrm{Sym}\left(V^\vee \oplus V[1]\right), Q\right)$$

where Q is the induced differential. This complex satisfies $\mathrm{H}^0 = \mathcal{O}(\mathrm{Crit}(S))$, so it is a derived replacement for the critical locus.

For a more general S (at least quadratic) we can split it up as $S = S^{\text{free}} + I$ where S^{free} *is* quadratic and I is a functional with only cubic or higher terms. The BV-complex is

$$(\mathrm{Sym}(V^\vee \oplus V[1]), Q + \{I, -\})\,. \tag{2}$$

Again, one checks that $\mathrm{H}^0 = \mathcal{O}(\mathrm{Crit}(S))$. Note that this complex is equal to functions on the graded vector space $T^*[-1]V = V \oplus V^\vee[-1]$ with some non-trivial differential determined by S. The bracket $\{-,-\}$ of degree -1 comes from the pairing between V and $V^\vee$ and has the structure of a (shifted) Poisson bracket. This bracket is present on the space of polyvector fields on any manifold and is known as the Schouten-Nijenhuis bracket.

We consider a generalization of the above constructions to infinite dimensional vector spaces.

There are two things that we need to be careful of in this more general case:

(1) All vector spaces carry a topology. Functionals will mean functions on the vector space that are continuous for this topology.
(2) All vector spaces will be spaces of sections of certain sheaves on a manifold. The notion of *locality* discussed in Lecture 1 will be critical for the definition of action functionals of classical field theories.

Example 1. *Let M be a smooth manifold equipped with a Riemannian metric g and consider the space of smooth functions on M, $V = C^\infty(M)$. Define the functional S on $C^\infty(M)$ by*

$$S(\varphi) = \frac{1}{2}\int_M \varphi \mathrm{D}\varphi$$

where D *denotes the Laplacian on M times the volume form. I.e. we view it as an operator*

$$\mathrm{D} : C^\infty(M) \to \mathrm{Dens}(M) \ \ , \ \ \varphi \mapsto (\Delta_g\varphi)\mathrm{dvol}_g.$$

Clearly, S is a quadratic functional. Note that the functional S is local, i.e. it belongs to the subspace of local functionals $S \in \mathcal{O}_{\mathrm{loc}}(C^\infty(M)) \subset \mathcal{O}(C^\infty(M))$ defined in Lecture 1.

Note that in infinite dimensions, the bracket $\{-,-\}$ is only partially defined: the bracket between arbitrary functionals is not well defined. When at least one of the functionals is local then the bracket does make sense.

We are now ready to make a general definition of a classical field theory in our formalism. Recall some of the structure from above:

(1) We want to study the critical locus of a functional on some (infinite dimensional) vector space of fields.
(2) The fields should exists locally on the manifold in which the field theory is defined. That is, they should form a sheaf. Moreover, classical functionals should respect this locality.
(3) The collection of functions on the space of fields should have a Poisson bracket of degree 1.

With this in mind we have the following definition from [2].

Definition 1. *A free BV-theory on a manifold M consists of the following:*

(1) *A $\mathbb{Z}$-graded vector bundle $\pi : E \to M$ of finite rank;*

(2) *A map*

$$\langle -, - \rangle : E \otimes E \to \mathrm{Dens}_M$$

of degree -1 that is graded antisymmetric and fiberwise nondegenerate.

(3) *A square-zero differential operator $Q : \mathcal{E} \to \mathcal{E}$ of cohomological degree 1 that is skew self-adjoint for $\langle -, - \rangle$.*

We assume that the complex $(\mathcal{E}(M), Q)$ is elliptic.

A general BV-theory is a free BV-theory together with a local functional $I \in \mathcal{O}^+_{\mathrm{loc}}(\mathcal{E})$ of degree zero that satisfies the classical master equation

$$QI + \frac{1}{2}\{I, I\} = 0.$$

Given this data we can define the analogous BV complex as in (2). We denote

$$\mathrm{Obs}^{cl}_{\mathcal{E}}(M) := (\mathrm{Sym}(\mathcal{E}(M)^\vee), Q + \{I, -\})$$

which we will also refer to as the *global classical observables*. Note that $\{I, -\}$ is well defined as I is local, and that the operator $Q + \{I, -\}$ squares to zero by the classical master equation.

We now turn to the *quantum BV-formalism*: an approach to the path integral in QFT. More precisely, the quantum BV-formalism is a tool to make sense of expectation values of *observables* of a quantum field theory. If $S : \mathcal{E} \to \mathbb{C}$ is the action functional, an *observable* O is a function on $\mathrm{Crit}(S)$. I.e., a measurement of the physical system. It's expectation value is

$$\langle O \rangle := \frac{1}{Z_S} \int_{\varphi \in \mathcal{E}} O(\varphi) e^{-S(\varphi)/\hbar} \mathrm{D}\varphi.$$

Here $e^{-S(\varphi)/\hbar}\mathrm{D}\varphi$ is thought of a probability measure on the space of fields. The normalization Z_S is the *partition function* of the quantum field theory and equals $\langle 1 \rangle$, the expectation of the unit observable. Just as in the classical approach to the BV-formalism, there is a complex that encodes this approach to integration.

We will motivate the definition of the quantum BV-complex by means of a finite dimensional example. Let M be a closed, oriented, smooth, finite-dimensional manifold of dimension n. Let $\mu \in \Omega^n_M$ be a top form, which we think of a probability density on M. Normalize the image of μ in cohomology $[\mu] = \int_M \mu \in \mathrm{H}^n_{dR}(M)$ to be 1. Note that $\mathrm{H}^n_{dR}(M)$ is one-dimensional in our case. Contraction with μ defines an isomorphism of graded vector spaces

$$i_\mu : \mathrm{PV}^{\#}(M) \xrightarrow{\cong} \Omega^{n-\#}(M).$$

which we use to pull-back the de Rham differential to poly-vector fields which we denote div_μ. This operator on poly-vector fields is known as a *divergence* operator. The complex $(\mathrm{PV}^*(M), \mathrm{div}_\mu)$ is the simplest example of a *quantum BV-complex*. The incarnation of integration in the BV-complex is simple:

Proposition 1. *Given a function $f : M \to \mathbb{R}$, the cohomology class $[f]_{\mathrm{BV}}$ in $\mathrm{H}^0(\mathrm{PV}^*(M), \mathrm{div}_\mu)$ satisfies*

$$[f]_{\mathrm{BV}} = \langle f \rangle_\mu [1]_{\mathrm{BV}}.$$

The goal is to equip the BV-complex for a general field theory $(\mathcal{E}, Q, \langle -, - \rangle, I)$ on M

$$(\mathcal{O}(\mathcal{E}), Q + \{I, -\}) = (\mathrm{Sym}(\mathcal{E}^\vee), Q + \{I, -\})$$

with a type of divergence operator that encodes integration. This is the *BV-Laplacian.* In the case of a general field theory the naive definition of the BV-laplacian above is ill-posed. The central idea in [2] is to use the effective approach formulated in [1] to come up with a regularized version of quantum BV-complex.

References

[1] K. Costello, *Renormalization and effective field theory*, AMS, 2011.
[2] K. Costello and O. Gwilliam, *Factorization algebras in quantum field theory, Volume I & II*, Cambridge University Press (submitted), 2015.

Higher enveloping algebras

Ben Knudsen

We discuss the following result.

Theorem. *Let M be a manifold and k a field of characteristic zero. There is an adjunction*

$$\mathrm{Shv}^{\mathrm{loc}}(M, \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Ch}_k)) \rightleftarrows \mathrm{Fact}^{\mathrm{loc}}_{\mathrm{nu}}(M, \mathrm{Ch}_k)$$

between the ∞-category of locally constant sheaves of dg Lie algebras over k and that of locally constant, nonunital factorization algebras valued in k-chain complexes. Moreover, the value of the left adjoint U_M of this adjunction is given by

$$U_M(L) \simeq CE(L_c),$$

where $(-)_c$ denotes the functor of compactly supported sections and CE the homological Chevalley-Eilenberg complex.

In the case $M = \mathbb{R}^n$, this result specializes to provide an adjunction between dg Lie algebras and (nonunital) dg E_n-algebras. Such an adjunction is available through the results of [4]; the advantage of our approach is the "Poincaré-Birkhoff-Witt" description of the higher enveloping algebra in terms of the Chevalley-Eilenberg complex. This description affords great computational opportunities, some of which are explored in [2] and [6].

The idea of the proof, inspired by [1], is that factorization algebras may be modeled as Lie algebras for an exotic monoidal structure. Specifically, we pass among three models:

$$\left\{ \begin{array}{c} \text{monoidal} \\ \text{model} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{cocommutative} \\ \text{model} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{Lie} \\ \text{model} \end{array} \right\}.$$

The first model views factorization algebras as symmetric monoidal functors out of a certain partially ordered set of multi-Euclidean neighborhoods in M, which carries the “partially defined” symmetric monoidal structure of disjoint union. These symmetric monoidal functors are modeled by certain “factorizable” cocommutative coalgebras using the theory of Day convolution of [5], providing the first equivalence, and the second equivalence is given by Koszul duality, following [3].

References

[1] A. Beilinson and V. Drinfeld. Chiral Algebras. Mat. Zametiki 5 (1969), 227-231. American Mathematical Society Colloquium Publications 51 (2004).

[2] G. Drummond-Cole and B. Knudsen. Betti numbers of configuration spaces of surfaces. In preparation.

[3] J. Francis and D. Gaitsgory. Chiral Koszul duality. Sel. Math. (N.S.) 18 (2012) no. 1, 27-87.

[4] B. Fresse. Koszul duality of E_n-operads. Sel. (N.S.) 17 (2011), no. 2, 363-434.

[5] S. Glasman. Day convolution for ∞-categories. arXiv:1308.4940v3.

[6] B. Knudsen. Betti numbers and stability for configuration spaces via factorization homology. arXiv:1405.6696.

