Mathematisches Forschungsinstitut Oberwolfach

Report No. 34/2016

DOI: 10.4171/OWR/2016/34

Calculus of Variations

Organised by
Simon Brendle, New York
Alessio Figalli, Austin
Robert L. Jerrard, Toronto
Neshan Wickramasekera, Cambridge UK

10 July – 16 July 2016

Abstract. The Calculus of Variations is subject with a long and distinguished history, a great deal of diverse current activity, and close connections to other fields such as geometry and mathematical physics. The July 2016 workshop on the Calculus of Variations presented research that resolved long-standing conjectures, shed new light on classical results, pointed toward new research directions, and displayed progress on a range of aspects, classical and otherwise, of the Calculus of Variations.

Mathematics Subject Classification (2010): 49-xx, 35Jxx, 53Cxx, 58-xx.

Introduction by the Organisers

The workshop *the Calculus of Variations* featured 22 talks that presented research on a variety of topics connected to variational problems, including minimal or constant mean curvature surfaces, inequalities and symmetry, gradient and other flows, among others. This research was motivated by questions in geometry, analysis, statistical mechanics, data science, partial differential equations, and materials science, among other fields. The workshop was attended by 50 participants, of whom 18 were graduate students or postdoctoral fellows. It was organised by Simon Brendle (New York), Alessio Figalli (Zürich), Robert Jerrard (Toronto), and Neshan Wickramasekera (Cambridge).

Among the major highlights of the meeting, Jean Dolbeault spoke about striking work on the sharp constants in the critical and sub-critical Caffarelli-Kohn-Nirenberg inequalities. These results complete a long effort of many researchers to characterize exactly when extremals are radially symmetric. Another high point of the conference was the talk of Guido de Phillippis on the singular structure of Radon measures that belong to the kernel of a linear differential operator. This work sheds a new light on classical results in geometric measure theory, such as Alberti's rank-1 theorem, and has various other important applications, including a weak converse to Rademacher's Theorem. Another striking work presented at the conference was Aaron Naber's result that establishes, in great generality, energy quantization and rectifiability of the defect measure associated to sequences of Yang-Mills connections.

Surfaces of constant or vanishing mean curvature are a subject at the heart of the calculus of variations, and were the focus of a number of talks. Brian White discussed which sets can occur as curvature blow-up sets of sequences of embedded minimal disks, thereby providing a partial converse to a deep theorem of Colding and Minicozzi. Costante Bellettini presented a regularity and compactness theory for stable constant-mean-curvature hypersurfaces that generalises earlier results for stable minimal hypersurfaces; this new theory is formulated in the full generality of codimension 1 integral varifolds and gives sharp regularity conclusions making no hypothesis on the singular set beyond two necessary structural conditions. Otis Chodosh reported on recent work on the uniqueness of large isoperimetric surfaces in asymptotically flat 3-manifolds. Andrea Mondino discussed the existence of optimal shapes for the isoperimetric-isodiametric inequality; this involves minimizing the product of surface area and radius subject to a volume constraint. Spencer Becker-Kahn presented results on the asymptotic behavior of two-valued Lipschitz minimal graphs of arbitrary dimension and codimension that are not assumed to satisfy any stability condition. Eleonora Cinti discussed quantitative flatness results and perimeter estimates for nonlocal minimal surfaces in low dimensions.

A number of talks addressed questions related to inequalities, sharp constants, symmetry, and stability. Among the highlights on these topics are the talk by Francesco Maggi which discussed sharp stability results for the euclidean concentration inequality and droplet formation in statistical mechanics, and the talk by Brian Krummel which discussed stability for Almgren's isoperimetry principle, giving sharp estimates on the Fraenkel asymmetry and Hausdorff distance between the unit sphere in $(n+1)$-dimensional Euclidean space and a closed n-dimensional hypersurface with mean curvature at most n.

The Calculus of Variations is intimately connected to the study of gradient flows, and through them to other geometric evolution problems. Michael Struwe spoke about a variety of results concerning a supercritical nonlinear heat equation connected to long-standing problems in minmax theory. These included a novel monotonicity formula, small data global well-posedness in an optimal space, and some results about blow for large data. Yoshi Tonegawa presented recent work that, in the case of hypersurfaces, substantially strengthens Brakke's foundational results on existence of weak solutions of the mean curvature flow for rough initial data. John Lott described the construction of a singular Ricci flow, obtained as a limit in a suitable sense of Ricci flow with surgeries on increasingly small scales, partially answering a question of Perelman. Peter Topping presented the elegant proof of a new sharp $L^1 - L^\infty$ smoothing estimate for Ricci flow on surfaces, together with a parallel result for the logarithmic fast diffusion equation in 2 dimensions. Pei-Ken Hung presented new results about the asymptotic behavior of inverse mean curvature flow in hyperbolic space. Maria Colombo spoke about an extension of the DiPerna-Lions theory to the case of vector fields with fast growth, which she uses to establish existence of weak solutions of the Vlasov-Poisson equation for general inital data.

Striking new developments connected to various classical issues in the calculus of variations were presented in several talks. Filippo Santambrogio discussed a Γ-convergence result that establishes the validity of a new phase field approximation to the Steiner problem in the plane, and hence of associated numerical algorithms. Radu Ignat presented very refined results that derive a reduced free energy characterizing the interaction of domain walls in a critically-scaled nonlocal variational problem arising in micromagnetics. Susanna Terracini spoke about strong partial regularity results for shape optimization problems involving a combination of eigenvalues and a volume constraint.

Finally, connections between the calculus of variations and stochastic analysis appeared in a couple of very interesting talks. Charles Smart described the continuum limit of an algorithm that involves "convex hull peeling" for random point clouds. And in a very different direction, Robert Haslhofer presented deep results on Ricci curvature and martingales, showing that bounded Ricci curvature may be characterized in terms of a generalized Bochner formula on path space.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

Workshop: Calculus of Variations

Table of Contents

Abstracts

An optimal local well-posedness result for the supercritical Lane–Emden heat flow

Michael Struwe
(joint work with Simon Blatt)

Let Ω be a smoothly bounded domain in $\mathbb{R}^n$, $n \geq 3$, and let $T > 0$. Given initial data u_0, we consider the Lane-Emden heat flow

$$(1) \qquad u_t - \Delta u = |u|^{p-2}u \text{ on } \Omega \times [0,T[,\ u = 0 \text{ on } \partial\Omega \times [0,T[,\ u\big|_{t=0} = u_0$$

for a given exponent $p > 2^* = 2n/(n-2)$, that is, in the "supercritical" regime.

An important feature of equation (1) is the following scaling property. Whenever u is a solution of (1) on Ω, then for any $R > 0$, any $x_0 \in \mathbb{R}^n$ the function

$$(2) \qquad u_{R,x_0}(x,t) = R^{-\alpha}u(R^{-1}(x-x_0), R^{-2}t),\ \alpha = 2/(p-2),$$

is a solution of (1) on the scaled domain $\Omega_{R,x_0} := \{x \in \mathbb{R}^n;\ R^{-1}(x-x_0) \in \Omega\}$.

As observed by Matano-Merle [14, p. 1048], the initial value problem (1) may be ill-posed for certain data $u_0 \in H_0^1 \cap L^p(\Omega)$. However, in [4], Section 6.5, [5], Remark 3.3, Simon Blatt and the author had shown that the Cauchy problem (1) is globally well-posed for suitably small data u_0 belonging to the Morrey space $H_0^{1,\mu} \cap L^{p,\mu}(\Omega)$, where $\mu = \frac{2p}{p-2} < n$ is the natural Morrey exponent compatible with (2). Here, $f \in L^{p,\lambda}(\Omega)$ if

$$\|f\|^p_{L^{p,\lambda}(\Omega)} := \sup_{x_0 \in R^n,\ r>0} r^{\lambda-n} \int_{B_r(x_0)\cap\Omega} |f|^p dx < \infty,$$

where $B_r(x_0)$ denotes the Euclidean ball of radius $r > 0$ centered at x_0. Moreover, we write $f \in L_0^{p,\lambda}(\Omega)$ whenever $f \in L^{p,\lambda}(\Omega)$ satisfies

$$\sup_{x_0 \in R^n,\ 0<r<r_0} r^{\lambda-n} \int_{B_r(x_0)\cap\Omega} |f|^p dx \to 0 \text{ as } r_0 \downarrow 0.$$

In recent joint work [6] with Simon Blatt we go one step further and show that problem (1) even is well-posed for suitably small data $u_0 \in L^{2,\lambda}(\Omega) \supset L^{p,\mu}(\Omega)$, where $\lambda = \frac{4}{p-2}$, thus considerably improving on the results of Brezis-Cazenave [7] or Weissler [18] for initial data in L^q, $q \geq n(p-2)/2$. In fact, the following holds.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a smoothly bounded domain, $n \geq 3$. There exists a constant $\varepsilon_0 > 0$ such that for any function $u_0 \in L^{2,\lambda}(\Omega)$ satisfying $\|u_0\|_{L^{2,\lambda}} < \varepsilon_0$ there is a unique global smooth solution u to* (1) *on $\Omega \times]0,\infty[$.*

The smallness condition can be somewhat relaxed.

Theorem 2. *Let $u_0 \in L^{2,\lambda}(\Omega)$ and suppose there is a number $R > 0$ such that*

$$\sup_{x_0 \in R^n,\ 0<r<R} r^{\lambda-n} \int_{B_r(x_0)\cap\Omega} |u_0|^2 dx \leq \varepsilon_0^2,$$

where $\varepsilon_0 > 0$ *is as determined in Theorem 1. Then there exists a unique smooth solution* u *to* (1) *on an interval* $]0, T_0[$*, where* $T_0/R^2 = C(\varepsilon_0/\|u_0\|_{L^{2,\lambda}}) > 0$.

In particular, for any $u_0 \in L_0^{2,\lambda}(\Omega)$ *there exists a unique smooth solution* u *to* (1) *on some time interval* $]0, T[$*, where* $T = T(u_0) > 0$.

Our results are similar to results of Taylor [17] who demonstrated local and global well-posedness of the Cauchy problem for the equation

$$u_t - \Delta u = DQ(u) \text{ on } \Omega \times [0, T[,$$

for suitably small initial data $u\big|_{t=0} = u_0$ in a Morrey space, where D is a linear differential operator of first order and Q is a quadratic form in u as in the Navier-Stokes system. However, similar to the work of Koch-Tataru [13] on the Navier-Stokes system, in our treatment of (1) we are able to completely avoid the use of pseudodifferential operators in favor of simple integration by parts and Banach's fixed-point theorem. By a different method, the analogue of Theorem 1 also for unbounded domains was recently demonstrated by Souplet [15].

The study of the initial value problem for (1) for non-smooth initial data is motivated by the question whether a solution u of (1) blowing up at some time $T < \infty$ can be extended as a partially regular weak solution of (1) on a time interval $]0, T_1[$ for some $T_1 > T$, still satisfying the monotonicity formula [5], Proposition 3.1, on $]0, T_1[$. In the notation of [5], for any $x_1 = 0 \in \Omega$ and any $0 < T < t_1 < T_1$ we choose (x_1, t_1) as center of scaling and define the scaled energy function

$$H^\varphi(R) = D^\varphi(R) + F_p^\varphi(R) + \frac{1}{p-2}\left(\frac{d}{dR}\big(RF_2^\varphi(R)\big) - A_2^\varphi(R)\right)$$

as in (2.13) in [5], involving the scaled Dirichlet energy and the scaled L^2- and L^p-norms of u, respectively, with a smooth cut-off function φ and the heat kernel as a natural weight, as in [16]. Given a sufficiently small number $R > 0$ we have $\varphi \equiv 1$ on $B_R(0)$. Also choosing $t_1 = t + R^2$, upon integrating $H^\varphi(r)$ in $0 < r < R$ we are then able to bound

$$R^{\lambda-n} \int_{\Omega_R(x_1)} |u(T)|^2\, dx \le CF_2^\varphi(R) \le C$$

with constants $C > 0$ independent of x_1 and $R > 0$; that is, $u(T) \in L^{2,\lambda}(\Omega)$.

Hence, the regularity assumption $u_0 \in L^{2,\lambda}(\Omega)$ is necessary from this point of view and cannot be weakened.

Our results in [6] also show that the condition $u(T) \in L^{2,\lambda}(\Omega)$ in general is not sufficient for continuation and that a smallness condition as in Theorem 2 is needed. To see this, recall that equation (1) may be interpreted as the negative gradient flow of the energy

$$E(u) = E_\Omega(u) = \int_\Omega \big(\frac{1}{2}|\nabla u|^2 - \frac{1}{p}|u|^p\big)dx.$$

As observed by Ball [2], Theorem 3.2, sharpening an earlier result of Kaplan [12], for data u_0 with $E(u_0) < 0$ the solution to (1) blows up in finite time.

In a way similar to the proof of the ill-posedness result for the Lane-Emden flow on $\mathbb{R}^n$ by Galaktionov-Vazquez [10], Theorem 10.4, we combine this observation with the scaling property (2) of equation (1) to obtain data $u_0 \in L^{2,\lambda}$ leading to instantaneous complete blow-up. Clearly we may assume that $0 \in \Omega$.

Theorem 3. *There is $M > 0$ such that for every initial data $0 \leq u_0 \in C^0(\Omega \setminus \{0\})$ satisfying*

$$\liminf_{x \to 0} \big(u_0(x) - M|x|^{-\alpha}\big) > 0, \ \ \alpha = 2/(p-2),$$

the Cauchy problem for (1) *with data u_0 only admits $u \equiv \infty$ as solution for $t > 0$.*

Can one show that (at least for sufficiently small exponents $p > 2^*$) we may choose $M = \alpha(n-2-\alpha) =: c_*$, where $u_*(x) := c_*|x|^{-\alpha}$ solves the time-independent equation (1) on $\mathbb{R}^n$? – Is it possible to show that a smooth solution u of (1) on $[0, T[$ blowing up at time $T > 0$ with bounded energy $|E(u(t))| \leq C < \infty$ for $0 < t < T$ always has a "trace" $u(T) \in L_0^{p,\lambda}(\Omega)$?

References

[1] D. R. Adams, *A note on Riesz potentials*, Duke Math. J. 42 (1975), no. 4, 765–778.

[2] J. M. Ball, *Finite time blow-up in nonlinear problems.* Nonlinear evolution equations (Proc. Sympos., Univ. Wisconsin, Madison, Wis., 1977), pp. 189–205, Publ. Math. Res. Center Univ. Wisconsin, 40 , Academic Press, New York-London, 1978.

[3] P. Baras and L. Cohen, *Complete blow-up after T_{max} for the solution of a semilinear heat equation*, J. Funct. Anal. 71 (1987), no. 1, 142–174.

[4] S. Blatt and M. Struwe, *An analytic framework for the supercritical Lane-Emden equation and its gradient flow*, Int. Math. Res. Notices 2015, no. 9 (2015), 2342–2385.

[5] S. Blatt and M. Struwe, *Boundary regularity for the supercritical Lane-Emden heat flow*, Calc. Var. 54 (2015), no. 2, 2269–2284. Publisher's erratum, Calc. Var. 54 (2015), 2285.

[6] S. Blatt and M. Struwe, *Well-posedness of the supercritical Lane-Emden heat flow in Morrey spaces*, arXiv:1511.07685, to appear in ESAIM:COCV (special issue in honor of Jean-Michel Coron).

[7] H. Brezis and T. Cazenave, *A nonlinear heat equation with singular initial data*, J. Anal. Math. 68 (1996), 277–304.

[8] A. Friedman, *Partial differential equations.* Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London, 1969.

[9] H. Fujita, *On the blowing up of solutions of the Cauchy Problem for $u_t = \Delta u + u^{1+\alpha}$.* J. Fac. Sci. Univ. Tokyo Sect. I, 13 (1996), 109–124.

[10] V. A. Galaktionov and J. L. Vazquez, *Continuation of blowup solutions of nonlinear heat equations in several space dimensions*, Comm. Pure Appl. Math. 50 (1997), no. 1, 1–67.

[11] D. D. Joseph and T. S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rational Mech. Anal. 49, 1972/73, 241–269.

[12] S. Kaplan, *On the growth of solutions of quasi-linear parabolic equations.* Comm. Pure. Appl. Math. 16 (1963), 305–330.

[13] H. Koch and D. Tataru, *Well-posedness for the Navier-Stokes equations.* Adv. Math. 157 (2001), no. 1, 22–35.

[14] H. Matano and F. Merle, *Classification of type I and type II behaviors for a supercritical nonlinear heat equation*, J. Funct. Anal. 256 (2009), no. 4, 992-1064.

[15] P. Souplet, *Morrey spaces and classification of global solutions for a supercritical semilinear heat equation in $\mathbb{R}^n$*, arXiv:1604.01667.

[16] M. Struwe, *On the evolution of harmonic maps in higher dimensions.* J. Differential Geom. 28 (1988), no. 3, 485–502.

[17] M. E. Taylor, *Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations*, Comm. Partial Differential Equations 17 (1992), no. 9-10, 1407–1456.
[18] F. B. Weissler, *Existence and non-existence of global solutions for a semilinear heat equation*, Israel J. Math. 38 (1981), no. 1-2, 29–40.

Flow of nonsmooth vector fields and the Vlasov-Poisson system

Maria Colombo
(joint work with Luigi Ambrosio, Alessio Figalli)

Given a vector field $\boldsymbol{b} : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$, we consider the ordinary differential equation for the flow of $\boldsymbol{b}$

$$\begin{cases} \partial_t \boldsymbol{X}(t,x) = \boldsymbol{b}(t, \boldsymbol{X}(t,x)) & \forall t \in (0,T) \\ \boldsymbol{X}(0,x) = x \end{cases} \tag{1}$$

where $x \in \mathbb{R}^d$. We also introduce the related Cauchy problem for the continuity equation, namely

$$\begin{cases} \partial_t u(t,x) + \operatorname{div}(\boldsymbol{b}(t,x) u(t,x)) = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{2}$$

where $(t,x) \in (0,\infty) \times \mathbb{R}^d$, $u \in \mathbb{R}$, recalling that the continuity equation is also equivalent to the transport equation

$$\partial_t u(t,x) + \boldsymbol{b}(t,x) \cdot \nabla u(t,x) = 0$$

in the case of divergence-free vector fields.

The theory of DiPerna-Lions, introduced in the seminal paper [10], provides existence and uniqueness of a suitable flow under weak regularity assumptions on $\boldsymbol{b}$, for instance when $\boldsymbol{b}(t,\cdot)$ is Sobolev [10] or BV [1] and satisfies global bounds on the divergence (see also [4] for an overview on the topic). More precisely, we introduce the notion of flow introduced by DiPerna and Lions and later slightly modified by Ambrosio (we follow the latter axiomatization): given a Borel vector field $\boldsymbol{b} : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$, a Regular Lagrangian Flow $\boldsymbol{X} : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is a function such that

- for a.e. x, the curve $\boldsymbol{X}(\cdot, x)$ is absolutely continuous and solves (1) for a.e. t;
- there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^d} \phi(\boldsymbol{X}(t,x))\, dx \le C \int_{\mathbb{R}^d} \phi(y)\, dy \qquad \text{for all } \phi \in C_c(\mathbb{R}^d) \text{ nonnegative.} \tag{3}$$

The existence and uniqueness results of DiPerna and Lions for Sobolev vector fields could be considered as a weak Cauchy-Lipschitz theory for ODE's with Lipschitz vector fields, with some remarkable differences. Indeed, the Cauchy-Lipschitz theory is not only pointwise but also purely local, meaning that existence and uniqueness for small intervals of time depend *only* on local regularity properties of the vector fields $\boldsymbol{b}(t,x)$. On the other hand, not only the DiPerna-Lions theory is an almost everywhere theory (and this really seems to be unavoidable) but also the existence results for the flow depend on *global* in space growth estimates on $|\boldsymbol{b}|$, the most typical one being

$$\frac{|\boldsymbol{b}(t,x)|}{1+|x|} \in L^1\big((0,T);L^1(\mathbb{R}^d)\big) + L^1\big((0,T);L^\infty(\mathbb{R}^d)\big). \tag{4}$$

We fill this gap, by developing a *local theory of flows for nonsmooth vector fields.* We assume that the vector field $\boldsymbol{b}$ satisfies the local integrability property $\boldsymbol{b} \in L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^d)$, a local one-sided bound on the distributional divergence, and the property that the continuity equation with velocity $\boldsymbol{b}$ is well-posed in the class of nonnegative bounded and compactly supported functions. This last assumption is fulfilled in many cases of interest, for instance for locally $W^{1,1}$ or BV vector fields as well as some vector fields whose gradient is the singular integral of a finite measure [10, 1, 7, 5] and it is known to be deeply linked to the uniqueness of the flow. Under these three assumptions we prove in [2, Theorem 5.2] existence of a unique *maximal regular* flow $\boldsymbol{X}(t,x)$, defined up to a maximal time $T_{\boldsymbol{X}}(x)$ which is positive $\mathcal{L}^d$-a.e.. Here "regular" refers to a natural adaptation of (3); "maximal" means that we cannot continuously extend $\boldsymbol{X}(\cdot,x)$ after $T_{\boldsymbol{X}}(x)$ in view of

$$\limsup_{t\uparrow T_{\boldsymbol{X}}(x)} |\boldsymbol{X}(t,x)| = \infty \qquad \text{for } \mathcal{L}^d\text{-a.e. } x\in\{T_{\boldsymbol{X}}<\infty\}. \tag{5}$$

Uniqueness of the maximal regular flow follows basically from the "probabilistic" techniques developed in [1], which allow one to transfer uniqueness results at the level of the continuity equation into uniqueness results at the level of the ODE. Existence is obtained by approximation with smooth vector fields; the main new difficulty here is that even if we truncate $\boldsymbol{b}$ by multiplying it by a $C^\infty_c(\mathbb{R}^d)$ cut-off function, the resulting vector field has not divergence in L^∞ (just L^1, actually, when $|\boldsymbol{b}_t| \notin L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$), hence the standard ideas are not applicable. Besides existence and uniqueness, the maximal regular flow $\boldsymbol{X}$ and its existence time enjoys further properties, such as a natural semigroup property.

Existence and uniqueness of a maximal regular flow can be applied to describe the Lagrangian structure of weak solutions to the transport equations, and in particular it has interesting consequences on kinetic equations such as the Vlasov-Poisson system. It appears, for instance, in plasma physics to describe the evolution of charged particles under their self-consistent electric field, and in astrophysics to describe the motion of galaxy clusters under their gravitational field. This equation describes the evolution of a nonnegative distribution function $f:(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty)$ through the system

$$\begin{cases} \partial_t f_t + v\cdot\nabla_x f_t + E_t\cdot\nabla_v f_t = 0 & \text{in } (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d \\ \rho_t(x) = \displaystyle\int_{\mathbb{R}^d} f_t(x,v)\,dv & \text{in } (0,\infty)\times\mathbb{R}^d \\ E_t(x) = \sigma\, c_d \displaystyle\int_{\mathbb{R}^d} \rho_t(y)\frac{x-y}{|x-y|^d}\,dy & \text{in } (0,\infty)\times\mathbb{R}^d. \end{cases} \tag{6}$$

Here $f_t(x,v)$ stands for the density of particles having position x and velocity v at time t, $\rho_t(x)$ is the distribution of particles in the physical space, $E_t = \sigma\,\nabla(\Delta^{-1}\rho_t)$ is the force field, $c_d > 0$ is a dimensional constant, and $\sigma \in \{\pm 1\}$. The case $\sigma = 1$ corresponds to electrostatic forces between charged particles with the same sign (repulsion) while $\sigma = -1$ corresponds to the gravitational case (attraction).

