Mathematisches Forschungsinstitut Oberwolfach

Report No. 28/2016

DOI: 10.4171/OWR/2016/28

Geometrie

Organised by
John Lott, Berkeley
André Neves, London
Iskander Taimanov, Novosibirsk
Burkhard Wilking, Münster

5 June – 11 June 2016

Abstract. The workshop *Geometry*, organized by John Lott (Berkeley), André Neves (London), Iskander Taimanov (Novosibirsk) and Burkhard Wilking (Münster) was well attended with over 53 participants with broad geographic representation from all continents. Compared to previous meetings there were for example quite a few young Brazilian postdocs at the meeting. The emphasize on min-max problems and related fields was somewhat increased.

Mathematics Subject Classification (2010): 51-XX.

Introduction by the Organisers

The format of the meeting consisted of 18 one hour talks and four half hour after-dinner talks. The after-dinner talks were given by PhD students and recent PhDs. The schedule left lots of room for discussions in between talks.

Six of the talks were related to geometric flows. Gerhard Huisken investigated the mean curvature flow with surgery in 3-dimensional manifolds. Carlo Sinestrari reported on progress on convergence results of the volume preserving curvature flow for hypersurfaces in Euclidean space. Ramiro Lafuente studied the Ricci flow of homogeneous spaces resulting in a nearly fully understanding of the dynamical properties of the ODE in the solvable case. Renato Bettiol presented some calculations on the Ricci flow on 4-dimensional cohomogeneity one manifolds. Felix Schulze explained some new existence results for Ricci flows coming out of singular spaces which has been a recurrent theme in previous workshops. Namely, he considered as initial space a singular Riemannian manifold with isolated conical singularities. Finally, Robert Haslhofer explained how the mean curvature flow with surgery can be used to show that the moduli space of 2-convex embedded spheres in Euclidean space is connected.

Two talks generalized well-known results of Yau from the (smooth) Kähler case to a more general setting. Song Sun discussed how one can generalize Yau's solution of the Calabi conjecture to certain singular Calabi Yau varieties. Ben Weinkove discussed the complex Monge-Ampere equations of Hermitian, Gauduchon and balanced metrics.

There were 4 talks involving min-max methods. Yevgeny Liokumovich presented an analogue of Weyl's law for the spectrum of the Laplacian of Riemannian manifolds. Therein the p-th eigenvalue of the Laplacian is replaced by the p-width of the manifold, the volume of a minmal hypersurface obtained by a min-max method applied to the p-th cohomology group of the space of $(n-1)$ cycles in the underlying manifold. Rafael Montezuma used min-max methods to construct minimal hypersurfaces in certain noncompact manifolds. Nicolau Sarquis Aiex addressed the question whether for a manifold with an analytic metric of positive Ricci curvature the space of embedded minimal hypersurfaces is non-compact. Daniel Ketover explained how min-max methods can be used to explain the existence of a sequence of minimal surfaces in S^3 converging to the double of the clifford torus.

An important problem in constructing minimal hypersurfaces using min-max methods is to establish and use index estimates, which three speakers addressed. Alessandro Carlotto gave effective index estimates of minimal hypersurfaces via Euclidean isometric embeddings. Ivaldo Nunes reported on stable constant mean curvature surfaces with free boundary. He ruled a potentially exceptional case from earlier work. Davi Maximo investigated the compactness properties of minimal surfaces in Euclidean space with an a priori bound on the index. Roughly they are the same as in the stable case except there are possibly finitely many exceptional points were the convergence is weaker.

Compactness and convergence problems for a sequence of manifolds were at the core of two other talks. Dorothea Jansen investigated collapsing sequences of manifolds with lower Ricci curvature bound. Despite the fact that no form of a fibration theorem is available she showed that the diameter of a typical fiber is well defined up to some uniform factor. Shouhei Honda explained that for a converging sequence of noncollapsed manifolds with bounded Ricci curvature, the spectrum of the Hodge-Laplacian on 1-forms converges as well.

Low eigenvalues of the Laplace operator played an essential role in two talks. Ursula Hamenstädt addressed the several questions in how the first eigenvalue of a hyperbolic three-manifold relates to its volume and its Heegaard genus. Guofang Wei proved optimal gap estimates between the first two eigenvalues of the Laplacian for a convex domain of the sphere.

The remaining three talks were given by Kerin, Hingston and Wickramasekera. Martin Kerin showed that each of the 28 oriented diffeomorphism classes of 7-dimensional spheres admits a metric with nonnegative sectional curvature, which has previously only been known for those which are S^3-bundles over S^4. A crucial step of the proof is to show that each exotic 7-sphere is the total space of a Seifert S^3-bundle over S^4 endowed with an orbifold metric. Nancy Hingston reported on various loop products with applications to bounds for the number of closed geodesics. Neshan Wickramasekera presented a regularity and compactness theory of CMC hypersurfaces in Riemannian manifolds.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

Workshop: Geometrie

Table of Contents

Abstracts

Convergence results for volume preserving flows of convex hypersurfaces

Carlo Sinestrari

In this talk we consider the evolution of a closed convex hypersurface in Euclidean space whose speed is given by a power of the mean curvature, plus an additional nonlocal term which keeps the enclosed volume constant. More precisely, let $F_0 : \mathcal{M} \to \mathbb{R}^{n+1}$ be a smooth embedding of a closed n-dimensional manifold $\mathcal{M}$, with $n \geq 1$, such that $\mathcal{M}_0 := F_0(\mathcal{M})$ is a convex hypersurface. For a given $k \in (0, +\infty)$, we consider the family of immersions $F : \mathcal{M} \times [0, T) \to \mathbb{R}^{n+1}$ which satisfies

$$\frac{\partial F}{\partial t}(p,t) = [-H^k(p,t) + \phi(t)]\nu(p,t), \tag{1}$$

$$F(p,0) = F_0(p). \tag{2}$$

Here H and ν denote the mean curvature and the outer normal vector of the evolving hypersurface $\mathcal{M}_t = F(\mathcal{M}, t)$, and the function ϕ is defined by

$$\phi(t) = \frac{1}{|\mathcal{M}_t|} \int_{\mathcal{M}_t} H^k d\mu. \tag{3}$$

This choice of ϕ is such that the volume of the domain enclosed by $\mathcal{M}_t$ remains constant, and the above flow is called the volume-preserving flow by powers of the mean curvature. In the literature, flows of the form (1) are more commonly studied without the $\phi(t)$ term; we call this case the standard H^k-flow. Flows with speed H^k are natural generalizations of the classical mean curvature flow, corresponding to the case $k = 1$. The standard H^k-flow was first analyzed by Schulze [12, 13], who later obtained a very interesting application of this evolution to isoperimetric inequalities in Euclidean and Riemannian spaces [14].

A classical result by Huisken [8] asserts that every compact convex hypersurface evolving by the standard mean curvature flow shrinks to a point in finite time and becomes spherical under rescaling, a behaviour which is usually called convergence "to a round point". Since then, many authors have investigated whether the same result holds for flows where the speed is given by a general symmetric, positively homogeneous function of the principal curvatures. For volume preserving flows, the corresponding expected result is that the evolution of a convex hypersurface is defined for all times and converges to a sphere as $t \to \infty$. Until now, these properties have been proved for fairly general speeds provided the homogeneity degree is 1, see [6] for the standard case and [11] for volume preserving flows. For a general homogeneity, the behaviour is less understood. If the degree is greater than one, the convergence to a round point of the standard flow has been proved for a large class of speeds under more restrictive assumptions on the initial hypersurface, see [5] and the references therein. Roughly speaking, one requires that the principal curvatures λ_i satisfy a pinching condition of the form $\lambda_i \geq cH$

for a suitably large constant $c > 0$. Similar results have been proved also in the volume preserving case, see [7]. For general convex initial data and $k > 1$, the pinching of the curvatures can become worse under the flow, see Section 5 in [4], so that the usual methods cannot be extended. Therefore, convergence results are known only in the case of curves $n = 1$, or for some particular speeds in the case $n = 2$ where other techniques have been discovered, see [1, 15]. If $k < 1$, the convergence to a round point of the standard flow is even known to be false in some cases, see e.g. [3]. For instance, if $n = 1$ and $k = 1/3$ then the flow has an affine-invariance property, which implies in particular that ellipses are homothetically shrinking solutions of the flow and do not become round with the evolution.

The result we present in this talk is the following, see [16].

Theorem. *Let $F_0 : \mathcal{M} \to \mathbb{R}^{n+1}$ be a smooth embedding such that $F_0(\mathcal{M})$ is a closed convex hypersurface with strictly positive principal curvatures. Then the flow* (1)-(2)*, with ϕ given by* (3)*, has a smooth solution, which is defined for all $t \in [0, \infty)$, is convex for all t, and converges smoothly to a sphere as $t \to +\infty$.*

Such a result is well known for the mean curvature flow $k = 1$, both standard and volume preserving, see [8, 9]. By contrast, as recalled above, if $k \neq 1$ the available theorems either require some additional curvature pinching assumption on the initial value, or they only hold for dimension $n \leq 2$. Let us describe these results in more detail. In the case of the standard H^k-flow with $k > 1$ in general dimension, Schulze [13] has proved convergence to a round point assuming a suitable pinching condition on the initial data. An analogous result has been obtained by the author and Cabezas Rivas [7] in the volume preserving case. In the case $n = 2$, Schnürer and Schulze [13] have proved that if $1 \leq k \leq 5$ then general convex surfaces converge to a round point of the standard H^k-flow. We remark that our theorem holds for any $k > 0$. This is not a contradiction with the above recalled counterexample for the standard flow of curves; in fact, the invariance under affine transformation if $k = 1/3$ is broken by the volume preserving term.

In contrast to the above quoted papers [7, 13], our result does not use any pinching condition on the solution. The proof relies instead on the monotonicity of the isoperimetric ratio of the region enclosed by the evolving hypersurface, which is a peculiar property of the volume preserving flow compared to the standard one. A crucial step in the proof is provided by a property of convex sets observed in [2, 10], stating that a bound on the isoperimetric ratio implies a bound on the ratio between outer and inner radius. From this, we can obtain curvature bounds and long time existence of the flow by standard arguments.

It is possible to generalize this result to volume preserving flows with speeds given by fairly general functions $\Phi(H)$ of the mean curvature, with Φ increasing (work in progress with M.C. Bertini). A future hope is to extend these techniques to speed given by other curvature functions, or to apply these convergence results to obtain isoperimetric inequalities.

References

[1] B. Andrews, *Gauss curvature flow: the fate of the rolling stones*, Invent. Math. **138** (1999), 151–161.
[2] B. Andrews, *Volume-preserving anisotropic mean curvature flow*, Indiana Univ. Math. J. **50** (2001), 783–827.
[3] B. Andrews, *Non-convergence and instability in the asymptotic behaviour of curves evolving by curvature*. Comm. Anal. Geom. **10** (2002), 409–449.
[4] B. Andrews, *Moving surfaces by non-concave curvature functions*, Calc. Var. Partial Differential Equations **39** (2010), 649–657.
[5] B. Andrews, J. McCoy, *Convex hypersurfaces with pinched principal curvatures and flow of convex hypersurfaces by high powers of curvature*, Trans. Am. Math. Soc. **364** (2012), 3427–3447.
[6] B. Andrews, J. McCoy, Y. Zheng, *Contracting convex hypersurfaces by curvature*, Calc. Var. Partial Differential Equations **47** (2013), 611–665.
[7] E. Cabezas Rivas, C. Sinestrari, *Volume preserving flows by powers of the* m^{th} *mean curvature*, Calc. Var. Partial Differential Equations **38**, (2010) 441–469.
[8] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differ. Geom. **20** (1984), 237–266.
[9] G. Huisken, *The volume preserving mean curvature flow*, J. Reine Angew. Math. **382** (1987), 35–48.
[10] G. Huisken, C. Sinestrari, *Convex ancient solutions of the mean curvature flow*, J. Differ. Geom. **101** (2015) 267–287.
[11] J.A. McCoy, *Mixed volume preserving curvature flows*, Calc. Var. Partial Differential Equations **24** (2005), 131–154.
[12] F. Schulze, *Evolution of convex hypersurfaces by powers of the mean curvature*, Math. Z. **251** (2005), 721–733.
[13] F. Schulze, *Convexity estimates for flows by powers of the mean curvature* (with an appendix by O. Schnürer and F. Schulze), Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **5** (2006), 261–277.
[14] F. Schulze, *Nonlinear evolution by mean curvature and isoperimetric inequalities*, J. Differ. Geom. **79** (2008), 197–241.
[15] O. C. Schnürer, *Surfaces contracting with speed* $|A|^2$, J. Differ. Geom. **71** (2005), 347–363.
[16] C. Sinestrari, *Convex hypersurfaces evolving by volume preserving curvature flows*, Calc. Var. Partial Differential Equations **54** (2015), 1985-1993.

Geometric invariants of closed hyperbolic 3-manifolds

Ursula Hamenstädt
(joint work with Hyungryul Baik, Ilya Gekhtman)

The goal of the talk is to relate the smallest positive eigenvalue $\lambda_1(M)$ of the Laplace operator of a closed hyperbolic 3-manifold M to the volume of M.

There are two classical bounds for $\lambda_1(M)$. A lower bound is due to Schoen and states that

$$\lambda_1(M) \geq \frac{q_1}{\mathrm{vol}(M)^2},$$

where $q_1 > 0$ is a universal constant (in general, this constant depends on the geometry and the dimension which are both fixed for the manifolds we discuss).

An upper bound is obtained from the *Cheeger constant* $h(M)$ of M. This Cheeger constant is minimal ratio of the area of ∂M_0 over the volume of M_0, where $M_0 \subset M$ is a manifold with boundary ∂M_0 and $\mathrm{vol}(M_0) \leq \frac{1}{2}\mathrm{vol}(M)$. Buser showed that

$$\lambda_1(M) \leq q_2(h(M) + h(M)^2),$$

where as before, $q_2 > 0$ is a universal constant.

A closed oriented 3-manifold can be glued from two *handlebodies*. Such a handlebody of genus $g \geq 0$ is the thickening of an embedded bouquet of g circles in $\mathbb{R}^3$. Its boundary is a surface of genus g. Identifying the boundaries of two such handlebodies with an orientation reversing diffeomorphism yields a closed oriented 3-manifold, and every closed oriented 3-manifold can be obtained in this way. The largest Euler characteristic of the boundary of a handlebody involved in such a decomposition is called the *Heegaard Euler characteristic* $\chi_H(M)$ of M. Lackenby proved that

$$h(M) \leq \frac{4\pi|\chi_H(M)|}{\mathrm{vol}(M)}.$$

These results combined then show that for every $g > 0$ there exists a constant $q = q(g) > 0$ with the following property. Let M be a closed hyperbolic 3-manifold of Heegaard genus at most g; then

$$\frac{q^{-1}}{\mathrm{vol}(M)^2} \leq \lambda_1(M) \leq \frac{q}{\mathrm{vol}(M)}.$$

A closed 3-manifold M is a *mapping torus* if there is a closed surface S and a diffeomorphism $\phi : S \to S$ such that $M = S \times [0,1]/\sim$ where $(x,1) \sim (\phi(x),0)$. If the genus of S equals g, then the Heegaard genus of M is at most $2g-1$. Furthermore, Agol proved that any closed hyperbolic 3-manifold has a finite cover which is a mapping torus. The following result [1, 2] gives a sharp estimate for the first eigenvalue of mapping tori.

Theorem. *For every $g \geq 2$ there exists a constant $c(g) > 0$ with the following property.*

(1) *Let M be a hyperbolic mapping torus of genus g; then*

$$\lambda_1(M) \leq \frac{c(g)\log \mathrm{vol}(M)}{\mathrm{vol}^{2^{2g-2}/(2^{2g-2}-1)}}.$$

(2) *There exists a sequence M_i of hyperbolic mapping tori of genus g with $\mathrm{vol}(M_i) \to \infty$ and*

$$\lambda_1(M_i) \geq \frac{c(g)^{-1}}{\mathrm{vol}(M_i)^{2^{2g-2}/(2^{2g-2}-1)}}.$$

The diffeomorphism type of a mapping torus of a surface of genus g only depends on the isotopy class of the diffeomorphism which defines the mapping torus. Thus mapping tori of genus g are classified up to diffeomorphism by elements of the *mapping class group* $\mathrm{Mod}(S)$. As $\mathrm{Mod}(S)$ is a finitely generated group, we can use random walks on $\mathrm{Mod}(S)$ to speak about *random mapping tori*. More precisely, the random walk defines a sequence P_n of probability measures on $\mathrm{Mod}(S)$ and hence on mapping tori. We say that a property Q holds for a random mapping torus if $P_n\{M \text{ has } Q\} \to 1$ $(n \to \infty)$.

For random mapping tori, we can say more [1].

Theorem. *There exists a number* $\hat{c}(g) > 0$ *such that for a random mapping torus of genus* g *we have*

$$\lambda_1(M) \leq \frac{\hat{c}(g)}{\mathrm{vol}(M)^2}.$$

References

[1] H. Baik, I. Gekhtman, U. Hamenstädt, *The smallest positive eigenvalue of fibered hyperbolic 3-manifolds*, preprint 2016.
[2] A. Lenzhen, J. Souto, *to appear*

Non-negative sectional curvature on exotic 7-spheres

Martin Kerin
(joint work with Sebastian Goette, Krishnan Shankar)

In 1956, Milnor discovered that there exist exotic spheres in dimension 7, that is, manifolds which are homeomorphic, but not diffeomorphic, to the standard sphere $\mathbf{S}^7$ [5]. He constructed examples of such manifolds as $\mathbf{S}^3$-bundles over $\mathbf{S}^4$, and these have come to be known as the Milnor spheres.

Within a few years, it had been shown that there are, in fact, 28 distinct oriented diffeomorphism types among manifolds homeomorphic to $\mathbf{S}^7$, that these form the cyclic group $\mathbb{Z}_{28}$ under the connected-sum operation, and that the Milnor spheres (which include $\mathbf{S}^7$) achieve 16 of these oriented diffeomorphism types (see [1,6,7]). If one forgets the orientation, there are 15 diffeomorphism types in total, 11 of which are achieved by the Milnor spheres.

In 1974, Gromoll and Meyer [3] found an example of an exotic Milnor sphere which admits a metric of non-negative sectional curvature. Much later, in 2000, Grove and Ziller [4] demonstrated that all Milnor spheres admit such a metric. By modifying the construction of Grove and Ziller, we have shown that there is a six-parameter family of 7-manifolds, the members of which are not constructed as $\mathbf{S}^3$-bundles over $\mathbf{S}^4$ and can be equipped with a metric of non-negative sectional curvature. It is not difficult to identify subfamilies comprising manifolds homeomorphic to $\mathbf{S}^7$ and, as a consequence, we obtain the following result.

Theorem. *All exotic* 7*-dimensional spheres admit a metric of non-negative sectional curvature.*

The major difficulty lies in identifying which diffeomorphism types have been obtained and, in particular, verifying that the four non-Milnor exotic spheres occur. In order to do so, we have modified the methods used in [2] to compute the Eells-Kuiper invariant [1] for each space.

