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Nonlinear Evolution Problems

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Abstract. The main themes of this workshop were geometric evolution equations and dispersive equations, including nonlinear wave and Schrödinger equations. Altogether there were 21 talks, presented by leading specialists from all over the world.

Mathematics Subject Classification (2010): 35L70, 35Q55, 47J35, 53C44, 74J30.

Introduction by the Organisers

There was a wide spectrum of topics discussed at this year's workshop on "Nonlinear Evolution Problems" that, however, all can be grouped into the main themes of geometric evolution equations or dispersive equations, including nonlinear wave and Schrödinger equations. Altogether there were 21 talks, presented by international specialists from Australia, Canada, Germany, Great Britain, Italy, France, Switzerland, and the United States. There was a large percentage of female participants in our meeting; five of our speakers were women. Moreover, a number of speakers were only a few years past their Ph.D.

Each morning, three 45-minute lectures were delivered, and on average two in the afternoon, thus leaving ample time for in-depth discussion among the participants of our meeting. This of course meant to make difficult choices; in particular, when different groups of people at our conference had studied and solved the same problem, it was not always easy to decide whom to give the chance to present their work. The final format of our meeting, however, seems to have satisfied everyone present. As a novel feature, moreover, we had asked a group of three specialists in the study of turbulence to organize an afternoon session, starting with an exposition of the theory of energy cascades and weak turbulence and leading to some more advanced recent results. Also this seems to have been a great success.

In geometric evolution equations, the prominent themes were the Ricci flow, highlighted by Haslhofer's talk on his very recent work with Naber on a definition of weak Ricci flows, harmonic map heat flow, 4-dimensional Yang Mills flow, and the mean curvature flow and their variants, where Rupflin presented the long-sought after global existence result in her work with Topping on Teichmüller harmonic map flow, and where Waldron's talk gave promise that the long-standing problem of global smooth existence for the Yang Mills flow on 4-manifolds might soon be resolved.

The talks by Krieger on stable blow-up in critical nonlinear wave equations and by Dalibard–Roux on the Prandtl equation for boundary layers in fluids confirmed our belief that the many different nonlinear evolution problems discussed at our meetings have many subtle features in common and are amenable to similar techniques. Indeed, in both their work the modulation method of Merle–Raphaël is used that had also been employed by Raphaël–Schweyer in their work on precise blow-up rates and stable blow-up regimes for the 2-dimensional harmonic map heat flow, once again underscoring the importance of a meeting joining these communities.

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Workshop: Nonlinear Evolution Problems

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Abstracts

A priori estimates for a fully nonlinear flow of two-convex hypersurfaces

Gerhard Huisken
(joint work with Simon Brendle)

We consider $F_0 : M^n \to (N^{n+1}, \bar{g})$, a smooth, closed and embedded hypersurface in a smooth Riemannian manifold without boundary, where $n \geq 3$. It is well-known that in case the hypersurface is convex and the ambient manifold has non-negative sectional curvature the hypersurface can be smoothly contracted to a point by using the harmonic mean curvature flow [1]. We say that the hypersurface is *2-convex* if the eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ of the second fundamental form satisfy $\lambda_1 + \lambda_2 > 0$. We then consider the second order, fully non-linear evolution system $\frac{d}{dt}F = -G\nu$, where ν is the (outer) unit normal to the hypersurface and the normal velocity G is the harmonic mean of the 2-sums of the principal curvatures:

$$G = \Big(\sum_{i<j} \frac{1}{\lambda_i + \lambda_j} \Big)^{-1}.$$

This flow has a smooth solution at least for short time in this class and has the property that 2-convexity can be controlled from below provided that the ambient curvature tensor satisfies $\bar{R}_{ikik} + \bar{R}_{jkjk} \geq 0$ in any orthonormal frame. This distinguishes the flow in a crucial way from mean curvature flow, where 2-convexity is only preserved in locally symmetric spaces.

The lecture shows how crucial estimates needed for the surgery approach of Huisken-Sinestrari [6] established for mean curvature flow in the Euclidean case can be replaced by new estimates in this fully non-linear case. When combined with the mean curvature flow result for embedded mean-convex hypersurfaces in 3-manifolds in [4] the following result is established for all $n \geq 2$:

Theorem[5] *Let $M_0 = \partial\Omega_0$ be a closed, embedded, 2-convex hypersurface of dimension $n \geq 2$ in a compact Riemannian manifold. Given any $T > 0$, there exists a surgically modified flow with velocity G which starts from M_0 and is defined on the time interval $[0, T)$. Moreover, if the ambient manifold satisfies $\bar{R}_{ikik} + \bar{R}_{jkjk} \geq 0$ at each point in Ω_0, then the flow becomes extinct in finite time.*

Only in the case $n = 2$ the long-time behavior is clarified even without any curvature condition on the ambient manifold, see [4]. It remains an open problem, how the fully nonlinear flow behaves for large times in the general case.

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Horizontal curves of metrics and applications to geometric flows

Melanie Rupflin
(joint work with Peter M. Topping)

On a surface M there are three basic ways to evolve a metric; by conformal change, by pull-back with diffeomorphisms and by horizontal curves, moving orthogonally to the first two types of evolution. In this talk we discussed the fine convergence properties of horizontal curves and their role in the analysis of finite time singularities of Teichmüller harmonic map flow.

A horizontal curve $(g(t))_{t\in[0,T)}$ on a closed oriented surface M of genus $\gamma \geq 2$ can be characterised as a curve of hyperbolic metrics so that at each time $\partial_t g(t)$ is given as real part of a holomorphic quadratic differential $\Psi(t)$.

It is well understood that the only way a general sequence g_i of hyperbolic metrics can degenerate is by collapsing simple closed geodesics and stretching out the surrounding "collars" to become infinitely long and thin as $i \to \infty$, and that the Deligne-Mumford compactness theorem allows to obtain a hyperbolic punctured limiting surface after passing to a subsequence and pulling-back by diffeomorphisms. For applications of horizontal curves to the study of geometric flows this information is far from sufficient; not only would one expect the whole curve of metrics to converge without having to pull-back by diffeomorphisms but additionally one would like to know where the metric will have essentially settled down to its limit by time $t < T$ as opposed to regions where the metric still has to do an infinite amount of stretching on $[t, T)$.

The results obtained in the joint work [3] with Peter Topping allow us to describe these different regions in terms of the length $\mathcal{L}(t) = \int_t^T \|\partial_t g\|$ of the restriction of g to $[t, T)$ and give a quantitative version of having smooth local convergence to a limit h away from the pinching set $F := \{p \in M : \liminf_{t\nearrow T} \mathrm{inj}_{g(t)}(p) = 0\}$, which we can furthermore characterise as

$$F = \bigcap_{t<T} \{p \in M : \mathrm{inj}_{g(t)}(p) \leq (K\mathcal{L}(t))^2\},$$

for any $K \geq K_0 = K_0(\gamma)$. Roughly speaking, we obtain that if $\delta(t)$ converges to zero more slowly than $\mathcal{L}(t)^2$ as $t \nearrow T$ then the metric $g(t)$ will have essentially settled down to its limit h on $\delta(t_0)$-thick$(M, g(t))$ by time t_0. To be more precise, given any $\delta > 0$ and t_0 sufficiently close to T so that $(2K_0\mathcal{L}(t_0))^2 < \delta$, we obtain C^k bounds on $\partial_t g$ which allow us to prove that for any $s, t \in [t_0, T)$

$$\|g(t) - h\|_{C^k(\delta\text{-thick}(M,g(s)),g(s))} \leq C\delta^{-\frac{1}{2}}\mathcal{L}(t),$$

where C depends only on k and the genus of M. We refer to [3, Theorem 1.2] for a more detailed result.

As an application of the theory of horizontal curves we discussed finite time singularities of Teichmüller harmonic map flow, which is a natural gradient flow of the Dirichlet energy that evolves both a map u from M to some closed target (N, g_N) and a hyperbolic metric g on M so as to reduce the energy of the map as quickly as possible. Previous joint work with P. Topping, see [1] and the reference therein, shows that if the flow admits a global solution then it decomposes the given initial map into a union of branched minimal immersions and curves.

The only thing that can stop the flow from existing for all times is a finite-time degeneration of the metric component. In [2] we find a canonical way of flowing beyond such a singular time, thus allowing us to obtain global weak solutions for arbitrary initial data, and analyse the fine structure of such singularities in order to prove a "no-loss-of-topology" result.

A key ingredient in this analysis are the estimate on horizontal curves described above as they imply that outside the regions where the metric has essentially settled down to its limit, and where consequently the behaviour of the map component is similar to the one of the classical harmonic map flow, the metric is sufficiently collapsed so that the maps can be viewed as almost harmonic maps from longer and longer (euclidian) cylinders. As a consequence, we cannot loose "unstructured" energy down degenerating collars and can account for all the "lost topology" in terms of curves and "bubbles", i.e. maps $\omega\colon S^2 \to N$ which are harmonic and thus (possibly branched) minimal immersions themselves.

The upshot of these and our previous results is

> *Any smooth map is decomposed by the Teichmüller harmonic map flow into a finite collection of branched minimal immersion from closed Riemann surfaces and can be reconstructed from these minimal immersions together with connecting curves.*

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Weak solutions for the Ricci flow

Robert Haslhofer
(joint work with Aaron Naber)

We introduce a new class of estimates for the Ricci flow, and use them both to characterize solutions of the Ricci flow and to provide a notion of weak solutions for the Ricci flow in the nonsmooth setting.

As a motivation, let us first explain the much easier task of characterizing supersolutions of the Ricci flow. Let $(M, g_t)_{t\in I}$ be a one-parameter family of Riemannian manifolds. We consider the heat equation $(\partial_t - \Delta_{g_t})w = 0$ on our evolving manifolds $(M, g_t)_{t\in I}$. For every $s, T \in I$ with $s \leq T$ and every smooth function u with compact support, we write $P_{sT}u$ for the solution at time T with initial condition u at time s, i.e. $(P_{sT}u)(x) = \int_M u(y)\, H(x, T\,|\, y, s) dV_s(y)$, where $H(x, T\,|\, y, s)$ is the heat kernel with pole at (y, s). We write $d\nu_{(x,T)}(y, s) = H(x, T\,|\, y, s)dV_s(y)$.

Proposition ([1]). *The following are equivalent:*

(1) $\partial_t g_t \geq -2Rc_{g_t}$
(2) $|\nabla P_{sT}u| \leq P_{sT}|\nabla u|$
(3) $|\nabla P_{sT}u|^2 \leq P_{sT}|\nabla u|^2$
(4) $\int_M u^2 \log u^2\, d\nu \leq 4(T-s)\int_M |\nabla u|^2 d\nu$
(5) $\int_M (u-\bar{u})^2\, d\nu \leq 2(T-s)\int_M |\nabla u|^2 d\nu.$

In essence, the proposition follows easily from the parabolic Bochner-formula

$$(\partial_t - \Delta)|\nabla u|^2 = 2\langle \nabla u, \nabla(\partial_t - \Delta)u\rangle - 2|\nabla^2 u|^2 - (\partial_t g + 2\mathrm{Rc})(\nabla u, \nabla u).$$

To characterize solutions of the Ricci flow, and not just supersolutions, we prove infinite-dimensional generalizations of the above estimates. Let $(M, g_t)_{t\in I}$ be a smooth family of Riemannian manifolds. Let $\mathcal{M} = M \times I$ be its space-time with the usual space-time connection, i.e. $\nabla_t Y = \partial_t Y + \frac{1}{2}\partial_t g_t(Y, \cdot)^{\sharp_{g_t}}$. For each $(x, T) \in \mathcal{M}$, we consider the based path space $P_{(x,T)}\mathcal{M}$ consisting of all space-time curves of the form $\{\gamma_\tau = (x_\tau, T-\tau)\}_{\tau\in[0,T]}$, where $\{x_\tau\}_{\tau\in[0,T]}$ is a continuous curve in M with $x_0 = x$. Let $\Gamma_{(x,T)}$ be the Wiener measure of Brownian motion on our evolving family of manifolds based at (x, T), i.e. the probability measure uniquely characterized by the following property. If $e_{\sigma_1,\ldots,\sigma_k}\colon P_{(x,T)}\mathcal{M} \to M^k$, $\gamma \mapsto (x_{\sigma_1}, \ldots, x_{\sigma_k})$, is the evaluation map at $0 \leq \sigma_1 \leq \ldots \leq \sigma_k \leq T$ then

$$e_{\sigma_1,\ldots,\sigma_k *} d\Gamma_{(x,T)}(y_1, \ldots, y_k) = d\nu_{(x,T)}(y_1, s_1)\cdots d\nu_{(y_{k-1},s_{k-1})}(y_k, s_k),$$

where $s_i = T - \sigma_i$. Path space can be equipped with two natural notions of gradient, the τ-parallel gradient $\nabla^{\|}_\tau$ and the Malliavin gradient $\nabla^{\mathcal{H}}$. We have

$$|\nabla^{\|} F|(\gamma) = \sup\{D_V F(\gamma)\,|\, V_\sigma = 1_{\{\sigma\geq\tau\}} P_\sigma^{-1} v_0\,,\, v_0 \in T_x M\,,\, |v_0| = 1\},$$

and

$$|\nabla^{\mathcal{H}} F|(\gamma) = \sup\{D_V F(\gamma)\,|\, V_\sigma = P_\sigma^{-1} v_\sigma\,,\, v \in H^1([0,T], T_x M)\,,\, v_0 = 0\},$$

where $P_\sigma\colon (T_{x_\sigma}M, g_{T-\sigma}) \to (T_x M, g_T)$ denotes stochastic parallel transport.

Our main theorem characterizes solutions of the Ricci flow in terms of certain sharp estimates on path space.

Theorem ([1]). *The following are equivalent:*

(1) $\partial_t g_t = -2Rc_{g_t}$
(2) $|\nabla_x \int_{P_T\mathcal{M}} F d\Gamma_{(x,T)}| \leq \int_{P_T\mathcal{M}} |\nabla^{\|} F| \, d\Gamma_{(x,T)}$
(3) $\int_{P_T\mathcal{M}} \frac{d[F^\bullet]_\tau}{d\tau} d\Gamma_{(x,T)} \leq 2 \int_{P_T\mathcal{M}} |\nabla^{\|}_\tau F|^2 \, d\Gamma_{(x,T)}$
(4) $\int_{P_T\mathcal{M}} (F^2)^{\tau_2} \log (F^2)^{\tau_2} - (F^2)^{\tau_1} \log (F^2)^{\tau_1} d\Gamma_{(x,T)}$
$\leq 4 \int_{P_T\mathcal{M}} \langle F, \mathcal{L}_{\tau_1,\tau_2} F\rangle d\Gamma_{(x,T)}$
(5) $\int_{P_T\mathcal{M}} (F^{\tau_2} - F^{\tau_1})^2 \, d\Gamma_{(x,T)} \leq 2 \int_{P_T\mathcal{M}} \langle F, \mathcal{L}_{\tau_1,\tau_2} F\rangle \, d\Gamma_{(x,T)}$

Here, F^τ denotes the martingale induced by $F \in L^2(P_T\mathcal{M}, \Gamma_{(x,T)})$, and $\mathcal{L}_{\tau_1,\tau_2}$ denotes the $[\tau_1, \tau_2]$-part of the Ornstein-Uhlenbeck operator $\mathcal{L} = \nabla^{\mathcal{H}*}\nabla^{\mathcal{H}}$. The estimates from the theorem are infinite-dimensional generalizations of the estimates from the proposition. In the very special case of 1-point test functions, i.e. test functions of the form $F(\gamma) = u(\gamma(t_1))$ for some $u : M \to \mathbb{R}$, our infinite dimensional estimates reduce to the finite-dimensional estimates from the proposition. Of course, there are many more test functions on path space, and this is one of the reasons why our infinite-dimensional estimates are strong enough to characterize solutions of the Ricci flow, and not just supersolutions.