The Vlasov-Poisson system has a transport structure: the first equation in (6) can be rewritten as

$$\partial_t f_t + \mathrm{div}_{x,v}(\boldsymbol{b}_t f_t) = 0$$

for the divergence-free vector field $\boldsymbol{b}_t(x,v) = (v, E_t(x))$. Hence, when the solutions is sufficiently smooth, f_t is transported along the characteristics of the vector field $\boldsymbol{b}_t(x,v) := (v, E_t(x))$. However, when dealing with weak or renormalized solutions (in the sense of [9] or [3, Definition 2.1]), it is not clear whether such a vector field defines a flow on the phase-space, and one loses the relation between the Eulerian and Lagrangian picture. We show that the Lagrangian picture is still valid even for weak/renormalized solutions ([3, Theorem 2.2]), and that the concepts of renormalized and Lagrangian solutions are equivalent. As a consequence, we show that, in the repulsive case with $d \leq 4$, renormalized solutions with finite energy are transported by a global flow (see [3, Corollary 2.3] and [6] for an analogous statement in dimension $d = 3$). In particular, they preserve all the natural Casimir invariants such as $t \to \int_{\mathbb{R}^d} f_t \log f_t$. A second consequence is the global existence of renormalized/Lagrangian solutions under minimal assumptions on the initial data (see [3, Theorem 2.6 and Corollary 2.7]), that extends the huge literature on existence of classical and weak solutions (see [11, 12] for some established contributions).

References

[1] L. Ambrosio: *Transport equation and Cauchy problem for BV vector fields.* Invent. Math., **158** (2004), 227–260.

[2] L. Ambrosio, M. Colombo and A. Figalli: *Existence and uniqueness of Maximal Regular Flows for non-smooth vector fields.* Arch. Ration. Mech. Anal., to appear. UPDATE

[3] L. Ambrosio, M. Colombo and A. Figalli: *On the Lagrangian structure of transport equations: the Vlasov-Poisson system.* Preprint (2014).

[4] L. Ambrosio and G. Crippa: *Existence, uniqueness, stability and differentiability properties of the flow associated to weakly differentiable vector fields.* Lecture Notes of the Unione Matematica Italiana, **5** (2008), 3–54.

[5] A. Bohun, F. Bouchut and G. Crippa: *Lagrangian flows for vector fields with anisotropic regularity.* Preprint (2014).

[6] A. Bohun, F. Bouchut and G. Crippa: *Lagrangian solutions to the Vlasov-Poisson equation with L^1 density.* Preprint (2014).

[7] F. Bouchut and G. Crippa: *Lagrangian flows for vector fields with gradient given by a singular integral.* J. Hyperbolic Differ. Equ., **10** (2013), 235–282.

[8] M. Colombo: *Flows of non-smooth vector fields and degenerate elliptic equations.* PhD Thesis (2015).

[9] R. J. DiPerna and P.-L. Lions: *Solutions globales d'équations du type Vlasov-Poisson.* (French) [Global solutions of Vlasov-Poisson type equations] C. R. Acad. Sci. Paris Sér. I Math., **307** (1988), 655–658.

[10] R. J. DiPerna and P.-L. Lions: *Ordinary differential equations, transport theory and Sobolev spaces.* Invent. Math., **98** (1989), 511–547.

[11] P.-L. Lions and B. Perthame: *Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system.* Invent. Math., **105** (1991), 415–430.
[12] K. Pfaffelmoser: *Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data.* J. Differential Equations, **95** (1992), 281–303.

Phase-field approximation of the Steiner problem.

Filippo Santambrogio
(joint work with Matthieu Bonnivard, Antoine Lemenant)

The Steiner Problem is a classical question (see, for instance, [7]) in graph optimization: given points $x_0, x_1, \dots, x_N \in \mathbb{R}^d$, find a compact connected 1D set S containing them, with minimal length:

$$\min \quad \left\{ \mathcal{H}^1(S) \ : \ S \supset \{x_0, x_1, \dots, x_N\}, \ S \text{ compact and connected} \right\}.$$

We know that such an optimal set exists, that it is a finite union of segments, with no cycles and with triple junctions at 120°. Yet, the number of possible topological configurations of this tree exploses with N and an exact search is formulated by computer scientist (in a discrete setting on a network) as an NP-hard problem.

In the talk we propose a new approach, where this minimization problem is approximated by a family of more standard minimization problem in the calculus of variations (see [3, 8]). The same approach can also be adapted to other length-penalized problems, such as the average distance problem (see [5]): here we look again for a set S, but instead of imposing that it should contain some given points, we only want it to be as close as possible to them, or to a given distribution of points in a domain Ω. If this distribution is a density f, we solve

$$\min \quad \left\{ F(S) := \int_\Omega \operatorname{dist}(x, S) f(x) dx + \lambda \mathcal{H}^1(S) \ : \ S \text{ compact and connected} \right\},$$

where $\lambda > 0$ gives a penalization on the length (which can be replaced by $\mathcal{H}^1(S) \leq L$).

The starting point to approximate the above problems are two well-known Γ-convergence results.

The celebrated result from [9] states that the functionals

$$F_\varepsilon(u) = \frac{1}{\varepsilon} \int_\Omega W(u) dx + \varepsilon \int_\Omega |\nabla u|^2 dx$$

where $W(0) = W(1) = 0$ and $W > 0$ sur $\mathbb{R} \setminus \{0, 1\}$, $\Gamma-$converge (see [4, 6]) to the functional

$$F(u) = \begin{cases} c\text{Per}(A) & \text{if } u = I_A, \ I_A \in BV \\ +\infty & \text{otherwise,} \end{cases}$$

where Per is the perimeter in the BV sense and $c = 2\int_0^1 \sqrt{W}$ is a constant depending on W. This result can be used, and has been used (for instance in [11]) to produce efficient numerical methods for problems involving the perimeter. Physically, the role of u is to provide a smooth transition from the phase $\{u = 0\}$ to the phase $\{u = 1\}$. This very same approach can be used to tackle the Steiner problem in the very particular case where the problems x_i lie on the boundary of a convex set in $\mathbb{R}^2$. Indeed, in this case, using the information that the optimal network has no loops, the problem becomes a partition problem where the goal is to minimize the perimeter.

Yet, in the most general case, the approximation that we propose recalls more the so-called Ambrosio-Tortorelli approximation for Mumford-Shah functional rather than the above Modica-Mortola approximation (also known as Allen-Cahn, or Cahn-Hilliard). From [1], we know that

$$F_\varepsilon(u,v) = \frac{1}{\varepsilon}\int_\Omega (1-v)^2 dx + \varepsilon \int_\Omega |\nabla v|^2 dx + \int_\Omega v|\nabla u|^2 dx$$

Γ-converges to the functional

$$F(u,v) = \begin{cases} \int_\Omega |\nabla u|^2 dx + \mathcal{H}^{d-1}(J_u) & \text{if } u \in SBV \text{ and } v = 1, \\ +\infty & \text{otherwise,} \end{cases}$$

where J_u is the jump set of u. This result can be used to approximate the Mumford-Shah problem (adding a penalization for data recovery, of the form $\int_\Omega |u-g|^2$, see [10]).

In [8] and [3] we studied a new strategy to approximate length-penalized problems. The novelty was to add a term taking care of the connexity constraint, relying on the weighted geodesic distance d_φ, defined as

$$d_\varphi(x,y) := \inf\left\{\int_\gamma \varphi(x) d\mathcal{H}^1(x);\ \gamma \text{ curve in } \Omega \text{ connecting } x \text{ and } y\right\}.$$

The distance d_φ can be treated numerically by the so-called *fast-marching* method [12] and a recent improvement of this algorithm (see [2]) is able to compute at the same time d_φ and its gradient with respect to φ, which is useful when optimizing w.r.t. φ. We then consider

$$F_\varepsilon(\varphi) := \frac{1}{2\varepsilon}\int_\Omega (1-\varphi)^2 dx + \frac{\varepsilon}{2}\int_\Omega |\nabla\varphi|^2 + \frac{1}{c_\varepsilon}\sum_{i=1}^N d_\varphi(x_i, x_1),$$

among all functions $\varphi \in H^1(\Omega)$ such that $\varphi = 1$ on $\partial\Omega$ and $\varepsilon \le \varphi \le 1$ (here c_ε is an arbitrary sequence of positive numbers tending to 0 with ε and, anyway, the corresponding term only provides the connectedness constraint at the limit). If, for $\varepsilon > 0$ we call φ_ε a minimizer of F_ε (or an almost-minimizer, as existence of minimizers without adding extra penalization terms is a tricky question), we consider $d_{\varphi_\varepsilon}(\,\cdot\,, x_1)$, which are Lip_1 and converge, up to subsequences, to a certain function d. Then the set $K := \{d = 0\}$ is compact, connected and is a solution to the Steiner Problem.

A variant for the average distance problem also exists, which amounts to considering

$$F_\varepsilon(v,\varphi) = \int_\Omega |v|dx + \frac{\lambda}{2\varepsilon}\int_\Omega (1-\varphi)^2 dx + \frac{\lambda\varepsilon}{2}\int_U |\nabla\varphi|^2 dx + \frac{1}{\sqrt{\varepsilon}}\int_\Omega d_\varphi(x,x_0)d\big|\nabla\cdot v + f\big|(x) + |\nabla\cdot v|(\Omega)$$

and allows to approximate (we omit details, see [3]) the problem

$$\min\left\{F(S) := \int_\Omega \mathrm{dist}(x,S)f(x)dx + \lambda\mathcal{H}^1(S)\ :\ S \text{ compact and connected}, x_0\in S\right\}.$$

References

[1] L. Ambrosio and V. M. Tortorelli, *Approximation of functionals depending on jumps by elliptic functionals via* Γ*-convergence*, Comm. Pure Appl. Math. **43** (8): 999–1036, 1990.

[2] F. Benmansour, G. Carlier, G. Peyré and F. Santambrogio, *Fast Marching Derivatives with Respect to Metrics and Applications*, Numerische Mathematik, **116**, no 3, 357–381, 2010

[3] M. Bonnivard, A. Lemenant and F. Santambrogio, *Approximation of length minimization problems among compact connected sets*, SIAM J. Math. An., **47**(2), 1489–1529, 2015.

[4] A. Braides, *Approximation of free-discontinuity problems*, LNM **1694**, Springer, 1998.

[5] G. Buttazzo, E. Oudet and E. Stepanov, *Optimal Transportation Problems with Free Dirichlet Regions*, Progress in Nonlinear Diff. Eq. Appl. **51**, 41–65, Birkhauser, 2002.

[6] G. Dal Maso: *An Introduction to* Γ*−convergence*, Birkhauser, Basel, 1992.

[7] E. N. Gilbert and H. O. Pollak, *Steiner minimal trees*, SIAM J. Appl. Math. **16**, 1–29, 1968.

[8] A. Lemenant, F. Santambrogio, *A Modica-Mortola approximation for the Steiner Problem*, C. R. Math. Acad. Sci. Paris, **352** (5), 451–454, 2014.

[9] L. Modica and S. Mortola, *Un esempio di* Γ*-convergenza*, Boll. Un. Mat. Ital. B **14** (1), 285–299, 1977.

[10] D. Mumford and J. Shah, *Optimal Approximations by Piecewise Smooth Functions and Associated Variational Problems*, Comm. Pure Appl. Math. **42** (5) (1989) 577–685.

[11] E. Oudet, *Approximation of partitions of least perimeter by* Γ*−convergence : around Kelvin's conjecture*, Exp. Math. **20**, No. 3 (2011), 260–270.

[12] J. A. Sethian, *Level Set Methods and Fast Marching Methods*, Cambridge, 1999.

Stable constant mean curvature varifolds: regularity and compactness theory in codimension one

Costante Bellettini

(joint work with Neshan Wickramasekera)

The talk reports on a recent joint work [1] by N. Wickramasekera and the author. The hypersurfaces under consideration are stable critical points of the area functional under the constraint of fixed enclosed volume. These objects have been widely studied in the smooth setting (and the criticality requirement is equivalent to the well-known constant mean curvature condition). The scope of this work is to provide a sharp regularity and compacness theory in a more general setting (namely that of codimension 1 varifolds, as opposed to smooth hypersurfaces) under suitable (and easily checkable) structural conditions and imposing the variational assumptions in the mildest possible fashion. Such a theory is desirable in order to have a suitable class of varifolds to employ in variational problems. Below we are going to state the results precisely and comment shortly upon them.

1. Statements of the main results

We need to introduce two special types of singularities that play a key role in our analysis and in the main theorems.

Definition (classical singularity). *A point $p \in spt\|V\|$ is a* classical singularity *of V if there are $\alpha \in (0,1)$ and $\sigma > 0$ such that $spt\|V\| \cap B^{n+1}_\sigma(p)$ is the union of three or more embedded $C^{1,\alpha}$ hypersurfaces with boundary having common boundary S (containing p), meeting pairwise only along S and such that at least two of the hypersurfaces meet transversely at p.*

Definition (two-fold touching singularity). *A point $p \in spt\|V\|$ is a* two-fold touching singularity *of V (we will write $p \in Sing_T V$) if there are $\sigma > 0$, an affine hyperplane L through p and two $C^{1,\alpha}$ functions $u_1, u_2 : L \to L^\perp$ such that*

$$\operatorname{spt}\|V\| \cap B^{n+1}_\sigma(p) = (\operatorname{graph} u_1 \cup \operatorname{graph} u_2) \cap B^{n+1}_\sigma(p).$$

with $u_1 \leq u_2$, $u_1(p) = u_2(p)$, $Du_1(p) = Du_2(p)$ but $u_1 \not\equiv u_2$.

The structural assumptions in the main theorems only require a certain (necessary) control on these two very special types of singularities: nothing else is assumed on the singular set (in particular there is no smallness requirement on the singular set of V).

Theorem 1 (regularity for stable CMC integral varifolds). *Let $n \geq 2$ and let V be an integral n-varifold in an open set $\mathcal{U} \subset \mathbb{R}^{n+1}$ that satisfies the following assumptions.*

1. *the generalized[1] mean curvature $\vec{H}$ is in $L^p(\|V\|)$ for some $p > n$*
2. *there are no classical singularities in V*
3. *for every $p \in Sing_T V$ there exists a neighbourhood $B^{n+1}_\rho(p)$ such that $\mathcal{H}^n\left(\{\Theta = \Theta(p)\} \cap B^{n+1}_\rho(p)\right) = 0$*
4. *for each orientable portion of the $C^{1,\alpha}$ embedded part of sptV there exists a choice of orientation such that the portion is critical for the area measure under volume-preserving variations*
5. *gen regV, i.e. the C^2-immersed part of sptV, is stable for the area measure under volume-preserving variations[2]*

[1] This assumption, widely used thanks to Allard's theorem, guarantees the validity of the monotonicity formula for the mass ratio and the existence for every $p \in \operatorname{spt}\|V\|$ of the density $\Theta(p) := \lim_{\rho\to 0} \frac{\|V\|(B^{n+1}_\rho(p))}{\omega_n \rho^n}$.

[2] The fact that gen reg$V$ is a CMC C^2-immersion (possibly with several connected components) is not an assumption here, it is an immediate consequence of assumption 4. Only the stability is an assumption.

Then there exists a closed set $\Sigma \subset spt\|V\|$ of Hausdorff dimension at most $n-7$ such that $spt\|V\| \setminus \Sigma$ locally near each point is either an embedded C^2 disk or the union of precisely two embedded C^2 disks intersecting tangentially; moreover, there is a constant $\lambda \in \mathbb{R}$ such that the mean curvature vector of V is $\lambda\nu$ on $spt\|V\| \setminus \Sigma$ (here ν denotes a choice of normal on $spt\|V\| \setminus \Sigma$).

The above regularity result is completed by the following

Theorem 2 (compactness for stable CMC integral varifolds). *Let $\{V_j\}_{j\in\mathbb{N}}$ be integral n-varifolds in the open set $\mathcal{U} \subset \mathbb{R}^{n+1}$, satisfying assumptions (1)-(5) from Theorem 1 with V replaced by V_j and H replaced by H_{V_j}.*

If $\limsup_{j\to\infty} \|V_j\|(K) < \infty$ for each compact $K \subset U$ and if $\limsup_{j\to\infty} |H_{V_j}| < \infty$ (note that $|H_{V_j}|$ is constant for each j by the above theorem), then there is an integral n-varifold V in U satisfying (1)-(5) and a subsequence $\{j'\}$ such that $V_{j'} \to V$ as varifolds in U.

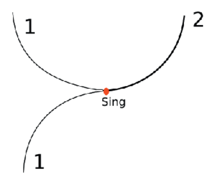
2. Comments and remarks

Let us begin by explaining why the two *structural assumptions* are necessary.

Hypothesis (2) cannot be dropped, unless one aims for a weaker regularity result. For example, consider two intersecting unit spheres crossing along a circle.

Assumption (3) is automatically satisfied in the case when the varifold under consideration is the reduced boundary of a set with finite perimeter (this follows from De Giorgi's rectifiability theorem).

For general varifolds, if hypothesis (3) is dropped, then C^2 regularity is false and one could hope for $C^{1,1}$ regularity. To see this, consider the 1-dimensional example in the picture (this could be made two-dimensional in $\mathbb{R}^3$ by taking the cartesian product with an interval). In this picture, each arc is a piece of a unit circle, and the numbers $1, 2$ denote the multiplicity on an arc. The singular point is a touching singularity (and not a classical singularity, since no pair of arcs meet transversely). All other assumptions of Theorem 1 are satisfied but the regularity here is $C^{1,1}$.



Let us now shortly comment on the *variational hypotheses* (4), (5): they are made respectively on the $C^{1,\alpha}$ embedded part and on the C^2 immersed part; no variational hypothesis is made across the entire (a priori potentially large) singular set. These features are in principle very useful for the applications of the theorem to variational contexts.

It is worthwile noting that Theorems 1 and 2 generalise the corresponding results established in [3] for codimension 1 integral varifolds V (with no classical singularities) that are stationary and stable with respect to the area functional for unconstrained ambient variations (i.e. minimal hypersurfaces); in this case, stationarity is equivalent to requiring $H_V = 0$, and it follows from the Hopf boundary point lemma that hypothesis (2) in Theorem 1 is redundant. In [3], stationarity was assumed everywhere, while in [1] it is only assumed on the embedded part; moreover in [1] one only needs to assume stability for volume preserving variations. The impact of he regularity theory from [3] (and its adaption [4] to varifolds arising as limiting interfaces of the Allen-Cahn functional) has had a strong impact on the fundamental geometric question about the existence of a minimal hypersurface in an arbitrary Riemannian manifold. The (affirmative) answer to this question was known by the works of Almgren-Pitts and Schoen-Simon in the early 80's and a new proof has recently been provided in [2] exploiting the results of [3] and [4]. The hope is that the results in [1] will likewise play a fundamental role in the unresolved geometric question about the existence of a hypersurfaces with prescribed constant mean curvature in an arbitrary Riemannian manifold.

References

[1] C. Bellettini and N. Wickramasekera, *Regularity and compactness for stable CMC codimension 1 integral varifolds*, manuscript in preparation.

[2] M. Guaraco, *Min-max for phase transitions and the existence of embedded minimal hypersurfaces*, arXiv, 2015.

[3] N. Wickramasekera, *A general regularity theory for stable codimension 1 integral varifolds*, Ann. of Math. 179 (2014), 843–1007.

[4] Y. Tonegawa and N. Wickramasekera, *Stable phase interfaces in the van der Waals-Cahn-Hilliard theory*, J. Reine Ang. Mat. (2012), 191–210.

Ricci flow through singulariites

John Lott

(joint work with Bruce Kleiner)

Theorem 1. *Any compact Riemannian* 3*-manifold is the initial time slice of a Ricci flow through singularities.*

The Ricci flow equation is $\frac{dg}{dt} = -2Ric(g)$, where $g(t)$ is a smooth 1-parameter family of Riemannian metrics. Given a normalized initial Riemannian metric g_0 on a compact 3-manifold M, the ensuing Ricci flow will generally encounter finite time singularities. It has been a longstanding problem to continue the flow through singularities. The proof of Theorem 1 goes by regularizing the Ricci flow, in order to ameliorate the singularities, and then taking the regularization parameter to zero. The regularized Ricci flow is the Hamilton-Perelman Ricci flow with surgery. The regularization parameter is Perelman's function $\delta : [0, \infty) \to (0, \infty)$, which determines the scale at which time-t surgeries are performed. Perelman showed that there is a function $\overline{\delta}$ with $\lim_{t \to \infty} \overline{\delta}(t) = 0$ so that the Ricci flow with surgery exists whenever $\delta \le \overline{\delta}$. Perelman raised the question of whether there is a limit of the Ricci flows with surgery, as one takes δ to zero [2]. Theorem 1 follows from the following more precise statement [1].

Theorem 2. *If $\delta_j : [0, \infty) \to (0, \infty)$ is a sequence of decreasing functions with $\delta_j \le \overline{\delta}$, and $\lim_{j\to\infty} \delta_j(0) = 0$, then after passing to a subsequence the ensuing Ricci flows with surgery (with initial metric g_0) converge to a Ricci flow through singularities.*

The proof of Theorem 2 is via a spacetime approach and a new compactness result for possibly incomplete Riemannian manifolds. Given the Ricci flow with surgery, with parameter δ_j and initial metric g_0, we construct a spacetime $\mathcal{M}^j$ with an adapted Riemannian metric G^j. Given $\overline{R} < \infty$, if R^j denotes the spatial scalar curvature of $\mathcal{M}^j$ then we show that after passing to a subsequence of j's, the sublevel sets $\{x \in \mathcal{M}^j : R(x) \le \overline{R}\}$ converge. Theorem 2 then follows by taking $\overline{R}$ to infinity and applying a diagonal argument.

The limit space $\mathcal{M}^\infty$ is an example of a singular Ricci flow. We give the general definition of singular Ricci flows and prove various properties of them, such as

(1) Volume evolution formula and L^1-boundedness of the scalar curvature on any time slab.
(2) L^p-boundedness of the scalar curvature on any time slice, for any $p < 1$.
(3) Countability of the number of static worldlines that do not extend backward from a given time slice to the initial time slice.

References

[1] B. Kleiner and J. Lott, *Singular Ricci flows I*, http://arxiv.org/abs/1408.2271 (2014).
[2] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, https://arxiv.org/abs/math/0211159 (2002).

Quantitative flatness results and BV estimates for nonlocal minimal surfaces

Eleonora Cinti
(joint work with Joaquim Serra, Enrico Valdinoci)

We establish quantitative properties for stable sets of a nonlocal perimeter functional. Although our results hold for very general —possibly anisotropic and not scaling invariant— functionals, for the sake of simplicity, we focuse on the case of the fractional s-perimeter, which was introduced in [2].