References

[1] J. Eells and N. Kuiper, *An invariant for certain smooth manifolds*, Ann. Mat. Pura Appl. **60** (1962), 93–110.
[2] S. Goette, *Adiabatic limits of Seifert fibrations, Dedekind sums and the diffeomorphism type of certain 7-manifolds*, J. Eur. Math. Soc. **16** (2014), 2499–2555.
[3] D. Gromoll and W. Meyer, *An exotic sphere with nonnegative sectional curvature*, Ann. of Math. **100** (1974), 401–406.
[4] K. Grove and W. Ziller, *Curvature and symmetry of Milnor spheres*, Ann. of Math. **152** (2000), 331–367.
[5] J. Milnor, *On manifolds homeomorphic to the 7-sphere*, Ann. of Math. **64** (1956), 399–405.
[6] J. Milnor, *Differentiable manifolds which are homotopy spheres*, mimeographed notes, Princeton University, Princeton, N.J. 1959.
[7] S. Smale, *Generalized Poincaré's conjecture in dimensions greater than four*, Ann. of Math. **74** (1961), 391–406.

Min-max minimal hypersurfaces in noncompact manifolds

Rafael Montezuma

The main result mentioned in this talk is a theorem on existence of closed, embedded, smooth minimal hypersurfaces in certain non-compact spaces.

Minimal surfaces, the extremizers of the area functional, are among the most important topics in differential geometry. Euler and Lagrange were the first to consider minimal surfaces, proving that if the graph of a C^2 function u is minimal in the Euclidean space, then u satisfies a second order elliptic quasilinear partial differential equation. Later, Meusnier discovered a geometric characterization for these surfaces, by the vanishing of the mean-curvature. These two points of view explain why minimal surfaces are natural objects of study in geometric analysis.

Henceforth, many mathematicians contributed to the development of the theory and applied its ideas to settle deep problems and establish beautiful results in geometry. For instance, minimal surfaces have a strong link to problems involving the scalar curvature of three-manifolds. The proof of the positive mass conjecture in general relativity by Schoen and Yau [14] is one of the most important examples of this connection. Some other important recent works using minimal surfaces are the proof of the finite time extinction of the Ricci flow with surgeries starting at a homotopy 3-sphere by Colding and Minicozzi [6] and [7], the proof of the Willmore Conjecture and the existence of infinitely many closed minimal hypersurfaces in closed manifolds with positive Ricci curvature by Marques and Neves [9] and [10].

The existence of minimal submanifolds with some specific properties plays a fundamental role in the development of the theory. The most natural way to produce minimal surfaces is by minimizing the area functional in a fixed class. This idea was applied in many different settings.

In the topology of 3-manifolds, there are two important types of surfaces, namely the Heegaard splittings and the incompressible surfaces. The latter can occur as minimal surfaces produced by minimization processes, and for this reason, are stable; i.e., the Morse index is equal to zero. It is also possible to apply variational methods to construct higher index minimal surfaces. There are two basic approaches: applying Morse theory to the energy functional on the space of maps from a fixed surface, such as in the works of Sacks and Uhlenbeck, Micallef and Moore and Fraser, or via a min-max argument for the area functional over classes of sweepouts. In some cases, these methods can be applied to realize Heegaard splittings as embedded minimal surfaces, see [13] and [5].

This min-max technique was inspired by the work of Birkhoff [4] on the existence of simple closed geodesics in Riemannian 2-spheres, a question posed by Poincaré.

In higher dimensions, the original method was introduced in [2] and [12] between the 1960's and 1980's. It has been used recently by Marques and Neves to answer deep questions in geometry, see [9] and [10]. The method consists of applications of variational techniques for the area functional. It is a powerful tool in the production of unstable minimal surfaces in closed manifolds.

In this talk, we deal with a new min-max construction of minimal hypersurfaces and apply the technique to obtain existence results in non-compact manifolds.

There is no immersed closed minimal surface in the Euclidean space $\mathbb{R}^3$. This fact illustrates the existence of obstructions for a Riemannian manifold to admit closed minimal hypersurfaces. In the Euclidean space, this obstruction can be seen as a simple application of the maximum principle.

The main result in this talk is:

Theorem 1. *Let (N^n, g) be a complete non-compact Riemannian manifold of dimension $n \leq 7$. Suppose:*

- *N contains a bounded open subset Ω, such that $\overline{\Omega}$ is a manifold with smooth and strictly mean-concave boundary, and*
- *N is thick at infinity.*

Then, there exists a closed embedded minimal hypersurface $\Sigma^{n-1} \subset N$. Moreover, the obtained hypersurface intersects Ω.

The thickness assumption can be loosely phrased as follows: "The decay of the geometric objects at infinity is at most polynomial". For instance, manifolds of bounded geometry and manifolds with ends asymptotic to right cylinders are thick at infinity.

In the recent paper [8], Collin, Hauswirth, Mazet and Rosenberg prove that any complete non-compact hyperbolic three-dimensional manifold of finite volume admits a closed embedded minimal surface. These manifolds have a different behavior at infinity from those considered in Theorem 1.

The hypothesis involving the mean-concave bounded domain Ω comes from the theory of closed geodesics in non-compact surfaces. In 1980, Bangert proved the existence of infinitely many closed geodesics in a complete Riemannian surface M of finite area and homeomorphic to either the plane, or the cylinder or the Möbius band, see [3]. The first step in his argument is to prove that the finiteness of the area implies the existence of locally convex neighborhoods of the ends of M.

To prove Theorem 1, we developed a min-max method that is adequate to produce minimal hypersurfaces with intersecting properties. Let (M^n, g) be a closed Riemannian manifold and Ω be an open subset of M. Consider a homotopy class Π of one-parameter families of codimension-one submanifolds sweeping M out. For each given sweepout $S = \{\Sigma_t\}_{t\in[0,1]} \in \Pi$, we consider the number

$$L(S,\Omega) = \sup\{\mathcal{H}^{n-1}(\Sigma_t) : \Sigma_t \text{ intersects } \overline{\Omega}\},$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure associated with the Riemannian metric. Define the width of Π with respect to Ω to be

$$L(\Pi,\Omega) = \inf\{L(S,\Omega) : S \in \Pi\}.$$

Then, we prove:

Theorem 2. *Let (M^n, g) be a closed Riemannian manifold, $n \leq 7$, and Π be a non-trivial homotopy class of sweepouts. Suppose that M contains an open subset Ω, such that $\overline{\Omega}$ is a manifold with smooth and strictly mean-concave boundary. There exists a stationary integral varifold Σ whose support is a smooth embedded closed minimal hypersurface intersecting Ω and with $||\Sigma||(M) = \boldsymbol{L}(\Pi,\Omega)$.*

The intersecting condition in Theorem 2 is optimal in the sense that it is possible that the support of the minimal surface Σ is not entirely in $\overline{\Omega}$. We illustrate this fact with an example of a mean-concave subset of the unit three-sphere $S^3 \subset \mathbb{R}^4$ containing no great sphere. For each $0 < t < 1$, consider the subset of S^3 given by

$$\Omega(t) = \{(x,y,z,w) \in S^3 : x^2 + y^2 > t^2\}.$$

It is not hard to see that no $\Omega(t)$ contains great spheres. Moreover, the boundary of $\Omega(t)$ is a constant mean-curvature torus in S^3. If $0 < t < 1/\sqrt{2}$, the mean-curvature vector of $\partial\Omega(t)$ points outside $\Omega(t)$. In this case, $\Omega = \Omega(t)$ is a mean-concave subset of S^3 that contains no great sphere.

References

[1] Almgren, F., *The homotopy groups of the integral cycle groups.* Topology (1962), 257–299.
[2] Almgren, F., *The theory of varifolds.* Mimeographed notes, Princeton (1965).
[3] Bangert, V., *Closed geodesics on complete surfaces.* Math. Ann. 251 (1980), no. 1, 83–96.
[4] Birkhoff, G. D., *Dynamical systems with two degrees of freedom.* Trans. Amer. Math. Soc. 18 (1917), no. 2, 199–300.
[5] Colding, T. and De Lellis, C., *The min-max construction of minimal surfaces,* Surveys in Differential Geometry VIII , International Press, (2003), 75–107.
[6] Colding, T. and Minicozzi, W., *Estimates for the extinction time for the Ricci flow on certain 3-manifolds and a question of Perelman.* J. Amer. Math. Soc. 18 (2005), 561–569.
[7] Colding, T. and Minicozzi, W., *Width and finite extinction time of Ricci flow.* Geom. Topol. 12 (2008), no. 5, 2537–2586.
[8] Collin, P., Hauswirth, L., Mazet, L. and Rosenberg, H., *Minimal surfaces in finite volume non compact hyperbolic 3-manifolds.* preprint, arXiv:1405.1324 [math.DG]
[9] Marques, F. C. and Neves, A., *Min-max theory and the Willmore conjecture.* Ann. of Math. (2) 179 (2014), no. 2, 683–782.
[10] Marques, F. C. and Neves, A., *Existence of infinitely many minimal hypersurfaces in positive Ricci curvature.* preprint, arXiv:1311.6501 [math.DG]
[11] Montezuma, R., *Min-max minimal hypersurfaces in non-compact manifolds* Journal of Differential Geometry (accepted for publication).
[12] Pitts, J., *Existence and regularity of minimal surfaces on Riemannian manifolds,* Mathematical Notes 27, Princeton University Press, Princeton, (1981).

[13] Pitts, J. and Rubinstein, J., *Applications of minimax to minimal surfaces and the topology of 3-manifolds.* In Miniconference on geometry and partial differential equations, 2 (Canberra, 1986), volume 12 of Proc. Centre Math. Anal. Austral. Nat. Univ., p. 137-170. Austral. Nat. Univ., Canberra, 1987.

[14] Schoen, R. and Yau, S.T., *On the proof of the positive mass conjecture in general relativity.* Comm. Math. Phys. 65, 45–76 (1979).

Mean curvature flow with surgery in 3-manifolds

Gerhard Huisken
(joint work with Simon Brendle)

The lecture describes geometric aspects of joint work with Simon Brendle (Stanford) on the motion of closed 2-dimensional hypersurfaces in a closed smooth Riemannian 3-manifold in direction of the mean curvature vector. It is assumed that the surfaces are embedded and have positive mean curvature.

We show that the flow has a solution that is smooth except at finitely many surgery times and either becomes extinct in finite time or converges smoothly in infinite time to a weakly stable minimal surface, which has genus no larger than the initial surface.

Applications include asymptotically flat 3-manifolds, where initial surfaces taken as large coordinate spheres lead to a complete sweep-out of the 3-manifold all the way up to the outermost horizon.

Essential ingredients of the proof include a convexity estimate (joint with Sinestrari 1999), a non-collapsing estimate by Brendle improving a previous estimate by B. Andrews and an estimate on the self-improvement of shrinking necks. Finally, gradient estimates for the curvature by Ecker-Huisken, Haslhofer-Kleiner are combined with a pseudo-locality result to control first and second derivatives of curvature, preparing for the surgery construction.

References

[1] B. Andrews, *Non-collapsing in mean-convex mean curvature flow*, Geom. Topol. 16, 1413–1418 (2012).

[2] S. Brendle, *A sharp bound for the inscribed radius under mean curvature flow*, Invent. Math. 202, 217–237 (2015).

[3] S. Brendle, *An inscribed radius estimate for mean curvature flow in Riemannian manifolds*, to appear in Ann. Scuola Norm. Sup. Pisa.

[4] S. Brendle, *A monotonicity formula for mean curvature flow with surgery*, preprint (2013) arXiv:1312.0262[math.DG].

[5] S. Brendle and G. Huisken, *Mean curvature flow with surgery of mean convex surfaces in three-manifolds*, 21pp., (2015), to appear JEMS.

[6] S. Brendle and G. Huisken, *Mean curvature flow with surgery of mean convex surfaces in R3*, Invent. Math. 203:615–654, 2016.

[7] R. Haslhofer and B. Kleiner, *Mean curvature flow of mean convex hypersurfaces*, (2013), arXiv:1304.0926[math.DG]

[8] G. Huisken and C. Sinestrari. *Mean curvature flow with surgeries of twoconvex hypersurfaces.* Inventiones mathematicae, 175:137–221, 2009.

Loop Products and Closed Geodesics

Nancy Hingston
(joint work with Mark Goresky, Alexandru Oancea, Hans-Bert Rademacher, Nathalie Wahl)

We report on recent results with Nathalie Wahl in string topology. However because this is a geometry meeting, most of the lecture will be devoted to geometric motivation, the relationship between products and iteration of closed geodesics, and the principle of Poincaré Duality in the free loop space.

A metric on a compact manifold M of dimension $n > 2$ gives rise to a length function on the free loop space ΛM whose critical points are the closed geodesics on M in the given metric. Morse theory gives a link between Hamiltonian dynamics and the topology of loop spaces, between iteration of closed geodesics and the algebraic structure given by the Chas-Sullivan product on the homology of ΛM. Poincaré Duality reveals the existence of a related product on the cohomology of ΛM. A number of known results on the existence of closed geodesics are naturally expressed in terms of nilpotence of products. We use products to prove a resonance result for the loop homology of spheres. There are interesting consequences for the length spectrum, and related results in Floer and contact theory.

The best function for Morse theory on $\Lambda = \Lambda M$ is not the length ℓ or the energy $E : \Lambda \to \mathbb{R}$,

$$E(\gamma) = \int |\dot{\gamma}|^2 dt,$$

but $F = \sqrt{E}$. (If $\gamma \in \Lambda$, $F(\gamma) \geq \ell(\gamma)$ with equality if and only if γ is parameterized proportional to arclength.) Let $X \in H_*(\Lambda M)$ be a homology class. We define the *critical level of* X as

$$cr(X) = \inf\{a \in \mathbb{R} : X \text{ is supported on } \{\gamma \in \Lambda : F(\gamma) \leq a\}\}.$$

This critical level is always the length of a closed geodesic on M; Morse theory gives a rough correspondence

$$H_k(\Lambda M) \approx \text{ critical points of index } k.$$

The main difficulty with using this correspondence to find closed geodesics on M is that the *iterates* of a closed geodesic γ (indistinguishable from the point of view of geometry) appear as different critical points in Λ, with different length and different index. An algebraically minded geometer is led to the following question: *Is there an operation on homology that corresponds to iteration of critical points?* Matthias Schwarz long ago suggested a connection between products and iteration.

The original loop product is the Pontryagin product

$$\cdot_{PP} : H_i(\Omega) \otimes H_j(\Omega) \to H_{i+j}(\Omega)$$

induced from the concatenation product on the based loop space $\Omega = \Omega M$.

The Chas-Sullivan product

$$\cdot_{CS} : H_i(\Lambda) \otimes H_j(\Lambda) \to H_{i+j-n}(\Lambda)$$

was introduced in 1999, with a more rigorous definition given by Cohen and Jones. Current work with Nathalie Wahl includes a simpler definition.

The author and Mark Goresky used the principle of Poincaré duality to produce "dual" products on cohomology

$$\begin{aligned} \cdot_{GH} &: H^i(\Omega) \otimes H^j(\Omega) \to H^{i+j+n-1}(\Omega) \\ \cdot_{GH} &: H^i(\Lambda) \otimes H^j(\Lambda) \to H^{i+j+n-1}(\Lambda) \end{aligned}$$

The principle of Poincaré duality on loop spaces does not have a rigorous statement as yet, but the idea is that the loop spaces look roughly the same when the function F is replaced by the upside-down function $-F$. The products on the free loop space satisfy the dual fundamental inequalities

$$\begin{aligned} cr(X \cdot_{CS} Y) &\leq cr(X) + cr(Y) \quad \text{if } X, Y \in H_*(\Lambda) \\ cr(x \cdot_{GH} y) &\geq cr(x) + cr(y) \quad \text{if } x, y \in H^*(\Lambda) \end{aligned}$$

and there are similar inequalities on Ω. (Note the change in direction of the inequality!) It turns out that the homology products are nontrivial, and model the local geometry in cases where the index growth is minimal, and the cohomology products are nontrivial and model the local geometry in cases where the index growth is maximal.

Application, joint work with Rademacher: Fix a metric (Riemannian or Finsler) on S^n. Fix a coefficient field. The points $(cr(X), \deg(X))$, where $X \in H_*(\Lambda)$, lie at a bounded distance from a line through the origin in the (ℓ, d) plane. The units of the slope are "conjugate points per unit length". The proof uses both products.

Recent developments on string topology:
(joint with Alexandru Oancea) Chas-Sullivan-type products on path spaces.
(joint with Nathalie Wahl) lift of the homology coproduct; higher order products and formulas involving the product and coproduct.

Open questions:

(1) Give a rigorous statement of the Poincaré duality principle for loop spaces.
(2) For the loop coproduct $\vee$ on $H_*(\Lambda M)$: Show that, if $X \in H_*(\Lambda M)$ has a representative consisting of simple loops, then $\vee X = 0$.

References

[1] M. Chas, D. Sullivan, *String Topology*, preprint: math.GT/9911159.
[2] R. L. Cohen, J. D. S. Jones, *A homotopy theoretic realization of string topology*, Math. Annalen, vol. 324, 773–798 (2002).
[3] M. Goresky, N. Hingston, *Loop products and closed geodesics*, Duke Math. J. **150** (2009), no. 1, 117–209.

[4] N. Hingston, A. Oancea, *The space of paths in complex projective space with real boundary conditions*, Proceedings of the Gökova Geometry-Topology Conference 2014, 192–233, Gökova Geometry/Topology Conference (GGT), Gökova, 2015.
[5] N. Hingston, H. B. Rademacher, *Resonance for loop homology of spheres*, J. Differential Geom. **93** (2013), no. 1, 133–174.
[6] N. Wahl, C. Westerland, to appear.

Singularities of Kähler-Einstein metrics

Song Sun
(joint work with Hans-Joachim Hein, University of Maryland)

I shall describe the main result through the following example. Let X_f be a smooth hypersurface in $\mathbb{P}^{n+1}$ defined by a homogeneous polynomial of degree $n+2$. A well-known theorem of Yau [7] asserts that there is a unique Ricci-flat Kähler metric ω_f in the Kähler class $[\omega_{FS}|_{X_f}]$. Now let f vary in the space of all degree $n+2$ homogeneous polynomials, so that X_f becomes singular. Fix an $\tilde{f}$ such that $X_{\tilde{f}}$ has at worst nodal singularities. The latter are by definition isolated singularities locally isomorphic to $S := \{z_1^2 + \cdots + z_{n+1}^2 = 0\} \subset \mathbb{C}^{n+1}$, and these are the most generic singularities. Suppose $x_1, \cdots, x_N$ are the singularities of such an $X_{\tilde{f}}$. We are interested in understanding the limits of the Ricci-flat metrics ω_f as f tends to $\tilde{f}$. The following are known

(1) (Eyssidieux-Guedj-Zeriahi [4]): There is a unique Kähler current $\omega_{\tilde{f}}$ on $X_{\tilde{f}}$ in the class $\omega_{FS}|_{X_{\tilde{f}}}$, which is smooth and Ricci-flat on $X_{\tilde{f}} \setminus \{x_1, \cdots, x_N\}$, and has locally continuous potential near each x_i;
(2) (Rong-Zhang [6]): As f tends to $\tilde{f}$, (X_f, ω_f) converges in the Gromov-Hausdorff sense to the metric completion of $(X_{\tilde{f}} \setminus \{x_1, \cdots, x_N\}, \omega_{\tilde{f}})$, and the latter space is topologically homeomorphic to $X_{\tilde{f}}$. The last statement uses [2].
(3) (Donaldson-Sun [3]): There is a unique tangent cone at each x_i, which is naturally an affine algebraic variety, and there is an algebro-geometric description of this tangent cone in terms of filtrations of the local ring of holomorphic functions at x_i.

Now (1) provides a weak solution, (2) enables the use of the Riemannian convergence theory and (3) builds connections with algebraic geometry. A folklore question is to understand the precise behavior of the solution $\omega_{\tilde{f}}$ near each x_i. This can be regarded as studying the "regularity problem", just like in many other geometric analytic context. Our result achieves this by proving that the metric is asymptotic to a model conical metric in a polynomial rate.

