Mathematisches Forschungsinstitut Oberwolfach

Report No. 30/2016

DOI: 10.4171/OWR/2016/30

Hyperbolic Techniques in Modelling, Analysis and Numerics

Organised by
Rinaldo M. Colombo, Brescia
Philippe G. LeFloch, Paris
Christian Rohde, Stuttgart

19 June – 25 June 2016

Abstract. Several research areas are flourishing on the roots of the breakthroughs in conservation laws that took place in the last two decades. The meeting played a key role in providing contacts among the different branches that are currently developing. All the invitees shared the same common background that consists of the analytical and numerical techniques for nonlinear hyperbolic balance laws. However, their fields of applications and their levels of abstraction are very diverse.

The workshop was the unique opportunity to share ideas about analytical issues like the fine-structure of singular solutions or the validity of entropy solution concepts. It turned out that generalized hyperbolic techniques are able to handle the challenges posed by new applications. The design of efficient structure preserving methods turned out to be the major line of development in numerical analysis.

Mathematics Subject Classification (2010): 35L65, 74J40.

Introduction by the Organisers

The workshop *Hyperbolic Techniques in Modelling, Analysis and Numerics*, organized by Rinaldo M. Colombo (Brescia), Phillipe G. LeFloch (Paris) and Christian Rohde (Stuttgart) welcomed 46 invitees from eight different countries. The group of attendants included besides internationally renowned researchers doctoral students and young postdocs. The program consisted of longer comprehensive lectures but also of a small number of short presentations given by young researchers. The general topic of the workshop circled around the mathematical theory of hyperbolic partial differential equations, in particular of balance laws, which has seen an astonishing development in the last two decades. The progress in analysis and numerics was mainly driven by challenges from continuum mechanics, a prominent role being played by shock waves in gas dynamics. Modeling through hyperbolic partial differential equations has now become a cornerstone in many other branches of science, for instance wherever nonlinear transport phenomena occur. Many new models have been derived, which, in turn, pose completely new questions to the mathematical theory and to the numerical analysis of hyperbolic equations. Given this background, the meeting tried to exploit and foster all possible synergies. Apparently, new joint cooperations can be tracked back to the interaction during the workshop week.

An interesting new development in the field is the characterization of singularity development and transport in different instances of nonlinear wave equations by hyperbolic techniques. Alberto Bressan developped a program to describe the fine structure of sets of generic singularities in a wide class of wave equations. Stefano Bianchini presented new results on a detailed description of the entropy dissipation connected to the emergence of shock waves. Stefano Modena showed how a Lagrangian approach can be used to achieve a deeper insight in the behaviour and the structure of the solutions to hyperbolic conservation laws. Wave breaking in the Hunter-Saxton system was the topic of Anders Nordli. The understanding of the interaction of dispersive approximations and shock waves is far from being a settled problem. Michael Shearer reported on various phenomena related to Korteweg- and Boussinesq-type equations. Sylvie Benzoni demonstrated how modulation theory can help to understand discrete wave motion. Using variational time discretization Michael Westdickenberg gave an existence proof for measure valed solutions of the full Euler system including a characterization of the entropy dissipation. Eitan Tadmor showed how to derive new BV-estimates for the pressureless Euler equations in multiple space dimensions. The talk of Jan Giesselmann on relative entropies for Hamiltonian systems like Euler–Korteweg equations fitted also in this context. Christian Klingenberg broached the issue of the effect of different linearization levels in numerical schemes for multidimensional Euler equations. Even linear wave equations with rough coefficients can pose major difficulties to numerical discretisation methods as was shown by Franziska Weber. This applies even more for hyperbolic systems with uncertainty. Alina Chertock proposed a splitting method for the efficient stochastic Galerkin discretization of Euler systems. The challenges of low Mach number scenarios in astrophysical flows have also been discussed by Christian Klingenberg.

The analytical study of singular limits and associated numerical questions on the design of structure-preserving schemes for asymptotic regimes provided the joint chord for another block of contributions. The study of the zero-viscosity limit, i.e. the passage from a parabolic regularization towards a hyperbolic limit problem, is a seminal topic in the field. Driven by the needs of applied sciences, a much wider variety of asymptotic scenarios is analyzed currently. In this context, Andrea Corli gave a presentation on the study of nonlinear diffusion approximations. Within the workshop, Gianluca Crippa devoted his talk to the passage from non-local to local hyperbolic balance laws. Numerical aspects of non-local evolution equations have been the topic of Elena Rossi. Graziano Guerra presented rigorous results for the compressible-incompressible passage covering discontinuous solutions. On the numerical side a novel class of asymptotic preserving finite-volume schemes that deal efficiently with weakly compressible low Mach number flows has been advocated by Mária Lukáčová-Medviďová. Manuel Torrilhon reviewed the analysis and numerics for the whole hierarchy of moment systems for the Boltzmann equations. The construction of new stable discretization schemes for moment equation was the topic of the lecture given by Philippe Helluy. Konstantina Trivisa discussed related model hierarchies to describe phenomena of collective self-organization.

As mentioned before, new applications have been always a driving force for the field. Phase transition in compressible two-phase flows has been studied by Ferdinand Thein generalizing the classical Riemann solver concept. Athanasios Tzavaras showed how mixed type models can help to understand shear band instabilities. The control of hyperbolic transport systems for population dynamics has been discussed by Mauro Garavello. The study of in particular hyperbolic evolutions on manifolds and networks is still an emerging research field. This has been addressed in the presentation of Raul Borsche on chemotactic movement on graphs and the lecture of Helge Holden about a hyperbolic transport equation which allows the numerical verification of the well-known Braess paradoxon in traffic flow. Francesca Marcellini showed how hyperbolic Riemann solver techniques can be used to understand the behavior of road traffic at junctions.
Finally the workshop was closed with an overview talk also given by Helge Holden who summarized the state of the art in the field and proposed a number of new challenges.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Alina Chertock in the "Simons Visiting Professors" program at the MFO.

Workshop: Hyperbolic Techniques in Modelling, Analysis and Numerics

Table of Contents

Abstracts

On modulated equations for Hamiltonian PDEs

Sylvie Benzoni-Gavage
(joint work with C. Mietka, L. Miguel Rodrigues)

The zero dispersion limit in dispersive perturbations of hyperbolic PDEs is a challenging topic, which is well understood only for the Korteweg-de Vries equation [9, 10, 11, 12], and, to some extent, the cubic Schrödinger equation [8]. For more general equations like the Euler–Korteweg (EK) system or even generalized KdV equations (gKdV), a preliminary route consists in investigating *modulated equations.*

We have undertaken to explore this route for a general class of *Hamiltonian PDEs* that contains EK and gKdV, the former itself containing the fluid formulation of nonlinear Schrödinger equations (NLS) and various other models of mathematical physics.

These Hamiltonian PDEs are of the form

$$\partial_t \mathbf{U} = \partial_x(\mathbf{B}\,\delta\mathcal{H}(\mathbf{U}))\,, \tag{1}$$

where the unknown $\mathbf{U}$ is (possibly) vector-valued, $\mathbf{B}$ is a symmetric and nonsingular matrix, and $\delta\mathcal{H}(\mathbf{U})$ denotes the variational derivative of an energy $\mathcal{H} = \mathcal{H}(\mathbf{U}, \mathbf{U}_x)$ depending on $\mathbf{U}$ and its spatial derivative $\mathbf{U}_x$. In practice, we restrict to a framework that is compatible with the examples mentioned here above, in which $\mathbf{U}$ is either scalar-valued (*e.g.* for gKdV) or with values in $\mathbb{R}^2$ (*e.g.* for EK), and the energy $\mathcal{H}$ depends only on the first derivative of a single component of $\mathbf{U}$, in a quadratic manner.

Formally, the zero dispersion limit of the system (1) is obtained by substituting the 'standard' energy $E(U) := \mathcal{H}(\mathbf{U}, 0)$ for $\mathcal{H}(\mathbf{U}, \mathbf{U}_x)$, which yields the first order system of conservation laws

$$\partial_t \mathbf{U} = \partial_x(\mathbf{B}\,\nabla_{\mathbf{U}} E(\mathbf{U}))\,. \tag{2}$$

When linearized about a 'stable' constant state $\mathbf{U}_0$ this system admits harmonic waves that propagate at a speed independent of their frequency, namely one of the characteristic speeds. By contrast, linear waves associated with (1) are dispersive, since the dispersion relation involves the differential operator $\delta^2\mathcal{H}(\mathbf{U}_0)$, which is 'generically' of second order, instead of the matrix $\nabla^2_{\mathbf{U}} E(\mathbf{U}_0)$ for (2).

As far as nonlinear systems are concerned, the well-known shock waves propagated by (2) are expected to have dispersive counterparts that are oscillatory, unsteady solutions to (1). These dispersive patterns are referred to as *dispersive shocks*, and have been an active field of research for the last decades, especially from the physical point of view, see for instance [13].

Since the seminal work of Whitham in the late 1960s [14], *modulated equations* have been viewed as governing the propagation of oscillatory wave trains, and more precisely the evolution of averaged quantities associated with wave trains.

In particular, dispersive shocks were first interpreted in the light of modulated equations by Gurevich and Pitaevskii [7] in 1973. This was for KdV, and by extension we speak of the Gurevich–Pitaevski problem for the determination of dispersive shocks associated with any dispersive equation.

However, there has been some fuzziness in the terminology, in that the term dispersive shock can be meant to describe either exact solutions to the original dispersive PDEs or some (supposedly) approximate solutions to these PDEs that are actually associated with exact, rarefaction wave solutions to modulated equations. The former are in most cases – meaning, apart from the KdV case, basically – far from being known to exist. As regards the latter, their understanding has been improved thanks to a breakthrough by El [4] and subsequent work by himself and co-authors. See for instance the review paper by El and Hoefer [5] and references therein. Nevertheless, there is not yet a rigorous proof of the existence of these idealized dispersive shocks as rarefaction wave solutions to modulated equations associated with dispersive PDEs that are not completely integrable.

We have been working on modulated equations associated with abstract systems of the form (1) with the aim of filling this gap. So far, we have pointed out a simple set of coordinates in which modulated equations are in closed form and that sheds new light on the Gurevich–Pitaevskii problem.

To explain briefly how these modulated equations look like, let us recall that the building blocks of modulated equations are *periodic travelling wave* solutions to (1). If (1) has N equations, a periodic travelling wave is 'generically' parametrized by $(N+2)$ parameters. A 'slowly' modulated wave train is a perturbation of a periodic travelling wave obtained by letting parameters vary on large time scales and large space scales. Modulated equations are partial differential equations for the slowly varying parameters, obtained by averaging over a period of the underlying wave.

In their basic form it is not obvious that modulated equations are in closed form. It turns out that a suitable set of coordinates is given by

- the local wave number, denoted by k,
- the average value of the wave profile $\underline{\mathbf{U}}$, denoted by $\mathbf{M}$,
- another scalar dependent variable that can be expressed in a simple manner in terms of k, $\mathbf{M}$, and the average value of the *momentum*, or Benjamin's impulse, along the wave.

To be more precise, this last variable reads

$$\alpha := \frac{1}{k}\left(\langle \mathcal{Q}(\underline{\mathbf{U}})\rangle - \mathcal{Q}(\mathbf{M})\right),$$

where $\mathcal{Q}(\mathbf{U}) := \frac{1}{2}\mathbf{U}^T\mathbf{B}^{-1}\mathbf{U}$ is the momentum, and $\langle \mathcal{Q}(\underline{\mathbf{U}})\rangle$ denotes its average value along the wave. Introducing, in addition, the averaged energy $H := \langle \mathcal{H}(\underline{\mathbf{U}}, \underline{\mathbf{U}}_x)\rangle$, we have shown that it can be viewed as a function of $(k, \alpha, \mathbf{M})$, and that the modulated equations associated with (1) read

$$\left\{ \begin{aligned} &\partial_t k = \partial_x(\partial_\alpha H)\,, \\ &\partial_t \alpha = \partial_x(\partial_k H)\,, \\ &\partial_t \mathbf{M} = \partial_x(\mathbf{B}\,\nabla_{\mathbf{M}} H)\,. \end{aligned} \right. \tag{3}$$

It is to be noted that this formulation was obtained in the special case of the Euler–Korteweg system in Lagrangian coordinates by Gavrilyuk and Serre in [6] (also see [1] for further explanations). The appealing form (3) of modulated equations has a number of implications investigated in the forthcoming paper [2]. Also see [3] for an earlier study of modulated equations associated with an abstract system of the form (1).

References

[1] Sylvie Benzoni Gavage. Planar traveling waves in capillary fluids. *Differential Integral Equations*, 26(3-4):433–478, 2013.

[2] S. Benzoni-Gavage, C. Mietka and L. M. Rodrigues Modulated equations of Hamiltonian PDEs and dispersive shocks. In preparation.

[3] S. Benzoni-Gavage, P. Noble, and L. M. Rodrigues. Slow modulations of periodic waves in Hamiltonian PDEs, with application to capillary fluids. *J. Nonlinear Sci.*, 24(4):711–768, 2014.

[4] G. A. El. Resolution of a shock in hyperbolic systems modified by weak dispersion. *Chaos*, 15(3):037103, 21, 2005.

[5] G. A. El and M. A. Hoefer. Dispersive shock waves and modulation theory. Physica D, doi:10.1016/j.physd.2016.04.006, 2016.

[6] S. L. Gavrilyuk and D. Serre. A model of a plug-chain system near the thermodynamic critical point: connection with the Korteweg theory of capillarity and modulation equations.

[7] A. V. Gurevich and L. P. Pitaevskiĭ. Nonstationary structure of a collisionless shock wave. *Soviet Phys. JETP*, 38(3), 1974.

[8] S. Jin, C. D. Levermore and D. W. McLaughlin The semiclassical limit of the defocusing NLS hierarchy. *Comm. Pure Appl. Math.*, 52(5), 613–654, 1999.

[9] P. D. Lax and C. D. Levermore. The small dispersion limit of the Korteweg-de Vries equation. I. *Comm. Pure Appl. Math.*, 36(3):253–290, 1983.

[10] P. D. Lax and C. D. Levermore. The small dispersion limit of the Korteweg-de Vries equation. II. *Comm. Pure Appl. Math.*, 36(5):571–593, 1983.

[11] P. D. Lax and C. D. Levermore. The small dispersion limit of the Korteweg-de Vries equation. III. *Comm. Pure Appl. Math.*, 36(6):809–829, 1983.

[12] P. D. Lax, C. D. Levermore, and S. Venakides. The generation and propagation of oscillations in dispersive initial value problems and their limiting behavior. In *Important developments in soliton theory*, Springer Ser. Nonlinear Dynam., pages 205–241. Springer, Berlin, 1993.

[13] Wenjie Wan, Shu Jia and Jason W. Fleischer. Dispersive superfluid-like shock waves in nonlinear optics. *Nature Physics*, 3(1):46–51, 2007.

[14] G. B. Whitham. *Linear and nonlinear waves*. John Wiley & Sons Inc., New York, 1999. Reprint of the 1974 original.

Concentration of entropy dissipation for L^∞-entropy solutions of scalar conservation laws in one-space dimension

Stefano Bianchini
(joint work with Elio Marconi)

We consider the following problem: let u be a bounded entropy solution to the scalar conservation law

$$u_t + f(u)_x = 0, \qquad u \in [-M, M],\ f : \mathbb{R} \to \mathbb{R} \text{ smooth}, \tag{1}$$

with initial datum $u_0(x)$. Being an entropy solution, by definition for all convex entropies η it holds in distributions

$$\eta(u)_t + q(u)_x \leq 0, \tag{2}$$

where $q'(u) = f'(u)\eta'(u)$ is the entropy flux. In particular the r.h.s. of (2) is a negative locally bounded measure μ_η, with the additional property that $\mu_\eta(B) = 0$ for all Borel sets B such that $\mathcal{H}^1(B) = 0$: this last property is a consequence of being the divergence of an L^∞ vector field.

For BV solutions, Volpert's formula together with the definition of the entropy flux q gives that

$$\begin{aligned}
&\eta(u)_t + q(u)_x \\
&\quad = \eta'(u)\big(D_t^{\mathrm{cont}} u + f'(u) D_x^{\mathrm{cont}} u\big) \\
&\qquad + \sum_{i\in\mathbb{N}} \bigg\{ -\dot\gamma_i(t)\Big[\eta(u(t,x+)) - \eta(u(t,x-))\Big] \\
&\qquad\qquad + \Big[q(u(t,x+)) - q(u(t,x-))\Big]\bigg\} g_i(t)\mathcal{H}^1 \llcorner_{\mathrm{Graph}(\gamma_i)} \\
&\quad = \sum_{i\in\mathbb{N}} \bigg\{ -\dot\gamma_i(t)\Big[\eta(u(t,x+)) - \eta(u(t,x-))\Big] \\
&\qquad\qquad + \Big[q(u(t,x+)) - q(u(t,x-))\Big]\bigg\} g_i(t)\mathcal{H}^1 \llcorner_{\mathrm{Graph}(\gamma_i)},
\end{aligned}$$

where

(1) $D^{\mathrm{cont}} u = (D_t^{\mathrm{cont}} u, D_x^{\mathrm{cont}} u)$ is the continuous part of the measure Du,
(2) $u(t, x\pm)$ is the right/left limit of $u(t)$ at the point x,
(3) the curves γ_i are such that

$$D^{\mathrm{jump}} u = \sum_i \big(u(t,x+) - u(t,x-)\big) \begin{pmatrix} 1 \\ -\dot\gamma_i(t) \end{pmatrix} g_i(t)\mathcal{H}^1 \llcorner_{\mathrm{Graph}(\gamma_i)}.$$

In short we will say that *the entropy dissipation is concentrated*, meaning that the measure μ_η is concentrated on a $\mathcal{H}^1$-rectifiable set J. A simple superposition argument implies that J can be chosen to be independent on η.

For general L^∞-entropy solutions, in the case the flux is uniformly convex, the solution is BV for all positive times due to Oleinik estimate [9]

$$D_x u(t) \leq \frac{\mathcal{L}^1}{ct},$$

and then the above computation applies.

For more general flux functions, in [8] it has been proved that under the assumption that f has finitely many inflection points (together with a regularity assumption on the local behavior of f about an inflection point), then again the entropy is concentrated: here the set J is the set where the characteristic speed $f'(u(t,x))$ jumps, which has been proved to be a BV function in [4]. However for general flux f it can be shown that f' is not BV.

The main result is the following:

Theorem. *If u is a bounded entropy solution of a scalar conservation law, then the entropy dissipation is concentrated.*

No assumptions on the flux function f have been made, i.e. it can have flat parts of Cantor-like sets where $f'' = 0$. Such a statement is a corollary of a detailed description of the regularity of bounded entropy solutions, description which is at the core of this analysis.

References

[1] C. Bardos, A. Y. le Roux and J.-C. Nédélec, *First order quasilinear equations with boundary conditions*, Comm. Partial Differential Equations, 4 (1979), pp. 1017–1034.

[2] B. Ben Moussa and A. Szepessy, *Scalar conservation laws with boundary conditions and rough data measure solutions*, Methods Appl. Anal., 9 (2002), pp. 579–598.

[3] S. Bianchini and E. Marconi, *On the concentration of entropy for scalar conservation laws*, Discrete Contin. Dyn. Syst. Ser. S, 9 (2016), pp. 73–88.

[4] K. S. Cheng, *A regularity theorem for a nonconvex scalar conservation law*, J. Differential Equations, 61 (1986), pp. 79–127.

[5] C. M. Dafermos, *Hyperbolic conservation laws in continuum physics*, vol. 325 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, third ed., 2010.

[6] R. J. DiPerna, *Measure-valued solutions to conservation laws*, Arch. Rational Mech. Anal., 88 (1985), pp. 223–270.

[7] S. N. Kružkov, *First order quasilinear equations with several independent variables.*, Mat. Sb. (N.S.), 81 (123) (1970), pp. 228–255.

[8] C. D. Lellis and T. Riviere, *Concentration estimates for entropy measures*, Journal de Mathématiques Pures et Appliquées, 82 (2003).

[9] O. A. Oleĭnik, *Discontinuous solutions of non-linear differential equations*, Amer. Math. Soc. Transl. (2), 26 (1963), pp. 95–172.

[10] F. Otto, *Initial-boundary value problem for a scalar conservation law*, C. R. Acad. Sci. Paris Sér. I Math., 322 (1996), pp. 729–734.

[11] A. Szepessy, *Measure-valued solutions of scalar conservation laws with boundary conditions*, Arch. Rational Mech. Anal., 107 (1989), pp. 181–193.

Kinetic and macroscopic models for chemotaxis on networks

Raul Borsche

We are interested in cell movement on networks. This can be described by different models along the edges of the network, which are supplemented by coupling conditions at the nodes. In this work we develop coupling conditions for different chemotaxis models. The starting point is the description of cell movement along the edges by a kinetic model. Since the equation is hyperbolic, it is sufficient to focus on a single junction from which arbitrary networks can be constructed. Consider for $i \in \{1, \ldots, N\}$

$$\partial_t f_i + \frac{1}{\epsilon} v.\partial_x f_i = -\frac{\lambda}{\epsilon^2}\left(f_i - \frac{\rho}{2}\right) + \frac{1}{2\epsilon}\alpha v \overline{\partial_x m_i} \rho$$

$$\partial_t m_i - D(\partial_{xx}) m_i = \gamma_\rho \rho_i - \gamma_m m_i \ ,$$

where $\overline{\partial_x m} = \frac{\partial_x m}{\sqrt{1+|\partial_x m|^2}}$ and $\rho = \int_{-1}^{1} f(t,x,v)dv$. $f(t,v,x)$ is the density of cells at time $t \in [0,T]$, location $x \in \mathbb{R}^+$ and with velocity $v \in [-1,1]$. $m(t,x)$ is quantifying the chemoattractant emitted by the cells. At the boundary at $x = 0$ the values of $f(t,0,v)$ for $v > 0$ have to be prescribed. At a junction we determine these values from the outgoing quantities $f(t,0,v)$ for $v < 0$ in the following way

$$f^+ = Af^- \ , \ v > 0, \tag{1}$$

where $f^+ = f(t,0,v)$ and $f^- = f(t,0,-v)$.

From the kinetic model different macroscopic models can be derived [1]. In the following we consider a model hierarchy as shown in figure 1.

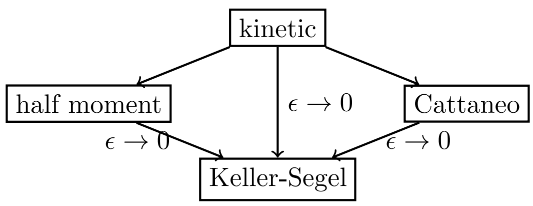

Figure 1. Hierarchy of models describing chemotaxis.

The half moment model is obtained by integrating the kinetic equation with respect to half spaces $v > 0$ and $v < 0$ and defining the macroscopic quantities

$$\rho^- = \int_{-1}^{0} f(v)dv \ , \quad \rho^+ = \int_{0}^{1} f(v)dv \ , \quad q^- = \int_{-1}^{0} vf(v)dv \ , \quad q^+ = \int_{0}^{1} vf(v)dv \ .$$

This set of equations can be closed using the ansatz functions $f(v) = a^+ + vb^+, \ v \geq 0, \ f(v) = a^- + vb^-, \ v \leq 0$, which leads to a hyperbolic model system for four unknowns. The corresponding coupling conditions are obtained by integrating (1) for positive velocities, as well as their first moment.

For the Cattaneo model consider the integral over the full velocity space with

$$\rho(x,t) = \int_{-1}^{1} f(x,t,v)dv \ , \qquad q(x,t) = \frac{1}{\epsilon}\int_{-1}^{1} vf(x,t,v)dv$$

and the linear closure ansatz $f(x,t,v) = \frac{1}{2}\rho(x,t) + \epsilon\frac{3}{2}vq(x,t)$. For the coupling conditions the closure is inserted into (1) and integrated over positive velocities. Thus the coupling conditions depend on the choice of the closure. The resulting equations are of the same form as the coupling conditions proposed by [2].