We start by recalling its definition. Let $s \in (0,1)$. Given a bounded domain $\Omega \subset \mathbb{R}^n$, we define the fractional s-perimeter of a measurable set $E \subset \mathbb{R}^n$ relative to Ω as

$$P_{s,\Omega}(E) := L_s(E \cap \Omega, \mathcal{C}E \cap \Omega) + L_s(E \cap \Omega, \mathcal{C}E \setminus \Omega) + L_s(E \setminus \Omega, \mathcal{C}E \cap \Omega),$$

where $\mathcal{C}E$ denotes the complement of E in $\mathbb{R}^n$ and the interaction L_s of two disjoint measurable sets A, B is defined by

$$L_s(A,B) := \int_A \int_B \frac{dx\, d\bar{x}}{|x-\bar{x}|^{n+s}}.$$

Roughly speaking, this s-perimeter captures the interactions between a set E and its complement. These interactions occur in the whole of the space and are weighted by a (homogeneous and rotationally invariant) kernel with polynomial decay.

Here, we present two types of results for *stable sets* of the fractional perimeter, that is, sets for which the second variation of the fractional perimeter functional is nonnegative:

- BV estimates (universal bounds for the classical perimeter) and sharp energy estimates;
- quantitative flatness results.

The first result gives a uniform BV estimate for stable sets (see Theorem 1.1 in [5]).

Theorem 1. *Let $s \in (0,1)$, $R > 0$ and E be a stable set in the ball B_{2R} for the nonlocal s-perimeter functional. Then, the classical perimeter of E in B_R is bounded by CR^{n-1}, where C depends only on n and s.*

Moreover, the s-perimeter of E in B_R is bounded by CR^{n-s}.

We observe that the estimate for the fractional perimeter of minimizers can be easily proven using a comparison argument. Here the difficulty relies on the fact that we are assuming stability and not minimality.

To better appreciate Theorem 1 let us compare it with the best known similar results for classical minimal surfaces. A universal perimeter estimate for (local) stable minimal surfaces is only known for the case of *two-dimensional* stable minimal surfaces that are *simply connected* and *immersed* in $\mathbb{R}^3$. Conversely, the perimeter estimate in our Theorem 1 holds in every dimension and without topological constraints. In fact, an estimate exactly like ours can not hold for classical stable minimal surfaces since a large number of parallel planes is always a classical stable minimal surface with arbitrarily large perimeter in B_1.

Having a universal bound for the classical perimeter of embedded minimal surfaces in every dimension $n \geq 4$ would be a decisive step towards proving the following well-known and long standing conjecture: *The only stable embedded minimal (hyper)surfaces in $\mathbb{R}^n$ are hyperplanes as long as the dimension of the ambient space is less than or equal to 7.* Indeed, it would open the door to use the monotonicity formula to prove that blow-downs of stable surfaces are stable minimal cones —which are completely classified. On the other hand, without a universal perimeter bound, the sequence of blow-downs could have perimeters converging to ∞. In the same direction, we believe that our result in Theorem 1 can be used to reduce the classification of stable s-minimal surfaces in the whole $\mathbb{R}^n$ to the classification of stable cones —although by now this classification of cones is only known for $n = 2$ (or for $n \leq 7$, but s sufficiently close to 1, see [7] and [4]).

Since it is well-known [6, 3, 1] that the classical perimeter is the limit as $s \uparrow 1$ of the nonlocal s-perimeter (suitably renormalized), it is natural to ask whether our results give some informations in the limit case $s = 1$. Unfortunately, our proof relies strongly on the nonlocal character of the s-perimeter and the constant C appearing in Theorem 1 blows up as $s \uparrow 1$.

We next give our quantitative flatness estimate in dimension $n = 2$ for the case of the s-perimeter (see Theorem 1.3 in [5]). It states that stable sets in a large ball B_R are close to being a halfplane in B_1, with a quantitative control on the measure of the symmetric difference that decays to 0 as $R \to \infty$.

Theorem 2. *Let the dimension of the ambient space be equal to* 2*. Let* $\mathbb{R} \geq 2$ *and* E *be a stable set in the ball* B_R *for the* s*-perimeter.*

Then, there exists a halfplane $\mathfrak{h}$ *such that* $|(E \triangle \mathfrak{h}) \cap B_1| \leq CR^{-s/2}$.

Moreover, after a rotation, we have that $E \cap B_1$ *is the graph of a measurable function* $g : (-1, 1) \to (-1, 1)$ *with* $\operatorname{osc} g \leq CR^{-s/2}$ *outside a "bad" set* $\mathcal{B} \subset (-1, 1)$ *with measure* $CR^{-s/2}$.

The previous result provides a quantitative version of the classification result in [7] which says that if E is a minimizer of the s-perimeter in any compact set of $\mathbb{R}^2$, then it is necessarily a halfplane. Moreover, Theorem 2 extends this classification result to the class of stable sets.

The proofs of our main results have, as starting point, a nontrivial refinement of the variational argument introduced by Savin and one of the authors in [7, 8] to prove that halfplanes are the only cones minimizing the s-fractional perimeter in every compact set of $\mathbb{R}^2$. Namely, we consider perturbations $E_{R,t}$ of a minimizer E which coincide with E outside B_R and are translations $E + t\boldsymbol{v}$ of E in $B_{R/2}$ —with "infinitesimal" $t > 0$. A first step in the proof is estimating how much $P_{s,B_R}(E_{R,t})$ differs from $P_{s,B_R}(E)$ depending on R. By exploiting the nonlocality of the perimeter functional, the previous control on $P_{s,B_R}(E_{R,t}) - P_{s,B_R}(E)$ is translated into a control on the minimum between $|E_{R,t} \setminus E|$ and $|E \setminus E_{R,t}|$. We emphasize that we always use *arbitrarily small* perturbations of our set E. That is why we can establish some results for *stable* sets.

References

[1] L. Ambrosio, G. De Philippis, and L. Martinazzi, *Gamma-convergence of nonlocal perimeter functionals*, Manuscripta Math. 134 (2011), no. 3-4, 377–403.

[2] L. Caffarelli, J.-M. Roquejoffre, and O. Savin, *Nonlocal minimal surfaces*, Comm. Pure Appl. Math. 63 (2010), 1111–1144.

[3] L. Caffarelli and E. Valdinoci, *Uniform estimates and limiting arguments for nonlocal minimal surfaces*, Calc. Var. Partial Differential Equations 41 (2011), no. 1-2, 203–240.

[4] L. Caffarelli and E. Valdinoci, *Regularity properties of nonlocal minimal surfaces via limiting arguments*, Adv. Math. 248 (2013), 843–871.

[5] E. Cinti, J. Serra, and E. Valdinoci, *Quantitative flatness results and BV estimates for stable nonlocal minimal surfaces*, submitted. (Available at https://arxiv.org/abs/1602.00540).

[6] J. Dávila, *On an open question about functions of bounded variation*, Calc. Var. Partial Differential Equations 15 (2002), no. 4, 519–527.

[7] O. Savin and E. Valdinoci, *Regularity of nonlocal minimal cones in dimension 2*, Calc. Var. Partial Differential Equations, 48 (2013), no. 1–2, 33–39.

[8] O. Savin and E. Valdinoci, *Some monotonicity results for minimizers in the calculus of variations*, J. Funct. Anal. 264 (2013), no. 10, 2469–2496.

The limit shape of convex peeling

Charles K. Smart
(joint work with Jeff Calder)

Convex peeling provides a natural generalization of order statistics to higher dimensions. To be more precise, suppose $X \subseteq \mathbb{R}^d$ is finite define

$$K_0 = \text{hull}(X) \quad \text{and} \quad K_{n+1} = \text{hull}(X \cap \text{int}(K_n)),$$

stopping when $\text{int}(K_n)$ is empty. The descending chain of convex hulls

$$K_0 \supseteq K_1 \supseteq \cdots \supseteq K_n$$

is called the convex peeling of X. The centroid of K_n is a natural choice for the median of multidimensional data.

We are interested the convex peeling of a generic or random set X. An early result along these lines is the following.

Theorem (Dalal [1])**.** *If $X_m \subseteq B_1$ consists of m points sampled uniformly at random from the unit ball, then $\mathbb{E}[\#$ of convex peels$] \sim m^{2/(d+2)}$.*

To sharpen this result, we define the height function

$$h_m = \sum_k \mathbf{1}_{K_k},$$

which is the sum of the indicator functions of the peels. When m is large, the height function h_m approximates the solution of a partial differential equation. For example, we have the following scaling limit result.

Theorem. *There is a constant $\alpha_d > 0$ such that, for all $\varepsilon > 0$,*

$$\lim_{m\to\infty} \mathbb{P}[\sup |\alpha_d m^{-2/(d+2)} h_m - h| > \varepsilon] = 0,$$

where $h \in C(\bar{B}_1)$ is the unique semiconcave viscosity solution of

$$\begin{cases} Dh \cdot \text{Adj}(D^2 h) Du = 1 & \text{in } B_1 \\ h = 0 & \text{on } \partial B_1. \end{cases}$$

In fact, we prove more, obtaining exponential tail bounds for point clouds sampled from arbitrary continuous densities. Moreover, since

$$Dh \cdot \text{Adj}(D^2 h) Du = |Dh|^{d+1} \kappa_G,$$

where κ_G is the Gauss curvature of the super level sets of h, the convex peeling can be interpreted as Gauss curvature flow.

To prove this theorem, we adapt the Martingale approach used by Armstrong and Cardaliaguet [2] to homogenize forced mean curvature motion. This requires the identification of a natural scale-invariant problem and a way of localizing the evolution. For the former, we consider peeling a Poission cloud inside an infinite paraboloid. For the latter, we sharpen the original estimates of Dalal. The Martingale argument implies homogenization of the scale-invariant problem. One concludes the theorem by invoking the uniqueness of viscosity solutions via the perturbed test function method.

References

[1] K. Dalal, *Counting the onion*, Random Structure & Algorithms **24** (2004), no. 2, 155-165
[2] S. Armstrong and P. Cardaliaguet, *Stochastic homogenization of quasilinear Hamilton-Jacobi equations and geometric motions*, preprint, arXiv:1504.02045

Sharp stability for the Euclidean concentration inequality and droplets formation in statistical mechanics

Francesco Maggi
(joint work with Eric A. Carlen, Alessio Figalli, Connor Mooney)

The starting point of this study if the analysis of liquid-vapor phase transitions in a model from statistical mechanics, based on the minimization of the *Gates-Penrose-Lebowitz* (GPL) free energy

$$GPL(u) = \frac{1}{2}\int_{T_L} dx \int_{T_L} J(|x-y|)\,|u(x)-u(y)|^2\,dy + \int_{T_L} W(u)\,.$$

Here T_L denotes a n-dimensional flat torus of side length L, $J(r)$ is a bounded decreasing interaction kernel with compact support on $[0,1]$ ($L \gg 1$) such that $\int_{\mathbb{R}^n} J(|x|)\,dx = 1$, $u : T_L \to (-1,1)$ represents a particle-hole density, and $W : (-1,1) \to [0,\infty)$ is an even, smooth, double-well potential, with $W(\pm m_0) = 0$ and $W''(m_0) > 0$ for some $m_0 \in (0,1)$. We stress that the length scale L is large compared to the length scale of the interaction kernel, which was set to unit by requiring $\operatorname{spt} J = [0,1]$.

Given a volume fraction $m \in (-1,1)$, one minimizes $GPL(u)$ under the constraint that $L^{-n}\int_{T_L} u = m$. Very much like in the case of the Cahn-Hilliard free energy, the double-well favors two constant states (namely, $u \equiv m_0$ and $u \equiv -m_0$) and the interaction energy penalizes variations. In particular, when $m = \pm m_0$ there is no doubt that the constant states are the unique minimizers. For other volume fractions m we expect to see a competition between the two terms in the energy, leading to transition profiles u between a m_0-phase and a $(-m_0)$-phase.

Because of this competition, in both models, one expects the formation of almost spherical "droplets", whenever $m \in (-m_0, m_0)$ and L is large enough. An heuristic analysis shows that this should also happen when $m \to \pm m_0$ as $L \to \infty$, and precisely for $m = -m_0 + K\,L^{-n/(n+1)}$ with K larger than some critical K_*. This kind of study for the Cahn-Hilliard model has been addressed, independently, in [3, 4]. There are two significant differences between the Cahn-Hilliard and the GPL models: first, since the interaction kernel J is not singular, minimizers of

GPL possess no smoothness property, and second, because of the statistical origin of the model, one is actually interested in understanding all near-minimizers of GPL, as the most likely observed states of the system. On a deeper level, almost sphericity of droplets is related to the Euclidean isoperimetric inequality in the Cahn-Hilliard case, and to the Euclidean concentration inequality in the GPL case. As explained below, a quantitative analysis of near-minimizers is definitely subtler for Euclidean concentration than for Euclidean isoperimetry.

Why round droplets? Guessing that near-minimizing u are sharp transitions between the constant densities m_0 and $-m_0$, concentrated along the boundary of $\{u \geq m_0\}$, and with $\operatorname{diam}(\{u \geq m_0\})$ way smaller than L, one should be able to argue as if $T_L \approx \mathbb{R}^n$. On the whole space, it makes sense to compare u by its spherically symmetric decreasing rearrangement u^*, whose super-level sets are balls with same volume as the corresponding super-level sets of u. This equimensurability property guarantees that $\int_{\mathbb{R}^n} g(u) = \int_{\mathbb{R}^n} g(u^*)$ for every $g : \mathbb{R} \to \mathbb{R}$, and thus, by combining the identity

$$GPL(u) = \int_{\mathbb{R}^n} u^2 - \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} J(|x-y|)\, u(x)\, u(y)\, dy$$

(recall that $\int_{\mathbb{R}^n} J(|x|)\, dx = 1$) with the *Riesz rearrangement inequality*

$$\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} J(|x-y|)\, u(x)\, u(y)\, dy \leq \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} J(|x-y|)\, u^*(x)\, u^*(y)\, dy$$

we deduce that $GPL(u) \geq GPL(u^*)$. In particular, if u is a minimizer or a near-minimizer, so it is u^*. The quantitative analysis of radially decreasing near-minimizers of the GPL model in the spherical droplet regime has been addressed in [1, 2, 5]. The next step is thus understanding how far a generic near minimizer u is from being almost spherical, i.e. how to control the distance of u from u^* in terms of $GPL(u) - GPL(u^*)$. Considering the discussion of equality cases in the Riesz rearrangement inequality can be addressed in terms of a discussion of equality cases for the Euclidean concentration inequality, there are two main problems to address:

(i) provide a stability estimate for the Euclidean concentration inequality;
(ii) exploit such an estimate to obtain a robust improvement of the Riesz rearrangement inequality.

Both problems are addressed in the joint paper [6] with Eric Carlen, by exploiting suitable geometric arguments. These results pave the a way to a quantitative description of every near-minimizer of the GPL free energy in the droplet regime.

The paper [6] also indicate some interesting problems in the theory of geometric inequalities. For example, the arguments presented in [6] are not sufficient to produce a *sharp* stability estimates for Euclidean concentration. From the mathematical viewpoint, this last problem is particularly interesting because it seems out of reach for all the three different approaches developed in proving the closely related sharp stability estimate for Euclidean isoperimetry [13, 10, 7]. A new approach is thus required, and this is the content of the joint paper [9] with Alessio Figalli and Connor Mooney.

Let us recall that the *Euclidean concentration inequality* states that if E is a subset of $\mathbb{R}^n$, E^* is a ball with same volume as E, and $N_r(E) = \{x \in \mathbb{R}^n : \mathrm{dist}(x, E) < r\}$ denotes the r-neighborhood of E, then

$$|N_r(E)| \geq |N_r(E^*)|\,, \qquad \forall r > 0\,, \tag{1}$$

with equality if and only if, up to a zero volume set, E is a ball. The main result proved in [9] is the existence of $c(n) > 0$ such that whenever $|E| = |B|$, then there exists $x \in \mathbb{R}^n$ with

$$\max\Big\{r, \frac{1}{r}\Big\}\Big(\frac{|N_r(E)|}{|N_r(E^*)|} - 1\Big) \geq c(n)\,|E\Delta(x+B)|^2\,, \qquad \forall r > 0\,. \tag{2}$$

The factor $\max\{r, r^{-1}\}$ is needed for the inequality to be true, as otherwise the left-hand side of the inequality tends to 0 as $r \to 0^+$ or $r \to +\infty$. Notice also that in the limit $r \to 0^+$, (2) implies the sharp quantitative isoperimetric inequality: there exists $c_*(n) > 0$ such that whenever $|E| = |B|$, then there exists $x \in \mathbb{R}^n$ with

$$P(E) - P(B) \geq c_*(n)\,|E\Delta(x+B)|^2\,, \tag{3}$$

provided $P(E)$ denotes the perimeter of E (i.e., the $(n-1)$-dimensional measure of the boundary of E).

The approach to (3) developed in [13] is based on dimension induction through the localization of the isoperimetric deficit $P(E) - P(B)$ on hyperplane slices of E. This kind of argument, clearly, does not combine smoothly with the nonlocal nature of the operation of forming the Minkowski sum $N_r(E) = E + B_r$. Although one can use localization by slicing and dimension induction to obtain *non-sharp* quantitative versions of the Brunn-Minkowski inequality, see [8], it seems quite hard to optimize this approach to the extent of proving sharp inequalities. The mass transportation approach to (3) developed in [10] can be used to prove (2) in the special case that E is convex. This is already detailed in [10] and, with a more direct argument, in [11]. Extending this analysis to the case when E is non-convex seems hard because it would require, for example in the case $r = 1$ and with T denoting the Brenier map between E and B, to control the distance of E from its convex envelope in terms of the non-negative quantity $|S(E)| - |N_1(B)|$, where $S = \mathrm{Id} + T$. Finally, the quite versatile approach to (3) proposed in [7] is based on the regularity theory for local minimizers of the perimeter functional, an ingredient that is completely missing when the functional under consideration is the volume of the r-neighborhood of a set.

The proof of (2) given in [9] is based on two separate arguments, one degenerating as r becomes larger, the one valid only if r is large enough. Both arguments move from a "regularization by viscosity" procedure based on taking an envelope of E by balls of radius r contained in its complement. The estimate degenerating for r large is obtained by combining the strong form of (2) obtained in [12] with the reduction to this notion of r-convex envelope. In large r-regime, one shows by a geometric construction that any set with $|E| = |B|$ and $r\,(|N_r(E)|/|N_r(B)| - 1)$ small enough must have positive reach of order one in the sense of Federer. The proof is then completed by combining the Steiner-Federer formula for sets of positive reach with (3).

References

[1] G. Alberti, *Some remarks about a notion of rearrangement*, Ann. Scuola Norm. Sup. Pisa Cl. Sci XXIX, 457–472 (2000).
[2] G. Alberti and G. Bellettini, *A nonlocal anisotropic model for phase transitions Part i: the optimal profile problem*, Math. Annalen. **310**, (1998), 527–560.
[3] G. Bellettini, M. S. Gelli, S. Luckhaus and M. Novaga, *Deterministic equivalent for the Cahn-Hilliard energy of a scaling law in the Ising model*, Calc. of Variations and PDE, **26**, no 4, (2006), 429–445
[4] E. A. Carlen, M. C. Carvalho, R. Esposito, J. L. Lebowitz and R. Marra, *Droplet minimizers for the Cahn Hilliard free energy functional*, J. Geometric Analysis **16** (2006), 233–264.
[5] E. A. Carlen, M. C. Carvalho, R. Esposito, J. L. Lebowitz and R. Marra, *Droplet minimizers for the Gates- Lebowitz-Penrose free energy functional*, Nonlinearity **22**(2009), 2919–2952.
[6] E. A. Carlen and F. Maggi, *Stability for the Brunn-Minkowski and Riesz rearrangement inequalities, with applications to Gaussian concentration and finite range non-local isoperimetry*, to appear on Canadian J. Math.
[7] M. Cicalese and G. P. Leonardi, *A selection principle for the sharp quantitative isoperimetric inequality*. Arch. Ration. Mech. Anal. **206** (2012), no. 2, 617–643.
[8] A. Figalli and D. Jerison, *Quantitative stability for the Brunn-Minkowski inequality*, preprint arXiv:1502.06513.
[9] A. Figalli, F. Maggi and C. Mooney, *The sharp quantitative Euclidean concentration inequality*, preprint arXiv:1601.04100.
[10] A. Figalli, F. Maggi and A. Pratelli, *A mass transportation approach to quantitative isoperimetric inequalities*. Invent. Math. **182** (2010), no. 1, 167–211.
[11] A. Figalli, F. Maggi and A. Pratelli, *A refined Brunn-Minkowski inequality for convex sets.* Ann. Inst. H. Poincar Anal. Non Linéaire **26** (2009), no. 6, 2511–2519.
[12] N. Fusco and V. Julin, *A strong form of the quantitative isoperimetric inequality*. Calc. Var. Partial Differential Equations **50** (2014), no. 3–4, 925–937.
[13] N. Fusco, F. Maggi, A. Pratelli, *The sharp quantitative isoperimetric inequality*, Ann. of Math. **168** (2008), 941–980.

Asymptotic behavior of the inverse mean curvature flows in the hyperbolic spaces

Pei-Ken Hung
(joint work with Mu-Tao Wang)

The solution of the inverse mean curvature flow is a family of smooth maps $F_t : \Sigma^{n-1} \to M^n$ satisfying the evolution equation

$$\frac{\partial F_t}{\partial t} = \frac{\nu}{H},$$

where H is the mean curvature and ν is the unit outer normal of $\Sigma_t = F_t(\Sigma)$. Geroch [2] introduced this parabolic flow and discovered that the Hawking mass of a surface is monotone nondecreasing along the flow provided the scalar curvature of M is nonnegative. Jang-Wald [5] observed that if there is a smooth solution of the inverse mean curvature flow which starts from the horizon and exists for all time, then the Penrose inequality follows the Geroch monotonicity. However, the last assumption is not always satisfied. In general the mean curvature H may go to zero in finite time and the smooth flow doesn't exist anymore. Nevertheless, Huisken and Ilmanen [3] developed the level set formulation of the inverse mean curvature flow and proved the Geroch monotonicity holds under the weak setting. As a result, they gave the first proof of the Riemannian Penrose inequality.

One of the ingredient of Huisken-Ilmanen's proof is that when t is large, Σ_t will be "round" enough. This implies that the Hawking mass of surfaces are bounded from above by the ADM mass of M. The corresponding asymptotic behavior in the smooth setting was proved by Gerhardt [1] when the ambient manifold is $\mathbb{R}^n$. Precisely, he showed that suppose Σ_0^{n-1} is a mean convex and star-shaped hypersurface in $\mathbb{R}^n$, then the solution of inverse mean curvature flow Σ_t^{n-1} exists for all time, remains star-shaped. Furthermore, the rescaled metric $\exp(-\frac{2t}{n-1})g_t$ converges smoothly to a round metric on S^{n-1}.

In the asymptotically hyperbolic setting, however, the limit of the rescaled metric is not necessarily round. This phenomenon was first pointed out by Neves [6]. He gave examples in Anti-deSitter-Schwarzschild spaces.