[7] J. Lurie. Higher algebra. Version dated September 14, 2014. Manuscript available at math.harvard.edu/$\sim$lurie.

E_n-algebras associated to secondary polytopes

Mikhail Kapranov

(joint work with Yan Soibelman)

Factorization algebras are algebraic descriptions of field theories. It is known [1, 4] that factorization algebras on the $\mathbb{R}^n$ are in equivalence with E_n-algebras, where E_n is the chain operad of the little n-cubes operad of J.-P. May. Since the shifted Lie operad is embedded into the homology operad of E_n, a E_n-algebra gives, in particular, a (homotopy) Lie algebra (a Lie_∞-algebra for short).

We construct a class of combinatorial examples of E_n-algebras corresponding to secondary polytopes [2]. To each finite set of points $A \subset \mathbb{R}^n$ in general position one associates the secondary polytope $\Sigma(A)$ whose vertices correspond to regular triangulations of the polytope $Q = Conv(A)$ into simplices with vertices in A. These polytopes have a remarkable factorization property: each face of $\Sigma(A)$ is itself a product of several secondary polytopes $\Sigma(A_i)$. This factorization property allows us to construct a E_n-algebra.

The corresponding Lie algebras have been introduced in [3]. In the case $n = 1$ we obtain the fact (pointed out and used in [3]) that the corresponding Lie_∞-algebras come from A_∞-algebras by an appropriate analog of the formula $[a, b] = ab - ba$.

The definition of a factorization algebra is, in its original form, not operadic: factorization algebras are co-sheaves on the appropriate Grothendieck topology associated to $\mathbb{R}^n$ which satisfy the property that certain open embeddings induce quasi-isomorphisms. We use an analog of the Dwyer-Kan localization procedure for operads to convert this definition into a more direct construction of the E_n-operad itself. This allows us to use the factorization properties of secondary polytopes to construct E_n-algebras.

References

[1] K. Costello, O. Gwylliam. *Factorization algebras in quantum field theory.* Book draft, available at www.math.northwestern.edu/~costello/factorization.pdf.

[2] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky. *Discriminants, Resultants and Multidimensional Determinants,* Birkhäuser, Boston, 1994.

[3] M. Kapranov, M. Kontsevich, Y. Soibelman, *Algebra of the infrared and secondary polytopes*, arXiv:1408.2673.

[4] J. Lurie. *Derived Algebraic Geometry VI: E_k Algebras.* arXiv:0911.0018.

Lecture Series: Observables in the effective BV-formalism; Talk 3: Effective BV-quantization

Si Li

One intepretation of BV-quantization is a general approach to quantize gauge theories. As we saw in the last lecture one of the difficulties in physical/geometric applications of quantum gauge theories is the fact that the space of fields is infinite dimensional.

One incarnation of this is the so-called ultra-violet divergence which was briefly mentioned last time. Suppose $(\mathcal{E}, Q, \langle -, - \rangle)$ is a free classical BV-theory. The (-1)-shifted symplectic pairing $\langle -, - \rangle$ induces a partially defined Poisson bracket on $\mathcal{O}(\mathcal{E}) = \mathrm{Sym}(\mathcal{E}(M)^\vee)$. It is partially defined because the dual $\mathcal{E}(M)^\vee$ involves distributional sections and one cannot multiply such elements. Moreover, the naive definition of the BV-laplacian

$$\Delta|_{\mathrm{Sym}^{=2}} = \{-, -\}$$

is also ill-defined. In general, the naive definition of the BV-laplacian is by contraction with the element in $\bar{\mathcal{E}} \otimes \bar{\mathcal{E}}$ determined by the pairing.

The usual fix of this problem by physicists is the method of *renormalization.* In this talk, we discuss a homotopic approach to the effective renormalization of quantum gauge theories as developed by Kevin Costello in [1].

The basic idea is to use the homotopy equivalence between distributions and smooth functions to regularize the BV quantization formalism into homotopic families.

Suppose $(\mathcal{E}(M), Q)$ is an arbitrary *elliptic complex* on a manifold M. This means that $\mathcal{E}(M)$ is the global sections of some $\mathbb{Z}$-graded sheaf, Q is a differential operator of degree $+1$ of square zero, and that the induced complex is elliptic. For instance, any free BV-theory gives such an object. One can also consider the induced complex $(\bar{\mathcal{E}}(M), Q)$ where the bar denotes distributional sections.

A famous result of Atiyah-Bott [3] states that there is a homtopy equivalence between the smooth sections and distributional sections

$$(\mathcal{E}(M), Q) \simeq (\bar{\mathcal{E}}(M), Q).$$

A lift of a distributional section to a smooth section is sometimes called a *regularization.*

The pairing of a free BV-theory determines an element $K_0 \in \bar{\mathcal{E}} \otimes \bar{\mathcal{E}}$ of degree one. According to the above we can choose a regularization

$$K_r = K_0 + QP_r$$

where $K_r \in \mathcal{E} \otimes \mathcal{E}$ is smooth. In particular, contraction with K_r

$$\Delta_r := \partial_{K_r} : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$$

is well-defined.

Suppose r, r' are two regularizations

$$K_0 = K_r + QP_r = K_{r'} + QP_{r'}.$$

Then, $K_r - K_{r'} = Q(P_r^{r'})$ for some element $P_r^{r'} \in \mathcal{E} \otimes \mathcal{E}$ of degree zero. Note that $P_r^{r'}$ is smooth.

The main idea here is that $P_r^{r'}$ is an instance of the propogator from the effective construction of local functionals. The operator $e^{\hbar \partial_{P_r^{r'}}}$ intertwines the differential:

$$e^{\hbar \partial_{P_r^{r'}}}(Q + \hbar \Delta_r) = (Q + \hbar \Delta_{r'}) e^{\hbar \partial_{P_r^{r'}}}.$$

Using this, we can "homtopy transfer" the interaction $I \in \mathcal{O}(\mathcal{E})$ via

$$I[r] = e^{\hbar P_0^r} e^{I/\hbar}.$$

This is precisely the expansion in terms of Feynman weights $I[L] = W(P_0^L, I)$ given in Lecture 1 in the case that the regularization is "length scale". This type of regularization is defined in terms of heat kernels as in [1].

Definition 1. *([2]) An effective BV-quantum field theory based on $(\mathcal{E}, Q, \langle -, - \rangle)$ consists of the following data:*

(1) *For each regularization r we have a functional*

$$I[r] \in \mathcal{O}(\mathcal{E})[[\hbar]].$$

Moreover, $I[r]$ must be at least cubic.

(2) *Given r, r' then $I[r]$ must be related by RG-flow*

$$I[r] = W(P(r', r), I[r']).$$

(3) *For each r, $I[r]$ must satisfy the scale r quantum master equation*

$$QI[r] + \hbar \Delta_L I[r] + \frac{1}{2}\{I[r], I[r]\}_r = 0.$$

(4) *Locality axiom garaunteeing that in the limit as $r \to 0$ the functionals $I[r]$ become local.*

The limit of $I[r] \mod \hbar$ exists and is local, which is denoted $I \in \mathcal{O}_{loc}(\mathcal{E})$. Moreover, it determines a classical field theory for the same underlying free BV-theory. Such a QFT is called a *quantization* of I.

Given a QFT we can defined the following *quantum BV-complex.* For each regularization r define

$$\mathrm{Obs}^q(M)[r] := (\mathrm{Sym}(\mathcal{E}(M)^\vee)[[\hbar]], Q + \hbar\Delta_r + \{I[r], -\}_r)\,.$$

It is called the complex of *global observables* associated to the regularization r. Moreover, the homotopy $P_r^{r'}$ defines a homotopy equivalence

$$\mathrm{Obs}^q(M)[r] \simeq \mathrm{Obs}^q(M)[r']$$

for any regularizations r, r'.

References

[1] K. Costello, *Renormalization and effective field theory*, AMS, 2011.
[2] K. Costello and O. Gwilliam, *Factorization algebras in quantum field theory, Volume I & II*, Cambridge University Press (submitted), 2015.
[3] M.F. Atiyah and R. Bott, *A Lefshetz fixed point formula for elliptic complexes*, Ann. of Math. (2), **86** (1967), 374-407. MR 0212836 (35 #3701).

Lecture Series: Observables in the effective BV-formalism, Talk 4: Factorization algebras: examples and constructions

Ryan Grady

In this talk we presented some basics of factorization algebras as defined by Costello and Gwilliam in [2]. Further, we explained some examples and constructions of factorization algebras coming from sheaves of differential graded Lie algebras.

Let M be a manifold, a *prefactorization algebra* $\mathcal{F}$ on M, taking values in vector spaces, is a rule that assigns to each open $U \subset M$ a vector space $\mathcal{F}(U)$ along with the following maps and combatibilities.

(1) For each inclusion $U \subset V$, a linear map $m_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$;
(2) For each finite collection of pairwise disjoint open sets $\{U_i\}$ with $U_i \subset V$, a linear map $m_V^{U_1,\dots,U_n} : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V)$;
(3) The maps satisfy the obvious compatibility condition, i.e., if $U_{i,1} \sqcup \cdots \sqcup U_{i,n} \subset V_i$ and $V_1 \sqcup \cdots \sqcup V_k \subset W$, then the following diagram commutes.

$$\begin{tikzcd} \otimes_{i=1}^k \otimes_{j=1}^{n_i} \mathcal{F}(U_j) \arrow[rr] \arrow[dr] & & \otimes_{i=1}^k \mathcal{F}(V_i) \arrow[dl] \\ & \mathcal{F}(W) & \end{tikzcd}\,.$$

Note that $\mathcal{F}(\emptyset)$ is necessarily a commutative algebra. A prefactorization algebra $\mathcal{F}$ is *unital* if $\mathcal{F}(\emptyset)$ is a unital commutative algebra.