Finally the Keller-Segel model is obtained by considering the limit $\epsilon \to 0$ in the above models. Also the coupling conditions converge in all three cases to those conditions investigated in [4]. In numerical examples this convergence is analyzed numerically on networks using asymptotic preserving schemes [3]. Properties like the conservation of mass or positivity at the node persist from the kinetic level to the Keller-Segel equation.

In a future work we plan to investigate models assuring positive values for the densities and to extend this procedure to other phenomena on networks.

References

[1] N. Bellomo, A. Bellouquid, J. Nieto, J. Soler, *On the asymptotic theory from microscopic to macroscopic growing tissue models: an overview with perspectives* Math. Models Methods Appl. Sci. **22** no. 1 (2012), 1130001

[2] F.R. Guarguaglini and R. Natalini, *Global smooth solutions for a hyperbolic chemotaxis model on a network'*, SIAM J. Math. Anal. **47** no. 6 (2015), 4652–4671

[3] S. Jin, L. Pareschi and G. Toscani, *Diffusive relaxation schemes for multiscale discrete-velocity kinetic equations*, SIAM J. Numer. Anal. **35** no. 6 (1998), 2405–2439

[4] R. Borsche, S. Göttlich, A. Klar, P. Schillen, *The scalar Keller-Segel model on networks*, Math. Models Methods Appl. Sci. no. 2 **24** (2014) ,221–247

Generic singularities of solutions to some nonlinear wave equations

Alberto Bressan

For a wide class of nonlinear hyperbolic PDEs, it is well known that solutions can develop singularities within finite time.

In general, the structure of the set where the solution is not smooth can be extremely complicated. However, at least in the case of one space dimension, it is reasonable to expect that for generic initial data the solution develops singularities only along a finite set of points or curves in t-x space. Here "generic" should be understood in a topological sense, i.e., for all initial data in the intersection of countably many open dense sets in the space $\mathcal{C}^k(I\!R)$, for a suitable $k \geq 1$. Three main settings will be considered here.

1) Hyperbolic systems of conservation laws:

$$u_t + f(u)_x \ = \ 0. \tag{1}$$

For a scalar conservation law, Schaeffer [13] has shown that, for an open dense set of initial data in $\mathcal{C}^3$, the solution contains finitely many shocks, on any bounded region of the t-x plane. This result has been extended in [9] to a particular class of hyperbolic systems, where shock and rarefaction curves coincide. On the other hand, for 3×3 system, the recent analysis in [8] shows that a wide class of solutions can develop infinitely many jumps in finite time.

At the present time, the problem of generic regularity remains open for 2×2 systems of conservation laws, such as isentropic gas dynamics. Indeed, for such systems one conjectures that, for a generic initial data in $\mathcal{C}^3(I\!R,\, I\!R^2)$, the solution remains smooth outside a locally finite family of shock curves. We recall that, for 2×2 systems, a detailed description of the formation of new shocks was provided in [12].

2) The Burgers-Hilbert equation:

$$u_t + (u^2/2)_x \;=\; H[u], \tag{2}$$

where the right hand side contains the Hilbert transform of u. This equation was derived in [1] as a model of nonlinear waves with constant frequency. For initial data in $H^2(I\!R)$, the local existence and uniqueness of the solution to (2) was proved in [11]. Global existence in the space $\mathbf{L}^2(I\!R)$ was recently proved in [5]. However, the uniqueness and continuous dependence of these general solutions remains an open problem. Here the main difficulty stems from the fact that Burgers' equation generates a semigroup which is contractive in the space $\mathbf{L}^1(I\!R)$, but only Hölder continuous in $\mathbf{L}^2(I\!R)$. On the other hand, the Hilbert transform is a linear isometry in $\mathbf{L}^2$, but it is not continuous as a map from $\mathbf{L}^1$ into itself.

The detailed asymptotic structure of solutions of (2) near a shock has been studied in [6]. At the present time it remains to understand how new shocks are formed, and whether generic initial data yield solutions with finitely many shock curves in the t-x plane.

3) Variational wave equations:

$$u_{tt} - c(u)(c(u)u_x)_x \;=\; 0. \tag{3}$$

Here the wave speed $c(\cdot)$ is a smooth map, taking strictly positive values. As shown in [7], this equation can be rewritten as a first order semilinear system, by a suitable transformation of dependent and independent variables. This provides a method to construct global solutions to (3), for any initial data

$$(u(0,\cdot),\, u_t(0,\cdot)) \;=\; (u_0, u_1) \;\in\; H^1(I\!R)\times \mathbf{L}^2(I\!R).$$

Uniqueness of conservative solutions has recently been proved in [3].

For this equivalent semilinear system, smooth initial data yield globally smooth solutions. All singularities in the solution to the original wave equation arise from the change of variables.

The generic regularity of solutions to (3) is now well understood. Using Thom's transversality theorem and ideas from [10], it was shown in [2] that generic solutions of this wave equation are smooth outside a locally finite number of curves in the t-x plane. An asymptotic description of all types of singularities that can occur in a generic solution is given in [4].

References

[1] J. Biello and J. K. Hunter, *Nonlinear Hamiltonian waves with constant frequency and surface waves on vorticity discontinuities*, Comm. Pure Appl. Math. **63** (2009), 303–336.
[2] A. Bressan and G. Chen, *Generic regularity of conservative solutions to a nonlinear wave equation*, Ann. Inst. H.Poincaré Anal. Nonlin., to appear.
[3] A. Bressan, G. Chen, and Q. Zhang, *Unique conservative solutions to a variational wave equation*, Archive Rational Mech. Anal. **217** (2015), 1069–1101.
[4] A. Bressan, T. Huang, and F. Yu, *Structurally stable singularities for a nonlinear wave equation*, Bull. Inst. Math. Acad. Sinica, **10** (2015), 449–478.
[5] A. Bressan and K. Nguyen, *Global existence of weak solutions for the Burgers-Hilbert equation*, SIAM J. Math. Anal. **46** (2014), 2884–2904.
[6] A. Bressan and T. Zhang, *Piecewise smooth solutions to the Burgers-Hilbert equation*, Comm. Math. Sci., to appear.
[7] A. Bressan and Y. Zheng, *Conservative solutions to a nonlinear variational wave equation*, Comm. Math. Phys. **266** (2006), 471–497.
[8] L. Caravenna and L. Spinolo, *Schaeffer's regularity theorem for scalar conservation laws does not extend to systems*, Indiana Univ. Math. J., to appear.
[9] C. Dafermos and X. Geng, *Generalized characteristics uniqueness and regularity of solutions in a hyperbolic system of conservation laws*, Ann. Inst. H. Poincaré Anal. Non Linéaire **8** (1991), 231–269.
[10] J. Damon, *Generic properties of solutions to partial differential equations*, Arch. Rational Mech. Anal. **140** (1997) 353–403.
[11] J. K. Hunter and M. Ifrim, *Enhanced life span of smooth solutions of a Burgers-Hilbert equation*, SIAM J. Math. Anal. **44** (2012), 2039–2052.
[12] D.-X. Kong, *Formation and propagation of singularities for 2×2 quasilinear hyperbolic systems*, Trans. Amer. Math. Soc. **354** (2002) 3155–3179.
[13] D. Schaeffer, *A regularity theorem for conservation laws*, Adv. in Math. **11** (1973), 368–386.

An operator splitting based stochastic Galerkin method for nonlinear systems of conservation laws with uncertainty

Alina Chertock

(joint work with Shi Jin, Alexander Kurganov)

We introduce a flux-splitting based stochastic Galerkin methods for nonlinear systems of hyperbolic conservation/ balance laws with random inputs. The method uses a generalized polynomial chaos approximation in the stochastic Galerkin framework (referred to as the gPC-SG method). It is well-known that such approximations for nonlinear system of hyperbolic conservation laws do not necessarily yield globally hyperbolic systems: the Jacobian may contain complex eigenvalues and thus trigger instabilities and ill-posedness.

In this talk, we present a systematic way to overcome this difficulty. The main idea is to split the underlying system of conservation laws into a linear hyperbolic system, and a nonlinear degenerated hyperbolic system which can be solved successively as scalar conservation laws with variable coefficients and source terms. The gPC-SG method, when applied to each of these subsystems, result in globally hyperbolic systems. The performance of the new gPC-SG method will be illustrated on a number of numerical examples including the compressible Euler equations [1] and the Saint-Venant system of shallow water equations [2].

References

[1] A. Chertock, S. Jin and A. Kurganov, *An operator splitting based stochastic Galerkin method for the one-dimensional compressible Euler equations with uncertainty*, submitted.
[2] A. Chertock, S. Jin and A. Kurganov, *A well-balanced operator splitting based stochastic Galerkin method for the one-dimensional Saint-Venant system with uncertainty*, submitted.

Semi-wavefronts in models of collective movements with density-dependent diffusivity

Andrea Corli
(joint work with Lorenzo di Ruvo and Luisa Malaguti)

We consider the scalar parabolic equation

$$\rho_t + f(\rho)_x = \left(D(\rho)\rho_x\right)_x + g(\rho), \qquad (x,t) \in \mathbb{R} \times [0,+\infty), \tag{1}$$

where $f \in C^1[0,\overline{\rho}]$, $f(0) = 0$, $g \in C[0,\overline{\rho}]$ and $D \in C^1[0,\overline{\rho}]$, for some $\overline{\rho} > 0$. On the diffusivity we assume that $D(\rho) > 0$ for $\rho \in (0,\overline{\rho})$, allowing however that D can vanish at either 0 or $\overline{\rho}$, or even at both points. About the forcing term g we assume $g(\rho) > 0$ for $\rho \in [0,\overline{\rho})$ but $g(\overline{\rho}) = 0$.

The reaction-diffusion-convection equation (1), with D as above, models several physical and biological phenomena; probably the most known of them is the fluid flow through porous media. However, our main source of inspiration has been the appearance of (1) with $g = 0$ in the framework of collective movements, namely, traffic flows and crowd dynamics [2]. As far as regards model (1), the source term g can be thought as modeling diffused entries [1].

We are concerned with traveling-wave solutions of (1), namely, special solutions of (1) of the form $\rho(x,t) = \varphi(x - ct)$. In this case the profile φ must satisfy the ordinary differential equation

$$\left(D(\varphi)\varphi'\right)' + \left(c - f'(\varphi)\right)\varphi' + g(\varphi) = 0. \tag{2}$$

Under the previous assumptions, one easily understands that we are faced to two main difficulties: the degeneracy of the diffusivity and the fact that equation (1) has only one equilibrium point. Indeed, we prove [3] that the latter excludes the possibility of traveling waves defined in the whole of $\mathbb{R}$.

In [3] we prove the existence of semi-wavefront solutions [5] for every wave speed c. We also give precise results about the slopes of the profiles when they reach 0 and characterize their strict monotony through suitable assumptions on the source term g. We fully discuss as well the singular case when D is no more differentiable at 0 but $\dot{D}(0) = \pm\infty$. Furthermore, in the case $D(\overline{\rho}) = 0$ we also analyze the possibility of *sharp* (i.e., non smooth) semi-wavefront solutions [4] and completely characterize when this occurs.

The key remark we exploit in the proof is that every profile $\varphi = \varphi(\xi)$ is strictly monotone in the region where $0 \leq \varphi(\xi) < \overline{\rho}$; hence, it is invertible there, with inverse function $\xi = \xi(\varphi)$, $\varphi \in [0, \overline{\rho})$. This allows us to reduce the second-order equation (2) to a first-order equation by defininig $z(\varphi) := D(\varphi)\varphi'(\xi(\varphi))$, $\varphi \in (0, \overline{\rho})$. Then z satisfies the singular equation [5, 6]

$$\dot{z}(\varphi) = h(\varphi) - c - \frac{D(\varphi)g(\varphi)}{z(\varphi)}, \qquad \varphi \in (0, \overline{\rho}). \tag{3}$$

We look for solutions of (3) vanishing at $\overline{\rho}$; among them, the possibility that they also vanish at 0 (when also D does) provides informations about the slope of the profile at 0.

References

[1] P. Bagnerini, R. M. Colombo and A. Corli, *On the role of source terms in continuum traffic low models*, Math. Comput. Model., **44**(9-10) (2006), 917–930.

[2] L. Bruno, A. Tosin, P. Tricerri and F. Venuti*Non-local first-order modelling of crowd dynamics: a multidimensional framework with applications*, Appl. Math. Model., **35**(1) (2011), 426–445.

[3] A. Corli and L. Malaguti, *Semi-wavefront solutions in models of collective movements with density-dependent diffusivity*, Submitted (2016).

[4] A. Corli, L. di Ruvo and L. Malaguti, *Sharp profiles in models of collective movements*, Submitted (2016).

[5] B. H. Gilding and R. Kersner, *Travelling waves in nonlinear diffusion-convection reaction*, Birkhăuser Verlag, Basel, 2004.

[6] L. Malaguti and C. Marcelli, *Finite speed of propagation in monostable degenerate reaction-diffusion-convection equations*, Adv. Nonlinear Stud., **5**(2) (2005), 223–252.

The nonlocal-to-local limit for conservation laws

Gianluca Crippa

(joint work with Maria Colombo, Laura V. Spinolo)

Nonlocal conservation laws appear in the modeling of a large number of phenomena, for instance in the study of traffic problems. Given a convolution kernel ρ_ε we focus on the following Cauchy problem:

$$\begin{cases} \partial_u u_\varepsilon + \partial_x\big((u_\varepsilon * \rho_\varepsilon)\, u_\varepsilon\big) = 0 \\ u_\varepsilon(t=0) = \bar{u}\,. \end{cases} \tag{1}$$

For any given $\varepsilon > 0$ the Cauchy problem (1) is well posed, see for instance [3]. In fact, well posedness holds in much larger generality: instead of (1), it is possible to consider multidimensional systems, general nonlinearities in the second term, and measure solutions. However, for simplicity of exposition, we restrict our attention to the case of (1).

In [1] the question of the behavior of the solution u_ε of (1) when $\varepsilon \downarrow 0$ was raised. Supported by some numerical experiments, the authors were led to conjecture that the unique solution u_ε of (1) converges to the unique entropic solution u of Burgers' equation

$$\begin{cases} \partial_t u + \partial_x(u^2) = 0 \\ u(t=0) = \bar{u}\,. \end{cases} \tag{2}$$

From an analytical point of view this question is very challenging. In terms of a priori estimates, we easily see that (1) conserves the L^1 norm uniformly with respect to ε, while the L^∞ norm and the BV norm may blow up when $\varepsilon \downarrow 0$. The uniform bound in L^1 gives weak compactness of u_ε in the sense of measures. However this is not sufficient in order to pass to the limit in the nonlinearity.

A first result regarding the convergence was provided in [4]: if the solution u of (2) is sufficiently regular in $[0,T[\times\mathbb{R}$, and if the kernel ρ_ε is even (i.e., $\rho_\varepsilon(-x) = \rho_\varepsilon(x)$ for all x), then u_ε converges to u in $[0,T[\times\mathbb{R}$.

In our work we investigate the question of the convergence in the case of non smooth solutions. We construct the following three counterexamples to the convergence of u_ε to u.

(a) If ρ_ε is even and $\bar{u}$ changes sign, then in general u_ε does not converge weakly to u. To show this, we consider

$$\bar{u} = \begin{cases} 1 & \text{for } x < 0 \\ -1 & \text{for } x > 0. \end{cases}$$

We can check that u_ε remains odd for any $t > 0$. This implies that $(u_\varepsilon * \rho_\varepsilon)(0) = 0$ at any time. It follows that the total "mass" of u_ε on $\{x < 0\}$ and on $\{x > 0\}$ are conserved for all times, while the entropic solution u "loses mass" on the zero-speed shock placed at $x = 0$. This is incompatible with the weak convergence of u_ε to u.

(b) If $\bar{u}$ is nonnegative and ρ_ε is supported on $\{x < 0\}$, then in general u_ε does not converge weakly to u. We can consider

$$\bar{u} = \begin{cases} 1 & \text{for } x < 0 \\ 0 & \text{for } x > 0. \end{cases}$$

The entropic solution u of (2) consists of a shock with speed 1. On the other hand, for any $\varepsilon > 0$, we can check that $u_\varepsilon \equiv 0$ on $\{x > 0\}$ for any $t > 0$. This is incompatible with the weak convergence of u_ε to u.

(c) If $\bar{u}$ is nonnegative and ρ_ε is even, then in general u_ε does not converge strongly in $L^{1+\nu}$ to u for any $\nu > 0$. The argument is based on the conservation in time, when ρ_ε is even, of the quantity

$$\int_{\mathbb{R}} u_\varepsilon \log u_\varepsilon \, dx \tag{3}$$

for any nonnegative solution u_ε of (1). Choosing

$$\bar{u} = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } 0 < x < 1 \\ 0 & \text{for } x > 1, \end{cases}$$

we deduce from (3) that $\int u_\varepsilon \log u_\varepsilon \, dx = 0$ for any $t > 0$ and $\varepsilon > 0$. If one had $u_\varepsilon \to u$ in $L^{1+\nu}$, it would follow that $\int u \log u \, dx = 0$. However, the entropic solution u of (2) consists of a rarefaction fan and an entropic shock, so that $\int u \log u \, dx$ has to be strictly negative.

In addition, we study the nonlocal-to-local convergence in presence of viscosity in both (1) and (2). In detail, for $\mu > 0$ we consider

$$\begin{cases} \partial_u u_{\varepsilon,\mu} + \partial_x\big((u_{\varepsilon,\mu} * \rho_\varepsilon)\, u_{\varepsilon,\mu}\big) = \mu \partial_{xx} u_{\varepsilon,\mu} \\ u_{\varepsilon,\mu}(t=0) = \bar{u} \end{cases} \tag{4}$$

and

$$\begin{cases} \partial_t u_\mu + \partial_x(u_\mu^2) = \mu \partial_{xx} u_\mu \\ u_\mu(t=0) = \bar{u}\,. \end{cases} \tag{5}$$

This is relevant both theoretically and in connection to the numerical experiments in [1] (which may be influenced by the presence of numerical viscosity). Extending a previous result in [2] (restricted to smooth solutions and specific to the viscous Burgers' equation) we show that the unique solution $u_{\varepsilon,\mu}$ of (4) converges to the unique solution u_μ of (5) strongly in L^2, for any $\bar{u} \in L^2 \cap L^\infty$. This result does not require assumptions neither on the sign of $\bar{u}$ nor on the symmetry of ρ_ε.

We can summarize our results in the following convergence scheme:

$$\begin{array}{ccc} u_{\varepsilon,\mu} & \xrightarrow{(A)} & u_\mu \\ \Big\downarrow{\scriptstyle (B)} & & \Big\downarrow{\scriptstyle (C)} \\ u_\varepsilon & \xrightarrow{(D)} & u \end{array}$$

Regarding the two vanishing viscosity convergences (B) and (C), we observe that (C) is the classical result by Kružkov for scalar conservation laws, while (B) can be easily proved by establishing uniform (in μ) L^∞ estimates on the solution $u_{\varepsilon,\mu}$ of (4) and observing that the convolution with ρ_ε (with $\varepsilon > 0$ fixed) improves the convergence as $\mu \downarrow 0$ from weak to strong in the term $u_{\varepsilon,\mu} * \rho_\varepsilon$. Our results show that the convergence in (A) holds, while in general the convergence in (D) does not hold.

References

[1] P. Amorim, R. M. Colombo & A. Teixeira, *On the numerical integration of scalar nonlocal conservation laws*, ESAIM Math. Model. Numer. Anal. 49 (2015), 19–37.

[2] P. Calderoni & M. Pulvirenti, *Propagation of chaos for Burgers' equation*, Annales de l'Institut Henri Poincaré (A) Physique théorique **39** (1983), 85–97.

[3] G. Crippa & M. Lécureux-Mercier, *Existence and uniqueness of measure solutions for a system of continuity equations with non-local flow*, Nonlinear Differential Equations and Applications NoDEA **20** (2013), no. 3, 523–537.

[4] K. Zumbrun, *On a nonlocal dispersive equation modeling particle suspensions*, Quart. Appl. Math. **57** (1999), 573–600.

Control problems for structured population dynamics

Mauro Garavello
(joint work with Rinaldo M. Colombo)

This note deals with control problems for a biological resource, breed in order to make a profit. More precisely, assume there is a population of *juveniles*, whose density is described by the function $J = J(t, a)$, and whose evolution is described by a renewable equation; see [1, 2, 3, 9, 10]. Here t is the time, while a denotes the biological age. At a certain age $\bar{a} > 0$, the individuals of the J population are selected either for reproduction purposes or directed to the market. The functions $S = S(t, a)$ and $R = R(t, a)$ denote the densities, respectively, of the population to be sold and of the population used for reproduction. The selling of the S individuals happen at the ages $\bar{a}_1, \ldots, \bar{a}_N$, where $N \in \mathbb{N} \setminus \{0\}$ and $\bar{a} < \bar{a}_1 < \cdots < \bar{a}_N$. Thus, the dynamics of the structured (J, S, R) population is described by the following nonlocal system of renewal equations

$$\left\{\begin{array}{ll}
\partial_t J + \partial_a J = d_J(t,a)\, J & t \geq 0,\ a \in [0, \bar{a}] \\
\partial_t S + \partial_a S = d_S(t,a)\, S & t \geq 0,\ a \geq \bar{a},\ a \notin \{\bar{a}_1, \cdots, \bar{a}_N\} \\
\partial_t R + \partial_a R = d_R(t,a)\, R & t \geq 0,\ a \geq \bar{a} \\
S(t, \bar{a}) = \eta\, J(t, \bar{a}) & t \geq 0 \\
R(t, \bar{a}) = (1 - \eta)\, J(t, \bar{a}) & t \geq 0 \\
J(t, 0) = \displaystyle\int_{\bar{a}}^{+\infty} w(\alpha)\, R(t, \alpha) d\alpha & t \geq 0 \\
S(t, \bar{a}_i +) = \theta_i\, S(t, \bar{a}_i -) & t \geq 0\,, i = 1, \ldots, N \\
J(0, a) = J_o(a) & a \in [0, \bar{a}] \\
S(0, a) = S_o(a) & a \in [\bar{a}, +\infty[\\
R(0, a) = R_o(a) & a \in [\bar{a}, +\infty[\ ,
\end{array}\right. \tag{1}$$

where d_J, d_S, and d_R are mortality functions, $w = w(a)$ is a fertility function, and J_o, S_o, and R_o are the initial conditions; see also [4, 5, 8]. For further structured population models, we refer for instance to [3, 6, 7, 10]. Moreover, we consider the maps $\eta = \eta(t)$, and $\theta_i = \theta_i(t)$ $(i \in \{1, \ldots, N\})$ as control functions. Here η is responsible for the selection of the individuals at the age $\bar{a}$, while $1 - \theta_i$ is the fraction of the S population which is sold at age $\bar{a}_i$. Note that the new juveniles individuals depend on the R population in a nonlocal way.

The profit $\mathcal{P}$ of the biological resource is given by

$$\mathcal{P}(\eta, \theta; T) = \mathcal{I}(\eta, \theta; T) - \mathcal{C}(\eta, \theta; T)\,, \tag{2}$$

where $T > 0$ is the time horizon, while the income $\mathcal{I}$ and the cost $\mathcal{C}$ are defined as

$$\mathcal{I}(\eta, \theta; T) = \sum_{i=1}^{N} \int_0^T P_i\left(t, (1 - \theta_i(t))\ S(t, \bar{a}_i -)\right) dt\,, \tag{3}$$

$$\begin{aligned}\mathcal{C}(\eta,\theta;T) = &\int_0^T \int_0^{\bar a} C_J\left(t,a,J(t,a)\right) da dt + \int_0^T \int_{\bar a}^{+\infty} C_S\left(t,a,S(t,a)\right) da dt \\ &+ \int_0^T \int_{\bar a}^{+\infty} C_R\left(t,a,R(t,a)\right) da dt\,.\end{aligned} \tag{4}$$

As regards the income $\mathcal{I}$, each map $P_i = P_i(t,s)$ is the price due to selling the individuals at age $\bar a_i$. The maps $C_u(t,a,w)$ in (4), for $u \in \{J,S,R\}$, are the cost for maintaining the u population of age a at time t.