Theorem (N). Let M^3 be a Anti-deSitter-Schwarzschild manifold with positive mass parameter. There exists a star-shaped mean convex closed surface Σ_0 in M^3 that has the following property. Let Σ_t be the inverse mean curvature flow of Σ_0, and $|\Sigma_t|$ and g_t be the area of Σ_t and the induced metric on Σ_t, respectively. As $t \to \infty$, $|\Sigma_t|^{-1} g_t$ converges to a metric on S^2 that is not of constant curvature.

Roughly speaking, he set the initial surface with Hawking mass larger than the mass of the ambient manifold. The Geroch monotonicity preserves this inequality and then the limiting metric would not be round. This observation rules out the possibility of using the inverse mean curvature to prove Penrose inequality with negative cosmological constant.

We are interested in the case of hyperbolic space $\mathbb{H}^3$. We construct examples of inverse mean curvature flow in $\mathbb{H}^3$ similar to ones given by Neves.

Theorem 1. *There exists a star-shaped mean convex closed surface Σ_0 in $\mathbb{H}^3$ that has the following property. Let Σ_t be the inverse mean curvature flow of Σ_0, and $|\Sigma_t|$ and g_t be the area of Σ_t and the induced metric on Σ_t, respectively. As $t \to \infty$, $|\Sigma_t|^{-1} g_t$ converges to a metric on S^2 that is not of constant curvature.*

We recall the Hawking mass of a closed embedded surface Σ in $\mathbb{H}^3$:

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma (H^2 - 4) d\mu \right).$$

By the Gauss equation one can rewrite it as:

$$m_H(\Sigma) = -\sqrt{\frac{|\Sigma|}{16\pi}}\frac{1}{8\pi}\int_\Sigma |\mathring{A}|^2 d\mu,$$

where $\mathring{A}$ is the traceless part of the second fundamental form. In particular, Hawking mass is always nonpositive. Furthermore, its limit along the inverse mean curvature flow is always zero and gives no information of the limiting metric. Therefore, we consider a modified quantity

$$\tilde{m}(\Sigma) = -|\Sigma|\int_\Sigma |\mathring{A}|^2 d\mu.$$

Denote by σ the standard metric on S^2 and by D the covariant derivative with respect to σ. If the rescaled metric $|\Sigma_t|^{-1}g_t$ converges to $e^{2f}\sigma$, the limit of $\tilde{m}(\Sigma_t)$ is given by

$$\lim_{t\to\infty}\tilde{m}(\Sigma_t) = -\int_{S^2} e^{2f}d\mu_\sigma \int_{S^2} |\mathring{D}^2 e^{-f}|^2_\sigma d\mu_\sigma,$$

which is zero if and only if $e^{2f}\sigma$ is of constant curvature. The evolution equation of $\tilde{m}(\Sigma_t)$ is

$$\frac{d}{dt}\tilde{m}(\Sigma_t) = |\Sigma_t|\int_{\Sigma_t}\frac{|\nabla H|^2}{H^2}d\mu_t,$$

which is still monotone but is not in favor of our proof. Fortunately, we can estimate the growth rate of $\tilde{m}(\Sigma_t)$ directly and then we are able to construct our examples. We also remark that in higher dimensions, the naive generalization

$$Q(\Sigma^{n-1}) := |\Sigma|^{-\frac{n-5}{n-1}}\int_\Sigma |\mathring{A}|^2 d\mu$$

plays the same role as $\tilde{m}(\Sigma)$ and Theorem 1 holds for $\mathbb{H}^n,\ n \geq 3$.

References

[1] C. Gerhardt, *Flow of nonconvex hypersufaces into spheres*, J. Differential Geom. **32** (1990), 299–314

[2] R. Geroch, *Energy extraction*, Ann. New York Acad. Sci. **224** (1973), 108–117.

[3] G. Huisken and T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom. **59** (2001), 353–437

[4] P.K. Hung and M.T. Wang, *Inverse mean curvature flows in the hyperbolic 3-space revisited*, Calc. Var. Partial Differ. Equ. **54** (2015), 119–126

[5] P.S. Jang and R.M. Wald, *The Positive Energy Conjecture and the Cosmic Censor Hypothesis*, J. Math. Phys. **18** (1977), 41–44.

[6] A. Neves, *Insufficient convergence of inverse mean curvature flow on asymptotically hyperbolic manifolds*, J. Differential Geom. **84** (2010), 191–229

Helicoid-Like Minimal Surfaces

Brian White

Consider a simple closed curve Γ_n in the boundary of a solid cylinder $\mathbf{B} \times \mathbf{I} \subset \mathbf{R}^3$ consisting of:

(1) a diameter at the top,
(2) a (non-parallel) diameter at the bottom,
(3) two helical curves, each winding around approximately n times.

What embedded minimal disks does Γ_n bound?

Of course this is the classical Plateau Problem for the curve Γ_n. I was led to these particular curves in an effort to understand the Colding-Minicozzi theory and, indeed, understanding the Plateau Problem for these curves has led to interesting examples illustrating that theory.

Note that the curve Γ_n bounds a portion H_n of a helicoid (Figure 1). That helicoidal surface H_n is the *unique* minimal disk with boundary Γ_n that has 180° symmetry about Z. For n large, H_n is *not* area minimizing, because

$$\operatorname{area}(H_n) \to \infty,$$

whereas the least area surface has area less than 1/2 the area of the boundary of $\mathbf{B} \times \mathbf{I}$.

(Indeed, for large n, a minimizing disk is a slight perturbation of one of the two components of $\partial(\mathbf{B} \times \mathbf{I}) \setminus \Gamma_n$. That is, it looks like a horizontal half disk at the top, a ribbon (near $(\partial\mathbf{B}) \times \mathbf{I}$) winding from top to bottom, and a horizontal half disk at the bottom.)

Thus Γ_n bounds at least 3 minimal disks: H_n and two area minimizing disks.

In fact, Γ_n bounds *many* embedded minimal disks:

Theorem. *The curve* Γ_n *bounds disks of index* 0, 1, 2, 3,..., $\operatorname{index}(H_n)$, *and*

$$\operatorname{index}(H_n) \geq n + O(1).$$

Thus the number of surfaces grows *at least* like $2n$.

Conjecture. *The number of minimal disks grows at least like* 2^n.

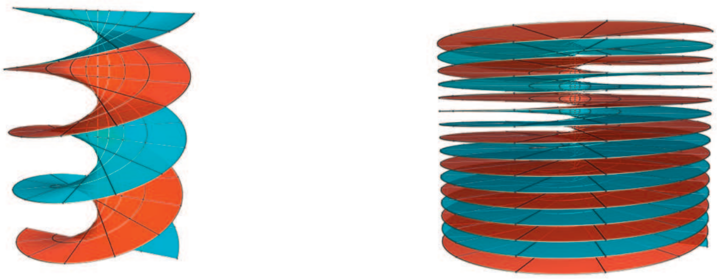
1. A surprising dichotomy

Consider a minimal embedded disk D with boundary Γ_n (where n is large.) I claim that either D is very similar to a helicoid, or D is very different from a helicoid.

Here "similar to a helicoid" means "nearly horizontal away from Z". Figure 1 illustrates that for large n, the helicoidal surface H_n is very nearly horizontal once one moves a little away from the axis.

Definition. *If* $M \subset \mathbf{R}^3$, $\operatorname{slope}(M)$ *denotes the* sup *of the slope of* $\operatorname{Tan}(M, x)$ *(with respect to the horizontal) for* $x \in M$.

Thus if M contains any point with a vertical tangent plane, then $\operatorname{slope}(M) = \infty$ (even if much of M is nearly horizontal). If $\operatorname{slope}(M) \leq \epsilon$, then all tangent planes to M have slope at most ϵ.



n=4 n=16

Figure 1

Theorem (Dichotomy Theorem, weak version)**.** *Let $0 < r < 1$. Let D_n be a minimal embedded disk with boundary Γ_n. After passing to a subsequence, either*

$$\text{slope}(D_n \setminus Z(r)) \to 0 \qquad \textit{(the "good", or helicoidal, case)}$$

or

$$\text{slope}(D_n \setminus Z(r)) \to \infty \qquad \textit{(the "bad", or non-helicoidal, case).}$$

Theorem (Dichotomy Theorem, Strong version)**.** *Let $0 < \epsilon < r < 1$. (One should think of ϵ as very small and of r as very close to 1.) Let $C < \infty$. Suppose D_n is a minimal embedded disk with boundary Γ_n. If*

$$\text{slope}(D_n \setminus Z(r)) \leq C < \infty,$$

then

$$\text{slope}(D_n \setminus Z(\epsilon)) \to 0.$$

Thus if one has even a little control (slopes bounded by C) near the boundary of the cylinder $\mathbf{B} \times \mathbf{R}$, then one gets extremely good control (slopes bounded by ϵ) everywhere except in a small neighborhood of the axis.

2. Remarks

(1) In the Dichotomy Theorem, the curved portions of Γ_n need not be helices: it is enough that
$$\text{slope}(\Gamma_n) \to 0.$$
(2) The Dichotomy Theorem (for curves Γ_n as in (1)) becomes *false* if upper and lower diameters are parallel.
(3) The Riemannian metric on $\mathbf{B} \times \mathbf{I}$ need not be Euclidean: we just need the horizontal disks $z =$ constant to be minimal and to be perpendicular to Z. (In this case the hypothesis that the upper and lower diameters be non-parallel should be replaced by the hypothesis that that the z-axis is the only curve joining the upper and lower diameters that is perpendicular to every slice $z =$ constant.)

3. Existence

For a general Riemannian metric as above, do good disks and bad disks exist (for n large)? For bad disks, the answer is yes. For example, the area minimizing disk is bad (since good disks have huge area). For good disks, the answer is also yes, but it is much more subtle.

Theorem (Existence Theorem)**.** *Fix a small $\epsilon > 0$. For all sufficiently large n, there exist D with $\partial D = \Gamma_n$ such that*

$$\text{slope}(D \setminus Z(\epsilon)) < \epsilon.$$

Indeed, the number of such D (suitably counted) is odd.

Suitably counted means counted with an appropriate multiplicity; the situation is analogous to the statement that every degree n polynomial has n complex roots, if one counts roots suitably.

The proof of the existence theorem is by the continuity or degree-theoretic method, as pioneered by Tomi and Tromba in minimal surface theory. In the Euclidian case, the Existence Theorem is easy: Γ_n bounds the good surface H_n, and all other minimal disks it bounds occur in pairs (by the 180° symmetry about Z.) As we deform the metric from Euclidean to the metric we want, various bifurcations can happen. For example, two surfaces can come together and annihilate each other. Note this does not change the mod 2 number of surfaces. In such a pair annihilation, it is impossible for one of the surfaces to be good and the other bad, since good and bad surfaces are far apart from each other by the Dichotomy Theorem. Thus in fact the mod 2 number of *good* surfaces does not change during the deformation.

4. Application to Colding-Minicozzi Theory

Let U be an open ball in $\mathbf{R}^3$ or, more generally, a open subset of a Riemannian 3 manifold such that is homeomorphic to a ball. Consider a foliation $\mathcal{F}$ of U by properly embedded minimal disks.

(Examples of such foliations are easy to make. For example, consider any solution $f : D \subset \mathbf{R}^2 \to \mathbf{R}$ of the minimal surface equation, and let $\mathcal{F}$ be of the graph of $\mathcal{F}$ together with the vertical translates of that graph.)

Let C be an orthogonal trajectory to the foliation. We may assume that C intersects every leaf; otherwise replace $\mathcal{F}$ by the set of leaves that intersect C and U by the union of those leaves.

Theorem (Examples Theorem)**.** *There exist minimal embedded disks $D_n \subset U_n$ where $U_1 \subset U_2 \subset \dots$ is an exhaustion of U by open subsets such that the curvatures of the D_n blow up at every point p of C (i.e., for every $p \in C$, there are $p_n \in D_n$ converging to C with $|A(D_n, p_n)| \to \infty$) and such that the D_n converge smoothly in compact subset of $U \setminus C$ to the foliation $\mathcal{F}$.*

This theorem is an easy consequence of the Existence Theorem. (If the leaves of the foliation are flat disks in Euclidean $\mathbf{R}^3$, Meeks and Weber proved this by a very different method.)

Acknlowledgement: The research was supported by NSF grants DMS–1105330 and DMS–1404282.

Interaction energy of domain walls of logarithmically decaying tails in a nonlocal variational model

Radu Ignat

(joint work with Roger Moser)

The model. For $\alpha \in (0, \pi)$, consider maps $m = (m_1, m_2) : (-1, 1) \to \mathbb{S}^1$ with

$$m_1(-1) = m_1(1) = \cos\alpha. \tag{1}$$

For $\varepsilon > 0$, consider the energy

$$E_\varepsilon(m) = \varepsilon \int_{-1}^{1} |m'|^2 \, dx_1 + \int_{\mathbb{R}^2_+} |\nabla u|^2 \, dx,$$

where $u : \mathbb{R}^2_+ \to \mathbb{R}$ is determined (up to a constant) by the boundary value problem

$$\begin{aligned} \Delta u &= 0 && \text{in } \mathbb{R}^2_+, \\ \frac{u}{x_2} &= -m_1' && \text{on } \mathbb{R} \times \{0\}, \end{aligned}$$

where m_1 is extended by $\cos\alpha$ outside of $(-1, 1)$ and $x = (x_1, x_2)$. The energy E_ε can be written as a strictly convex functional in m_1:

$$E_\varepsilon(m) = \varepsilon \int_{-1}^{1} \frac{(m_1')^2}{1 - m_1^2} \, dx_1 + \|m_1\|^2_{\dot{H}^{1/2}(\mathbb{R})}.$$

This represents a simplified version of the free energy of a magnetisation vector field m in a thin film of a ferromagnetic material (for more details on the model, see e.g. [3]) and u is called the stray field potential.

Néel walls. We are interested in transition layers corresponding to rotations between $(\cos\alpha, \pm\sin\alpha)$ and $(\cos\alpha, \mp\sin\alpha)$ on the unit circle $\mathbb{S}^1$. Such a transition is called Néel wall and is typically a two-length scale object (a core and two logarithmically decaying tails) with an energy E_ε of order $\pi\gamma_\pm^2/|\log\varepsilon|$ as $\varepsilon \to 0$ (see [5]). Here, $\gamma_\pm = \pm 1 - \cos\alpha$ stands for the height of the transition in m_1 when m_1 passes through ± 1.

We are particularly interested in the interaction of several transitions (see Figure 1). For fixed $-1 < a_1 < \cdots < a_N < 1$ and $d_n \in \{\pm 1\}$, $n = 1, \ldots, N$, set

$$M(a, d) = \Big\{ m : (-1, 1) \to \mathbb{S}^1 \text{ with (1) and } m_1(a_n) = d_n \text{ for } 1 \le n \le N \Big\}.$$

Note that minimizers of E_ε over $M(a, d)$ exist and have a unique component m_1 that is smooth away from the positions a_n, $1 \le n \le N$.

Main result. We estimate the minimal energy E_ε required for a profile in $M(a, d)$.

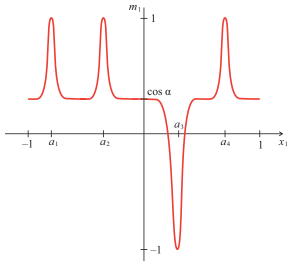


FIGURE 1. Several Néel walls of positions a_n, $1 \le n \le 4$.

Theorem 1 (Ignat–Moser [4])**.** *As $\varepsilon \to 0$, we have*

$$\inf_{M(a,d)} E_\varepsilon = \pi \sum_{n=1}^{N} \frac{\gamma_n^2}{\log \frac{1}{\delta}} + \frac{W(a,d)}{\left(\log \frac{1}{\delta}\right)^2} + o\left(\frac{1}{\left(\log \frac{1}{\delta}\right)^2}\right)$$

where $\delta = \varepsilon |\log \varepsilon|$, $\gamma_n = d_n - \cos\alpha$ and

$$W(a,d) = \sum_{n=1}^{N} \left(e(d_n) - \pi\gamma_n^2 \log(2 - 2a_n^2)\right) - \pi \sum_{n=1}^{N} \sum_{k \ne n} \gamma_k \gamma_n \log\left(\frac{1 + \sqrt{1 - \rho(a_k, a_n)}}{\rho(a_k, a_n)}\right)$$

where $e(\pm 1) > 0$ and $\rho(a_k, a_n) = \frac{|a_k - a_n|}{1 - a_k a_n}$.

In analogy to the theory of Ginzburg-Landau vortices (see [1]), we call $W(a,d)$ the renormalised energy for the N walls placed at $a = (a_1, \ldots a_N)$ with signs $d = (d_1, \ldots, d_N)$. As the theorem shows, $W(a,d)$ represents the next-to-leading order term in the expansion of $\inf_{M(a,d)} E_\varepsilon$ in $1/|\log \delta|$. This is an improvement of the result in [2] giving only the first leading order term of E_ε.

We now briefly discuss how the above expression comes about. Suppose that for a given $a \in A_N$, we study minimisers m of E_ε in $M(a,d)$. When ε is small, we expect to have a typical Néel wall profile near each of the points $a_1, \ldots, a_N$ with the prescribed signs $d_1, \ldots, d_N$, and the full transition layer m is essentially a superposition of all of these. We can think of a Néel wall as consisting of two parts: a small core around a_n and two logarithmically decaying tails. In our situation, the walls are confined in the relatively short interval $(-1,1)$ and each tail will interact with the other walls and with the boundary as well. We can then account for the full energy $\inf_{M(a,d)} E_\varepsilon$ (at leading and next-to-leading order) as follows.

Core energy. The core of each wall requires a certain amount of energy, namely $\frac{e(\pm 1)}{\left(\log \frac{1}{\delta}\right)^2}$ for a positive and a negative wall, respectively. The constants $e(\pm 1)$ represent the rescaled energy of the core profile as $\varepsilon \to 0$. This is the only term where we have a contribution from the Dirichlet integral of m and it appears only at next-to-leading order in the full energy. All the remaining terms below come from the stray field energy alone.

Tail energy. The two tails of the wall at a_n give rise to the energy $\frac{\pi\gamma_n^2}{\log\frac{1}{\delta}}$. This is the leading order term of the full energy.

Tail-boundary interaction. Moving a wall relative to the boundary points ± 1 will deform the tail profile, resulting in a change of the energy. This phenomenon gives rise to the energy $\frac{\pi\gamma_n^2 \log(2-2a_n^2)}{\left(\log\frac{1}{\delta}\right)^2}$ for the wall at a_n. (The sign here is not a mistake; it is the opposite of the sign of the corresponding expression in Theorem 1.) This means that the tails are attracted by the boundary, in the sense that the energy decreases if a_n approaches ± 1.

Tail-tail interaction. There is an energy contribution coming from reinforcement or cancellation between the stray fields generated by different walls. For the walls at a_k and a_n with $k \neq n$, this amounts to

$$\frac{\pi\gamma_k\gamma_n}{\left(\log\frac{1}{\delta}\right)^2}\log\left(\frac{1+\sqrt{1-\varrho(a_k,a_n)^2}}{\varrho(a_k,a_n)}\right).$$

(Again we have the opposite sign relative to the above theorem.) A conclusion is that the tails of two walls attract each other if they have opposite signs and repel each other if they have the same sign.

Tail-core interaction. Since the profile of a Néel wall decays only logarithmically, it will change the turning angle of the neighbouring walls slightly. This has an effect on the energy as well (at the next-to-leading order). Indeed, the tail of the wall at a_k and the core of the wall at a_n with $k \neq n$ lead to a contribution of

$$-\frac{2\pi\gamma_k\gamma_n}{\left(\log\frac{1}{\delta}\right)^2}\log\left(\frac{1+\sqrt{1-\varrho(a_k,a_n)^2}}{\varrho(a_k,a_n)}\right).$$

We also have an interaction between the two tails of a wall and its own core: if $k = n$, then we obtain the energy $-\frac{2\pi\gamma_n^2 \log(2-2a_n^2)}{\left(\log\frac{1}{\delta}\right)^2}$. This is twice the size of the terms from the tail-boundary interaction and tail-tail interaction, but with the opposite signs, resulting in a net repulsion between walls of opposite signs and a net attraction between walls of the same sign. Furthermore, we have a net repulsion of the walls by the boundary.

Notwithstanding the term ‘energy’ used in this description, strictly speaking, these are energy differences and therefore some of them may be negative. All except one of these contributions occur similarly in the theory of Ginzburg-Landau vortices. The core-tail interaction, on the other hand, is new and more delicate to handle.

References

[1] F. Béthuel, H. Brezis, F. Hélein, *Ginzburg-Landau vortices*, Birkhäuser Boston Inc., 1994.

[2] A. DeSimone, R. V. Kohn, S. Müller and F. Otto, *Repulsive interaction of Néel walls, and the internal length scale of the cross-tie wall*, Multiscale Model. Simul. **1** (2003), 57–104.

[3] A. DeSimone, R. V. Kohn, S. Müller and F. Otto, *Recent analytical developments in micromagnetics*, The Science of Hysteresis , Vol. 2, 269–381, Elsevier Academic Press 2005.

[4] R. Ignat and R. Moser, *Interaction energy of domain walls in a nonlocal Ginzburg-Landau type model from micromagnetics*, Arch. Ration. Mech. Anal. **221** (2016), 419-485.
[5] C. Melcher, *The logarithmic tail of Néel walls*, Arch. Ration. Mech. Anal. **168** (2003), 83–113.

Ricci curvature and martingales

Robert Haslhofer
(joint work with Aaron Naber)

The main goal of this talk, based on [1], is to explain how bounded Ricci curvature can be understood by analyzing the evolution of martingales on path space, generalizing the well known and important principles of how lower bounds on Ricci curvature can be understood by analyzing the heat flow.

To put things into context, let us recall that the starting point for most of the analysis on spaces with Ricci curvature bounded below, say by a constant $-\kappa$, is the classical Bochner inequality

$$\tfrac{1}{2}\Delta|\nabla u|^2 \geq \langle \nabla\Delta u, \nabla u\rangle + |\nabla^2 u|^2 - \kappa|\nabla u|^2 . \tag{1}$$

Using the Bochner inequality it is a simply exercise to show that Ricci bounded below by $-\kappa$ is equivalent to several other geometric-analytic estimates, e.g. the following sharp gradient estimate for the heat flow

$$|\nabla H_t u| \leq e^{\frac{\kappa}{2}t} H_t|\nabla u| . \tag{2}$$

In contrast to the well developed theory of Ricci curvature bounded below, until recently there was no characterization available at all for spaces with bounded Ricci curvature. This characterization problem has been solved recently by Naber [3]. The key insight was that to understand two-sided bounds for Ricci curvature, and not just lower bounds, one should do analysis on path space PM, instead of analysis on M. By definition, given a complete Riemannian manifold M, its path space $PM = C([0,\infty), M)$ is the space of continuous curves in M. Path space comes equipped with a family of natural probability measures, the Wiener measure Γ_x of Brownian motion starting at $x \in M$. Path space also comes equipped with a natural one parameter family of gradients, the t-parallel gradients $\nabla_t^{||}$ $(t \geq 0)$. Using this framework, it was proved in [3] that the Ricci curvature of M is bounded by a constant κ if and only if the sharp gradient estimate

$$\left|\nabla_x \int_{PM} F\, d\Gamma_x\right| \leq \int_{PM} \left(|\nabla_0^{||}F| + \int_0^\infty \frac{\kappa}{2} e^{\kappa t/2}|\nabla_t^{||}F|\, dt\right) d\Gamma_x \tag{3}$$

holds for all test functions $F : PM \to \mathbb{R}$. In the simplest case of one-point test functions, i.e. functions of the form $F(\gamma) = u(\gamma(t))$ where $u : M \to \mathbb{R}$ and t is fixed, the infinite dimensional gradient estimate (3) reduces to the finite dimensional gradient estimate (2). The gradient estimate (3) can be used to define a weak notion of Ricci curvature for metric measure spaces.