Finally, let us briefly indicate how the above characterization of solutions of the Ricci flow can be used to provide a notion of weak solutions for the Ricci flow [2]. We consider 1-parameter Hausdorff semigroups of metric-measure spaces $\mathcal{M}$ equipped with a linear heat flow. We call $\mathcal{M}$ a weak solution of the Ricci flow if and only if the infinite dimensional gradient estimate

$$\Big|\nabla_x \int_{P_T\mathcal{M}} F \, d\Gamma_{(x,T)}\Big| \leq \int_{P_T\mathcal{M}} \big|\nabla^{\|} F\big| \, d\Gamma_{(x,T)}$$

holds. In particular, our solutions are super-solutions in the sense of Sturm [5]. We establish various geometric and analytic estimates for our weak solutions. In particular, one of our applications concerns a question of Perelman about limits of Ricci flows with surgery [4]. Namely, the metric completion of the space-time of Kleiner-Lott [3], which they obtained as a limit of Ricci flows with surgery where the neck radius is sent to zero, is a weak solution in our sense.

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Harmonic Ricci Flow on surfaces

Reto Buzano
(joint work with Melanie Rupflin)

Let $g(t)$ be a family of smooth Riemannian metrics on an n-dimensional closed manifold M. Moreover, given a smooth closed Riemannian manifold (N, g_N) of arbitrary dimension, let $\phi(t)$ be a family of smooth maps from M to N. Then $(g(t), \phi(t))$ is called a solution of the volume preserving Harmonic Ricci Flow (or Ricci Flow coupled with Harmonic Map Heat Flow), if it satisfies

$$(1) \qquad \begin{cases} \partial_t g = -2\,\mathrm{Ric}_g + 2\alpha\, d\phi \otimes d\phi + \dfrac{2}{n} g \displaystyle\fint_M \Big(\mathrm{R}_g - \alpha |d\phi|_g^2\Big) d\mu_g =: T(g, \phi), \\ \partial_t \phi = \tau_g(\phi). \end{cases}$$

Here, Ric_g and R_g denote the Ricci and scalar curvatures of (M, g), α is a (possibly time-dependent) positive coupling constant, and $\tau_g(\phi) = \mathrm{tr}_g(\nabla d\phi)$ is the tension field of ϕ.

The Harmonic Ricci Flow was introduced in [4], with some special cases previously studied in [2, 3]. Some of the key properties of this flow are that on the one hand, in special situations, it behaves less singular than the two flows considered separately, while on the other hand most of the Ricci Flow techniques carry over almost directly to the coupled system. Therefore, this relatively new flow has gained the attention of many authors recently, studying the flow usually in general dimensions. In this talk, we consider the special case where the domain manifold M is a surface of positive genus $\gamma > 0$, a situation in which much stronger results can be obtained. In particular, we will explain that at a finite singular time of the flow, both the map *and* the metric component must blow up simultaneously.

Theorem 1 (Theorem 1.2. of [1])**.** *Let M be a closed surface and let (g, ϕ) be a solution of (1) defined and smooth on a maximal time interval $[0, T)$ and with a smooth coupling function α that is bounded away from zero. If $T < \infty$, then*

$$\limsup_{t \nearrow T} \max_{x \in M} \big|K_{g(t)}(x)\big| = \infty \qquad \textit{and} \qquad \limsup_{t \nearrow T} \max_{x \in M} \frac{1}{2}\big|d\phi(x,t)\big|^2_{g(t)} = \infty,$$

where K_g denotes the Gauss curvature of (M, g).

If the coupling constant α is chosen large enough, such finite time singularities cannot happen. We prove the following theorem.

Theorem 2 (Theorem 1.1. of [1])**.** *Let $\alpha(t) \in [\underline{\alpha}, \bar{\alpha}]$ be a smooth coupling function, where $0 < \underline{\alpha} \leq \bar{\alpha} < \infty$ and*

$$(2) \qquad \underline{\alpha} > 2 \max \big\{ K(\tau) \mid \tau \subset T_p N \text{ two-plane, } p \in N \big\}.$$

Here, $K(\tau)$ denotes the sectional curvature of the target manifold (N, g_N) at a point p in direction τ. Then every solution (g, ϕ) of (1) with a two-dimensional domain manifold is defined and smooth for all times $t \geq 0$.

Theorem 2 follows directly from Theorem 1, as the assumption (2) prevents $|d\phi|_g^2$ from blowing up. In fact, by the Bochner-formula, $|d\phi|_g^2$ is uniformly bounded (in space and time) in terms of its initial value and $\bar{\alpha}$, $\underline{\alpha}$ satisfying (2). Moreover, in [4, Corollary 5.3], we showed that if the coupling constant $\alpha(t)$ is smooth and bounded away from zero, then a concentration of $|d\phi|_g^2$ cannot happen as long as the curvature of $g(t)$ stays bounded. Thus in order to prove Theorem 1, we need to show that in the case of a two-dimensional domain, the converse holds as well, that is, it is not possible for the (Gauss) curvature $K_{g(t)}$ of $g(t)$ to blow up while $|d\phi|_g^2$ remains bounded. In other words, both Theorems 1 and 2 are consequences of the following main result, which is equivalent to Proposition 1.3. in [1].

Theorem 3 (Proposition 1.3. of [1]). *Let* (g, ϕ) *be a solution of* (1) *and assume that on an interval* $[0, T)$, $T < \infty$, *we have*

$$\sup_{x \in M, t \in [0,T)} \frac{1}{2} \big| d\phi(x,t) \big|_{g(t)}^2 < \infty. \tag{3}$$

Then both the curvature and the injectivity radius of $g(t)$ *are uniformly bounded,*

$$\sup_{x \in M, t \in [0,T)} \big| K_{g(t)}(x) \big| < \infty, \qquad \textit{and} \qquad \inf_{t \in [0,T)} \operatorname{inj}(M, g(t)) > 0,$$

and thus the solution (g, ϕ) *of* (1) *can be extended smoothly past time* T.

The main idea used to prove Theorem 3 is that one can always split a flow of metrics on a surface into a *conformal* part, the *pull-back by diffeomorphisms* and a *horizontal* movement. More precisely, there exist a family of smooth diffeomorphisms f_t of M, a smooth function $u(t)$ and a horizontal curve $g_0(t)$, such that

$$g(t) = f_t^* \big(e^{2u(t)} g_0(t) \big). \tag{4}$$

We then first show that the evolution of the underlying conformal structure, described by the horizontal curve $g_0(t)$, is well controlled and in particular that the injectivity radius of $g_0(t)$ is a priori bounded away from zero on any given time interval of finite length by the theory of Rupflin and Topping on Teichmüller Harmonic Map Flow, see in particular [5, 6] and references therein.

Next, we analyse the evolution of the conformal factor $u(t)$ following the approach of Struwe [7], i.e. by studying the Liouville energy

$$E_L(t) := \frac{1}{2} \int_M \Big(\big| du(t) \big|_{g_0(t)}^2 + 2\bar{K} u(t) \Big) d\mu_{g_0(t)}, \tag{5}$$

where $\bar{K}$ is the average Gauss curvature of (M, g). The main differences to the Ricci Flow case studied in [7] are that the background metric g_0 is not fixed, but an evolving horizontal curve, and that the evolution equation for $u(t)$ contains various extra terms stemming from the map component of the flow (1) as well as the diffeomorphisms f_t. Nevertheless, we can still derive bounds on the Liouville energy in this more complicated situation and they in turn then yield estimates on the H^1-norm of u. In a further step, we also derive H^2-bounds on u with respect to the evolving background metric $g_0(t)$, before setting up a bootstrapping scheme to obtain higher regularity estimates and conclude in particular the claimed curvature and injectivity radius bounds for $g(t)$.

Once uniform bounds on the curvature, the injectivity radius and the energy density are known, a solution (g, ϕ) of (1) can always be smoothly extended by standard arguments – compare with Section 6 of [4] where the corresponding result was proven in detail for the non-renormalised Harmonic Ricci Flow in arbitrary dimension.

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On stability properties of type II solutions

Joachim Krieger

The energy critical nonlinear wave equation

$$(-\partial_{tt} + \triangle)u = -u^5 \tag{1}$$

on $\mathbb{R}^{3+1}$, the standard Minkowski space of spatial dimension three, has recently received a lot of attention, as it serves as a key model for other more geometric/physical field theories, such as energy critical Wave Maps or the critical Yang-Mills equations. The problem (1) admits a rough dichotomy of its solutions into two kinds, those of *type I*, and those of *type II.* The latter are characterised by the the property that if J is the maximal time interval on which the Shatah-Struwe energy class solution u is defined, then

$$\sup_{t\in J}\big\|\nabla_{t,x}u(t,\cdot)\big\|_{L^2_x} < +\infty.$$

On the other hand, the solution is of type I if

$$\sup_{t\in J}\big\|\nabla_{t,x}u(t,\cdot)\big\|_{L^2_x} = +\infty.$$

We note that the presence of these two essentially distinct types of dynamics is linked to the fact that the conserved energy

$$E(u) = \int_{\mathbb{R}^3}\left[\frac{1}{2}\big|\nabla_{t,x}u\big|^2 - \frac{1}{6}u^6\right]dx$$

is not positive definite. Other more geometric models, such as Wave Maps or Yang-Mills, have positive definite preserved energy and hence all their solutions are type II. This has motivated a strong interest recently in characterising all type II solutions of (1). This has been approached in two ways: on the one hand, a program led by Duyckaerts-Kenig-Merle [5]–[8] has aimed at giving an abstract soliton resolution type result of all possible type II solutions. This has been achieved in complete generality provided the solutions are restricted to the radial class in [8], and without radiality but only along a sequence of times in [9]. Remarkably, in the radial case, all type II solutions can be described asymptotically as superpositions of dynamically re-scaled ground state profiles

$$W(x) = \frac{1}{\left(1 + \frac{|x|^2}{3}\right)^{\frac{1}{2}}}$$

plus a more regular or scattering error term.

On the other hand, there has been a lot of interest in constructing explicit type II solutions of various types, as in [21, 11, 19, 4], and understanding the stability properties of these. In fact, for all type II solutions with exactly one bulk term W and sufficiently small error, it was shown in [17] that such such solutions are all unstable in that perturbations away from a co-dimension one Lipschitz hyper surface lead either to solutions scattering to zero or solutions blowing up. In particular, we have the following result (we use the notation $W_\lambda(x) = \lambda^{\frac{1}{2}} W(\lambda x)$).

Theorem 1 (K.–Nakanishi–Schlag '13)**.** *Let*

$$u(t,x) = W_{\lambda(t)}(x) + v(t,x) \tag{2}$$

be a type II blow up solution for (1)*, with* $\lim_{t\to T} \lambda(t) = +\infty$*, and such that*

$$\sup_{t\in I} \big\| \nabla_{t,x} v(t,\cdot) \big\|_{L^2_x} \le \delta \ll 1$$

for some sufficiently small $\delta > 0$*, where as usual* I *denotes the maximal life span of the Shatah-Struwe solution* u*. Also, assume that* $t_0 \in I$*. Then there exists a co-dimension one Lipschitz manifold* Σ *in a small neighbourhood of the data* $\big(u(t_0,\cdot), u_t(t_0,\cdot)\big) \in \Sigma$ *in the energy topology* $\dot H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ *and such that initial data* $\big(u_0, u_1\big) \in \Sigma$ *result in a type II solution, while initial data*

$$\big(u_0, u_1\big) \in B_\delta \setminus \Sigma,$$

where $B_\delta \subset \dot H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ *is a sufficiently small ball centred at* $\big(u(t_0,\cdot), u_t(t_0,\cdot)\big)$*, either lead to blow up in finite time, or solutions scattering to zero, depending on the "side of* Σ*" these data are chosen from.*

In fact, the solutions "above" Σ which blow up can be seen to be type I in a certain generalised sense, see [22].

Singular type II solutions as in (2) were constructed explicitly in [21, 19] with $\lambda(t) = t^{-1-\nu}$, $\nu > 0$ but otherwise arbitrary. The preceding theorem then naturally suggests the question whether *perturbations of such solutions along the hyper surface* Σ result in a similar type of blow up. Our main result, obtained in joint ongoing work with W. Schlag, is that this is in fact the case provided we impose a co-dimension one condition on such perturbations along Σ:

Theorem 2 ([20]). *There is $\nu_0 > 0$ sufficiently small, such that the following holds: Let u_ν, $0 < \nu < \nu_0$ be one of the solutions constructed in* [21, 19], *on a time slice $(0, t_0] \times \mathbb{R}^3$, with $0 < t_0 \ll 1$ sufficiently small. Then there exists a co-dimension one Lipschitz hyper surface Σ_0 in a Hilbert space S which is essentially $\big(H^{\frac{3}{2}+}_{rad}(\mathbb{R}^3) \cap \{\phi_d\}^\perp\big) \times \big(H^{\frac{1}{2}+}_{rad}(\mathbb{R}^3) \cap \{\phi_d\}^\perp\big)$, and a positive $\delta_1 \ll 1$, such that for any $(u_0, u_1, \gamma) \in \Sigma_0 \cap B_{\delta_1, S}(0) \times (-\delta_1, \delta_1)$ and suitable Lipschitz functions*

$$\gamma_{1,2} \colon S \cap B_{\delta_1,S}(0) \times (-\delta_1, \delta_1) \longrightarrow \mathbb{R},$$

the solution of (1) *with data*

$$\begin{aligned} u[t_0] &:= u_\nu[t_0] + \big(u_0, u_1\big) + \big(\gamma\phi_d + \gamma_1(u_0, u_1, \gamma)\phi_d, \gamma_2(u_0, u_1, \gamma)\phi_d\big) \\ &\in \big(H^{1+}_{rad}(\mathbb{R}^3) \times H^{0+}_{rad}(\mathbb{R}^3)\big) \cap \Sigma \end{aligned}$$

exists on $I = (0, t_0]$ and can be written in the form

$$u(t, x) = W_{\lambda(t)} + v_1(t, x),\ \lambda(t) = t^{-1-\nu}$$

with $(v_1, v_{1,t}) \in H^{1+\frac{\nu}{2}-} \times H^{\frac{\nu}{2}-}$ on each time slice $t = t_1 \in I$, and furthermore

$$\big(E_{loc}(v)\big)(t) := \int_{|x| \le t} \frac{1}{2} \big|\nabla_{t,x} v_1\big|^2\, dx \longrightarrow 0$$

as $t \to 0$. Thus for small enough $\nu > 0$, the solutions constructed in [21, 19] *are stable under perturbations along a co-dimension two manifold in a suitable topology.*

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Asymptotic stability of solitons for the Zahkarov-Kuznetsov equation

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(joint work with Claudio Muñoz, Didier Pilod, and Gideon Simpson)

We consider the Zahkarov-Kuznetsov equation in dimension $d \geq 2$

$$\text{(ZK)} \qquad \partial_t u + \partial_{x_1}\big(\Delta u + u^2\big) = 0,$$

where $u = u(x,t)$ is a real-valued function, $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{d-1}$ and $t \in \mathbb{R}$. (ZK) was introduced by Zakharov and Kuznetsov [5] to describe the propagation of ionic-acoustic waves in uniformly magnetized plasma in dimensions 2 and 3. (ZK) was derived from the Euler-Poisson system with magnetic field in the long wave limit was carried out by Lannes, Linares and Saut [6]; and from the Vlasov-Poisson system in a combined cold ions and long wave limit, by Han-Kwan [4].

The Cauchy problem for (ZK) in $H^s(\mathbb{R}^d)$ has been extensively studied: let us mention the currently optimal results for local well posedness in dimension 2, for $s > 1/2$ by Grünrock and Herr [3], and by Molinet and Pilod [10] (solutions in $H^1(\mathbb{R}^2)$ are global); and in dimension 3 for $s > 1$ by Ribaud and Vento [11].

We are interested in studying the flow of (ZK) around special travelling wave solutions called solitons. They are solutions fo the form

$$Q_c(x_1 - ct, x_2) \quad \text{with} \quad Q_c(x) \underset{|x|\to+\infty}{\longrightarrow} 0\,, \quad c > 0$$

(the travelling speed must lie along the privileged direction x_1 if one expects the travelling wave to be in H^1), where $Q_c(x) = c^{1/(p-1)}Q(c^{1/2}x)$ and $Q > 0$ satisfies

$$-\Delta Q + Q - Q^p = 0.$$

A. de Bouard [1] proved that the L^2-subcritical solitons are orbitally stable (using concentrate compactness *à la* Cazenave-Lions). The main result of [2] is the *asymptotic stability* of solitons of (ZK) in the case $d = 2$.