1. Main results

Fix $T > 0$, $\kappa \in \mathbb{N} \setminus \{0\}$, and introduce the notation $\mathbb{R}^+ = [0,+\infty[$, $I_J = [0,\bar a]$, $I_S = I_R = [\bar a,+\infty[$, $I_T = [0,T]$. Consider the following assumptions.

(A): For $u = J,S,R$, the mortality functions d_u satisfy

$$d_u \in (C^1 \cap L^\infty)(I_T \times I_u;\mathbb{R}) \qquad \text{and} \qquad \sup_{t\in\mathbb{R}^+} \tilde v\left(d_u(t,\cdot)\right) < +\infty\,,$$

while the fertility function w belongs to $C^1_c([\bar a,+\infty[\,;\mathbb{R}^+)$.

(ID): $J_o \in \mathrm{BV}(I_J;\mathbb{R}^+)$, $S_o \in (L^1 \cap \mathrm{BV})(I_S;\mathbb{R}^+)$, $R_o \in (L^1 \cap \mathrm{BV})(I_R;\mathbb{R}^+)$.

(P): $P \in L^\infty_{loc}([0,\bar a] \times \mathbb{R}^+;\mathbb{R})$ and $P_i \in L^\infty_{loc}(I_T \times \mathbb{R}^+;\mathbb{R})$ for $i = 1,\ldots,N$. Moreover, the map $j \to P(a,j)$, respectively $s \to P_i(t,s)$ for $i = 1,\ldots,N$, is a polynomial of degree at most κ in j for all $a \in [0,\bar a]$, respectively in s for $t \in \mathbb{R}^+$.

(C): $C_u \in L^1_{loc}(I_T \times I_u \times \mathbb{R};\mathbb{R})$ and the map $v \to C_u(t,a,v)$ is a polynomial of degree at most κ in v, for $u = J,S,R$.

The following results gives the well posedness of (1) in L^1.

Theorem 1.1 ([5, Theorem 2.1])**.** *Assume **(A)** and **(ID)**. For any $\eta \in \mathrm{BV}(I_T;[0,1])$ and $\theta \in \mathrm{BV}(I_T;[0,1]^N)$, system (1) admits a unique solution such that, for every $t \in I_T$, $J(t,a) \geq 0$ for every $a \in I_J$, and $S(t,a) \geq 0$, $R(t,a) \geq 0$ for every $a \geq \bar a$. Moreover, there exists a function $\mathcal{K} \in C^0(I_T;\mathbb{R}^+)$, with $\mathcal{K}(0) = 0$, dependent only on g_J, g_S, g_R, d_J, d_S, d_R and w such that for all initial data (J'_o,S'_o,R'_o) and (J''_o,S''_o,R''_o) and for all controls η', η'', θ' and θ'', the corresponding solutions (J',S',R') and (J'',S'',R'') to (1) satisfy, for every $t \in I_T$, the stability estimate:*

$$\begin{aligned}&\|J'(t) - J''(t)\|_{L^1(I_J;\mathbb{R})} + \|S'(t) - S''(t)\|_{L^1(I_S;\mathbb{R})} + \|R'(t) - R''(t)\|_{L^1(I_R;\mathbb{R})} \\ \leq &\mathcal{K}(t)\left(\|J'_o - J''_o\|_{L^1(I_J;\mathbb{R})} + \|S'_o - S''_o\|_{L^1(I_S;\mathbb{R})} + \|R'_o - R''_o\|_{L^1(I_R;\mathbb{R})}\right) \\ &+ t\,\mathcal{K}(t)\left(\|J'_o - J''_o\|_{L^\infty(I_J;\mathbb{R})} + \|S'_o - S''_o\|_{L^\infty(I_S;\mathbb{R})} + \|R'_o - R''_o\|_{L^\infty(I_R;\mathbb{R})}\right) \\ &+ \mathcal{K}(t)\left(\|\eta' - \eta''\|_{L^\infty([0,t];\mathbb{R})} + \|\theta' - \theta''\|_{L^\infty([0,t];\mathbb{R}^N)}\right)\,.\end{aligned}$$

The next result explains the dependence of the solution to (1) with respect to the controls.

Theorem 1.2. *Pose conditions **(A)**, **(ID)**. Let* (J,S,R) *be the solution to* (1) *corresponding to the piecewise controls*

$$\eta(t) = \sum_{k=1}^{m} \eta_k \, \chi_{[k-1,k[}(t)\,, \qquad \theta_i(t) = \sum_{k=1}^{m} \theta_i^k \, \chi_{[k-1,k[}(t) \tag{5}$$

for $m \in \mathbb{N} \setminus \{0\}$, $i = 1, \ldots, N$, $t \in [0,T]$, $T = m$, *where the control parameters* η_k *and* θ_i^k *belong to the real interval* $[0,1]$. *Then, for all* t *and* a,

(1) the quantities $J(t,a)$, $R(t,a)$ *and* $S(t,a)$ *are multiaffine in* η_k;
(2) the quantities $J(t,a)$, $R(t,a)$ *do not depend on* θ_i^k;
(3) the quantity $S(t,a)$ *is multiaffine in* θ_i^k.

The following result is a direct consequence of Theorem 1.2 and of assumptions **(P)** and **(C)**.

Corollary 1.3. *Pose conditions **(A)**, **(ID)**, **(P)** and **(C)**. Choose controls* η *and* θ_i *as in (5). Then, the net profit* $\mathcal{P}$, *defined in* (2), *is polynomial in* η *and* θ_i *of degree at most* κ *in each of the (scalar) variables* $\eta_1, \ldots, \eta_m, \theta_1^k, \ldots, \theta_N^k$ *separately. Moreover, globally, it is a polynomial of degree at most* $\kappa\, m$ *in* $\eta_1, \ldots, \eta_m$ *and of degree at most* $\kappa\, m\, N$ *in* $\theta_1^k, \ldots, \theta_N^k$.

Acknowledgments. The work was partial supported by the INdAM-GNAMPA 2016 project *Balance Laws: Theory and Applications*.

References

[1] A. S. Ackleh and K. Deng. A nonautonomous juvenile-adult model: well-posedness and long-time behavior via a comparison principle. *SIAM J. Appl. Math.*, 69(6):1644–1661, 2009.
[2] A. S. Ackleh, K. Deng, and X. Yang. Sensitivity analysis for a structured juvenile–adult model. *Comput. Math. Appl.*, 64(3):190–200, 2012.
[3] À. Calsina and J. Saldaña. Basic theory for a class of models of hierarchically structured population dynamics with distributed states in the recruitment. *Math. Models Methods Appl. Sci.*, 16(10):1695–1722, 2006.
[4] R. M. Colombo and M. Garavello. Stability and optimization in structured population models on graphs. *Mathematical Biosciences and Engineering*, 12(2):311–335, 2015.
[5] R. M. Colombo and M. Garavello. Control of biological resources on graphs. *ESAIM: COCV, to appear*, 2016.
[6] O. Diekmann, M. Gyllenberg, J. A. J. Metz, and H. R. Thieme. On the formulation and analysis of general deterministic structured population models. I. Linear theory. *J. Math. Biol.*, 36(4):349–388, 1998.
[7] J. Z. Farkas and T. Hagen. Stability and regularity results for a size-structured population model. *J. Math. Anal. Appl.*, 328(1):119–136, 2007.
[8] M. Garavello. Optimal control in renewable resources modeling. *Bulletin of the Brazilian Mathematical Society, New Series*, 47(1):347–357, 2016.
[9] B. Perthame. *Transport equations in biology.* Frontiers in Mathematics. Birkhauser Verlag, Basel, 2007.
[10] G. F. Webb. *Theory of nonlinear age-dependent population dynamics*, volume 89 of *Monographs and Textbooks in Pure and Applied Mathematics.* Marcel Dekker, Inc., New York, 1985.

Relative Entropy for Hamiltonian Flows in Gas Dynamics

Jan Giesselmann
(joint work with Corrado Lattanzio, Athanasios E. Tzavaras)

We study systems of partial differential equations having the form

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \operatorname{div}_x(\rho u) = 0 \\ &\frac{\partial u}{\partial t} + (u \cdot \nabla_x) u = -\nabla_x \frac{\delta \mathcal{E}}{\delta \rho}(\rho) \end{aligned} \qquad x \in \mathbb{R}^d\,,\ t > 0\,, \tag{1}$$

where $\rho \geq 0$ is a density obeying the conservation of mass, u is a velocity and $m = \rho u$ a momentum flux. The evolution of u results from a functional $\mathcal{E}(\rho)$ on the density and $\frac{\delta \mathcal{E}}{\delta \rho}$ denotes the generator of the directional derivative of that functional. In case of irrotational flows the dynamics (1), indeed, have a Hamiltonian structure, i.e. defining $\mathcal{H}(\rho, u) := \mathcal{E}(\rho) + \int \frac{1}{2}\rho |u|^2 \, dx$ equation (1) is equivalent to

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ u \end{pmatrix} = \begin{pmatrix} 0 & -\operatorname{div}_x \\ -\nabla_x & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta \mathcal{H}}{\delta \rho} \\ \frac{\delta \mathcal{H}}{\delta u} \end{pmatrix} + \begin{pmatrix} 0 \\ u \times \operatorname{curl}_x u \end{pmatrix}.$$

In the non-irrotational case there is a discrepancy, but this discrepancy is compatible with conservation of energy. Solutions of (1) formally satisfy

$$\frac{d}{dt}\left(\int \tfrac{1}{2} \rho |u|^2 \, dx + \mathcal{E}(\rho) \right) = 0\,.$$

Depending on the selection of the functional $\mathcal{E}(\rho)$ several models of interest fit under this framework. These include the equations of isentropic gas dynamics for

$$\mathcal{E}(\rho) = \int h(\rho)\, dx$$

with given energy density function $h : [0, \infty) \to [0, \infty)$; the Euler-Poisson system (*e.g.* [6]) for

$$\begin{aligned} &\mathcal{E}(\rho) = \int \left(h(\rho) - \tfrac{1}{2}\rho c \right) dx\,, \\ &\text{where } c \text{ is the solution of } \quad -\Delta_x c + \beta c = \rho - <\rho>\,, \\ &\beta \geq 0 \text{ is a constant and } <\rho> \text{ denotes the mean of } \rho; \end{aligned} \tag{2}$$

the system of quantum hydrodynamics (*e.g.* [1]) for

$$\mathcal{E}(\rho) = \int h(\rho) + \frac{1}{2\rho} |\nabla_x \rho|^2 \, dx;$$

and the Euler-Korteweg system (*e.g.* [5]), for

$$\mathcal{E}(\rho) = \int h(\rho) + \frac{C_\kappa}{2} |\nabla_x \rho|^2 \, dx \quad \text{where } C_\kappa > 0 \text{ is a constant.} \tag{3}$$

Our goal is to use the formal structure (1) in order to obtain a relative entropy identity. Depending on the choice of energy density h, most (but not all) of the problems above are generated by convex functionals. Thus, it is natural to use the quadratic part of the Taylor expansion of the functional $\mathcal{E}(\rho)$,

$$\mathcal{E}(\rho|\bar{\rho}) := \mathcal{E}(\rho) - \mathcal{E}(\bar{\rho}) - \big\langle \frac{\delta \mathcal{E}}{\delta \rho}(\bar{\rho}), \rho - \bar{\rho} \big\rangle \,, \tag{4}$$

for comparing two states ρ and $\bar{\rho}$. This definition involves the directional derivative of $\mathcal{E}(\rho)$ in the direction $(\rho - \bar{\rho})$ and provides a functional which we call *relative potential energy*. We combine it with the *relative kinetic energy*

$$K(\rho, m|\bar{\rho}, \bar{m}) = \int \frac{\rho}{2}\Big|\frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}}\Big|^2 dx \tag{5}$$

as a measure for the distance between two solutions (ρ, m) and $(\bar{\rho}, \bar{m})$.

In order to obtain a useful relative entropy identity we need to assume existence of a stress tensor (functional) $S(\rho)$ satisfying

$$-\rho \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} = \nabla_x \cdot S \,. \tag{6}$$

Hypothesis (6) holds for all the above examples. It gives a meaning to the notion of weak solution for (1) as it induces a conservative form.

Under this condition the structure (1) allows us to (formally) derive the following relative energy identity for solutions (ρ, m), $(\bar{\rho}, \bar{m})$ of (1):

$$\begin{aligned} &\frac{d}{dt}\big(\mathcal{E}(\rho|\bar{\rho}) + K(\rho, m|\bar{\rho}, \bar{m})\big) \\ &\qquad = \int \nabla_x \bar{u} : S(\rho|\bar{\rho})\, dx - \int \rho \nabla_x \bar{u} : (u - \bar{u}) \otimes (u - \bar{u})\, dx \,, \end{aligned} \tag{7}$$

where the relative stress functional is defined by

$$S(\rho|\bar{\rho}) := S(\rho) - S(\bar{\rho}) - \big\langle \frac{\delta S}{\delta \rho}(\bar{\rho}), \rho - \bar{\rho} \big\rangle \,. \tag{8}$$

Formula (7) is similar to the well known relative entropy formulas first obtained in the works of Dafermos [2, 3] and DiPerna [4] which have been used successfully in many contexts. However, equation (7) has a different origin from all these calculations: it is based on the abstract Hamiltonian flow structure (1) while the latter are based on the thermodynamical structure induced by the Clausius-Duhem inequality. This difference notwithstanding, formula (7) and the formulas obtained in [3, 7] are similar in that they allow for a mechanical interpretation of the relative mechanical stress and the relative convective stress.

While formula (7) seems quite simple the actual formulas in specific examples are cumbersome. Moreover, (for pairs of weak solutions (ρ, m) and classical solutions $(\bar{\rho}, \bar{m})$) the derivation of (7) needs to be justified in specific models (e.g. Euler-Korteweg) which is technically quite intricate. Details can be found in [8]. The framework described here can, in particular, be applied for obtaining weak-strong uniqueness for the models under consideration as long as the potential energy is strictly convex. We refer to [8] for the precise statements. In case of the Euler-Korteweg model (3) we even obtain weak-strong uniqueness for certain non-convex energy densities h, since the gradient terms in the energy can be used to compensate for the lack of convexity, see [9] for details.

References

[1] P. Antonelli and P. Marcati, *The quantum hydrodynamics system in two space dimensions*, Comm. Math. Physics **203** (2012), 499–527.

[2] C. M. Dafermos, *The second law of thermodynamics and stability*, Arch. Rational Mech. Anal. **70** (1979), 167–179.

[3] C. M. Dafermos, *Stability of motions of thermoelastic fluids*, J. Thermal Stresses **2** (1979), 127–134.

[4] R. J. DiPerna, *Uniqueness of solutions to hyperbolic conservation laws*, Indiana Univ. Math. J. **28** (1979), 137–188.

[5] J. E. Dunn and J. Serrin, *On the thermomechanics of interstitial working*, Arch. Rational Mech. Anal. **88** (1985), 95–133.

[6] T. Luo and J. Smoller, *Existence and non-linear stability of rotating star solutions of the compressible Euler-Poisson equations*, Arch. Ration. Mech. Anal. **191** (2009), 447–496.

[7] C. Lattanzio and A. E. Tzavaras, *Relative entropy in diffusive relaxation*, SIAM J. Math. Anal. **45** (2013), 1563–1584.

[8] J. Giesselmann, C. Lattanzio and A. E. Tzavaras, *Relative energy for the Korteweg theory and related Hamiltonian flows in gas dynamics*, ArXiv Preprint.

[9] J. Giesselmann and A. E. Tzavaras, *Relative energy for Korteweg models: Weak-strong uniqueness and vanishing capillarity limit*, In Preparation.

Uniqueness for a non–linear 1D compressible to incompressible limit in the non smooth case

Graziano Guerra

(joint work with Rinaldo M. Colombo)

The compressible to incompressible limit is widely studied in the literature, see for instance the review [9] and the references therein. The classical setting considers regular solutions, whose existence is proved only for a finite time, to the compressible equations. As the Mach number vanishes, these solutions are proved to converge to the solutions to the incompressible system. Here we are concerned with the isentropic 1D system of Euler equations and with its compressible to incompressible limit. In particular we want to study this limit for non smooth solutions defined for all times. In a 1D setting, an incompressible fluid behaves like a solid since its speed is constant in space. In [3] the full 1D non–isentropic Euler equations in all the real line are considered. The authors prove rigorously that the second order coefficients of the asymptotic expansion of the solution in the (small) Mach number satisfy the linear acoustic system. Here we consider instead two compressible immiscible fluids and let only one of the two become incompressible. More precisely we consider a 1D volume of a compressible inviscid fluid, say the *liquid*, that fills the segment of a tube $[a(t), b(t)]$ and is surrounded by another compressible fluid, say the *gas*, filling the rest of the tube. We assume that the gas obeys a fixed pressure law $\overline{P}\,(\rho)$, while for the liquid we assume a one parameter family of pressure laws $\overline{P}_\kappa\,(\rho)$ such that $\overline{P}'_\kappa\,(\rho) \to +\infty$ as $\kappa \to 0$. The total mass of the liquid is fixed: $\int_{a(t)}^{b(t)} \rho(t,x)\ dx = m$. Since the two fluids are immiscible, Lagrangian coordinates are a natural choice: $z(t,x) = \int_{a(t)}^{x} \rho(t,\xi)\ d\xi, \quad \tau = \frac{1}{\rho}, \quad P(\tau) = \overline{P}\left(\frac{1}{\tau}\right),$ with τ being the specific volume. In these coordinates, the liquid occupies the fixed region $[0,m]$ and the interfaces at $z = 0$ and $z = m$ become stationary in time. For this problem it is also convenient to write the equations in terms of the pressure, so we introduce the inverses of the pressure laws, now seen as functions of the pressure: $\mathcal{T}(p) = P^{-1}(p), \quad \mathcal{T}_\kappa(p) = P_\kappa^{-1}(p), \quad \mathcal{T}'_\kappa(p) \xrightarrow{\kappa\to 0} 0$. The isentropic Euler equations for the two interacting fluids in these new variables become:

$$(1) \qquad \begin{cases} \partial_t \mathcal{T}_\kappa(z,p) - \partial_z v = 0 \\ \partial_t v + \partial_z p = 0\,, \end{cases} \qquad \mathcal{T}_\kappa(z,p) = \begin{cases} \mathcal{T}_\kappa(p) & \text{for } z \in]0,m[\\ \mathcal{T}(p) & \text{for } z \notin]0,m[\,. \end{cases}$$

Given two functions $(p,v) \in \mathbf{L}^1(\mathbb{R},\mathbb{R}^2)$, introduce now the weighted total variation: $\mathrm{TV}_\kappa(p,v) = \mathrm{TV}(p,\mathbb{R}) + \mathrm{TV}(v,\mathbb{R}\setminus]0,m[) + \frac{1}{\kappa}\mathrm{TV}(v,]0,m[)$. The following two theorems where proved in [8] with the assumption of a linear pressure in the liquid region and in [5] without this linearity assumption.

Theorem 1.1. *Given a positive constant $c > 0$, there exist $\delta, \Delta, L > 0$ independent of κ such that if $\mathrm{TV}_\kappa(p_o^\kappa, v_o^\kappa) < \delta$ and $p_o^\kappa \geq c$ hold, then the Cauchy problem for* (1) *with (p_o^κ, v_o^κ) as initial data has an entropy solution (p^κ, v^κ) defined for all times $t \geq 0$.*

Theorem 1.2. *Given a positive constant $c > 0$, fix an initial data (p_o, v_o) such that $\mathrm{TV}(p_o) + \mathrm{TV}(v_o) \leq \delta$, $p_o \geq c$, $v_o(z) = \tilde{v}$ for all $z \in]0,m[$, then for any $\kappa \in]0,1[$ there exists an entropic solution (p_o^κ, v_o^κ) to the Cauchy problem for* (1) *with initial data (p_o, v_o). Define the specific volume as $\tau^\kappa(t,z) = \mathcal{T}_\kappa(z, p^\kappa(t,z))$, then as $\kappa \to 0$, up to subsequences, we have the following convergence results.*

$$\begin{matrix} \tau^\kappa(t,\cdot) \to \bar{\tau} \\ v^\kappa(t,\cdot) \to v_l(t) \end{matrix} \ \text{in } \mathbf{L}^1(]0,m[)\,, \qquad \begin{matrix} \tau^\kappa(t,\cdot) \to \tau^*(t,\cdot) \\ v^\kappa(t,\cdot) \to v^*(t,\cdot) \\ p^\kappa(t,\cdot) \to p^*(t,\cdot) \end{matrix} \ \text{in } \mathbf{L}^1(\mathbb{R}\setminus[0,m])\,.$$
$$p^\kappa(\cdot,\cdot) \overset{*}{\rightharpoonup} p_l(\cdot,\cdot) \ \text{in } \mathbf{L}^\infty(]0,m[\times\mathbb{R}^+)\,,$$

Moreover, the limits $v_l(t)$, $(p^*, v^*)(t,z)$ are entropy solutions [1] *to*

$$(2) \qquad \begin{cases} \left.\begin{matrix}\partial_t \mathcal{T}(p^*) - \partial_z v^* = 0 \\ \partial_t v^* + \partial_z p^* = 0\end{matrix}\right\} z \notin [0,m] \\ m\frac{d}{dt}v_l(t) = p^*(t,0-) - p^*(t,m+) \\ v_l(t) = v^*(t,0-) = v^*(t,m+), \end{cases} \qquad \begin{cases} p^*(0,z) = p_o(z) \\ v^*(0,z) = v_o(z) \\ v_l(0) = \tilde{v}. \end{cases}$$

The limit pressure is given by $p_l(t,z) = \left(1 - \frac{z}{m}\right)p^*(t,0-) + \frac{z}{m}p^*(t,m+)$ a.e. $t \geq 0$, $z \in [0,m]$.