Theorem ([5])**.** *There are a neighborhood U of the vertex in the* Stenzel cone*, which is an explicit Ricci-flat Kähler cone metric $\omega_S = \frac{i}{2}\partial\bar{\partial}(\sum_{k=1}^{n+1}|z_k|^2)^{(n-1)/n}$ on S, a neighborhood V_i of x_i, and a holomorphic equivalence $P_i : U \to V_i$, $C > 0$, $d > 0$ such that for all $k \geq 0$,*

$$|\nabla^k_{\omega_S}(P^*\omega_{\tilde{f}} - \omega_S)|_{\omega_S} \leq C r^{d-k},$$

where $r := (\sum_{k=1}^{n+1} |z_k|^2)^{(n-1)/2n}$ *is the distance function to the vertex on* S.

When $n = 2$ this follows easily from a simple extension of Yau's theorem. Indeed it is well known that the metric near the singularities is of "orbifold" type and the Stenzel cone is a flat cone in this case. When $n \geq 3$ this is no longer the case since the Stenzel cone is not flat. The above theorem provides the first known examples of compact Ricci-flat metrics with isolated conical, but non-orbifold singularities. A more general statement is true, but the proof of the special case above is not much simpler.

This result also provides singular models for certain glueing construction. Notice the corresponding complete model, namely *asymptotically conical Calabi-Yau manifolds*, has been well-studied, see for example [1].

References

[1] R. Conlon, H. Hein. *Asymptotically conical Calabi-Yau manifolds, I* Duke Math. J. 162 (2013), 2855-2900.

[2] S. Donaldson, S. Sun, *Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry.* Acta Math, 2015.

[3] S. Donaldson, S. Sun, *Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry, II*, arXiv:1507.05082. To appear in J. Differential Geom.

[4] P. Eyssidieux, V. Guedj, A. Zeriahi. *Singular Kähler-Einstein metrics.* J. Amer. Math. Soc. 22 (2009), no. 3, 607-639.

[5] H-J. Hein, S. Sun. *Calabi-Yau manifolds with isolated conical singularities.* Preprint.

[6] X-C. Rong, Y-G. Zhang. *Continuity of extremal transitions and flops for Calabi-Yau manifolds, with an appendix written by M. Gross.* J. Differential Geom. 89 (2011), no. 2, 233–269.

[7] S-T. Yau. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Communications on Pure and Applied Mathematics 31 (3): 339–411.

Immortal homogeneous Ricci flows

Ramiro Lafuente

(joint work with Christoph Böhm)

Consider an unnormalized Ricci flow solution $(M^n, g(t))$ which is *immortal*, that is, it exists for all times $t \in [0, \infty)$ without encountering any singularities.

Question. *What can be said about the asymptotic behavior of* $g(t)$ *as* $t \to \infty$*?*

Much progress to answer this question has been made in the case of closed 3-manifolds by Lott [6], Lott-Sesum [7], Bamler [1], among others. Under some conditions it can be shown that the parabolic blow-downs $g_s(t) := \frac{1}{s} g(s \cdot t)$ lifted to the universal cover subconverge to an expanding *homogeneous* Ricci soliton, as $s \to \infty$. The situation in higher dimensions seems to be much less understood.

In this talk we address the question in the case of homogeneous manifolds. This seems to be a natural setting to be explored: on one hand, the Ricci flow equation reduces to a nonlinear ODE, thus a better understanding of its dynamical properties can be hoped. On the other hand, there are vast families of examples of non-gradient expanding homogeneous Ricci solitons, which are especially relevant for the long-time behavior of immortal solutions as depicted in the 3-dimensional case. Our main result states that these self-similar solutions are indeed the only possible limits.

Theorem ([2]). *For any immortal homogeneous Ricci flow solution, any sequence of blow-downs subconverges to an expanding homogeneous Ricci soliton, in the sense of Riemannian groupoids.*

In the presence of a uniform lower bound for the injectivity radius, the convergence is in fact in the C^∞ Cheeger-Gromov topology, that is, smooth up to pull-back by diffeomorphisms. It is important to notice that a large family of examples of immortal homogeneous Ricci flows is given by solutions starting at a left-invariant metric on a solvable Lie group [4]. In this case, a stronger statement holds: Ricci soliton metrics are global attractors among all left-invariant metrics on a fixed solvable group.

Theorem ([2]). *On a simply-connected solvable Lie group S admitting a left-invariant Ricci soliton metric g_{sol}, any scalar curvature normalized Ricci flow solution starting at any other left-invariant metric on S converges to g_{sol} in the Cheeger-Gromov sense.*

The improvement to Cheeger-Gromov convergence is possible due to a new injectivity radius estimate for solvmanifolds. If moreover g_{sol} is Einstein, then the convergence takes place in the C^∞ topology, without the need of pulling back by diffeomorphisms. Since the Ricci flow preserves the isometry group, this yields as an immediate consequence the fact that Einstein solvmanifolds have maximal isometry group among all left-invariant metrics on the same solvable group, a result recently obtained by Gordon and Jablonski [3] using different methods.

The proof of the above results has various essential ingredients. Our approach consists in studying an ODE for Lie brackets, called the *bracket flow*, introduced by Lauret in [5], whose solutions are in one-to-one correspondence with homogeneous Ricci flow solutions. An important point in this setting is that the same homogeneous geometry may correspond to several different brackets, because there might be different groups acting transitively by isometries on the same space. This is reflected in the fact that along a Ricci flow solution, the algebraic data might diverge while the geometry stays bounded – we call this *algebraic collapsing*. If this is the case, the bracket flow then fails to reveal the true geometric limit of the solution. To overcome this issue, we study the convergence of Killing fields along a Cheeger-Gromov-convergent sequence of Riemannian manifolds. This allows us to show that the dimension of the isometry group of a sequence which algebraically collapses must be strictly bigger at the limit.

Once the algebraic collapsing is ruled out, the next step is to find a scale-invariant quantity which is monotone along the flow, and constant precisely on Ricci solitons – that is, a *Lyapunov* function. This is obtained by exploiting a close link between the Ricci curvature of a homogeneous space and a certain moment map associated with the representation of the real reductive group $\mathrm{GL}_n(\mathbb{R})$ on the vector space of skew-symmetric bilinear forms $\Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$. It turns out that the tools developed by Geometric Invariant Theory to study such actions can be adapted to this situation, providing the desired estimates for the Ricci curvature.

Finally, the uniqueness and global stability in the case of solvmanifolds follow from studying the linearization of the flow at a homogeneous Ricci soliton. Certain uniqueness properties enjoyed by the above mentioned moment map are also needed.

To conclude this note, let us mention that it would be very interesting to determine if these methods can be applied to study the question of *existence* of homogeneous Ricci solitons on a given homogeneous space. Indeed, all known examples so far occur in solvable groups, and in fact no immortal homogeneous Ricci flow solutions are known on simply-connected homogeneous space with a non-trivial compact factor in its topology, for example among left-invariant metrics on a non-compact simple Lie group such as $\mathrm{SL}(3,\mathbb{R})$.

References

[1] R. Bamler, *Long-time behavior of 3 dimensional Ricci flow – Introduction*, arXiv:1411.6658.

[2] C. Böhm and R. Lafuente, *Immortal homogeneous Ricci flows*, preprint (2016).

[3] C. Gordon, M. Jablonski, *Einstein solvmanifolds have maximal symmetry*, arXiv:1507.08321.

[4] R. Lafuente, *Scalar curvature behavior of homogeneous Ricci flows*, J. Geom. Anal. **25** (2015), 2313–2322.

[5] J. Lauret, *Ricci flow of homogeneous manifolds*, Math. Z. **274** (2013), no. 1–2, 373–403.

[6] J. Lott, *Dimensional reduction and the long-time behavior of Ricci flow*, Comm. Math. Helv. **85** (2010), 485–534.

[7] J. Lott and N. Sesum, *Ricci flow on three-dimensional manifolds with symmetry*, Comm. Math. Helv. **89** (2014), no. 1, 1–32.

Cohomogeneity one Ricci flow and nonnegative curvature

Renato G. Bettiol

(joint work with Anusha M. Krishnan)

We prove that S^4, $\mathbb{C}P^2$, $S^2 \times S^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ have metrics with nonnegative sectional curvature that immediately acquire negatively curved tangent planes when evolved via Ricci flow [1]. By taking products with spheres, it follows that Ricci flow does not preserve nonnegative sectional curvature on closed manifolds of any dimension ≥ 4. The above are the first compact 4-dimensional examples exhibiting such behavior, which was previously known for closed manifolds of dimension at least 6 [4], and non-compact manifolds [10].

The metrics we consider were introduced by Grove and Ziller [6], and support a large isometry group (acting with cohomogeneity one), which reduces the Ricci flow equation to a PDE in only 2 variables, one for time and one for space. This makes its analysis more accessible, and allows us to explicitly show that the first variation of the sectional curvature of certain initially flat planes is negative.

Analogous cohomogeneity one frameworks were previously employed by other authors, e.g., to construct non-homogeneous Ricci solitons [5] and study the asymptotics of neckpinches without rotational symmetry [7]. It is our hope that the unifying viewpoint of *cohomogeneity one Ricci flow* will be more systematically studied, mirroring the current study of *homogeneous Ricci flow* pioneered by Böhm, Lafuente, Lauret, and others. In some sense, this would be the "next step" in a symmetry program approach to understanding Ricci flow, and there is a wealth of questions that naturally arise and remain to be answered, including:

(1) When do diagonal cohomogeneity one metrics remain diagonal?
(2) Which singularities may develop along cohomogeneity one Ricci flow?
(3) Understand the behavior of immortal cohomogeneity one Ricci flows, especially in relation with results on existence and non-existence of cohomogeneity one Einstein metrics and Ricci solitons [2, 3, 5], as well as cohomogeneity one manifolds whose principal orbits are homogeneous spaces that do not admit homogeneous Einstein metrics.

References

[1] R. G. Bettiol and A. M. Krishnan, *Four-dimensional cohomogeneity one Ricci flow and nonnegative sectional curvature*, preprint. arXiv:1606.00778
[2] C. Böhm, *Inhomogeneous Einstein metrics on low-dimensional spheres and other low-dimensional spaces.*, Invent. Math. 134 (1998), no. 1, 145–176.
[3] C. Böhm, *Non-existence of cohomogeneity one Einstein metrics.* Math. Ann. 314 (1999), no. 1, 109–125.
[4] C. Böhm and B. Wilking, *Nonnegatively curved manifolds with finite fundamental groups admit metrics with positive Ricci curvature*, Geom. Funct. Anal., 17 (2007), 665–681.
[5] A. S. Dancer and M. Y. Wang, *On Ricci solitons of cohomogeneity one*, Ann. Global Anal. Geom., 39 (2011), 259–292.
[6] K. Grove and W. Ziller, *Curvature and symmetry of Milnor spheres*, Ann. of Math. (2), 152 (2000), 331–367.
[7] J. Isenberg, D. Knopf, and N. Sesum, *Ricci flow neckpinches without rotational symmetry*, preprint. arXiv:1312.2933
[8] L. Ni, *Ricci flow and nonnegativity of sectional curvature*, Math. Res. Lett., 11 (2004), 883–904.

The space of min-max hypersurfaces for analytic metrics with Ric > 0

Nicolau Sarquis Aiex

In [4] F. C. Marques and A. Neves have shown the existence of infinitely many embedded minimal hypersurfaces in a closed manifold with positive Ricci curvature. Their result is divided in two cases: when $\omega_p < \omega_{p+1}$ for all p or the equality case $\omega_p = \omega_{p+1}$, for some p.

In the first case the minimal hypersurfaces they obtain are geometrically distinct because they must have different areas. However, nothing is known about their topological types, a priori they could all be the same surface with distinct embeddings. For example, in the 3-torus it is possible to find a sequence of embedded 2-tori with area tending to infinity.

In the second case the hypersurfaces given by their proof actually have constant area, so they could all be the same embedding under isometries. Take the round 3-sphere as an example. As it is known, in this case $\omega_1 = \omega_2$, so their construction is actually giving us the 3-parameter of S^2 in the equator, all of which are isometric.

It would be interesting to know whether in the second case the minimal hypersurfaces in [4] are isometrically distinct. To answer this one could analyse either how the index or the area changes along the space of minimal hypersurfaces. It turns out that a bound on both the index and the area is sufficient to have compactness, as it was proven by B. Sharp in [9]. With that in mind, a non-compactness result would imply that either the index or the area of minimal hypersurfaces must be unbounded.

We are interested in showing that the space of minimal hypersurfaces is non-compact when the metric is analytic with positive Ricci curvature. The idea of the proof is the following. First we show that if we have compactness then there exists $N > 0$ so that $\omega_p < \omega_{p+N}$. Now the result follows as in [4] because we are able to obtain an increasing subsequence of the width spectrum with the number of parameters growing linearly.

The first step is based on the ideas of Lusternik-Schnirelmann category theory. In their context they are able to obtain results on the topology of the critical set whenever one has equality $\omega_p = \omega_q$. However, their method only works for smooth functions in Banach manifolds so we need a careful adaptation to our setting. The second step follows from the asymptotic behaviour of the width proved first by M. Gromov (see [2]).

References

[1] O. Cornea et al - *Lusternik-Schnirelmann Category*, Math. Surveys and Monographs 103, AMS (2003)

[2] L. Guth - *Minimax Problems Related to Cup Powers and Steenrod Squares*, arxiv:0702066v2 (2008)

[3] F. C. Marques, A. Neves - *Min-Max theory and the Willmore conjecture*, Ann. of Math. 179, No. 2, p.683-782 (2014)

[4] F. C. Marques, A. Neves - *Existence of infinitely many minimal hypersurfaces in positive Ricci curvature*, arXiv:1311.6501 (2013)

[5] F. C. Marques, A. Neves - *Morse index and multiplicity of min-max minimal hypersurfaces*, arXiv:1512.06460 (2015)

[6] J. T. Pitts - *Existence and regularity of minimal surfaces on Riemannian manifolds*, Mathematical Notes 27, Princeton University Press, Princeton (1981)

[7] R. Schoen, L. Simon - *Regularity of stable minimal hypersurfaces*, Comm. Pure Appl. Math. 34, p.741-797 (1981)

[8] L. Simon - *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems*, Ann. of Math. 118, p.525–571 (1983)

[9] B. Sharp - *Compactness of minimal hypersurfaces with bounded index*, arXiv:1501.02703v1 (2015)

Spectral convergence under bounded Ricci curvature

Shouhei Honda

Let n be a positive integer, let K be a positive real number, let X_i be a sequence of n-dimensional compact Riemannian manifolds with

$$|\mathrm{Ric}_{X_i}| \le K, \tag{1}$$

and let X be the Gromov-Hausdorff limit compact metric space with $\dim_H X = n$, where $\dim_H$ is the Hausdorff dimension. It is known in [1] by Cheeger-Colding that (X_i, H^n) measured Gromov-Hausdorff converge to (X, H^n), where H^n is the n-dimensional Hausdorff measure.

A main result in [7] states that we have

$$\lim_{i\to\infty} \lambda_m^{H,1}(X_i) = \lambda_m^{H,1}(X) < \infty \tag{2}$$

for all m, where $\lambda_m^{H,1}$ is the m-th eigenvalue of the Hodge Laplacian $\Delta_{H,1} = \delta d + d\delta$ acting on 1-forms, and the Hodge Laplacian $\Delta_{H,1}$ on X is as a metric measure space (X, H^n) introduced in [4] by Gigli. Note that the statement of (2) includes the discreteness and the unboundedness of the spectrum of $\Delta_{H,1}$ on X which are also new. We call (2) the spectral convergence of $\Delta_{H,1}$ for short.

As a corollary of (2) we have the upper semicontinuity of the 1st Betti numbers:

$$\limsup_{i\to\infty} b_1(X_i) \le b_1^{\mathrm{har}}(X) < \infty,$$

where $b_1^{\mathrm{har}}(X)$ is the dimension of the space of harmonic 1-forms on X. Moreover $\lim_{i\to\infty} b_1(X_i) = b_1^{\mathrm{har}}(X)$ holds if and only if a uniform spectral gap of $\Delta_{H,1}$ exists, i.e.

$$\liminf_{i\to\infty} \nu_{H,1}(X_i) > 0, \tag{3}$$

where $\nu_{H,1}$ is the first positive eigenvalue of $\Delta_{H,1}$. For $b_1^{\mathrm{har}}(X)$, note that the finiteness is also new, that it coincides with the 1st Betti number as a metric masure space introduced in [4]. However it is unknown whether it coincides with the ordinary one $b_1(X)$. It is conjectured that (3) holds and that $b_1^{\mathrm{har}}(X)$ coincides with $b_1(X)$, i.e.

$$\lim_{i\to\infty} b_1(X_i) = b_1(X). \tag{4}$$

We now recall related Sormani-Wei's work on the behavior of revised fundamental groups with respect to the Gromov-Hausdorff topology. They proved in [10] that for a Gromov-Hausdorff noncollapsed convergent sequence of n-dimensional compact Riemannian manifolds M_i with a uniform Ricci bound from below to an n-dimensional compact metric space Y, we have

$$\pi_1(M_i)/F_i \simeq \overline{\pi}_1(Y)$$

for all sufficiently large i, where F_i is a finite subgroup of $\pi_1(M_i)$ and $\overline{\pi}_1(Y)$ is the revised fundamental group of Y which is introduced by them. In particular if Y is semi-locally simply connected, then $\lim_{i\to\infty} b_1(M_i) = b_1(Y)$ holds. However, even for X, it is unknown whether X is semi-locally simply connected. The difficulty is in the geometric structure of the singular set of the limit space. The conjecture (4) means that Sormani-Wei's geometric approach might be compatible with our analytic approach.

We now turn to the spectral convergence. It is worth pointing out that for $k \geq 2$ we can not expect the spectral convergence of the Hodge Laplacian $\Delta_{H,k}$ acting on k-forms even in the case when the sequence consists of Einstein manifolds. Because the spectral convergence of $\Delta_{H,k}$ implies the upper semicontinuity of the k-th Betti numbers. However, Kobayashi-Todorov gave in [8] a Gromov-Hausdorff noncollapsed convergent sequence of (real) 4-dimensional compact Ricci flat Kähler manifolds to a 4-dimensional compact orbifold such that the second Betti numbers are strictly decreasing. In this meaning, (2) might be sharp.

On the other hand, it is also proven in [7] that for the connection Laplacian $\Delta_{C,k} = \nabla^*\nabla$ acting on k-forms, the spectral convergence holds for all k, i.e.

$$\lim_{i\to\infty} \lambda_m^{C,k}(X_i) = \lambda_m^{C,k}(X)$$

for all m, where $\lambda_m^{C,k}$ is the m-th eigenvalue of $\Delta_{C,k}$.

These spectral convergence are generalization of the spectral convergence of the Laplacian Δ acting on functions proved by Cheeger-Colding in [2]. The main difference is that the noncollapsed assumption is essential in the case of differential forms. In fact, it is easy to check that for a collapsing 2-tori, $\mathbf{S}^1(1) \times \mathbf{S}^1(\epsilon_i) \to \mathbf{S}^1(1)$ as $\epsilon_i \to 0$, although the spectral convergence of Δ holds, but the spectral convergence of $\Delta_{H,1} = \Delta_{C,1}$ does not hold.