Theorem 1 (Asymptotic stability)**.** *Assume $d = 2$. Let $c_0 > 0$. For any $\beta > 0$, there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon \leq \varepsilon_0$ and $u \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^2))$ is a solution of* (ZK) *satisfying*

$$\|u(0) - Q_{c_0}\|_{H^1} \leqslant \varepsilon,$$

then the following holds true.

There exist $c_+ > 0$ with $|c_+ - c_0| \leq K_0\varepsilon$, for some positive constant K_0 independent of ε_0, and $\rho = (\rho_1, \rho_2) \in C^1(\mathbb{R}, \mathbb{R}^2)$ such that

$$u(\cdot, t) - Q_{c_+}(\cdot - \rho(t)) \to 0 \quad \text{in } H^1(x_1 > \beta t) \quad \text{as} \quad t \to +\infty, \tag{1}$$

$$\rho_(t) \to (c_+, 0) \quad \text{as} \quad t \to +\infty. \tag{2}$$

In fact, the convergence (1) can also be obtained in regions of the form

$$\mathcal{AS}(t,\theta) := \big\{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_1 - \beta t + (\tan\theta)\, x_2 > 0\big\}, \ \text{where } \theta \in (-\tfrac{\pi}{3}, \tfrac{\pi}{3})\,.$$

The maximal angle $\frac{\pi}{3}$, which appears in the crucial monotonicity formula (in physical space), also occurs in Fourier space: for example, when proving the following Strichartz estimate – used in [10] to improve the well-posed results for ZK at low regularity –

$$\big\||K(D)|^{\frac{1}{8}} e^{-t\partial_{x_1}\Delta}\varphi\big\|_{L^4_{xt}} \leqslant C\|\varphi\|_{L^2}\,,$$

where $|K(D)|^{\frac{1}{8}}$ is the Fourier multiplier associated to the symbol $|K(k_1, k_2)|^{\frac{1}{8}} = |3k_1^2 - k_2^2|^{\frac{1}{8}}$. Observe that the multiplier $|K(k_1, k_2)|^{\frac{1}{8}}$ cancels out along the cone $|k_2| = \tan(\frac{\pi}{3})|k_1|$.

The above result follows the framework developed by Martel and Merle [7, 8] for (gKdV), and which we extend to higher space dimension. The proof does neither rely on the structure of the nonlinearity $\partial_{x_1}(u^2)$ of (ZK) – which is *not* integrable

– neither on the dimension d. Indeed, it could be extended to different d and p as long as

- local well posedness holds in H^1 (not available in dimension 3).
- a sign condition holds, namely that $\langle \mathcal{L}^{-1}\Lambda Q, \Lambda Q\rangle_{L^2} < 0$. Here $\mathcal{L} = -\Delta + 1 - pQ^{p-1}$ is the linearized operator around Q, and $\Lambda Q = \frac{d}{dc}Q_c\big|_{c=1}$ is the scaling operator.

The above sign spectral condition ensures a certain coercivity in a crucial Virial identity: it has been numerically checked in dimension 2 for $p < 2.1491$ and in dimension 3 for $p < 1.8333$. We however believe that the coercivity could be obtained for a large class of p (observe nonetheless that (ZK) is L^2 critical when $d = 3$, so that solitons are expected to be unstable).

The tools developed for Theorem 1 also apply in the context of a sum of decoupled solitons: they allow to show stability and asymptotic stability of multisolitons in the sense of (1) and (2).

Let us finally mention a few open problems. We would be interested in understanding the behavior in the whole space (i.e. convergence in $H^1(\mathbb{R}^d)$ in (1)): this would require improved dispersion estimates and revisit the theory of global well posedness for small data. A second natural question regards the long time dynamics of large solution in dimension 3. More precisely, we conjecture the existence of finite-time blowup solution for 3D (ZK), as it is expected in the L^2 critical context. Also, (ZK) admits another nonlinear solution: the line soliton. Although it does not lie in $H^1(\mathbb{R}^d)$, this object is worth studying: this was initiated by Mizumachi [9] for the Kadomtsev-Petviashvili II equation, another extension of (KdV) in dimension 2.

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Global regularity and scattering for energy critical geometric wave equations

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(joint work with Daniel Tataru)

This extended abstract describes the recent work of the author and D. Tataru on the $(4+1)$-dimensional (massless) Maxwell–Klein–Gordon equation [4, 5, 6]. The main result fits in the broader context of establishing large data global regularity and scattering for geometric wave equations in the energy critical case, which has seen tremendous progress recently [2, 18, 7, 8, 9].

Here, by the Maxwell–Klein–Gordon (MKG) equation, we mean the minimally coupled system of an electromagnetic field, described by the Maxwell equation, and a massless[1] scalar field, described by the wave equation. More precisely, we say that a pair (A, ϕ) of a real-valued 1-form $A = A_\alpha \, \mathrm{d}x^\alpha$ and a $\mathbb{C}$-valued function ϕ on $\mathbb{R}^{1+d}$ is a solution to the MKG equation if[2]

$$\text{(MKG)} \qquad \begin{cases} \partial^\mu F[A]_{\nu\mu} = \mathrm{Im}(\phi \overline{\mathbf{D}_\nu \phi}), \\ \mathbf{D}^\mu \mathbf{D}_\mu \phi = 0, \end{cases}$$

where $\mathbf{D}_\mu = \partial_\mu + i A_\mu$ and $F[A]_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Observe that this equation is invariant under the scaling $(A, \phi) \mapsto (\lambda^{-1} A, \lambda^{-1}\phi)(\lambda^{-1} t, \lambda^{-1} x)$ $(\lambda > 0)$.

The initial value problem for (MKG) alone is not formally well-posed due to gauge invariance, i.e., if (A, ϕ) is a solution to (MKG) then so is $(\tilde{A}, \tilde{\phi}) = (A - \mathrm{d}\chi, e^{i\chi}\phi)$ for any real-valued function χ. To fix this issue, we impose[3] in addition to (MKG) the *Coulomb gauge* condition $\sum_{j=1}^4 \partial_j A_j = 0$.

Of fundamental importance in the study of large data solutions is the *conserved energy* for (MKG), which is a nonnegative quantity that takes the form

$$\mathcal{E}_{\{t\} \times \mathbb{R}^d}[A, \phi] = \int_{\{t\} \times \mathbb{R}^d} \frac{1}{2} \sum_{\mu < \nu} \left| F[A]_{\mu\nu} \right|^2 + \frac{1}{2} \sum_\mu |\mathbf{D}_\mu \phi|^2 \, \mathrm{d}x.$$

We consider the *energy critical* case $d = 4$, when the conserved energy is invariant under the scaling of (MKG). In general, this is the borderline case in which

[1] Strictly speaking, from the point of view of terminology it would be more apt to take a *massive* scalar field governed by the Klein–Gordon equation. The convention we use here is, however, standard in the literature. Our interest in the massless case comes from the viewpoint that (MKG) is a simpler model for the Yang–Mills equation, which is massless.

[2] We adopt the standard conventions of using the Minkowski metric $\mathbf{m} = \mathrm{diag}(-1, +1, \cdots, +1)$ to raise and lower indices, and summing up repeated upper and lower indices.

[3] Thanks to the linearity of the formula $\tilde{A} = A - \mathrm{d}\chi$, which holds since the gauge group $U(1)$ for (MKG) is abelian, this choice of gauge results in no loss of any generality; see [4, Section 3].

finite time blow-up may be possible (e.g., wave maps into $\mathbb{S}^2$). We show that, nevertheless, (MKG) is globally regular for any finite energy data. More precisely:

Theorem 1 ([4, 5, 6])**.** *The initial value problem for* (MKG) *on* $\mathbb{R}^{1+4}$ *in the Coulomb gauge is globally well-posed for any initial data with finite energy. Moreover, such solutions scatter as* $t \to \pm\infty$.

For the precise statement, we refer to [6, Theorem 1.3].

We remark that, around the same time as our work, this conclusion was also reached independently by Krieger–Lührmann [2]. Their method of proof is, however, different. While [2] uses the celebrated concentration compactness/rigidity method of Kenig–Merle (first adapted for a geometric wave equation in [3]), our work follows the scheme developed by Sterbenz–Tataru [7, 8], in which Theorem 1 is derived as a consequence of the following "bubbling" result:

Theorem 2. *Let* (A, ϕ) *be a finite energy (well-posed) solution to* (MKG) *on* $\mathbb{R}^{1+4}$ *in the Coulomb gauge with maximal future lifespan* $[0, T_+)$*. Then either* $T_+ = \infty$ *and the solution scatters as* $t \to \infty$*, or there exists a sequence of translations and rescalings of* (A, ϕ) *which converges strongly to a nontrivial finite energy stationary*[4] *solution to* (MKG) *in* $H^1_{loc}((-1, 1) \times \mathbb{R}^4)$.

It can be shown that (MKG) on $\mathbb{R}^{1+4}$ does *not* admit any nontrivial stationary solutions with finite energy [6, Section 7]; therefore, Theorem 2 implies Theorem 1. A virtue of our approach is that, as in the case of wave maps [7, 8], it is naturally adapted to establishing a *threshold theorem* (i.e., global well-posedness and scattering for data with energy below every nontrivial static solution), whose proof is currently an open problem for the energy critical Yang–Mills equation, which resembles (MKG).

Due to space constraint, we will not attempt to present the proof in any detail. For an extended summary of the proof, we refer the reader to [6, Sections 2 & 3]. Here, we will content ourselves with discussion of two key ingredients of the proof.

The first important ingredient is a monotonicity formula, or a Morawetz-type estimate, for (MKG) that is analogous to an estimate of Grillakis for wave maps [1] used by Sterbenz–Tataru [8]. Given an interval I, let C_I be the truncated cone $\{(t, x) : |x| \leq t, \, t \in I\}$, and let $\partial C_I = \{(t, x) : |x| = t, \, t \in I\}$ be its lateral boundary. For simplicity, we state a version of the monotonicity formula under the assumption that the energy flux vanishes on ∂C_I:

Proposition 3 ([6, Section 5])**.** *Let* (A, ϕ) *be a solution to* (MKG) *on the cone* C_I *with* $I = [t_1, t_2]$*, whose energy flux on* ∂C_I *is zero. Then for some non-negative density* ${}^{(X_0)}P_0[A, \phi](t, x)$*, we have*

$$\int_{S_{t_2}} {}^{(X_0)}P_0 + \iint_{C_I} \left(|\iota_X F|^2 + \left| \mathbf{D}_X \phi + \frac{1}{\rho} \phi \right|^2 \right) \frac{\mathrm{d}t \, \mathrm{d}x}{\rho} = \int_{S_{t_1}} {}^{(X_0)}P_0$$

where $\rho = \sqrt{t^2 - |x|^2}$, $X = \frac{x^\mu}{\rho} \partial_\mu$ *and* $S_t = (\{t\} \times \mathbb{R}^4) \cap C_I$.

[4] We say that a solution (A, ϕ) to (MKG) is stationary if $\iota_Y F = 0$ and $\mathbf{D}_Y \phi = 0$ for some constant time-like vector field Y.

To illustrate how this is used, let us consider the finite time blow up case. By [4, Theorem 1.1] and finite speed of propagation, it suffices to restrict our attention to the domain of influence of an energy concentration point, which we may assume to be $C_{(0,1]}$ after applying symmetries of (MKG). In general, the energy flux on $C_{(0,t]}$ monotonically decreases to zero as $t \to 0$; for simplicity, let us assume that it vanishes *exactly* on $C_{(0,1]}$. It can be shown that $\limsup_{t\to 0+} \int_{S_t} {}^{(0)}P_0 \leq CE$, where E is the conserved energy of (A, ϕ). By Proposition 3, we see that

$$\iint_{C_{(0,1]}} \left(|\iota_X F|^2 + \left| \mathbf{D}_X \phi + \frac{1}{\rho}\phi \right|^2 \right) \frac{\mathrm{d}t\,\mathrm{d}x}{\rho} \leq CE < \infty.$$

When restricted to a time-like cone $\{|x| \leq \gamma t\}$ $(0 < \gamma < 1)$, note that $\rho = O(t)$ as $t \to 0+$, which shows that the quantity in the parentheses vanish (in an integrated sense) as $t \to 0$. Ultimately, this decay is what allows us to conclude that the bubbles extracted in Theorem 2 are stationary.

Another key ingredient of our proof is the following analogue of the theorem of Sterbenz–Tataru for wave maps [7], which may be viewed as a refined continuation criterion:

Theorem 4 ([5, Theorems 1.5, 1.6])**.** *Let (A, ϕ) be a solution to* (MKG) *on $I \times \mathbb{R}^4$ in the Coulomb gauge with energy E. There exists a function $\epsilon(E) > 0$ such that if*

$$\sup_{k\in\mathbb{Z}} 2^{-2k} \left\| P_k(\nabla_{t,x}\phi) \right\|_{L^\infty(I\times\mathbb{R}^4)} \leq \epsilon(E), \tag{1}$$

where $(P_k)_{k\in\mathbb{Z}}$ are the usual dyadic Littlewood–Paley projections, then (A, ϕ) can be continued past finite endpoints of I and scatters towards infinite endpoints of I.

The difficulty, and usefulness, of this theorem lies in the fact that (1) is quite a weak condition; in particular, the energy E of the solution may be quite large. Proof of Theorem 4 requires combination of many ideas, such as induction on energy, paradifferential renormalization, function spaces, null structure of (MKG) in the Coulomb gauge etc.

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Effective dynamics of nonlinear Schrödinger equations on large domains

Zaher Hani
(joint work with T. Buckmaster, P. Germain, and J. Shatah)

The purpose of this note is to report on the work [1], where we consider the nonlinear Schrödinger equation

$$\text{(NLS)} \qquad \begin{cases} -i\partial_t u + \Delta u = \pm|u|^p u & \text{set on } (t,x) \in \mathbb{R}\times\mathbb{T}_L^n \\ \quad u(t=0) = u_0 \end{cases}$$

Here $p \geq 2$ is an even integer and $\mathbb{T}_L^n$ is the box $[0,L]^n$ with periodic boundary conditions.

As is well-known, the long-time behavior of solutions for such equations is more or less well-understand on the spatial domain $\mathbb{R}^n$ (at least for small localized data); however the situation is markedly different on bounded domains. There, a very rich set of dynamics can be observed even starting from very small data; ranging from quasi-periodic solutions to solutions whose energy cascades between characteristically different length scales. Our aim is to better understand the question of long-time dynamics by deriving effective equations for it when L is very large.

Time scale heuristics. The problem of describing the effective dynamics for a nonlinear dispersive equation posed on a very large domain is quite fundamental for mathematical and physical reasons. To explain this point, let us start with a generic nonlinear dispersive equation posed on a large domain $\mathcal{D}_L \subset \mathbb{R}^n$ with characteristic size L (for example $D_L = LD_0$ for some closed set D_0 with smooth boundary). Such an equation can be written as

$$(1) \qquad \mathcal{L}u = \mathcal{N}_{p+1}(u) + h.o.t. \qquad x \in \mathcal{D}_L$$

where $\mathcal{L}$ is the linear dispersive part of the equation, and we have written the nonlinearity as a sum of term of degree $p+1$ with $p>0$, and another higher order term ($h.o.t.$) that vanishes to higher order as $u \to 0$.

The question that we ask is whether we can describe the effective dynamics of this equation when L is very large. This is relevant for instance in studying water waves in the ocean. As a first guess, one might think that the effective dynamics is given by the same equation (1) posed on Euclidean space $\mathbb{R}^n$. We call this the Euclidean approximation, but it is only relevant on certain time scales for which the solution does not feel the difference between $\mathcal{D}_L$ and $\mathbb{R}^n$. To understand the range of validity of the Euclidean approximation one needs to compare two time scales:

- The nonlinear times scale T_{nl}: before which the nonlinearity has negligible effect. It is not hard to see that for initial data of size ϵ and a nonlinearity of degree $p+1$, the nonlinear time scale is $T_{nl} \sim \epsilon^{-p}$.
- The Euclidean time scale $T_{\mathcal{E}}$ which is the time it takes the solution to feel the effect of the boundary. Assuming that at the linear level, wave packets at frequency scale ~ 1 move at speed ~ 1, then one can heuristically argue that a scale-1 wave packet localized in the interior of $\mathcal{D}_L$ would take time $O(L)$ to feel the effect of the boundary. Therefore $T_{\mathcal{E}} \sim L$.