While [3] gives a way to generalize certain estimates for lower Ricci curvature on M to estimates for bounded Ricci curvature on PM, e.g. the finite dimensional gradient estimate (2) to the infinite dimensional gradient estimate (3), what hasn't been answered yet is the following question:

Is there any way to generalize the Bochner inequality (1) *from M to PM?*

This question has been the guiding principle for the present work. Given that the Bochner formula is the starting point for most of the theory of lower Ricci, such a generalization is clearly valuable for the theory of bounded Ricci curvature.

The first main point we wish to explain is that martingales on PM are the correct generalization of the heat flow on M. Recall that a *martingale* on P_xM is a Σ_t-adapted integrable stochastic process $F_t : P_xM \to \mathbb{R}$ such that

$$F_{t_1} = E_x[F_{t_2} \,|\, \Sigma_{t_1}] \qquad (t_1 \leq t_2). \tag{4}$$

Here, the right hand side denotes the conditional expectation value on P_xM given the σ-algebra Σ_{t_1}, of events which are observable until time t_1. The simplest examples of martingales on path space have the form

$$F_t(\gamma) = \begin{cases} H_{T-t}u(\gamma(t)), & \text{if } t < T \\ u(\gamma(T)), & \text{if } t \geq T, \end{cases} \tag{5}$$

where $u : M \to \mathbb{R}$ and T are fixed, and thus are indeed given by the (backwards) heat flow on M.

We found that the correct generalization of the Bochner formula (1) on M is given by a certain evolution equation for martingales on PM. To get there, we start with by reformulating the martingale representation theorem and the Clark-Ocone formula in the form

$$dF_t = \langle \nabla^{||}_t F_t, dW_t \rangle \,. \tag{6}$$

Expressed this way, we can view the martingale equation as an evolution equation on path space. We then proceed by computing various evolution equations for associated quantities on path space. In particular, if $F_t : P_xM \to \mathbb{R}$ is a martingale on path space, and $s \in \mathbb{R}$ is fixed, then its s-parallel gradient $\nabla^{||}_s F_t : P_xM \to T_xM$ satisfies the stochastic equation

$$d\nabla^{||}_s F_t = \langle \nabla^{||}_t \nabla^{||}_s F_t, dW_t \rangle + \frac{1}{2}\mathrm{Ric}_t(\nabla^{||}_t F_t)\, dt + \nabla^{||}_s F_s \,\delta_s(t) dt \,, \tag{7}$$

where $\langle \mathrm{Ric}_t(X), Y \rangle = \mathrm{Ric}(P_t^{-1}X, P_t^{-1}Y)$ and $P_t = P_t(\gamma) : T_{\gamma(t)}M \to T_xM$ is stochastic parallel transport. Combining this with the Ito formula we obtain

$$\begin{aligned} d|\nabla^{||}_s F_t|^2 &= \langle \nabla^{||}_t |\nabla^{||}_s F_t|^2, dW_t \rangle \\ &\quad + |\nabla^{||}_t \nabla^{||}_s F_t|^2 dt + \mathrm{Ric}_t\big(\nabla^{||}_t F_t, \nabla^{||}_s F_t\big)\, dt + |\nabla^{||}_s F_s|^2 \delta_s(t) dt, \end{aligned} \tag{8}$$

which is the correct generalization of the Bochner formula to path space.

We will now discuss four applications of our calculus on path space. First, it yields a shorter proof of the characterizations from [3]. For illustration, if $\mathrm{Ric} = 0$

then by (7) the process $t \mapsto |\nabla_s^{\|} F_t|$ is a submartingale. Thus, by the very definition of a submartingale we get

$$|\nabla_s^{\|} F_t| \leq E_x \left[|\nabla_s^{\|} F_T| \,\middle|\, \Sigma_t \right] \qquad (t \leq T). \tag{9}$$

Taking the limit $T \to \infty$, and specializing to $s = t = 0$, this implies the $\kappa = 0$ case of the infinite dimensional gradient estimate (3):

$$\left| \nabla_x \int_{PM} F \, d\Gamma_x \right| \leq \int_{PM} |\nabla_0^{\|} F| \, d\Gamma_x. \tag{10}$$

Other characterizations, and estimates for $\kappa \neq 0$, can be proven with similar ease.

Second, the gradient estimate (3) can be strengthened to the family of estimates

$$|\nabla_s^{\|} F_t| \leq E \left[|\nabla_s^{\|} F| + \frac{\kappa}{2} \int_t^\infty e^{\frac{\kappa}{2}(r-t)} |\nabla_r^{\|} F| \, dr \,\middle|\, \Sigma_t \right]. \tag{11}$$

Third, we obtain new characterizations of bounded Ricci curvature. In particular, $|\mathrm{Ric}| \leq \kappa$ is equivalent to the full Bochner inequality on path space

$$\begin{aligned} d|\nabla_s^{\|} F_t|^2 \geq{}& \langle \nabla_t^{\|} |\nabla_s^{\|} F_t|^2, dW_t \rangle \\ &+ |\nabla_t^{\|} \nabla_s^{\|} F_t|^2 dt - \kappa |\nabla_t^{\|} F_t| |\nabla_s^{\|} F_t| \, dt + |\nabla_s^{\|} F_s|^2 \delta_s(t) dt, \end{aligned} \tag{12}$$

as well as the weak Bochner inequality on path space

$$d|\nabla_s^{\|} F_t| \geq \langle \nabla_t^{\|} |\nabla_s^{\|} F_t|, dW_t \rangle \; - \tfrac{\kappa}{2} |\nabla_t^{\|} F_t| \, dt. \tag{13}$$

Forth, we obtain new Hessian estimates for martingales on the path space of manifolds with bounded Ricci curvature, e.g.

$$\begin{aligned} \int_{PM} |\nabla_s^{\|} F_s|^2 \, d\Gamma_x &+ \int_0^T \int_{PM} |\nabla_t^{\|} \nabla_s^{\|} F_t|^2 \, d\Gamma_x \, dt \\ &\leq e^{\frac{\kappa}{2}(T-s)} \int_{PM} \left(|\nabla_s^{\|} F|^2 + \frac{\kappa}{2} \int_s^T e^{\frac{\kappa}{2}(t-s)} |\nabla_t^{\|} F|^2 \, dt \right) d\Gamma_x. \end{aligned} \tag{14}$$

Combined with Doob's inequality this generalizes the classical $L^\infty H^1 \cap L^2 H^2$ estimate for the heat flow on M.

The methods can also be adapted to the time-dependent setting, and thus also provide a useful tool for the study of Ricci flow in the framework of [2].

References

[1] R. Haslhofer and A. Naber, *Ricci curvature and martingales*, in preparation.
[2] R. Haslhofer and A. Naber, *Characterizations of the Ricci flow*, JEMS (to appear).
[3] A. Naber, *Characterizations of bounded Ricci curvature on smooth and nonsmooth spaces.*

Symmetry by flow

Jean Dolbeault

(joint work with Maria J. Esteban, Michael Loss and Matteo Muratori)

With the norms $\|w\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \big(\int_{\mathbb{R}^d} |w|^q\,|x|^{-\gamma}\,dx\big)^{1/q}$, let us consider the family of *Caffarelli-Kohn-Nirenberg inequalities* introduced in [2] and given by

$$\|w\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \le \mathsf{C}_{\beta,\gamma,p}\,\|\nabla w\|^{\vartheta}_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)}\,\|w\|^{1-\vartheta}_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)} \tag{1}$$

in a suitable functional space $\mathrm{H}^p_{\beta,\gamma}(\mathbb{R}^d)$ obtained by completion of smooth functions with support in $\mathbb{R}^d\setminus\{0\}$, w.r.t. the norm given by $\|w\|^2 := (p_\star-p)\,\|w\|^2_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)} + \|\nabla w\|^2_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)}$. Here $\mathsf{C}_{\beta,\gamma,p}$ denotes the optimal constant, the parameters β, γ and p are subject to the restrictions

(2)

$$d\ge 2\,,\quad \gamma-2<\beta<\frac{d-2}{d}\,\gamma\,,\quad \gamma\in(-\infty,d)\,,\quad p\in(1,p_\star]\quad \text{with } p_\star:=\frac{d-\gamma}{d-\beta-2}$$

and the exponent $\vartheta=\frac{(d-\gamma)\,(p-1)}{p\,(d+\beta+2-2\,\gamma-p\,(d-\beta-2))}$ is determined by the scaling invariance.

Equality in (1) is achieved by Aubin-Talenti type functions

$$w_\star(x)=\big(1+|x|^{2+\beta-\gamma}\big)^{-1/(p-1)}\quad \forall\,x\in\mathbb{R}^d$$

if we know that *symmetry* holds, that is, if we know that the equality is achieved among radial functions. However, depending on the parameters, to decide whether a minimizer has the full symmetry or not can be difficult. To show that symmetry is broken one can minimize the functional in the *class of symmetric functions* and then check whether the value of the functional can be lowered by perturbing the minimizer away from the symmetric situation. This is the method that has been used to establish that *symmetry breaking* occurs in (1) if

$$\gamma<0 \quad\text{and}\quad \beta_{\mathrm{FS}}(\gamma)<\beta<\frac{d-2}{d}\,\gamma \tag{3}$$

where

$$\beta_{\mathrm{FS}}(\gamma):=d-2-\sqrt{(\gamma-d)^2-4\,(d-1)}\,.$$

In the critical case $p=p_\star$, the method was implemented by F. Catrina and Z.-Q. Wang in [3], and the sharp result has been obtained by V. Felli and M. Schneider in [6]. The same condition was recently obtained in the subcritical case $p<p_\star$, in [1]. Throughout this report, by *critical* we simply mean that $\|w\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)}$ scales like $\|\nabla w\|_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)}$. One has to observe that proving symmetry breaking by establishing the linear instability is a *local* method, which is based on a painful but rather straightforward linearization around the special function $w_\star$.

A real difficulty occurs when the minimizer in the symmetric class is stable, *i.e.*, all local perturbations that break the symmetry increase the energy: in our case, non-radial perturbations. To establish the optimal symmetry range in (1), and thus determine the sharp constant in the Caffarelli-Kohn-Nirenberg inequalities whenever the optimal functions are radially symmetric, a new method had to be designed. What has been proved in [4] in the critical case $p = p_\star$, and extended in [5] to the sub-critical case $1 < p < p_\star$, is that the symmetry breaking range given in (3) is optimal: symmetry holds in the complementary region of the admissible parameters.

Theorem 1. [4, 5] *Under Condition* (2) *assume that either* $\gamma \ge 0$ *or* $\beta \le \beta_{\rm FS}(\gamma)$ *if* $\gamma < 0$. *Assume that* $d \ge 2$. *Then all positive solutions in* $\mathrm{H}^p_{\beta,\gamma}(\mathbb{R}^d)$ *to*

$$-\,div\left(|x|^{-\beta}\,\nabla w\right) = |x|^{-\gamma}\left(w^{2p-1} - w^p\right) \quad in \quad \mathbb{R}^d\setminus\{0\}\,. \tag{4}$$

are radially symmetric and, up to a scaling and a multiplication by a constant, equal to $w_\star$.

The main ideas of the proof can be summarized into a three steps scheme.

1) The first step is based on a change of variables which amounts to rephrase our problem in a space of higher, *artificial dimension* $n > d$ (here n is a dimension at least from the point of view of the scaling properties), or to be precise to consider a weight $|x|^{n-d}$ which is the same in all norms. With

$$\alpha = 1 + \tfrac{\beta-\gamma}{2} \quad \text{and} \quad n = 2\,\tfrac{d-\gamma}{\beta+2-\gamma}\,,$$

we claim that Inequality (1) can be rewritten for a function $v(|x|^{\alpha-1}\,x) = w(x)$ as

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)} \le \mathsf{K}_{\alpha,n,p}\,\|\mathsf{D}_\alpha v\|^{\vartheta}_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\,\|v\|^{1-\vartheta}_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)} \quad \forall\, v \in \mathrm{H}^p_{d-n,d-n}(\mathbb{R}^d)\,,$$

with the notations $s = |x|$, $\omega = \frac{x}{s}$ and $\mathsf{D}_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s}, \frac{1}{s}\,\nabla_\omega v\right)$.

2) Let us consider the derivative of a generalized *Rényi entropy power* functional

$$\mathcal{G}[u] := \left(\int_{\mathbb{R}^d} u^m\,d\mu\right)^{\sigma-1}\int_{\mathbb{R}^d} u\,|\mathsf{D}_\alpha \mathsf{P}|^2\,d\mu$$

where $\sigma = \frac{2}{d}\,\frac{1}{1-m} - 1$. Here P is the *pressure* variable $\mathsf{P} := \frac{m}{1-m}\,u^{m-1}$ while m and p are related by $p = \frac{1}{2\,m-1}$. Next we consider the fast diffusion equation

$$\frac{\partial u}{\partial t} = \mathcal{L}_\alpha\,u^m \quad \text{with} \quad \mathcal{L}_\alpha\,u = -\,\mathsf{D}_\alpha^*\,\mathsf{D}_\alpha\,u = \alpha^2\left(u'' + \tfrac{n-1}{s}\,u'\right) + \tfrac{1}{s^2}\,\Delta_\omega\,u \tag{5}$$

in the subcritical range $1 - 1/n < m < 1$ and in the critical case $m = 1 - 1/n$. The key computation is the proof that

$$\begin{aligned}
&-\frac{d}{dt}\,\mathcal{G}[u(t,\cdot)]\left(\int_{\mathbb{R}^d} u^m\,d\mu\right)^{1-\sigma}\\
&\ge (1-m)\,(\sigma-1)\textstyle\int_{\mathbb{R}^d} u^m\left|\mathcal{L}_\alpha\,\mathsf{P} - \frac{\int_{\mathbb{R}^d} u\,|\mathsf{D}_\alpha \mathsf{P}|^2\,d\mu}{\int_{\mathbb{R}^d} u^m\,d\mu}\right|^2 d\mu\\
&\quad + 2\textstyle\int_{\mathbb{R}^d}\left(\alpha^4\left(1-\frac{1}{n}\right)\left|\mathsf{P}'' - \frac{\mathsf{P}'}{s} - \frac{\Delta_\omega\,\mathsf{P}}{\alpha^2\,(n-1)\,s^2}\right|^2 + \frac{2\,\alpha^2}{s^2}\left|\nabla_\omega \mathsf{P}' - \frac{\nabla_\omega \mathsf{P}}{s}\right|^2\right) u^m\,d\mu\\
&\quad + 2\textstyle\int_{\mathbb{R}^d}\left((n-2)\left(\alpha_{\rm FS}^2 - \alpha^2\right)|\nabla_\omega \mathsf{P}|^2 + c(n,m,d)\,\frac{|\nabla_\omega \mathsf{P}|^4}{\mathsf{P}^2}\right) u^m\,d\mu =: \mathcal{H}[u]
\end{aligned}$$

for some numerical constant $c(n,m,d)>0$. Hence if $\alpha \le \alpha_{\rm FS} := \sqrt{(d-1)/(n-1)}$, the r.h.s. in $\mathcal{H}[u]$ vanishes if and only if P is an affine function of $|x|^2$.

3) This method has a hidden difficulty. In the above computation, many integrations by parts have to be performed, which require a sufficient decay of the function u and of its derivatives as $|x| \to +\infty$ and also, because of the weight, good properties as $x \to 0$. So far, such properties are not known for a general solution of (5). However, we may consider a positive solution to (4) and, up to the above changes of variables, take the corresponding function u as an initial datum for (5). On the one hand, since u is a critical point of $\mathcal{G}$ under mass constraint, we know that $\frac{d}{dt}\,\mathcal{G}[u(t,\cdot)]=0$ at $t=0$. On the other hand, because u solves an elliptic PDE, it is possible to establish all regularity and decay estimates that are needed to do the integrations by parts, hence $\mathcal{H}[u]=0$. In that way we conclude that w is equal to $w_\star$ up to a scaling and a multiplication by a constant, if $\beta \le \beta_{\rm FS}(\gamma)$.

Applying the flow at $t=0$ to a critical point amounts to write the Euler-Lagrange equation and test it with $\mathcal{L}_\alpha\, u^m$. In other words, what we write is

$$0=\int_{\mathbb{R}^d} \mathrm{d}\mathcal{G}[u]\cdot\mathcal{L}_\alpha\, u^m\, d\mu \ge \mathcal{H}[u] \ge 0$$

where the last inequality holds because $\mathcal{H}[u]$ is the integral of a sum of squares (with nonnegative constants in front of each term). If we undo the change of variables, our method amounts to rewrite (4) as

$$\frac{(p-1)^2}{p\,(p+1)}\, w^{1-3p}\, \mathrm{div}\left(|x|^{-\beta}\, w^{2p}\, \nabla w^{1-p}\right) + |\nabla w^{1-p}|^2 + |x|^{-\gamma}\left(c_1\, w^{1-p} - c_2\right) = 0$$

for some constants c_1, c_2 and test it against $|x|^\gamma\, \mathrm{div}\left(|x|^{-\beta}\, \nabla w^{1+p}\right)$.

References

[1] M. Bonforte, J. Dolbeault, M. Muratori and B. Nazaret, *Weighted fast diffusion equations (Part I): Sharp asymptotic rates without symmetry and symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities.* hal-01279326 & arxiv:1602.08315, February 2016.

[2] L. Caffarelli, R. Kohn and L. Nirenberg, *First order interpolation inequalities with weights*, Compositio Math., 53 (1984), pp. 259–275.

[3] F. Catrina and Z.-Q. Wang, *On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions*, Comm. Pure Appl. Math., 54 (2001), pp. 229–258.

[4] J. Dolbeault, M. J. Esteban and M. Loss, *Rigidity versus symmetry breaking via nonlinear flows on cylinders and Euclidean spaces*, to appear in Inventiones Mathematicae, Springer Link (2016).

[5] J. Dolbeault, M. J. Esteban, M. Loss and M. Muratori, *Symmetry for extremal functions in subcritical Caffarelli-Kohn-Nirenberg inequalities.* hal-01318727 & arxiv:1605.06373, May 2016.

[6] V. Felli and M. Schneider, *A note on regularity of solutions to degenerate elliptic equations of Caffarelli-Kohn-Nirenberg type*, Adv. Nonlinear Stud., 3 (2003), pp. 431–443.

On the structure of $\mathscr{A}$-free measures and applications.

Guido De Philippis
(joint work with Filip Rindler)

Let us consider the following problem: Let $\mathscr{A}$ be a k'th-order linear constant-coefficient PDE operator acting on $\mathbb{R}^m$-valued functions:

$$\mathscr{A}u = \sum_{|\alpha|\le k} A_\alpha \partial^\alpha u \qquad \text{for all } u \in C^\infty(\Omega;\mathbb{R}^m)$$

where $A_\alpha \in \mathbb{R}^{n\times m}$ are matrices and $\partial^\alpha = \partial_1^{\alpha_1}\dots\partial_d^{\alpha_d}$, for every multindex $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}^d$.

Question 1. *Let $\mu \in \mathcal{M}(\Omega,\mathbb{R}^m)$ be a $\mathbb{R}^m$-valued Radon measure on an open set $\Omega \subset \mathbb{R}^d$ and let us assume that μ is $\mathscr{A}$-free, i.e. that it soves the following system of linear PDE in the sense of distribution:*

$$\mathscr{A}\mu = \sum_{|\alpha|\le k} A_\alpha \partial^\alpha \mu = 0 \qquad \text{in } \mathcal{D}'(\Omega;\mathbb{R}^n). \tag{1}$$

What can be said about the singular part (w.r.t. $\mathcal{L}^d$) of μ?

Besides its own theoretical interest, understanding the structure of singularities of PDE constrained measures turns out to have several (sometimes surprising) applications in the Calculus of Variations and in Geometric Measure Theory, which we describe below.

In answering to Question 1 a prominent role is played by the *wave cone* associated with the differential operator $\mathscr{A}$:

$$\Lambda_{\mathscr{A}} = \bigcup_{|\xi|=1} \operatorname{Ker} \mathbb{A}^k(\xi) \subset \mathbb{R}^m \qquad \text{with} \qquad \mathbb{A}^k(\xi) = (2\pi \mathrm{i})^k \sum_{|\alpha|=k} A_\alpha \xi^\alpha,$$

and we have set $\xi^\alpha = \xi_1^{\alpha_1}\cdots\xi_d^{\alpha_d}$.

Roughly speaking, $\Lambda_{\mathscr{A}}$ contains all the amplitudes along which the system (1) is *not elliptic*. Indeed if we assume that $\mathscr{A}$ is homogeneous, $\mathscr{A} = \sum_{|\alpha|=k} A_\alpha\partial^\alpha$, then it is immediate to check that $\lambda \in \mathbb{R}^m$ belongs to $\Lambda_{\mathscr{A}}$ if and only if there exists a non zero $\xi \in \mathbb{R}^d \setminus \{0\}$ such that $\lambda h(x\cdot\xi)$ is $\mathscr{A}$-free for all $h : \mathbb{R} \to \mathbb{R}$. In other words "one dimensional" oscillations and concentrations are possible only if the amplitudes belongs to the wave cone.

Since the singular part of a measure can be thought as containing "condensed" concentrations, it is quite natural to conjecture that for $|\mu|^s$-almost everywhere the polar vector $d\mu/d|\mu|$ shall belong to $\Lambda_{\mathscr{A}}$. This is indeed the main result of [9]:

Theorem 1. *Let $\Omega \subset \mathbb{R}^d$ be an open set, let $\mathscr{A}$ be a k'th-order linear constant-coefficient differential operator as above, and let $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ be an $\mathscr{A}$-free Radon measure on Ω with values in $\mathbb{R}^m$. Then,*

$$\frac{d\mu}{d|\mu|}(x) \in \Lambda_{\mathscr{A}} \qquad \text{for } |\mu|^s\text{-a.e. } x \in \Omega.$$

As mentioned above Theorem 1 has some interesting applications in the Calculus of Variations and in Geometric Measure Theory. The first one is a new proof of the celebrated Alberti's rank-one Theorem for vector valued BV functions and its extension to BD, the class of functions of bounded deformation. Let us recall here that for a L^1 function u,

$$u \in BV(\mathbb{R}^d, \mathbb{R}^\ell) \iff Du \text{ is a } \mathbb{R}^\ell \otimes \mathbb{R}^d \text{ -valued Radon measure,}$$

and

$$u \in BD(\mathbb{R}^d) \iff Eu := Du + (Du)^T \text{ is a } \mathbb{R}^d \otimes \mathbb{R}^d\text{-valued Radon measure.}$$

We note that $BV(\mathbb{R}^d, \mathbb{R}^d) \subset BD(\mathbb{R}^d)$ and that, due to the failure of Korn's inequality [14], the inclusion is strict.