System (2) is a system of PDE and ODE coupled through the boundary values of the solutions to the PDE. The well posedness for this kind of systems was proved in [1] while a characterization of their solutions in terms of tangent vectors is given in [4]. This characterization is able to ensure the uniqueness of the compressible to incompressible limit obtained in Theorem 1.2. To show this, we recall some definitions and results of [4] adapted to system (2). On the set $Y = \mathbb{R} \times \left(\mathbf{L}^1 \cap \boldsymbol{BV}\right)\left(\mathbb{R} \setminus [0,m], \mathbb{R}^2\right)$ introduce the metric

$$d\left(\left(v_{l,1},(p_1,v_1)\right),\left(v_{l,2},(p_2,v_2)\right)\right) = |v_{l,1} - v_{l,2}| + \|(p_1,v_1) - (p_2,v_2)\|_{\mathbf{L}^1(\mathbb{R}\setminus[0,m],\mathbb{R}^2)}\,.$$

For any $u_o = (v_{l,o},(p_o,v_o)) \in Y$ with $|v_{l,o}| + \mathrm{TV}\,(p_o,v_o)$ sufficiently small, introduce the Lipschitz curve leaving u_o:

$$\mathcal{F}(h)\left(v_{l,o},(p_o,v_o)\right) = \left(v_{l,o} + h\left(p^{\sigma-} - p^{\sigma+}\right), S_h\left(\tilde{p},\tilde{v}\right)|_{\mathbb{R}\setminus[0,m]}\right), \quad h \geq 0. \tag{3}$$

Here $p^{\sigma-}$ is the unique value of the pressure such that the state $(p_o,v_o)(0-)$ can be connected to $(p^{\sigma-},v_{l,o})$ with a wave of the first family, while $p^{\sigma+}$ is the unique value of the pressure such that the state $(p^{\sigma+},v_{l,o})$ can be connected to $(p_o,v_o)(m+)$ with a wave of the second family. The existence and uniqueness of these two states is ensured by [6, Lemma 4.1]. S_h is the Standard Riemann Semigroup [2, Definition 9.1] on all the real line generated by $\partial_t \mathcal{T}(p) - \partial_z v = 0$, $\partial_t v + \partial_z p = 0$. The initial data $(\tilde{p},\tilde{v})$ is given by $(\tilde{p},\tilde{v})(z) = (p_o,v_o)(z)$ for $z \in \mathbb{R} \setminus [0,m]$; $(\tilde{p},\tilde{v})(z) = (p^{\sigma-},v_{l,o})$ for $z \in [0,m/2[$; $(\tilde{p},\tilde{v})(z) = (p^{\sigma+},v_{l,o})$ for $z \in [m/2,m]$. With these definitions, [4, Theorem 3] applied to system (2) becomes:

Theorem 1.3. *There exists a positive δ, a set of initial data*

$$X \supset \left\{u = (v_l,(p,v)) \in \mathbb{R} \times \left(\mathbf{L}^1 \cap \boldsymbol{BV}\right)\left(\mathbb{R}\setminus[0,m],\mathbb{R}^2\right) : \ |v_l| + \mathrm{TV}\,(p,v) < \delta\right\}$$

and a Lipschitz continuous local semigroup [4, Definition 2] *$\mathcal{S}$ on X, such that*

- *(i) $\forall u_o \in X$, the map $u(t) = \mathcal{S}_t u_o$ is a solution to* (2) *with initial datum u_o;*
- *(ii) $\forall u_o \in X$, the map $h \to \mathcal{S}_h u_o$ is first order tangent to $h \to \mathcal{F}(h)u_o$ defined in* (3) *at u_o, in the sense that $\lim_{h\to 0+} \frac{1}{h}\, d\left(\mathcal{S}_h u_o, \mathcal{F}(h)u_o\right) = 0$;*
- *(iii) $\mathcal{S}$ is unique up to the domain;*
- *(iv) any Lipschitz curve $u : [0,T] \to X$ first order tangent to $\mathcal{F}$ at any point, coincides with the semigroup trajectories: $u(t) = \mathcal{S}_t u(0)$, $\forall t \in [0,T]$.*
- *(v) $\mathcal{S}_t u_o$ is defined as long as it does not leave the domain X, in particular if $\mathcal{S}_t u_o \in X$ for any $u_o \in X$ and any $t \geq 0$, $\mathcal{S}_t$ is a global semigroup.*

Using this last result, we are able to show that the compressible to incompressible limit of Theorem 1.2 is unique. Indeed in [7] the following theorem is proved.

Theorem 1.4. *For any initial data $u_o = (v_{l,o},(p_o,v_o)) \in X$, the compressible to incompressible limit $u^*(t) = (v_l(t),(p^*,v^*)(t,\cdot))$, satisfies $\lim_{h\to 0} \frac{d(\mathcal{F}(h)u^*(t),u^*(t+h))}{h} = 0$, and consequently it coincides with a trajectory of the semigroup in Theorem 1.3.*

Since from Theorem 1.2 we know that $(v_l(t),(p^*,v^*)(t))$ is defined for all times $t \geq 0$, as a consequence we also have:

Corollary 1.5. *The trajectories of the local semigroup of Theorem 1.3 are defined for all times $t \geq 0$, hence $\mathcal{S}_t$ is a global semigroup defined on X.*

References

[1] R. Borsche, R. M. Colombo, and M. Garavello. Mixed systems: ODEs - balance laws. *J. Differential Equations*, 252(3):2311–2338, 2012.

[2] A. Bressan. *Hyperbolic systems of conservation laws*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.

[3] G.-Q. Chen, C. Christoforou, and Y. Zhang. Continuous Dependence of Entropy Solutions to the Euler Equations on the Adiabatic Exponent and Mach Number. *Arch. Rational Mech. Anal.*, 189:97–130, 2008.

[4] R. M. Colombo and G. Guerra. Characterization of the Solutions to ODE–PDE Systems, Submitted, 2016.

[5] R. M. Colombo and G. Guerra. BV Solutions to 1D Isentropic Euler Equations in the Zero Mach Number Limit, *Journal of Hyperbolic Differential Equations*, to appear, 2016.

[6] R. M. Colombo and G. Guerra. On general balance laws with boundary, *J. Differential Equations*, 248(5):1017–1043, 2010.

[7] R. M. Colombo and G. Guerra. Uniqueness of the 1D Compressible to Incompressible Limit, In preparation, 2016.

[8] R. M. Colombo, G. Guerra, and V. Schleper. The compressible to incompressible limit of 1d Euler Equations: the non smooth case. *Arch. Ration. Mech. Anal.*, 219(2):701–718, 2016.

[9] S. Schochet. The mathematical theory of low Mach number flows. *M2AN Math. Model. Numer. Anal.*, 39(3):441–458, 2005.

Stability analysis of an implicit lattice Boltzmann scheme

Philippe Helluy

1. Introduction

Lattice kinetic models are essential in computational fluid dynamics. They are the key ingredient of the Lattice Boltzmann Method (LBM). The idea is to construct a kinetic interpretation of a hyperbolic system of conservation laws with a minimal set of velocities. In this report we analyze the D1Q3 lattice kinetic model, which is the simplest kinetic model representing the isothermal Euler equations. We show that it is unstable but that it can be made stable if the transport step is solved with an implicit scheme. The unknown of the D1Q3 model is a three-dimensional distribution function $f(x,t) \in \mathbb{R}^3$, where $x \in \mathbb{R}$ and $t \in [0,T]$ are respectively the space and time variable. The distribution function satisfies transport equations with a BGK relaxation source term [1]

$$f_t^i + v_i f_x^i = \frac{1}{\varepsilon}(M(f)^i - f^i), \quad i = 1 \dots 3, \tag{1}$$

where we have noted partial derivatives with indices ($f_t = \partial_t f$ for instance). The kinetic velocity takes only three values

$$v = (-\lambda, 0, \lambda),$$

where λ is a positive real number. The fluid macroscopic variables are the density $\rho(x,t)$, the momentum $q(x,t)$ and the momentum flux $z(x,t)$. As usual the fluid velocity is defined by

$$u = q/\rho.$$

The macroscopic variables are recovered by computing discrete moments of f

$$\begin{pmatrix} \rho \\ q \\ z \end{pmatrix} = P \begin{pmatrix} f^1 \\ f^2 \\ f^3 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 & 1 \\ -\lambda & 0 & \lambda \\ \lambda^2 & 0 & \lambda^2 \end{pmatrix}.$$

The constant sound speed of the isothermal fluid is denoted by $c > 0$. The discrete Maxwellian state $M(f)$ is then given by

$$M(f) = \frac{1}{\lambda^2} \begin{pmatrix} \rho u(u-\lambda)/2 + c^2\rho/2 \\ \rho(\lambda^2 - u^2 - c^2) \\ \rho u(u+\lambda^2)/2 + c^2\rho/2 \end{pmatrix} \tag{2}$$

in such a way that

$$PM(f) = \begin{pmatrix} \rho \\ q \\ \rho u^2 + c^2\rho \end{pmatrix}.$$

Multiplying the kinetic equation (1) by P we obtain

$$\begin{aligned} \rho_t + q_x &= 0, \\ q_t + z_x &= 0, \\ z_t + \lambda^2 q_x &= \frac{1}{\varepsilon}(q^2/\rho + c^2\rho - z). \end{aligned} \tag{3}$$

When $\varepsilon \to 0$, then formally $f = M(f)$ and from (1) we see that ρ and u satisfy the isothermal Euler equations

$$\begin{aligned} \partial_t\rho + \partial_x(\rho u) &= 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + c^2\rho) &= 0. \end{aligned} \tag{4}$$

The model (1), (1) is thus a minimalistic abstract kinetic interpretation of the isothermal Euler equation. It is also denoted as the "D1Q3" model in the lattice-Boltzmann community [3]. It can be extended to higher dimensions. For instance, in two or three dimensions it becomes the D2Q9 or D3Q27 models.

2. Numerical method and asymptotic expansion

A traditional method for solving numerically (1) is the first order Lie splitting algorithm. For applying one time step of the splitting algorithm, we start from a state that is close to equilibrium: $f = M(f) + O(\varepsilon)$. We first apply the free transport equation for a duration of Δt

$$f_t + v \cdot f_x = 0.$$

Then in a second stage of the same duration Δt we apply the local BGK return to equilibrium

$$f_t = \frac{1}{\varepsilon}(M(f) - f).$$

In the case of the D1Q3 model, this approach can lead to instabilities that are sometimes observed in LBM simulations [2]. Therefore, we replace the exact transport step by a first order implicit solver in time. Assuming high precision of the solver in the x variable the effect of the implicit solver can be modeled by

$$\frac{f(x,t)-f(x,t-\Delta t)}{\Delta t}+vf_x(x,t)=0. \tag{5}$$

By a Taylor expansion, we find the equivalent equation of the implicit solver (5)

$$f_t+vf_x-\frac{\Delta t}{2}v^2f_{xx}=O(\Delta t^2). \tag{6}$$

In a second step, we solve the differential equation exactly

$$f_t=\frac{1}{\varepsilon}(M(f)-f)=\frac{M-I}{\varepsilon}f.$$

This is easy because during the relaxation step ρ, q, and thus $M(f)$, are constant.

In the following, ε is a small parameter, but we assume that the vector field M is restricted to a manifold of f's on which

$$\frac{M-I}{\varepsilon}f=O(1). \tag{7}$$

In the literature this hypothesis is often formulated by saying that f remains close to a Maxwellian state and that the initial data are "well-prepared". Hypothesis (7) is crucial because it will allow us to apply the Baker-Campbell-Hausdorf (BCH) formula with the good ordering for estimating the equivalent equation of the splitting algorithm. Let us also point out that we assume that (7) remains true even if $\varepsilon\sim\Delta t$ or $\varepsilon\sim\Delta t^2$ for instance. For a more precise analysis of this hypothesis, we refer to [4] (Section VI.3 pages 388–392).

In the Lie formalism, one time-step of the splitting scheme can be written

$$\varphi(\tau)=\exp(\tau\frac{M-I}{\varepsilon})\exp(\tau(-v\partial_x+\frac{1}{2}\tau v^2\partial_{xx}))+O(\tau^3).$$

Now we apply the BCH formula

$$\exp(A)\exp(B)=\exp(A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]+[B,[B,A]])+\cdots).$$

We obtain

$$\varphi(\tau)=\exp(\tau L)+O(\tau^3),$$

with

$$L=-v\partial_x+\frac{1}{2}\tau v^2\partial_{xx}+\frac{M-I}{\varepsilon}+\frac{1}{2}\tau\left[\frac{M-I}{\varepsilon},-v\partial_x\right].$$

Therefore at second order in time, the equivalent equation of the scheme is

$$f_t+vf_x-\frac{\Delta t}{2}v^2f_{xx}-\frac{1}{2}\Delta t\left[\frac{M-I}{\varepsilon},-v\partial_x\right]f=\frac{M(f)-f}{\varepsilon}.$$

For expressing the Lie bracket in a more convenient way, we introduce the matrix

$$V=\begin{bmatrix}-\lambda & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \lambda\end{bmatrix}.$$

Then the Lie bracket becomes

$$\begin{aligned}[M-I,-v\partial_x]\,f &= -V\partial_x M(f)+M'(f)V\partial_x f\\ &= (M'V-VM')\partial_x f\end{aligned}$$

Now we go back to variables (ρ,q,z). After some computations, we find that

$$P\,[M-I,-v\partial_x]\,P^{-1}\begin{bmatrix}\rho\\ q\\ z\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ -c^2+u^2 & -2u & 1\\ 0 & c^2-\lambda^2-u^2 & 2u\end{bmatrix}\partial_x\begin{bmatrix}\rho\\ q\\ z\end{bmatrix}.$$

We obtain the equivalent equations solved by the splitting algorithm at order 2 in Δt

$$\begin{aligned}&\rho_t+q_x-\frac{\Delta t}{2}z_{xx}=0,\\ &q_t+z_x-\frac{\Delta t}{2}\lambda^2 q_{xx}=\frac{\Delta t}{2\varepsilon}\left((u^2-c^2)\rho_x-2uq_x+z_x\right),\\ &\partial_t z+\lambda^2\partial_x q-\frac{\Delta t}{2}\lambda^2 z_{xx}=\frac{1}{\varepsilon}(q^2/\rho+c^2\rho-z)+\frac{\Delta t}{2\varepsilon}\left((c^2-\lambda^2-u^2)q_x+2uz_x\right).\end{aligned}\tag{8}$$

On this equation we will now assume that $1\gg\Delta t>\varepsilon$. We freeze Δt and perform a Chapman-Enskog expansion when $\varepsilon\to 0$. The second equation implies that when $\varepsilon\to 0$

$$z_x=(c^2-u^2)\rho_x+2uq_x=(q^2/\rho+c^2\rho)_x+O(\varepsilon)$$

and is thus redundant with

$$z=q^2/\rho+c^2\rho+O(\varepsilon).$$

The third equation in (8) gives

$$z=q^2/\rho+c^2\rho-\varepsilon\left(\partial_t z+\lambda^2\partial_x q\right)+\frac{\Delta t}{2}\left((c^2-\lambda^2-u^2)q_x+2uz_x\right)+O(\varepsilon\Delta t).$$

We need to rewrite the factor in ε:

$$\partial_t z+\lambda^2\partial_x q,$$

with only spatial derivatives. At leading order we have

$$\begin{aligned}\partial_t z &= \partial_t\left(q^2/\rho+c^2\rho\right)+O(\varepsilon+\Delta t)\\ &= \frac{2q}{\rho}q_t-\frac{q^2}{\rho^2}\rho_t+c^2\rho_t+O(\varepsilon+\Delta t).\end{aligned}$$

But $q_t=-z_x+O(\varepsilon+\Delta t)$ and $\rho_t=-q_x+O(\varepsilon+\Delta t)$ thus

$$z_t=-2uz_x+u^2q_x-c^2q_x+O(\varepsilon+\Delta t).$$

Then

$$z_t=-2u\left(2uq_x+(c^2-u^2)\rho_x\right)+(u^2-c^2)q_x+O(\varepsilon+\Delta t).$$

Finally

$$z_t+\lambda^2 q_x=\left(-3u^2+\lambda^2-c^2\right)q_x-2u\left(c^2-u^2\right)\rho_x+O(\varepsilon+\Delta t).$$

We then obtain the equivalent viscous equation of the splitting method

$$\rho_t + (\rho u)_x = \kappa \frac{\Delta t}{2} z_{xx},$$
$$(\rho u)_t + \left(\rho u^2 + c^2\rho\right)_x = \kappa \frac{\Delta t}{2} \lambda^2 q_{xx} + D_x,$$

with $\kappa = 1$ (effect of the implicit solver) or $\kappa = 0$ (exact transport solver) and

$$D = (\varepsilon + \frac{\Delta t}{2}) \left(\left(\lambda^2 - c^2 - 3u^2\right) q_x + 2u \left(u^2 - c^2\right) \rho_x \right).$$

3. Stability analysis

Now we want to analyze the entropy stability of the second order term when $\varepsilon \to 0$. For this, we define

$$w = \begin{pmatrix} \rho \\ q \end{pmatrix}, \quad F(w) = \begin{pmatrix} q \\ \frac{q^2}{\rho} + c^2\rho \end{pmatrix},$$

$$A(w) = \begin{pmatrix} \kappa(c^2 - u^2) & 2\kappa u \\ 2u\left(u^2 - c^2\right) & (1+\kappa)\lambda^2 - c^2 - 3u^2 \end{pmatrix},$$

and thus second order equivalent equations become

$$w_t + F(w)_x = \frac{\Delta t}{2} \left(A(w) w_x\right)_x \tag{9}$$

An entropy of the Euler equations is

$$S(w) = \frac{q^2}{2\rho} + c^2 \rho \ln \rho.$$

We know that with this choice there exists an entropy flux $G(w)$ such that

$$S'F' = G'.$$

Multiplying (9) on the left by $S'(w)$, integrating by part in x and neglecting boundary terms, we obtain the entropy dissipation balance

$$\frac{d}{dt} \int_x S = -\frac{\Delta t}{2} \int_x w_x \cdot S''(w) A(w) w_x.$$

A sufficient condition for entropy dissipation is thus that $E(w) = S''(w)A(w)$ is a positive matrix. The D1Q3 model is generally used for subsonic flows. When $\kappa = 0$ (no numerical viscosity) $E(w)$ has always a negative eigenvalue and the scheme is thus unstable. The negative eigenvalue has a minimal modulus if $\lambda = \sqrt{3}c$ and is then of order $O(u^6)$. It justifies the fact that the scheme can, however, be applied in practice on relatively coarse meshes for low Mach number flows. When $\kappa \neq 0$ Taylors expansions in u give

$$\rho^2 \det(E(w)^T + E(w)) = -4\,c^6 + 8\,\lambda^2 c^4 + O\left(u^2\right),$$
$$\rho \operatorname{Tr}(E(w)^T + E(w)) = 2\,c^4 - 2\,c^2 + 4\,\lambda^2 + O(u^2).$$

If λ is large enough, the scheme is thus stable for low Mach flows.

References

[1] Prabhu Lal Bhatnagar, Eugene P Gross, and Max Krook. A model for collision processes in gases. i. small amplitude processes in charged and neutral one-component systems. *Physical review*, 94(3):511, 1954.

[2] François Dubois. Stable lattice Boltzmann schemes with a dual entropy approach for monodimensional nonlinear waves. *Computers & Mathematics with Applications*, 65(2):142–159, 2013.

[3] YH Qian, Dominique d'Humières, and Pierre Lallemand. Lattice BGK models for Navier-Stokes equation. *EPL (Europhysics Letters)*, 17(6):479, 1992.

[4] G Wanner and E Hairer. *Solving ordinary differential equations, stiff and differential-algebraic problems, second edition.*

Burgers meets Braess

Helge Holden

(joint work with Rinaldo M. Colombo)

The talk is based on the recent paper [2] where we introduce a framework to study the possible occurrence of the Braess paradox on a traffic network, where the flow on each road is described using so-called traffic hydrodynamics [5, 6].

The Braess paradox was introduced by D. Braess in 1968 in [1], where he described a simple network with traffic flow in which one had the paradoxical situation that the addition of a new road to the network, could make the travel times worse for all. The paradox has been studied extensively, and turns up not only in traffic flow, but also in mesoscopic electron systems, and in mechanical springs. Real-world examples include Seoul, Stuttgart, and New York City. The list of relevant literature is too vast to be included here; see [2] and the references therein.

The modeling of dense traffic flow by a hydrodynamic approach where cars are represented by their density, and the dynamics is described by the conservation of the number of cars, was introduced by Lighthill and Whitham [5] and Richards [6] in 1955–56. It has been studied extensively, and was extended to a network of traffic in [4], see also [3].

Let us briefly describe the model we study. We consider unidirectional traffic flow on a network of roads. The traffic dynamics on each road is given by $\rho_t + (\rho\, v(\rho))_x = 0$ where ρ is the density of vehicles and $v = v(\rho)$ is the velocity, which is considered to be a decreasing function of the density. The resulting flux function $f(\rho) = \rho\, v(\rho)$ is a concave function which satisfies $f(0) = f(\rho_{\max}) = 0$ where $\rho_{\max}$ denotes the maximum capacity of the road. There is a maximum of the flow at a point ρ_m with $\rho_m \in (0, \rho_{\max})$. We only study stationary flow in the uncongested phase, i.e., $\rho \in (0, \rho_m)$. These assumptions vastly simplify the analysis.

Instead of giving the general formulation, we will here only present a simple example. Consider the roads given on Figure 1. The network is given by two routes, denoted α and β, connecting A and B. The route α consists of roads a and b, the route β consists of roads c and d. Roads a and d are identical, and similarly for roads b and c. The velocity of roads a and d is given by $\ln(1+\rho)/\rho$, while

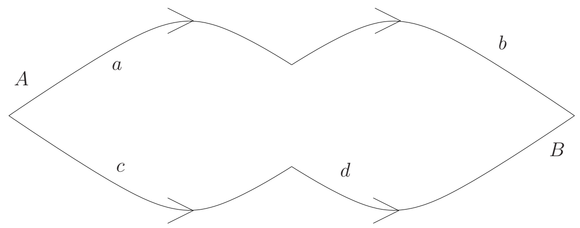

FIGURE 1. Network consisting of two routes connecting A to B.

on roads b and c the velocity is constant, denoted V. All roads have unit length. Let there be a constant inflow at A of cars given by ϕ. We need to determine the fraction $\theta \in [0,1]$ of cars that choose route α (which clearly implies that the fraction $1-\theta$ follows route β). The total travel times read

$$\tau_\alpha(\theta) = \tau_a(\theta) + \tau_b(\theta) \quad \text{and} \quad \tau_\beta(1-\theta) = \tau_a(1-\theta) + \tau_b(1-\theta)\,.$$

in obvious notation, and the mean travel time of the network equals

$$T(\theta) = \theta\tau_\alpha(\theta) + (1-\theta)\tau_\beta(1-\theta),$$

and the name of the game is to determine the minimum of T. The symmetry of the problem clearly implies that the equilibrium, i.e., when the travel time along α equals that of β, occurs when $\theta = 1/2$, which is a global minimum as well as a local Pareto optimum (i.e., no perturbation will reduce all travel times) and Nash equilibrium (no driver would benefit for making any local change).

Now add a new road e of unit length, see Figure 2. We denote by γ the route

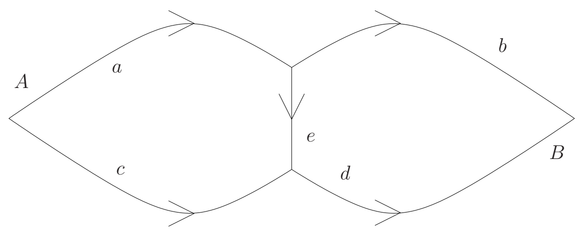

FIGURE 2. A network consisting of three routes α, β, and γ connecting A to B.

connecting a, e, and d, and assume that the velocity along road c is given by another constant v. Denote by θ_1 and θ_2 the fraction of cars taking routes α and β, respectively. The fraction that uses route γ is $1-(\theta_1+\theta_2)$. Naturally $\theta_1, \theta_2, (\theta_1+\theta_2) \in [0,1]$. Now the travel times read

$$\begin{aligned}
\tau_\alpha(\theta_1,\theta_2) &= \tau_a(1-\theta_2) + \tau_b(\theta_1),\\
\tau_\beta(\theta_1,\theta_2) &= \tau_b(\theta_2) + \tau_a(1-\theta_1),\\
\tau_\gamma(\theta_1,\theta_2) &= \tau_a(1-\theta_2) + \tau_e(1-\theta_1-\theta_2) + \tau_a(1-\theta_1),
\end{aligned}$$

and the average total travel time

$$T(\theta_1, \theta_2) = \theta_1 \, \tau_\alpha(\theta_1, \theta_2) + \theta_2 \, \tau_\beta(\theta_1, \theta_2) + (1 - \theta_1 - \theta_2) \, \tau_\gamma(\theta_1, \theta_2) \, .$$

Then, one can show, see [2, Theorem 2.7], that provided

$$\frac{e^\phi - 1}{\phi} < \frac{1}{V} - \frac{1}{v} < \frac{2}{\phi}(e^\phi - e^{\phi/2}),$$

the above network example will display the Braess paradox, that is, the addition of road e will make travel times worse than in the case without road e. More precisely, the point with $\theta_1 = \theta_2 = 0$ is the unique Nash point for the network with five roads. At the same time the corresponding travel time $\tau_\gamma(0, 0)$ is worse than the global optimal configuration for the network with four roads.