Comparing those two time scales, one obtains that the Euclidean approximation is only relevant in the regime when $T_{nl} \leq T_{\mathcal{E}}$ (equivalently $L > \epsilon^{-p}$) and over time scales $\lesssim L$. Therefore, the natural question becomes

Question: What happens after the Euclidean approximation breaks? Can we still describe the effective dynamics for very large L?

Rough statement of the results. Answering the above question necessitates specifying the boundary conditions for the problem (1). At a linear level, this tells us how a wave packet is reflected once it reaches the boundary. In [1], we give a positive answer to this question for the nonlinear Schrodinger equation on the box $\mathbb{T}_L^n$ with periodic boundary conditions. Earlier results for $n = 2$ and $p = 2$ appeared in [2].

Roughly speaking, the results in [1] and [2] state that there exists another much longer time scale (compared to T_{nl} and $T_{\mathcal{E}}$), which we call the *resonant time scale* T_R, where one can still describe the effective dynamics precisely by an equation on $\mathbb{R}^n$ called the *continuous resonant equation.* Up to possible logarithmic factors of L (only in dimension $n = 2$), $T_R \sim \frac{L^2}{\epsilon^2} \gg T_{nl}, T_{\mathcal{E}}$.

Formal derivation of the effective equation. For simplicity of presentation, we restrict to the case $p = 2$, and choose the defocusing + sign of the nonlinearity in (NLS) (which has little role in what follows). We start with an ansatz $u = \epsilon v$ to emphasize the size of the initial data under consideration. The equation satisfied by v is given by

$$-i\partial_t v + \Delta v = \epsilon^2 |v|^2 v \qquad \text{set on } (t,x) \in \mathbb{R} \times \mathbb{T}_L^n$$

We expand $v(t,x) = \frac{1}{L^n} \sum_{K \in \mathbb{Z}_L^n} \widehat{v}_K(t) e(K \cdot x)$, where $K \in \mathbb{Z}_L^n = \left(\frac{\mathbb{Z}}{L}\right)^n$ and $e(\alpha) = e(2\pi i \alpha)$, and the define $a_K(t) = e(-|K|^2 t)\widehat{u}_K(t)$. The equation satisfied by b_K reads

$$-i\partial_t a_K = \frac{\epsilon^2}{L^{2n}} \sum_{\mathcal{S}(\mathcal{K})} a_{K_1} \overline{a_{K_2}} a_{K_3} e(2\pi \Omega(K) t), \tag{2}$$

where

$$\mathcal{S}(K) = \{(K_1, K_2, K_3) \in (\mathbb{Z}_L^n)^3 \mid K_1 - K_2 + K_3 - K = 0\}$$
$$\Omega(K) = |K_1|^2 - |K_2|^2 + |K_3|^2 - |K|^2.$$

We now split the above right-hand side into resonant interactions (for which $\Omega = 0$) and nonresonant interactions:

$$(3) \qquad -i\partial_t a_K = \underbrace{\frac{\epsilon^2}{L^{2n}} \sum_{\substack{\mathcal{S}(K)\\ \Omega(K)=0}} a_{K_1}\overline{a_{K_2}}a_{K_3}}_{\text{resonant interactions}} + \underbrace{\frac{\epsilon^2}{L^{2n}} \sum_{\substack{\mathcal{S}(K)\\ \Omega(K)\neq 0}} a_{K_1}\overline{a_{K_2}}a_{K_3} e(\Omega(K)t)}_{\text{nonresonant interactions}}.$$

The expectation is now that

- If ϵ is small enough, the non-resonant interactions become dynamically irrelevant. This requires tools from the theory of dynamical systems, namely normal forms, and it implies that the dynamics are well-approximated by the resonant contributions only. Applying one normal forms transformation justifies this approximation under the restrictive condition $\epsilon < L^{-1-}$ (cf. [2]). In [1] we apply a large (but finite) number of normal forms transformations in order to justify this resonant approximation under the relatively mild condition $\epsilon < L^{-\gamma}$ for any $\gamma > 0$. The upshot is that effectively, the dynamics of $a_K(t)$ are given by

$$-i\partial_t a_K = \frac{\epsilon^2}{L^{2n}} \mathcal{T}_L(a,a,a) \qquad \text{where} \quad \mathcal{T}_L(a,a,a) = \sum_{\substack{\mathcal{S}(K)\\ \Omega(K)=0}} a_{K_1}\overline{a_{K_2}}a_{K_3}$$

- If L is large enough, one can try to approximate the resonant sum above by an integral in a matter similar to how Riemann sums are approximated by integrals. However, the fact that the set $\mathcal{R}(K)$ is defined by nonlinear restrictions on $(K_1, K_2, K_3) \in (\mathbb{Z}^n_L)^3$, leads to one of the deep problems in analytic number theory. The main tool here is the Hardy-Littlewood circle method, of which we rely on some relatively recent refinements (e.g. double Kloosterman refinement) culminating in [3]. We shall not go into the details of this here, but the end result is that there exists a normalization factor $Z(L)$, such that the resonant sum converges to an explicit integral operator $\mathcal{T}$ as follows: If f is a sufficiently smooth and decaying

$$\frac{1}{Z(L)} \mathcal{T}_L(f,f,f) \overset{L\to\infty}{\longrightarrow} \mathcal{T}(f,f,f) \quad \text{with} \quad Z_n(L) = \begin{cases} \frac{1}{\zeta(2)} L^2 \log L & \text{if } n = 2 \\ \frac{\zeta(n-1)}{\zeta(n)} L^{2n-2} & \text{if } n \geq 3 \end{cases}$$

where $\zeta(\cdot)$ is the Riemann zeta function.

We show that these expectations are fulfilled for initially smooth and localized data, and therefore, that the limiting dynamics of a_K (up to rescaling time by a factor $\frac{L^{2n}}{Z_n(L)\epsilon^2}$) is given by the "Continuous Resonant" equation

$$\text{(CR)} \qquad -i\partial_t g(t,\xi) = \mathcal{T}\left(g(t,\cdot), g(t,\cdot), g(t,\cdot)\right)(t,\xi) \qquad \xi \in \mathbb{R}^n.$$

A more precise statement of the main result. We state the main result of [1] for dimensions $n \geq 3$. The statement for $n = 2$ is more technical to state here.

Theorem 1 (Buckmaster, Germain, H., Shatah '16)**.** *Let $n \geq 3$ and $\gamma > 0$ be arbitrary. Suppose that $g(t,\xi)$ is a sufficiently "nice" solution*[1] *of the* (CR) *equation on an interval $[0, M]$ (M arbitrary). Suppose we start with an NLS solution such that $a_K(0) = g_0(K)$. If L is large enough, and if $\epsilon < L^{-\gamma}$, then*

$$\left\| a_K(t) - g\Big(\frac{t}{T_R}, K\Big) \right\|_{L^2 \cap L^\infty} = o(1)_{L\to\infty}.$$

for all $0 \leq t \leq MT_R$ where $T_R = \frac{\zeta(n)}{\zeta(n-1)} \left(\frac{L^2}{\epsilon^2}\right)$.

Final remarks. The result when $n = 2$ was first proved in [2] under the restriction $\epsilon < L^{-1-\gamma}$ and where $o(1) = (\log L)^{-(1-\gamma)}$. In contrast, the $o(1)$ in the above theorem is polynomially decaying in L. We improve on this logarithmic error bound when $n = 2$ in [1], by identifying the logarithmically decaying correction term which allows to make an approximation result with polynomially decaying error term (in L).

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Onsager's Conecture and Kolmogorov's 1941 Theory

Tristan Buckmaster

(joint work with Camillo De Lellis, Philip Isett, Nader Masmoudi, László Székelyhidi Jr., Vlad Vicol)

We consider the *incompressible* Euler equations on the 3-dimensional torus:

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = 0 \\ \operatorname{div} v = 0 \end{cases}, \tag{1}$$

where here v is a vector field representing the velocity of the fluid and p is the pressure.

A fundamental feature of turbulent flow is that of dissipation of kinetic energy [22, 19, 16], where given a solution to (1), its *kinetic energy* is defined to be

$$E(t) := \frac{1}{2} \int |v(x,t)|^2 \, dx.$$

A simple calculation however yields the conservations of energy for any smooth solution of (1). This formal calculation does not however hold for distributional solutions to Euler as is demonstrated by the paradoxical solution of Scheffer (cf.

[1] (CR) is locally well-posed in appropriate spaces for any dimension, and is globally regular for $n = 2, 3, 4$.

[23, 8, 9]). In his famous note [22] on statistical hydrodynamics, Lars Onsager conjectured the following dichotomy:

Conjecture 1 (Onsager's conjecture)**.**

(a) *Any weak solution v belonging to the Hölder space C^θ for $\theta > \frac{1}{3}$ conserves the energy.*

(b) *For any $\theta < \frac{1}{3}$ there exist weak solutions $v \in C^\theta$ which do not conserve the energy.*

Part (a) of this conjecture has since been resolved: it was first considered by Eyink in [13] following Onsager's original calculations and later proven by Constantin, E and Titi in [7] (see also [12, 5]). Indeed in [7], Constantin, E and Titi proved the following stronger result:

Theorem 1. *For any $\varepsilon > 0$, every weak solution $v \in C([0,T]; L^2(\mathbb{T}^3))$ of* (1) *belonging to the space $L^3([0,T]; B_3^{\frac{1}{3}+\varepsilon,\infty}(\mathbb{T}^3))$, conserves its total kinetic energy.*

Part (b) however remains an open conjecture. The first constructions of non-conservative $\frac{1}{10} - \varepsilon$ Hölder-continuous weak solutions appeared in work of De Lellis and Székelyhidi Jr. [11], which itself was based on their earlier seminal work [10] where continuous weak solutions were constructed. In the works of Buckmaster, Isett, De Lellis and Székelyhidi Jr. [2, 17, 18, 4] a number of new ideas in order improve the Hölder exponent to $1/5 - \varepsilon$. Specifically, the following result was proved:

Theorem 2. *Assume $e\colon [0,1] \to \mathbb{R}$ is a* strictly positive *smooth function. Then there exists a continuous vector field $v \in C^{\frac{1}{5}-\varepsilon}(\mathbb{T}^3 \times [0,1])$ which solves* (1) *in the sense of distributions and is such that $E(t) = e(t)$.*

In view of Theorem 1, one could however speculate that the *threshold* for energy conservation should in fact be $L^3([0,T]; B_3^{\frac{1}{3}+\varepsilon,\infty}(\mathbb{T}^3))$. Indeed such space fits naturally in the context of Kolmogorov K41 theory [19, 21, 20, 16]. Kolmogorov K41 theory predicts that for homogeneous, isotropic turbulence, the dissipation rate is non-vanishing in the inviscid limit. In particular, let us define the *structure functions* for homogeneous, isotropic turbulence by

$$S_p(\ell) := \left\langle \left| v(x+\hat{\ell}) - v(x) \right|^p \right\rangle,$$

where $\langle \cdot \rangle$ denotes an ensemble average and $\hat{\ell}$ is a spatial vector of length ℓ. Then Kolmogorov's famous four-fifths law can be stated as

$$S_3(\ell) \sim -\frac{4}{5}\varepsilon_d \ell \ , \tag{2}$$

where here ε_d denotes the mean energy dissipation per unit mass. More generally, Kolmogorov's scaling laws can be stated as

$$S_p(\ell) = C_p \varepsilon_d^{\frac{p}{3}} \ell^{\frac{p}{3}} \ , \tag{3}$$

for any positive integer p.

A well known consequence of the above scaling law for the case $p = 2$ is the Kolmogorov spectrum, also known as the $\frac{5}{3}$ law, which postulates a scaling relation on the *energy spectrum* of a turbulent flow (cf. [16, 15]). Written in terms of Littlewood-Paley theory (cf. [6]) the $\frac{5}{3}$ law can be written as

$$E(\lambda) \sim \varepsilon^{\frac{2}{3}} \lambda^{-\frac{5}{3}} \tag{4}$$

where $E(\kappa) = \frac{1}{2} \frac{d}{d\kappa} \langle |u_{<\kappa}|^2 \rangle$.

Since we are concerned with individual realizations and not statistical averages, it is interesting to note that in the work [14], Eyink provides analytical evidence that suggests at the inviscid limit the $\frac{4}{5}$ law should hold with just local space-time averaging and angular averaging over the direction of the separation vector. This viewpoint has both numerical and experimental support [24]. We are naturally lead to the following weak version of Onsager's conjecture:

Conjecture 2 (Kolmogorov-Onsager conjecture)**.** *For any $\theta < \frac{1}{3}$, there exists weak solutions $v \in C([0,1]; L^2(\mathbb{T}^3))$ to the Euler equations* (1) *belonging to the Besov space $L^3([0,1], B_3^{\theta,\infty}(\mathbb{T}^3))$ which do not conserve their kinetic energy.*

Building on the arguments of [2], in the work [1], the author proved the existence of non-conservative $\frac{1}{5} - \varepsilon$ Hölder-continuous weak solutions that are for almost every time $\frac{1}{3} - \varepsilon$ Hölder-continuous:

Theorem 3. *There exists weak solutions $v \in C^{\frac{1}{5}-\varepsilon}(\mathbb{T}^3 \times [0,1])$ to* (1) *such that for times t outside a set of Hausdorff dimension strictly less than* 1 *we have that $v(\cdot, t)$ is Hölder $C^{\frac{1}{3}-\varepsilon}$ continuous.*

Introducing a complicated dynamical systems like argument as well as a complementary bookkeeping scheme, in [3], Buckmaster, De Lellis and Székelyhidi Jr. further pushed the ideas of [1] in order to prove the following theorem:

Theorem 4. *There exists weak solutions $v \in L^1([0,1], C^{\frac{1}{3}-\varepsilon}(\mathbb{T}^3)) \cap C(\mathbb{T}^3 \times [0,1])$ to* (1) *that do no conserve kinetic energy.*

In work in progress, Buckmaster, Nader Masmoudi and Vlad Vicol, are attempting to introduce a number of new ideas with the aim of closing the gap on Conjecture 2. In [3], sharper temporal cut-offs were introduced to successfully trade time integrability for spatial regularity. In the new work, new ideas will be introduced to trade spatial integrability for spatial regularity. The aim is to construct weak non-conservative solutions with a Kolmogorov-like spectrum. In particular, we aim at proving the following theorem:

Theorem 5. *There exists weak solutions $v \in L^\infty([0,1]; H^{\frac{1}{3}-\varepsilon}(\mathbb{T}^3))$ to* (1) *that do not conserve kinetic energy.*

Curiously, the spaces $L^3([0,1], B_3^{\theta,\infty}(\mathbb{T}^3))$ of Conjecture 2 are interpolation spaces of the spaces considered in Theorem 4 and Theorem 5

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Nonlinear echoes and Landau damping with insufficient regularity

Jacob Bedrossian

In this talk we discussed recent developments on Landau damping in the Vlasov-Poisson equations. We focused on the recent proof [1], by the speaker, that Mouhot and Villani's theorem on Landau damping near equilibrium in $\mathbb{T}_x \times \mathbb{R}_v$ [3] cannot in general be extended to finite regularity. This is demonstrated by constructing a sequence of homogeneous background distributions and arbitrarily small perturbations in H^s which deviate arbitrarily far from free transport for long times (in a sense to be made precise). The density experiences a sequence of nonlinear oscillations that damp at a rate which is arbitrarily slow compared to the predictions of the linearized Vlasov equations. The nonlinear instability is due to the repeated re-excitation of a resonance known as a plasma echo. The results hold for a specific, small background distribution, but include both electrostatic and gravitational interactions. The recent results of Nader Masmoudi, Clement Mouhot, and the speaker [2] were also briefly discussed to contrast the unconfined and confined cases. In particular, we emphasized the importance that infinite regularity plays in the confined case whereas the unconfined case in three dimensions does not require infinite regularity.