A fundamental example is the factorization algebra on $\mathbb{R}$ determined by an associative algebra A. In this example, each open interval (a, b) is assigned the algebra A, the map induced by an inclusion $(a, b) \subset (c, d)$ is the identity and the map induced by including disjoint intervals is determined by the multiplication in

A. That the compatibility condition (3) holds follows from the associativity of the multiplication.

The preceding example is universal for prefactorization algebras on $\mathbb{R}$ which are *locally constant*, i.e., the map induced by an inclusion of intervals is an isomorphism.

Proposition 1. *Let $\mathcal{F}$ be a locally constant, unital prefactorization algebra on $\mathbb{R}$ taking values in vector spaces. Then $\mathcal{F}(\mathbb{R})$ has the structure of an associative algebra.*

Alternatively, one can define prefactorization algebras on M valued in a multicategory C as functors $\mathcal{F} : \mathsf{Disj}_M \to \mathsf{C}$, where Disj_M is the multicategory with objects the connected open subsets of M and morphisms corresponding to inclusions of pairwise disjoint collections of opens into another open set. There is an associated symmetric monoidal category $\mathbf{\mathsf{SDisj}}_M$ and for any symmetric monoidal category $\mathsf{C}^{\otimes}$, a prefactorization algebra valued in $\mathsf{C}^{\otimes}$ is a symmetric monoidal functor $\mathcal{F} : \mathbf{\mathsf{SDisj}}_M \to \mathsf{C}$.

Prefactorization algebras have a flavor similar to precosheaves. It is often useful for objects to satisfy descent or a local-to-global property, e.g., cosheaves, and such prefactorization algebras are called *factorization algebras*.

Definition 1. *Let U be an open set. A collection of open sets $\mathfrak{U} = \{U_i\}$ is a Weiss cover of U if for any finite collection of points $\{x_1, \dots, x_k\}$ in U, there is an open set $U_i \in \mathfrak{U}$ such that $\{x_1, \dots, x_k\} \subset U_i$.*

The Weiss covers define a Grothendieck topology on the category of open subsets of a space M which is called the *Weiss topology*. A Weiss cover is a cover in the traditional sense, but typically contains an enormous number of open sets. Given a manifold M of dimension n, there are several ways to construct a Weiss cover of M. For instance, the collection of all open sets in M diffeomorphic to a disjoint union of finitely many copies of the open n-disk forms a Weiss cover.

Definition 2. *A prefactorization algebra $\mathcal{F}$ on M is a factorization algebra if $\mathcal{F}$ is a cosheaf with respect to the Weiss topology.*

Generalizing the proposition of the previous section, factorization algebras (valued in cochain complexes) on $\mathbb{R}^n$ resemble E_n algebras (algebras over the operad of little n-disks). In fact, E_n algebras form a full subcategory of factorization algebras on $\mathbb{R}^n$: those that are locally constant, i.e., those for which an inclusion of open discs induces a quasi-isomorphism. The following theorem of Lurie [4] makes this claim precise (see also the work of Matsuoka [4]).

Theorem 1. *There is an equivalence of $(\infty, 1)$-categories between E_n algebras and locally constant factorization algebras on $\mathbb{R}^n$.*

Let E be a vector bundle on M and let $\mathcal{E}$ denote the sheaf of sections. Similarly, let $\mathcal{E}_c$ denote the cosheaf of compactly supported sections. It is easy to verify that the symmetric algebra of a cosheaf is a prefactorization algebra, it is more difficult to check the local-to-global (factorization) property. However, Costello and Gwilliam prove that both $\mathrm{Sym}\mathcal{E}_c$ and the completed version $\widehat{\mathrm{Sym}}\mathcal{E}_c$ form factorization algebras on M.

The preceding construction can be bootstrapped to the case of Chevalley-Eilenberg chains/cochains of a sheaf of differential graded Lie algebras. If $\mathcal{L}$ is such a sheaf, we will denote Chevalley-Eilenberg chains by $C_*(\mathcal{L})$ and cochains by $C^*(\mathcal{L})$.

Theorem 2. *Let L be a local dg Lie algebra on M. Then for $U \subset M$ an open set, the assignemnts*

$$\mathbb{U}\mathcal{L} : U \mapsto C_*(\mathcal{L}_c(U)) \quad \textit{and} \quad \mathbb{O}\mathcal{L} : U \mapsto C^*(\mathcal{L}(U))$$

define factorization algebras.

As a simple example, let $\mathfrak{g}$ be an ordinary Lie algebra and consider the sheaf of differential graded Lie algebras on $\mathbb{R}$ given by $\mathfrak{g}_{\mathbb{R}} \overset{def}{=} \Omega^*_{\mathbb{R}} \otimes \mathfrak{g}$, where Ω^* denotes differential forms. By the preceding theorem $\mathbb{U}\mathfrak{g}_{\mathbb{R}}$ is a factorization algebra on $\mathbb{R}$ valued in complexes. Passing to cohomology, we obtain a locally constant factorization algebra on $\mathbb{R}$ valued in vector spaces. Hence, by the proposition above $H^*(\mathbb{U}\mathfrak{g}_{\mathbb{R}})$ corresponds to an associative algebra; one can identify this algebra as the universal enveloping algebra $U\mathfrak{g}$.

References

[1] K. Costello and O. Gwilliam, *Factorization algebras in quantum field theory, Volume I & II*, Cambridge University Press (submitted), 2015.

[2] P. Boavida de Brito and M. Weiss, *Manifold calculus and homotopy sheaves*, Homology Homotopy Appl., 15 (2), 361-383.

[3] J. Lurie, *Higher algebra*

[4] T. Matsuoka, *Descent and Koszul duality for factorization algebras*

Cyclotomic Structures and Factoization Homology

Thomas Nikolaus

(joint work with Peter Scholze)

Let R be a ring, we want to compute algebraic K-theory groups. This turns out to be very hard, thus one tries to approximate those by more computable invariants. One has the following square

$$\begin{tikzcd} \mathrm{K}_*(R) \arrow[r] & \mathrm{TC}_*(R) \arrow[r] \arrow[d] & \mathrm{CH}^-_*(R) \arrow[d] \\ & \mathrm{THH}_*(R) \arrow[r] & \mathrm{HH}_*(R) \end{tikzcd}$$

all of whose corners we explained in the talk. The hochschild homology groups $\mathrm{HH}_*(R)$ are given by the homology groups of the Hochschild chain complex which is given by the factorization homology

$$\mathrm{HH}(R) := \int_{S^1} R[0] \in \mathrm{Ch}_{\mathbb{Z}}$$

where $R[0] \in \mathrm{Ch}_{\mathbb{Z}}$ is considered as a chain complex concentrated in degree 0. The topological Hochschild homology groups $\mathrm{THH}_*(R)$ are given by the homotopy groups of the topological Hochschild homology spectrum

$$\mathrm{THH}(R) := \int_{S^1} HR \in \mathrm{Sp}$$

where $HR \in \mathrm{Sp}$ is the Eilenberg-MacLane spectrum associated to R.

By functoriality of factorization homology the Hochschild chain complex (as well as the topological Hochschild homology spectrum) carries an action by the circle group $\mathbb{T}$. The negative cyclic homology chain complex is defined as the homotopy fixed point chain complex for this action:

$$\mathrm{CH}^-(R) := \mathrm{HH}(R)^{h\mathbb{T}}$$

For topological cyclic homology (which should really be called negative topological cyclic homology) one has to take an additional structure besides the $\mathbb{T}$-action on $\mathrm{THH}(R)$ into account: the cyclotomic structure. For the formulation of the cyclotomic structure we will use the notion of Tate spectrum X^{tG} for a finite G-action on a spectrum X.

Proposition 2.

- *For every spectrum X and every prime p there is 'diagonal' map $\Delta : X \to (X \otimes ... \otimes X)^{tC_p}$ which is natural in X.*
- *Let R be an $\mathbb{E}_n$-ringspectrum and $E \xrightarrow{C_p} M$ be a principal C_p-bundle over a framed n-manifolds M. Then there is a map*

$$\int_M R \to \left(\int_E R\right)^{tC_p}$$

which is natural in M and for $M = \mathbb{R}^n$ given by the diagonal Δ as above

It is a remarkable fact that the last proposition is not correct in the category of chain complexes. That is the reason that one has to work in spectra to see the cyclotomic structure. Using the p-fold self-covers of the circle we get the following immediate corollary.

Corollary 3. *For every ring spectrum R the spectrum $\mathrm{THH}(R)$ has the following structure:*

- *An action by the circle group $\mathbb{T}$*
- *For every prime p an $\mathbb{T}$-equivariant map*

$$\varphi_p : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$$

where the action on the target uses the identification $\mathbb{T}/C_p \cong \mathbb{T}$.

This structure is what we call a cyclotomic structure. Given this structure extra structure on THH we can give the following formula for topological cyclic homology where we assume for simplicty that everything is p-completed:

$$\mathrm{TC}(R) := \mathrm{fib}\left(\mathrm{THH(R)}^{h\mathbb{T}} \xrightarrow{\mathrm{can}-\varphi_p} \left(\mathrm{THH(R)}^{tC_p}\right)^{h\mathbb{T}/C_p}\right)$$

Here can denotes the map induced on homotopy $\mathbb{T}/C_p$-fixed points from the canonical map $\mathrm{THH}(R)^{hC_p} \to \mathrm{THH}(R)^{tC_p}$.

Theorem 1. *This definition of topological cyclic homology is for a connective ring spectrum R equivalent to the old one (as given by Bökstedt-Hsian-Madsen).*

More generally we prove that the ∞-category of connective cyclotomic spectra as defined this way is equivalent to the ∞-category underlying the classical description of cyclotomic spectra using genuine equivariant homotopy theory (in the incarnation given by Blumberg-Mandell).

Our main result allows to give simpler descriptions and computations for a lot of results in the area, in particular of the cyclotomic trace.