Finally we give a remark on curvature tensors on X. It is proven in [7] that the Riemannian curvature tensor R_X, the Ricci tensor Ric_X, and the scalar curvature s_X are well-defined on X in some weak sense. In particular we see that

$$\Delta_{H,1}\omega = \Delta_{C,1}\omega + \mathrm{Ric}_X(\omega^*, \cdot)$$

on X and that the behavior of the scalar curvature with respect to the Gromov-Hausdorff topology gives a positive answer to a question by Lott given in [9]. Note that from the regularity theory of X, it is known that we can not define the curvature tensors in the ordinary way of Riemannian geometry.

The key technical tools in order to prove the results above are the regularity of the regular set of X established in [1], the solution of the codimension 4-conjecture and several estimates given in [3], and the L^p-convergence of tensor fields and their properties given in [5, 6].

It seems to be an interesting open problem whether these observations (except for curvature tensors) can be justified if we replace (1) by a weaker condition;

$$\mathrm{Ric}_{X_i} \geq -K.$$

References

[1] J. Cheeger and T. H. Colding: On the structure of spaces with Ricci curvature bounded below, I, J. Differential Geom. 45 (1997), 406–480.

[2] J. Cheeger and T. H. Colding: On the structure of spaces with Ricci curvature bounded below, III, J. Differential Geom. 54 (2000), 37–74.

[3] J. Cheeger and A. Naber, Regularity of Einstein manifolds and the codimension 4 conjecture, Ann. of Math. 182 (2015), 1093–1165.
[4] N. Gigli: Nonsmooth differential geometry-An approach tailored for spaces with Ricci curvature bounded from below, arXiv preprint, arXiv:1407.0809, to appear in Mem. of Amer. Math. Soc.
[5] S. Honda: Ricci curvature and L^p-convergence, J. Reine Angew. Math. 705 (2015), 85–154.
[6] S. Honda: Elliptic PDEs on compact Ricci limit spaces and applications, arXiv preprint, arXiv:1410.3296v5 (2014), to appear in Mem. of Amer. Math. Soc.
[7] S. Honda: Spectral convergence under bounded Ricci curvature, arXiv preprint, arXiv:1510.05349.
[8] R. Kobayashi and A. Todorov, Polarized period map for genaralized $K3$ surfaces and the moduli of Einstein metrics, Tohoku Math J. 39 (1987), 145–151.
[9] J. Lott, Remark about the spectrum of the p-form Laplacian under a collapse with curvature bounded below, Proc. Amer. Math. Soc. 132 (2003), 911–918.
[10] C. Sormani and G. Wei, Hausdorff convergence and universal covers, Trans. Amer. Math. Soc. 353 (2001), 3585–3602.

Weyl's law for widths of Riemannian manifolds

Yevgeny Liokumovich
(joint work with Fernando C. Marques, André Neves)

Let M be a compact Riemannian manifold of dimension $n+1$. The eigenvalues of the Laplace-Beltrami transform Δ on M have the following variational characterization. Let $V = W^{1,2}(M)\backslash\{0\}$ and consider the Rayleigh quotient $E : V \to [0,\infty]$, $E(f) = \int_M \frac{|\nabla f|^2}{f^2} dV$. The functional E is homogeneous, $E(af) = E(f)$, so it descends to the quotient $P = V/\{\text{lines in V}\} \simeq \mathbb{R}P^\infty$. Then

$$\lambda_p = \inf_{P_p \subset P} \sup_{f \in P_p} E(f),$$

where the infimum runs over all linear subspaces of P of dimension p.

Weyl's law states that the eigenvalues $\{\lambda_p\}$ have an asymptotic behaviour which only depends on the volume of M:

$$\lim_{p\to\infty} \lambda_p p^{-\frac{2}{n+1}} = \alpha(n) vol(M)^{-\frac{2}{n+1}},$$

where $\alpha(n) = 4\pi^2 vol(B)^{-\frac{2}{n+1}}$ and B is the unit disc in $\mathbb{R}^{n+1}$.

Gromov ([3], [4, Section 8], [5, Section 5.2], [6]) proposed studying widths of Riemannian manifolds as a non-linear analog of the spectral problem on M. The definition of width is similar to the above min-max characterization of the eigenvalues, but with the space of cycles on M as the underlying space and the mass as the energy. We will work with the spaces $\mathcal{Z}_n(M;\mathbb{Z}_2)$ of mod 2 flat n-cycles in M and $\mathcal{Z}_{k,\mathbb{R}}(M,\partial M;\mathbb{Z}_2)$ of relative mod 2 flat cycles whenever M has boundary.

By the work of Almgren [1] there is a weak homotopy equivalence $\mathcal{Z}_n(M;\mathbb{Z}_2) \simeq \mathcal{Z}_{k,\mathbb{R}}(M,\partial M;\mathbb{Z}_2) \simeq \mathbb{R}P^\infty$. We say that a family of cycles $\{z_t\}$ is a p-sweepout if it represents a non-trivial element in $H_p(\mathcal{Z}_{k,\mathbb{R}}(M,\partial M;\mathbb{Z}_2);\mathbb{Z}_2)$.

The p-width of M of dimension n, denoted by $\omega_p(M)$, is defined as the infimum over all real numbers w, such that there exists a p-sweepout $\{z_t\}$ with the mass of z_t at most w. A well-written exposition of widths and their properties can be found in the introduction of Guth's paper [7]. The widths arise as volumes of minimal hypersurfaces obtained via Almgren-Pitts min-max theory. Recently, the analysis of widths has been used by Marques and Neves to prove existence of infinitely many minimal hypersurfaces in manifolds with positive Ricci curvature [11].

Gromov's insightful idea is that using the cohomology structure of $\mathbb{R}P^\infty$ many properties of the spectrum of the Laplacian can be extended to widths.

In [2] and [9] upper bounds for widths of Riemannian manifolds were obtained that depend on the conformal class of the manifold. These results were inspired by Gromov's analogy and the upper bounds obtained by Korevaar for the eigenvalues of the Laplacian [8].

Gromov conjectured ([4, 8.4]) that the non-linear spectrum $\{\omega_p(M)\}$ has asymptotic behaviour similar to the Weyl's law. The proof of this conjecture can be found in [10].

Theorem (L, Marques, Neves, 2016)**.** *Let M be a compact Riemannian manifold with (possibly empty) boundary. There exists a constant $a(n)$, which depends only on the dimension, such that* $\lim_{p\to\infty} \omega_p(M)p^{-\frac{1}{n+1}} = a(n)vol(M)^{\frac{n}{n+1}}$.

We list some open questions, related to this result. The first question is to compute the constants $a(n)$. This is unknown even in the simplest case $n = 1$. Potential candidates for the asymptotically optimal families of sweepouts include nodal sets of eigenfunctions on the flat disc or the round sphere, or zero sets of harmonic polynomials on the flat disc.

The second question is whether the argument for widths of Riemannian manifolds can be extended to higher codimension. Namely, is it true that for a compact Riemannian manifold

$$\lim_{p\to\infty} \omega_p^k(M)p^{-\frac{n+1-k}{n+1}} = a(n,k)vol(M)^{\frac{k}{n+1}}$$

for $k < n$? In the case of higher codimension, the cohomology ring of the space of relative cycles is richer (see [7]) and so another question would be to understand the asymptotic limit for the widths associated with Steenrod powers.

Finally, is it possible to use min-max argument for p-parameter families of cycles to construct minimal surfaces, which become equidistributed in the manifold as $p \to \infty$, like nodal domains of eigenfunctions? (cf. Conjectures in Section 9 of [11]).

References

[1] F. Almgren, *The homotopy groups of the integral cycle groups,* Topology (1962), 257–299.

[2] P. Glynn-Adey, Y. Liokumovich, Width, Ricci curvature and minimal hypersurfaces, to appear in J. Different. Geom.

[3] M. Gromov, *Dimension, nonlinear spectra and width,*. Geometric aspects of functional analysis,(1986/87), 132–184, Lecture Notes in Math., 1317, Springer, Berlin, 1988.

[4] M. Gromov, *Isoperimetry of waists and concentration of maps,* Geom. Funct. Anal. 13 (2003), 178–215.

[5] M. Gromov, *Singularities, expanders and topology of maps. I. Homology versus volume in the spaces of cycles.*
[6] M. Gromov, *Morse spectra, homology measures, spaces of cycles and parametric packing problem*, preprint, 2015.
[7] L. Guth, *Minimax problems related to cup powers and Steenrod squares,* Geom. Funct. Anal. 18 (2009), 1917–1987.
[8] N. Korevaar, *Upper bounds for eigenvalues of conformal metrics.* J. Different. Geom. 37, 1, 73–93, 1993.
[9] Y. Liokumovich, *Families of short cycles on Riemannian surfaces*, Duke Math. J. **165** (2016), no. 7, 136–1379.
[10] Y. Liokumovich, F. C. Marques, A. Neves, *Weyl law for the volume spectrum*, in preparation.
[11] F. C. Marques, A. Neves, *Existence of infinitely many minimal hypersurfaces in positive Ricci curvature,* arXiv:1311.6501 [math.DG] (2013).

The moduli space of 2-convex embedded spheres

Robert Haslhofer
(joint work with Reto Buzano, Or Hershkovits)

To put things into context, let us start with a general discussion of the moduli space of embedded n-spheres in $\mathbb{R}^{n+1}$, i.e. the space

$$\mathcal{M}_n = \mathrm{Emb}(S^n, \mathbb{R}^{n+1})/\mathrm{Diff}(S^n).$$

In 1959, Smale proved that the space of embedded circles in the plane is contractible [3], i.e.

$$\mathcal{M}_1 \simeq *.$$

In particular, the assertion $\pi_0(\mathcal{M}_1) = 0$ is equivalent to the smooth version of the Jordan-Schoenflies theorem, and the assertion $\pi_1(\mathcal{M}_1) = 0$ is equivalent to Munkres' theorem that $\mathrm{Diff}_+(S^2)$ is path-connected [4].

Moving to $n = 2$, Smale conjectured that the space of embedded 2-spheres in $\mathbb{R}^3$ is also contractible, i.e. that

$$\mathcal{M}_2 \simeq *.$$

In 1983, Hatcher proved the Smale conjecture [5]. The assertion $\pi_0(\mathcal{M}_2) = 0$ is equivalent to Alexander's strong form of the three dimensional Schoenflies theorem [6]. The assertion $\pi_1(\mathcal{M}_2) = 0$ is equivalent to Cerf's theorem that $\mathrm{Diff}_+(S^3)$ is path-connected [7], which had wide implications in differential topology.

For $n \geq 3$ not a single homotopy group of $\mathcal{M}_n$ is known. Most importantly:

The naive guess that $\mathcal{M}_n \simeq *$ for all n is completely false.

Indeed, if $\mathcal{M}_n$ were contractible for every n, then arguing as in [5] we could infer that $\mathcal{D}_n := \mathrm{Diff}(D^{n+1} \,\mathrm{rel}\, \partial D^{n+1}) \simeq *$ for every n. However, it is known that $\mathcal{D}_n$ has non-vanishing homotopy groups for every $n \geq 4$. Even more strikingly, for every $n \geq 6$ there are infinitely many i such that $\pi_i(\mathcal{D}_n) \neq 0$ [8].

In the view of the topological complexity of $\mathcal{M}_n$ for general n, it is an interesting question whether one can still derive some positive results on the space of embedded n-spheres under some curvature conditions. Such results would show that all the non-trivial topology of $\mathcal{M}_n$ is caused by embeddings of S^n that are geometrically very far away from the canonical one. Motivated by the topological classification result from [9], we consider 2-convex embeddings, i.e. embeddings such that the sum of the smallest two principle curvatures is positive. Clearly, 2-convexity is preserved under reparametrizations. We can thus consider the subspace

$$\mathcal{M}_n^{\text{2-conv}} \subset \mathcal{M}_n$$

of 2-convex embedded n-spheres in $\mathbb{R}^{n+1}$. We propose the following higher dimensional Smale type conjecture.

Conjecture. *The moduli space of 2-convex embedded n-spheres in $\mathbb{R}^{n+1}$ is contractible, for every dimension n, i.e.*

$$\mathcal{M}_n^{\textit{2-conv}} \simeq *.$$

We recently confirmed the π_0-part of the conjecture:

Theorem (Buzano-Haslhofer-Hershkovits [1])**.** *The moduli space of 2-convex embedded n-spheres in $\mathbb{R}^{n+1}$ is path-connected, for every dimension n, i.e.*

$$\pi_0(\mathcal{M}_n^{\textit{2-conv}}) = 0.$$

To the best of our knowledge, our theorem is the first topological result about a moduli space of embedded spheres for any $n \geq 3$ (except of course for the moduli space of convex embedded spheres, which is easily seen to be contractible).

Our proof uses mean curvature flow with surgery. Surgery for 2-convex mean curvature flow has been implemented first by Huisken-Sinestrari [9], and more recently by Haslhofer-Kleiner [10] and Brendle-Huisken [2]. We use the approach from [10]. Besides being comparably short, this approach has the advantage that it works in every dimension and that it comes with the canonical neighborhood theorem [10, Thm. 1.22], which is quite crucial for our topological application.

Given a two 2-convex embedded sphere $M_0 \subset \mathbb{R}^{n+1}$, we consider its mean curvature flow with surgery $\{M_t\}_{t\in[0,\infty)}$ as provided by the existence theorem from [10, Thm. 1.21]. There are finitely many times where suitable necks are replaced by standard caps and/or where connected components with specific geometry and topology are discarded. The flow always becomes extinct in finite time $T < \infty$.

We first analyze the discarded components. By the canonical neighborhood theorem [10, Thm. 1.22] and the topological assumption on M_0, each connected component which gets discarded is either a convex sphere of controlled geometry or a capped off chain of ε-necks. This information is enough to construct an explicit path in $\mathcal{M}_n^{\text{2-conv}}$ connecting any discarded component to a round sphere.

We then prove by backwards induction on the surgery times that at each time every connected component is isotopic via 2-convex embeddings to what we call a marble tree. Roughly speaking, a marble tree is a connected sum of spheres (marbles) along admissible curves (strings), that does not contain any loop.

At the extinction time, by the above discussion, every connected component is isotopic via 2-convex embeddings to a round sphere, i.e. a marble tree with just a single marble and no strings. The key for the induction step is to glue together the isotopies of the pieces. To this end, we prove the existence of a connected sum operation that preserves 2-convexity and embeddedness, and that is continuous for families of suitable gluing configurations. Roughly speaking, the key for the gluing is to choose the string radius r_s much smaller than the trigger scale H_{trig}^{-1} associated to the flow with surgery, so that the different scales barely interact.

Finally, we prove by induction on the number of marbles that every marble tree is isotopic via 2-convex embeddings to a round sphere.

References

[1] R. Buzano, R. Haslhofer, O. Hershkovits, *The moduli space of 2-convex embedded spheres*, in preparation.

[2] S. Brendle, G. Huisken, *Mean curvature flow with surgery of mean convex surfaces in* $\mathbb{R}^3$, Invent. Math. 203, 615–654, 2016.

[3] S. Smale, *Diffeomorphism of the 2-sphere*, Proc. Amer. Math. Soc. 10, 621–626, 1959.

[4] J. Munkres, *Differentiable isotopies on the 2-sphere*, Michigan Math. J. 7, 193–197, 1960.

[5] A. Hatcher, *A proof of the Smale conjecture,* $\mathrm{Diff}(S^3) \simeq \mathrm{O}(4)$, Ann. of Math. 117, 553–607, 1983.

[6] J. Alexander, *On the subdivision of 3-space by a polyhedron*, P.N.A.S. 10, 6–8, 1924.

[7] J. Cerf, *Sur les diffeomorphismes de la sphere de dimension trois* ($\Gamma_4 = 0$), Springer Lecture Notes 53, 1968.

[8] D. Crowley, T. Schick, *The Gromoll filtration, KO-characteristic classes and metrics of positive scalar curvature*, Geom. Topol. 17, 1773–1789, 2013.

[9] G. Huisken, C. Sinestrari, *Mean curvature flow with surgeries of two-convex hypersurfaces*, Invent. Math. 175, 137–221, 2009.

[10] R. Haslhofer, B. Kleiner, *Mean curvature flow with surgery*, arXiv:1404.2332, 2014.

Regularity of stable codimension 1 constant-mean-curvature varifolds

Neshan Wickramasekera

(joint work with Costante Bellettini)

1. Introduction

The talk reported on the recent joint work [5] of Costante Bellettini and the author. The work considers codimension 1 integral n-varifolds V on an open subset $U \subset \mathbb{R}^{n+1}$ that have generalized mean curvature locally in L^p for some $p > n$ and, away from the singularities and with respect to a choice of orientation, are stationary and stable with respect to the area functional (with multiplicity) for ambient deformations that are "volume preserving." The main result (Theorem 1 below) gives two structural conditions (hypotheses (a) and (b) of Theorem 1) on such a varifold V that imply that its support, away from a closed set of Hausdorff dimension at most $n - 7$, is locally either a single smoothly embedded constant-mean-curvature (CMC) disk or precisely two smoothly embedded CMC disks intersecting tangentially, with the value of the scalar mean curvature the same constant everywhere. Simple examples show that neither of the two structural conditions can be dropped; see remarks (1) and (3) following Theorem 1. The work also establishes compactness of locally uniformly area-bounded subsets of the set of codimension 1 integral varifolds on U satisfying these hypotheses with a uniform bound on the mean curvature (Theorem 2 below).

There are two important features of the theorems with regard to the hypotheses: the first is that each of the two structural hypotheses rules out or controls a type of singularity that is formed when regular, embedded pieces of the varifold come together regularly in a certain way. Beyond that, no hypothesis is made concerning the singular set, which a priori could be very large. Indeed, there is no known reason why the singular set must a priori even have zero n-dimensional measure. The second feature, mentioned above also, is that in the presence of the structural conditions, it suffices to verify the stationarity and stability hypotheses on the regular part of the varifold; see hypotheses (c) and (d) of Theorem 1. Indeed, to have a natural notion of enclosed volume (so that volume preserving deformations make sense) without assuming the varifold is a boundary, one needs a choice of unit normal which is facilitated if only the regular part is involved.

All structural and variational hypotheses are therefore, in principle, easily checkable in an application. Moreover, the resulting theory is local, and allows for higher multiplicity. It is of course applicable under global hypotheses such as when the hypersurface is the multiplicity 1 varifold associated with the boundary of a Caccioppoli set, in which case one of the structural conditions is automatically satisfied. See Corollary 1.

This work generalizes the earlier work [9] establishing regularity and compactness for stable minimal hypersurfaces. A motivating factor for the work [5] is its potential applicability to the question of existence of a CMC hypersurface in a compact Riemannian manifold with a prescribed value of the mean curvature. In the case of zero mean curvature, the affirmative answer to this is a long known theorem due to the combined work of Almgren, Pitts and Schoen–Simon, for which an elegant new proof using the Allen–Cahn functional has recently been given by Guaraco ([6]) based upon a result of Tonegawa and the author ([8]); the work [9] plays a considerable role in this new approach via its use in [8]. For similar applications to existence of CMC hypersurfaces with prescribed mean curvature, it is of course necessary to extend the results of [5] to the Riemannian setting, but this is expected to be a routine exercise in view of the local nature of the theory and the robustness of the techniques used in [5].

2. Variational hypotheses in the smooth setting

Since in Theorems 1 and 2 the variational hypotheses only concern the regular part of the varifold, it is useful to first consider the classical (i.e. C^2) setting, where V corresponds to an embedded, oriented C^2 hypersurface $M \subset U$ with $\partial M \cap U = \emptyset$, and with ν a continuous choice of unit normal on M. Write

$$\mathcal{A}(M) = \mathcal{H}^n(M),$$

$$vol\,(M) = \frac{1}{n+1}\int_M x \cdot \nu(x)\, d\mathcal{H}^n(x).$$

Note that $vol\,(M)$ is the enclosed volume in case M is the boundary of a bounded, open set $\Omega \subset \mathbb{R}^{n+1}$ and ν is the outward pointing unit normal. Assume that $\mathcal{A}\,(M) < \infty$ and $vol\,(M) < \infty$.