If the distribution function F is written as $F(t,x,v) = f^0(v) + h(t,x,v)$, where h is assumed to be a mean-zero fluctuation, then the Vlasov equations for h are

$$\begin{cases} \partial_t h + v \cdot \nabla_x h + E(t,x) \cdot \nabla_v \left(f^0 + h\right) = 0, \\ E(t,x) := -(\nabla_x W *_x \rho)(t,x), \\ \rho(t,x) := \int_{\mathbb{R}} h(t,x,v)dv, \\ h(t=0,x,v) = h_{\rm in}(x,v). \end{cases} \tag{1}$$

The potential W describes the mean-field interaction between particles; we will consider:

$$\widehat{W}(k) = \zeta |k|^{-2}, \quad k \neq 0, \tag{2}$$

with $\zeta \in \{-1, +1\}$; -1 corresponds to gravitational interactions in stellar mechanics and $+1$ corresponds to electrostatic interactions between electrons in a quasi-neutral plasma (after making an electrostatic approximation and neglecting collisions and ion acceleration).

Denote $\mathcal{T}_t$ as the free transport group:

$$h \circ \mathcal{T}_t = h(x + tv, v). \tag{3}$$

Then, the $h = h_{in} \circ \mathcal{T}_{-t}$ solves the free transport equation

$$\begin{cases} \partial_t h + v \cdot \nabla_x h = 0 \\ h(0,x,v) = h_{in}(x,v). \end{cases} \tag{4}$$

By direct computation, one verifies that the Fourier transform satisfies

$$\widehat{h_{in} \circ \mathcal{T}_{-t}}(t,k,\eta) = \widehat{h_{in}}(k, \eta + kt).$$

In 1946, Landau observed, that the linearized Vlasov equations (with f^0 Maxwellian) with analytic initial data predicts the following for some $\lambda, c > 0$ provided $x \in \mathbb{T}^d$:

$$\|e^{\lambda|\nabla|}\left(h(t)\circ\mathcal{T}_t - h_\infty\right)\|_{L^2} \lesssim \|e^{(\lambda+c)|\nabla|}\left(\langle v\rangle^d h_{in}\right)\|_{L^2} e^{-\frac{1}{2}ct}. \tag{5}$$

The decay of the electric field was experimentally confirmed in [4], and is now known as *Landau damping* and is somewhat analogous to scattering in dispersive equations. That all initial data which is small enough, in a suitable sense, exhibits Landau damping for $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$ was first proved by Mouhot and Villani [3], provided one takes the initial data in Gevrey-ν for some ν close to 1. Moreover, Mouhot and Villani predicted from nonlinear heuristics based on the so-called *plasma echoes* that something may go wrong due to nonlinear effects if one tries to take $\nu > 3$. The results of [3] were later extended to cover the predicted range of $\nu \in [1, 3)$ in [7]. Plasma echoes were discovered and isolated experimentally in [5]; they are a kind of resonance associated with the so-called Orr mechanism. See e.g. [6, 1] for more discussion and references.

Several works have explored the unusually stringent regularity requirement, specifically, the results [8, 9, 10, 2]. However, are all in settings that either avoid, or suppress in some way, the nonlinear echoes. In [1], we showed that in the original setting studied by Mouhot and Villani [3], small perturbations h in (1), in general, do *not* behave like the linearized Vlasov equations if the initial condition is only assumed to be small in a Sobolev space. Hence, for long times, the linearization is not valid even for arbitrarily small data and the results of [3, 2] do not extend to finite regularity results on $\mathbb{T}_x \times \mathbb{R}_v$.

Consider the following background density:

$$f^0(v) = \frac{4\pi\delta}{(1+v^2)},$$

where $0 < \delta \ll 1$ will be chosen small later. The full statement of the theorem is then given as:

Theorem 1 (Nonlinear echoes in Sobolev spaces)**.** *Let $R \geq 1$, $p \in (0,1)$ be arbitrary, and suppose $\widehat{W}(k) = \pm|k|^{-1-\gamma_0}$ with $\gamma_0 \geq 1$. There exists $\sigma_0(R) \gg R$ such that for all $\sigma \geq \sigma_0$, there is a constant $\epsilon_0(R,\sigma) \ll 1$ such that for all $\epsilon \leq \epsilon_0$ and $0 < \delta \leq \epsilon^p$, there exists a real analytic h_{in} with $f^0 + h_{in}$ strictly positive and h_{in} satisfying the quantitative bound*

$$\|\langle v\rangle h_{in}\|_{H^\sigma} \leq \epsilon \tag{6}$$

but such that at some finite time $t_\star = t_\star(\epsilon, R)$ satisfying $\epsilon t_\star \to \infty$ as $\epsilon \to 0$, the solution to (1) *satisfies the following for all $z \geq 0$:*

$$\|h(t_\star)\circ\mathcal{T}_{t_\star}\|_{H^{\sigma-R+z}} \gtrsim t_\star^z \gg \epsilon^{-z}, \tag{7a}$$

$$\|E(t_\star)\|_{L^2} \gtrsim t_\star^{R-\sigma}. \tag{7b}$$

A sketch of the proof was also briefly discussed. The proof is based on finding an approximate solution which exhibits the echo instability and then proving a kind of stability result to deduce that the true solution stays nearby. The latter step is significantly more difficult though neither are non-trivial.

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A mathematical proof of boundary layer separation

Anne-Laure Dalibard
(joint work with Nader Masmoudi)

The Prandtl equation was first derived by Ludwig Prandtl in 1904, in his presentation at the Third International Mathematics Congress in Heidelberg. The Prandtl equation describes the motion of a fluid with small viscosity in the vicinity of a solid wall, and is obtained by passing to the limit (formally) in the Navier-Stokes system as the viscosity goes to zero, after an appropriate rescaling of the normal variable. We refer the interested reader to [1, 2] for more detail. We are interested here in a stationary version of this equation in which the exterior pressure gradient has a positive sign, namely

$$\text{(P)}\qquad \begin{aligned} uu_x + vu_y - u_{yy} &= -1, \quad x>0,\ y>0,\\ u_x+v_y &= 0, \qquad x>0,\ y>0,\\ u_{|x=0}=u_0, \quad u_{|y=0} &= 0,\ \lim_{y\to\infty} u(x,y) = u_E(x), \end{aligned}$$

with $u_E(x)=\sqrt{2(x_0-x)}+U_0$, for some $x_0>0$, $U_0>0$, and $u_0\in\mathcal{C}^{2,\alpha}_b(\mathbb{R})$ is such that $u_0(0)=0$, $u_0'(0)>0$, $\lim_{y\to\infty}u_0(y)=u_E(0)>0$. We assume furthermore that u_0 is increasing in y.

Under such assumptions, it is known since the works of Oleinik (see [1, Theorem 2.1.1]) that there exists $x^*>0$ such that equation (P) has a unique smooth positive solution u in $[0,x^*)$. Additionally, $u(x,y)$ is increasing in y for all $x\in[0,x^*)$.

In the case of equation (P), i.e. with an adverse pressure gradient, it is commonly believed that $x^* < +\infty$: the solution cannot be extended beyond x^* by the means of Oleinik's theorem. In that case, using the monotony assumption on u_0, it follows that

$$\frac{\partial u}{\partial y}(x^*, 0) = 0. \tag{1}$$

In the physics literature (see for instance the seminal work of Goldstein [3], followed by the one of Stewartson [4]), this condition is used as a characterization of the "separation point". *The main goal of the present work is to rigorously prove the existence of the separation point and to have a precise quantitative description of the solution near the separation.* The first computational works on this subject go back to Goldstein [3] and Landau [5, Chapter 4, §40]. In particular, Goldstein uses an asymptotic expansion in self-similar variables to compute the profile of the singularity close to the separation point. These computations are later extended by Stewartson [4]. However, these calculations are formal; furthermore, some of the coefficients of the asymptotic expansion cannot be computed by either method. Independently, Landau proposes another characterization of the separation point, and gives an argument suggesting that $\partial_y u_{|y=0} \sim \sqrt{x^* - x}$ close to the separation point.

On the other hand, in the paper [2] Weinan E announces a result obtained in collaboration with Luis Caffarelli. This result states, under some structural assumption on the initial data, that the existence time x^* of the solutions of (P) in the sense of Oleinik is finite, and that the family $u_\mu := \frac{1}{\sqrt{\mu}} u(\mu(x^* - x), \mu^{1/4} Y)$ is compact in $\mathcal{C}(\mathbb{R}^2_+)$. Moreover, the author states two technical Lemmas playing a key role in the proof. However, to the best of our knowledge, the complete proof of this result was never published.

The goal of the present paper is to give a more quantitative version of the compactness result announced by E and Caffarelli, and to retrieve rigorously Goldstein's singularity. More specifically, we identify a class of initial data for which separation occurs, in the sense that $\lim_{x \to x^*} \partial_y u(x, 0) = 0$ for some $x^* > 0$ depending on u_0, and we compute the cancellation rate of $\partial_y u_{|y=0}$ within this class.

Let us now be more precise about our result. If u is a solution of (P), we set $\lambda(x) := \partial_y u(x, y = 0)$, $b = -2\lambda' \lambda^3$, and we define an approximate solution in the form

$$u^{\text{app}}(x, y) = \lambda(x)\ y + \frac{y^2}{2} - a_4 b \lambda^{-2} y^4 - a_7 b^2 \lambda^{-5} y^7 \text{ for } y \leq \lambda(x)^{1/3}.$$

Our assumptions on the initial data are as follows:

- (H1) Monotony: u_0 is increasing in y;
- (H2) There exists $c_0 > 0$ such that $c_0^{-1} \leq -\partial_y^4 u_0(0) \leq c_0$;
- (H3) Boundedness of the second derivative: there exists constants $C_1, C_2 > 0$ such that

$$\sup(-C_1, -C_2 y^2) \partial_{yy} u_0(y) - 1 \leq 0;$$

(H4) Small perturbation of the approximate solution: there exists a constant C_0 and a parameter $\eta > 0$ such that

$$E(u - u^{\text{app}}; x = 0) \leq C\lambda_0^\eta$$

for some $\eta > 0$, where $E(u; x)$ is some explicit weighted Sobolev norm.

Theorem 1. *Assume that the hypotheses (H1)-(H4) on the initial data u_0 are satisfied. There exists $\eta_0 > 0$ such that if $\eta > \eta_0$, there exists $\delta_\eta > 0$ such that if $\lambda_0 < \delta_\eta$, the solution of* (P) *with $u_{|x=0} = u_0$ has a separation point at $x^* = O(\lambda_0^2)$. Moreover, there exists a constant $C > 0$ such that*

$$\lambda(x) \sim C\sqrt{x^* - x}.$$

The proof of Theorem 1 relies on two main ingredients. On the one hand, one must build an approximate solution with good stability properties. On the other hand, we prove error estimates, which rely very strongly on the structure of the equation.

Concerning the first point, the choice of the approximate solution is inspired from arguments developed for the study of singularities and blow-up phenomena in the nonlinear Schrdinger equation. These arguments are explained formally in the book by Catherine and Pierre-Louis Sulem [6], and were made rigorous by Franck Merle and Pierre Raphaël for (see for instance [7]). The idea is to use the scaling invariance of the equation to perform a change of variables using a parameter depending intrinsically on the solution of the equation (in our case, the parameter is $\lambda(x)$). We then construct approximate solutions thanks to a Taylor expansion, and we choose the approximate solution with the least possible growth at infinity; the latter condition is crucial in order to obtain stability.

In a second step, we use the transport-diffusion nature of the equation to prove the stability of the approximate solutions constructed in the first step. We emphasize that the energy estimates derived here are new, to our knowledge, and rely strongly on the structure of the equation. In order to control some nonlinear terms, we also need L^∞ estimates on the solution and its derivatives, which rely on a careful use of the maximum principle. The error estimates obtained in this way dictate the asymptotic law of the "modulation rate", i.e. λ_x.

Eventually, we close the estimates thanks to a bootstrap argument, and translate the stability result back in the original variables. The details of the proof will be given in [8].

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Invariant measures and long time dynamics for periodic NLS

Andrea R. Nahmod

The purpose of this talk is to describe recent work in two directions. One direction concerns almost sure global well posedness for periodic NLS and a new probabilistic propagation of regularity result (joint with G. Staffilani). This work is intimately connected to Bourgain's approach and work in the mid 90's to almost sure global well-posedness – for dispersive PDE – via the existence of an associated invariant Gibbs measure ("stationary equilibrium state"). The other direction aims at contributing to our understanding of the so called transfer of energy/energy cascades phenomena for periodic NLS. We describe recent work (joint with Z. Hani, J. Mattingly, L. Rey-Bellet and G. Staffilani) where we construct a unique invariant ergodic non-equilibrium measure associated to (a finite subset of) the resonant NLS ("stationary non-equilibrium state").

We first review the main ideas behind Bourgain's work[1] to invariant Gibbs measures and almost sure global well-posedness for Hamiltonian dispersive PDE. We then describe our probabilistic propagation regularity result which allow us to close an important gap between the deterministic global well-posedness (gwp) theory and the a.s gwp one proved by Bourgain for the defocusing cubic NLS on $\mathbb{T}^2$ and for the focusing quintic NLS on $\mathbb{T}^1$. More precisely, we prove that the cubic defocusing NLS is a.s gwp in $H^s(\mathbb{T}^2)$, $s > 0$ and that the quintic focusing NLS is a.s gwp in $H^s(\mathbb{T})$, $s > \frac{1}{2}$. For example in 2D, deterministic methods yield local well posedness for $s > 0$ (Bourgain) and gwp for $s > 2/3$ (De Silva-Pavlovic-Staffilani-Tzirakis) via the so called I-method of *almost conservation laws*. Data randomization and the invariance of the Gibbs measure yield a.s. gwp in $H^{-\epsilon}$ (Bourgain). Our result thus fill the gap a.s. for $0 < s \leq 2/3$. Our theorem is not trivial since any $\Sigma \subset H^s$, $s > 0$, is such that for the Gibbs measure μ one has $\mu(\Sigma) = 0$.

For nonlinear dispersive and wave equations, proving the existence of large data global in time flows, at a critical or supercritical regularity level, is a challenging question which is not made any easier by assuming higher regularity of the initial data. To prove global large data well-posedness results one has to start with data at the regularity level of some conserved quantity such as, for example, the mass (L^2) or the Hamiltonian (H^1). It is only after one has proved such global

[1] After the works of Lebowitz-Rose-Speer and of Zhidkov for Hamiltonian PDE and of Glimm-Jaffe and others for the ϕ^4 model.

result, that smooth global solutions can be obtained by a standard preservation of regularity argument based on differentiation of the equation. This is a purely deterministic approach. The procedure we implement in our proof is not based on differentiation of the equation as in the deterministic preservation of regularity argument. Our key idea instead is to suitably decompose the data into a term that is close to the support of the invariant measure in the rougher topology, and a smoother remainder term to which now deterministic arguments can be applied. Then, a non-deterministic perturbation argument is used to conclude. The argument is rather general and we expect it is applicable to other problems for which an almost sure global well-posedness is proved using an invariant Gibbs measure (or an almost invariant measure weighted Wiener one).