BV algebras in causal QFT

Kasia Rejzner

In my talk I presented an axiomatic approach to perturbative QFT on Lorentzian manifolds, called perturbative Algebraic Quantum Field Theory. In this framework a model is constructed as a functor from an appropriate category of spacetimes to the category of topological $*$-algebras. This functor has to satisfy some axioms, the most important being causality. This approach gives a new perspective on BV quantization, since it allows to obtain the quantum master equation intrinsically in the Lorentzian setting, without invoking path integrals.

1. Causal structure

Let $\mathcal{M} = (M, g)$ be a d-dimensional spacetime, i.e. a smooth d-dimensional manifold with the metric g of signature $(+, -, \dots, -)$. We assume $\mathcal{M}$ to be oriented, time-oriented and globally hyperbolic (i.e. it admits foliation with Cauchy hypersurfaces). To make this concept clear let me recall a few important definitions in Lorentzian geometry.

Definition 1. *Let $\gamma : \mathbb{R} \supset I \to M$ be a smooth curve in M, for I an interval in $\mathbb{R}$ and let $\dot{\gamma}$ be the vector tangent to the curve. We say that γ is*

- *timelike, if $g(\dot{\gamma}, \dot{\gamma}) > 0$,*
- *spacelike, if $g(\dot{\gamma}, \dot{\gamma}) < 0$,*
- *lighlike (null), if $g(\dot{\gamma}, \dot{\gamma}) = 0$,*
- *causal, if $g(\dot{\gamma}, \dot{\gamma}) \geq 0$.*

The classification of curves defined above is referred to as the *causal structure.*

Definition 2. *Given the global timelike vector field u (the time orientation) on M, a causal curve γ is called future-directed if $g(u, \dot{\gamma}) > 0$ all along γ. It is past-directed if $g(u, \dot{\gamma}) < 0$.*

Definition 3. *A causal curve is **future inextendible** if there is no $p \in M$ such that:*

$$\forall U \subset M \, \text{open neighborhoods of } p, \; \exists t' \; \text{s.t. } \gamma(t) \in U \; \forall t > t' \,.$$

Definition 4. ***A Cauchy hypersurface** in $\mathcal{M}$ is a smooth subspace of $\mathcal{M}$ such that every inextendible causal curve intersects it exactly once.*

Let **Loc** be the category where objects are connected, (time-)oriented globally hyperbolic spacetimes of given dimension and morphisms are isometric embeddings that preserve orientations and the causal structure. The latter means that for an embedding to be a morphism of **Loc**, it cannot create new causal links. More precisely, let $\chi : \mathcal{M} \to \mathcal{N}$, for any causal curve $\gamma : [a, b] \to N$, if $\gamma(a), \gamma(b) \in \chi(M)$ then for all $t \in]a, b[$ we require: $\gamma(t) \in \chi(M)$.

We can extend **Loc** to a monoidal category $\mathbf{Loc}^{\otimes}$ by allowing for objects that are disjoint unions of objects in **Loc**. The relevant monoidal structure is the disjoint union $\sqcup$.

Let **Alg** be the category of nuclear, topological locally convex unital $*$-algebras (i.e. algebras with the involution operation $*$), where morphisms are injective, continuous and they preserve all the relevant algebraic structures. We can equip **Alg** with a monoidal structure provided by the completed tensor product $\widehat{\otimes}$. The resulting monoidal category is denoted by $\mathbf{Alg}^{\otimes}$.

In a similar way, we define **Vec** to be the category of nuclear, topological locally convex vector spaces, with injective morphisms and **CAlg** will be the category of commutative, nuclear, topological locally convex unital $*$-algebras (i.e. algebras with the involution operation $*$), where morphisms are also assumed to be injective. For future reference, we will denote the forgetful functors from **Alg** and **CAlg** to **Vec** by the same symbol $\mathfrak{v}$.

We are now ready to define what is meant by a locally covariant quantum field theory in our setting.

Definition 5. *A locally covariant quantum field theory (LCQFT) is a functor $\mathfrak{A} : \mathbf{Loc} \to \mathbf{Alg}$ such that*

(1) **Einstein causality***: for $\chi_i : \mathcal{M}_i \to \mathcal{M}$, $\chi_i \in \hom(\mathbf{Loc})$, $i = 1, 2$, with the property that $\chi_1(\mathcal{M}_1)$ is spacelike to $\chi_2(\mathcal{M}_2)$, we have*

$$[\mathfrak{A}\chi_1(\mathfrak{A}(\mathcal{M}_1)), \mathfrak{A}\chi_2(\mathfrak{A}(\mathcal{M}_2))] = \{0\} \,,$$

(2) **Time-slice axiom***: Let $\chi : \mathcal{M} \to \mathcal{N} \in \hom(\mathbf{Loc})$ be such that $\chi(\mathcal{M})$ contains a neighborhood of a Cauchy surface $\Sigma \subset \mathcal{N}$. Then $\mathfrak{A}\chi$ is an isomorphism.*

Remark 1. The first requirement can be rephrased as the condition that $\mathfrak{A}$ can be extended to a symmetric monoidal functor from $\mathbf{Loc}^{\otimes}$ to $\mathbf{Alg}^{\otimes}$, as discussed in [BFIR14].

2. Constructing Models

In order to construct LCQFT models, one can use methods of perturbative algebraic quantum field theory (pAQFT). For more details see [Rej16]. Here we sketch out the main steps on the example of scalar field theory. The (off-shell) configuration space in this case is $\mathfrak{E}(\mathcal{M}) = \mathcal{C}^\infty(M, \mathbb{R})$. Consider $\mathcal{C}^\infty(\mathfrak{E}(\mathcal{M}), \mathbb{C})$, the space of smooth functionals on the configuration space.

Definition 6. *Local functionals are smooth functionals such that for every $\varphi \in \mathfrak{E}(\mathcal{M})$ there exists $k \in \mathbb{N}$ and f, a compactly supported function on the jet bundle, such that*

$$F(\varphi) = \int_M f(j^k(\varphi)) d\mu_g \,,$$

where $j^k_x(\varphi)$ is the k-th jet of φ at point x and $d\mu_g(x) \doteq \sqrt{-g} d^d x$. The space of local functionals is denoted by $\mathfrak{F}_{\mathrm{loc}}(\mathcal{M})$.

Definition 7. *We define the space $\mathfrak{F}(\mathcal{M})$ of multilocal functionals as the algebraic completion of $\mathfrak{F}_{\mathrm{loc}}(\mathcal{M})$ under the point-wise product $\cdot$ defined by*

$$(F \cdot G)(\varphi) \doteq F(\varphi) G(\varphi) \,.$$

The assignment of spaces of multilocal functionals to objects in **Loc** is functorial and the corresponding functor from **Loc** to **CAlg** is denoted by $\mathfrak{F}$. We can think of $\mathfrak{F}(\mathcal{M})$ as the space of (off-shell) classical observables.

Now we want to perform the quantization. We start with the free theory $\mathfrak{A}_0$ (with the non-commutative product $\star$), constructed by means of deformation quantization from the classical theory of the free scalar field. The basic ingredient in this construction is dS_0, a 1-form on $\mathfrak{E}(\mathcal{M})$ that gives the equations of motion

$$dS_0(\varphi) = 0 \,. \tag{1}$$

For the free scalar field we have $dS_0(\varphi) = -(\Box + m^2)\varphi$, where $\Box$ is the wave operator.

In order to define interacting fields, we need first to make sense of formal S-matrices corresponding to possible local interactions (these are realized as local functionals). To this end, we need to introduce the *time-ordered product*.

Definition 8. *A time-ordered product is realized as a triple $(\mathfrak{A}_T, \xi, \mathcal{T})$ where $\mathfrak{A}_T$ is a functor from* **Loc** *to* **CAlg**, *ξ is a natural transformation $\xi : \mathfrak{v} \circ \mathfrak{A}_T \dot{\to} \mathfrak{v} \circ \mathfrak{A}$ and $\mathcal{T}$ is a natural transformation from $\mathfrak{F}$ to $\mathfrak{A}_T$ that provides the equivalence between the time-ordered product $\cdot_{\mathcal{T}}$ of $\mathfrak{A}_T$ and the classical product $\cdot$ of $\mathfrak{F}$, i.e.*

$$F \cdot_{\mathcal{T}} G \doteq \mathcal{T}_{\mathcal{M}}(\mathcal{T}^{-1}_{\mathcal{M}} F \cdot \mathcal{T}^{-1}_{\mathcal{M}} G) \,,$$

where $F, G \in \mathfrak{A}_T(\mathcal{M})$ and we require the following properties to hold: let $\psi_i : \mathcal{M}_i \to \mathcal{M}$, $i = 1, 2$,

(1) *if $\psi_1(\mathcal{M}_1) \prec \psi_2(\mathcal{M}_2)$ then*

$$\xi_{\mathcal{M}} \circ m_{\mathcal{T}} \circ (\mathfrak{A}_T \psi_2 \otimes \mathfrak{A}_T \psi_1) = m_\star \circ (\mathfrak{A}\psi_2 \circ \xi_{\mathcal{M}_2} \otimes \mathfrak{A}\psi_1 \circ \xi_{\mathcal{M}_1}) \,,$$

(2) *if* $\psi_2(\mathcal{M}_2) \prec \psi_1(\mathcal{M}_1)$ *then*

$$\xi_{\mathcal{M}} \circ m_{\mathcal{T}} \circ (\mathfrak{A}_T\psi_2 \otimes \mathfrak{A}_T\psi_1) = m_\star \circ (\mathfrak{A}\psi_1 \circ \xi_{\mathcal{M}_1} \otimes \mathfrak{A}\psi_2 \circ \xi_{\mathcal{M}_2})\,,$$

where $m_{\mathcal{T}}/m_\star$ *is the multiplication with respect to the time-ordered/star product and the relation "*$\prec$*" means "not later than" i.e. there exists a Cauchy surface in* $\mathcal{M}$ *that separates* $\psi_1(\mathcal{M}_1)$ *and* $\psi_2(\mathcal{M}_2)$ *and in the first case* $\psi_1(\mathcal{M}_1)$ *is in the future of this surface and in the second case it's in the past.*

The above definition expresses the fact that the time-ordered product coincides with the star product when the arguments are time-ordered. Note that it actually fixes the time-ordered product of observables localized in disjoint regions. The remaining freedom to define $\cdot_{\mathcal{T}}$ for arguments localized in overlapping regions can be further constrained by some additional requirements [BDF09], so that the construction of the time-ordered product of n arguments is reduced to the extension problem for distributions defined everywhere outside the thin diagonal of M^n. This extension process is called *Epstein-Glaser renormalization*.