It was conjecture by Ambrosio and De Giorgi in [4] and proved by Alberti in [1] that for a BV function the singular part (w.r.t. $\mathcal{L}^d$) of Du has a *rank-one* structure, namely

$$\frac{dD^s u}{d|Du|}(x) = a(x) \otimes b(x) \qquad \text{for } |D^s u|\text{-a.e. } x.$$

The natural (and relevant for applications, see [10, 11]) generalisation of the above property for BD-functions is the following:

Question 2. *Is it true that for a function of bounded deformation $u \in BD(\mathbb{R}^d)$*

$$\frac{dE^s u}{d|Eu|}(x) = a(x) \odot b(x) \qquad \text{for } |E^s u|\text{-a.e. } x?$$

Here $a \odot b = a \otimes b + b \otimes a$.

Both Alberti's rank one Theorem and a positive answer to Question 2 are simple consequences of Theorem 1. We have indeed the following:

Theorem 2. *Let $\Omega \subset \mathbb{R}^d$ be an open set, then:*

(i) *For every $u \in BV(\Omega; \mathbb{R}^\ell)$*

$$\frac{dD^s u}{d|Du|}(x) = a(x) \otimes b(x) \qquad \text{for } |D^s u|\text{-a.e. } x.$$

(ii) *For every $u \in BD(\Omega)$*

$$\frac{dE^s u}{d|Eu|}(x) = a(x) \odot b(x) \qquad \text{for } |E^s u|\text{-a.e. } x.$$

Proof. Since $\mu = Du$ is curl-free,

$$0 = \operatorname{curl}(\mu) = \Big(\partial_i \mu_j^k - \partial_j \mu_i^k\Big)_{i,j=1,\dots,d;\; k=1,\dots,\ell}.$$

point (i) above follows from

$$\Lambda_{\text{curl}} = \big\{a \otimes \xi : a \in \mathbb{R}^\ell,\, \xi \in \mathbb{R}^d \setminus \{0\}\big\}.$$

In the same way if $\mu = Eu$, then it satisfies the Saint-Venant compatibility conditions:

$$0 = \operatorname{curl}\operatorname{curl}(\mu) := \left(\sum_{i=1}^{d} \partial_{ik}\mu_i^j + \partial_{ij}\mu_i^k - \partial_{jk}\mu_i^i - \partial_{ii}\mu_j^k\right)_{j,k=1,\dots,d}.$$

It is now a direct computation to check that

$$\Lambda_{\operatorname{curl}\operatorname{curl}} = \big\{a \odot \xi : a \in \mathbb{R}^d,\, \xi \in \mathbb{R}^d \setminus \{0\}\big\}.$$

□

A quite surprising application of Theorem 1 concerns the study of the sharpness of Rademacher's Theorem. Let us recall that Rademacher's Theorem asserts that a Lipschitz function $f \in \operatorname{Lip}(\mathbb{R}^d, \mathbb{R}^\ell)$ is diffferentiable $\mathcal{L}^d$-almost everywhere. A natural question, which has attracted the attention of several researchers, is to understand how sharp is this result. Namely:

Question 3 (Weak converse of Rademacher Theorem). *Let $\nu \in \mathcal{M}_+(\mathbb{R}^d)$ be a positive Radon measure such that every Lipschitz function is differentiable ν-almost everywhere. Is ν necessarily absolutely continuous with respect to the Lebesgue measure?*

We refer to [2, 3] for a detailed account on the history of this problem and of the related problem concerning the *strong converse* of Rademacher Theroem.

The link between Question 3 and Question 1 is due to the beautiful work of Alberti and Marchese, [3], see Theorem 1.1 and Corollary 6.5 there and [9, Lemma 3.1].

Lemma. *Let $\nu \in \mathcal{M}_+(\mathbb{R}^d)$ be a positive Radon measure, then the following are equivalent*

(i) *Every Lipschitz function is differentiable ν-almost everywhere.*
(ii) *There exists d $\mathbb{R}^d$-valued measures $\mu_1, \dots, \mu_d \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d)$ with measure valued divergence* $\operatorname{div} \mu_i \in \mathcal{M}(\mathbb{R}^d; \mathbb{R})$*, such that $\nu \ll |\mu_i|$ for $1 = 1, \dots, d$ and*

$$\operatorname{span}\left\{\frac{d\mu_1}{d|\mu_1|}(x), \dots, \frac{d\mu_d}{d|\mu_d|}(x)\right\} = \mathbb{R}^d \qquad \text{for } \nu \text{ a.e. } x. \tag{2}$$

With the above Lemma at hand a positive answer to Question 3 follows straightforwardly from Theorem 1. Indeed let ν be a measure such that Lipschitz functions are differentiable ν-almost everywhere an let μ_i the measures provided by Lemma . Let us consider the matrix-valued measure

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_d \end{pmatrix} \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d \otimes \mathbb{R}^d).$$

and note that $\operatorname{div}\boldsymbol{\mu} \in \mathcal{M}(\mathbb{R}^d;\mathbb{R}^d)$, where div is the row-wise divergence operator. Since, by direct computation,

$$\Lambda_{\mathrm{div}} = \big\{M \in \mathbb{R}^d \otimes \mathbb{R}^d \text{ such that } \operatorname{rank} M \le d-1\big\},$$

Theorem 1 implies that $\operatorname{rank}(d\boldsymbol{\mu}/d|\boldsymbol{\mu}|) \le d-1$ for $|\boldsymbol{\mu}|^s$-almost every point. Hence, by (2), $\nu \perp |\boldsymbol{\mu}|^s$. On the other hand, since $\nu \ll |\mu_i|$ for all $i = 1, \dots, d$, $\nu^s \ll |\boldsymbol{\mu}|^s$. These two facts then implies that $\nu^s = 0$ as desired.

Let us conclude by mentioning that the weak converse of Rademacher Theorem, i.e. a positive answer to Question 3, has important implications concerning the structure of Ambrosio–Kirchheim metric currents, [5], and the structure of the so called Lipschitz differentability spaces, [7, 13]. In particular it allows to prove the top-dimensional case of the flat chain conjecture proposed by Ambrosio and Kirchheim in [5], see [9, Theorem 1.15] and [15], and to provide a positive answer to a conjecture raised by Cheeger [7, Conjecture 4.63], see [6, 12, 13] and [8]. We refer the reader to the above mentioned references for more details.

References

[1] G. Alberti, *Rank one property for derivatives of functions with bounded variation* , Proc. Roy. Soc. Edinburgh Sect. A, 123:239–274, 1993.

[2] G. Alberti, M. Csörnyei, and D. Preiss, *Structure of null sets in the plane and applications*, In Proceedings of the Fourth European Congress of Mathematics (Stockholm, 2004), pages 3–22. European Mathematical Society, 2005.

[3] G. Alberti and A. Marchese, *On the differentiability of Lipschitz functions with respect to measures in the Euclidean space*, Geom. Funct. Anal., 2015, To appear.

[4] L. Ambrosio and E. De Giorgi, *Un nuovo tipo di funzionale del calcolo delle variazioni*, Atti Acc. Naz. dei Lincei, Rend. Cl. Sc. Fis. Mat. Natur., 82:199–210, 1988.

[5] L. Ambrosio and B. Kirchheim, *Currents in metric spaces*, Acta Math., 185:1–80, 2000.

[6] D. Bate, *Structure of measures in Lipschitz differentiability spaces* , J. Amer. Math. Soc., 28:421–482, 2015.

[7] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal., 9:428–517, 1999.

[8] G. De Philippis, A. Marchese and F. Rindler, *On a conjecture of Cheeger*, Preprint.

[9] G. De Philippis and F. Rindler, *On the structure of $\mathscr{A}$-free measures and applications*, Ann. of Math., To appear.

[10] G. De Philippis and F. Rindler, *Characterization of Young measures generated by functions of bounded deformation*, Preprint 2016.

[11] G. Francfort, A. Giacomini and J.J. Marigo, *The taming of plastic slips in von mises elasto-plasticity*, Interfaces Free Bound., 17:497–516, 2015.

[12] J. Gong, *Rigidity of derivations in the plane and in metric measure spaces*, Illinois J. Math., 56:1109–1147, 2012.

[13] S. Keith, *A differentiable structure for metric measure spaces*, Adv. Math., 183:271–315, 2004.

[14] D. Ornstein, *A non-inequality for differential operators in the L_1 norm*, Arch. Rational Mech. Anal., 11:40–49, 1962.

[15] A. Schioppa, *Metric currents and Alberti representations*, Preprint 2016.

On a isoperimetric-isodiametric inequality

Andrea Mondino
(joint work with Emanuele Spadaro)

One of the oldest questions of mathematics is the *isoperimetric* problem: *What is the largest amount of volume that can be enclosed by a given amount of area?* A related classical question is the *isodiametric* problem: *What is the largest amount of volume that can be enclosed by a domain having a fixed diameter?*

In this seminar we present results contained in [5] where we address a mix of the previous two questions, namely we investigate the following mixed isoperimetric-isodiametric problem: *What is the largest amount of volume that can be enclosed by a domain having a fixed product of diameter and boundary area?*

Of course, if we ask the three above questions in the Euclidean space, the answer is given by the round balls of the suitable radius; but, of course, the situation in non-flat geometries is much more subtle. We start by recalling classical material on the isoperimetric problem which motivated our investigation on the mixed isoperimetric-isodiametric one.

The solution of the isoperimetric problem in the Euclidean space $\mathbb{R}^n$ can be summarized by the classical isoperimetric inequality

(1)
$$n\,\omega_n^{\frac{1}{n}}\,\mathrm{Vol}(\Omega)^{\frac{n-1}{n}} \leq \mathcal{A}(\partial\Omega)\,, \quad \text{for every } \Omega \subset \mathbb{R}^n \text{ open subset with smooth boundary,}$$

where $\mathrm{Vol}(\Omega)$ is the n-dimensional Hausdorff measure of Ω (i.e. the "volume" of Ω), $\mathcal{A}(\partial\Omega)$ is the $(n-1)$-dimensional Hausdorff measure of $\partial\Omega$ (i.e. the "area" of $\partial\Omega$), and $\omega_n := \mathrm{Vol}(B^n)$ is the volume of the unit ball in $\mathbb{R}^n$. As it is well known, the regularity assumption on Ω can be relaxed a lot (for instance (1) holds for every set Ω of finite perimeter) but let us not enter in technicalities here since we are just motivating our problem.

As anticipated above, we will not deal with the isoperimetric problem itself but we will focus on a mixed isoperimetric-isodiametric problem. Let us start by stating the Euclidean mixed isoperimetric-isodiametric inequality which will act as model for this seminar. Given a bounded open subset $\Omega \subset \mathbb{R}^n$ with smooth boundary, by the divergence theorem in $\mathbb{R}^n$, we have

$$n\,\mathrm{Vol}(\Omega) \leq \mathrm{rad}(\Omega)\,\mathcal{A}(\partial\Omega), \tag{2}$$

where $\mathrm{rad}(\Omega)$ is the radius of the smallest ball of $\mathbb{R}^n$ containing Ω. It is easy so see that inequality (2) is *sharp* and *rigid*; indeed, equality occurs if and only if Ω is a round ball in $\mathbb{R}^n$.

In sharp contrast with the classical isoperimetric problem, where both problems are still open in the general case, it is not difficult to show that the inequality (2) holds in Cartan-Hadamard spaces (i.e. simply connected Riemannian manifolds with non-positive sectional curvature) and on minimal submanifolds of $\mathbb{R}^n$. Even if the validity of inequality (2) in such spaces is probably known to experts, we included it as motivation and also because the equality case for minimal submanifolds presents an interesting link with free-boundary minimal surfaces: equality is attained in (2) if and only if the minimal submanifold is a free boundary minimal surface in a Euclidean ball. If on one hand the negative curvature gives a stronger isoperimetric-isodiametric inequality, on the the other hand we show that non-negative Ricci curvature forces metric balls to satisfy a weaker isoperimetric-isodiametric inequality. The precise statement is the following comparison result.

Theorem 1. *Let (M^n, g) be a complete (possibly non compact) Riemannian n-manifold with non-negative Ricci curvature. Let $B_r \subset M$ be a metric ball of volume $V = \mathrm{Vol}_g(B_r)$, and denote with $B^{\mathbb{R}^n}(V)$ the round ball in $\mathbb{R}^n$ having volume V. Then*

(3)
$$\mathrm{rad}(B_r)\ \mathcal{A}(\partial B_r) = r\ \mathcal{A}(\partial B_r) \leq n \,\mathrm{Vol}_g(B_r) = \mathrm{rad}_{\mathbb{R}^n}(B^{\mathbb{R}^n}(V))\ \mathcal{A}_{\mathbb{R}^n}(\partial B^{\mathbb{R}^n}(V)).$$

Moreover equality holds if and only if B_r is isometric *to a round ball in the Euclidean space $\mathbb{R}^n$.*

Remark 1. Since by Bishop-Gromov volume comparison we know that if $Ric_g \geq 0$ then for every metric ball $B_r(x_0) \subset M$ it holds $\mathrm{Vol}_g(B_r(x_0)) \leq \omega_n r^n = \mathrm{Vol}_{\mathbb{R}^n}(B_r^{\mathbb{R}^n})$, it follows that

$$\mathrm{rad}(B_r(x_0)) \geq \mathrm{rad}_{\mathbb{R}^n}(B^{\mathbb{R}^n}(V)),$$

where $B^{\mathbb{R}^n}(V)$ is a Euclidean ball of volume $V = \mathrm{Vol}_g(B_r(x_0))$. Therefore Theorem 1 in particular implies that $\mathcal{A}(\partial B_r(x_0)) \leq \mathcal{A}_{\mathbb{R}^n}(\partial B^{\mathbb{R}^n}(V))$, but is a strictly stronger statement which at best of our knowledge is original. The aforementioned counterpart of Theorem 1 for the isoperimetric problem was proved instead by Morgan-Johnson [3, Theorem 3.5] for compact manifolds and extended to non-compact manifolds in [4, Proposition 3.2]. □

We then investigate the existence of optimal shapes in a general Riemannian manifold (M, g). More precisely, given a measurable subset $E \subset M$ we denote with $\mathcal{P}(E)$ its perimeter and define its extrinsic radius as

$$\mathrm{rad}(E) := \inf \{ r > 0 \ :\ \mathrm{Vol}_g(E \setminus B_r(z_0)) = 0 \text{ for some } z_0 \in M \},$$

where $B_r(z_0)$ denotes the open metric ball with center z_0 and radius $r > 0$. We consider the following minimization problem: for every fixed $V \in (0, \mathrm{Vol}_g(M))$,

$$\min \Big\{ \mathrm{rad}(E)\, \mathcal{P}(E) \ :\ E \subset M,\ \mathrm{Vol}_g(E) = V \Big\}, \tag{4}$$

and call the minimizers of (4) isoperimetric-isodiametric sets (or regions). To best of our knowledge this is first time such a problem is considered in literature.
As it happens also for the isoperimetric problem, by using classical compactness and lower semicontinuity results, it is not difficult to see that if the ambient manifold is compact then for every volume there exists an isoperimetric-isodiametric region but if the ambient space is non-compact the situation changes dramatically. Indeed we show that in complete minimal submanifolds with planar ends (like

the helicoid) and in asymptotically locally Euclidean Cartan-Hadamard manifolds there exists no isoperimetric-isodiametric region of positive volume. On the other hand, we show that in C^0-locally asymptotically Euclidean manifolds with non negative Ricci curvature for every volume there exists an isoperimetric-isodiametric region.

Finally we discuss the optimal regularity for isoperimetric-isodiametric regions under suitable assumptions on regularity of the enclosing ball. We first observe that outside of the contact region with the minimal enclosing ball B, such sets are locally minimizers of the perimeter under volume constraint. Therefore by classical results (see, for example, [2, Corollary 3.8]) in the interior of B the boundary of the region is a smooth hypersurface outside a singular set of Hausdorff co-dimension at least 8.

The rest of the seminar is devoted to prove the optimal regularity at the contact region. We first show that isoperimetric-isodiametric regions are almost-minimizers for the perimeter and therefore, by a result of Tamanini [6] their boundaries are $C^{1,1/2}$ regular. Then, by means of geometric comparisons and sharp first variation arguments, we show that the mean curvature of the boundary of an isoperimetric-isodiametric region is in L^∞ with explicit estimates. Finally we establish the optimal $C^{1,1}$ regularity stated below.

Theorem 2. *Let $E \subset M$ be an isoperimetric-isodiametric set and $x_0 \in M$ be such that $\mathrm{Vol}_g(E \setminus B_{\mathrm{rad}(E)}(x_0)) = 0$. Assume that $B := B_{\mathrm{rad}(E)}(x_0)$ has smooth boundary. Then, there exists $\delta > 0$ such that $\partial E \setminus B_{\mathrm{rad}(E)-\delta}(x_0)$ is $C^{1,1}$ regular.*

An essential ingredient in the proof of Theorem 2 is to show that the boundary of E leaves the obstacle at most quadratically. Then the conclusion will follow by combining Schauder estimates outside of the contact region with the general fact that functions which leave the first order approximation quadratically are $C^{1,1}$. Although the techniques are inspired by the ones introduced in the study of the classical obstacle problem (cf., for example, [1]), here we treat the geometric case of the area functional in a Riemannian manifold with volume constraints and we take several short-cuts thanks to some specifically geometric arguments, such as the theory of almost minimizers.

Remark 2. We expect the $C^{1,1}$ regularity to be optimal, because in general the continuity of the second fundamental form of ∂E across the free boundary of ∂E fails. This is indeed true for the simplest case of the classical obstacle problem.

References

[1] L. Caffarelli, *The obstacle problem revisited*, J. Fourier Anal. Appl. 4 (1998), no. 4-5, 383–402.

[2] F. Morgan, *Regularity of isoperimetric hypersurfaces in Riemannian manifolds*, Trans. Amer. Math. Soc., Vol. 355, (2003), no. 12, 5041–5052.

[3] F. Morgan and D. L. Johnson, *Some sharp isoperimetric theorems in Riemannia manifolds*, Indiana Univ. Math. J., Vol. 49, (2000), 1017–1042.

[4] A. Mondino and S. Nardulli, *Existence of isoperimetric regions in non-compact Riemannian manifolds under Ricci or scalar curvature conditions*, Comm. Anal. Geom., Vol. 24, Num. 1, 115–138, (2016).

[5] A. Mondino and E. Spadaro, *On a isoperimetric-isodiametric inequality*, preprint arXiv:1603.05263.
[6] I. Tamanini, *Boundaries of Caccioppoli sets with Hölder-continuous normal vector,* J. Reine Angew. Math. 334 (1982), 27–39.

The isoperimetric problem for large volumes in asymptotically flat 3-manifolds

Otis Chodosh
(joint work with Michael Eichmair, Yuguang Shi, Haobin Yu)

A 3-manifold is *asymptotically flat* if for some compact set K,

$$M \setminus K \cong \{x \in \mathbb{R}^3 : |x| > 1\} \qquad \text{and} \qquad g_{ij} = \delta_{ij} + O(|x|^{-\tau})$$

for $\tau > \frac{1}{2}$ (along with derivatives). We also include the assumption that the scalar curvature is integrable and that there are no closed minimal surfaces in (M, g) other than ∂M (which is required to be minimal, if non-empty). Such (M, g) arise naturally as study of initial data sets for the Einstein equations in general relativity. Our main theorem is:

Theorem 1 ([3])**.** *Assume that (M, g) is asymptotically flat and has non-negative scalar curvature. If (M, g) is not isometric to flat Euclidean space $\mathbb{R}^3$, then there exists a unique isoperimetric region Ω_V containing volume V for all V sufficiently large.*

We mention also that Theorem 1 in the case that g is additionally C^0-asymptotic to the Schwarzschild metric, i.e.,

$$g_{ij} = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} + O(|x|^{-2}),$$

(but without the assumption that the scalar curvature is non-negative) was proven by M. Eichmair and J. Metzger [4] building on an ingenious idea of H. Bray [1]. Moreover, G. Huisken has proposed a novel concept of isoperimetric mass, and used mean curvature flow to prove sharp estimates for the isoperimetric defect for large volumes in asymptotically flat three manifolds.

An interesting feature of Theorem 1 is the global nature of the isoperimetric problem. Indeed, the theorem is false (in this generality) if "isoperimetric region" is replaced by "volume preserving stable constant mean curvature surface" (see Appendix A in [3] for references concerning the study of stable constant mean curvature surfaces in asymptotically flat manifolds).

We mention here a related result (due to the author and M. Eichmair) of a similarly global nature. A proof is included in [2].

Theorem 2. *Suppose that (M, g) is asymptotically flat with non-negative scalar curvature. If (M, g) contains a non-compact area-minimizing boundary, then (M, g) is isometric to $(\mathbb{R}^3, \bar{g})$.*

This resolves a conjecture of R. Schoen. As in Theorem 1, this result is false if "area-minimizing" is weakened to "stable."

References

[1] H. Bray, *The Penrose inequality in general relativity and volume comparison theorems involving scalar curvature*, Ph.D. thesis, Stanford Univ., Stanford, CA, 1997.

[2] A. Carlotto, O. Chodosh, M. Eichmair, *Effective versions of the positive mass theorem*, to appear in Invent. Math.

[3] O. Chodosh, M. Eichmair, Y. Shi, H. Yu, *Isoperimetry, scalar curvature, and mass in asymptotically flat Riemannian 3-manifolds*, preprint, available at http://arxiv.org/abs/1606.04626.

[4] M. Eichmair, J. Metzger, *Large isoperimetric surfaces in initial data sets*, J. Differential Geom. **94** 159–186.

Singularities of Minimal Two-Valued Graphs

Spencer T. Becker-Kahn

(joint work with Neshan Wickramasekera)

The problem of finding and studying critical points of the n-dimensional area functional in an $(n+k)$-dimensional Riemannian manifold is an old and difficult one. It subsumes, for example, the problem of understanding geodesics on a closed surface and that of finding closed minimal hypersurfaces. It was with this problem in mind that Almgren began to develop the theory of a class of objects which he called *integral varifolds.* Two of the most important facts about integral varifolds are a) The area functional is continuous (not just lower semicontinuous) with respect to the (weak) convergence of varifolds. And b) There is a straightforward compactness theorem for integral varifolds. This means for example that the sequence obtained by taking a standard catenoid in $\mathbf{R}^3$ and shrinking by a factor $1/j$ converges in the sense of varifolds, as $j \to \infty$, to a multiplicity two copy of a single plane.

An integral varifold that is a critical point of the area functional is called a *stationary varifold.* Almgren was able to prove the following theorem.

Theorem 1 (Almgren, Corollary 15.2 of [2])**.** *Let M be a compact N-dimensional Riemannian manifold. For every $1 \leq n \leq N-1$, there exists an n-dimensional stationary integral varifold V in M.*

Almgren's result is unsatisfactory because stationary integral varifolds are very weak objects that can potentially have bad singularities and it would appear to be difficult to analyze the stationary varifold that this theorem gives you. (An integral n-varifold V is a pair (M, θ) where M is a countably n-rectifiable set and $\theta : M \to \mathbb{Z}_{\geq 0}$ is a locally $\mathcal{H}^n$-integrable function called the *multiplicity.* We say V is *stationary* in U if

$$\int_U \operatorname{div}_{T_x M} \left(\Psi(x)\right) \theta(x) d\mathcal{H}^n(x) = 0$$

for all smooth, compactly supported vector fields $\Psi : U \to \mathbf{R}^{n+k}$).