On $\mathbb{R}^d$ scattering/asymptotic stability results, when available, tell us that there cannot be any n forward / backward cascades. In other words, the energy (kinetic or mass) while remaining conserved – does not move its concentration zones from low to high frequencies or vice versa. On compact domains, asymptotic stability results around equilibrium solutions (e.g. zero solution) are lost and out of equilibrium dynamics are expected. The question is how to analytically describe this expected out-of-equilibrium behavior. Bourgain proposed to study the growth of higher Sobolev norms since its growth gives us a quantitative estimate for how much of the support of $|\widehat{u}|^2$ has transferred from the low to the high frequencies while maintaining constant mass and energy (forward cascade). Bourgain's unbounded orbits/infinite cascade conjecture asks whether there exist global solutions to the cubic NLS whose $H^s(\mathbb{T}^d)$ norm (some $s \gg 1$) grows indefinitely in time: $\limsup_{t\to\infty} \|u\|_{H^s} = +\infty$? After work by Bourgain ('95–'97) and by Kuksin ('97), further progress was made by by Colliander-Keel-Staffilani-Takaoka-Tao [CKSTT], Hani, Gérard-Grellier and Guardia-Kaloshin ('10–'12). The conjecture however remains a very difficult question. An intermediate problem between: 1) The existence of (equilibrium) invariant Gibbs measures and 2) Bourgain's unbounded orbits conjecture/understanding of out-of-equilibrium dynamics for NLS is the study of the existence and uniqueness of non-equilibrium invariant measures. The latter has an interest in its own right for example, in connection with the theory of weak/wave turbulence. But even for stochastically forced systems, proving the existence and uniqueness of non-equilibrium invariant measures is very hard in the context of PDEs. On the other hand, we have recent developments in understanding analogous questions for some ODE systems modeling *heat transfer* in a chain of oscillators from the works of Eckmann, Pillet and Rey-Bellet ('99), Rey-Bellet and Thomas ('00–02') up to more recent progress by Hairer and Mattingly ('07) for a finite collection of anharmonic oscillators with nearest neighbor couplings (classical Hamiltonian system) put into contact with two heat baths at different temperatures[2]. We try to use those developments in order to shed some light on the non-equilibrium dynamics for (resonant) NLS. Our point of departure is the reduced *toy model* first derived by [CKSTT] where interactions in its Hamiltonian H depend not just on relative distance like in anharmonic oscillator

[2]The interaction with heat baths is modeled by standard Langevin dynamics.

but also on the momenta of each particle and that of its neighbors. We attach the first and last modes c_1 and c_n to two heat baths at temperatures T_1 and T_n respectively.[3] If $T_1 = T_n = T$ then we are at equilibrium and we can prove that $\exp(-H/2T)\, dc\, d\bar{c}$ is an invariant Gibbs measure. Our interest then is in $T_1 < T_n$. The question becomes: does there exist a unique smooth ergodic nonequilibrium invariant measure? One expects an initial distribution of system to converge to a (stationary) nonequilibrium state in which energy/matter is flowing. In joint work with Z. Hani, J. Mattingly, Luc Rey-Bellet and G. Staffilani, we consider the case of $n = 3$ and study the existence of a unique ergodic non-equilibrium invariant measure with estimates on the rate of convergence. The heart of the matter lies in proving the existence part. To that effect we construct a continuous and piecewise C^2 Lyapunov function V with compact level sets, that penalizes the region where the second mode is small as well as regions of high energies. Such construction gives an upper bound on the hitting time of the good region G (compact set) where the dynamics spends most of time. The natural candidate is to use a coercive conserved quantity of the original Hamiltonian system such as $V = e^M$, where M is the "energy". Such function however does not work in the whole space: we need to chop our phase space in several regions and solve suitable Poisson equations for V with suitable boundary values satisfying certain 'convexity' conditions (cf. Herzog-Mattingly). A delicate study of the behavior of the phases is fundamental. Once existence is established, uniqueness and ergodicity of the invariant measure follow from a controllability lemma for the deterministic system showing one can access any region of phase space plus Stroock-Varadhan theorem.

On the long-term dynamics of water wave models

Alexandru D. Ionescu
(joint work with Y. Deng, B. Pausader, F. Pusateri)

The evolution of an inviscid perfect fluid that occupies a domain $\Omega_t \subset \mathbb{R}^n$, for $n \geq 2$, at time $t \in \mathbb{R}$, is described by the free boundary incompressible Euler equations. If v and p denote the velocity and the pressure of the fluid (with constant density equal to 1) at time t and position $x \in \Omega_t$, these equations are

$$(1) \qquad (\partial_t + v \cdot \nabla)v = -\nabla p - g e_n, \qquad \nabla \cdot v = 0, \qquad x \in \Omega_t,$$

where g is the gravitational constant. The first equation in (1) is the conservation of momentum equation, while the second is the incompressibility condition. The free surface $S_t := \partial\Omega_t$ moves with the normal component of the velocity according to the kinematic boundary condition

$$(2) \qquad \partial_t + v \cdot \nabla \text{ is tangent to } \bigcup\nolimits_t S_t \subset \mathbb{R}^{n+1}_{x,t}.$$

The pressure on the interface is given by

$$(3) \qquad p(x,t) = \sigma\kappa(x,t), \qquad x \in S_t,$$

[3] Mechanism to stochastically add and dissipate energy from the system in a controlled way.

where κ is the mean-curvature of S_t and $\sigma \geq 0$ is the surface tension coefficient. At liquid-air interfaces, the surface tension force results from the greater attraction of water molecules to each other than to the molecules in the air.

In the case of irrotational flows, $\operatorname{curl} v = 0$, one can reduce (1)-(3) to a system on the boundary. Indeed, assume also that $\Omega_t \subset \mathbb{R}^n$ is the region below the graph of a function $h\colon \mathbb{R}^{n-1}_x \times I_t \to \mathbb{R}$, that is

$$\Omega_t = \{(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R} \, : y \leq h(x,t)\} \quad \text{and} \quad S_t = \{(x,y) : y = h(x,t)\}.$$

Let Φ denote the velocity potential, $\nabla_{x,y}\Phi(x,y,t) = v(x,y,t)$, for $(x,y) \in \Omega_t$. If $\phi(x,t) := \Phi(x,h(x,t),t)$ is the restriction of Φ to the boundary S_t, the equations of motion reduce to the following system for the unknowns $h,\phi\colon \mathbb{R}^{n-1}_x \times I_t \to \mathbb{R}$:

$$(4) \quad \begin{cases} \partial_t h = G(h)\phi, \\ \partial_t \phi = -gh + \sigma \operatorname{div}\Big[\dfrac{\nabla h}{(1+|\nabla h|^2)^{1/2}}\Big] - \dfrac{1}{2}|\nabla \phi|^2 + \dfrac{(G(h)\phi + \nabla h \cdot \nabla \phi)^2}{2(1+|\nabla h|^2)}. \end{cases}$$

Here

$$G(h) := \sqrt{1+|\nabla h|^2}\mathcal{N}(h),$$

and $\mathcal{N}(h)$ is the Dirichlet-Neumann map associated to the domain Ω_t. Roughly speaking, one can think of $G(h)$ as a first order, non-local, linear operator that depends nonlinearly on the domain.

The main theorem we proved in [3, 4] concerns the gravity-capillary water waves system, which corresponds to $g > 0, \sigma > 0$, in dimension $n = 3$. In this case h and ϕ are real-valued functions defined on $\mathbb{R}^2 \times I$.

To state our main theorem we need some more notation. The rotation vector-field $\Omega := x_1\partial_{x_2} - x_2\partial_{x_1}$ commutes with the linearized system. For $N \geq 0$ let H^N denote the standard Sobolev spaces on $\mathbb{R}^2$. More generally, for $N, N' \geq 0$ and $b \in [-1/2, 1/2]$, $b \leq N$, we define the norms

$$(5) \qquad \|f\|_{H^{N',N}_\Omega} := \sum_{j \leq N'} \|\Omega^j f\|_{H^N}, \qquad \|f\|_{\dot{H}^{N,b}} := \big\|(|\nabla|^N + |\nabla|^b) f\big\|_{L^2}.$$

Theorem 1 (Global Regularity)**.** *Assume that $g, \sigma > 0$, $\delta > 0$ is sufficiently small, and N_0, N_1, N_3, N_4 are sufficiently large (for example $\delta = 1/2000$, $N_0 := 4170$, $N_1 := 2070$, $N_3 := 30$, $N_4 := 70$). Assume that the data (h_0, ϕ_0) satisfies*

$$(6) \qquad \begin{aligned} &\|\mathcal{U}_0\|_{H^{N_0} \cap H^{N_1,N_3}_\Omega} + \sup_{2m+|\alpha| \leq N_1+N_4} \|(1+|x|)^{1-50\delta} D^\alpha \Omega^m \mathcal{U}_0\|_{L^2} = \varepsilon_0 \leq \overline{\varepsilon_0}, \\ &\mathcal{U}_0 := (g - \sigma\Delta)^{1/2} h_0 + i|\nabla|^{1/2}\phi_0, \end{aligned}$$

where $\overline{\varepsilon_0}$ is a sufficiently small constant. Then, there is a unique global solution $(h,\phi) \in C\big([0,\infty) : H^{N_0+1} \times \dot{H}^{N_0+1/2,1/2}\big)$ of the system (4), with $(h(0),\phi(0)) = (h_0,\phi_0)$. In addition

$$(7) \qquad (1+t)^{-\delta^2} \|\mathcal{U}(t)\|_{H^{N_0} \cap H^{N_1,N_3}_\Omega} \lesssim \varepsilon_0, \qquad (1+t)^{5/6-3\delta^2} \|\mathcal{U}(t)\|_{L^\infty} \lesssim \varepsilon_0,$$

for any $t \in [0,\infty)$, where $\mathcal{U} := (g - \sigma\Delta)^{1/2} h + i|\nabla|^{1/2}\phi$.

In other words, small, smooth, localized, and irrotational data lead to global solutions of the gravity-capillary water wave model.

Superficially, this theorem could look similar to other global regularity results for water waves, such as those of [9] and [10]. However, the situation we consider in [3, 4] is substantially more difficult, due to the combination of the following factors:

- Strictly less than $|t|^{-1}$ pointwise decay of solutions. In our case, the linear dispersion relation is $\Lambda(\xi) = \sqrt{g|\xi| + \sigma|\xi|^3}$ and the best possible pointwise decay, even for solutions of the linearized equation corresponding to Schwartz initial data, is $|t|^{-5/6}$.
- Large set of time resonances. In certain cases one can overcome the slow pointwise decay using the method of normal forms of Shatah. The critical ingredient needed is the absence of time resonances (or at least a suitable "null structure" of the quadratic nonlinearity matching the set of time resonances). Our system, however, has a full (codimension 1) set of time resonances.

We remark that all the previous work on long term solutions of water waves models was under the assumption that either $g = 0$ or $\sigma = 0$. This is not coincidental: in these cases the combination of slow decay and full set of time resonances described above was not present. More precisely, in all the previous global results in 3 dimensions, [5, 13, 6] it was possible to prove $1/t$ pointwise decay of the nonlinear solutions and combine this with high order energy estimates with slow growth. On the other hand, in all the two-dimensional models [12, 9, 10, 1, 2, 7, 8, 11] there were no significant time resonances for the quadratic terms, and normal form analysis could be performed.

To address these issues, in these papers we use a combination of improved energy estimates and Fourier analysis. The main components of our analysis are:

(1) The energy estimates, which are used to control high Sobolev norms and weighted norms (corresponding to the rotation vector-field). They rely on several new ingredients, most importantly on a *strongly semilinear structure* of the space-time integrals that control the increment of energy, and on a *restricted nondegeneracy condition* of the time resonant hypersurfaces. The strongly semilinear structure is due to an algebraic correlation between the size of the multipliers of the space-time integrals and the size of the modulation, and is related to the Hamiltonian structure of the original system.

(2) The dispersive estimates, which lead to decay and rely on a partial bootstrap argument in a suitable *Z norm*. We analyze carefully the Duhamel formula, in particular the quadratic interactions related to the slowly decaying frequencies and to the *set of space-time resonances*. The choice of the Z norm in this argument is very important; we use an atomic norm, based on a space-frequency decomposition of the profile of the solution, which depends in a significant way on the location and the shape of the space-time resonant set, thus on the quadratic part of the nonlinearity.

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Conserved energies in completely integrable PDEs

Daniel Tataru

(joint work with Herbert Koch)

We first consider the (de)focusing cubic Nonlinear Schrödinger equation (NLS)

$$iu_t + u_{xx} \pm 2u|u|^2 = 0, \qquad u(0) = u_0,$$

and the complex (de)focusing modified Korteweg-de Vries equation (mKdV)

$$u_t + u_{xxx} \pm 2(|u|^2 u)_x = 0, \qquad u(0) = u_0,$$

with real or complex solutions in one space dimension on the real line.

These are part of an infinite family of commuting Hamiltonian flows, where each of the Hamiltonians can be viewed as conservation laws for each of the flows. The symplectic form is

$$\omega(u, v) = \Im \int u\bar{v}\, dx$$

and first several Hamiltonians are as follows:

$$
\begin{aligned}
H_0 &= \int |u|^2 \, dx, \\
H_1 &= \frac{1}{i} \int u \partial_x \bar{u} \, dx, \\
H_2 &= \int |u_x|^2 + |u|^4 \, dx, \\
H_3 &= i \int u_x \partial_x \overline{u}_x + \frac{3}{2} |u|^2 u \partial_x \bar{u} \, dx, \\
H_4 &= \int |u_{xx}|^2 + 2||u|_x^2|^2 + u^2 (\bar{u}_x)^2 + (\bar{u}_x)^2 u^2 + \frac{3}{2} |u|^6 \, dx.
\end{aligned}
$$

Both of these equations have $\dot{H}^{-\frac{1}{2}}$ as a scale invariant critical Sobolev space. On the other hand the (current) Sobolev local well-posedness threshold is $s \geq 0$ for NLS, respectively $s \geq \frac{1}{4}$ for mKdV. Then the following questions are natural:

- Is NLS locally well-posed for $-\frac{1}{2} < s < 0$? Is MKdV locally well-posed for $-\frac{1}{2} < s < \frac{1}{4}$?
- Assuming the data is in H^s for $s > -\frac{1}{2}$, are the H^s norms of the solutions globally bounded in H^s?

In this work we address the second question above, leaving the first one open. Better than proving uniform estimates, we are in fact constructing new conservation laws which are equivalent to the H^s norms of the solutions.

Before stating our main result some preliminary explanations are needed. Our construction of the conserved energies is based on the scattering transform associated to these problems. This is defined in terms of the spectral problem for the corresponding Lax operator

$$
L = i \begin{pmatrix} \partial_x & -u \\ \bar{u} & -\partial_x \end{pmatrix}
$$

Here one looks for Jost solutions for the ode

$$
\begin{cases} \frac{d\psi_1}{dx} = -i\xi\psi_1 + u\psi_2 \\ \frac{d\psi_2}{dx} = i\xi\psi_2 + \bar{u}\psi_1 \end{cases}
$$

which have the form

$$
\begin{aligned}
\psi_l(\xi, x, t) &= \begin{pmatrix} e^{-i\xi x} \\ 0 \end{pmatrix} + o(1) \quad \text{ as } x \to -\infty, \\
\psi_l(\xi, x, t) &= \begin{pmatrix} T^{-1}(\xi) e^{-i\xi x} \\ R(\xi) T^{-1}(\xi) e^{i\xi x} \end{pmatrix} + o(1) \quad \text{ as } x \to \infty,
\end{aligned}
$$

Here $R(\xi)$ is called the reflection coefficient, and $T(\xi)$ is the transmission coefficient. Their time evolution along the NLS flow is given by $\dot{R} = i\xi^2 R$, $\dot{T} = 0$. The map $u \to R$ can be roughly viewed as a nonlinear Fourier transform, which conjugates the NLS/mKdV flows to the (Fourier transform of) the corresponding linear flows.

The functions R and T are related by $|R|^2 + |T|^2 = 1$ in the defocusing case, respectively $|T|^2 - |R|^2 = 1$ in the focusing case.

Our conserved energies are defined in terms of the transmission coefficient T, precisely $|T|$. This does not carry the full scattering information, but is conserved along the flow. Further, unlike R, it admits a holomorphic (meromorphic in the focusing case) extension to the upper half-space, defined via the same ode's as above. This is important as the reflexion coefficient R on the real line is not well defined for merely L^2 type data, but the transmission coefficient T in the upper half-space is defined. Of course, any function of $|T|$ is conserved long the flows; the challenge is to relate well chosen functions of $|T|$ to the Sobolev norms of u.

We further note that the poles of T in the upper half-space are associated to NLS/mKdV solitons, and that the poles together with their residue information are also included in the scattering transform.

Now we are ready to state our main result:

Theorem 1. *For each $s > -\frac{1}{2}$ and both for the focusing and defocusing case the energy functionals E_s are globally defined*

$$E_s : H^s \to \mathbb{R}$$

with the following properties:

1. *E_s is conserved along the NLS and mKdV flow.*
2. *If[1] $\|u\|_{L^2+DU^2} \leq 1$ then*
$$\big|E_s(u) - \|u\|^2_{H^s}\big| \lesssim \|u\|^2_{L^2+DU^2}\|u\|^2_{H^s}.$$
3. *The map*
$$H^\sigma \times (-\tfrac{1}{2}, \sigma] \ni (u,s) \to E_s(u)$$
is analytic in $u \in H^\sigma$ in the defocusing case. In the focusing case it is analytic provided $\frac{i}{2}$ is not an eigenvalue for L, and it is continuous in $u \in H^\sigma$ in general. It is also continuous in s, and analytic in s for $s < \sigma$.