Clearly, the occurrence of the Braess paradox is unwanted, and a natural question is to inquire if one can enforce a *control* on road e that removes the paradox. We show in [2, Theorem 3.1] that by enforcing a suitable speed limit on road e, the Braess paradox does not occur.

References

[1] D. Braess, *Über ein Paradoxon aus der Verkehrsplanung*, Unternehmensforschung **1968** (12), 258–268. English translation: *On a paradox of traffic planning.* Transp. Science **2005** (39), 446–450.

[2] R. M. Colombo and H. Holden, *On the Braess paradox with nonlinear dynamics*, J. Optim. Theory Appl. **168** (2016), DOI 10.1007/s10957-015-0729-5.

[3] M. Garavello and B. Piccoli, *Traffic Flow on Networks*, American Institute of Mathematical Sciences, 2006.

[4] H. Holden and N. H. Risebro, *A mathematical model of traffic flow on a network of unidirectional roads.* SIAM J. Math. Anal. **1995** (26), 999–1017.

[5] M. J. Lighthill and G. B. Whitham, *On kinematic waves. II. A theory of traffic flow on long crowded roads.* Proc. Roy. Soc. London. Ser. A. **1955**(229), 317–345.

[6] P. I. Richards, *Shock waves on the highway*, Operations Res. **1956** (4), 42–51.

Towards a numerical solver for the multi-dimensional Euler equations

Christian Klingenberg

(joint work with Wasilij Barsukow)

Our goal is to develop a numerical method for the two dimensional compressible inviscid Euler equations. To that end we consider the linearized Euler equations:

$$\begin{aligned} v_t + \nabla p &= 0 \\ p_t + \nabla \cdot v &= 0 \quad . \end{aligned}$$

The motivation for studying this is that for the multi-dimensional Euler equations acoustics and advection are different. Here we begin by studying acoustics via the linearized Euler equations.

To solve these equations we can represent the solution via a closed formula for any given initial value problem. Note that these equations can be written as

$$\begin{aligned} p_{tt} - \Delta p &= 0 \\ v_{tt} - \nabla\nabla \cdot v &= 0 \quad , \end{aligned}$$

in other words even though the pressure satisfies a wave equations the velocity satisfies a more complicated equation. The closed form solution for v uses a Green's function and is more complicated than Kirchhoff's or Hadamard's formula.

This closed form solution can now be used to study the evolution of piecewise constant initial data on a rectangular mesh in two space dimensions. This will ensure that there are no numerical artifacts in the evolution that falsify the solution. So our two-dimensional algorithm consists of projecting the data to piecewise constants on rectangles, followed by the exact evolution of this data for a sufficiently small time step and finally projecting it back to piecewise constants. This leads to a poor algorithm, whose poor quality can not be attributed to an approximation of the evolution. The poor quality is seen by the smearing out of shocks that depend on their orientation with respect to the grid and also that this algorithm lacks the ability to solve low Mach number flow. Note that even for these linearized equations it is possible to have a notion of a Mach number. As this number goes to zero on obtains an incompressible equation.

We conclude that a two dimensional algorithm can not be based on reconstructing the data as piecewise discontinuous data. Our next step will be to base a new algorithm on the reconstruction to piecewise continuous elements.

We acknowledge helpful discussions on this topic with Phil Roe.

Asymptotic preserving IMEX finite volume schemes for singular limits of weakly compressible flows

Mária Lukáčová-Medviďová
(joint work with G. Bispen, L. Yelash)

In the case of weakly compressible flows the magnitude of flow velocity $\mathbf{u}$ is much smaller than the sound speed c, which results in the so-called low Mach number flows. Here the Mach number is a reference number defined as $M = \frac{|\mathbf{u}|}{c}$. Such flows arise in many applications, such as meteorology, combustion or astrophysics. Since the resulting problem is stiff, it is a well-known fact that a naive discretization would require that the spatial and the temporal steps, Δx and Δt, need to be reduced simultaneously as the Mach number $M \to 0$. Clearly, this is non-affordable computationally.

In our talk we have presented new IMEX finite volume schemes for the Euler equations with the gravity source term that are based on the so-called acoustic/advection splitting strategy. More precisely, we split the whole nonlinear system of the Euler equations into a stiff linear part governing fast acoustic and gravity waves and a non-stiff nonlinear part that models slow nonlinear advection effects, see also our recent papers [2, 3, 4, 11]. For time discretization we have used higher order globally stiffly accurate IMEX schemes and approximate stiff linear operator implicitly and the non-stiff nonlinear operator explicitly, see, e.g. ASR(2,2,2) [1]. Consequently, we can efficiently resolve slow nonlinear dynamics due to advection effects.

Our main goal is to analyse the asymptotic preserving properties of these methods and show that a suitable splitting into the linear stiff subsystem for acoustic/gravity waves and the nonlinear non-stiff subsystem for the advection combined with the IMEX FV discretization yields asymptotic preserving schemes. The concept of the *asymptotic preserving schemes* has been firstly introduced by Jin et al., see [8], [9] and the references therein: a numerical scheme is called asymptotic preserving if it is uniformly consistent as a singular limit parameter, e.g. the Mach number, approaches its limit. In particular, the scheme reduces to a consistent approximation of the limit equation. In our recent paper [5] we have analysed both the *asymptotic consistency* as well as *asymptotic stability* of our IMEX FV schemes.

In particular, using the theory of circulant matrices we are able to investigate the matrix properties of the resulting discrete system. We show that its inverse acts as an orthogonal projection on null spaces of the corresponding discrete operators appearing at the right hand side of the discrete system. Consequently, we obtain that new solution at the time step t_{n+1} satisfy the expected asymptotic properties. More precisely, they are of order $\mathcal{O}(M^2)$. This leads to the consistency result: *Numerical solution yields a consistent approximation of the limiting incompressible Euler equations in the singular limit as* $M \to 0$. Furthermore, using the energy method and the equivalence of discrete norms we are also able to prove that the numerical solution is uniformly stable with respect to M, if a fixed mesh is used.

We also refer to a recent work of Kaiser et al. [10], where asymptotic consistency of the so-called RS IMEX schemes for the isentropic Euler equations has been studied. Note that RS IMEX schemes are strongly related to our IMEX FV schemes; both IMEX methods use analogous splitting and may differ in the choice of a reference solution or an equilibrium solution. In [2, 3] we have analysed asymptotic consistency and accuracy of the IMEX FV for the shallow water equations with a bottom topography source term, which are mathematically equivalent to the isentropic Euler equations used in [10].

In order to preserve equilibria of the underlying hyperbolic balance laws on the discrete level a special treatment of zero-order source terms is required, which yields the well-balanced schemes, see, e.g., [6], [7] and the references therein. Our schemes are *well-balanced* as well. Indeed, they preserve a particular underlying equilibrium by the construction, since time update is realized only for perturbations of the underlying equilibrium state.

Our numerical experiments presented at the end of the talk clearly demonstrated the uniform order of convergence with respect to M, as far as advective effects are dominant. If acoustic waves are also present in the solution, the scheme is convergent uniformly with respect to M, but the order of convergence is recovered only if the space discretization parameter is small enough to resolve fast acoustic waves.

References

[1] U.M. Ascher, S.J. Ruuth, and R.J. Spiteri, *Implicit-explicit Runge-Kutta methods for time-dependent partial differential equations*, Appl. Numer. Math. **25**(2-3) (1997), 151–167.

[2] G. Bispen, *IMEX finite volume schemes for the shallow water equations*, PhD-thesis, University of Mainz, 2015.

[3] G. Bispen, K. R. Arun, M. Lukáčová-Medvid'ová, S. Noelle, *IMEX large time step finite volume methods for low Froude number shallow water flows*, Comm. Comput. Phys. **16** (2014), 307–347.

[4] G. Bispen, M. Lukáčová-Medvid'ová, L. Yelash, *IMEX finite volume evolution Galerkin schemes for three-dimensional weakly compresible flows*, Proceedings of Algoritmy, Eds. K. Mikula et al., (2016).

[5] G. Bispen, M. Lukáčová-Medvid'ová, L. Yelash, *Asymptotic preserving IMEX finite volume schemes for low Mach number Euler equations with gravitation*, submitted (2016).

[6] P. Chandrashekar, C. Klingenberg, *A second order well-balanced finite volume scheme for Euler equations with gravity*, SIAM J. Sci. Comput. **37** (2015), B382-B402.

[7] A. Chertock, S. Cui, A. Kurganov, S. Ozcan and E. Tadmor, *Well-balanced central-upwind schemes for the Euler equations with gravitation*, submitted (2015).

[8] J. Haack, S. Jin, and J.-G. Liu, *An all-speed asymptotic-preserving method for the isentropic Euler and Navier-Stokes equations*, Commun. Comput. Phys. **12** (2012), 955–980.

[9] S. Jin, *Asymptotic preserving (AP) schemes for multiscale kinetic and hyperbolic equations: a review*, Riv. Mat. Univ. Parma **3** (2012), 177–216.

[10] K. Kaiser, J. Schütz, R. Schöbel, S. Noelle, *A new stable splitting for the isentropic Euler equations*, IGPM report 442, RWTH Aachen University (2016).

[11] M. Lukáčová-Medvid'ová, A. Müller, V. Wirth, and L. Yelash, *Adaptive discontinuous evolution Galerkin method for dry atmospheric flow.* J. Comput. Phys. **268** (2014), 106–133.

A traffic model with phase transitions at a junction

Francesca Marcellini

(joint work with Mauro Garavello)

We consider the Phase Transition traffic model in [6], based on a non-smooth 2×2 system of conservation laws,

$$(1) \qquad \begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho, w)\right) = 0 \\ \partial_t (\rho\, w) + \partial_x \left(\rho\, w\, v(\rho, w)\right) = 0 \end{cases} \qquad \text{with} \qquad v = \min \left\{V_{\max}, w\, \psi(\rho)\right\} ,$$

where ρ is the traffic density, $w = w(t,x)$ is the maximal speed of each driver, ψ is a $\mathbf{C^2}$ function and $V_{\max}$ is a uniform bound on the speed. This is a macroscopic description displaying 2 phases, the *Free* phase F and *Congested* phase C, described by the sets

$$F = \left\{(\rho, w) \in [0, R] \times [\check{w}, \hat{w}] \colon v(\rho, \rho w) = V_{\max}\right\},$$
$$C = \left\{(\rho, w) \in [0, R] \times [\check{w}, \hat{w}] \colon v(\rho, \rho w) = w\, \psi(\rho)\right\} ,$$

where R is the maximal traffic density. This model is an extension of the classical Lighthill-Whitham [12] and Richards [14] model and it falls into the class of second order traffic models introduced by Aw and Rascle in [1] and independently by

Zhang in [15]. In 2002, Colombo proposed the first second order model with two different phases in [4, 5]. See also the phase transition models in [2, 10, 13].

1. The Riemann Problem at a Junction

We propose a Riemann solver at a junction for the model in (1) which conserves the number of cars and also the maximal speed w of each vehicle, see [8]. Note that w is a peculiar characteristic of (1), being a specific feature of every single driver.

We consider a junction with n incoming arcs $I_1, ..., I_n$ and m outgoing arcs $I_{n+1}, ..., I_{n+m}$, where each incoming arc is given by $I_i = \left]-\infty, 0\right]$ and each outgoing arc is $I_j = \left[0, +\infty\right[$, see [3, 7, 9, 11]. On each arc we consider the phase transition model in (1) with the change of variable $\eta = \rho w$; we get a system where the conserved variables are ρ and η and the speed is $v(\rho, \eta) = \min\left\{V_{\max}, \frac{\eta}{\rho}\,\psi(\rho)\right\}$.

We consider the following Riemann problem

$$
\begin{cases}
\begin{cases} \partial_t \rho + \partial_x\left(\rho\, v(\rho,\eta)\right) = 0 \\ \partial_t \eta + \partial_x\left(\eta\, v(\rho,\eta)\right) = 0 \end{cases} & (\rho,\eta) \in I_i \\
\begin{cases} \partial_t \rho + \partial_x\left(\rho\, v(\rho,\eta)\right) = 0 \\ \partial_t \eta + \partial_x\left(\eta\, v(\rho,\eta)\right) = 0 \end{cases} & (\rho,\eta) \in I_j \\
(\rho_i, \eta_i)(0,x) = (\bar\rho_i, \bar\eta_i) \\
(\rho_j, \eta_j)(0,x) = (\bar\rho_j, \bar\eta_j)\,,
\end{cases}
\tag{2}
$$

where $(\bar\rho_i, \bar\eta_i) \in F \cup C$ are the initial data in each incoming arc I_i, $i = 1, ..., n$, and $(\bar\rho_j, \bar\eta_j) \in F \cup C$ are the initial data in each outgoing arc I_j, $j = 1, ..., m$.

We define the concept of Riemann solver at a generic junction.

Definition 1.1. *A Riemann solver at a junction is a function*

$$
\begin{array}{rcl}
\mathcal{RS}_J : \prod_{i=1}^{n+m} (F \cup C) & \longrightarrow & \prod_{i=1}^{n+m} (F \cup C) \\
\left((\rho_1,\eta_1), \cdots, (\rho_{n+m}, \eta_{n+m})\right) & \longmapsto & \left((\rho_1^*,\eta_1^*), \cdots, (\rho_{n+m}^*, \eta_{n+m}^*)\right)
\end{array}
$$

satisfying the following properties.

1. *The consistency condition holds, i.e.:*
$$\mathcal{RS}_J\left((\rho_1^*,\eta_1^*), \cdots, (\rho_{n+m}^*, \eta_{n+m}^*)\right) = \left((\rho_1^*,\eta_1^*), \cdots, (\rho_{n+m}^*, \eta_{n+m}^*)\right)\,.$$
2. *For every $i \in \{1, \ldots, n\}$, the Riemann problem in* (2) *with initial data $(\rho,\eta)(0,x) = (\rho_i, \eta_i)$, with $x < 0$, is solved with waves with negative speed.*
3. *For every $i \in \{n+1, \ldots, n+m\}$, the Riemann problem in* (2) *with initial data $(\rho,\eta)(0,x) = (\rho_i, \eta_i)$, with $x > 0$, is solved with waves with positive speed.*
4. *The traffic distribution*
$$A \begin{bmatrix} \rho_1^*\, v(\rho_1^*, \eta_1^*) \\ \vdots \\ \rho_n^*\, v(\rho_n^*, \eta_n^*) \end{bmatrix} = \begin{bmatrix} \rho_{n+1}^*\, v(\rho_{n+1}^*, \eta_{n+1}^*) \\ \vdots \\ \rho_{n+m}^*\, v(\rho_{n+m}^*, \eta_{n+m}^*) \end{bmatrix}$$

holds, where $A = (\alpha_{i,j})_{i=1,\ldots,n;\, j=n+1,\ldots,n+m}$*, whose coefficients indicate the percentage of traffic that passes from* I_i *to* I_j*, with* $\sum_{j=n+1}^{n+m} \alpha_{ij} = 1$.

(5) *The mass conservation holds, i.e.* $\sum_{i=1}^{n} \rho_i^* v(\rho_i^*, \eta_i^*) = \sum_{i=n+1}^{n+m} \rho_i^* v(\rho_i^*, \eta_i^*)$.

(6) *The distribution of the maximal speed holds, i.e.:*

$$w_{n+1}^* = \frac{1}{\sum_{i=1}^{n} \alpha_{i,n+1}\gamma_i^*} \left[\alpha_{1,n+1}\gamma_1^* w_1^* + \ldots + \alpha_{n,n+1}\gamma_n^* w_n^*\right],$$

$$\vdots$$

$$w_{n+m}^* = \frac{1}{\sum_{i=1}^{n} \alpha_{i,n+m}\gamma_i^*} \left[\alpha_{1,n+m}\gamma_1^* w_1^* + \ldots + \alpha_{n,n+m}\gamma_n^* w_n^*\right],$$

where $w_i^* = \frac{\eta_i^*}{\rho_i^*}$ *and* $\gamma_i^* = \rho_i^* v(\rho_i^*, \eta_i^*)$ *for every* $i \in \{1, \ldots, n+m\}$.

For special junctions, the cases of $1 \times m$ and 2×1 junctions, we prove that the Riemann solver is well defined. The following result holds (see [8] for the proof).

Theorem 1.2. *Under the assumptions*

(H-1): $R, \breve{w}, \hat{w}, V_{\max}$ *are positive constants, with* $\breve{w} < \hat{w}$*;* $\breve{w}$ *and* $\hat{w}$ *are the minimum, respectively, maximum, of the maximal speeds of each vehicle;*

(H-2): $\psi \in \mathbf{C^2}([0,R];[0,1])$ *with* $\psi(0) = 1$*,* $\psi(R) = 0$*,* $\psi'(\rho) \leq 0$ *and* $\frac{d^2}{d\rho^2}(\rho\,\psi(\rho)) \leq 0$*, for all* $\rho \in [0,R]$*;*

(H-3): $\breve{w} > V_{\max}$*;*

(H-4): *the waves of the first family in* C *have negative speed,*

the Riemann solver $\mathcal{RS}_J$ *for the cases of* $1 \times m$ *and* 2×1 *junctions, constructed as in* [8, Section 4, Section 5]*, satisfies all the conditions of Definition 1.1 and produces a solution to the Riemann problem (2).*

Remark 1.3. *We note that the distribution of the maximal speed in (6) of Definition 1.1 is given by*

$$w_2^* = \ldots = w_{1+m}^* = \bar{w}_1,$$

in the case of $1 \times m$ *junction and is given by*

$$w_3 = \frac{\gamma_1}{\gamma_1 + \gamma_2}\bar{w}_1 + \frac{\gamma_2}{\gamma_1 + \gamma_2}\bar{w}_2,$$

where $\gamma_1 = \rho_1 v(\rho_1, \eta_1)$ *and* $\gamma_2 = \rho_2 v(\rho_2, \eta_2)$*, in the case of* 2×1 *junction, see* [8].

Acknowledgments. The author thanks Rinaldo M. Colombo for useful discussions. The author was partial supported by the INdAM-GNAMPA 2016 project "Balance Laws: Theory and Applications".

References

[1] A. Aw, and M. Rascle, *Resurrection of "second order" models of traffic flow*, SIAM J. Appl. Math. **60** (2000), 916–938.

[2] S. Blandin, D. Work, P. Goatin, B. Piccoli, and A. Bayen, *A general phase transition model for vehicular traffic*, SIAM J. Appl. Math. **63** (2011), 107–127.

[3] G.M. Coclite, M. Garavello, B. Piccoli *Traffic flow on a road network*, SIAM J. Math. Anal. **36** (2005), 1862–1886.

[4] R.M. Colombo, *Hyperbolic phase transitions in traffic flow*, SIAM J. Appl. Math. **63** (2002), 708–721.
[5] R.M. Colombo, *Phase transitions in hyperbolic conservation laws*, In Progress in analysis, Vol. I, II (Berlin, 2001). World Sci. Publ., River Edge, NJ, (2003), 1279–1287.
[6] R.M. Colombo, F. Marcellini, and M. Rascle, *A 2-phase traffic model based on a speed bound*, SIAM J. Appl. Math. **70** (2010), 2652–2666.
[7] M. Garavello, B. Piccoli, *Models for Vehicular Traffic on Networks*, volume 9 of AIMS Series on Appl. Math. American Institute of Mathematical Sciences (AIMS), Springfield, MO (2016).
[8] M. Garavello, F. Marcellini, *The Riemann Problem at a Junction for a Phase-Transition Traffic Model*, Preprint (2016).
[9] M. Garavello, B. Piccoli, *Traffic flow on networks*, volume 1 of AIMS Series on Appl. Math. American Institute of Mathematical Sciences (AIMS), Springfield, MO (2006).
[10] P. Goatin, *The Aw-Rascle vehicular traffic flow model with phase transitions*, Math. Comput. Modelling **44** (2006), 287–303.
[11] H. Holden, N.H. Risebro*A mathematical model of traffic flow on a network of unidirectional roads*, SIAM J. Math. Anal. **4** (1995), 999–1017.
[12] M.J. Lightill, and G.B. Witham, *On kinematic waves. II. A theory of traffic flow on long crowded roads*, Proc. Roy. Soc. London. Ser. A. **229** (1955), 317–345.
[13] F. Marcellini, *Free-Congested and Micro-Macro Descriptions of Traffic Flow*, Discrete Contin. Dynam. Systems-Series S-AIMS **7** (2014), 543-556.
[14] P.I. Richards, *Shock waves on the highway*, Operations Res. **4** (1956), 42–51.
[15] H.M. Zhang, *A non-equilibrium traffic model devoid of gas-like behavior*, Transportation Research Part B: Methodological **36** (2002), 275–290.

Lagrangian structure of BV solutions for hyperbolic systems of conservation laws

Stefano Modena
(joint work with Stefano Bianchini)

One of the key observations in Fluid Dynamics is that the fluid flow can be described from two different (and in some sense complementary) points of view: the Lagrangian point of view (in which the trajectory in space-time of each single fluid particle is tracked) and the Eulerian point of view (in which one looks at fluid motion focusing on fixed locations in the space through which the fluid flows as time passes). Such key observation has been successfully applied to the analysis of some particular partial differential equations (among all, the transport equation and the Euler equation), leading to important theoretical results. For instance, in the linear transport equation

$$\begin{cases} \partial_t v(t,x) + b(t,x) \cdot \nabla_x v(t,x) &= 0, \\ v(t,0) &= \bar{v}(x), \end{cases} \tag{1}$$

where

$v : [0,\infty) \times \mathbb{R}^d \to \mathbb{R},$ is the unknown,

$b : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is a given, incompressible vector field, $\mathrm{div}_x b(t,x) = 0,$

the solution to (1) presents a strong connection with the Lagrangian flow $\mathtt{x} : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ generated by the ODE

$$\begin{cases} \dfrac{\partial \mathtt{x}}{\partial t}(t,y) &= b(t,\mathtt{x}(t,y)), \\ \mathtt{x}(0,y) &= y. \end{cases} \tag{2}$$

Indeed, under suitable regularity assumptions, it can be proved that the function $v(t,x)$ implicitly defined by

$$v(t,\mathtt{x}(t,y)) = v(0,y) = \bar{v}(y) \tag{3}$$

provides the unique solution to the Cauchy problem (1).

In a joint work [2] in preparation with Stefano Bianchini (see also [7]) we show that also the hyperbolic system of conservation laws in one space variable in its most general form

$$\begin{cases} \partial_t u + \partial_x F(u) = 0, \\ u(0,x) = \bar{u}(x), \end{cases} \qquad u = u(t,x) \in \mathbb{R}^N,\ t \geq 0,\ x \in \mathbb{R},\ \text{Tot.Var.}(\bar{u}) \ll 1, \tag{4}$$

can be analyzed from a Lagrangian point of view. Here $F : \mathbb{R}^N \to \mathbb{R}^N$ is a generic smooth function, which is only assumed to be *strictly hyperbolic*, i.e. its differential $DF(u)$ has N distinct real eigenvalues in each point of its domain. No convexity assumption on F is made. We also consider small initial data (Tot.Var.$(\bar{u}) \ll 1$), because this is the setting where the well-posedness of the Cauchy problem (4) is established (see for instance [1, 3, 5] and the references therein).

There are basically two different motivations for studying hyperbolic systems of conservation laws from a Lagrangian point of view. On one side there is a purely theoretical motivation: our theory provides a suitable extension in a more general setting to two well known theories developed in the past years. In [6] Tai-Ping Liu proposed a *wave-tracing* algorithm for tracing the trajectory in time of each wavefront present in a Glimm approximate solution to (4). Our Lagrangian approach provides the counterpart of Liu's wave-tracing for an exact (and not approximate) solution to (4). In [4] Constantine Dafermos introduced the notion of generalized characteristics. His theory works when the flux F is genuinely non linear; our approach is a generalization of Dafermos' theory in the case when F is only strictly hyperbolic, without any convexity assumption.