Among many other interesting questions that Almgren's result raises, we have the fundamental issue of regularity: *Does a stationary varifold corresponded to a smooth minimal submanifold $\mathcal{H}^n$-almost everywhere?* In complete generality, little more is known than Allard's theorem ([1]) which implies that the singular set is closed and nowhere dense. This regularity result comes from a so-called ϵ-regularity theorem which implies that on a stationary varifold, any point at which there is a multiplicity 1 tangent plane is a regular point. What this shows is that the obstruction to almost everywhere regularity is precisely the set of points at which there is a higher multiplicity tangent plane. Roughly speaking, the work discussed in this talk pertains to a program aimed at beginning to understand multiplicity two singular behaviour for stationary varifolds in arbitrary codimension and without the assumption of stability or area-minimization.

We work in the class of Lipschitz two-valued graphs: To understand a two-valued graph, one might think that "above each point in the domain, there are exactly two points when counted with multiplicity". A *minimal two-valued graph* is a stationary integral varifold in $B^n_2(0) \times \mathbf{R}^k$ that is associated to the graph of a two-valued Lipschitz function. While the assumption may appear to be restrictive, I would emphasize that many of the most interesting and challenging types of behaviour are still present. Consider the following three examples:

1) Let $\mathbf{C} = |P_1| + |P_2|$, where P_1, P_2 are n-dimensional subspaces such that $\dim(P_1 \cap P_2) = n - 1$. The varifold $\mathbf{C}$ is a minimal two-valued graph with $\mathcal{H}^{n-1}(\text{sing } \mathbf{C}) > 0$, which shows that working in this class provides no free pass on the size of the singular set.

2) Consider the irreducible holomorphic variety $I := \{(z, w) \in \mathbb{C} \times \mathbb{C} : z^2 = w^3\} \subset \mathbf{R}^4$, which is two-dimensional and has sing $I = \{0\}$. Such a variety is area-minimizing (because it is 'calibrated') and so the associated varifold is indeed stationary. This is a minimal $C^{1,1/2}$ two-valued graph (of $w \mapsto \pm w^{3/2}$) and has a multiplicity two tangent plane at the origin, so working in this class does not evade this type of branch point singularity.

3) Let $\eta : \mathrm{S}^3 \to \mathrm{S}^2$ denote the Hopf map. The homogeneous degree one function $f : \mathbf{R}^4 \to \mathbf{R}^3$ given by $f(x) = \frac{\sqrt{5}}{2}|x|\eta\left(\frac{x}{|x|}\right)$ for $x \neq 0$ is a Lipschitz weak solution to the minimal surface system on $\mathbf{R}^4$ with an isolated singularity at the origin. So the varifold associated to graph f is a cone that is not made from planes or half-planes like in the previous two examples. This shows that working just with graphs does not preclude this kind of, shall we say, 'exotic' singular behaviour.

One can establish relatively straightforwardly that for a minimal two-valued graph V, we have that sing V is the disjoint union $\mathfrak{B} \cup \mathcal{T} \cup \mathcal{E}$ where

- At $x_0 \in \mathfrak{B}$, there exists a multiplicity 2 tangent plane.
- $\dim \mathcal{T} \leq n - 1$ and at $x_0 \in \mathcal{T}$, there exists transverse tangent cone made from planes or half-planes.
- $\dim \mathcal{E} \leq n - 3$ and at $x_0 \in \mathcal{E}$, every tangent cone is 'exotic' (not made from planes or half-planes).

The work of my thesis is a local analysis near points of $\mathcal{T}$ and the work-in-progress that is a continued collaboration with my advisor Neshan Wickramasekera is a local analysis near points of $\mathfrak{B}$. Some of the key results about the set $\mathcal{T}$ can be summarized in the following brief statement (see the paper for much more detailed statements):

Theorem 2 ([3])**.** *At each point $x_0 \in \mathcal{T}$, there is a unique tangent cone $\mathbf{C}_{x_0}$ and a neighbourhood of x_0 in which* sing V *is contained in an m-dimensional $C^{1,\alpha}$ submanifold, where $m \leq n-1$.*

Part of the significance of this work comes from the fact that in arbitrary codimension - and in the purely stationary case - there are extremely few cases in which any detailed analysis of the singular set has been possible (*e.g.* gaining precise asymptotics on approach to singularities, studying the geometry and structure of the singular set or proving uniqueness of tangent cones). Historically, analyzing any kind of branch-point singularities (even in the area-minimizing or stable cases) has also been very hard, but we believe that it will be possible in this case:

Work in progress. *At every point $x_0 \in \mathfrak{B}$, the tangent cone is unique (and equal to a multiplicity two plane).*

One of the reasons that analyzing branch point singularities is hard is that they are, by definition, singular points at which the tangent cone is regular and so on some level the singular behavior is necessarily more subtle. We will draw from the works of Wickramasekera on stable hypersurfaces ([7], [8], [9]) which in some sense have established a philosophy that, in certain cases, enables a detailed analysis of branch points via the so-called 'blow-up' method (the method centers around an analysis of the linearization of the minimal surface operator and in essence goes back to De Giorgi ([4]) although it is in [1] that it first appears in a more pertinent form). The general analytic framework in both Wickramasekera's work and in my previous work is directly inspired by the work of Simon [5].

Finally, we mention one conjecture that all of this might be building towards:

Conjecture. $\dim_{\mathcal{H}}(\mathfrak{B}) \leq n-2$.

Proving this would imply that $\dim_{\mathcal{H}} \operatorname{sing} V \leq n-1$. This would make minimal two-valued graphs one of only a few classes in which an optimal dimension estimate on the size of the singular set has been obtained. (It is worth noting that in [6], Simon and Wickramasekera use primarily PDE-based arguments to show that the branch set of a $C^{1,\alpha}$ minimal two-valued graph is at most $(n-2)$-dimensional).

References

[1] W. K. Allard, *On the first variation of a varifold*, Ann. of Math. **95** (1972), no. 3, 417–491.

[2] F. J. Almgren, *The theory of varifolds: A variational calculus in the large for the k-dimensional area integrand*, mimeographed notes.

[3] S. T. Becker-Kahn, *Transverse singularities of minimal two-valued graphs in arbitrary codimension*, J. Differential Geom. (to appear).

[4] E. De Giorgi, *Frontiere orientate di misura minima*, Editr. tecnico scientifica, 1961.

[5] L. Simon, *Cylindrical tangent cones and the singular set of minimal submanifolds*, J. Differential Geom. **38** (1993), no. 3, 585–652.
[6] L. Simon and N. Wickramasekera, *A frequency function and singular set bounds for branched minimal immersions*, Comm. Pure Appl. Math. **69** (2016), no. 7, 1213–1258.
[7] N. Wickramasekera, *A rigidity theorem for stable minimal hypercones*, J. Differential Geom. **68** (2004), no. 3, 433–514.
[8] N. Wickramasekera, *A regularity and compactness theory for immersed stable minimal hypersurfaces of multiplicity at most 2*, J. Differential Geom. **80** (2008), no. 1, 79.
[9] N. Wickramasekera, *A general regularity theory for stable codimension 1 integral varifolds*, Ann. Math. **179** (2014), no. 3, 843–1007.

Isoperimetry with upper mean curvature bounds and sharp stability estimates

Brian Krummel
(joint work with Francesco Maggi)

Motivated by capillarity-type problems, in our recent work of [7], we consider the structure of hypersurfaces with almost constant mean curvature (almost CMC). For a bounded, open subset $\Omega \subset \mathbb{R}^{n+1}$ with a smooth boundary, we define the *CMC deficit* $\delta_{\rm cmc}(\Omega)$ by

$$\delta_{\rm cmc}(\Omega) = \left\| \frac{H_\Omega}{H_0} - 1 \right\|_{L^\infty(\partial\Omega)} \quad \text{where} \quad H_0 = \frac{n\,P(\Omega)}{(n+1)\,|\Omega|}$$

and where H_Ω denotes the mean curvature of $\partial\Omega$ computed with respect to the outward unit normal to Ω and $P(\,\cdot\,)$ denotes perimeter. Note that if $\partial\Omega$ is CMC, then $H_\Omega = H_0$. We say that $\partial\Omega$ is almost CMC if $\delta_{\rm cmc}(\Omega)$ is small.

Previous work by Ciraola and Maggi [3] showed that if $\partial\Omega$ is almost CMC, $H_0 = n$, and $P(\Omega) \leq (L+\tau)\,P(B_1)$ for an integer $L \geq 1$ and $\tau \in (0,1)$, then $\partial\Omega$ can be represented as a $C^{1,\alpha}$ graph over a union of at most L tangent unit balls away from spherical caps where the tangent balls touch and with estimates. However, the estimates of [3] were not optimal. Ciraolo and Vezzoni [5] showed that if $\partial\Omega$ is almost CMC, $|\Omega| = |B_1|$, and Ω satisfies an interior/exterior ball condition of radius $\rho > 0$ at each point of $\partial\Omega$, then $\mathrm{hd}(\partial\Omega, \partial B_1(x_0)) \leq C(n, P(\Omega), \rho)\,\delta_{\rm cmc}(\Omega)$ for some $x_0 \in \mathbb{R}^{n+1}$. This estimate is optimal. However, the interior/exterior ball condition is too restrictive for the study of critical points of capillarity-type energies since the uniform ball condition prevents bubbling phenomena; for instance, consider the surface obtained by truncating and then smoothly completing an unduloid with very thin necks. As a step towards obtaining sharp estimates for almost CMC hypersurfaces close to a union of tangent unit balls, we will prove sharp estimates for almost CMC hypersurfaces close to a single unit ball without the interior/exterior ball condition.

Our approach involves an isoperimetric principle due to Almgren [1], which in codimension one can be stated as follows:

Almgren's isoperimetric principle. *If $\Omega \subset \mathbb{R}^{n+1}$ is a bounded, open subset with a smooth boundary such that $H_\Omega \leq n$ at each point of $\partial\Omega$, then $P(\Omega) \geq P(B_1)$ with equality if and only if Ω is a unit ball.*

This isoperimetric principle arises from Almgren's proof of the sharp isoperimetric inequality in codimension > 1 and can be stated more generally for weak notions of submanifolds, namely rectifiable varifolds, in arbitrary codimension. In codimension one, mean curvature can be represented by a scalar quantity and we assume a one-sided bound $H_\Omega \leq n$ with no lower bound on H_Ω. Almgren's proof of the isoperimetric principle yields the quantitative information that $\delta(\Omega) = P(\Omega) - P(B_1)$ is given by

$$\delta(\Omega) = \int_{\partial A \cap \partial\Omega} \left(\left(\frac{H_\Omega}{n} \right)^n - K_\Omega \right) + \int_{\partial A \cap \partial\Omega} \left(1 - \left(\frac{H_\Omega}{n} \right)^n \right) + \mathcal{H}^n(\partial\Omega \cap \partial A),$$

where A is the convex hull of Ω and each integrand is nonnegative. This naturally leads to the question of whether one can use this quantitative information to address stability for Almgren's isoperimetric principle: *If $\Omega \subset \mathbb{R}^{n+1}$ is a bounded, open subset with a smooth boundary such that $H_\Omega \leq n$ on $\partial\Omega$ and $\delta(\Omega) = P(\Omega) - P(B_1)$ is small, must Ω be close to a unit ball? Can we obtain sharp estimates on* $\mathrm{hd}(\partial\Omega, \partial B_1)$ *and $|\Omega \,\Delta\, B_1|$ in terms of $\delta(\Omega)$?*

A key obstruction to stability for Almgren's isoperimetric principle is that, since we do not assume any lower bound on H_Ω, Ω could have tiny holes, e.g. $\Omega = B_1 \setminus B_\varepsilon$, or thin tentacles protruding into Ω. As a result, we could have $\mathrm{hd}(\partial\Omega, \partial B_1) \approx 1$ despite $\delta(\Omega)$ being small. In [7, Theorem 1.2], we remove the holes and tentacles by using Almgren's isoperimetric principle to show that the total perimeter and volume of the holes is $\leq C(n)\,\delta(\Omega)$ and by replacing Ω with a solution E to an obstacle problem, minimize $P(E) + n\,|E|$ amongst E with $\Omega \subseteq E$, as the minimizer E satisfies $-n \leq H_E \leq n$ a.e. on ∂E and has total perimeter and volume outside of Ω that is $\leq C(n)\,\delta(\Omega)$.

Now assuming that $-n \leq H_\Omega \leq n$ on $\partial\Omega$, one can argue using Allard regularity that if $\delta(\Omega)$ is sufficiently small then, up to translating, $\Omega = \{(1+u(x))\,x : x \in \mathbb{S}^n\}$ for some $u \in C^1(\mathbb{S}^n)$ with $\|u\|_{C^1} \leq \varepsilon(n)$ small. Having shown this, we obtain the following sharp stability estimates for Almgren's isoperimetric inequality.

Theorem 1 (Theorem 1.1 of [7]). *If $\Omega \subseteq \mathbb{R}^{n+1}$ is a bounded, open set with smooth boundary such that $H_{\partial\Omega} \leq n$ on $\partial\Omega$ and $\delta(\Omega) \leq \delta_0(n)$, then there exists $x_0 \in \mathbb{R}^{n+1}$ such that*

$$|\Omega \,\Delta\, B_1(x_0)| + \sup\{s > 0 : \Omega \subseteq B_{1+s}(x_0)\} \leq C(n)\,\delta(\Omega).$$

Observe that, by contrast, the isoperimetric inequality has a different stability estimate, $|\Omega \,\Delta\, B_1(x_0)|^2 \leq C(n)\,\delta(\Omega)$.

Theorem 2 (Theorem 1.5 of [7]). *If $\Omega \subseteq \mathbb{R}^{n+1}$ is a bounded, open set with smooth boundary such that $H_{\partial\Omega} \leq n$ on $\partial\Omega$, $\partial\Omega = \{(1 + u(x))\,x : x \in \mathbb{S}^n\}$ for* *some* $u \in C^1(\mathbb{S}^n)$ *with* $\|u\|_{C^1} \le \varepsilon(n)$*, and* $\int_{\partial\Omega} x \, d\mathcal{H}^n(x) = 0$*, then*

$$\mathrm{hd}(\partial\Omega, \mathbb{S}^n) \le \begin{cases} C(1)\,\delta(\Omega) & \text{if} n = 1 \\ C(2)\,\delta(\Omega)\,\log(C(2)/\delta(\Omega)) & \text{if} n = 2 \\ C(n)\,\delta(\Omega)^{\frac{1}{n-1}} & \text{if} n \ge 3. \end{cases}.$$

Moreover, there exists an example showing that this estimates is sharp.

The proofs of Theorems 1 and 2 use a series expansion argument based on [6], which when combined with elliptic estimates and an interpolation inequality of Fuglede [6, Lemma 1.4] yields the estimates in Theorems 1 and 2. The series expansion argument additionally shows that the average of u dominates the other Fourier coefficients of u so that

$$0 < c(n)\,\delta(\Omega) \le \int_{\mathbb{S}^n} u \le C(n)\,\delta(\Omega).$$

Since taking u to be constant corresponds to scaling the unit sphere, we interpret this as meaning that we must scale the unit sphere outward while deforming it into $\partial\Omega$ in order to preserve $H_\Omega \le n$. Note that, by contrast, for the isoperimetric inequality one typically fixes the volume of Ω and consequently the average of u is negligible, i.e. $\int_{\mathbb{S}^n} u = O(\|u\|^2_{W^{1,2}})$.

We construct the example showing that the Hausdorff distance estimates in Theorem 2 are sharp as follows. Rescale the unit sphere by $1 + t$ for $t > 0$ small. Push the rescaled sphere in at the north pole to form a tentacle as a surface of revolution. Up to radius r_1 from the axis of symmetry for the tentacle, the profile the tentacle will be a solution to an ordinary differential equation and the tentacle roughly behaves like the graph of a fundamental solution for the Laplacian. We cut this portion of the tentacle off at a radius r_1 where its gradient relative to the unit sphere equals a small constant $\mu > 0$ and then cap off the tentacle with a spherical cap.

Our approach also yields the following sharp estimates for almost CMC hypersurfaces close to a single tangent ball, see [7]. Let $\partial\Omega = \{(1 + u(x))\,x : x \in \mathbb{S}^n\}$ for some $u \in C^1(\mathbb{S}^n)$ with $\|u\|_{C^1}$ small and $\int_{\partial\Omega} x\,d\mathcal{H}^n(x) = 0$. If $\|H_\Omega - n\|_{L^2(\partial\Omega)}$ is sufficiently small, then $\|u\|_{W^{1,2}} \le C(n)\,\|H_\Omega - n\|_{L^2(\partial\Omega)}$. This result is of particular interest since it may have applications to convergence to equilibrium in geometric flows, see for instance [4] for this kind of application of stability theorems to Yamabe-type fast diffusion equations). If additionally

$$\delta^{(p)}_{\mathrm{cmc}}(\Omega) = H_0^{-1} \max\left\{\|(H_0 - H_\Omega)^+\|_{L^\infty(\partial\Omega)}, \|(H_\Omega - H_0)^+\|_{L^p(\partial\Omega)}\right\}$$

is sufficiently small for $p \ge 2$ when $n = 2$ and $p > n/2$ when $n \ge 3$, then $\mathrm{hd}(\partial\Omega, \mathbb{S}^n) \le C(n,p)\,\delta^{(p)}_{\mathrm{cmc}}(\Omega)$.

References

[1] F. Almgren, *Optimal isoperimetric inequalities*, Indiana Univ. Math. J., 35(3):451–547, 1986.
[2] L. Caffarelli, *The obstacle problem revisited*, J. Fourier Anal. Appl., 4(4-5):383–402, 1998.
[3] G. Ciraolo and F. Maggi, *On the shape of compact hypersurfaces with almost constant mean curvature*, 2015.

[4] G. Ciraolo, A. Figalli, and F. Maggi, *A quantitative analysis of metrics on $\mathbb{R}^n$ with almost constant positive scalar curvature, with applications to yamabe and fast diffusion flows*, 2016.
[5] G. Ciraolo and L. Vezzoni, *A sharp quantitative version of Alexandrov's theorem via the method of moving planes*, Preprint arXiv:1501.07845, 2015.
[6] B. Fuglede, *Stability in the isoperimetric problem for convex or nearly spherical domains in* $\mathbb{R}^n$, Trans. Amer. Math. Soc., 314:619–638, 1989.
[7] B. Krummel and F. Maggi, *Isoperimetry with upper mean curvature bounds and sharp stability estimates*, Preprint arXiv:1606.00490, 2016.
[8] I. Tamanini, *Boundaries of caccioppoli sets with hoelder-continuous normal vector*, J. Reine Angew. Math., 334:27–39, 1982.

Sharp local smoothing estimates for the Ricci flow on surfaces.

Peter M. Topping
(joint work with Hao Yin)

Consider the logarithmic fast diffusion equation

$$\partial_t u = \Delta \log u \tag{1}$$

where u is a positive function on a two-dimensional domain. This equation has an extensive literature when posed on the plane, and many beautiful results. The logarithm makes the equation nonlinear, but in a very special way. The geometric reason why this particular choice of equation is so natural comes from the fact that it describes locally the Ricci flow on surfaces. In this two-dimensional situation, the Ricci flow is a one parameter family of Riemannian metrics $g(t)$ that is governed by the nonlinear evolution equation

$$\frac{\partial g}{\partial t} = -2Kg$$

where K is the Gauss curvature of g. In local isothermal coordinates x, y, we can write $g = u.(dx^2 + dy^2)$, and then u can be seen to satisfy the logarithmic fast diffusion equation.

We are interested in posing the Ricci flow with very rough data. For smooth initial data, there is an extensive literature, with the ultimate result that we have existence of a unique instantaneously complete solution for completely arbitrary (smooth) initial data, with that solution existing for a definite period of time, normally for all time (see [1] and [3]). To make a Ricci flow with very general (e.g. locally L^1) initial data, there is an obvious strategy that is to approximate the initial data by smooth initial data, each of which is then Ricci flowed with the existing theory, and then to try to pass to a limit of the flows. To have any hope of getting reasonable convergence of these smooth flows to the desired flow, we need uniform C^k estimates at later times, and by standard parabolic theory, we can obtain these if we are able to derive uniform L^∞ estimates on $\log u$, i.e uniform estimates on u from above and below by positive numbers depending on the time at which we demand an estimate, but independent of how good an approximation of the initial data we have taken. Lower bounds on u are relatively easy to obtain using the global theory in [1], but upper bounds are more tricky. Indeed, for L^1

initial data, we would normally ask for an L^∞ estimate for u that is depending only on the L^1 norm of u and the times t. However, such estimates are known to be impossible (see [4] and the references therein).

The purpose of my talk was to explain the following theorem that circumvents the problem that we have just described. Intuitively, instead of taking L^1 data and a time t as input and giving out an upper bound in return (which is impossible, as we have just mentioned) we essentially take the L^1 data and the desired upper bound as input and give in return the time t by which this upper bound is achieved.

Theorem 1 (joint with Hao Yin)**.** *Suppose $u : B \times [0,T) \to (0,\infty)$ is a smooth solution to the equation*

$$\partial_t u = \Delta \log u \tag{2}$$

on the unit ball $B \subset \mathbb{R}^2$, and suppose that $u_0 := u(0) \in L^1(B)$. Then for all $\delta > 0$ (however small) and for any $k \geq 0$ and any time $t \in [0,T)$ satisfying

$$t \geq \frac{\|(u_0 - k)_+\|_{L^1(B)}}{4\pi}(1+\delta), \qquad \text{we have} \qquad \sup_{B_{1/2}} u(t) \leq C(t+k),$$

for some constant $C < \infty$ depending only on δ.

In the talk, we described a more geometric version of the theorem, and explained the proof. A significant element here is that the estimate is purely local, and can be used to study noncompact Ricci flows. However, a key ingredient in the proof is derived from a paper addressing the closed case for arbitrary dimensional Kähler Ricci flow, of Guedj and Zeriahi [2]. We also explained in the talk that a sharp form of the classical $L^p - L^\infty$ smoothing estimate for $p > 1$ follows immediately from our work. See [4] for further information and references.

References

[1] G. Giesen and P. M. Topping, *Existence of Ricci flows of incomplete surfaces*, Comm. Partial Differential Equations, **36** (2011) 1860–1880.

[2] V. Guedj and A. Zeriahi, *Regularizing properties of the twisted Kähler-Ricci flow*, To appear, J. Reine Angew. Math.

[3] P. M. Topping, *Uniqueness of instantaneously complete Ricci flows*, Geometry and Topology **19** (2015) 1477–1492.

[4] P. M. Topping and H. Yin, *Sharp decay estimates for the logarithmic fast diffusion equation and the Ricci flow on surfaces*, Preprint.