Also part of our result are the following trace formulas:

Theorem 2. *For all $u \in H^s$ the limit of $\mp \log |T|$ (signs correspond to the defocusing/focusing case) exists as a positive measure, and the following trace formulas hold with absolute convergence in all sums and integrals. In the defocusing case we have*

$$\begin{aligned} E_s &= \int (1+\xi^2)^s(-\Re \ln T(\xi/2))\, d\xi \\ &= 4\sin(\pi s)\int_1^\infty (\tau^2-1)^s\Big[\Re \ln T(i\tau/2) + \frac{1}{2\pi}\sum_{j=0}^N (-1)^j H_{2j}\tau^{-2j-1}\Big]\, d\tau \\ &\quad + \sum_{j=0}^N \binom{s}{j} H_{2j} \end{aligned}$$

[1]Here the space $L^2 + DU^2$ is a convenient proxy for the obvious but ill-behaved choice $H^{-\frac{1}{2}}$.

and in the focusing case

$$E_s = \int (1+\xi^2)^s \Re \ln T(\xi/2)\, d\xi + 2\sum_k m_k \Xi(2z_k)$$
$$= 4\sin(\pi s)\int_1^\infty (\tau^2-1)^s \Big[-\Re \ln T(i\tau/2) + \frac{1}{2\pi}\sum_{j=0}^{N}(-1)^j H_{2j}\tau^{-2j-1}\Big]\, d\tau$$
$$+ \sum_{j=0}^{N}\binom{s}{j} H_{2j}$$

Here $2z_k$ *and* m_k *denote the poles/multiplicities of* T *in the upper half-space and*

$$\Xi_s(z) = \Im \int_0^z (1+\zeta^2)^s\, d\zeta.$$

As a last remark, our ideas easily carry over to the real KdV problem on the real line; full results are included in the paper. Independent work of Killip-Visan-Zhang is also devoted to the same question for the KdV problem.

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Critical Half-Wave Problems

Enno Lenzmann

We consider two critical evolution problems of half-wave type. As a first example, we discuss the energy-critical half-wave maps equation given by

$$\partial_t u = u \wedge |\nabla| u, \quad u\colon [0,T)\times\mathbb{R} \to \mathbb{S}^2. \tag{1}$$

Here $\wedge$ denotes the cross product in $\mathbb{R}^3$, whereas $|\nabla| \equiv \sqrt{-\Delta}$ stands for the square root of the Laplacian on $\mathbb{R}$; and $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ denotes the unit sphere in $\mathbb{R}^3$. Apart from its criticality and its physical relevance (e. g. integrable spin chains), an interesting feature of (1) is that the problem has traveling solitary wave solutions of the form $u(t,x) = u_v(x - vt)$, where $v \in \mathbb{R}$ denotes a velocity parameter and the profile $u_v : \mathbb{R} \to \mathbb{S}^2$ satisfies

$$-v\partial_x u_v = u_v \wedge |\nabla| u_v. \tag{2}$$

As our main result in [1], we classify all solutions $u_v \in \dot{H}^{\frac{1}{2}}$ by showing that $u_v \equiv \text{const}$ if $|v| \geq 1$ holds, whereas u_v is explicitly given by suitable finite non-trivial Blaschke products when $|v| < 1$. Moreover, we prove that the energy of the profiles u_v, with $|v| < 1$, is given by the formula

$$E(u_v) = \frac{1}{2} \int_{\mathbb{R}} \left| |\nabla|^{\frac{1}{2}} u_v \right|^2 dx = (1 - v^2)\pi d, \tag{3}$$

where $d \in \mathbb{N}$ is an integer that corresponds to a topological degree of the maps $u_v : \mathbb{R} \to \mathbb{S}^2$. The proof of our main results uses minimal surface theory and complex analysis.

As a second critical half-wave problem, we discuss the cubic half-wave equation on the real line given by

$$i\partial_t \psi = |\nabla|\psi - |\psi|^2 \psi, \quad \psi : [0,T) \times \mathbb{R} \to \mathbb{C}. \tag{4}$$

In [2], we construct $H^{\frac{1}{2}}$-small global-in-time solutions that exhibit transient turbulent behavior in H^s with $s > \frac{1}{2}$. The proof is a tour-de-force argument using an approximate two-soliton ansatz and closeness of this multi-soliton ansatz (in some certain sense) to the dynamics of the cubic Szegö equation on the real line.

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Asymptotic behavior of the nonlinear Schrödinger equation with harmonic trapping

Laurent Thomann

(joint work with Pierre Germain and Zaher Hani)

We consider the cubic nonlinear Schrödinger equation with harmonic trapping on $\mathbb{R}^D$ $(1 \leq D \leq 5)$. In the case when all but one directions are trapped (a.k.a "cigar-shaped" trap), following the approach of [4], we prove modified scattering and construct modified wave operators for small initial and final data respectively. The asymptotic behavior turns out to be a rather vigorous departure from linear scattering and is dictated by the resonant system of the NLS equation with full trapping on $\mathbb{R}^{D-1}$. In the physical dimension $D = 3$, this system turns out to be exactly the (CR) equation derived and studied in [1, 2, 3]. The special dynamics of the latter equation, combined with the above modified scattering results, allow to justify and extend some physical approximations in the theory of Bose-Einstein condensates in cigar-shaped traps.

Our aim is to study the long-time behavior of the cubic nonlinear Schrödinger equation with harmonic trapping given by

$$(1) \qquad (i\partial_t - \Delta_{\mathbb{R}^D} + \sum_{j=1}^{D} \omega_j x_j^2)U = \kappa_0 |U|^2 U, \qquad (x_1, \dots, x_d) \in \mathbb{R}^D,$$

with a particular emphasis on the anisotropic limit $\omega_1 = 0 < \omega_2 = \ldots = \omega_D$. Here ω_j signifies the frequency of the harmonic trapping in the $j-$th direction and $\kappa_0 \neq 0$.

The motivation for this study is two-fold: On the one hand, we aim at justifying some approximations done in the physics literature that allow reducing the dynamics of (1) in the highly anisotropic setting (a.k.a. cigar-shaped trap) to that of the homogeneous (i.e. with no trapping) 1D cubic NLS equation. Such approximations, often referred to as the "quasi-1D dynamics" [6], allow access to the complete integrability theory of the 1D cubic NLS equation along with its plethora of special solutions that give theoretical explanations of fundamental phenomena in Bose-Einstein condensates. On the other hand, from a purely mathematical point of view, the analysis falls under the recent progress and interest in understanding the asymptotic behavior of nonlinear dispersive equations in the presence of a confinement. Such a confinement can come from the compactness (or partial compactness) of the domain or via a trapping potential. In either case, this leads to the complete or partial loss of dispersive decay of linear solutions, and consequently complicating and diversifying the picture of long-time dynamics. In this line, using tools developed for the study of long-time dynamics of nonlinear Schrödinger equations on product spaces, we will be able to describe the asymptotic dynamics and show that they exhibit highly nonlinear behavior in striking contrast to linear scattering. As a consequence of this description, we get the general extension of the "quasi-1D approximation" mentioned above to cases when higher and multiple energy levels of the harmonic trap are excited.

A Bose-Einstein condensate (BEC for short) is an aggregate of matter (Bosons) which appears at very low temperature and which is due to the fact that all particles are in the same quantum state. Their existence was predicted by Bose in 1924 for photons and by Einstein in 1925 for atoms, and they were experimentally observed in 1995 by W. Ketterle, A. Cornell and C. Wieman who were awarded a Nobel Prize shortly after, in 2001, for this achievement. This observation was followed by a burst of activity in the theoretical and experimental study of BEC which constitutes a rare manifestation of a quantum phenomenon which shows through at a macroscopic level. For an nice introduction to this topic we refer to the book [6] and to [7].

In the physical space $\mathbb{R}^3$, BEC can be realized by trapping particles using a magnetic trap which is modelled in the mean-field theory by the harmonic potential term in (1). The wave function $U(t, x, y_1, y_2)$ of the particles in (1) (with $D = 3$) can be interpreted as the probability density of finding particles at point $(x, y_1, y_2) \in \mathbb{R}^3$ and time $t \in \mathbb{R}$. The sign $\kappa_0 = +1$ or -1 depends on whether the Boson interaction is attractive (focusing case) or repulsive (defocusing).

In the case when $\omega_1 \ll \omega_2 = \omega_3$, the harmonic trap is often described as "cigar-shaped", and we will be interested in this case. This regime is of great importance from the physical point of view as it allows for a "dimensional reduction" in which the condensate is described by better-understood lower-dimensional dynamics. More precisely, a naturally adopted approximation of (1) is obtained by going to the anisotropic limit and setting $\omega_1 = 0$ (which is justified for x not too large) and non-dimensionalizing $\omega_2 = \omega_3 = 1$. Then, the resulting equation is

$$(i\partial_t - \Delta_{\mathbb{R}^3} + y_1^2 + y_2^2)U = \kappa_0 |U|^2 U, \qquad (x, y_1, y_2) \in \mathbb{R}^3. \tag{2}$$

In this context, (see for instance [8] or [6, paragraph 1.3.2]) physicists often adopt an Ansatz of the form

$$U(t,x,y) \sim \psi(t,x) e^{2it} e^{-|y|^2/2}; \qquad y = (y_1, y_2), \tag{3}$$

which leads them through a multiple time-scale expansion to the 1D-dynamics obeyed by $\psi(t,x)$. This dynamics is given by none other than the one dimensional Schrödinger equation

$$\begin{cases} (i\partial_t - \partial_x^2)\psi = \kappa_0 \lambda_0 |\psi|^2 \psi, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ \qquad \psi(0,x) = \varphi(x). \end{cases} \tag{4}$$

This equation is obtained by projecting the nonlinear term in the Ansatz equation on the groundstate $g_0(y) = e^{-|y|^2/2}$ of the harmonic oscillator $-\Delta_{\mathbb{R}^2} + |y|^2$, thus $\lambda_0 = \|g_0\|^4_{L^4(\mathbb{R}^2)} / \|g_0\|^2_{L^2(\mathbb{R}^2)} = 1/2$.

One consequence of our work is a justification of the approximation (3) for large times, as well as the correct extension of that approximation when higher and/or multiple energy levels of the quantum harmonic oscillator are excited. We give the relevant result concerning the approximation (3) and refer to the next section for more general and precise results. Denote by $\mathcal{S}(\mathbb{R})$ the set of the Schwartz functions, then

Theorem 1. *Let $\varphi \in \mathcal{S}(\mathbb{R})$ be small enough, and let ψ be the solution of (4). Then there exists a solution $U \in \mathcal{C}\big([0,+\infty); L^2(\mathbb{R} \times \mathbb{R}^2)\big)$ of (2) such that*

$$\big\| U(t,x,y) - \psi(t,x) e^{2it} e^{-\frac{1}{2}|y|^2} \big\|_{L^2(\mathbb{R}\times\mathbb{R}^2)} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow +\infty.$$

Moreover, the function U can be chosen to be axisymmetric: $U(t,x,y) = \widetilde{U}(t,x,|y|)$ for some $\widetilde{U}$.

This shows that the 1D dynamics of (4) can be embedded in the 3D dynamics of (2), a reduction, known as *quasi 1D dynamics*, which is at the basis of the theoretical explanation of many fundamental phenomena in Bose-Einstein condensates. Physicists arrive at it using some multiple time-scale approximations, and use it afterwards to transfer information from the well-understood and completely integrable dynamics of (4) to that of (1).

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Ancient Solutions of geometric flows

Panagiota Daskalopoulos

We study ancient of eternal solutions to geometric flows such as the Mean Curvature flow, the Ricci flow or the Yamabe flow. We establish uniqueness results about these solutions under certain natural geometric assumptions such as curvature bounds or non-collapsing conditions. We also construct new ancient solutions from the gluing of one or more solitons. We present recent work and open research directions.

Type-II singularities of two-convex immersed mean curvature flow

Theodora Bourni
(joint work with Mat Langford)

The main aim of this work is to show that any strictly mean convex translator of dimension $n \geq 3$ which admits a cylindrical estimate and a corresponding gradient estimate is rotationally symmetric. As a consequence, we deduce that any translating solution of the mean curvature flow which arises as a blow-up limit of a two-convex mean curvature flow of compact immersed hypersurfaces of dimension $n \geq 3$ is rotationally symmetric. The proof is rather robust, and applies to a more general class of translator equations. As a particular application, we prove an analogous result for a class of flows of embedded hypersurfaces which includes the flow of two-convex hypersurfaces by the two-harmonic mean curvature. In what follows we give a more precise description of the presented results.

We are interested in hypersurfaces $X\colon M^n \to \mathbb{R}^{n+1}$ satisfying the *translator* equation

$$\vec{H} = T^{\perp} \tag{T}$$

for some constant vector $T \in \mathbb{R}^{n+1}$, where, given a local choice of unit normal field ν, $\vec{H} = -H\nu$ is the mean curvature vector of the immersion with respect to the choice of mean curvature $H = \operatorname{div}\nu$, and $\perp$ denotes the projection onto the normal bundle. We call such immersions *translators*. Up to a time-dependent tangential reparametrization, the family $\{X(\cdot,t)\}_{t\in\mathbb{R}}$ of immersions $X(\cdot,t)\colon M^n \to \mathbb{R}^{n+1}$ defined by $X(x,t) := X(x) + tT$ satisfies the mean curvature flow

$$\partial_t X(\cdot,t) = \vec{H}(\cdot,t), \tag{MCF}$$

where $\vec{H}(\cdot,t)$ is the mean curvature vector of $X(\cdot,t)$. We therefore also refer to solutions of (T) as *translating solutions of the mean curvature flow*. It is well-known that translating solutions arise as blow-up limits of the mean curvature flow about type-II singularities [7, 11].

Probably the most well-known translator is the Grim Reaper curve Γ, which is the graph of the function $x \mapsto -\log\cos x$, $x \in (-\pi/2, \pi/2)$. In dimensions $n \geq 2$, there exists a strictly convex, rotationally symmetric translator asymptotic to a paraboloid, which is commonly referred to as the "bowl" [2, 6]. In a remarkable study of convex ancient graphical solutions of the mean curvature flow, X.-J. Wang showed that any strictly convex, entire translator in dimension two is rotationally symmetric, and hence the bowl [15]. Moreover, in every dimension $n \geq 3$, he constructed strictly convex, entire examples without rotational symmetry.

In the setting of two-convex (that is, $\kappa_1 + \kappa_2 > 0$, where $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n$ denote the principal curvatures) mean curvature flow in dimensions $n \geq 3$, the far-reaching theory of Huisken and Sinestrari [12, 11, 13] shows that regions of high curvature are either uniformly convex and cover a whole connected component of the surface, or else they contain regions which are very close, up to rescaling, to cylindrical segments $[-L, L] \times S^{n-1}$. This suggests that the translating blow-up limits which arise at type-II singularities might be rotationally symmetric. We note that this is true (in dimensions $n \geq 3$) for two-convex self-shrinking solutions which arise as blow-up limits of the mean curvature flow with type-I curvature blow-up since the only possibilities are shrinking spheres $S^n_{\sqrt{-2nt}}$ and cylinders $\mathbb{R} \times S^{n-1}_{\sqrt{-2(n-1)t}}$ [10].

Recently, Haslhofer [8] proved that this is true in the embedded case (even in dimension 2), his proof relying crucially on the non-collapsing theory of [4] and [9]. In fact, he shows that any strictly convex, uniformly two-convex translator which is non-collapsing is necessarily rotationally symmetric. In the immersed setting, we no longer have a non-collapsing property; however, by the work of Huisken and Sinestrari [13], we have a cylindrical estimate and a corresponding gradient estimate. Motivated by Haslhofer's result and the Huisken–Sinestrari theory, we prove the following.