On the other side, there is a more applicative reason. We think that our Lagrangian approach can lead to a deeper understanding of the behavior and of the structure of the solutions to (4). In particular we aim to use our Lagrangian approach to study the fine structure of the solution (its regularity, its behavior near a point of "strong interaction", the stability of the shocks and so on).

The introduction of the precise notion of *Lagrangian representation* for the solution to the general system (4) is beyond the scope of this paper. For this reason we prefer here to give such definition in the context of a single scalar conservation law (i.e. $N = 1$). In such simplified setting, the definition reads as follows.

Definition 1.1. *A* Lagrangian representation *of the solution of* (4) *in the case* $N = 1$ *is a* 3*-tuple* $(\mathcal{W}, \mathtt{x}, \rho)$ *where*

$\mathcal{W} \subseteq \mathbb{R}$ *is an interval, called the* set of waves *or* set of particles,

$\mathtt{x} : [0, \infty) \times \mathcal{W} \to \mathbb{R}$ *is the* position function,

$\rho : [0, \infty) \times \mathcal{W} \to [-1, 1]$ *is the* density function,

such that

(a) some regularity assumptions on $\mathtt{x}$, ρ *hold; more precisely*
- *the map* $t \mapsto \mathtt{x}(t, w)$ *is Lipschitz for each fixed* w*; the map* $w \mapsto \mathtt{x}(t, w)$ *is increasing for each fixed* t*;*
- *the distributional derivative* $D_t\rho$ *of* ρ *with respect to the time is a finite Radon measure;*

(b) for a.e. wave w *and a.e. time* t*, it holds* $\frac{\partial \mathtt{x}}{\partial t}(t, w) = \lambda(t, \mathtt{x}(t, w))$*, where*

$$\lambda(t,x) = \begin{cases} f'(u(t,x)) & \text{if } x \mapsto u(t,x) \text{ is continuous at } x, \\ \frac{f(u(t,x+)) - f(u(t,x-))}{u(t,x+) - u(t,x-)} & \text{if } u \text{ has a jump at } x; \end{cases}$$

(c) for a.e. time t*, it holds* $D_x u(t) = \mathtt{x}(t)_\sharp(\rho(t)\mathcal{L}^1)$*.*

The main theorem we prove in [2], [7] is the following.

Theorem 1.2. *There exists (at least) one Lagrangian representation for the solution to the general strictly hyperbolic system* (4)*.*

We would like to stress once again that, even if we gave above the definition of Lagrangian representation only for a scalar conservation law, in [2], [7] we provide the correct definition and prove Theorem 1.2 for a general system of N conservation laws.

Instead of presenting a sketch of the proof of Theorem 1.2, we prefer to conclude this extended abstract trying to give an "interpretation" of Definition 1.1.

First of all there is a set of (infinitesimal) waves (or particles) $\mathcal{W}$. Each particle $w \in \mathcal{W}$ moves along a Lipschitz trajectory $\mathtt{x}(\cdot, w)$, which satisfies a suitable ODE, see Property (b) above. The monotonicity of $w \mapsto \mathtt{x}(t, w)$ for fixed time t implies that two different particles can have the same position at a given time, but they can not cross each other. Finally Property (c) means that, at each fixed time t, starting from $\mathtt{x}$ and ρ we are able to reconstruct the (distributional derivative of the) solution $u(t, \cdot)$, so that $\mathtt{x}$ and ρ are enough to determine u.

It is instructive to make a comparison with the analysis made at the beginning about the transport equation (1). Exactly as in (1), also for the system of conservation laws (4) we are able to construct a *flow* $\mathtt{x}(t, w)$, which satisfies the "correct" ODE (compare Property (b) in Definition 1.1 with equation (2)) and along which the initial datum is transported. However, differently from the transport equation where the vector field b is incompressible, the system of conservation laws (4) is far from satisfying any "incompressibility" property: this is clear even when one tries to solve a scalar conservation law through the method of characteristics and sees that two distinct characteristics can collide in a finite time. In our Lagrangian framework, this reflects on the fact that, as we have already pointed out, two waves can have the same position at the same time (i.e. they can *interact*), and such interactions can create or cancel waves. This is a well known behavior of conservation laws and a main source of difficulties in their analysis. The map ρ is exactly the tool we introduce in order to keep track of all these phenomena: in our setting, equation (3) (where, roughly speaking, all the particles have the same density) is substituted by Property (c) above, (where each particle is counted with its density $\rho(t, w)$).

References

[1] S. Bianchini, A. Bressan, *Vanishing viscosity solutions of nonlinear hyperbolic systems*, Annals of Mathematics **161** (2005), 223–342.

[2] S. Bianchini, S. Modena, *Lagrangian representation for solution to general systems of conservation laws*, in preparation (2016).

[3] A. Bressan, *Hyperbolic Systems of Conservation Laws. The One Dimensional Cauchy problem*, Oxford University Press (2000).

[4] C. Dafermos, *Generalized characteristics in hyperbolic systems of conservation laws*, Arch. Rational Mech. Anal. **107** (1989), 127–155.

[5] J. Glimm, *Solutions in the Large for Nonlinear Hyperbolic Systems of Equations*, Comm. Pure Appl. Math. **18** (1965), 697–715.

[6] T.P. Liu, *The deterministic version of the Glimm scheme*, Comm. Math. Phys., **57** (1977), 135–148.

[7] S. Modena, *Interaction functionals, Glimm approximations and Lagrangian structure of BV solutions for Hyperbolic Systems of Conservation Laws*, Ph.D. thesis (2015), Sissa Digital Library.

On α-dissipative solutions of the two-component Hunter–Saxton equation

Anders Nordli

(joint work with Katrin Grunert)

The two-component Hunter–Saxton system is a generalization of the Hunter–Saxton equation, given by

$$u_t(x,t) + u(x,t)u_x(x,t) = \frac{1}{4}\bigg(\int_{-\infty}^{x} (u_x(z,t)^2 + \rho(z,t)^2)\,\mathrm{d}z$$

$$- \int_{x}^{\infty} (u_x(z,t)^2 + \rho(z,t)^2)\,\mathrm{d}z\bigg), \tag{1a}$$

$$\rho_t(x,t) + (u(x,t)\rho(x,t))_x = 0. \tag{1b}$$

Solutions of (1) experience wave breaking which means that u_x tends pointwise to $-\infty$ in finite time while u stays continuous. At wave breaking the energy density, $u_x(x,t)^2 + \rho(x,t)^2$, concentrates into a measure. What amount of the concentrated energy to keep can be chosen freely, and hence there is no uniqueness of weak solutions. To select a solution we let α be a Lipschitz continuous function, $\alpha : \mathbb{R} \to [0, 1)$, and choose to retain an $(1 - \alpha)$-fraction of the energy concentrated at the point of wave breaking.

The method of characteristics can be applied to obtain an equivalent system in Lagrangian coordinates. The new system is given by

$$\begin{aligned} y_t(\xi,t) &= U(\xi,t), \\ U_t(\xi,t) &= \frac{1}{2}V(\xi,t) - \frac{1}{4}\lim_{\xi\to\infty} V_\infty(\xi,t), \\ H_t(\xi,t) &= 0, \\ r_t(\xi,t) &= 0, \end{aligned}$$

where the definition of V determines which weak solution we get. The choice of α gives

$$V_\xi(\xi,t) = \begin{cases} H_\xi(\xi,0), & \text{if there has been no wave breaking,} \\ (1-\alpha(\bar{y})H_\xi(\xi,0), & \text{if there was wavebreaking at } \bar{y}. \end{cases}$$

Existence and uniqueness of solutions of the equivalent system can be shown by a contraction argument. The solution of the equivalent system can be mapped back to Eulerian coordinates to prove existence of α-dissipative weak solutions of (1).

References

[1] J.K. Hunter and R. Saxton, *Dynamics of director fields*, SIAM J. Appl. Math. **51(6)** (1991), 1498–1521.
[2] M.V. Pavlov, *The Gurevich–Zybin system*, J. Phys. A **38(17)** (2005), 3823–3841.
[3] M. Wunsch, *On the Hunter–Saxton system*, Discrete Contin. Dyn. Syst. Ser. B **12(3)** (2009), 647–656.
[4] K. Grunert and A. Nordli, *A Lipschitz metric for α-dissipative solutions of the two-component Hunter–Saxton system*, In preparation.

Non local mixed systems and IBVPs for balance laws

Elena Rossi
(joint work with Rinaldo M. Colombo, Veronika Schleper)

We couple a non local balance law with a parabolic equation, obtaining the following new class of mixed hyperbolic–parabolic systems:

$$\begin{cases} \partial_t u + \nabla\cdot(u\,v(w)) = (\alpha\,w - \beta)u \\ \partial_t w - \mu\Delta w = (\gamma - \delta\,u)w \end{cases} \tag{1}$$

The idea behind is to describe two competing populations, predators and prey, characterised by their density, called respectively u and w: the first evolves according to the balance law, the second diffuses according to the parabolic equation. The source terms of this system, motivated by Lotka–Volterra equations, describe the feeding.

The main feature of the mixed system (1) lies in the drift term v in the balance law: v is chosen to be a non local function of the prey density w. This allows to model the fact that predators can feel the presence of prey also from far away. A possible choice of the functional v is the following

$$v(w) = \kappa \frac{\nabla(w * \eta)}{\sqrt{1 + \|\nabla(w * \eta)\|^2}}, \tag{2}$$

where $\kappa > 0$ and η is a positive smooth mollifier. When v is chosen as in (2), predators direct their movement towards the regions where the concentration of prey is higher.

Solutions (u, w) to system (1) are sought in the space $(\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R}) \times (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^n; \mathbb{R})$, thus in a space different from that considered usually for parabolic equations. In [3] the basic well posedness results for the mixed system (1) are proven: in particular [3, Theorem 2.2] ensures existence, uniqueness, continuous dependence of the solution on the initial data, $\mathbf{L}^1$ and $\mathbf{L}^\infty$ estimates.

The analytic structure suggests that a reliable numerical procedure should be devised to study qualitatively the solutions to (1). In [6] we derive an algorithm to numerically solve the coupled system: the parabolic part is approximated by an explicit finite–difference method, while an *ad hoc* adaptation of Lax–Friedrichs method with dimensional splitting is used for the hyperbolic part. Both source terms are treated by operator splitting, using a second order Runge–Kutta method.

The convergence of the numerical algorithm is proven in [6, Theorem 4.1]: the hyperbolic variable u converges strongly in $\mathbf{L}^1$, the parabolic one w converges weakly* in $\mathbf{L}^\infty$. The proof relies on a careful tuning between the integration methods and it exploits strongly the non locality of the convective part in the balance law.

The algorithm has been implemented in a series of `Python` scripts. Using them, qualitative properties of the solutions are investigated. We observe the formation of a discrete, quite regular pattern: while prey diffuse, predators accumulate on the vertices of a regular lattice, see [3, Figure 3.3] and [6, Figure 5]. In [2, Section 2] we try to change some parameters of the system and see how this influences the pattern.

The analytic study of system (1) is on all $\mathbb{R}^n$. However, both numerical integrations and possible biological applications suggest that the boundary plays a relevant role. It would then be interesting to study the mixed system (1) in a bounded domain. As far as parabolic equations in bounded domain are concerned, in the literature many results can be found. However, known results for balance laws in bounded domains lack some estimates needed to deal with the coupling. Therefore, in [4] the focus is shifted to the following Initial Boundary Value Problem (IBVP) for a general balance law

$$\begin{cases} \partial_t u + \nabla \cdot f(t,x,u) = F(t,x,u) & (t,x) \in \mathrm{I} \times \Omega \\ u(0,x) = u_o(x) & x \in \Omega \\ u(t,\xi) = u_b(t,\xi) & (t,\xi) \in \mathrm{I} \times \partial\Omega, \end{cases} \tag{3}$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain and $\mathrm{I} = [0,T]$.

The key reference for the study of this IBVP is the fundamental paper [1]. However, there detailed estimates are given explicitly only in the case of homogeneous boundary conditions, that is $u_b = 0$. In [4] we go over this case, providing rigorous hypotheses and estimates and paying particular attention to the regularity and compatibility conditions. Then, we study also the general IBVP, with possibly non homogeneous boundary condition, and prove its well posedness: in [4, Theorem 2.7 and Theorem 4.3] we show the existence and uniqueness of an entropy solution to (3), $\mathbf{L}^\infty$ and TV estimates, and $\mathbf{L}^1$ Lipschitz continuity of the solution as a function of time, of the initial datum and of the boundary datum. Parabolic, hyperbolic and also elliptic techniques have been used to deal with (3).

What is now missing is the stability of the solution with respect to the flux and the source. As far as the latter is concerned, it might result from a careful use of the doubling of variables technique. However, the stability with respect to the flux appears to be a challenging problem. A first step in this direction has been made in [5]. Here, we consider the one dimensional IBVP for a conservation law with flux f that does not depend explicitly on the space variable:

$$\begin{cases} \partial_t u + \partial_x f(t,u) = 0 & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ u(0,x) = u_o(x) & x \in \mathbb{R}^+ \\ u(t,0) = u_b(t) & t \in \mathbb{R}^+ \end{cases} \tag{4}$$

Note that also the case of an unbounded domain, such as the half line, is now included. We prove the existence of a solution, exploiting the wave front tracking technique, and the stability of the solution with respect to the flux function.

This stability result is necessary to deal with the coupling of a balance law with a parabolic equation. However, its extension to the multidimensional case, but even to the one dimensional case with space dependent flux, is still an open problem.

References

[1] C. Bardos, A. Y. le Roux, and J.-C. Nédélec, *First order quasilinear equations with boundary conditions*, Comm. Partial Differential Equations **4(9)** (1979), 1017–1034.

[2] R. M. Colombo, F. Marcellini, and E. Rossi, *Biological and industrial models motivating nonlocal conservation laws: a review of analytic and numerical results*, Netw. Heterog. Media **11(1)** (2016), 49–67.

[3] R. M. Colombo and E. Rossi., *Hyperbolic predators vs. parabolic prey*, Commun. Math. Sci. **13(2)** (2015), 369–400.

[4] R. M. Colombo and E. Rossi., *Rigorous estimates on balance laws in bounded domains*, Acta Math. Sci. Ser. B Engl. Ed. **35(4)** (2015), 906–944.

[5] R. M. Colombo and E. Rossi., *Stability of the 1D IBVP for a non autonomous scalar conservation law*, submitted.

[6] E. Rossi and V. Schleper, *Convergence of a numerical scheme for a mixed hyperbolic–parabolic system in two space dimensions*, ESAIM Math. Model. Numer. Anal. **50(2)** (2016), 475-497.

Shock waves in the presence of dispersion

Michael Shearer
(joint work with Gennady El and Mark Hoefer)

Dispersive shock waves (DSW) of the KdV equation have a well-defined structure [3] that includes a modulated periodic wave train, led by a solitary wave. This structure is also seen in the modified KdV equation,

$$u_t + (u^3)_x = \mu u_{xxx}, \tag{1}$$

in which the flux $f(u) = u^3$ is non-convex [1]. Due to the non-convex (cubic) flux in the modified equation, the sign of the dispersion coefficient, μ, is important and there is a richer set of DSW for equation (1), including contact DSW (if $\mu < 0$) and kinks (if $\mu > 0$). We investigate the structure of these waves using the approach of Gurevich and Pitaevskii [3, 4] in which the analysis is simplified by the use of Riemann invariants for the Whitham modulation equations.

It is also instructive to compare solutions of (1) with those of the modified KdV-Burgers equation, which includes a diffusive term,

$$u_t + (u^3)_x = \nu u_{xx} + \mu u_{xxx}, \quad \nu > 0. \tag{2}$$

The structure of shock waves for (2) is quite different, and fits the classical conservation laws description of Lax and Oleinik if $\mu < 0$. However, for $\mu > 0$, there are undercompressive shocks, which are diffusive equivalents of the kink solutions of (1).

We are in the process of extending results to equations like the BBM equation, in which the dispersion has an evolutionary quality. In this analysis, we discovered a new phenomenon, that of stationary expansion shocks, for both the BBM equation, and for the Boussinesq system of equations.

Stationary shock solutions $u(x,t) = \mathrm{sgn}(x)A$ of the BBM equation

$$u_t + uu_x = \mu u_{xxt}, \tag{3}$$

are *expansive* if $A > 0$. When such a shock is approximated by a smooth initial function $u(x,0) = u_0(x) = \tanh(x/\epsilon)$ with $0 < \epsilon << 1$, and allowed to evolve in time, we observe numerically that the smoothed discontinuity persists. The magnitude decays like $1/t$ and a rarefaction wave develops. This structure is explained in [2] using matched asymptotic expansions in which the stationary shock is treated as an inner layer and the rarefaction is a simple wave attaching to the constants $\pm A$ in the far field.

A similar surprising phenomenon is observed in stationary solutions of a version of the Boussinesq equations of shallow water flow,

$$\begin{aligned} h_t + (uh)_x &= 0 \\ u_t + uu_x + h_x - \tfrac{1}{3}u_{xxt} &= 0. \end{aligned} \tag{4}$$

Smooth initial data approximating an expansion shock persists in time, weakening to give way to a rarefaction wave only algebraically in time. This structure is explained using the Riemann invariants, with the observation that one invariant is close to constant throughout, while the other invariant satisfies the BBM equation (3) to leading order.

Acknowledgements

We thank the MFO for the opportunity to share our results. The research was supported by the Royal Society International Exchanges Scheme IE131353, NSF CAREER grant DMS-1255422 (Hoefer), and NSF grant DMS-1517291 (Shearer).

References

[1] El, G.A., Hoefer, M.A. and Shearer, M. *Dispersive and diffusive-dispersive shock waves for nonconvex conservation laws.* SIAM Review, to appear. arXiv:150101681 [nlinPS]. 2015.

[2] El, G.A., Hoefer, M.A. and Shearer, M. *Expansion shock waves in regularized shallow-water theory.* Proc. R. Soc. A 472: 20160141. http://dx.doi.org/10.1098/rspa.2016.0141

[3] El, G.A. and Hoefer, M.A. *Dispersive shock waves and modulation theory.* Physica D, to appear. arXiv:160206163 [nlinPS]. 2016.

[4] Gurevich, A.V. and Pitaevskii, L.P. *Nonstationary structure of a collisionless shock wave.* Sov Phys JETP. 1974. 38(2), 291–297.

[5] Whitham, G.B. *Linear and nonlinear waves.* New York: Wiley, 1974.

On the two-dimensional pressure-less equations

Eitan Tadmor

We prove the existence of weak solutions for the two-dimensional pressure-less Euler equations. To this end we develop an L^1 framework of dual solutions for such equations. Their existence is realized as vanishing viscosity limits,

$$u_t^\epsilon + u^\epsilon \cdot \nabla_x u^\epsilon = \epsilon \Delta u^\epsilon.$$

The limit $\overline{u} := \lim_{\epsilon \downarrow 0} u^\epsilon$ follows from new BV estimates,

$$\|u^\epsilon(\cdot, t)\|_{BV} \leq Const. \|u_0(\cdot)\|_{BV},$$

derived by tracing the *spectral gap* of the velocity gradient matrix, $\nabla u = (\partial_{x_i} u_j)$

$$\|\eta(\cdot, t)\|_{L^1} \leq \|\eta_0(\cdot)\|_{L^1}, \qquad \eta(\cdot, t) := \lambda_2(\nabla u) - \lambda_1(\nabla u).$$

Analytical results for isothermal & adiabatic two phase flow with phase transition

Ferdinand Thein
(joint work with Maren Hantke)

We study compressible two phase flow governed by the Euler equations for a liquid and a vapor phase and allow for phase transition. In the work by Hantke et al. [1], existence and uniqueness results to the Riemann problem for the compressible isothermal Euler equations were shown. They considered two phase flows for a liquid and a vapor phase with and without phase transition. To close the system two (specific) linear equations of state were chosen to relate density and pressure inside the phases. In order to clearly distinguish their approach from other models for two phase flows which are currently studied, we shortly summarize important points of their model:

- *one* set of Euler equations for both phases and the two phase are distinguished by the equation of state
- the *sharp* interface between the phases is a non-classical shock
- the mass transfer across the interface is modeled via a kinetic relation which is consistent with the second law of thermodynamics

Based on this several questions arise. The first one considers the equations of state linking the pressure and the density. We want to generalize the result of [1] in the spirit of [2] to arbitrary (yet thermodynamically consistent) equations of state. The basic assumption for the equations of state is that they are suitable to solve the single phase Riemann problem. For the isothermal Euler equations

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} &= 0,\\ \frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2 + p)}{\partial x} &= 0.\end{aligned} \tag{$*$}$$

we have the following jump conditions at the interface, see [3],

$$0 = [\![\rho(u-w)]\!], \tag{1}$$

$$0 = \rho(u-w)[\![u]\!] + [\![p]\!] \tag{2}$$

and the entropy inequality for isothermal processes

$$0 \leq z[\![g + e_{kin}]\!]. \tag{3}$$

We use the following notation

- ρ denotes the mass density, u the velocity
- w denotes the speed of the phase boundary
- $z = -\rho(u-w)$ is the mass flux across the interface
- g is the Gibbs free energy and e_{kin} the kinetic energy

To complete the system one additional equation is needed. The missing equation is given by the kinetic relation and has to be chosen such that (3) is fulfilled. Hence we choose

$$z = \tau p_V [\![g + e_{kin}]\!] \tag{4}$$

with p_V being the pressure in the vapor phase and $\tau \in \mathbb{R}$ a strictly positive constant. Using standard thermodynamical properties of the Gibbs free energy and some (sufficient) assumptions we can prove that there exists a unique solution of the equations at the interface. Additionally we show that if the Riemann problem has a solution this solution is also unique. Furthermore we have for the solution that the phase boundary is subsonic and thus always lies between the two classical waves.
The second question that the work in [1] raises is how this result extends in the adiabatic (i.e. no heat flux) case. Therefore we consider the full system of Euler equations

$$(**) \qquad \begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} &= 0, \\ \frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2 + p)}{\partial x} &= 0, \\ \frac{\partial(E)}{\partial t} + \frac{\partial(u(E+p))}{\partial x} &= 0, \\ E &= \rho(e + \frac{1}{2}u^2). \end{aligned}$$

Now we consider the following conditions at the interface (cf. [4])

$$0 = [\![\rho(u-w)]\!], \tag{5}$$

$$0 = \rho(u-w)[\![u]\!] + [\![p]\!], \tag{6}$$

$$0 = \rho(u-w)[\![e + \frac{p}{\rho} + \frac{1}{2}(u-w)^2]\!], \tag{7}$$

$$0 \le \zeta_{PB} = \rho(u-w)[\![s]\!]. \tag{8}$$

Here s denotes the (specific) entropy. Now the kinetic relation is given by

$$z = -\rho(u-w) = -\tau p_V [\![s]\!]. \tag{9}$$

Here τ may depend on the Temperature and thus is in general not constant anymore. In order to solve the problem we first discuss the phase boundary. Therefore we prescribe the vapor phase on one side and show that there is a unique solution for the liquid phase on the other side of the phase boundary. The same must be done again for a prescribed liquid phase. In the case of evaporation, i.e. $z > 0$ we can show that under certain assumptions the interface conditions (5) - (7) connect two states uniquely if a solution exists. As in [1] we can only have subsonic solutions in the liquid phase. Hence we have two classical acoustic waves (one in each phase), a contact wave in the vapor phase and the phase boundary between the contact wave and the acoustic wave in the liquid phase.