Regularity of the optimal sets for spectral functionals

Susanna Terracini

(joint work with Dario Mazzoleni and Bozhidar Velichkov)

We deal with shape optimization problems for eigenvalues of the Dirichlet Laplacian, i.e.

$$\min\Big\{F\big(\lambda_1(\Omega),\dots,\lambda_k(\Omega)\big) \,:\, \Omega \subset \mathbb{R}^d,\ |\Omega| = 1\Big\}, \tag{1}$$

where $F : \mathbb{R}^k \to \mathbb{R}$ is a given continuous cost function, increasing in each variable, and $\lambda_i(\Omega)$, for $i = 1, \dots, k$, denotes the i^{th} eigenvalue of the Dirichlet Laplacian on Ω, (counted with the due multiplicity).

The optimization problems of the form (1) naturally arise in many physical contexts as, for example, heat diffusion or wave propagation inside a domain $\Omega \subset \mathbb{R}^d$. Despite of their simple formulation, these problems turn out to be rather challenging and their analysis usually relies on sophisticated variational techniques. Even the question of the existence of a minimizer for problem (1) was answered only recently in its whole generality (see [6] and [12]) in the class of *quasi-open* sets[1].

The regularity of the optimal sets or that of the corresponding eigenfunctions turn out to be a rather complicated question, due to the min-max nature of the spectral cost functionals, and was an open problem since the general Buttazzo-Dal Maso existence theorem. The only complete result on the regularity of the free boundary $\partial\Omega$ of the optimal set Ω concerns the minimizers of (1) for the easier functional λ_1 (under the additional constraint $\Omega \subset D$, where $D \subset \mathbb{R}^d$ is a bounded open set) and is due to Briançon and Lamboley ([2]) who proved that the free boundary of the optimal sets is smooth.

The study of optimal partition problems for eigenvalues of the Dirichlet Laplacian has provided better results for the regularity issue, for example in the recent work [14]. On the other hand, the regularity issue for shape optimization problems like (1) shows a strong link with the regularity of free boundary problems, which was studied e.g. in [1, 15] and this approach turns out to be more fruitful. We are concerned with a special case of problem (1), that is,

$$\min\Big\{\lambda_1(\Omega) + \cdots + \lambda_k(\Omega) \ : \ \Omega \subset \mathbb{R}^d, \text{ open }, \ |\Omega| = 1\Big\}, \tag{2}$$

which, as a simple scaling argument can show (see [9]), is equivalent to

$$\min\Big\{\lambda_1(\Omega) + \cdots + \lambda_k(\Omega) + \Lambda|\Omega| \ : \ \Omega \subset \mathbb{R}^d \text{ open}\Big\}, \tag{3}$$

where $\Lambda > 0$ is a Lagrange multiplier. Our results cn be easily extended to the case of the sum of powers of the first k eigenvalues. This is of great interest also from the point of view of applications to the Lieb-Thirring theory.

We summarize in the following theorem what is known for solutions of (3). The results were proved in [5, 4, 12].

Theorem 1 (D. Bucur, D. Mazzoleni, A. Pratelli and B. Velichkov, 2015)**.** *There exists an optimal set Ω^* for problem* (3) *in the class of quasi-open sets. Every optimal set has finite perimeter and bounded diameter (the boundedness constants depends only on d, k). Finally every optimal set has the first k eigenfunctions which can be extended in a Lipschitz continuous way in the whole $\mathbb{R}^d$ and hence every optimal set is open.*

[1] A quasi-open set is a level set $\{u > 0\}$ of a Sobolev function $u \in H^1(\mathbb{R}^d)$.

Our aim is to achieve a better regularity for the boundary of optimal sets and in order to carry out this work, we consider an equivalent problem in the "free-boundary" setting:

$$\mathcal{F}_0(W) = \sum_{i=1}^{k} \int_{\mathbb{R}^d} |\nabla w_i|^2 \, dx + \Lambda \left| \left\{ w_1^2 + \cdots + w_k^2 > 0 \right\} \right| . \tag{4}$$

Then the vector of normalized eigenfunctions $U = (u_1, \ldots, u_k)$ on the optimal set for (3) is a solution to the problem

$$\min \left\{ \mathcal{F}_0(W) \; : \; W = (w_1, \ldots, w_k) \in H_0^1(\mathbb{R}^d, \mathbb{R}^k), \; \int_{\mathbb{R}^d} w_i w_j \, dx = \delta_{ij} \right\}. \tag{5}$$

Our study of problem (5) can be seen as a vector-valued extension of the result by Weiss [15]. In particular, we have to take care of sign-changing functions, which is a main difference with respect to the classical works by Alt, Caffarelli and Weiss and the major difficulty of our work.

Our main result is the following ([13]).

Theorem 2 (D. Mazzoleni, S. Terracini and B. Velichkov, 2016)**.** *Let Ω_k^* be an optimal set for problem* (3)*. Then Ω_k^* is connected and $\partial\Omega_k^*$ is the disjoint union of $Reg(\partial\Omega_k^*) \cup Sing(\partial\Omega_k^*)$, such that $Reg(\partial\Omega_k^*)$ is $C^{1,\alpha}$ regular, while $dim_H(Sing(\partial\Omega_k^*)) \leq d - d^*$.*

The natural number $d^* \in [3, 7]$ is the smallest dimension at which minimizing free boundaries admit singular cones, for more details we refer to [8, 15].

In order to prove Theorem 2 we need an auxiliary regularity result, which better highlights the link with the free-boundary problem studied by Alt and Caffatelli [1]. We note that the extension to the vectorial case that we are able to prove still requires one function to have a positive trace (and so to be positive itself).

Theorem 3 (D. Mazzoleni, S. Terracini and B. Velichkov, 2016)**.** *Let $k \in \mathbb{N}$ and consider the problem*

$$\begin{aligned} \min \Big\{ \sum_{i=1}^{k} \int_{\mathbb{R}^d} |\nabla u_i|^2 \, dx + \Lambda | \cup_{i=1}^{k} \{u_i \neq 0\} =: D|, \; u_i \in H^1(\mathbb{R}^d), \\ u_i = \phi_i \text{ on } \partial D, \; \phi_i \in C^0(\partial D), \; \phi_1 > 0 \Big\}. \end{aligned} \tag{6}$$

Then every solution $U = (u_1, \ldots, u_k)$ has all the components which are Lipschitz continuous in $\mathbb{R}^d$ and moreover the free-boundary $\partial \cup_{i=1}^{k} \{u_i \neq 0\}$ is $C^{1,\alpha}$ regular up to a singular set which has Hausdorff dimension lower than $d - d^*$.

A major open problem is to prove Theorem 3 without the positivity hypothesis on ϕ_1 (and hence without a positive component u_1).

The proof of our result we mostly use the free-boundary approach for this shape optimization problem, suitably modifying many seminal ideas from [14, 1, 15]. To this aim, se will exploit monotonicity formulas from which we can perform a blow-up analysis. Another key point of our proof is the use of a "viscosity" approach, following mostly the result for a free boundary problem with only one positive function by De Silva [7]. Moreover, we need to use the theory of NTA domains in order to get, at the end, an optimality condition which involves only u_1 on the regular part of the free boundary, so that we can apply the classical results to get $C^{1,\alpha}$ regularity. Then the analysis of the dimension for the singular set follows as in [15, Section 4] by standard arguments.

References

[1] H. W. Alt and L. A. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math. **325** (1981), 105–144.

[2] T. Briançon and J. Lamboley, *Regularity of the optimal shape for the first eigenvalue of the Laplacian with volume and inclusion constraints*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (4) (2009), 1149–1163.

[3] D. Bucur and G. Buttazzo, *Variational Methods in Shape Optimization Problems*, Progress in Nonlinear Differential Equations **65**, Birkhäuser Verlag, Basel (2005).

[4] D. Bucur and D. Mazzoleni, *A surgery result for the spectrum of the Dirichlet Laplacian*, SIAM J. Math. Anal., **47** (6), (2015) 4451–4466.

[5] D. Bucur, D. Mazzoleni, A. Pratelli and B. Velichkov, *Lipschitz regularity of the eigenfunctions on optimal domains*, Arch. Ration. Mech. Anal. **216** (1) 117–151 (2015).

[6] G. Buttazzo, G. Dal Maso, *An existence result for a class of shape optimization problems*, Arch. Rational Mech. Anal. **122** (1993), 183–195.

[7] D. De Silva, *Free boundary regularity from a problem with right hand side*, Interfaces and Free Boundaries, **13** (2) (2011), 223–238.

[8] D. De Silva and D. Jerison, *A singular energy minimizing free boundary*, J. Reine Angew. Math. **635** (2009), 1–21.

[9] A. Henrot and M. Pierre, *Variation et Optimisation de Formes. Une Analyse Géométrique, Mathématiques & Applications* **48** , Springer-Verlag, Berlin (2005).

[10] D.S. Jerison and C.E. Kenig, *Boundary behavior of harmonic functions in nontangentially accessible domains*, Adv. Math. **46** (1), (1982) 80–147.

[11] C.E. Kenig and T. Toro, *Free boundary regularity for harmonic measures and Poisson kernels*, Ann. of Math., **150** (1999), 369–454.

[12] D. Mazzoleni and A. Pratelli, *Existence of minimizers for spectral problems*, J. Math. Pures Appl., **100** (3), (2013) 433–453.

[13] D. Mazzoleni, S. Terracini and B. Velichkov, *Regularity of the optimal sets for spectral functionals*, preprint 2016

[14] M. Ramos, H. Tavares and S. Terracini, *Existence and regularity of solutions to optimal partition problems involving Laplacian eigenvalues*, Arch. Rational Mech. Anal., **220** (2016) 363–443.

[15] G. S. Weiss, *Partial regularity for a minimum problem with free boundary*, J. Geom. Anal., **9** (2) (1999), 317–326.

Existence theorem on the mean curvature flow starting from closed rectifiable set

Yoshihiro Tonegawa

(joint work with Lami Kim)

A family of n-dimensional surfaces $\{\Gamma(t)\}_{t\geq 0}$ in $\mathbb{R}^{n+1}$ is called the mean curvature flow (abbreviated by MCF) if the velocity is equal to its mean curvature at each point and time. Given a smooth surface Γ_0, one can find a smoothly moving MCF starting from Γ_0 until some singularities such as vanishing or pinching occur. The presence of singularities necessitates a weak formulation of MCF, and there have been intensive research dealing with this aspect in the last few decades. Among such attempts, Brakke started a theory of MCF ("Brakke flow") inclusive of singularities in the framework of geometric measure theory in his seminal book [1]. In particular, he developed a general existence theory of MCF: starting from any integral varifold (with a minor technical restriction) of any codimension, he showed that there exists a family of varifolds satisfying the motion law of MCF in a weak sense and existing for all time. One major concern for the validity of his existence theorem is that the proof does not guarantee the non-triviality of the solution when Γ_0 is not a smooth surface. In [3], we rectify this point of non-triviality by introducing a few framework and also modifying Brakke's original argument. The main existence theorem of [3] may be stated roughly as follows.

Theorem 1. *Suppose that $\Gamma_0 \subset \mathbb{R}^{n+1}$ is a closed countably n-rectifiable set whose complement $\mathbb{R}^{n+1} \setminus \Gamma_0$ equals $\cup_{i=1}^N E_{0,i}$, where $E_{0,1}, \ldots, E_{0,N} \subset \mathbb{R}^{n+1}$ are mutually disjoint non-empty open sets and $N \geq 2$. Assume that the n-dimensional Hausdorff measure of Γ_0 is finite or grows at most exponentially near infinity. Then, for each $i = 1, \ldots, N$, there exists a family of open sets $\{E_i(t)\}_{t\geq 0}$ with $E_i(0) = E_{0,i}$ such that $E_1(t), \ldots, E_N(t)$ are mutually disjoint for each $t \geq 0$ and $\Gamma(t) := \cup_{i=1}^N \partial E_i(t)$ coincides with the space-time support of a nontrivial Brakke flow starting from Γ_0. Each $E_i(t)$ moves continuously in time with respect to the Lebesgue measure.*

We may regard each $E_i(t) \subset \mathbb{R}^{n+1}$ as a region of "i-th grain" at time t, and $\Gamma(t)$ as the "grain boundaries" which move by the mean curvature in a generalized sense. Some of $E_i(t)$ shrink and vanish, and some may grow and may even occupy the whole $\mathbb{R}^{n+1}$ in finite time. Note that the continuity of $E_i(t)$ guarantees that $\Gamma(t) \neq \emptyset$ at least for a short initial time interval, and $\Gamma(t) \neq \emptyset$ unless $E_i(t) = \mathbb{R}^{n+1}$ for some i.

Notion not stated clearly in Theorem 1 is that of Brakke flow, which is as follows. For simplicity, assume Γ_0 has a finite n-dimensional Hausdorff measure, $\mathcal{H}^n(\Gamma_0) < \infty$.

Definition. *A Brakke flow starting from Γ_0 is a family of n-varifolds $\{V_t\}_{t\geq 0}$ satisfying the following.*

(1) *$V_0 = |\Gamma_0|$ =unit density varifold induced from Γ_0.*
(2) *For $\mathcal{L}^1$ a.e. $t \in \mathbb{R}$, V_t is an integral varifold with L^2 generalized mean curvature vector $h(\cdot, V_t)$.*
(3) *$\|V_t\|(\mathbb{R}^n)$ is decreasing in t and $\int_0^\infty \int_{\mathbb{R}^{n+1}} |h(\cdot, V_t)|^2 \, d\|V_t\| dt \leq \mathcal{H}^n(\Gamma_0)$.*
(4) *For any $0 \leq t_1 < t_2 < \infty$ and $\phi \in C_c^1(\mathbb{R}^{n+1} \times \mathbb{R}^+; \mathbb{R}^+)$, we have*

$$\|V_t\|(\phi(\cdot, t))\Big|_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2} \int_{\mathbb{R}^{n+1}} \{\nabla\phi(\cdot, t) - \phi(\cdot, t)h(\cdot, V_t)\} \cdot h(\cdot, V_t) + \frac{\partial \phi}{\partial t}(\cdot, t) \, d\|V_t\| dt.$$

Here, $\|V\|$ is the weight measure of V.

The property (4) is a weak form of the motion law of MCF. The property (2) allows a possibility of having a higher (≥ 2) multiplicity representing a "folding" of surfaces (whether it happens or not is not clear). When the multiplicity stays 1 for a.e. $t > 0$, we say that the flow is a unit density flow. Almost everywhere regularity of unit density flow for general Brakke flow has been studied originally by Brakke and is recently completed by [2, 5, 4]. If the flow is a limit of smooth MCF, White's reguralityy theory [6] gives also the almost everywhere regularity. Theorem 1 includes as a part of theorem the existence of $\{V_t\}_{t\geq 0}$ satisfying the Definition 1. We may then define a Radon measure μ on $\mathbb{R}^{n+1} \times \mathbb{R}^+$ by $d\mu = d\|V_t\|dt$. The claim of Theorem 1 is that $\{x \in \mathbb{R}^{n+1} : (x,t) \in \operatorname{spt}\mu\} = \Gamma(t)$ for all $t > 0$.

The proof is divided roughly into two stages, one is a construction of time-discrete approximate flows, and the other is the proof of a suitable compactness theorem of varifolds suited for our purpose. In each time step of the construction, there are two different kinds of motions, one is a locally area-minimizing Lipschitz deformation and the other is a motion by smoothed mean curvature vector. There are a number of estimates measuring the errors of approximations. For the second stage, we prove an analogue of Allard's compactness theorem of integral varifolds. Here the difference is that we only have a control of smoothed mean curvature vectors of converging integral varifolds, not the exact mean curvature vectors. To supplement this point, we have a local area minimizing property in a small length scale. There are roughly three different length scales, grid size for time (very small), smoothing of mean curvature vectors (small) and area-minimizing (not so small). These differing length scales play an important role throughout the analysis.

References

[1] K. A. Brakke, *The motion of a surface by its mean curvature*, Mathematical Notes, **20** Princeton University Press, 1978.

[2] K. Kasai, Y. Tonegawa, *A general regularity theory for weak mean curvature flow*, Calc. Var. Partial Differential Equations **50** (2014) no. 1, 1–68.

[3] L. Kim, Y. Tonegawa, *On the mean curvature flow of grain boundaries*, to appear in Ann. Inst. Fourier (Grenoble), arXiv:1511.02572.

[4] A. Lahiri, *Regularity of the Brakke flow*, Dissertation, Freie Universität Berlin, 2014.

[5] Y. Tonegawa, *A second derivative Hölder estimate for weak mean curvature flow*, Advances in Calculus of Variations **7** (2014) no. 1, 91–138.

[6] B. White, *A local regularity theorem for classical mean curvature flows*, Ann. of Math. (2) **161** (2005), no. 3, 1487–1519.

Reporter: Florian Johne

Participants

Prof. Dr. Giovanni Alberti
Dipartimento di Matematica "L.Tonelli"
Università di Pisa
Largo Bruno Pontecorvo, 5
56127 Pisa
ITALY

Prof. Dr. Lisa Beck
Institut für Mathematik
Universität Augsburg
86135 Augsburg
GERMANY

Dr. Spencer T. Becker-Kahn
Department of Mathematics
Massachusetts Institute of Technology
Rm. 2-165
77 Massachusetts Avenue
Cambridge, MA 02139-4307
UNITED STATES

Tom Begley
Department of Pure Mathematics
and Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Dr. Costante Bellettini
Department of Pure Mathematics
and Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Prof. Dr. Simon Brendle
Department of Mathematics
Columbia University
New York, NY 10027
UNITED STATES

Prof. Dr. Yann Brenier
CMLS
École Polytechnique
Plateau de Palaiseau
91128 Palaiseau Cedex
FRANCE

Otis Chodosh
Department of Pure Mathematics and
Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Dr. Eleonora Cinti
Weierstraß-Institut für
Angewandte Analysis und Stochastik
Mohrenstrasse 39
10117 Berlin
GERMANY

Dr. Maria Colombo
Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190
8057 Zürich
SWITZERLAND

Dr. Sara Daneri
Mathematisches Institut
FAU Universität Erlangen-Nürnberg
Cauerstraße 11
91058 Erlangen
GERMANY

Prof. Dr. Manuel del Pino
Departamento de Ingenieria Matemática
F.C.F.M.
Universidad de Chile
Casilla 170, Correo 3
Santiago
CHILE

Prof. Dr. Guido De Philippis
SISSA
Office 547
Via Bonomea, 265
34136 Trieste
ITALY

Prof. Dr. Jean Dolbeault
CEREMADE
Université Paris Dauphine
Place du Marechal de Lattre de Tassigny
75775 Paris Cedex 16
FRANCE

Dr. Xavier Fernández-Real
Department of Mathematics
The University of Texas at Austin
1 University Station C1200
Austin, TX 78712-1082
UNITED STATES

Prof. Dr. Alessio Figalli
Department Mathematik
ETH Zürich
ETH-Zentrum, HG G 63.2
Rämistrasse 101
8092 Zürich
SWITZERLAND

Prof. Dr. Nicola Fusco
Dipartimento di Matematica e Applicazioni
Università degli Studi di Napoli Federico II
Via Cintia
80126 Napoli
ITALY

Dr. Robert Haslhofer
Department of Mathematics
University of Toronto
40 St George Street
Toronto, Ont. M5S 2E4
CANADA

Fritz L. Hiesmayr
Department of Pure Mathematics and Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Prof. Dr. Gerhard Huisken
Fachbereich Mathematik
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Pei-Ken Hung
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
UNITED STATES

Prof. Dr. Radu Ignat
Institut de Mathématiques de Toulouse
Université Paul Sabatier
118, route de Narbonne
31062 Toulouse Cedex 9
FRANCE

Prof. Dr. Robert L. Jerrard
Department of Mathematics
University of Toronto
40 St George Street
Toronto, Ont. M5S 2E4
CANADA

Florian Johne
Fachbereich Mathematik
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Dr. Martin Kell
Fachbereich Mathematik
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Dr. Brian J. Krummel
Department of Mathematics
The University of Texas at Austin
1 University Station C1200
Austin, TX 78712-1082
UNITED STATES

Dr. Xavier Lamy
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstrasse 22 - 26
04103 Leipzig
GERMANY

Tim Laux
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstrasse 22 - 26
04103 Leipzig
GERMANY

Prof. Dr. John Lott
Department of Mathematics
University of California, Berkeley
970 Evans Hall
Berkeley CA 94720-3840
UNITED STATES

Prof. Dr. Stephan Luckhaus
Mathematisches Institut
Universität Leipzig
Postfach 10 09 20
04109 Leipzig
GERMANY

Prof. Dr. Francesco Maggi
Department of Mathematics
The University of Texas at Austin
1 University Station C1200
Austin, TX 78712-1082
UNITED STATES

Prof. Dr. Ulrich Menne
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm
GERMANY

Dr. Andrea Mondino
Department of Mathematics
Warwick University
Gibbet Hill Road
Coventry CV4 7AL
UNITED KINGDOM

Prof. Dr. Aaron C. Naber
Department of Mathematics
Northwestern University
Lunt Hall
Evanston, IL 60208-2730
UNITED STATES

Dr. Mircea Petrache
Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie
4, Place Jussieu
75005 Paris Cedex
FRANCE

Prof. Dr. Tristan Rivière
Departement Mathematik
ETH-Zentrum
Rämistrasse 101
8092 Zürich
SWITZERLAND

Prof. Dr. Filippo Santambrogio
Laboratoire de Mathématiques d'Orsay
Université Paris-Sud
Centre d'Orsay
Batiment 425
91405 Orsay Cedex
FRANCE

Dr. Felix Schulze
Department of Mathematics
University College London
Gower Street
London WC1E 6BT
UNITED KINGDOM

Prof. Dr. Charles K. Smart
Department of Mathematics
Cornell University
436 Malott Hall
Ithaca, NY 14853-4201
UNITED STATES

Prof. Dr. Didier Smets
Laboratoire Jacques-Louis Lions
Université Paris 6
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. Emanuele Nunzio Spadaro
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstrasse 22 - 26
04103 Leipzig
GERMANY

Dr. Luca Spolaor
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstrasse 22 - 26
04103 Leipzig
GERMANY

Prof. Dr. Michael Struwe
Departement Mathematik
ETH-Zentrum
Rämistrasse 101
8092 Zürich
SWITZERLAND

Prof. Dr. Susanna Terracini
Dipartimento di Matematica
Università degli Studi di Torino
Via Carlo Alberto, 10
10123 Torino
ITALY

Prof. Dr. Yoshihiro Tonegawa
Department of Mathematics
Faculty of Science
Tokyo Institute of Technology
2-12-1, Ookayama, Meguro
Tokyo 152-8551
JAPAN

Prof. Dr. Peter M. Topping
Mathematics Institute
University of Warwick
Gibbet Hill Road
Coventry CV4 7AL
UNITED KINGDOM

Prof. Dr. Tatiana Toro
Department of Mathematics
University of Washington
Padelford Hall
Box 354350
Seattle, WA 98195-4350
UNITED STATES

Prof. Dr. Maria G. Westdickenberg
Lehrstuhl 1 für Mathematik
RWTH Aachen
52056 Aachen
GERMANY

Prof. Dr. Brian White
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
UNITED STATES

Prof. Dr. Neshan Wickramasekera
Department of Pure Mathematics
and Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Prof. Dr. Robert Young
Courant Institute of Mathematical
Sciences
New York University
251, Mercer Street
New York, NY 10012-1110
UNITED STATES