Using time-ordered products one can define the formal S-matrix corresponding to an interaction $V \in \mathfrak{F}_{\mathrm{loc}}(\mathcal{M})$. In the first step we identify $\mathcal{T}_{\mathcal{M}}V \in \mathfrak{A}_T(\mathcal{M})$ as the normal-ordered quantity, traditionally denoted by $:V:$. Next we define

$$\mathcal{S}(\lambda :V:) \doteq \xi_{\mathcal{M}} \circ e_{\mathcal{T}}^{i\lambda :V:/\hbar}\,,$$

where λ is the coupling constant and $e_{\mathcal{T}}$ denotes the exponential function where the product is taken to be the time-ordered product $\cdot_{\mathcal{T}}$. The S-matrix defined in this way is identified with an element of $\mathfrak{A}(\mathcal{M})[[\lambda]]((\hbar))$.

The interacting quantum field corresponding to a classical observable $F \in \mathfrak{F}_{\mathrm{loc}}(\mathcal{M})$ is then defined as

$$R_{\lambda V}(F) \doteq \left(\xi_{\mathcal{M}} \circ e_{\mathcal{T}}^{i\lambda :V:/\hbar}\right)^{\star -1} \star \left(\xi_{\mathcal{M}} \circ \left(e_{\mathcal{T}}^{i\lambda :V:/\hbar} \cdot_{\mathcal{T}} :F:\right)\right)\,,$$

where the first factor is the inverse with respect to the star product.

3. BV algebras

The BV formalism in physics becomes relevant when treating gauge theories. However, the basic mathematical structures appear already in the treatment of the scalar field. In the first step of our construction we consider $T^*[-1]\mathfrak{E}(\mathcal{M})$, the shifted cotangent bundle over $\mathfrak{E}(\mathcal{M})$ and $\mathfrak{BV}(\mathcal{M}) \doteq \mathcal{O}_{\mathrm{ml}}(T^*[-1]\mathfrak{E}(\mathcal{M}))$, the space of multilocal functionals on it (see [Rej16] for details). Since elements of $\mathfrak{BV}(\mathcal{M})$ can be identified with vector fields on $\mathfrak{E}(\mathcal{M})$, we can equip this space with the Schouten bracket, denoted by $\{.,.\}$. We define a differential δ_{S_0} on $\mathfrak{BV}(\mathcal{M})$ as insertion of the 1-form dS_0, i.e. $\delta_{S_0} \doteq -\iota_{dS_0}$. The 0-th cohomology of $(\mathfrak{BV}(\mathcal{M}), \delta_{S_0})$ characterizes the space of multilocal functionals on the space of solutions to (1).

In quantum theory we require that the formal S-matrix is invariant under δ_{S_0}, i.e. $\delta_{S_0}(\mathcal{S}(\lambda :V:)) = 0$, which is equivalent to the condition, called the *quantum master equation* (QME) that

$$\frac{1}{2}\{\lambda V, \lambda V\} + \delta_{S_0}\lambda V - i\hbar \bigtriangleup (\lambda V) = 0\,,$$

where $\triangle$ is the renormalized BV Laplacian [FR12]. The quantum BV operator is defined by

$$\hat{s} \doteq R_{\lambda V}^{-1} \circ \delta_{S_0} \circ R_{\lambda V}$$

and, provided the QME holds, it can be expressed as

$$\hat{s} = \delta_{S_0} + \{., \lambda V\} - i\hbar \triangle \ .$$

References

[BDF09] R. Brunetti, M. Dütsch, and K. Fredenhagen, *Perturbative algebraic quantum field theory and the renormalization groups*, Adv. Theor. Math. Phys. **13** (2009), no. 5, 1541–1599.

[BFIR14] R. Brunetti, K. Fredenhagen, P. Imani, and K. Rejzner, *The locality axiom in quantum field theory and tensor products of C^*-algebras*, Rev.Math.Phys **26** (2014), 1450010, [arXiv:math-ph/1206.5484].

[FR12] K. Fredenhagen and K. Rejzner, *Batalin-Vilkovisky formalism in perturbative algebraic quantum field theory*, Communications in Mathematical Physics **317** (2012), no. 3, 697–725.

[Rej16] K. Rejzner, *Perturbative Algebraic Quantum Field Theory. An introduction for Mathematicians*, Mathematical Physics Studies, Springer, 2016.

Lecture Series: Observables in the effective BV-formalism; Talk 5: The factorization algebra of observables

Brian Williams

We reach the main construction of this lecture series, which we will state as one of the central theorems of [1]

Theorem 1. [1] *Let M be a manifold. There is an assignment*

$$\mathrm{Obs}^q : \{\text{QFTs on } M\} \to \{\text{factorization algebras on } M\}$$

called the quantum observables.

There is a simpler construction at the classical level. Let us fix a classical BV-theory $(\mathcal{E}, \langle -, - \rangle, I)$. We have defined the global observables via the classical BV-complex

$$\mathrm{Obs}^{cl}(M) := (\mathrm{Sym}(\mathcal{E}(M)^\vee), Q + \{I, -\})\,.$$

Since $\mathcal{E}$ is a sheaf of sections of some vector bundle, it makes sense to consider, for each open U, the subcomplex

$$\mathrm{Obs}^{cl}(U) := (\mathrm{Sym}(\mathcal{E}(U)^\vee), Q + \{I, -\})$$

that we call the *classical observable supported on U*.

Proposition 1. *The assignment $U \mapsto \mathrm{Obs}^{cl}(U)$ defines a factorization algebra on M.*

In fact, this is a corollary of the $\mathbb{O}$-construction from the last talk, but we can be explicit. If $\sqcup_i U_i \to V$ is a disjoint union of open subsets inside of the open set V then we have a map

$$\mathcal{E}(V) \to \mathcal{E}(\sqcup_i U_i) = \oplus_i \mathcal{E}(U_i)$$

because $\mathcal{E}$ is a sheaf. Taking the duals and noticing that Sym is a symmetric monoidal functor we have a map

$$\otimes_i \mathrm{Sym}(\mathcal{E}(U_i)^\vee) \to \mathrm{Sym}(\mathcal{E}(V)).$$

That is, a map $\otimes_i \mathrm{Obs}^{cl}(U_i) \to \mathrm{Obs}^{cl}(V)$. One shows directly that this is a cochain map and defines the factorization structure maps.

Now, suppose we have a quantum field theory on M. This is the data of $(\mathcal{E}, Q, \langle -, - \rangle)$ together with a collection $\{I[r]\}$ of effective functionals that satisfy the RG-flow equation and the regularized quantum master equation. We have constructed the global quantum observables $\mathrm{Obs}^q(M)$. An element is a collection of functionals $\{O[r]\}$ where each $O[r] \in \mathrm{Obs}^q(M)[r]$ that are related by RG-flow.

To define the factorization algebra, we first need to define what we mean by a quantum observable $\{O[r]\}$ to be *supported* on an open set U. The naive definition used in the classical case does not work here: both the regularized BV-laplacian Δ_r and the Poisson bracket $\{-, -\}_r$ increase the support of an element $O[r]$ so that the total differential

$$\hat{Q}_r := Q + \hbar \Delta_r + \{I[r], -\}$$

also increases support. For instance, if $O[r]$ is in an element of subspace $\mathrm{Sym}(\mathcal{E}(U))[[\hbar]]$ then $\hat{Q}_r O[r]$ may not be.

Luckily, the magnitude in which $\hat{Q}_r$ does increase support is controllable. One says that a quantum observable $\{O[r]\}$ is *supported* on $U \subset M$ iff there exists a closed subset $K \subset U$ and a small enough regularization r such that

$$\mathrm{Supp}\, O[r] \subset K.$$

A main technical result of [1] is that if we have such an observable supported on U then $\hat{Q}_r$ applied to it is still supported on U. Thus we have defined the subcomplex

$$\mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(M)$$

of *observables* supported on U.

We now describe the structure maps of the factorization algebra. Focus on the case $U \sqcup U' \hookrightarrow V$ where U, U' are disjoint. We need to describe a map

$$\mathrm{Obs}^q(U) \otimes \mathrm{Obs}^q(U') \to \mathrm{Obs}^q(V).$$

Take quantum observables $\{O[r]\}$ and $\{O[r']\}$ supported on U, U' respectively. Viewing the functionals as elements of the symmetric algebra $\mathcal{O}(\mathcal{E})[[\hbar]]$ we may consider the product

$$O[r] \cdot O[r'] \in \mathcal{O}(\mathcal{E})[[\hbar]].$$

Theorem 2. *The following limit*

$$\lim_{r'\to 0} W_{r'}^{r}(O[r]\cdot O[r']) \in \mathcal{O}(\mathcal{E})[[\hbar]]$$

exists and will be denoted $(O\cdot O')[r]$.

We can then define the factorization product () by

$$\{O[r]\}\otimes\{O'[r]\}\mapsto\{(O\cdot O')[r]\}.$$

It is straightforward to check that this is a cochain map and satisfies the associativity and commutativity properties necessary to define a prefactorization map. A spectral sequence argument is needed to show that

$$\mathrm{Obs}^q : U \mapsto \mathrm{Obs}^q(U)$$

actually is a factorization algebra.