References

[1] M. Hantke, W. Dreyer, G. Warnecke, *Exact solutions to the Riemann problem for compressible isothermal Euler equations for two phase flows with and without phase transition*, Quarterly of Applied Mathematics **71** (2013), 509–540.
[2] R. Menikoff, B. Plohr, *The Riemann problem for fluid flow of real materials*, Rev. Mod. Phys. **61** (1989), 75–130.
[3] W. Dreyer, *On Jump Conditions at Phase Boundaries for Ordered and Disordered Phases*, WIAS Preprint (2003).
[4] F. Thein, M. Hantke, *Singular and selfsimilar solutions for Euler equations with phase transitions*, Bulletin of the Brazilian Mathematical Society, New Series **47** (2016), 779–786.

Model reduction through tangent spaces in kinetic gas theory

Manuel Torrilhon

We consider a kinetic transport equation in the form

$$\partial_t f + c_i \partial_{x_i} f = S(f) \tag{1}$$

for the probability density $f : \Omega \times \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}, (x,t,c) \mapsto f(x,t,c)$ with a spatial domain $\Omega \subset R^d$, $d \in \{1,2,3\}$ and $i = 1, ...d$. In the kinetic theory of monatomic gases the probability density describes the distribution of particle velocities such that $f(x,t,c)\, dcdx$ represents the number of particle in the phase space volume element $[x, x+dx] \times [c, c+dc]$. The operator $S(f)$ may be given by the Boltzmann collision integral or a kinetic model like the BGK-approximation. We assume that there is an equilibrium given by the Maxwell distribution $f_M(c; \rho, v, \theta)$, an isotropic Gaussian with density ρ, veloсty v, temperature θ and particle mass m that satisfies $S(f_M) = 0$.

A model reduction approach aims at the replacement

$$f(x,t,c) \;\; c \in \mathbb{R}^d \quad \longleftrightarrow \quad W(x,t) = \{w_\alpha(x,t)\}_{|\alpha|=1,2,...N} \tag{2}$$

that is, at (x,t) the description of the gas by a velocity distirbution function defined on a 3-dimensional space is replaced by a finite set of field variables $w_a(x,t)$ where α is a multiindex. This set may contain variables related to the fluid dynamic fields density, velocity and temperature, but could also contain further internal variables that describe the state of the gas. If we decide for such a reduced description it is necessary to formulate a reconstruction or prolongation operator $f^{(\star)}$ that computes a distribution function from w_a in the form

$$f(x,t,c) = f^{(\star)}(W(x,t), c)\,. \tag{3}$$

The function $f^{(\star)}$ is also called *model* or *closure relation.* To obtain evolution equations for the variables $W(x,t)$ we use a variational formulation of the kinetic equation (1) with test functions φ_α

$$\int_{\mathbb{R}^d} \varphi_\alpha(x,t,c) \left(\partial_t f^{(\star)}(W(x,t),c) + c_i \partial_{x_i} f^{(\star)}(W(x,t),c) \right) dc = \int_{\mathbb{R}^d} \varphi_\alpha(x,t,c)\, S\left(f^{(\star)}(W(x,t),c) \right) dc, \quad |\alpha| = 1,2,...N \tag{4}$$

where for generality the test function can depend on space and time as well. The general questions are now: What is a good model to choose for $f^{(\star)}$? Ideally, it should contain Maxwellians, but also approximates more general distributions with few parameters. What should be used as test functions φ_α? Ideally, these contain the monomials $1, c, c^2$ such that the physical conservation laws are part of the reduced equations. What can be said about the final evolution equations for $W(x,t)$? Ideally, we expect symmetric hyperbolic balance laws with an entropy as a natural nonlinear approximation for kinetic transport (1).

The model $f^{(\star)}\left(W(x,t),c\right)$ can be interpreted as a manifold to which the evolution of f is constraint due to the dimensional reduction. The tangent space of this manifold is given by the gradients

$$v_\beta\left(W\left(x,t\right),c\right) = \frac{\partial f^{(\star)}\left(W\left(x,t\right),c\right)}{\partial w_\beta} \tag{5}$$

and the differential is $df^{(\star)} = \sum_{\beta=1}^{N} v_\beta\left(W\left(x,t\right),c\right) dw_\beta$.With this the variational formulation simplifies to a first order, quasi-linear system of partial differential equations in the form

$$A^{(0)}_{\alpha\beta}(W)\partial_t w_\beta(x,t) + A^{(i)}_{\alpha\beta}(W)\partial_{x_i} w_\beta(x,t) = P_\alpha(W), \quad |\alpha| = 1,2,...N \tag{6}$$

with summation convention for β and coefficients ($c_0 = 1$)

$$A^{(i)}_{\alpha\beta}(W;x,t) = \int_{\mathbb{R}^d} c_i \varphi_\alpha(x,t,c) v_\beta(W,c) dc, \tag{7}$$

$$P_\alpha(W;x,t) = \int_{\mathbb{R}^d} \varphi_\alpha(x,t,c) S(f^{(\star)}(W,c)) dc. \tag{8}$$

With *tangent space reduction* the gradients are used as test functions $\varphi_\alpha(x,t,c) = v_\alpha(W(x,t),c)$, so that the matrices $A^{(i)}_{\alpha\beta}$ become symmetric and moreover $A^{(0)}_{\alpha\beta}$ is positive definite. Consequently, for any model the system is of Friedrichs type and indeed symmetric hyperbolic. However, due to the nonlinearity of the matrices $A^{(i)}_{\alpha\beta}$ the system can not be written in balance law form with a flux function in general. The existence of an entropy remains unclear and also the conservation laws may not be part of the system.

For the *linear model* $f^{(\star)}\left(W,c\right) = \sum_{\alpha=1}^{N} w_\alpha\left(x,t\right)\varphi_\alpha\left(c\right)$ with functions φ_α independent of physical space and time, the tangent space reduction leads to the usual Galerkin formulation where $\varphi_\alpha(c)$ are test and ansatz functions. In this case the matrices $A^{(i)}_{\alpha\beta}$ become constant and the system can be written in balance form, thus possesses an entropy which is equivalent to the L^2-norm of W. However, the linear model does not contain Maxwell distributions in a natural way and typically requires large N to achieve accurate approximations.

A typical choice for a model that contains Maxwellians is the *Hermite or Grad expansion* [2]

$$f^{(\star)}\left(W,c\right) = \sum_{\alpha=1}^{N} w_\alpha \psi_\alpha\left((c-v)/\theta\right) f_M(c;\rho,v,\theta) \tag{9}$$

with ψ_α chosen as d-variate polynomials and density, velocity and temperature are considered part of the variable set W. Polynomials are also used as test functions $\varphi_\alpha(W,c) = \psi_\alpha((c-v)/\theta)$ such that the reduced equations (4) represent moment equations including the conservation laws. Unfortunately, this model leads to a complicated expression for the gradients $v_\beta(W,c)$ and the system is not hyperbolic in general. However, the gradient can be artificially projected onto the model space by requiring

$$\frac{\partial f^{(\star)}(W,c)}{\partial w_\beta} \stackrel{!}{=} \sum_{\gamma=1}^{N} \psi_\gamma((c-v)/\theta)\; T_{\gamma\beta}(W)\; f_M(c;\rho,v,\theta) \tag{10}$$

with a regular matrix $T_{\gamma\beta}(W)$. This approach yields system matrices of the form

$$A^{(i)}_{\alpha\beta}(W) = \int_{\mathbb{R}^d} c_i \psi_\alpha((c-v)/\theta)\, \psi_\gamma((c-v)/\theta)\;\; f_M \;dc\;\; T_{\gamma\beta}(W) \tag{11}$$

which can be symmetrized by mutliplying with the transpose of $T_{\gamma\beta}(W)$ from the left. Consequently, the system is hyperbolic and contains the conservation laws, but as for the tangent space reduction it can not be written in balance law form and does not possess an entropy in general. The various projection approaches in [1] can be recast into the form (10).

In order to achieve a more tractable expression for the gradients v_β it is possible to use

$$\frac{\partial f^{(\star)}(W,c)}{\partial w_\beta} \stackrel{!}{=} \sum_{\gamma=1}^{N} \psi_\gamma(c)\; T_{\gamma\beta}(W)\; f^{(\star)}(W,c) \tag{12}$$

as an equation to determine the model $f^{(\star)}$. For simplicity we can set $T_{\gamma\beta} = \delta_{\gamma\beta}$ and find $f^{(\star)}(W(x,t),c) = \exp\left(\sum_{\alpha=1}^{N} w_\alpha(x,t)\,\psi_\alpha(c)\right)$ for the model. Similar to the Grad expansion polynomials are chosen for $\psi_\alpha(c)$, so that the model corresponds to the well-known *maximum-entropy distribution* [4, 5]. Using test functions $\varphi_\alpha(c) = \psi_\alpha(c)$ the system matrices have the form

$$A^{(i)}_{\alpha\beta}(W) = \int_{\mathbb{R}^d} c_i \varphi_\alpha(c)\varphi_\beta(c) f^{(\star)}(W(x,t),c)dc \tag{13}$$

which yield a symmetric hyperbolic system. This system can be written in balance law form and the Boltzmann entropy $\eta(W) = \int f^{(\star)}(W,c)\log f^{(\star)}(W,c)dc$ is also an entropy for the partial differential equation.

In [3] it was demonstrated that within the maximum-entropy approach the mapping between the coefficients W and the evolved moments is ill-posed even arbitrary close to a Maxwellian. As a result the approach yields a singular flux function for the moment equations. We show in [7] that this singularity leads to the existence of fast and small shock waves that allow to compute smooth shock profiles without violating the characteristic condition of [6].

References

[1] Y. Fan, J. Koellermeier, J. Li, R. Li, and M. Torrilhon. *Model Reduction of Kinetic Equations by Operator Projection.* Journal of Statistical Physics, pp. 1–30 (2015)
[2] H. Grad, *On the Kinetic Theory of Rarefied Gases,* Comm. Pure Appl. Math., pp. 331–407 (1949)
[3] M. Junk, *Domain of definition of Levermore's five-moment system*, J. Stat. Phys. **93**, pp.1143–67, (1998)
[4] M.N. Kogan, Rarefied Gas Dynamics, New York (1969)
[5] C.D. Levermore, *Moment closure hierarchies for kinetic theories*, J. Stat. Phys. **83**, pp.1021–1065 (1996)
[6] T. Ruggeri, *Breakdown of shock-wave-structure solutions* Phys. Rev. E, **47**(6), pp.4135–4140 (1993)
[7] R. P. Schaerer and M. Torrilhon. *On Singular Closures for the 5-Moment System in Kinetic Gas Theory.* Communications in Computational Physics, **17**(02):pp. 371–400 (2015)

On kinetic models for the collective self-organization of agents

Konstantina Trivisa

1. Mathematical modeling, wellposedness results, and investigation of hydrodynamic limits for kinetic flocking models.

Models describing collective self-organization of biological agents are currently receiving considerable attention. In this line of research we investigate a class of such models. The novel idea in (cf. Karper, Mellet and Trivisa [9, 10]) is the creation of a new model by combining the kinetic Cucker-Smale model with the strong local alignment term $\{\beta \mathrm{div}_v(f(u-v))\}$ which is obtained as the singular limit of a non-symmetric alignment term in [12]. That leads to a new kinetic flocking model able to treat both long range interactions (Cucker-Smale model) and short-range interactions (due to the strong local alignment term):

$$(1) \qquad f_t + v\cdot\nabla_x f + \mathrm{div}_v(fL|f|) + \beta \mathrm{div}_v(f(u-v))) = 0 \qquad \text{in } \mathbb{R}^d\times\mathbb{R}^d\times(0,T)$$

where $f := f(t,x,v)$ is the scalar unknown, $d\geq 1$ is the spatial dimension, and $\beta\geq 0$ is a constant.
The first two terms describe the free transport of the individuals, and the last two terms take into account the interactions between individuals, who try to align with their neighbors. The alignment operator L and the average local velocity u have the form

$$(2) \qquad L[f] = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_f(x,y)f(y,w)(w-v)dw\,dy\,, \qquad u(t,x) = \frac{\int_{\mathbb{R}^d} fv\,dv}{\int_{\mathbb{R}^d} f\,dv}$$

where the kernel K_f may depend on f and may not be symmetric in x and y. This model is endowed with some very desirable features: it has symmetry, it preserves the total *momentum* and has the advantage that it captures both *the* *long ranged dynamics and short ranged dynamics*. Thus it provides a correction of some earlier models in the literature [1, 2, 12].

1.1. **On strong local alignment in the kinetic Cucker-Smale model.** In this work [9] Trivisa et al. justify rigorously the model (1) by proving that "the correction term" proposed in [12] $\{f^r \tilde{L}^r[f^r]\}$ as correction to the kinetic Cucker-Smale model [1, 2] converges weakly to the strong local alignment term $\{f(u-v)\}$ when the radius of interaction goes to zero, $f^r \tilde{L}^r[f^r] \rightharpoonup f(u-v)$ as $r \to 0$. The analysis relies on a new velocity averaging lemma, delicate estimates and a refined covering lemma (cf. Lemma 2.5, [10]).

1.2. **Existence of solutions to kinetic flocking models.** This research establishes the global existence of weak solutions to a class of kinetic flocking equations. The models under consideration include the kinetic Cucker-Smale equation [1, 2] with possibly non-symmetric flocking potential, the Cucker-Smale equation with additional strong local alignment, and a model proposed by Motsch and Tadmor in [12] as correction to the Cucker-Smale model (cf. Ha and Tadmor [6]). The investigation of Trivisa et al. [9] provides the first rigorous existence result for a large class of kinetic flocking models. The main tools employed in the analysis are a new velocity averaging lemma (cf. Lemma 2.6, [9]) and the Schauder fixed point theorem.

1.3. **Hydrodynamic limit of the kinetic Cucker-Smale flocking model.** The starting point is the model (1) considered by Trivisa et al. [9, 10] significantly enhanced to include possible random effects, noise, and a confinement potential. The objective of this work is the rigorous investigation of the singular limit corresponding to strong noise and strong local alignment. The proof relies on the construction of a new relative entropy functional with suitable dissipation properties and the establishment of entropy inequalities which yield the appropriate convergence results. The resulting limiting system (cf. Section 3, Theorem 3.1 [12]), is an Euler-type flocking system.

References

[1] F. Cucker and S. Smale, On the mathematics of emergence, *Jpn. J. Math.* **2** (2007) 197–227.

[2] F. Cucker and S. Smale, Emergent behavior in flocks, *IEEE Trans. Automat. Control* **52** (2007) 852–862.

[3] T. Goudon, P.-E. Jabin and A. Vasseur, Hydrodynamic Limit for the Vlasov-Navier-Stokes Equations. Light Particles Regime, Indiana Univ. Math. J. **53 no 6** (2004) 1495–1516.

[4] T. Goudon, P.-E. Jabin and A. Vasseur, Hydrodynamic Limit for the Vlasov-Navier-Stokes Equations. Part II: Fine Particles Regime, Indiana Univ. Math. J. **53 no 6** (2004) 1517–1535.

[5] S.-Y. Ha, and J.-G. Liu. A simple proof of the Cucker-Smale flocking dynamics and mean-field limit. *Commun. Math. Sci. Vol.* **7**, No. **2**.:297–325.

[6] S.-Y. Ha, and E. Tadmor. From particle to kinetic and hydrodynamic descriptions of flocking. *Kinet. Relat. Models* **1** no. **3**: 415-435, 2008.

[7] M. Hairer and J. Mattingly, Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing. *Ann. of Math.* **(2) 164, no. 3**, (2006) 993–1032.

[8] M. Hairer and J. Mattingly, A theory of hypoellipticity and unique ergodicity for semilinear stochastic PDEs. *Electron. J. Probab.* **16** (2011), no. **23**, 658–738.

[9] T. Karper, A. Mellet, and K. Trivisa. Existence of weak solutions to kinetic flocking models. *SIAM. Math. Anal.* **45**(1): 215–243, 2013.
[10] T. Karper, A. Mellet, and K. Trivisa. On strong local alignment in the kinetic Cucker-Smale equation. *Springer Proc. Math. Stat.*, **49**, 227–242, 2014.
[11] T. Karper, A. Mellet and K. Trivisa, Hydrodynamic Limit of the Kinetic Cucker-Smale Flocking Model. *Math. Methods Appl. Sci.* **Vol. 25, No. 1** (2015), 131–163.
[12] S. Motsch and E. Tadmor. A new model for self-organized dynamics and its flocking behavior. *Journal of Statistical Physics, Springer*, **141 (5)**: 923–947, 2011.

Emergence of localizing solutions out of the competition of Hadamard instability and viscosity in plasticity

Athanasios E. Tzavaras
(joint work with Theodoros Katsaounis, Min-Gi Lee, Julien Olivier)

Shear bands are narrow zones of intense shear observed during plastic deformations of metals at high strain rates. As they often precede rupture their study has attracted attention as a mechanism of material failure [1]. We aim to reveal the onset of localization into shear bands using a simple model from viscoplasticity. We consider the system of partial differential equations

$$\begin{aligned} \gamma_t &= v_x, \\ v_t &= \left(\gamma^{-m} v_x^n\right)_x, \end{aligned} \tag{1}$$

which describes shear motions of a viscoplastic material and in terms of classification belongs to the class of hyperbolic-parabolic systems. γ is the plastic strain, v is the velocity in the shearing direction, and $m, n > 0$ are material parameters. The yield relation $\sigma = \gamma^{-m}\gamma_t^n$ characterizes the viscoplastic nature of materials: γ^{-m} accounts for plastic (net) strain softening and γ_t^n for strain-rate hardening.

For $n = 0$, the system (1) is elliptic in the t-direction and exhibits *Hadamard instability* - the catastrophic growth of oscillations for the linearized initial value problem - induced by the (net) strain-softening response. But when $n > 0$, the viscosity competes against this ill-posedness. The combination of the destabilizing effect of strain softening and the stabilizing effect of strain-rate hardening is conjectured to lead to localization of the strain in narrow zones called shear bands. It is at the core of a destabilizing mechanism proposed for more general models in the mechanics literature [5, 1] for the explanation of shear band formation.

To set the localization problem in the language of mathematical analysis, observe that (1) admits a class of solutions, that are valid for any values of the parameters m and n and describe uniform shearing

$$v_s(x) = x, \quad \gamma_s(t) = t + \gamma_0, \quad \sigma_s(t) = (t + \gamma_0)^{-m}. \tag{2}$$

The issue then becomes to examine whether small perturbations of the uniform shearing solutions develop nonuniformities that go astray or whether nonuniformities get suppressed resulting into stable response. Due to their time-dependent nature, the study of stability generally involves the behavior of non-autonomous systems. In the regime $n > m$, both linearized and nonlinear analyses indicate that the uniform shearing solutions are stable. On the complementary region $m > n$, an analysis of the linearized system of relative perturbations indicates instability of the uniform shearing solutions. Such results can be easily explained in the special case $\varphi(\gamma) = \frac{1}{\gamma}$ but retaining the dependence in n, by studying

$$v_t = \left(\frac{v_x^n}{\gamma}\right)_x, \quad \gamma_t = v_x \,. \tag{3}$$

This model has a special property: after considering a transformation to relative perturbations and a rescaling of variables,

$$\begin{aligned} v_x(x,t) =: u(x,t) = U(x,\tau(t))\,, \qquad & \gamma(x,t) = \gamma_s(t)\,\Gamma(x,\tau(t))\,, \\ \sigma(x,t) = \sigma_s(t)\,\Sigma(x,\tau(t)) \qquad \text{where} \qquad & \tau(t) = \log(1+\frac{t}{\gamma_0})\,, \end{aligned} \tag{4}$$

the problem of stability of the time-dependent uniform shearing solution is transformed into the problem of stability of the equilibrium $(\bar{U},\bar{\Gamma}) = (1,1)$ for the nonlinear but autonomous parabolic system

$$U_\tau = \Sigma_{xx} = \left(\frac{U^n}{\Gamma}\right)_{xx}, \quad \Gamma_\tau = U - \Gamma\,. \tag{5}$$

The following heuristic argument leads to a conjecture regarding the effect of rate sensitivity n on the dynamics: As time proceeds the second equation in (5), which is of relaxation type, relaxes to the equilibrium manifold $\{U = \Gamma\}$. Accordingly, the stability of (5) is determined by the equation describing the effective equation

$$U_\tau = \left(U^{n-1}\right)_{xx},$$

which is parabolic for $n > 1$ and backward parabolic for $n < 1$. This heuristic argument suggests instability in the regime $n < 1$.

In [3] we study the dynamics of (5) and provide an analysis of linearized stability. The linearized system around the equilibrium $(1,1)$ reads

$$\begin{aligned} \tilde{U}_\tau &= n\tilde{U}_{xx} - \tilde{\Gamma}_{xx}, \\ \tilde{\Gamma}_\tau &= \tilde{U} - \tilde{\Gamma}, \end{aligned} \tag{6}$$

and its dynamics can be analyzed via Fourier analysis. A complete picture emerges:

(a) For $n = 0$, high-frequency modes grow exponentially fast and indicate catastrophic growth and Hadamard instability.
(b) For $0 < n < 1$, the modes still grow and are unstable but at a tame growth rate; in this regime the behavior is that of Turing instability, familiar from problems in morphogenesis.
(c) For $n > 1$ strain-rate dependence is strong and stabilizes the motion.

In [4], we study the subtle mechanism of shear band formation in the nonlinear regime. We construct a class of self-similar solutions that exhibit localization in the regime $m > n$. We exploit the invariance properties of the system (1) and seek (following [2]) self-similar solutions of the form

$$\begin{aligned} \bar\gamma(t,x) &= (t+1)^a \bar\Gamma((t+1)^\lambda x), \\ \bar v(t,x) &= (t+1)^b \bar V((t+1)^\lambda x)\,, \end{aligned} \tag{7}$$

where $\xi = (t+1)^\lambda x$ is the similarity variable and $\lambda > 0$ is a parameter. The usual form of self-similar solutions for parabolic problems are generated for values of the parameter $\lambda < 0$ and capture the spreading effect associated with parabolic behavior. By contrast, we insist here on $\lambda > 0$ and study the existence of solutions focusing around the line $x = 0$ as time proceeds.

Parameters a and b are selected by

$$a^{\lambda,m,n} = \frac{2-n}{1+m-n} + \frac{2\lambda}{1+m-n}, \quad b^{\lambda,m,n} = \frac{1-m}{1+m-n} + \frac{1-m+n}{1+m-n}\lambda \tag{8}$$

and $(\bar\Gamma, \bar V)$ is constructed by solving the initial value problem for the singular system of ordinary differential equations

$$\begin{aligned} a^{\lambda,m,n}\bar\Gamma + \lambda\xi\bar\Gamma_\xi &= \bar V_\xi, \\ b^{\lambda,m,n}\bar V + \lambda\xi\bar V_\xi &= \big(\bar\Gamma^{-m}\bar V_\xi^n\big)_\xi, \end{aligned} \tag{9}$$

$$\bar\Gamma|_{\xi=0} = \bar\Gamma(0) > 0\,, \quad \bar V|_{\xi=0} = \bar U(0) > 0\,, \tag{10}$$

where $\bar\Gamma(0)$ and $\bar U(0)$ are positive parameters.

The invariance properties of the system (9) allows to de-singularize (9) and together with a nonlinear change of variables leads to reformulating the problem into an autonomous system of three first-order equations

$$\begin{aligned} \dot p &= p\Big(\frac{1}{\lambda}\big(r - \frac{2-n}{1+m-n}\big) - \frac{1-m+n}{1+m-n} \qquad +1 - q - \lambda pr\Big), \\ \dot q &= q\Big(\qquad\qquad +1 - q - \lambda pr\Big) + b^{\lambda,m,n}pr, \\ n\dot r &= r\Big(\frac{m-n}{\lambda}\big(r - \frac{2-n}{1+m-n}\big) + \frac{1-m+n}{1+m-n} \qquad -1 + q + \lambda pr\Big). \end{aligned} \tag{11}$$

The question of existence of a solution $(\bar V, \bar\Gamma)$ to (9) is reformulated to that of the construction of a suitable heteroclinic orbit for (11). In [3], we considered a system related to the case $m = 1$ and numerically constructed the heteroclinic orbit. In [4], we exploit the geometric theory of nonlinear perturbations, we construct a normally hyperbolic invariant manifold, and analyze the dynamics on that manifold to construct a suitable heteroclinic orbit. This provides a profile for the localizing solution of the form (7). As n is a small parameter, the system (11) admits both fast and slow time scales. Problems with multiple time scales are habitually found in multiple contexts, and one gets a clear geometric picture of the problem by analyzing the geometric picture in the phase space via the geometric singular perturbation theory. In [4] we present a novel application of the method to analyze the nonlinear competition of Hadamard instability with viscosity effected by strain-rate hardening in dynamic plasticity.