Definition. *M is* stationary in U with respect to $\mathcal{A}(\cdot)$ for $vol(\cdot)$ preserving deformations *if for each compact $K \subset U$ and each smooth map $\varphi : U \times (-\epsilon, \epsilon) \to U$, $\epsilon > 0$, with (i) $\varphi_t = \varphi(\cdot, t) : U \to U$ a diffeomorphism for each $t \in (-\epsilon, \epsilon)$, (ii) $\varphi_0 = identity$, (iii) $\varphi_t|_{U\setminus K} = identity|_{U\setminus K}$ for each $t \in (-\epsilon, \epsilon)$ and (iv) $vol\,(\varphi_t(M \cap K)) = vol\,(M \cap K)$ for each $t \in (-\epsilon, \epsilon)$, we have that $\frac{d}{dt}\big|_{t=0}\, \mathcal{A}(\varphi_t(M \cap K)) = 0$.*

Write H_M for the mean curvature vector of M. Let $\lambda \in \mathbb{R}$ be a constant, and let

$$\mathcal{J}_\lambda(M) = \mathcal{A}(M) + \lambda\, vol\,(M).$$

It is well-known, and is straightforward to verify, that the following statements are equivalent (see e.g. [4]):

(a) M is CMC with $H_M \cdot \nu = \lambda$.
(b) $\lambda = \frac{1}{\mathcal{A}(M)} \int_M H_M \cdot \nu \, d\mathcal{H}^n$, and M is stationary in U with respect to $\mathcal{A}(\cdot)$ for $vol(\cdot)$ preserving deformations.
(c) M is stationary in U with respect to $\mathcal{J}_\lambda(\cdot)$ for arbitrary deformations (i.e. for φ_t as above but not necessarily with $vol\,(\varphi_t(M \cap K)) = vol\,(M \cap K)\ \ \forall t$).

Definition. *An embedded CMC hypersurface M in U is stable if $\frac{d^2}{dt^2}\Big|_{t=0}\, \mathcal{A}(\varphi_t(M \cap K)) \geq 0$ for all compact $K \subset U$ and all $vol\,(\cdot)$ preserving φ_t as in the definition above. Stability of M is equivalent to the fact that*

$$\int_M |A|^2 \zeta^2 \, d\mathcal{H}^n \leq \int_M |\nabla \zeta|^2 \, d\mathcal{H}^n$$

for each $\zeta \in C_c^\infty(M)$ with $\int_M \zeta \, d\mathcal{H}^n = 0$, where A is the second fundamental form of M (see [4]).

3. Varifold setting and two special types of singularities

Now consider an integral n-varifold $V = (M, \theta)$ in U with generalized mean curvature vector H_V and associated weight measure $\|V\|$ (notation as in [7], except for $\|V\|$ which is denoted μ_V in [7]). This means that M is an $\mathcal{H}^n$ measurable, countably n-rectifiable subset of U, $\theta : M \to \mathbb{N}$ is a positive integer valued $\mathcal{H}^n$ measurable function on M, $\|V\| = \mathcal{H}^n \lfloor \widetilde{\theta}$ where $\widetilde{\theta} = \theta$ on M and $\widetilde{\theta} = 0$ in $U \setminus M$, $H_V \in L^1_{loc}(\|V\|)$ in U, and the formula

$$\int_M \operatorname{div}_M X \, d\|V\| = -\int_M H_V \cdot X \, d\|V\|$$

holds for every $X \in C_c^\infty(U; \mathbb{R}^{n+1})$. Here $\operatorname{div}_M X(x) = \sum_{j=1}^n \tau_j \cdot D_{\tau_j} X(x)$, where $\{\tau_1, \ldots, \tau_n\}$ is any orthonormal basis for the approximate tangent space T_xM and D_τ denotes the directional derivative in the direction τ. Note that when $V = (M, 1)$ with M an oriented C^2 hypersurface, the validity of this formula with

H_V equal to the classical mean curvature vector of M follows from the divergence theorem. In the varifold setting (where M is merely countably n-rectifiable), this is the defining formula for the generalized mean curvature vector H_V.

The basic goal of the work [5] is to find suitably general hypotheses that guarantee $\operatorname{spt}\|V\| \cap U$ is a CMC hypersurface of class C^2 (and hence of class C^∞ by elliptic PDE regularity theory).

To have any hope of regularity of $\operatorname{spt}\|V\| \cap U$, we need $H_V \in L^p_{loc}(\|V\|)$ for some $p > n$. Else $\operatorname{spt}\|V\| \cap U$ need not even be n-dimensional! (Ex: union of suitable countably many concentric circles around rational points in $\mathbb{R}^2$.) If on the other hand $H_V \in L^p_{loc}(\|V\|)$ in U for some $p > n$, then $\operatorname{spt}\|V\| \cap U$ is n-rectifiable, $\mathcal{H}^n\left(((\operatorname{spt}\|V\| \setminus M) \cup (M \setminus \operatorname{spt}\|V\|)) \cap U\right) = 0$ and the C^1 embedded part Ω_V of $\operatorname{spt}\|V\| \cap U$ is a relatively open, dense subset of $\operatorname{spt}\|V\| \cap U$. In fact Ω_V is of class $C^{1,1-\frac{n}{p}}$ if $n < p < \infty$. The condition $H_V \in L^p_{\text{loc}}(\|V\|)$ also implies (via the well-known approximate monotonicity formula for the area ratio) that the area density $\Theta(\|V\|, p) = \lim_{\rho \to 0} \frac{\|V\|(B^{n+1}_\rho(p))}{\omega_n \rho^n}$ exists for every $p \in U$, and that $\operatorname{spt}\|V\| \cap U = \{p \in U : \Theta(\|V\|, p) \geq 1\}$. Here ω_n denotes the volume of the unit ball in $\mathbb{R}^n$, and $B^{n+1}_\rho(p)$ is the open ball in $\mathbb{R}^{n+1}$ with centre p and radius ρ. These facts were all established in the landmark work of Allard ([1]) that extended earlier fundamental work of Almgren ([2]).

Definition. *For an integral n-varifold V as above, the singular set* $\operatorname{sing} V$ *is defined by* $\operatorname{sing} V = (\operatorname{spt}\|V\| \setminus \Omega_V) \cap U$, *where Ω_V is the set of points $p \in \operatorname{spt}\|V\| \cap U$ near which* $\operatorname{spt}\|V\|$ *is a C^1 embedded hypersurface.*

Note that a.e. regularity (i.e. the fact that $\mathcal{H}^n(\operatorname{sing} V) = 0$) does not follow from the assumption $H_V \in L^p_{\text{loc}}$; a construction due to Brakke ([3]) gives an integral 2-varifold V in $\mathbb{R}^3$ with $H_V \in L^\infty$ such that $\operatorname{sing} V$ has positive $\mathcal{H}^2$ measure.

Let us now introduce two special types of singularities that will play a key role in the main theorems described subsequently.

Definition. *A point $p \in \operatorname{spt}\|V\| \cap U$ is a* classical singularity *if there are $\alpha \in (0,1)$ and $\sigma > 0$ such that* $\operatorname{spt}\|V\| \cap B^{n+1}_\sigma(p)$ *is the union of three or more embedded $C^{1,\alpha}$ hypersurfaces-with-boundary having common boundary S containing p, meeting pairwise only along S, and with at least two of the hypersurfaces-with-boundary meeting transversely.*

Let $\operatorname{sing}_C V$ be the set of classical singularities of V.

Definition. *A point $p \in \operatorname{spt}\|V\| \cap U$ is a* touching singularity *of V if $p \notin \operatorname{sing}_C V \cup \Omega_V$ and there are $\sigma > 0$, an affine hyperplane L containing p, $\alpha \in (0,1)$ and two $C^{1,\alpha}$ functions $u_1, u_2 : L \to L^\perp$ such that* $\operatorname{spt}\|V\| \cap B^{n+1}_\sigma(p) = (\operatorname{graph} u_1 \cup \operatorname{graph} u_2) \cap B^{n+1}_\sigma(p)$.

(Note that it follows from the definition that $u_1(p) = u_2(p)$, $Du_1(p) = Du_2(p)$.)

Let $\operatorname{sing}_T V$ be the set of touching singularities of V.

4. Main theorems

The first main theorem of [5] is the following:

Theorem 1 (CMC REGULARITY THEOREM). *Let V be an integral n-varifold on an open subset $U \subset \mathbb{R}^{n+1}$, $n \geq 2$, with $H_V \in L^p_{loc}(\|V\|)$ for some $p > n$ and satisfying the following:*

Structural Hypotheses:

(a) $\mathrm{sing}_C V = \emptyset$;

(b) *For each $p \in \mathrm{sing}_T V$, there is $\rho > 0$ such that $\mathcal{H}^n(\{y : \Theta(\|V\|, y) = \Theta(\|V\|, p)\} \cap B^{n+1}_\rho(p)) = 0$;*

Variational Hypotheses:

(c) *stationarity: there is a continuous choice of unit normal ν on Ω_V (= the C^1 embedded part of $\mathrm{spt}\,\|V\|$) such that $W \equiv V \llcorner (U \setminus \mathrm{sing}\, V)$ is stationary in $U \setminus \mathrm{sing}\, V$ with respect to the area functional $\mathcal{A}(W) = \|W\|(U \setminus \mathrm{sing}\, V)$ for ambient deformations that leave $\mathrm{sing}\, V$ fixed and preserve $vol\,(W) = \frac{1}{n+1}\int x \cdot \nu \, d\|W\|$;*

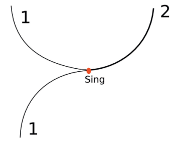
(d) *stability: the C^2 immersed part M of $\mathrm{spt}\,\|V\|$ (which, by virtue of (c) and the C^2 assumption, contains Ω_V and is a CMC hypersurface consisting locally of a single embedded disk or precisely two embedded disks intersecting tangentially), taken with multiplicity 1, is stable with respect to area for $vol\,(\cdot)$ preserving ambient deformations, or equivalently, the stability inequality $\int_M |A|^2\zeta^2 \leq \int_M |\nabla \zeta|^2$ holds for each $\zeta \in C^\infty_c(M)$ with $\int_M \zeta = 0$.*

Then there exists a closed set $\Sigma \subset \mathrm{spt}\,\|V\|$ with $\dim_{\mathcal{H}}(\Sigma) \leq n - 7$ such that $\mathrm{spt}\,\|V\| \setminus \Sigma$ locally near each point is either an embedded C^2 disk or the union of precisely two embedded C^2 disks intersecting tangentially; moreover, there is a constant $\lambda \in \mathbb{R}$ such that $H_V = \lambda\nu$ on $\mathrm{spt}\,\|V\| \setminus \Sigma$.

Remark. (1) *Hypothesis (a) cannot be dropped. Consider e.g. a piece of two intersecting unit spheres or cylinders.*

(2) *It suffices however to verify, in place of hypothesis (a), that there is a set $Z \subset \mathrm{spt}\,\|V\|$ (not required to be closed) with $\mathcal{H}^{n-1}(Z) = 0$ such that no point of $\mathrm{spt}\,\|V\| \setminus Z$ is a classical singularity. This fact follows directly from the definition of classical singularity, since if a point $p \in \mathrm{sing}_C V$, then there is an $(n-1)$-dimensional $C^{1,\alpha}$ submanifold S containing p such that every point of S is a classical singularity.*

(3) *If hypothesis (b) is dropped, then C^2 regularity is false. To see this, consider the example $\Gamma \times \mathbb{R}^{n-1}$, where Γ is the following 1-dimensional varifold in a suitable open subset of $\mathbb{R}^2$:*



In this picture, each arc is a piece of a unit circle, and the numbers 1, 2 denote the multiplicity on an arc. The singular point is a touching singularity (and not a classical singularity, since no pair of arcs meet transversely). The only hypothesis not satisfied by this example is (b). The support of the varifold depicted in this example is not the union of two C^2 graphs (it is however the union of a C^2 graph and a $C^{1,1}$ graph).

(4) *The two hypotheses (a), (b) rule out/control two types of "regular" singular structure of* $\operatorname{spt}\|V\|$. *The theorem makes no hypothesis concerning (a priori potentially large set of) singular points in a neighborhood of which nothing is known about the nature of* $\operatorname{spt}\|V\|$. *Moreover, the variational hypotheses (c), (d) are made respectively on the C^1 embedded part and the C^2 immersed part; no variational hypothesis is made across the entire (a priori potentially large) singular set. These features, as well as the one described in remark (2), are in principle very useful for the applications of the theorem.*

The following corollary follows directly from the theorem since under the hypotheses of the corollary, De Giorgi's rectifiability theorem implies that the structural condition (b) is automatically satisfied.

Corollary 1. *If V is the multiplicity 1 varifold associated with the boundary of a Caccioppoli set,* $\operatorname{sing}_C V = \emptyset$, *and if (c), (d) hold, then V satisfies the conclusions of the theorem.*

The second main theorem of [5] is the following compactness result:

Theorem 2 (CMC COMPACTNESS THEOREM). *Let (V_j) be a sequence of integral n-varifolds in open $U \subset \mathbb{R}^{n+1}$, $n \geq 2$, satisfying $H_{V_j} \in L^{p_j}_{loc}(\|V_j\|)$ for some $p_j > n$ and (a)–(d) as in the above theorem with $V = V_j$. If* $\limsup_{j\to\infty} \|V_j\|(K) < \infty$ *for each compact $K \subset U$ and* $\limsup_{j\to\infty} |H_{V_j}| < \infty$ *(note that $|H_{V_j}|$ is constant for each j by the above theorem), then there is an integral n-varifold V in U satisfying (a)–(d), and a subsequence $\{j'\}$ such that $V_{j'} \to V$ as varifolds in U.*

References

[1] W. K. Allard. *On the first variation of a varifold.* Ann. of Math. 95 (1972), 417–491.

[2] F. J. Almgren, Jr. *The theory of varifolds.* Mimeographed notes. Princeton (1965).

[3] K. Bakke. *The motion of a surface by its mean curvature.* Princeton University press, Princeton (1978).

[4] J. Barbosa, M. do Carmo. *Stability of hypersurfaces with constant mean curvature.* Math Z. 185 (1984), 339–353.

[5] C. Bellettini, N. Wickramasekera. *Regularity and compactness for stable codimension 1 CMC varifolds.* Manuscript in preparation.

[6] M. Guaraco. *Min-max for phase transitions and the existence of embedded minimal hypersurfaces.* arXiv, 2015.

[7] L. Simon. *Lectures on Geometric Measure Theory.* Centre for Mathematical Analysis, Australian National University, Camberra. (1984).

[8] Y. Tonegawa, N. Wickramasekera. *Stable phase interfaces in the van der Waals–Cahn–Hilliard theory.* J. reine ang. Mat. (2012), 191–210.

[9] N. Wickramasekera. *A general regularity theory for stable codimension 1 integral varifolds.* Ann. of Math. 179 (2014), 843–1007.

Existence of Typical Scales for Manifolds with Lower Ricci Curvature Bound

Dorothea Jansen

In 1991, Yamaguchi [6] proved that if a sequence of manifolds with uniform lower Ricci curvature bound collapses to a compact manifold with bounded sectional curvature, then these manifolds fibre over the limit manifold. In particular, the collapse happens along the fibres and, after rescaling, the manifolds converge to a product of a Euclidean and a compact space.

In 1992, Anderson [1] provided examples verifying that Yamaguchi's fibration theorem might fail if the manifolds only satisfy a lower Ricci curvature bound. Nevertheless, we are able to prove the following result, which is part of the author's Ph.D. thesis.

Theorem. *Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a collapsing sequence of pointed complete connected n-dimensional Riemannian manifolds which satisfy the uniform lower Ricci curvature bound $\mathrm{Ric}_{M_i} \geq -(n-1)$ and converge to a limit (X, p) of dimension $k < n$ in the measured Gromov-Hausdorff sense. Then for every $\varepsilon \in (0, 1)$ there exist a subset of good points $G_1(p_i) \subseteq B_1(p_i)$ satisfying*

$$\mathrm{vol}(G_1(p_i)) \geq (1 - \varepsilon) \cdot \mathrm{vol}(B_1(p_i)),$$

a sequence $\lambda_i \to \infty$ and a constant $D > 0$ such that the following holds: For any choice of base points $q_i \in G_1(p_i)$ and every sublimit (Y, q) of $(\lambda_i M_i, q_i)_{i\in\mathbb{N}}$ there is a compact metric space K of dimension $l \leq n - k$ with $\frac{1}{D} \leq \mathrm{diam}(K) \leq D$ such that Y splits isometrically as a product

$$Y \cong \mathbb{R}^k \times K.$$

Moreover, for any base points $q_i, q_i' \in G_1(p_i)$ such that, after passing to a subsequence, both $(\lambda_i M_i, q_i) \to (\mathbb{R}^k \times K, \cdot)$ and $(\lambda_i M_i, q_i') \to (\mathbb{R}^k \times K', \cdot)$ as before, $\dim(K) = \dim(K')$.

Essentially, this theorem states the following: For a set of good base points of large volume and after rescaling, any sublimit of the rescaled manifolds splits into a product of a Euclidean and a compact factor. The Euclidean dimension is independent of choice of the base points whereas the compact space may depend on the base points. However, all possible compact spaces satisfy the same diameter bounds and their dimensions do not depend on the choice of base points but only on the convergent subsequence in question.

Here, dimension is meant in the sense of Colding-Naber, cf. [4], i.e. $\dim(X)$ is the unique natural number k such that for almost all points $x \in X$ the tangent cone at x is unique and isometric to $\mathbb{R}^k$. Note that this dimension is at most the Hausdorff dimension of X and that equality is still an open problem.

In the special case of a sequence collapsing to a Euclidean space, the above result was already shown by Kapovitch and Wilking [5]. The proof of our result is in their spirit. Furthermore, we draw from the work of Cheeger-Colding [2, 3], Colding-Naber [4] and Kapovitch-Wilking [5].

References

[1] M. T. Anderson, *Hausdorff perturbations of Ricci-flat manifolds and the splitting theorem*, Duke Mathematical Journal **68** (1992), no. 1, 67–82.

[2] J. Cheeger and T. H. Colding, *Lower bounds of Ricci curvature and the Almost Rigidity of Warped Products*, Annals of Mathematics. Second Series **144** (1996), no. 1, 189–237.

[3] J. Cheeger and T. H. Colding, *On the structure of spaces with Ricci curvature bounded below. II*, Journal of Differential Geometry **54** (2000), no. 1, 13–35.

[4] T. H. Colding and A. Naber, *Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications*, Annals of Mathematics. Second Series **176** (2012), no. 2, 1173–1229.

[5] V. Kapovitch and B. Wilking, *Structure of fundamental groups of manifolds with Ricci curvature bounded below*, preprint (2011), arXiv:1105.5955v2.

[6] T. Yamaguchi, *Collapsing and Pinching Under a Lower Curvature Bound*, Annals of Mathematics. Second Series **133** (1991), no. 2, 317–357.