The connection with the classical observables is the following.

Theorem 1. [1] *Suppose* $\{I[r]\}$ *is a quantization of the classical theory* $I \in \mathcal{O}_{loc}(\mathcal{E})$. *Then* Obs^q *is a factorization algebra in* $\mathbb{C}[[\hbar]]$*-modules. Moreover, there is an isomorphism*

$$\mathrm{Obs}^q \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C} \cong \mathrm{Obs}^{cl}$$

between the reduction of the factorization algebra of quantum observables modulo $\hbar$*, and the factorization algebra of classical observables.*

References

[1] K. Costello and O. Gwilliam, *Factorization algebras in quantum field theory, Volume I & II*, Cambridge University Press (submitted), 2015.

Lecture Series: Observables in the effective BV-formalism; Talk 6: Some examples of BV-quantization

Si Li

We discuss two nontrivial examples of quantum field theories in the effective BV-formalism developed in [1]. Both of these examples involve quantum corrections at all loops.

The first is a one-dimensional σ-model, describing topological quantum mechanics discussed in [2]. The effective BV quantization is equivalent to the geometric model of Fedesov's Abelian connections on Weyl bundles. It leads to a simple approach to deformation quantization and algebraic index theorem on symplectic manifolds.

The second is a two-dimensional model describing chiral deformations of conformal field theories, which arises naturally from topological B-model on elliptic curves. Much of this work can be found in [3]. The effective BV-quantization is equivalent to integrability of infinitely many commuting chiral operators. The partition function is equivalent to the higher genus GW invariants on elliptic curves, which is an incidence of mirror symmetry.

References

[1] K. Costello and O. Gwilliam, *Factorization algebras in quantum field theory, Volume I & II*, Cambridge University Press (submitted), 2015.
[2] R. Grady, S. Li, Q. Li, *BV-quantization and the algebraic index*, arXiv:1507.01812 (submitted), 2015.
[3] S. Li, *Calabi-Yau Geometry and Higher Genus Mirror Symmetry*, Thesis, Harvard University, May 2011.

Factorisation structures on Hilbert schemes of points

Emily Cliff

Let X be a smooth projective surface over $\mathbb{C}$, and let

$$\mathrm{Hilb}_X = \bigsqcup_{n\geq 0} \mathrm{Hilb}_X^n$$

be the union of all Hilbert schemes of points in X.

Grojnowski [1] and Nakajima [2] proved that the Heisenberg Lie algebra modelled on the lattice $H^\bullet(X,\mathbb{Z})$ acts on the cohomology $H^\bullet(\mathrm{Hilb}_X)$, yielding a representation isomorphic to the Fock space of central charge 1. It follows for formal reasons, from the work of Frenkel, Lepowsky, and Meurmann [3], that $H^\bullet(\mathrm{Hilb}_X)$ has a canonical structure of vertex algebra, and hence that it can be used to construct a factorisation algebra, the *Heisenberg factorisation algebra* over any smooth complex curve C.

I would like to understand the relationship between the geometry of the Hilbert scheme and the resulting factorisation algebra over C. My strategy is to build a *factorisation space* over C using the Hilbert scheme of points, which can then be linearised in different ways to produce factorisation algebras. The hope is that one such way yields the Heisenberg factorisation algebra.

In this talk, I defined the notion of a factorisation space, and showed as an example how the Hilbert scheme of points of any smooth variety Y can be used to produce a factorisation space over Y. This is an accessible example of factorisation spaces over varieties of arbitrary dimension. On the other hand, it cannot be the sought-after factorisation space, even in the case that $Y = X$ is our projective surface, because that should live over an arbitrary curve C, not over X.

In the last part of the talk I sketched the construction of a factorisation space which does live over C. One key idea is to replace the Hilbert scheme of X by a subscheme $\widetilde{\mathrm{Hilb}}_{D\times X}$ of the Hilbert scheme of the formal threefold $D \times X$, where X is our projective surface and D is a formal one-dimensional disc. This $\widetilde{\mathrm{Hilb}}_{D\times X}$ is the largest open subscheme equipped with a well-behaved map to Hilb_X, and the fibres of this map are just affine spaces $\mathbb{C}^d$ for varying integers d. This allows us to define a subspace of the factorisation space defined above for the variety $Y = C \times X$; pushing forward this factorisation space yields a factorisation space over C, whose fibres are closely related to Hilb_X, as desired.

The next step in this project is of course to check whether this factorisation space can be linearised to recover the Heisenberg factorisation algebra.

References

[1] I. Grojnowski, *Instantons and affine algebras I. The Hilbert schemes and vertex operators*, Mathematical Research Letters, 3(2):275–291 (1996).
[2] H. Nakajima, *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, Annals of Mathematics, 145(2):379–388 (1997).
[3] I. Frenkel, J. Lepowsky, and A. Meurman, *Vertex operator algebras and the Monster*, volume 134 of *Pure and Applied Mathematics*, Academic Press, Inc., (1988).

Field theories with defects

Ingo Runkel

(joint work with Nils Carqueville)

Field theories with defects are field theories formulated on manifolds equipped with a stratification into submanifolds. This additional decoration leads to a number of interesting constructions, of which the *orbifold construction* is the focus of this report. The idea behind field theory with defects and orbifolds is very general. In the following, we outline the main ideas in somewhat vague terms, and indicate their precise form for two-dimensional functorial field theory, where the details have been worked out.

1. Topological defects

We would like to consider field theories which take as an input an oriented d-dimensional manifold (possibly with extra geometric structure, like a metric, a spin structure, ...), together with a stratification into oriented submanifolds. The strata carry a label chosen according to their dimension. That is, as an additional datum we fix sets

$$D_k \quad , \quad k = 0, \dots, d \ ,$$

and each connected component of a k-dimensional stratum is labelled by an element of D_k. This assignment of labels has to satisfy a consistency condition, phrased in terms of an additional datum which dictates how strata with different labels are allowed to meet. In codimension 1, this amounts to fixing two maps

$$s, t : D_{d-1} \to D_d \ ,$$

which specify that the top-dimensional strata on the two sides of a codimension 1 stratum with label $x \in D_{d-1}$ must be labelled by $s(x)$ and $t(x)$ (as decided by the orientations of the d- and $(d-1)$-dimensional strata). In higher codimension, the formulation becomes more technical, see [1, 2] for a discussion of codimension 2. We refer to the elements of the sets D_k as *defect conditions* (though for elements of D_d we will also use the term *world-volume phases*).

In functorial field theory, one collects the above manifolds with extra structure into a category of (collared, stratified) $(d-1)$-manifolds and stratified d-dimensional bordisms,

$$\mathrm{Bord}_{d-1,d}(D) \ ,$$

where D stands for the collection of sets $(D_k)_{k=0,\dots,d}$ and the maps describing the compatibility conditions. A field theory with defects is then a symmetric monoidal functor Z from $\mathrm{Bord}_{d-1,d}(D)$ to an appropriate target category, e.g. some version of topological vector spaces.

In the following we will restrict ourselves to a subclass of field theories with defects. Namely we require that the d-dimensional field theory depends on the stratification of the d-manifold only up to isotopy, i.e. it is constant on stratifications related by a diffeomorphisms in the connected component of the identity (rel boundary). In the metric case, this diffeomorphism only transports the stratification – it is neither required to be an isometry, nor does it transport the metric itself. We will call defects with this property *topological*.

In functorial field theory, and depending on preference, one can describe field theories with topological defects via an invariance condition on the functor or by defining an equivalence relation on d-dimensional decorated bordisms. Here we take the second point of view and arrive at a bordism category

$$\mathrm{Bord}^{\mathrm{top}}_{d-1,d}(D)\ .$$

Quite generally, given a field theory with topological defects, one would expect it to produce a *category of topological defect conditions*, that is, some version of a d-category whose objects are the set D_d – the possible world-volume phases – whose 1-morphisms are built from the codimension 1 defect conditions D_{d-1}, etc.

This has been made precise for $d = 2, 3$. One obtains a 2-category, respectively a Gray category, with certain types of duals, see [1, 2].

2. Generalised orbifolds as state sum constructions

State sum constructions are a way to produce examples of d-dimensional topological field theories. One fixes label sets D_k, $k = 0, \dots, d$, and a rule to assign a number $W_{C,\Lambda}$ to a simplicial complex C together with a labelling Λ of its dimension-k-faces by elements from D_k, $k = 0, \dots, d$. There may be constraints on which labellings are allowed, just as for the consistency requirement for defect labellings above.

Given a d-manifold M, one chooses a simplicial decomposition C and computes the number

$$Z(M, C) \ = \sum_{\text{allowed labellings } \Lambda \text{ of } C} W_{C,\Lambda}\ .$$

If one can find an assignment $(C, \Lambda) \mapsto W_{C,\Lambda}$ such that $Z(M, C)$ is independent of the choice of C, one has obtained an invariant for M.

Given a field theory (which may be metric-dependent) with topological defects, one may now try to implement the following idea [3]:

> *Try to define a new field theory by carrying out a state sum construction in terms of the topological defects of the given theory.*

We refer to this idea as the *generalised orbifold construction*. Let us elaborate this a little more. Firstly, instead of the sum over allowed labellings as in the state sum construction, it is much more convenient to think of the corresponding category $\mathcal{D}$ of topological defect conditions as being completed with respect to sums, and to consider sums of "elementary" defect conditions. One then needs to choose

- an object $a \in \mathcal{D}$ (the world-volume phase one wishes to orbifold),
- a 1-morphism $A : a \to a$ (the orbifolding defect),
- various higher coherence morphisms depending on the precise type of cell-decompositions one is working with.