References

[1] R.J.Clifton, J.Duffy, K.A.Hartley and T.G.Shawki, *On critical conditions for shear band formation at high strain rates*, Scripta Met., **18** (1984), 443–448.

[2] Th. Katsaounis, J. Olivier, and A.E. Tzavaras, *Emergence of coherent localized structures in shear deformations of temperature dependent fluids*, arXiv preprint arXiv:1411.6131, (2014).

[3] Th. Katsaounis, M.G. Lee, and A.E. Tzavaras, *Localization in inelastic rate dependent shearing deformations*, arXiv preprint arXiv:1605.04564, (2016).

[4] M.G. Lee, and A.E. Tzavaras, *Existence of localizing solutions in plasticity via the geometric singular perturbation theory*, preprint, (2016).

[5] C.Zener and J.H.Hollomon, *Effect of strain rate upon plastic flow of steel*, J. Appl. Physics, **15** (1944), 22–32.

Convergence rates of finite difference schemes for the linear transport and wave equation with rough coefficient

Franziska Weber

Propagation of acoustic waves in a heterogeneous medium plays an important role in many applications, for instance in seismic imaging in geophysics and in the exploration of hydrocarbons [1, 4]. This wave propagation can be modeled by the linear wave equation:

$$\frac{1}{c^2(x)}\partial_{tt}^2 p(t,x) - \Delta p(t,x) = 0, \quad (t,x) \in D_T, \tag{1a}$$

$$p(0,x) = p_0(x), \quad x \in D, \tag{1b}$$

$$\partial_t p(0,x) = p_1(x), \quad x \in D, \tag{1c}$$

where $D_T := [0,T] \times D$, $D \subset \mathbb{R}^d$, augmented with boundary conditions. Here, p is the acoustic pressure and the wave speed is determined by the coefficient $c^2 = c^2(x) > 0$. The coefficient c encodes information about the material properties of the medium. As an example, the coefficient c represents various geological properties when seismic waves propagate in a rock formation.

Under the assumption that the coefficient $c^2 \in C^{0,\alpha} \cap L^\infty(D)$ for some $\alpha > 0$ and that it is uniformly positive and bounded on D, and that the initial data $p_0 \in H^1(D)$ and $p_1 \in L^2(D)$, one can prove existence of a unique weak solution $p \in C^0([0,T]; H^1(D))$ with $\partial_t p \in C^0([0,T]; L^2(D))$ following classical energy arguments for linear partial differential equations. See for instance [11, Chapter III, Thm. 8.1, 8.2]. A smoother coefficient c and more regular initial data p_0, p_1 result in a more regular solution [11]. Many numerical methods are available for the approximation of the linear wave equation with inhomogeneous coefficient c, see for example [3, 8, 10], but the error estimates are often based on the assumption that the coefficient has sufficiently much regularity. Some exceptions are the works [5, 7, 6]. However, this regularity assumption is not always realized in practice, since, as mentioned before, the coefficient represents properties of the possibly very heterogeneous material in which the waves propagate. Moreover, the material properties can often only be determined by measurements. Such measurements are inherently uncertain. This uncertainty is modeled in a statistical manner by representing the material properties (such as rock permeability) as random fields. In particular, log-normal random fields are heavily used to model porous and other geophysically relevant media [4, 2]. Thus, the coefficient c is not smooth, not even continuously differentiable, see Figure 1, on the left, for an illustration of the coefficient c where the rock permeability is modeled by a log-normal random field (the figure represents a single realization of the field).

Closer inspection of the coefficients obtained in practice reveals that the material coefficient c is at most a Hölder continuous function, that is, $c \in C^{0,\alpha}$ for some $0 < \alpha < 1$.

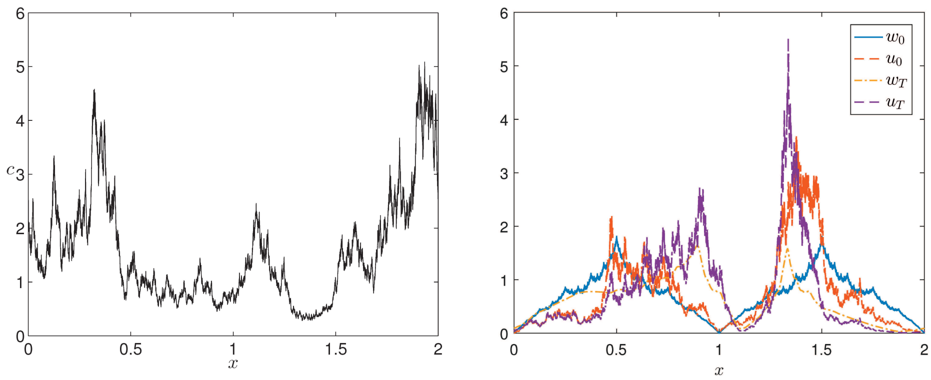

FIGURE 1. Left: A sample of a log-normally distributed random coefficient c ($\alpha = 1/2$). Right: Approximation of (2) by scheme (3) at time $T = 0$ and $T = 1$, number of grid points $N_x = 2^{14}$, $\gamma = 1/2$.

Given these facts, it makes sense to study (1) and its numerical approximation under the assumption that the coefficient c is only Hölder continuous.

In the talk, we therefore discussed the numerical approximation of the simpler model of the one dimensional transport equation

$$\partial_t u(t,x) + \partial_x (a(x)u(t,x)) = 0, \quad (t,x) \in [0,T] \times D, \tag{2}$$

for $a \in C^{0,\alpha}(D)$ positive, by a simple upwind finite difference scheme,

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a_j u_j^n - a_{j-1} u_{j-1}^n}{\Delta x} = 0, \quad 1 \leq j \leq N_x,\ 0 \leq n \leq N_T, \tag{3}$$

Δt, $\Delta x > 0$ are the discretization parameters. This scheme is stable and it is possible to show that the approximations are approximately Hölder continuous in time. Using an adaption of Kružkov's doubling of variables technique [9], one can then show the following convergence rate of the scheme:

Theorem 1.1. *Let $a \in C^{0,\alpha}(D)$ strictly positive. Denote $w := au$, where u is the solution of (2) and $w_{\Delta x} := a_{\Delta x} u_{\Delta x}$, where $u_{\Delta x}$ is the piecewise constant* *interpolation of the solutions u_j of (3) and $a_{\Delta x}$ the piecewise constant interpolation of a_j. Assume that the initial data $w_0 := au_0 \in L^p(D)$ and is Hölder continuous with exponent $\gamma_\infty \in (0,1]$. Then $w_{\Delta x}(t,\cdot)$ converges to $w(t,\cdot)$ at (at least) the rate*

$$\|(w - w_{\Delta x})(t,\cdot)\|_{L^p(D)} \leq C\,\Delta x^{(\gamma_\infty \alpha)/(\gamma_\infty \alpha + 2 - \gamma_\infty)}, \tag{4}$$

where C is a constant not depending on Δx, and where $p \in \{1,2\}$.

The details of the proof can be found in [12]. The techniques used to prove the convergence rate can also be extended to prove a rate of convergence for a finite difference scheme for the linear wave equation (1) in one space dimension. We note that the rate (4) depends explicitly on the Hölder regularity of the initial data and the coefficient a. Numerical experiments confirm that the rates of convergence are indeed quite low. However, we have not found an example yet that shows that (4) is sharp.

References

[1] B. Biondi. *3d Seismic Imaging: Three dimensional seismic imaging.* Society of Exploration Geophysicists, 2006.

[2] J. Fouque, J. Garnier, G. Papanicolaou, and K. Solna. *Wave propagation and time reversal in randomly layered media.* Springer Verlag, 2007.

[3] B. Gustafsson, H. O. Kreiss, and J. Oliger. *Time dependent problems and difference methods.* John Wiley and sons, 1995.

[4] L. T. Ikelle and L. Amundsen. *Introduction to Petroleum Seismology.* Society of Exploration Geophysicists, 2005.

[5] B. S. Jovanović, L. D. Ivanović, and E. E. Süli. Convergence of a finite-difference scheme for second-order hyperbolic equations with variable coefficients. *IMA J. Numer. Anal.*, 7(1):39–45, 1987.

[6] B. S. Jovanović and E. Süli. *Analysis of finite difference schemes*, volume 46 of *Springer Series in Computational Mathematics.* Springer, London, 2014. For linear partial differential equations with generalized solutions.

[7] V. Jovanović and C. Rohde. Finite-volume schemes for Friedrichs systems in multiple space dimensions: A priori and a posteriori error estimates. *Numerical Methods for Partial Differential Equations*, 21(1):104–131, 2005.

[8] H. O. Kreiss and J. Lorenz. *Initial boundary values problems and the Navier-Stokes equations*, volume 47 of *Classics in Applied Mathematics.* SIAM, 2004.

[9] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.

[10] S. Larsson and V. Thomee. *Partial Differential Equations with Numerical Methods*, volume 45 of *Texts in Applied Mathematics.* Springer, 2003.

[11] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I.* Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.

[12] F. Weber. Convergence rates of finite difference schemes for the linear advection and wave equation with rough coefficient. *ArXiv e-prints*, July 2016.

A Variational Time Discretization for Compressible Euler Equations

Michael Westdickenberg
(joint work with Fabio Cavalletti, Marc Sedjro)

The compressible Euler equations model the dynamics of compressible fluids such as gases. They form a system of hyperbolic conservation laws

$$\left.\begin{aligned} \partial_t\varrho + \nabla\cdot(\varrho\boldsymbol{u}) = 0 \\ \partial_t(\varrho\boldsymbol{u}) + \nabla\cdot(\varrho\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla\pi = 0 \\ \partial_t\varepsilon + \nabla\cdot\big((\varepsilon+\pi)\boldsymbol{u}\big) = 0 \end{aligned}\right\} \quad \text{in } [0,\infty)\times\mathbf{R}^d. \tag{1}$$

The unknowns $(\varrho, \boldsymbol{u}, \varepsilon)$ depend on time $t \in [0,\infty)$ and space $x \in \mathbf{R}^d$ and we assume that suitable initial data $(\varrho, \boldsymbol{u}, \varepsilon)(t = 0, \cdot) =: (\bar\varrho, \bar{\boldsymbol{u}}, \bar\varepsilon)$ is given. We will consider ϱ as a map from $[0,\infty)$ into the space of nonnegative, finite Borel measures, which we denote by $\mathscr{M}_+(\mathbf{R}^d)$. The quantity ϱ is called the density and it represents the distribution of mass in time and space. The first equation in (1) (the continuity equation) expresses the local conservation of mass, where

$$\boldsymbol{u}(t,\cdot) \in \mathscr{L}^2\big(\mathbf{R}^d, \varrho(t,\cdot)\big) \quad \text{for all } t \in [0,\infty) \tag{2}$$

is the Eulerian velocity field taking values on $\mathbf{R}^d$. The second equation in (1) (the momentum equation) expresses the local conservation of momentum $\boldsymbol{m} := \varrho\boldsymbol{u}$. The quantity ε is the total energy of the fluid and $\varepsilon(t,\cdot)$ is a measure in $\mathscr{M}_+(\mathbf{R}^d)$. The third equation in (1) expresses the local conservation of energy.

The quantity π in the momentum equation is the pressure. It is determined by the thermodynamic properties of the fluid. Three cases are of interest:

- The pressure vanishes (the pressureless gas case). Then the total energy of the fluid is simply the kinetic energy, and the third equation in (1) follows *formally* from the first two, by the chain rule.
- The pressure is a function of the density ϱ only because the thermodynamical entropy is constant throughout time and space (the isentropic case). Again the conservation of total energy follows formally from the continuity and the momentum equation.
- The pressure is a function of the density ϱ and total energy ε (the full Euler case). In this case, there is again an additional conservation law since the thermodynamical entropy *formally* satisfies a transport equation.

Even though system (1) formally conserves the total energy (being a Hamiltonian system), there actually is a dissipation of energy due to the nonsmoothness of the solutions: In the pressureless case, the modeling suggests a concentration of mass (sticky particle dynamics) by which the kinetic energy decreases. In the cases with pressure, solutions may form jump singularities along codimension-one manifolds, which are called shocks. Again total energy is dissipated in the process.

We consider a variational time discretization for the system of conservation laws (1) in the spirit of minimizing movements for curves of maximal slopes on metric spaces. We recall that for certain (possibly degenerate) parabolic equations, such as the porous medium equations, the solutions are curves on the space of nonnegative measures characterized by the requirement that at each time an energy (or entropy) functional is decreased at maximal rate (which also characterizes gradient flows). This comes with a natural time discretization, where in each timestep one tries to find the right balance between minimizing this energy functional and keeping the step short. For the porous medium equation the update length is measured using the Wasserstein distance. For the variational time discretization of (1) we proceed analogously: In each timestep we minimize the sum of the internal energy and of a new functional measuring the deviation of material point trajectories from straight paths. This functional thus measures the work required to accelerate material points. The minimization then boils down to maximizing the difference between a change in internal energy and work done, which in formal analogy to the fundamental laws of thermodynamics we interpret as maximizing the (suitably defined) entropy production. Since the internal energy only depends on some negative power of the determinant of the deformation gradient, it does not suggest any natural function space setting in which one can hope for compactness. To circumvent this problem, we minimize over the closed convex cone of *monotone* transport maps, which in particular guarantees the non-interpenetration of matter. Notice that for the porous medium equation, the relevant transport maps are *cyclically monotone* because those are the maps that solve the optimal transport problem that underlies the Wasserstein distance. In this case, the (cyclical) monotonicity follows implicitly from the choice of metric, whereas for (1) we make monotonicity an explicit constraint. This can be justified by the fact that in each timestep the transport maps are perturbations of the identity map, which is monotone.

Since monotone maps enjoy very good properties (they are of bounded variation locally, for example) one can prove the existence of a minimizer for each timestep. By a suitable interpolation in time, we obtain a family of approximate solutions to (1), parametrized by the timstep $\tau > 0$. We establish that as $\tau \to 0$, these approximate solutions converge (along a subsequence) to a measure-valued solution of (1). One crucial ingredient to the proof is a characterization of the polar cone of the cone of monotone maps: every element in the polar cone can be represented by the distributional divergence of a matrix field taking values in symmetric positive semidefinite matrices. This matrix field, which we call a stress tensor, therefore has exactly the same structure as the matrix field $\varrho \boldsymbol{u} \otimes \boldsymbol{u} + \pi \mathbf{1}$, which appears in the momentum equation (1). The momentum can be shown to be Lipschitz continuous with respect to a suitable Kantorovorich norm. We show that measure-valued solutions obtained from this variational time discretization satisfy an energy inequality pointwise: energy can only be dissipated at each point in space and time; there is no spontaneous generation of energy. Energy dissipation is given explicitly and conists of two parts: the first is related to the dissipation of energy along the discontinuities of the solution (the shocks), the other one is related to small-scale rotations, a dissipation mechanism particularly relevant to incompressible flows. The energy dissipation also controls the residual of the momentum equation, as measured in the Kantorovich (dual Lipschitz) norm.

References

[1] F. Cavalletti, M. Sedjro, M. Westdickenberg, *A Simple Proof of Global Existence for the 1D Pressureless Gas Dynamics Equations*, SIAM Math. Anal. **47** (2015), 66–79.

[2] F. Cavalletti, M. Sedjro, M. Westdickenberg, *A Variational Time Discretization for Compressible Euler Equations*, Preprint.

[3] F. Cavalletti, M. Westdickenberg, *The polar cone of the set of monotone maps*, Proc. Amer. Math. Soc. **143**, 781–787.

Reporter: Christian Rohde

Participants

Prof. Dr. Debora Amadori
Dipartimento di Matematica
Università degli Studi dell'Aquila
via Vetoio, 1
67010 L'Aquila (AQ)
ITALY

Prof. Dr. Sylvie Benzoni-Gavage
Institut Camille Jordan
Université Claude Bernard Lyon 1
43 blvd. du 11 novembre 1918
69622 Villeurbanne Cedex
FRANCE

Prof. Dr. Stefano Bianchini
SISSA - ISAS
Via Bonomea 265
34136 Trieste
ITALY

Dr. Raul Borsche
Fachbereich Mathematik
T.U. Kaiserslautern
Erwin-Schrödinger-Straße
67653 Kaiserslautern
GERMANY

Dr. Benjamin Boutin
UFR Mathématiques - IRMAR
Université de Rennes I
Campus de Beaulieu
35042 Rennes Cedex
FRANCE

Prof. Dr. Alberto Bressan
Department of Mathematics
Pennsylvania State University
University Park, PA 16802
UNITED STATES

Prof. Dr. Christophe Chalons
Laboratoire de Mathématiques
Université Versailles SQY
45, avenue des Etats Unis
78000 Versailles Cedex
FRANCE

Prof. Dr. Alina Chertock
Department of Mathematics
North Carolina State University
Campus Box 8205
Raleigh, NC 27695-8205
UNITED STATES

Prof. Dr. Rinaldo M. Colombo
Dipartimento di Matematica
Universita degli Studi di Brescia
Via Branze 38
25123 Brescia
ITALY

Prof. Dr. Andrea Corli
Dipartimento di Matematica
Universita di Ferrara
Via Machiavelli 35
44121 Ferrara
ITALY

Prof. Dr. Gianluca Crippa
Departement Mathematik & Informatik
Universität Basel
Spiegelgasse 1
4051 Basel
SWITZERLAND

Johannes Daube
Abtlg. für Angewandte Mathematik
Universität Freiburg
Hermann-Herder-Strasse 10
79104 Freiburg i. Br.
GERMANY

Dr. Carlotta Donadello
Maître de conférences
Laboratoire de Mathématiques
Université de Franche-Comté
16, route de Gray
25030 Besancon Cedex
FRANCE

Prof. Dr. Mauro Garavello
Dipartimento di Matematica e
Applicazioni
Universita di Milano-Bicocca
Edificio U5
via Roberto Cozzi 53
20125 Milano
ITALY

Dr. Jan Giesselmann
Institut für Angewandte Analysis
und Numerische Simulation
Universität Stuttgart
Pfaffenwaldring 57
70569 Stuttgart
GERMANY

Dr. Paola Goatin
INRIA - Team ACUMES
2004, route des Lucioles - BP 93
06902 Sophia Antipolis Cedex
FRANCE

Prof. Dr. Edwige Godlewski
Laboratoire Jacques-Louis Lions
Université Paris 6
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. Graziano Guerra
Dipartimento di Matematica e
Applicazioni
Universita di Milano-Bicocca
Edificio U5
via Roberto Cozzi 53
20125 Milano
ITALY

Prof. Dr. Philippe Helluy
Institut de Mathématiques
Université de Strasbourg
7, rue René Descartes
67084 Strasbourg Cedex
FRANCE

Prof. Dr. Helge Holden
Department of Mathematical Sciences
Norwegian University of Science &
Technology
A. Getz vei 1
7491 Trondheim
NORWAY

Prof. Dr. Axel Klar
Fachbereich Mathematik
T.U. Kaiserslautern
Erwin-Schrödinger-Straße
67653 Kaiserslautern
GERMANY

Prof. Dr. Christian Klingenberg
Institut für Mathematik
Universität Würzburg
Emil-Fischer-Strasse 40
97074 Würzburg
GERMANY

Prof. Dr. Dietmar Kröner
Abteilung für Angewandte Mathematik
Universität Freiburg
Hermann-Herder-Strasse 10
79104 Freiburg i. Br.
GERMANY

Prof. Dr. Philippe G. LeFloch
Laboratoire Jacques-Louis Lions
Université Paris 6
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. Tai-Ping Liu
Institute of Mathematics
Academia Sinica
No. 1, Sec. 4, Roosevelt Road
Taipei 10617
TAIWAN

Prof. Dr. Stephan Luckhaus
Mathematisches Institut
Universität Leipzig
Postfach 10 09 20
04109 Leipzig
GERMANY

Prof. Dr. Maria Lukacova-Medvidova
Mathematisches Institut
FB Mathematik/Physik/Informatik
Johannes-Gutenberg-Universität
55099 Mainz
GERMANY

Prof. Dr. Francesca Marcellini
Dipartimento di Matematica e Applicazioni
Universita di Milano-Bicocca
Edificio U5
via Roberto Cozzi 53
20125 Milano
ITALY

Dr. Stefano Modena
Fakultät für Mathematik & Informatik
Universität Leipzig
Augustusplatz 10/11
04109 Leipzig
GERMANY

Prof. Dr. Siegfried Müller
Institut f. Geometrie & Praktische Mathematik
RWTH Aachen
Templergraben 55
52062 Aachen
GERMANY

Anders S. Nordli
Department of Mathematical Sciences
Norwegian University of Science & Technology
A. Getz vei 1
7491 Trondheim
NORWAY

Prof. Dr. Gabriella A. Puppo
Dipartimento di Scienza e Alta Tecnologia
Università degli Studi dell'Insubria
Via Valleggio, 11
22100 Como
ITALY

Prof. Dr. Nils Henrik Risebro
Department of Mathematics
University of Oslo
P. O. Box 1053 - Blindern
0316 Oslo
NORWAY

Prof. Dr. Christian Rohde
Institut für Angewandte Analysis und Numerische Simulation
Universität Stuttgart
Pfaffenwaldring 57
70569 Stuttgart
GERMANY

Dr. Elena Rossi
Dipartimento di Matematica e Applicazioni
Universita di Milano-Bicocca
Edificio U5
via Roberto Cozzi 53
20125 Milano
ITALY

Prof. Dr. Giovanni Russo
Dipartimento di Matematica e Informatica
Citta Universitaria
Viale A. Doria, 6 - 1
95125 Catania
ITALY

Dr. Veronika Schleper
Institut für Angewandte Analysis und
Numerische Simulation
Universität Stuttgart
Pfaffenwaldring 57
70569 Stuttgart
GERMANY

Prof. Michael Shearer
Department of Mathematics
North Carolina State University
Campus Box 8205
Raleigh, NC 27695-8205
UNITED STATES

Prof. Dr. Eitan Tadmor
Center for Scientific Computation
and Mathematical Modeling (CSCAMM)
CSIC Building # 406
University of Maryland
Paint Branch Drive
College Park MD 20742-3289
UNITED STATES

Ferdinand Thein
Institut für Analysis und Numerik
Otto-von-Guericke-Universität
Magdeburg
Postfach 4120
39016 Magdeburg
GERMANY

Prof. Dr. Manuel Torrilhon
Aachen Institute for Advanced Study in
Computational Engineering Sciences
RWTH Aachen
Schinkelstrasse 2
52062 Aachen
GERMANY

Prof. Dr. Konstantina Trivisa
Department of Mathematics
University of Maryland
College Park, MD 20742-4015
UNITED STATES

Prof. Dr. Athanasios E. Tzavaras
Applied Mathematics and
Computational Sciences
King Abdullah University of Science &
Technology (KAUST)
Thuwal 23955-6900, Jeddah
SAUDI ARABIA

Prof. Dr. Gerald Warnecke
Institut für Analysis und Numerik
Otto-von-Guericke-Universität
Magdeburg
39016 Magdeburg
GERMANY

Franziska Weber
Seminar for Applied Mathematics
ETH Zürich
Rämistrasse 101
8092 Zürich
SWITZERLAND

Prof. Dr. Michael Westdickenberg
Institut für Mathematik
RWTH Aachen
Templergraben 55
52062 Aachen
GERMANY