On stable constant mean curvature with free boundary

Ivaldo Nunes

Let (M^{n+1}, g) be a compact Riemannian manifold with nonempty boundary. The free boundary problem for constant mean curvature hypersurfaces consists of finding critical points of the area functional among all compact hypersurfaces $\Sigma \subset M$ with $\partial\Sigma \subset \partial M$ which divides M into two subsets of prescribed volumes. Critical points for this problem are constant mean curvature hypersurfaces $\Sigma \subset M$ meeting ∂M orthogonally along $\partial\Sigma$ and they are known as *constant mean curvature hypersurfaces with free boundary.* For more details, see [2] and references therein.

When a constant mean curvature hypersurface $\Sigma \subset M$ with free boundary has nonnegative second variation of area for all preserving volume variations we call it *stable*. In the case that M^{n+1} is a closed ball $B \subset \mathbb{R}^{n+1}$ we have the following natural question:

Question. *Are the totally umbilical ones the only immersed stable constant mean curvature hypersurfaces with free boundary in B?*

In [3], Ros and Vergasta gave the following partial answer for $n = 2$.

Theorem 1 (Ros and Vergasta)**.** *Let $B \subset \mathbb{R}^3$ be a closed ball. If $\Sigma \subset B$ is an immersed orientable compact stable constant mean curvature surface with free boundary, then $\partial\Sigma$ is embedded and the only possibilities are*

(i) Σ *is a totally geodesic disk;*
(ii) Σ *is a spherical cap;*
(iii) Σ *has genus* 1 *with at most two boundary components.*

In [2], we improve the above result by proving that the possibility (iii) does not occur. More precisely, we prove:

Theorem 2. *Let $B \subset \mathbb{R}^3$ be a closed ball. If $\Sigma \subset B$ is an immersed orientable compact stable constant mean curvature surface with free boundary, then Σ has genus zero.*

As a corollary of Theorems 1 and 2 we have that the answer of the question above is *yes*, that is:

Corollary 1. *The totally umbilical disks are the only immersed orientable compact stable constant mean curvature surfaces with free boundary in a closed ball $B \subset \mathbb{R}^3$.*

Remark. *Corollary 1 can be regarded as the result analogous to Barbosa and do Carmo's theorem* [1] *for immersed closed stable CMC surfaces in the Euclidean space. Since the latter holds for any dimension, we should expect the same for stable CMC surfaces with free boundary.*

In fact, Theorem 2 is a consequence of the following more general result.

Theorem 3. *Let $\Omega \subset \mathbb{R}^3$ be a smooth compact convex domain. Suppose that the second fundamental form $\Pi^{\partial\Omega}$ of $\partial\Omega$ satisfies the pinching condition*

$$k\, h \leq \Pi^{\partial\Omega} \leq (3/2)\, k\, h \tag{1}$$

for some constant $k > 0$, where h denotes the induced metric on $\partial\Omega$. If $\Sigma \subset \Omega$ is an immersed orientable compact stable CMC surface with free boundary, then Σ has genus zero and $\partial\Sigma$ has at most two connected components.

The following proposition is the key fact used to prove the theorem above. It states that a stable constant mean curvature surface Σ with free boundary in a convex domain $\Omega \subset \mathbb{R}^3$ is always *strongly* stable (i.e. the second variation of area is nonnegative among *all* variations of Σ fixing $\partial\Sigma$) for the fixed boundary problem.

Proposition 1. *Let $\Omega \subset \mathbb{R}^3$ be a compact convex domain. If $\Sigma \subset \Omega$ is an immersed stable CMC surface with free boundary, then*

$$Q^0(\varphi, \varphi) = \int_\Sigma |\nabla\varphi|^2 - |A|^2\varphi^2\, da \geq 0 \tag{2}$$

for all φ such that $\varphi = 0$ on $\partial\Sigma$.

Let us give an idea of the proof of Theorem 3. We first note by Proposition 1 that the stability of Σ implies that the quadratic form given by the second variation of area is nonnegative for *all* functions φ such that $\varphi = 0$ on $\partial\Sigma$ regardless of whether it satisfies $\int_\Sigma \varphi\, da = 0$ or not. This fact allows us to apply a *modified* Hersch type balancing argument which gives a better control on the genus of Σ. More precisely, instead of attaching a conformal disk at any connected component of $\partial\Sigma$, as in [3], in order to find a conformal map $\psi = (\psi_1, \psi_2, \psi_3) : \Sigma \to \mathbb{S}^2$ such that $\int_\Sigma \psi_i\, da = 0$, for $i = 1, 2, 3$, and having Dirichlet energy less than $8\pi(1 + [(g + 1)/2])$, where g and $[x]$ stands for the genus of Σ and the greatest integer less than or equal to x, respectively, we use as test functions the coordinates of a conformal map

$\psi = (\psi_1, \psi_2, \psi_3) : \Sigma \to \mathbb{S}^2_+$ satisfying $\int_\Sigma \psi_i \, da = 0$ for $i = 1, 2$, and having Dirichlet energy less than or equal to $4\pi(g + r)$, where r denotes the number of connected components of $\partial\Sigma$. The key point here is that, since $\psi_3 = 0$ on $\partial\Sigma$, we are able to use ψ_3 as a test function because of Proposition 1.

References

[1] J. L. Barbosa and M. do Carmo *Stability of hypersurfaces of constant mean curvature.*, Math. Z. **185** (1984), pp. 339–353.

[2] I. Nunes, *On stable constant mean curvature surfaces with free boundary*, arXiv:1605.09625.

[3] A. Ros and E. Vergasta, *Stability for Hypersurfaces of Constant Mean Curvature with free boundary*, Geometriae Dedicata **56** (1995), 19–33.

Ricci flow from metrics with isolated conical singularities

Felix Schulze

(joint work with Panagiotis Gianniotis)

We consider consider smooth solutions $(M, g(t))_{t\in(0,T)}$ to Ricci flow

$$\frac{\partial}{\partial t} g = -2\mathrm{Ric}(g)\,,$$

starting from a closed initial manifold with isolated conical singularities. More precisely we consider the following class of initial manifolds:

Definition. *We say that (Z, g_Z) is a compact Riemannian manifold with isolated conical singularities at $\{z_i\}_{i=0}^Q \subset Z$ modelled on the cones $(C(X_i), g_{c,i} = dr^2 + r^2 g_{X_i})$, where (X_i, g_{X_i}) are smooth compact Riemannian manifolds, if:*

1. *$Z \setminus \{z_0, \ldots, z_Q\}$ is a smooth manifold and g_Z is a smooth Riemannian metric on Z.*
2. *The metric completion (Z, d_Z) of $(Z\setminus\{z_0, \ldots, z_Q\}, g_Z)$ is a compact metric space.*
3. *There exist maps $\phi_i : (0, r_0] \times X_i \to Z \setminus \{z_0, \ldots, z_Q\}$, $i = 0, \ldots, Q$, diffeomorphisms onto their image, such that*
$$\sum_{j=0}^{4} r^j |(\nabla^{g_{c,i}})^j (\phi_i^* g_Z - g_{c,i})|_{g_{c,i}} < k_Z(r),$$
for some function $k_Z : (0, r_0] \to \mathbb{R}^+$ with $\lim_{r\to 0} k_Z(r) = 0$.

The following is our main result.

Theorem. *Let (Z, g_Z) be a compact Riemannian manifold with isolated conical singularities at $\{z_i\}_{i=0}^Q \subset Z$ each modelled on a cone $(C(\mathbb{S}^{n-1}), g_{c,i} = dr^2 + r^2 g_i)$ and Riem$(g_i) \geq 1$.*

Then, there exists a smooth manifold M, a smooth Ricci flow $(g(t))_{t\in(0,T]}$ on M and a constant C_{Riem} with the following properties.

(1) $(M, d_{g(t)}) \to (Z, d_Z)$ *as* $t \to 0$*, in the Gromov-Hausdorff topology.*
(2) *There exists a map* $\Psi : Z \setminus \{z_0, \dots, z_Q\} \to M$*, diffeomorphism onto its image, such that* $\Psi^* g(t)$ *converges to* g_Z*, smoothly uniformly away from* z_i*, as* $t \to 0$.
(3) $\max_M |Riem(g(t))|_{g(t)} \leq C_{Riem}/t$ *for* $t \in (0, T]$.
(4) *There is a continuous function* r_M *on* M *with* $r_M \equiv 0$ *on* $(\mathrm{Im}\Psi)^c$*, such that*

$$\max_M \sum_{j=0}^{2} r_M^{j+2} |(\nabla^{g(t)})^j Riem(g(t))|_{g(t)} \leq C_{Riem},$$

for $t \in (0, T]$.
(5) *Let* $t_k \searrow 0$ *and* $p_k \in (\mathrm{Im}\Psi)^c \subset (M, d_{g(t_k)})$*. Suppose that* $p_k \to z_i$ *under the Gromov-Hausdorff convergence, as* $k \to \infty$*. Then*

$$(M, t_k^{-1} g(t_k t), p_k)_{t\in(0,t_k^{-1}T]} \to (N_i, g_{e,i}(t))_{t\in(0,+\infty)},$$

where $(N_i, g_{e,i}(t))_{t\in(0,+\infty)}$ *is the Ricci flow induced by the unique expander* (N_i, g_{N_i}, f_i) *with positive curvature operator that is asymptotic to the cone* $(C(X_i), g_{c,i})$.

In [4–6], M. Simon shows that one can construct a Ricci flow from a space which can be approximated by smooth 3-dimensional manifolds which are locally uniformly non-collapsed and where the curvature operator is locally uniformly bounded from below. This result has been applied by Lebedeva–Matveev–Petrunin–Shevchishin [3] to show that 3-dimensional polyhedral manifolds with nonnegative curvature in the sense of Alexandrov can be approximated by nonnegatively curved 3-dimensional Riemannian manifolds. The local estimates by Simon have very recently been extended by Cabezas-Rivas–Bamler–Lu–Wilking [7] to higher dimensions.

In our present work we desingularise the initial manifold (Z, g_Z) by glueing in expanding gradient solitons with positive curvature operator, each asymptotic to the cone at the singular point, at a small scale s. These expanding gradient solitons exist by work of Deruelle [1]. Localizing a recent stability result of Deruelle-Lamm [2] for such expanding solitons, we show that there exists a solution from the desingularized initial metric for a uniform time $T > 0$, independent of the glueing scale s. The solution starting at (Z, g_Z) is then obtained by letting $s \to 0$.

References

[1] Alix Deruelle, *Smoothing out positively curved metric cones by Ricci expanders*, Geom. Funct. Anal. **26** (2016), no. 1, 188–249. MR3494489

[2] Alix Deruelle and Tobias Lamm, *Weak stability of ricci expanders with positive curvature operator*, 2016, `arXiv:1601.00325`.

[3] Nina Lebedeva, Vladimir Matveev, Anton Petrunin, and Vsevolod Shevchishin, *Smoothing 3-dimensional polyhedral spaces*, Electron. Res. Announc. Math. Sci. **22** (2015), 12–19. MR3361404
[4] Miles Simon, *Ricci flow of almost non-negatively curved three manifolds*, J. Reine Angew. Math. **630** (2009), 177–217. MR2526789
[5] ———, *Ricci flow of non-collapsed three manifolds whose Ricci curvature is bounded from below*, J. Reine Angew. Math. **662** (2012), 59–94. MR2876261
[6] ———, *Ricci flow of regions with curvature bounded below in dimension three*, 2014, `arXiv:1407.1251v1`.
[7] Burkhardt Wilking, *Manifolds with almost nonnegative curvature operator*, 2016, talk at the workshop 'Geometric Flows in Riemannian and Complex Geometry', MSRI.

Minimal surfaces with bounded index

Davi Maximo
(joint work with Otis Chodosh, Dan Ketover)

We provide a precise local picture of how a sequence of embedded minimal surfaces with uniformly bounded index in a Riemannian manifold (M^3, g) can degenerate. Loosely speaking, our results show that embedded minimal surfaces with bounded index behave qualitatively like embedded stable minimal surfaces, up to controlled errors.

By the curvature estimates of Schoen [2], a sequence of stable (*i.e.*, with zero Morse index) minimal surfaces must have uniformly bounded second fundamental form. Thus, modulo subsequence, they converge locally smoothly to a smooth minimal lamination. If, however, the index along a sequence is merely assumed to be uniformly bounded then second fundamental can blow up.

The main result presented is a precise picture of how that can happen. We prove that there are functions $m(I)$ and $r(I)$ with the following property. Fix a closed three-manifold (M^3, g) and a natural number $I \in \mathbb{N}$. Then, if $\Sigma_j \subset (M, g)$ is a sequence of closed embedded minimal surfaces with

$$\operatorname{Index}(\Sigma_j) \leq I,$$

then after passing to a subsequence, there is $C > 0$ and a finite set of points $\mathcal{B}_j \subset \Sigma_j$ with cardinality $|\mathcal{B}_j| \leq I$ so that the curvature of Σ_j is uniformly bounded away from the set $\mathcal{B}_j$, i.e.,

$$|\mathrm{II}_{\Sigma_j}|(x) \min\{1, d_g(x, \mathcal{B}_j)\} \leq C,$$

but not at $\mathcal{B}_j$, i.e.,

$$\liminf_{j\to\infty} \min_{p\in\mathcal{B}_j} |\mathrm{II}_{\Sigma_j}|(p) = \infty.$$

Passing to a further subsequence, the points $\mathcal{B}_j$ converge to a set of points $\mathcal{B}_\infty$ and the surfaces Σ_j converge locally smoothly, away from $\mathcal{B}_\infty$, to some lamination $\mathcal{L} \subset M \setminus \mathcal{B}_\infty$. The lamination has removable singularities, i.e., there is a smooth lamination $\widetilde{\mathcal{L}} \subset M$ so that $\mathcal{L} = \widetilde{\mathcal{L}} \setminus \mathcal{B}_\infty$. Moreover, there exists $\varepsilon_0 > 0$ smaller than the injectivity radius of (M, g) so that $\mathcal{B}_\infty$ is $4\varepsilon_0$-separated and for any $\varepsilon \in (0, \varepsilon_0]$, taking j sufficiently large guarantees that

(1) Writing Σ_j' for the components of $\Sigma_j \cap B_{2\varepsilon}(\mathcal{B}_\infty)$ containing at least one point from $\mathcal{B}_j$, no component of Σ_j' is a topological disk, so we call Σ_j' the "neck components." They have the following additional properties:
   (a) The surface Σ_j' intersects $\partial B_\varepsilon(\mathcal{B}_\infty)$ transversely in at most $m(I)$ simple closed curves.
   (b) Each component of Σ_j' is unstable.
   (c) The genus of Σ_j' is bounded above by $r(I)$.
   (d) The area of Σ_j' is uniformly bounded, i.e.,

$$\limsup_{j\to\infty} \operatorname{Area}_g(\Sigma_j') \leq 2\pi m(I)\varepsilon^2(1+o(\varepsilon))$$

(2) Writing Σ_j'' for the components of $\Sigma_j \cap B_{2\varepsilon}(\mathcal{B}_\infty)$ that do not contain any points in $\mathcal{B}_j$, each component of Σ_j'' is a topological disk, so we call Σ_j'' the "disk components." Moreover, we have the following additional properties
   (a) The curvature of Σ_j'' is uniformly bounded, i.e.,

$$\limsup_{j\to\infty} \sup_{x\in\Sigma_j''} |\mathrm{II}_{\Sigma_j}|(x) < \infty.$$

   (b) Each component of Σ_j'' has area uniformly bounded above by $2\pi\varepsilon^2(1+o(\varepsilon))$.

As is clear from the proof, it would be possible to give explicit bounds for $m(I)$ and $r(I)$, if one desired.

A key application of our result is a prescription for performing "surgery" on a sequence of bounded index minimal surfaces so that their curvature remains bounded, while only changing the topology and geometry in a controllable way: There exist functions $\tilde{r}(I)$ and $\tilde{m}(I)$ with the following property. Fix a closed three-manifold (M^3, g) and suppose that $\Sigma_j \subset (M^3, g)$ is a sequence of closed embedded minimal surfaces with

$$\operatorname{Index}(\Sigma_j) \leq I.$$

Then, after passing to a subsequence, there is a finite set of points $\mathcal{B}_\infty \subset M$ with $|\mathcal{B}_\infty| \leq I$ and $\varepsilon_0 > 0$ smaller than the injectivity radius of (M, g) so $\mathcal{B}_\infty$ is $4\varepsilon_0$-separated, and so that for $\varepsilon \in (0, \varepsilon_0]$, if we take j sufficiently large then there exist embedded surfaces $\widetilde{\Sigma}_j \subset (M^3, g)$ satisfying:

(1) The new surfaces $\widetilde{\Sigma}_j$ agree with Σ_j outside of $B_\varepsilon(\mathcal{B}_\infty)$.
(2) The components of $\Sigma_j \cap B_\varepsilon(\mathcal{B}_\infty)$ that do not intersect the spheres $\partial B_\varepsilon(\mathcal{B}_\infty)$ transversely and those that are topological disks appear in $\widetilde{\Sigma}_j$ without any change.
(3) The curvature of $\widetilde{\Sigma}_j$ is uniformly bounded, i.e.

$$\limsup_{j\to\infty} \sup_{x\in\widetilde{\Sigma}_j} |\mathrm{II}_{\widetilde{\Sigma}_j}|(x) < \infty.$$

(4) Each component of $\widetilde{\Sigma}_j \cap B_\varepsilon(\mathcal{B}_\infty)$ which is not a component of $\Sigma_j \cap B_\varepsilon(\mathcal{B}_\infty)$ is a topological disk of area at most $2\pi\varepsilon^2(1+o(\varepsilon))$.

(5) The genus drops in controlled manner, i.e.,

$$\text{genus}(\Sigma_j) - \tilde{r}(I) \leq \text{genus}(\widetilde{\Sigma}_j) \leq \text{genus}(\Sigma_j).$$

(6) The number of connected components increases in a controlled manner, i.e.,

$$|\pi_0(\Sigma_j)| \leq |\pi_0(\widetilde{\Sigma}_j)| \leq |\pi_0(\Sigma_j)| + \tilde{m}(I).$$

(7) While $\widetilde{\Sigma}_j$ is not necessarily minimal, it is asymptotically minimal in the sense that $\lim_{j\to\infty} \|H_{\widetilde{\Sigma}_j}\|_{L^\infty(\widetilde{\Sigma}_j)} = 0$, where H denote the mean curvature.

The new surfaces $\widetilde{\Sigma}_j$ converge locally smoothly to the smooth minimal lamination $\widetilde{\mathcal{L}}$ defined previously. Thus, modulo controlled errors, *i.e.*, going from Σ_j to $\widetilde{\Sigma}_j$, the sequence behaves just like a sequence of stable minimal surfaces. This allows us to draw parallels between known theorems for stable minimal surfaces and minimal surfaces with bounded index. We refer to [1] for this and other related results - including some generalizations to dimensions 4,5,6,7.

References

[1] Otis Chodosh, Dan Ketover, Davi Maximo, *Minimal hypersurfaces with bounded index*, Preprint, 2015.

[2] Richard Schoen, *Estimates for stable minimal surfaces in three-dimensional manifolds*, Seminar on minimal submanifolds, Ann. of Math. Stud. **103**, Princeton Univ. Press, Princeton, NJ, 1983, pp. 111–126.

Complex Monge-Ampère equations on complex and almost complex manifolds

Ben Weinkove

(joint work with Jianchun Chu, Gábor Székelyhidi, Valentino Tosatti)

Let (M, J) be a compact complex manifold of complex dimension n. We say that a Riemannian metric g is *Hermitian* if $g(JX, JY) = g(X, Y)$ for all vectors X, Y. Associated to g is a 2-form ω defined by $\omega(X, Y) := g(JX, Y)$ (abusing terminology, we will refer to ω as a metric). If ω is closed then g is *Kähler*.