One then defines the new field theory Z_{orb} (initially without defects, though they can be added) in terms of the given field theory Z on a d-manifold M as

$$Z_{\mathrm{orb}}(M) \; = \; Z(M+ \text{ labelled cell decomposition}) \; ,$$

where the strata of the various dimensions are labelled by the above choices (and there is no sum). The above data is subject to the condition that $Z(M)$ must be independent of the choice of cell decomposition.

This idea has been made precise in functorial field theory in dimension 2 [4, 5], and is work in progress in dimension 3 [6]. The construction is such that the original state sum construction arises as a generalised orbifold of the trivial theory.

For example, in dimension 2 the required data and the consistency conditions it needs to satisfy can be phrased as "A needs to be a separable symmetric Frobenius algebra object in the endomorphism category of a", see [5] for details (and for the explanation of the name "orbifold" by exhibiting orbifolding by a group symmetry of a field theory as a special case).

One may ask if Z_{orb} is in some sense equivalent to the restriction of Z to d-manifolds with only a d-dimensional stratum, which is labelled by one fixed world-volume phase $x \in D_d$. If not, one could try to "add" these orbifolds as new world-volume phases to D_d.

Slightly less vaguely, supposing one succeeded in defining the d-category of topological defect conditions $\mathcal{D}$, one can try to construct its *orbifold completion* $\mathcal{D}_{\mathrm{orb}}$ whose objects would consist of an object $a \in \mathcal{D}$ together with a collection of morphisms satisfying the conditions imposed by the orbifold construction.

This as been made concrete in dimension 2 [5, 8]. We prove that given a pivotal idempotent complete bicategory $\mathcal{D}$, one obtains a new such category $\mathcal{D}_{\mathrm{orb}}$, the *orbifold completion of* $\mathcal{D}$, with

- objects: (a, A), where $a \in \mathcal{D}$ and A a separable symmetric Frobenius algebra object in $\mathcal{D}(a, a)$
- 1-morphisms $(a, A) \to (b, B)$: B-A-bimodules in $\mathcal{D}(a, b)$
- 2-morphisms: bimodule intertwiners

There is a natural full embedding $\mathcal{D} \to \mathcal{D}_{\mathrm{orb}}$, $a \mapsto (a, I_a)$, with $I_a : a \to a$ the unit 1-morphism, and we show that this embedding furnishes an equivalence $\mathcal{D}_{\mathrm{orb}} \to (\mathcal{D}_{\mathrm{orb}})_{\mathrm{orb}}$, justifying the name "completion".

As pointed out by Thomas Nikolaus and Kevin Walker during the workshop, this construction is similar to a higher analogue of an idempotent completion [7].

References

[1] A. Davydov, L. Kong, I. Runkel, *Field theories with defects and the centre functor*, in Hisham Sati, Urs Schreiber (eds.): *Mathematical Foundations of Quantum Field theory and Perturbative String Theory*, Proc. Symp. Pure Math., AMS, **83** (2011) 71–128, [1107.0495 [math.QA]].
[2] N. Carqueville, C. Meusburger, G. Schaumann, *3-dimensional defect TQFTs and their tricategories*, 1603.01171 [math.QA].
[3] S. Novak, I. Runkel, *Spin from defects in two-dimensional quantum field theory*, 1506.07547 [hep-th].
[4] J. Fröhlich, J. Fuchs, I. Runkel, C. Schweigert, *Defect lines, dualities, and generalised orbifolds*, conference proceedings 'XVI International Congress on Mathematical Physics' (Prague, August 2009) [0909.5013 [math-ph]].
[5] N. Carqueville, I. Runkel, *Orbifold completion of defect bicategories*, Quantum Topol. **7** (2016) 203 [arXiv:1210.6363 [math.QA]].
[6] N. Carqueville, I. Runkel, G. Schaumann, work in progress.
[7] J. Lurie, *Higher Topos Theory*, Princeton University Press **AM-170** (2009).
[8] N. Carqueville, I. Runkel, *Equivariant completion*, Oberwolfach Reports **10** (2013) 2510

Bott periodicity via quantum Hamiltonian reduction

Theo Johnson-Freyd

The famous Morita equivalence $\mathrm{Cliff}(8) \simeq \mathbb{R}$ appears in many contexts, most notably as a manifestation of the eight-fold Bott periodicity of KO. It can be explained in many ways. The goal of this talk is to give yet another explanation, this time in terms of (super) symplectic geometry.

Clifford algebras have a natural (super) symplectic interpretation. Let $\mathbb{R}^{0|n}$ denote the "odd" manifold with coordinate ring $\mathcal{C}^\infty(\mathbb{R}^{0|n}) = \bigwedge^\bullet \mathbb{R}^n = \mathbb{R}[x_1, \dots, x_n]$, where the coordinate functions are Grassmann variables, so that $x_i x_j = -x_j x_i$ and $x_i^2 = 0$. Odd manifolds admit a calculus of differential forms fully analogous to the even case with one notable exception: since x_i is odd, the one-form $\mathrm{d}x_i$ is even, and so $\mathrm{d}x_i \wedge \mathrm{d}x_j = \mathrm{d}x_j \wedge \mathrm{d}x_i$ with no sign, and $\mathrm{d}x_i^{\wedge 2} \neq 0$ (as it has no reason to vanish). In particular, $\mathbb{R}^{0|n}$ admits a *positive definite symplectic form* $\omega = \frac{1}{2}\sum_i \mathrm{d}x_i^{\wedge 2}$. Since $\mathbb{R}^{0|n}$ is a vector space and ω translation-invariant, $(\mathbb{R}^{0|n}, \omega)$ admits a *canonical quantization* to its Weyl algebra, which in this case is nothing but the *Clifford algebra* $\mathrm{Cliff}(n) = \mathbb{R}\langle x_1, \dots, x_n\rangle/(x_i x_j + x_j x_i = 2\delta_{ij})$.

Linear symplectic geometry can explain Morita equivalences. Suppose that (M, ω) is a symplectic vector space with Weyl algebra $\mathrm{Weyl}(M) = \bigoplus (M^*)^{\otimes n}/([x,y] = \{x,y\}, x, y \in M^*)$. Given a linear Lagrangian $L \subseteq M$ cut out by linear equations $L^\perp \subseteq M^*$, the corresponding *left Fock module* is $\mathrm{Fock}(L) = \mathrm{Weyl}(M)/L^\perp$. By construction, the commutant of $\mathrm{Weyl}(M)$ in $\mathrm{End}(L)$ is $\mathbb{R}$, and so up to issues of functional analysis that are absent in the purely-odd case, $\mathrm{Fock}(L)$ is a Morita trivialization of $\mathrm{Weyl}(M)$. This in particular "explains" the two-fold Bott periodicty of KU. Indeed, the complex Clifford algebra $\mathbb{C}\mathrm{liff}(2) = \mathrm{Cliff}(2) \otimes \mathbb{C}$ is the canonical quantization of the holomorphic symplectic manifold $\mathbb{C}^{0|2}$ with symplectic form $\frac{1}{2}(\mathrm{d}x^{\wedge 2} + \mathrm{d}y^{\wedge 2})$, which admits a holomorphic linear Lagrangian spanned by the lightlike vector $x+iy$. Linear symplectic geometry does not, however, explain any nontrivial Morita equivalences of real Clifford algebras, because the positive-definiteness of $\frac{1}{2}\sum_i \mathrm{d}x_i^{\wedge 2}$ prevents $\mathbb{R}^{0|n}$ from admitting Lagrangian sub-supermanifolds.

A *Hamiltonian action* of a connected and simply connected Lie group G on a symplectic manifold M is determined by a *comoment map* $\mu : \mathrm{Lie}(G) \to \mathcal{C}^\infty(M)$, considered as a Lie algebra under the Poisson bracket. The corresponding action is infinitesimally generated by the Hamiltonian vector fields $\{\mu(g), -\}, g \in \mathrm{Lie}(G)$. The *Hamiltonian reduction* $M//G$ is the space $\mu^{-1}(0)/G$ with coordinate ring $\big(\mathcal{C}^\infty(M)/(\mu(\mathrm{Lie}(G)))\big)^G$. Assuming the action of G on M is not too singular, $M//G$ is again a symplectic manifold and $\mu^{-1}(0)$ is a Lagrangian correspondence between M and $M//G$. The story of Hamiltonian reduction can be quantized. A *quantum Hamiltonian action* of G on an associative algebra A is determined by a map $\mu : \mathrm{Lie}(G) \to A$, considered as a Lie algebra under the commutator bracket; the corresponding action is infinitesimally generated by $[\mu(g), -], g \in \mathrm{Lie}(G)$. The *quantum Hamiltonian reduction* $A//G$ is the ring $\big(A/(\mu(\mathrm{Lie}(G)))\big)^G = \mathrm{End}_A\big(A/(\mu(\mathrm{Lie}(G)))\big)$, where $(\mu(\mathrm{Lie}(G)))$ now denotes the left ideal generated by the image of μ. By construction, the cyclic module $A/(\mu(\mathrm{Lie}(G)))$ is a bimodule between A and $A//G$. If the action is "not too singular," $A/(\mu(\mathrm{Lie}(G)))$ is a Morita equivalence.

When $M = \mathbb{R}^{0|n}$ with its positive-definite symplectic form, there is a subgroup of the full symplectomorphism group given by the linear symplectomorphisms $\mathrm{Sp}(0|n) \cong \mathrm{SO}(n)$. (The "metaplectic group" for $\mathbb{R}^{0|n}$ is $\mathrm{Spin}(n)$.) Thus representations of compact groups provide linear symplectic actions on odd symplectic manifolds, which are automatically Hamiltonian if the group is connected and simply connected. It is natural to focus on linear symplectomorphisms, as they canonically quantize. Linear actions never satisfy the Marsden–Weinstein condition — the classical moment map always has a quadratic singularity at the origin — but the quantum action is "not too singular" as soon as the reduction $\mathrm{Cliff}(n)//G$ is non-zero. The main results of the talk are the following calculations:

(1) $\mathrm{Cliff}(4)//\mathrm{Spin}(3) \cong \mathbb{H}$, the purely-even quaternion algebra, where $\mathrm{Spin}(3)$ acts on $\mathbb{R}^{0|4}$ via the real spin representation.
(2) $\mathrm{Cliff}(7)//G_2 \cong \mathrm{Cliff}(-1)$, the Clifford algebra with one generator and oppositive signature to that of $\mathrm{Cliff}(1)$, where the exceptional group G_2 acts on $\mathbb{R}^{0|7}$ via its defining representation.
(3) $\mathrm{Cliff}(8)//\mathrm{Spin}(7) \cong \mathbb{R}$, where $\mathrm{Spin}(7)$ acts on $\mathbb{R}^{0|8}$ via the real spin representation.