Theorem 1. *Let $X\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be a mean convex translator and $C_1 < \infty$ a constant such that the following hold:*

(1) *cylindrical estimate:* $|A|^2 - \frac{1}{n-1}H^2 < 0$

(2) *gradient estimate:* $|\nabla A|^2 \leq -C_1 \left(|A|^2 - \frac{1}{n-1}H^2\right) H^2$

where A is the second fundamental form of X. Then M^n is rotationally symmetric.

In fact (assuming $T = e_{n+1}$), we need only prove that the blow-down of $M^n_t := M^n + te_{n+1}$ is the shrinking cylinder $S^{n-1}_{\sqrt{2(n-1)(1-t)}} \times \mathbb{R}$, since this is enough to deduce rotational symmetry of M^n by Haslhofer's work in [8].

We remark that the cylindrical estimate implies uniform two-convexity, $\kappa_1 + \kappa_2 \geq \frac{1}{2(n-1)}H$ (see [14]). As a consequence, any type-II blow-up limit of a two-convex mean curvature flow in dimensions $n \geq 3$ is rotationally symmetric (even when the mean curvature flow is only immersed).

Corollary 1. *Suppose that $X\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, is a translator which arises as a proper blow-up limit of a two-convex mean curvature flow of immersed hypersurfaces. Then M^n is rotationally symmetric.*

We note that Corollary 1 fails in dimension 2 without some additional assumption, such as non-collapsing, to rule out the Grim plane $\mathbb{R}\times\Gamma$. This is in accordance with the type-I case, where the non-embedded Abresch–Langer planes $\mathbb{R}\times\gamma_{k,l}$ can arise [1].

Apart from dealing with blow-up limits of type-II singularities of two-convex mean curvature flows of *immersed* hypersurfaces, a further motivation for removing the (two-sided) non-collapsing assumption in Haslhofer's result was to study translating solutions of more general curvature flows, where (two-sided) non-collapsing will in general not hold. Let F be given by $F(x) = f(\vec{\kappa}(x))$ for some smooth function $f\colon \Gamma^n \subset \mathbb{R}^n \to \mathbb{R}$ of the principal curvatures $\vec{\kappa} := (\kappa_1, \ldots, \kappa_n)$ defined with respect to some choice of unit normal field ν. Then we can consider solutions $X\colon M^n \to \mathbb{R}^{n+1}$ of the fully non-linear translator equation

$$F = -\langle \nu, T\rangle \tag{FT}$$

for some $T \in \mathbb{R}^{n+1}$. We will call the function $f\colon \Gamma^n \to \mathbb{R}$ *admissible* if Γ^n is an open, symmetric cone and f is smooth, symmetric, monotone increasing in each variable and 1-homogeneous. These conditions on f are very natural: Indeed, smoothness and symmetry are needed to ensure that F is smooth, monotonicity ensures that (FT) is elliptic, and homogeneity ensures that F scales like curvature.

Just as for the mean curvature flow, the family $\{X(\cdot,t)\}_{t\in\mathbb{R}}$ of immersions $X(\cdot,t)\colon M^n \to \mathbb{R}^{n+1}$ defined by $X(x,t) := X(x) + tT$ satisfies, up to a time-dependent tangential reparametrization, the corresponding flow

$$\partial_t X(\cdot,t) = -F(\cdot,t)\nu(\cdot,t). \tag{F}$$

Moreover, if (F) admits an appropriate Harnack inequality (which is true under very mild concavity assumptions for f [3]) then solutions of (FT) arise as blow-up limits of positive speed solutions of (F) about type-II singularities in a completely analogous way to the case of mean convex mean curvature flow. If F also admits a strong maximum principle for the Weingarten tensor (which also holds under natural concavity conditions for f) then our proof of Theorem 1, with minor modification, applies to solutions of (FT). Similarly, we obtain a result corresponding to Corollary 1 for solutions of (FT) that arise as blow-up limits of solutions of the flow (F). We remark that the class of flows to which this corollary applies includes the flow of two-convex hypersurfaces by the two-harmonic mean curvature,

$$F := \left(\sum_{i<j} \frac{1}{\kappa_i + \kappa_j} \right)^{-1}, \tag{1}$$

and, for $n = 3$, the flows of positive scalar curvature hypersurfaces by either the square root of the scalar curvature or the ratio of scalar to mean curvature. This corollary, however, does not include any convex speeds, because, as yet, it is not known if they admit an appropriate gradient estimate (although an appropriate cylindrical estimate was proved in [5]).

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Yang-Mills flow in dimension four

Alexandru Waldron

The *Yang-Mills flow*

$$\partial_t A = -D_A^* F_A$$

is a basic evolution equation in differential geometry, moving a connection by the negative L^2 gradient of the Yang-Mills functional

$$\int_M |F_A|^2 \, dV.$$

It gained success in Donaldson's proof [2] of the Kobayashi-Hitchin correspondence for stable bundles over compact Kähler surfaces, which subsequently became the Donaldson-Uhlenbeck-Yau theorem.

Over a general compact Riemannian manifold, the flow is not understood well, except to say that it behaves perfectly in low dimensions ($n \leq 3$) and catastrophically in higher dimensions ($n \geq 5$). In dimension four it is necessary for singularities to form at infinite time, but was not known if they ever form within finite time, and this question has been the focus of my research.

There is a well-known analogy between Yang-Mills and harmonic map flow, especially in the critical dimensions of $n = 4$ and $n = 2$, respectively. In either setting, according to work of Struwe ([5], [6]), blowup is characterized by concentration of energy on a finite set of points. Harmonic map flow in the equivariant case was shown to blow up in finite time [1]; however it was found by Schlatter et. al. [4] that equivariant Yang-Mills flow

$$\partial_t f(r,t) = \partial_r^2 f + \frac{1}{r}\partial_r f - \frac{2}{r^2} f(f-1)(f-2)$$

only blows up exponentially, at infinite time. This contrast was studied by Grotowski and Shatah [3], where the coefficient of the zeroth-order term, rather than the particular structure of the nonlinearity, was identified as the distinguishing factor.

In my thesis ([8], [9]) I tried to understand the contrast geometrically.

Theorem. *If initially* $\|F_A^+\|_{L^2(M)} < \delta$*, a universal constant, then the Yang-Mills flow exists for all time.*

In the spirit of Eells-Sampson's Theorem, the proof relies on a pointwise bound (Bochner formula and Moser iteration) together with an integral bound (energy inequality). In this case a split Bochner formula is used to bound the self-dual curvature F^+, which allows a cutoff to be introduced in the energy inequality at a finite price. For harmonic map flow, under the analogy

$$F^+ \to \bar{\partial} u$$

the same statement is false even for equivariant data. This is because the norm of the cutoff $\|\nabla \varphi\|_{L^4}$ is scale-invariant in dimension four, but blows up in dimension two; in other words, in dimension four, there is extra room for a singularity to "decouple" from the bulk of the solution.

My thesis contains a variety of additional results on convergence at infinite time and asymptotic stability, enabled by the same estimates. The main convergence result is very similar to Topping's thesis [7], although the conclusions are markedly different. His result states that on S^2, within a certain energy of the topological minimum, infinite-time singularities are unique; whereas on S^4 they simply do not occur near the minimum energy.

Since proving the above Theorem more than three years ago, I have worked to find the most general result along the same lines, and by now have done so [10]. Rather than assuming separate control of F^+ or F^-, one considers the *stress-energy tensor*

$$S_{ij} = \langle F_{ik}, F_{jk} \rangle - \frac{1}{4} g_{ij} |F|^2 = 2 \langle F^+_{ik}, F^-_{jk} \rangle.$$

In the harmonic map context, this corresponds to the real part of the Hopf differential (and I thank Melanie Rupflin for reminding me of this after the lecture).

At length, having tried various methods to control S_{ij} directly, without success, I became convinced that the local energy inequality in my thesis should be replaced by a more subtle identity

$$\frac{1}{2} \frac{d}{dt} \int |F|^2 r^2 \varphi \, dV + \int |D^* F|^2 r^2 \varphi \, dV = \int X^i X^j S_{ij} \Delta \varphi \, dV. \tag{1}$$

In lieu of the split Bochner formula, a Hodge decomposition can be applied to the components of the full system

$$\begin{aligned} (\partial_t - \Delta) F &= F \# F \\ (\partial_t - \Delta) D^* F &= F \# D^* F. \end{aligned} \tag{2}$$

Using (1) together with completely sharp parabolic estimates for (2), it is (just) possible to control the blowup. The next step will be to analyze infinite-time singularities by adapting the strategy of Perelman's canonical neighborhood theorem, which requires several new elements.

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Ricci flow from spaces with isolated conical singularities

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(joint work with Felix Schulze)

The Ricci flow with rough initial conditions is an intensive field of research, motivated in part from the need to use Ricci flow as a tool to study spaces with low regularity, as well as from the problem of continuing the flow after a singularity develops.

Existing work deals with situations where the singular space is the Gromov-Hausdorff limit of a non-collapsed sequence of Riemannian manifolds with curvature bounded below in dimensions 2 and 3, as in [10, 11], or situations where the space in question is close, in the L^∞ or Gromov-Hausdorff sense to a smooth Riemannian manifold, as in [6, 9, 2].

In this work we investigate the existence of a smooth Ricci flow $(M, g(t))_{t\in(0,T]}$ starting from a compact Riemannian manifold (M, g_0) with isolated conical singularities. Such metrics are close to a cone $(C(X), g_c = dr^2 + r^2 g_X)$ near the singular point, where (X, g_X) is a smooth compact Riemannian manifold. Recent progress on the structure at infinity of shrinking Ricci solitons by Munteanu and Wang in [7, 8], suggests that some singularities of the Ricci flow indeed have such conical structure.

We obtain the following result:

Theorem 1 ([4]). *Let (M, g_0) have an isolated conical singularity at z_0 modeled on a cone $(C(S^{n-1}), g_c = dr^2 + r^2 g_{S^{n-1}})$, where $g_{S^{n-1}}$ is a metric on S^{n-1} satisfying $\mathrm{Rm}(g_{S^{n-1}}) \geq 1$. Then there exists a smooth Ricci flow $(M, g(t))_{t\in(0,T]}$ with the properties:*

- $|\mathrm{Rm}(g(t))|_{g(t)} \leq C/t$ *for* $t \in (0, T]$.
- $(M, d_{g(t)}) \to (M, d_{g_0})$ *as* $t \searrow 0$, *in the Gromov-Hausdorff topology.*
- *There is* $\Psi\colon M \setminus z_0 \to M$, *diffeomorphism onto its image, such that:*
  - $\Psi^* g(t) \to g_0$ *in smoothly locally away from* z_0.
  - *For every* $q \notin \mathrm{Im}\Psi$ *and* $\lambda_k \searrow 0$,

$$(M, \lambda_k^{-1} g(\lambda_k t), q)_{t\in(0,\lambda_k^{-1}T]} \to (N, h(t))_{t\in(0,+\infty)}$$

in the smooth Cheeger-Gromov topology, where $(N, h(t))_{t\in(0,+\infty)}$ *is generated by the unique expanding gradient Ricci soliton asymptotic to the cone* $(C(S^{n-1}), g_c)$.

The result of course extends to deal with any number of isolated conical singularities.

Recall that asymptotically conical expanding gradient Ricci solitons are triples (N, g_N, f), where (N, g_N) is a smooth, non-compact, complete Riemannian manifold asymptotic to a cone $(C(X), g_c = dr^2 + r^2 g_X)$ at infinity, and f is a smooth function satisfying

$$\mathrm{Hess}_{g_N} f = \mathrm{Ric}(g_N) + \frac{g_N}{2}.$$

Such metrics induce special solutions of the Ricci flow $(N, h(t))_{t\in(0,+\infty)}$ that evolve only under diffeomorphisms and scalings, and converge, as $t \searrow 0$, to the cone $(C(X), g_c)$. In particular, asymptotically conical expanders are of our interest because they are models of how to flow out of a conical singularity.

By the very interesting work of Deruelle [3], given any metric $g_{S^{n-1}}$ with $\mathrm{Rm}(g_{S^{n-1}}) \geq 1$, other than the one with constant curvature one, there exists a unique expander asymptotic to the cone $(C(S^{n-1}), g_c = dr^2 + r^2 g_{S^{n-1}})$ with positive curvature operator.

The main idea behind the proof of Theorem 1 is to desingularize (M, g_0) by gluing, close to the singularity, large pieces of an expander from [3], and then limit out the corresponding Ricci flows $(M, g_s(t))_{t\in(0,T_s]}$, as $s \searrow 0$. Note that the standard existence theory for the Ricci flow only gives $T_s \geq Cs$. Thus we would like to exploit stability properties of the expander in order to estimate $g_s(t)$ in the high curvature region, in combination with Perelman's pseudolocality theorem which controls the flow inside the almost conical region.

These stability properties take the form of Gaussian estimates for the Lichnerowicz heat kernel on an asymptotically conical expander with positive curvature operator, obtained recently by Deruelle and Lamm in [2]. A consequence of these estimates is the L^∞ stability of such expanders and a key part of our argument is to localize this L^∞ stability result.

It is important to remark that the curvature assumption on the cone is used only to apply the results from [2]. Moreover, our argument is applicable even for $g_{S^{n-1}}$ that is close to a metric with $\mathrm{Rm} \geq 1$.

Finally, we mention that a similar scheme, but with different methods, was employed by Neves, Ilmanen and Schulze in [5], and Begley and Moore in [1], to approach analogous questions for the network flow and Lagrangian mean curvature flow respectively.

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On uniqueness for the supercritical harmonic map heat flow

Pierre Germain

1. The set up

The harmonic map heat flow is a flow on maps u from $\mathbb{R}^d$ to a Riemannian manifold M, given by the L^2 gradient flow of the Dirichlet energy

$$E(u) = \int_{\mathbb{R}^d} |\nabla u|^2 \, dx.$$

In the particular case where the target manifold is the sphere $\mathbb{S}^m$, the equation reads

$$\partial_t u - \Delta u = |\nabla u|^2 u \quad \text{with} \quad u(t=0) = u_0.$$

We will consider the particular case where the target is the sphere $\mathbb{S}^d$, under the corotational assumption

$$u(t,x) = \begin{pmatrix} \cos(h(t,|x|)) \\ \sin(h(t,|x|)\frac{x}{|x|} \end{pmatrix},$$

where $h(t,r)$ is a scalar function of a (spatial) scalar variable. The equation becomes

$$(1) \qquad \partial_t h - \partial_r^2 h - \frac{d-1}{r}\partial_r h + \frac{d-1}{2r^2}\sin(2h) = 0 \quad \text{with} \quad h(t=0) = h_0.$$

2. The question of uniqueness

Local well-posedness can be proved for data in the Hölder space $\mathcal{C}^\alpha$, with $\alpha > 0$, or for small data in L^∞ [4]

For finite energy data $u_0 \in H^1$, it is possible to buid up weak solutions [2] which belong to the energy space $L^\infty H^1$, and satisfy the monotonicity formula due to Struwe.

However, these solutions are not unique as soon as the data is allowed to be large in L^∞: [3] builds up a weak solution which does not satisfy the monotonicity formula, and therefore cannot agree with the above.

This leads to the natural question [7]: does the monotonicity formula suffice to guarantee uniqueness?

3. Main result

We not state our main result, in joint work with T. Ghoul and H. Miura [6], following an earlier work with M. Rupflin [5], in a slighlty simplified form.

Assume that h_0 is a smooth, bounded function on $[0, \infty)$.

- For $3 \leq d \leq 6$, if $h_0(0)$ is sufficiently close to $\frac{\pi}{2}$, there exists two solutions h to (1), which both satisfy the monotonicity formula. However, uniqueness can be retrieved by considering solutions constructed through the penalization method (Ginzburg-Landau).
- For $d \geq 7$, weak solutions are always unique.

This result relies on the study of self-similar solutions of (1): solutions of the form $u(t, r) = \psi\left(\frac{r}{\sqrt{t}}\right)$. They satisfy the equation

$$\psi'' + \left(\frac{d-1}{\rho} + \frac{\rho}{2}\right)\psi' - \frac{d-1}{\rho^2}\sin(2\psi) = 0.$$

A very precise study of this ODE is required, followed by a nonlinear analysis to prove nonlinear stability of these self-similar solutions.

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