Yau's Theorem [20] states that one can prescribe the volume form of a Kähler metric on a compact Kähler manifold. More precisely, let (M, J, ω) be a compact Kähler manifold, and $F \in C^\infty(M)$ with $\int_M e^F \omega^n = \int_M \omega^n$. Then there exists a unique Kähler metric $\tilde{\omega} \in [\omega] \in H^2(M; \mathbb{R})$ solving

$$\tilde{\omega}^n = e^F \omega^n. \tag{1}$$

By the $\partial\overline{\partial}$-Lemma one can write any Kähler metric $\tilde{\omega} \in [\omega]$ as $\tilde{\omega} = \omega + \sqrt{-1}\partial\overline{\partial}u$ for a smooth real-valued function u. Hence (1) can be written as the following complex Monge-Ampère equation for u,

$$(\omega + \sqrt{-1}\partial\overline{\partial}u)^n = e^F\omega^n, \quad \omega + \sqrt{-1}\partial\overline{\partial}u > 0, \tag{2}$$

and the solution is unique if we impose the normalization condition $\sup_M u = 0$, say.

This talk concerns extensions of Yau's Theorem to the case when the manifold is not Kähler. There are many different ways to do this. We consider five questions, and describe what is known so far in each case.

(1) **Hermitian metrics.** Does (2) admit a solution when ω is only Hermitian?

The answer to this is yes, at least up to a scaling factor. It was shown by Cherrier [2] for $n = 2$ (and for $n > 2$ with some extra conditions) and in general by Tosatti and the author [14] that one can find a unique pair (u, b) where $u \in C^\infty(M)$ and $b \in \mathbb{R}$ such that

$$(\omega + \sqrt{-1}\partial\overline{\partial}u)^n = e^{F+b}\omega^n, \quad \omega + \sqrt{-1}\partial\overline{\partial}u > 0, \quad \sup_M u = 0.$$

(2) **Gauduchon metrics.** A metric ω is *Gauduchon* if

$$\partial\overline{\partial}(\omega^{n-1}) = 0.$$

Every Hermitian metric is conformal to such a metric [7]. Suppose (M, J, ω) is compact Gauduchon. In 1984 Gauduchon [8] asked the question: given $F \in C^\infty(M)$, can we find a Gauduchon metric $\tilde{\omega}$ solving

$$\tilde{\omega}^n = e^F\omega^n \ ?$$

This result was proved recently by Székelyhidi, Tosatti and the author [11] using a certain Monge-Ampère equation related to $(n-1)$-plurisubharmonic functions (see also [6, 9, 10, 16, 17]).

(3) **Balanced metrics.** A metric ω is *balanced* if $d(\omega^{n-1}) = 0$. This is a stronger condition than Gauduchon and such metrics do not always exist on complex manifolds. Suppose (M, J, ω) is compact balanced. It was asked by Fu-Wang-Wu [6]: given $F \in C^\infty(M)$, can one find a balanced metric $\tilde{\omega}$ solving

$$\tilde{\omega}^n = e^F\omega^n \ ?$$

This is still an open question. Some evidence for the conjecture is provided in [16] where it was shown that if M also admits a Kähler metric ω_0 then one can find a balanced metric $\tilde{\omega}$ with prescribed volume form (up to scaling) such that $\tilde{\omega}^{n-1} = \omega^{n-1} + \sqrt{-1}\partial\overline{\partial}(u\,\omega_0^{n-2})$. It was shown in [11] that this also holds for ω_0 Astheno-Kähler (in the sense of Jost-Yau), namely $\partial\overline{\partial}\omega_0^{n-2} = 0$.

(4) **Non-integrable almost complex structures - Gromov's question.** Let (M, J, ω) be a compact symplectic manifold with a compatible almost complex structure (also known as an *almost-Kähler* manifold). In the 1990's Gromov asked the following (see [4]): given $F \in C^\infty(M)$ with $\int_M e^F\omega^n = \int_M \omega^n$ does there exist $u \in C^\infty(M)$ solving

$$(\omega + dJdu)^n = e^F\omega^n, \quad \omega + (dJdu)^{(1,1)} > 0 \ ?$$

It was shown by Delanöe [4] and Wang-Zhu [18] that the answer to this question is in general no. However, in a recent work of Chu, Tosatti and the author [3] it was shown that the answer is yes if $dJdu$ is replaced by $(dJdu)^{(1,1)}$, up to multiplying by a scaling factor e^b, as in (1).

(5) **Non-integrable almost complex structures - Donaldson's question.** Now let (M, J, ω) be a compact symplectic 4-manifold with a compatible almost complex structure, and assume the topological condition $b_2^+(M) = 1$. Donaldson [5] asked: given $F \in C^\infty(M)$ with $\int_M e^F \omega^2 = \int_M \omega^2$ does there exist a symplectic form $\tilde{\omega} \in [\omega]$ with

$$\tilde{\omega}^2 = e^F \omega^2 \ ? \tag{3}$$

This is still an open question. Some estimates were established in [13,19], giving partial results in this direction. Buzano-Fino-Vezzoni [1] recently showed that on the Kodaira-Thurston manifold, which is non-Kähler, one can solve (3) assuming an S^1 symmetry of the initial data. This improved an earlier result of Tosatti and the author [15] where T^2 symmetry was used.

An affirmative answer to Donaldson's conjecture (or more precisely, a slight generalization of it) would imply Donaldson's "tamed to compatible" conjecture [5]. This states that if a symplectic 4-manifold has a symplectic form ω taming an almost complex structure (i.e. $\omega^{(1,1)} > 0$), then it should admit a symplectic form compatible with J. This conjecture has been proved by Taubes [12], by completely different methods, under some additional conditions.

References

[1] E. Buzano, A. Fino and L. Vezzoni, *The Calabi-Yau equation on the Kodaira-Thurston manifold, viewed as an S^1-bundle over a 3-torus*, J. Differential Geom. **101** (2015), no. 2, 175–195.

[2] P. Cherrier, *Équations de Monge-Ampère sur les variétés Hermitiennes compactes*, Bull. Sc. Math (2) **111** (1987), 343–385.

[3] J. Chu, V. Tosatti and B. Weinkove, *The Monge-Ampère equation for non-integrable almost complex structures*, preprint, arXiv: 1603.00706.

[4] P. Delanöe, *Sur l'analogue presque-complexe de l'équation de Calabi-Yau*, Osaka J. Math. **33** (1996), no. 4, 829–846.

[5] S.K. Donaldson, *Two-forms on four-manifolds and elliptic equations*, in Inspired by S.S. Chern, World Scientific, 2006.

[6] J. Fu, Z. Wang and D. Wu, *Form-type Calabi-Yau equations*, Math. Res. Lett. **17** (2010), no. 5, 887–903.

[7] P. Gauduchon, *Le théorème de l'excentricité nulle*, C. R. Acad. Sci. Paris Sér. A-B **285** (1977), no. 5, A387–A390.

[8] P. Gauduchon, *La 1-forme de torsion d'une variété hermitienne compacte*, Math. Ann. **267** (1984), 495–518.

[9] F.R. Harvey and H.B. Lawson, *Geometric plurisubharmonicity and convexity: an introduction*, Adv. Math. **230** (2012), no. 4-6, 2428–2456.

[10] D. Popovici, *Aeppli cohomology classes associated with Gauduchon metrics on compact complex manifolds*, Bull. Soc. Math. France **143** (2015), no. 4, 763–800.

[11] G. Székelyhidi, V. Tosatti and B. Weinkove, *Prescribing the volume form of a Gauduchon metric*, preprint, arXiv:1503.04491.

[12] C.H. Taubes, *Tamed to compatible: symplectic forms via moduli space integration.*, J. Symplectic Geom. **9** (2011), no. 2, 161–250.

[13] V. Tosatti, B. Weinkove and S.-T. Yau, *Taming symplectic forms and the Calabi-Yau equation*, Proc. London Math. Soc. **97** (2008), no.2, 401–424.
[14] V. Tosatti and B. Weinkove, *The complex Monge-Ampère equation on compact Hermitian manifolds*, J. Amer. Math. Soc. **23** (2010), no. 4, 1187–1195.
[15] V. Tosatti and B. Weinkove, *The Calabi-Yau equation on the Kodaira-Thurston manifold*, J. Inst. Math. Jussieu **10** (2011), no. 2, 437–447.
[16] V. Tosatti and B. Weinkove, *The Monge-Ampère equation for $(n-1)$-plurisubharmonic functions on a compact Kähler manifold*, preprint, arXiv:1305.7511.
[17] V. Tosatti and B. Weinkove, *Hermitian metrics, $(n-1,n-1)$ forms and Monge-Ampère equations*, preprint, arXiv:1310.6326.
[18] H. Wang and P. Zhu, *On a generalized Calabi-Yau equation*, Ann. Inst. Fourier (Grenoble) **60** (2010), no. 5, 1595–1615.
[19] B. Weinkove, *The Calabi-Yau equation on almost-Kähler four-manifolds*, J. Differential Geom. **76** (2007), no. 2, 317–349.
[20] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation*, I, Comm. Pure Appl. Math. **31** (1978), no.3, 339–411.

Sharp Fundamental Gap Estimate on Convex Domains of Sphere

Guofang Wei
(joint work with Shoo Seto, Lili Wang)

Given a bounded smooth domain Ω in a Riemannian manifold M^n, the eigenvalues of the Laplacian on Ω with respect to the Dirichlet and Neumann boundary conditions are given by $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \cdots \to \infty$ and $0 = \mu_0 < \mu_1 \leq \mu_2 \cdots \to \infty$, respectively. There are many works in estimating the eigenvalues, especially the first eigenvalues. Estimating the gap between the first two eigenvalues, the fundamental (or mass) gap,

$$\Gamma(\Omega) = \begin{cases} \lambda_2 - \lambda_1 > 0 & \text{Dirichlet boundary} \\ \mu_1 > 0 & \text{Neumann boundary} \end{cases}$$

of the Laplacian or more generally for Schrödinger operators is also very important both in mathematics and physics. In this talk, we study the problem of obtaining optimal upper and lower bounds of the gap.

For Neumann boundary condition, it is the same as estimating the first nontrivial eigenvalue. In this case, for a convex domain in a Riemannian manifold with Ricci curvature bounded from below, μ_1 is greater than or equal to that of a 1-dim model by Kröger [8] using Li-Yau type gradient estimate. Equality is achieved when the dimension is 1. Later Andrews and Clutterbuck proved it using modulus of continuity [3]. Sharp upper bound is obtained for bounded domains in rank one symmetric spaces [1]. Here equality is achieved by geodesic balls.

For Dirichlet boundary condition, a sharp upper bound for $\lambda_2 - \lambda_1$ has been obtained for domains in the space of constant sectional curvature in [5–7] in their solution of the Payne-Polya-Weinberger conjecture. The optimal bound is achieved by geodesic balls. For convex domains $\Omega \subset \mathbb{R}^n$ with diameter D, it was independently conjectured by van den Berg, Ashbaugh and Benguria, Yau in the 80's that the gap $\Gamma(\Omega)$ has the sharp lower bound of $\frac{3\pi^2}{D^2}$. The subject has a long history, see the excellent survey by Ashbaugh [4] for discussion of the fundamental gap and history up to 2006. We only mention that in the influential paper, Singer, Wong, Yau and Yau [12] showed that $\Gamma(\Omega) \geq \frac{\pi^2}{4D^2}$. Yu and Zhong improved this to $\frac{\pi^2}{D^2}$. Only in 2011, the conjecture was completely solved by B. Andrews and J. Clutterbuck in their celebrated work [2] by establishing a sharp log-concavity estimate for the first eigenfunction, see also [10]. For convex domains on a sphere, Lee and Wang [9] showed the gap is $\geq \frac{\pi^2}{D^2}$.

In this talk we give a sharp lower bound on the gap for convex domains on sphere. One of our main result is the following.

Theorem. *Let $\Omega \subset S^n$ be a strictly convex domain with diameter D, λ_i $(i = 1, 2)$ be the first two eigenvalues of the Laplacian on Ω with Dirichlet boundary condition. Then*

$$\Gamma(\Omega) = \lambda_2 - \lambda_1 \geq \bar{\lambda}_2(n, D) - \bar{\lambda}_1(n, D) \text{ if } D \leq \frac{\pi}{2}, \tag{1}$$

where $\bar{\lambda}_i(n, D)$ are the first two eigenvalues of the operator $\frac{d^2}{ds^2} - (n-1)\tan(s)\frac{d}{ds}$ on $[-\frac{D}{2}, \frac{D}{2}]$ with Dirichlet boundary condition. Furthermore,

$$\bar{\lambda}_2(n, D) - \bar{\lambda}_1(n, D) \geq 3\frac{\pi^2}{D^2} \quad \text{if } D < \pi,\ n \geq 3.$$

In fact we prove for $D \in (0, \pi)$ that $D^2\left(\bar{\lambda}_2(n, D) - \bar{\lambda}_1(n, D)\right)$ is increasing in D when $n > 3$, equals to constant $3\pi^2$ when $n = 1$ or 3, and is decreasing in D when $n = 2$. For fixed D, we expect that the gap increases when the dimension gets bigger.

Corollary. *Let $\Omega \subset S^n$ be a strictly convex domain with diameter $D \leq \frac{\pi}{2}$, λ_i $(i = 1, 2)$ be the first two eigenvalues of the Laplacian on Ω with Dirichlet boundary condition. Then $\lambda_2 - \lambda_1 \geq 3\frac{\pi^2}{D^2}$ when $n \geq 3$.*

Remark. *This estimate is optimal. The same estimates are true for Schrödinger operator of the form $-\Delta + V$, where $V \geq 0$ and convex.*

The key to proving (1) is the following log-concavity of the first eigenfunction.

Theorem. *Let $\Omega \subset S^n$ be a strictly convex domain with diameter $D \leq \frac{\pi}{2}$, $\phi_1 > 0$ be a first eigenfunction of the Laplacian on Ω with Dirichlet boundary condition. Then for all $x, y \in \Omega$, with $x \neq y$,*

$$\langle \nabla \log \phi_1(y), \gamma'(\tfrac{d}{2})\rangle - \langle \nabla \log \phi_1(x), \gamma'(-\tfrac{d}{2})\rangle \leq 2\left(\log \bar{\phi}_1\right)'\left(\frac{d(x, y)}{2}\right), \tag{2}$$

where γ is the unit normal minimizing geodesic with $\gamma(-\frac{d}{2}) = x$ and $\gamma(\frac{d}{2}) = y$, and $\bar{\phi}_1 > 0$ is a first eigenfunction of the operator $\frac{d^2}{ds^2} - (n-1)\tan(s)\frac{d}{ds}$ on $[-\frac{D}{2}, \frac{D}{2}]$ with Dirichlet boundary condition with $d = d(x, y)$. Dividing (2) by $d(x, y)$ and letting $d(x, y) \to 0$, we have $\nabla^2 \log \phi_1 \leq -\bar{\lambda}_1\, g_{S^n}$.

This improves early estimate of Lee and Wang [9] that $\nabla^2 \log \phi_1 \leq 0$.

In the proof we work on spaces with constant sectional curvature. In particular, our proof works for spheres and Euclidean spaces at the same time. Some of our results hold also for negative constant curvature. For the log-concavity estimate, the last step fails for negative curvature. In fact we have several more general estimate, including parabolic version [11].

With the super log-concavity estimate, we have the following gap comparison.

First we introduce the 1-dimensional model space for the n-dimensional manifolds with constant sectional curvature K. Consider the operator

$$L_{n,K,D}(\phi) = \phi'' - (n-1)\,\mathrm{tn}_K(s)\phi'$$

on $[-\frac{D}{2}, \frac{D}{2}]$ with Dirichlet boundary condition, where

$$\mathrm{tn}_K(s) = \begin{cases} \sqrt{K}\tan(\sqrt{K}s), & K > 0 \\ 0, & K = 0 \\ -\sqrt{-K}\tanh(\sqrt{-K}s) & K < 0. \end{cases}$$

Denote $\bar{\lambda}_1(n, D, K)$, $\bar{\lambda}_2(n, D, K)$ its first and second eigenvalues and $\bar{\phi}_1 > 0$ a first eigenfunction.

Theorem. *Let Ω be a bounded convex domain with diameter D in a Riemannian manifold M^n with $\mathrm{Ric}_M \geq (n-1)K$, ϕ_1 a positive first eigenfunction of the Laplacian on Ω with Dirichlet boundary condition. Assume ϕ_1 satisfies the log-concavity estimates*

$$\langle \nabla \log \phi_1(y), \gamma'(\tfrac{d}{2})\rangle - \langle \nabla \log \phi_1(x), \gamma'(-\tfrac{d}{2})\rangle \leq 2\left(\log \bar{\phi}_1\right)'\left(\frac{d(x,y)}{2}\right),$$

where γ is the unit normal minimizing geodesic with $\gamma(-\frac{d}{2}) = x$, $\gamma(\frac{d}{2}) = y$, and $d = d(x, y)$. Then we have the gap comparison

$$\lambda_2 - \lambda_1 \geq \bar{\lambda}_2(n, D, K) - \bar{\lambda}_1(n, D, K).$$

As another application of (2), we obtain the following estimate.

Proposition. *Let Ω be a strictly convex domain with diameter D in $\mathbb{M}^n_K$ with $K \geq 0$, $D < \frac{\pi}{2\sqrt{K}}$ when $K > 0$. Then the first Dirichlet eigenvalues of the Laplacian on Ω satisfy $\lambda_1 \geq \frac{n\pi^2}{D^2} + \frac{n(n-1)}{2}K$.*

References

[1] A. R. Aithal and G. Santhanam, *Sharp upper bound for the first non-zero Neumann eigenvalue for bounded domains in rank-1 symmetric spaces*, Trans. Amer. Math. Soc. **348** (1996), 3955–3965.

[2] B. Andrews and J. Clutterbuck, *P*roof of the fundamental gap conjecture, J. Amer. Math. Soc., **2**4 (2011), 899–916.

[3] B. Andrews and J. Clutterbuck, *S*harp modulus of continuity for parabolic equations on manifolds and lower bounds for the first eigenvalue, Anal. PDE, **6** (2013), 1013–1024.

[4] M. S. Ashbaugh, *The Fundamental Gap*, http://www.aimath.org/WWN/loweigenvalues/ (2006).

[5] M. S. Ashbaugh and R. D. Benguria, *A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions*, Ann. of Math. (2) **135** (1992), 601–628.

[6] ———, *A sharp bound for the ratio of the first two Dirichlet eigenvalues of a domain in a hemisphere of S^n*, Trans. Amer. Math. Soc. **353** (2001), 1055–1087.

[7] R. D. Benguria and H. Linde, *A second eigenvalue bound for the Dirichlet Laplacian in hyperbolic space*, Duke Math. J. **140** (2007), 245–279.

[8] P. Kröger, *On the spectral gap for compact manifolds*, J. Differential Geom. **36** (1992), 315–330.

[9] Y. I. Lee and A. N. Wang, *Estimate of $\lambda_2 - \lambda_1$ on spheres*, Chinese J. Math. **15** (1987), 95–97.

[10] L. Ni, *Estimates on the modulus of expansion for vector fields solving nonlinear equations*, J. Math. Pures Appl. (9) **99** (2013), 1–16 (English, with English and French summaries).

[11] S. Seto, L. Wang and G. Wei, *S*harp Fundamental Gap Estimate on Convex Domains of Sphere, arXiv:1606.01212.

[12] I. M. Singer, B. Wong, S.-T. Yau, and S. S.-T. Yau, *An estimate of the gap of the first two eigenvalues in the Schrödinger operator*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **12** (1985), 319–333.