For comparison, the vector representation of $\mathrm{Spin}(n)$ on $\mathbb{R}^{0|n}$ is always "too singular." The calculations are explicit. For example, $G_2 \subseteq \mathrm{SO}(7)$ is by definition the stabilizer of the cubic function $\epsilon = x_1x_2x_7 - x_1x_3x_6 - x_1x_4x_5 - x_2x_3x_5 + x_2x_4x_6 + x_3x_4x_7 + x_5x_6x_7$ on $\mathbb{R}^{0|7}$, and $\mathrm{Spin}(7) \subseteq \mathrm{SO}(8)$ is the stabilizer of the quartic $\epsilon(x_8 + x_1x_2x_3x_4x_5x_6x_7) \in \mathrm{Cliff}(8)$.

How does this story relate to twisted functorial field theories and factorization algebras? My hope is that it can be used to explain the 576-fold periodicity of TMF. There is a conjectural analogy due in part to Stolz and Teichner and in part to Douglas and Henriques relating:

1-dim $\mathcal{N} = 1$ SUSY QFT	KO	real Clifford algebras	$\mathrm{Cliff}(8) \simeq \mathbb{R}$
2-dim $\mathcal{N} = 1$ SUSY QFT	TMF	free fermion chiral CFTs	$\mathrm{Fer}(576) \simeq \mathbb{R}$

The existence of such an analogy is conjectural, and also the lower right box is conjectural. With luck, quantum Hamiltonian reduction could establish the Morita equivalence conjectured in the lower right box. This would provide supporting evidence for the table as a whole.

Extensions of Bordism Categories

Christopher Schommer-Pries

(joint work with Bruce Bartlet, Chris Douglas, Jamie Vicary)

The Reshetikhin-Turaev construction is a process which takes a modular tensor category and produces a 3-dimensional topological quantum field theory. Except that in most cases the topological field theory is not quite well defined for oriented manifolds. Rather there is a *projective anomaly*. To resolve the projective ambiguity the cobordisms must be equipped with additional structure. For example, for the RT construction authors have used p_1-structures (also known as Atiyah 2-framings), bounding manifolds, and other structures. Each such structure gives rise to a central extension of the mapping class group Γ_g

$$\mathbb{Z} \to \tilde{\Gamma}_g \to \Gamma_g$$

of a surface fo genus g.

For large genus we have $H^2(\Gamma_g; \mathbb{Z}) = \mathbb{Z}$, and this raises a question. Can every extension class form the mapping class group be realized by an extension of the bordism category? What is an extension of the bordism category and how can we classify them?

In this talk I will describe a systematic approach to the theory of both central and non-central extensions of symmetric monoidal categories and bicategories. I will apply this in the case of the bordism category and describe the relationship between these extensions and twisted field theories. I will end by sketching a computation which shows that the fundamental central extension of the mapping class group (corresponding to the generator of $H^2(\Gamma_g; \mathbb{Z}) = \mathbb{Z}$) cannot be realized as a central extension of the bordism category.

From state sums to fully extended TQFTs via fields and local relations

Kevin Walker

Let $\mathbf{k}$ be a commutative ring (such as the complex numbers) and let $Z(W)$ be a $\mathbf{k}$-valued invariant of $n+1$-manifolds W. Assume that $Z(W)$ is computable via a topologically invariant "state sum" – a sum, over all labelings of the cells of some/any cell decomposition of W, of the product of local $\mathbf{k}$-valued weights which depend only on labeled cells in some bounded region. Then one can construct from the data for the state sum a fully extended TQFT, with the original state sum playing the role of the path integral. Furthermore, this TQFT is constructed using the fields-and-local-relations approach, which is a more stringent requirement than merely satisfying a fully extended version of the Atiyah-Segal axioms (i.e. fields-and-local-relations implies A-S, but not vice versa).

From the A-S point of view, the target $n+1$-category for the above TQFT is [n-categories, n-category bimodules, 2nd-order bimodules, ...]. The result suggests (perhaps only weakly) that this target $n+1$-category is universal in some sense. Another way of looking at it is that A-S TQFTs which are not also field-and-local-relations TQFTs are slightly strange, in that their $n+1$-dimensional part cannot be computed via a state sum.

It is not hard to see that a state sum is equivalent to a tensor network, where the tensors and edge vector spaces depend functorially on bounded diameter sub-complexes of the cell decomposition of W. The first step in the proof is a coarse-graining argument, which allows us to assume that the tensors depend only on the combinatorial type of individual cells and their normal bundles.

We can now define a "field" on an n-manifold M to be a cell decomposition of M together with a vector in the tensor product of the edge vector spaces associated to edges (in the tensor network) which cross M. The local projections are constructed out of the tensor network associated to a $n+1$-ball (with cell decomposition). Proceeding in a similar manner in higher codimensions, we eventually construct all of the data needed for the fields-and-local-relations machinery.

Relative Calabi-Yau structures on dg functors and shifted Lagrangian structures on moduli of objects

Christopher Brav

(joint work with Tobias Dyckerhoff)

This is a report on a two-part project with Tobias Dyckerhoff, the first part of which has appeared as 'Relative Calabi-Yau structures' at arxiv:1606.00619. We work in the context of non-commutative algebraic geometry via dg categories, in which a finite type dg category is considered to be the bounded derived category of coherent sheaves on a putative finite dimensional non-commutative space and various complexes associated to the Hochschild complex are considered to encode information about differential forms on the non-commutative space. In particular, there is a notion of Calabi-Yau structure of dimension d on a finite type dg category

S, due to Ginzburg and Kontsevich, which is given by a class $\theta \in \mathrm{HC}^{-}_{d}(S)$ in negative cyclic homology satisfying a non-degeneracy condition that requires a certain induced map between the 'inverse dualising complex' and the 'diagonal bimodule' of the dg category be an equivalence. The two basic examples are the dg category $\mathrm{Loc}(M)$ of local systems of chain complexes on a compact oriented manifold M, where using a theorem of Goodwillie the choice of fundamental class for M induces a non-degenerate class in negative cyclic homology of the category of local systems, and the dg category $\mathrm{D}^{b}_{\mathrm{Coh}}(X)$ of bounded complexes of coherent sheaves on a separated Gorenstein scheme of finite type with trivial canonical line bundle ω_X, where the choice of trivialisation $\mathcal{O}_X \simeq \omega_X$ induces a non-degenerate class in negative cyclic homology of the bounded derived category of coherent sheaves.

Our first task was to formulate a relative notion of Calabi-Yau structure on a dg functor $S \to T$, which is given by a class $\theta \in \mathrm{HC}^{-}{}_{d}(T, S)$ satisfying a non-degeneracy condition that requires certain induced maps of bimodules to be equivalences. The two main examples are the induction of local systems from the boundary to the whole manifold for a manifold with boundary equipped with a relative orientation and the push-forward of bounded complexes of coherent sheaves along the inclusion of an anti-canonical divisor. A particular case is a relative Calabi-Yau structure of dimension d on the zero functor $0 \to T$, which is equivalent to an absolute Calabi-Yau structure of dimension d on T. (Recently Kontsevich has informed us that he and Vlassopoulos have arrived at the same definition and same examples of relative Calabi-Yau structure, with applications to topological field theory.) The main results of our paper 'Relative Calabi-Yau structures' are that cospans of dg functors equipped with relative Calabi-Yau structure can be composed, generalising and abstracting the gluing of oriented manifolds along common boundary components, and that the composition of cospans with relative Calabi-Yau structure can be used to endow certain topological Fukaya categories of surfaces with absolute Calabi-Yau structure.

In the second, forth-coming part of our project, we consider the derived moduli space $\mathcal{M}_T$ of finite dimensional modules for a dg category T, showing that a Calabi-Yau structure of dimension d on T induces on $\mathcal{M}_T$ a shifted symplectic form of degree $2 - d$ in the sense Pantev-Töen-Vaquié-Vezzosi, and similarly a dg functor $S \to T$ with relative Calabi-Yau structure of dimension d such that S carries a compatible absolute Calabi-Yau structure of dimension $d-1$ induces on the natural restriction morphism $\mathcal{M}_T \to \mathcal{M}_S$ a Lagrangian structure. This construction of shifted symplectic structures and shifted Lagrangian structures from data on dg categories recovers and generalises many of the examples constructed by Pantev-Töen-Vaquié-Vezzosi.

On formality of IBL infinity structures

Kenji Fukaya

In this talk I explained an idea that the involutive bi Lie infinitey algebra on the cyclic chain complex of the Fukaya category are supposed to be formal. (It means that all the bracket and cobracket are trivial.)

First I explained the definition of involutive bi Lie algebra. An involutive bi Lie infinity algebra is its infinity version. I next explained a result jointly obtained with Cielibak and Latschev that on cyclic bar complex of cyclic DGA of finite dimension we can define a structure of involutive bi Lie algebra.

I next explained its analogue of infinite dimension which is the de Rham complex. Then studying bordered pseudoholomorphic curve gives the deformation of the involutive bi Lie infiity algebra structure of cyclic Bar complex of de Rham complex is obtained.

Finally I explained why this structure is expected to be trivial, especially in the case when 'open-closed map' is an isomorphism.

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