Applications of min-max in topology and geometry

Daniel Ketover

The min-max theory developed by Almgren-Pitts ([A], [P]) and Simon-Smith [SS] in the 80s produces from each non-trivial homotopy class in the space of surfaces in a fixed 3-manifold a smooth embedded minimal surface. The main challenge is to control the geometry and topology of the minimal surfaces that arise in this way. A main difficulty is to rule out the min-max limit from having multiplicity as the presence of multiplicity could mean that different homotopy classes produce integer copies of the same minimal surface and do not actually produce new critical points of the area functional.

Together with F.C. Marques and A. Neves we introduced the "catenoid estimate" that allows us to rule out multiplicities in several situations. The point of this estimate is that the unstable catenoid joining two very close parallel circles in $\mathbb{R}^3$ exceeds the area of the two flat disks subquadratically in the separation between the circles. Precisely, the catenoid estimate is as follows (where $U(r,h)$ denotes the unstable catenoid joining two parallel circles of radius r and separation h):

Proposition (Catenoid estimate in $\mathbb{R}^3$ [KMN])**.** *For $r > 0$ there exists $h(r) > 0$ so that if $h < h(r)$ then*

$$|U(r,h)| \leq 2\pi r^2 + \frac{4\pi h^2}{(-\log h)}. \tag{1}$$

This estimate allows us to "open up necks" about an unstable minimal surface in a three-manifold while keeping areas below twice that of the unstable minimal surface. More precisely, if Σ is an unstable minimal surface, we can make a sweep-out beginning at $\partial T_\epsilon(\Sigma)$ and ending at a one-dimensional graph on Σ, with all areas in the sweepout less than twice that of Σ. This observation allows us to rule out multiplicity in many situations and can be thought of as a one-parameter version of the log-cutoff trick as we use logarithmically cut off parallel surfaces rather than gluing in the catenoid explicitly.

One application that I explained is a min-max construction of "doublings of minimal surfaces," for instance the doubling of the Clifford torus first produced by the gluing method of Kapouleas. We have the following

Theorem ([KMN]). *For each $g \geq 2$, there exists a closed embedded minimal surface Σ_g resembling a doubled Clifford torus in $\mathbb{S}^3$. The area of Σ_g is strictly less than $4\pi^2$ (twice the area of the Clifford torus C). Moreover $\Sigma_g \to 2C$ in the sense of varifolds as $g \to \infty$ and the genus of Σ_g also approaches infinity. The surfaces Σ_g arise as min-max limits for a suitable equivariant saturation of sweepouts of $\mathbb{S}^3$.*

Another application of the catenoid estimate is to rule out that the width of an orientable manifold with Ric > 0 is a non-orientable minimal surface with multiplicity two. We have the following

Theorem ([KMN]). *If M is a three-manifold with positive Ricci curvature, and Γ a Heegaard surface realizing the Heegaard genus of M, then Γ is isotopic to an embedded minimal surface Σ of index 1. Moreover, Σ is the min-max limit obtained via Heegaard sweepouts of M determined by Γ.*

References

[A] F.J. Almgren Jr. *The theory of varifolds.* Mimeographed notes, Princeton University, 1965.

[KMN] D. Ketover, F.C. Marques and A. Neves. *Catenoid estimate and its geometric applications,*

[P] J.T. Pitts. *Existence and regularity of minimal surfaces on Riemannian manifolds.* Mathematical Notes, vol 27. Princeton University Press, Princeton, N.J., 1981.

[SS] F. Smith. *On the existence of embedded minimal two spheres in the three sphere, endowed with an arbitrary riemannian metric.* PhD thesis, Supervisor: Leon Simon, University of Melbourne, 1982.

Effective index estimates via Euclidean isometric embeddings

Alessandro Carlotto

(joint work with Lucas Ambrozio, Ben Sharp)

Given a compact Riemannian manifold (X^{n+1}, g) of dimension at least three, possibly with non-empty boundary, and considered the class

$$\Lambda := \{\text{compact minimal hypersurfaces in } (X^{n+1}, g), \quad \text{possibly with suitable boundary conditions}\}$$

there has been, in recent years, significant interest in establishing *universal comparison theorems* relating the Morse index with the topological invariants of an element M^n in Λ. More concretely, one would like to prove algebraic inequalities relating $Ind(M^n)$ and the dimension of the real homology groups of M^n with coefficients that only depend on the ambient manifold (X^{n+1}, g). A prototypical example is provided by the work of Savo [9], who was able to show that in the round three-sphere S^3 for a minimal surface of genus γ

$$Ind(M^2) \geq \frac{\gamma}{2} + 4$$

thereby providing a remarkable improvement of the well-known general bound by Urbano [10] asserting that the index of non-equatorial minimal surfaces is at least five, with equality attained only by Clifford tori. In fact, similar results in the case of three-dimensional flat tori had been obtained by Ros [8], who pioneered the study of the relations between the Dirichlet energy of a one-form and the *average value* of the Jacobi form $Q(\cdot,\cdot)$ (namely: the quadratic form defining the second variation of the area functional) evaluated on all the components of the form in question along a suitable ambient frame (indeed a *Euclidean* frame $\{dx^1, dx^2, dx^3\}$ in the special setting he was dealing with).

Motivated by a conjecture of Schoen, presented in extended form in the ICM lectures of F. Marques and A. Neves [6, 7], in [1] we generalized the methods above to handle the general case of an isometrically embedded compact Riemannian manifold without boundary. Roughly speaking, we were able to prove that if (X^{n+1}, g) is *sufficiently positively curved* with respect to the L^2-norm of the second fundamental form II^X of an isometric embedding $(X^{n+1}, g) \hookrightarrow \mathbb{R}^d$ then the estimate

$$Ind(M^n) \geq \frac{2}{d(d-1)} b_1(M^n)$$

holds true for all minimal hypersurfaces in X^{n+1}, where $b_1(M^n)$ denotes the first Betti number of the closed hypersurface M^n. We refer the reader to Theorem A in [1] for a precise statement, but we shall limit ourselves to state here that the curvature condition in question is satisfied in many cases of geometric interest like, for instance, all rank one symmetric spaces, product of spheres and suitably pinched three-manifolds. What is peculiar of our approach is the fact that the inequality we need to check is an *open* condition, so that our methods apply to spaces that are neither symmetric nor rigid in any reasonable sense.

It turns out that these tools effectively apply also to the study of free boundary minimal hypersurfaces in compact Riemannian manifolds with boundary, and in fact provide new insights even in the case of smooth domains inside Euclidean spaces. Sticking for the sake of simplicity to such setting, let us state our main result.

Theorem. *Let Ω^{n+1} be a smooth, compact domain of the $(n+1)$-dimensional Euclidean space, $n \geq 2$ and let M^n be a compact, orientable, properly embedded free boundary minimal hypersurface in Ω^{n+1}.*

(1) *If Ω^{n+1} is strictly mean convex, then*

$$index(M^n) \geq \frac{2}{n(n+1)} dim H_1(M^n, \partial M^{n-1}; \mathbb{R}).$$

(2) *If Ω^{n+1} is strictly two-convex, then*

$$index(M^n) \geq \frac{2}{n(n+1)} \max\left\{dimH_1(M^n, \partial M^{n-1}; \mathbb{R}); dimH_{n-1}(M^n, \partial M^{n-1}; \mathbb{R})\right\}.$$

Now, in the basic case of surfaces inside three-dimensional mean convex domains we derive the general inequality

$$index(M^2) \geq \frac{1}{3}(2\gamma + r - 1)$$

which, in turn, has the following remarkable corollaries:

- the examples constructed by Fraser and Schoen [5], which have genus zero and an arbitrary number of boundary components, and the examples constructed by Folha, Pacard and Zolotareva [3], which have genus one and an arbitrarily large number of boundary components, have their Morse indices growing linearly with the number of boundary components (in particular, this ensures the existence of free boundary minimal surfaces of arbitrarily large index in the unit ball);
- if Ω^3 is strictly convex, then by [4] any sequence $\{M_i^2\}$ of compact, properly embedded free boundary minimal surfaces in Ω^3 that has uniformly bounded index has a subsequence converging smoothly and graphically to a compact properly embedded free boundary minimal surface M^2 in Ω^3.

In general, the case $n > 2$ is more delicate, but we can still get various results of geometric interest. For instance, it is still true that in a mean convex domain $index(M^n) \geq 2(r-1)/n(n+1)$ so that we immediately see that a free boundary, stable minimal hypersurface must have only one boundary component.

The results for general ambient manifolds can be briefly summarized as follows: assume that Ω^{n+1} is mean convex (resp. two-convex) and that for every non-zero vector field X on M^n,

$$\begin{aligned}&\int_M \left[tr_M(Rm^\Omega(\cdot, X, \cdot, X)) + Ric^\Omega(N,N)|X|^2\right] d\mu \\ &> \int_M \left[(|II^\Omega(\cdot, X)|^2 - |II^\Omega(X,N)|^2) + (|II^\Omega(\cdot, N)|^2 - |II^\Omega(N,N)^2|)|X|^2\right] d\mu\end{aligned}$$

(where Rm^Ω denotes the Riemann curvature tensor of Ω^{n+1}, II^Ω denotes the second fundamental form of Ω^{n+1} in $\mathbb{R}^d$, $II^{\partial\Omega}$ denotes the second fundamental form of $\partial\Omega$ in Ω, and N is a local unit normal vector field on M^n), then the conclusion of (1) (resp. (2)) holds true for M^n with coefficient $2/d(d-1)$ in lieu of $2/n(n+1)$. Once again, the condition above can be checked pointwise for a wide class of ambient spaces (the upper hemisphere being a basic example).

Our proofs strongly rely on the Euclidean isometric embedding for providing a global, parallel frame $\{\theta_1, \ldots, \theta_d\}$ so that, set $u_{ij} := \langle N \wedge \omega^\sharp, \theta_i \wedge \theta_j\rangle$ and using the Bochner identity combined with the Gauss equations one can single out curvature criteria (only involving Rm^Ω and II^Ω, thus independent of the second fundamental form of M^n inside Ω^{n+1}) which ensure

$$\sum_{i<j} Q(u_{ij}, u_{ij}) < 0$$

for all harmonic 1-forms, possibly subject to suitable boundary conditions in the free boundary case (in fact, different boundary conditions will correspond to spaces of forms that are isomorphic either to $H_1(M, \partial M; \mathbb{R})$ or to $H_{n-1}(M, \partial M; \mathbb{R})$). For instance, when proving part (1) above in the Euclidean setting, such curvature criterion is precisely just the strict mean convexity of the boundary of Ω^{n+1}.

At that stage, one simply considers the map

$$\begin{array}{cccc} \Phi: & \mathcal{H}^1_{bc}(M,g) & \to & \mathbb{R}^{n(n+1)k/2} \\ & \omega & \mapsto & \left[\int_M \langle N \wedge \omega^\sharp, \theta_i \wedge \theta_j \rangle \phi_q d\mu\right], \end{array}$$

where $\phi_1, \ldots, \phi_k$ are the eigenfunctions associated to negative eigenvalues for the second variation of the area functional. If, by contradiction, the index inequality were false, then this map would have a non-trivial kernel and hence we would have for each $1 \leq i < j \leq d$

$$0 > Q(u_{ij}, u_{ij}) \geq \lambda_{k+1} \int_M u_{ij}^2 d\mu \geq 0$$

which is impossible and thus completes the argument.

References

[1] L. Ambrozio, A. Carlotto and B. Sharp, *Comparing the Morse index and the first Betti number of minimal hypersurfaces*, preprint (arXiv:1601.08152).

[2] L. Ambrozio, A. Carlotto and B. Sharp, *Index estimates for free boundary minimal hypersurfaces*, preprint (arXiv:1605.09704).

[3] A. Folha, F. Pacard and T. Zolotareva, *Free boundary minimal surfaces in the unit 3-ball*, preprint (arXiv:1502.06812).

[4] A. Fraser and M. Li, *Compactness of the space of embedded minimal surfaces with free boundary in three-manifolds with nonnegative Ricci curvature and convex boundary*, J. Differential Geom. **96** (2014), no. 2, 183-200.

[5] A. Fraser and R. Schoen, *Sharp eigenvalue bounds and minimal surfaces in the ball*, Invent. Math. **203** (2016), no. 3, 823-890.

[6] F. Marques, *Minimal surfaces - variational theory and applications*, Proceedings of the International Congress of Mathematicians, Seoul 2014.

[7] A. Neves, *New applications of Min-max Theory*, Proceedings of the International Congress of Mathematicians, Seoul 2014.

[8] A. Ros, *One-sided complete stable minimal surfaces*, J. Differential Geom. **74** (2006), no. 1, 69-92.

[9] A. Savo, *Index bounds for minimal hypersurfaces of the sphere*, Indiana Univ. Math. J. **59** (2010), no. 3, 823-837.

[10] F. Urbano, *Minimal surfaces with low index in the three-dimensional sphere*, Proc. Amer. Math. Soc. **108** (1990), no. 4, 989-992.

Reporter: Franziska Beitz

Participants

Prof. Dr. Rui Albuquerque
C I M A
Universidade de Evora
Rua Romao Ramalho, 59
7000-671 Evora
PORTUGAL

Dr. Lucas Ambrozio
Department of Mathematics
Imperial College London
Huxley Building
180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Prof. Dr. Sigurd B. Angenent
Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Drive
Madison, WI 53706-1388
UNITED STATES

Franziska Beitz
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Dr. Renato G. Bettiol
Department of Mathematics
David Rittenhouse Laboratory
University of Pennsylvania
209 South 33rd Street
Philadelphia, PA 19104-6395
UNITED STATES

Prof. Dr. Christoph Böhm
Mathematisches Institut
Universität Münster
48149 Münster
GERMANY

Prof. Dr. Esther Cabezas-Rivas
Institut für Mathematik
Goethe-Universität Frankfurt
Postfach 111932
60325 Frankfurt am Main
GERMANY

Prof. Dr. Alessandro Carlotto
Institute for Theoretical Studies
ETH Zürich, CLV Building
HG G 61.2
Clausiusstrasse 47
8092 Zürich
SWITZERLAND

Prof. Dr. Xianzhe Dai
Department of Mathematics
University of California at
Santa Barbara
South Hall
Santa Barbara, CA 93106
UNITED STATES

Prof. Dr. Anand N. Dessai
Département de Mathématiques
Université de Fribourg
Perolles
Chemin du Musée 23
1700 Fribourg
SWITZERLAND

Dr. Fernando Galaz-Garcia
Institut für Algebra u. Geometrie
Karlsruher Institut für Technologie
Englerstrasse 2
76131 Karlsruhe
GERMANY

Prof. Dr. Alexander A. Gayfullin
Department of Geometry & Topology
Steklov Mathematical Institute
Gubkina Str. 8
119 991 Moscow GSP-1
RUSSIAN FEDERATION

Dr. Nadine Große
Mathematisches Institut
Universität Freiburg
Eckerstrasse 1
79104 Freiburg i. Br.
GERMANY

Prof. Dr. Ursula Hamenstädt
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Robert Haslhofer
Department of Mathematics
University of Toronto
40 St George Street
Toronto, Ont. M5S 2E4
CANADA

Dr. Martin Herrmann
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Prof. Dr. Nancy Hingston
Department of Mathematics
The College of New Jersey
2000 Pennington Road
Ewing, NJ 08628
UNITED STATES

Dr. Shouhei Honda
Mathematical Institute
Faculty of Science
Tohoku University
6-3 Aoba
Sendai 980-8578
JAPAN

Prof. Dr. Sergei V. Ivanov
St. Petersburg Branch of Steklov
Mathematical Institute of
Russian Academy of Science
Fontanka 27
St. Petersburg 191 023
RUSSIAN FEDERATION

Dorothea Jansen
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Dr. Martin Kerin
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Dan Ketover
Department of Mathematics
Imperial College London
South Kensington Campus
London SW7 2AZ
UNITED KINGDOM

Prof. Dr. Dan Knopf
Mathematics Department
The University of Texas at Austin
1 University Station C 1200
Austin TX 78712-0257
UNITED STATES

Dr. Ramiro Lafuente
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Dr. Yevgeny Liokumovich
Department of Mathematics
Imperial College London
South Kensington Campus
Huxley Building
London SW7 2AZ
UNITED KINGDOM

Prof. Dr. John Lott
Department of Mathematics
University of California, Berkeley
970 Evans Hall
Berkeley CA 94720-3840
UNITED STATES

Matthias Ludewig
Institut für Mathematik
Universität Potsdam
Postfach 601553
14415 Potsdam
GERMANY

Prof. Dr. Alexander Lytchak
Mathematisches Institut
Universität zu Köln
Weyertal 86 - 90
50931 Köln
GERMANY

Henrik Matthiesen
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Dr. Davi Máximo
Department of Mathematics
Stanford University
450 Serra Mall; Bldg. 380
Stanford CA 94305-2125
UNITED STATES

Dr. Ricardo Mendes
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Dr. Rafael Montezuma
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Artem Nepechiy
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Prof. Dr. André A. Neves
Department of Mathematics
Imperial College London
South Kensington Campus
London SW7 2AZ
UNITED KINGDOM

Prof. Dr. Lei Ni
Department of Mathematics
University of California, San Diego
La Jolla, CA 92093-0112
UNITED STATES

Dr. Ivaldo Nunes
Instituto de Matematica Pura e Aplicada
I M P A
Jardim Botanico
Estrada Dona Castorina 110
Rio de Janeiro 22460-320
BRAZIL

Prof. Dr. Anton Petrunin
Department of Mathematics
Pennsylvania State University
University Park, PA 16802
UNITED STATES

Dr. Marco Radeschi
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Saskia Roos
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Nicolau Sarquis Aiex
Department of Mathematics
Imperial College London
Huxley Building
South Kensington Campus
London SW7 2AZ
UNITED KINGDOM

Dr. Felix Schulze
Department of Mathematics
University College London
Gower Street
London WC1E 6BT
UNITED KINGDOM

Prof. Dr. Miles Simon
Institut für Analysis und Numerik
Otto-von-Guericke-Universität
Magdeburg
Universitätsplatz 2
39106 Magdeburg
GERMANY

Prof. Dr. Carlo Sinestrari
Dipartimento di Matematica
Universita di Roma "Tor Vergata"
Via della Ricerca Scientifica 1
00133 Roma
ITALY

Dr. Stephan Stadler
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstrasse 39
80333 München
GERMANY

Prof. Dr. Karl-Theodor Sturm
Institut für Angewandte Mathematik
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Song Sun
Department of Mathematics
Stony Brook University
Stony Brook, NY 11794-3651
UNITED STATES

Prof. Dr. Iskander A. Taimanov
Sobolev Institute of Mathematics
Siberian Branch of the Russian Academy
of Sciences
Koptyuga Prospect N4
630 090 Novosibirsk
RUSSIAN FEDERATION

Prof. Dr. Peter M. Topping
Mathematics Institute
University of Warwick
Gibbet Hill Road
Coventry CV4 7AL
UNITED KINGDOM

Prof. Dr. Guofang Wei
Department of Mathematics
University of California at Santa Barbara
South Hall
Santa Barbara, CA 93106
UNITED STATES

Prof. Dr. Ben Weinkove
Department of Mathematics
Northwestern University
Lunt Hall
2033 Sheridan Road
Evanston, IL 60208-2730
UNITED STATES

Prof. Dr. Neshan Wickramasekera
Department of Pure Mathematics
and Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Prof. Dr. Burkhard Wilking
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Prof. Dr. Zhou Zhang
School of Mathematics and Statistics
The University of Sydney
Sydney NSW 2006
AUSTRALIA