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C*-Algebras

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Abstract. The field of operator algebras is a flourishing area of mathematics with strong ties to many other areas including functional/harmonic analysis, topology, (non-commutative) geometry, group theory and dynamical systems. The C^*-Algebra workshop at Oberwolfach brings together leading experts and young researchers in all subjects where C^*-algebras play a major role. The main goal of this meeting is to foster contacts and collaborations between researchers from different directions, as well as to highlight the main developments in the field.

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Introduction by the Organisers

The perhaps most important development in operator algebras the last years is the almost completion of the so-called *Elliott program*, that seeks to classify simple nuclear C^*-algebras by a simple invariant (called the *Elliott invariant*) that foremost consists of K-theory. This program was implicity initiated in 1959 with Glimm's classification of matricial C^*-algebras (also called Uniformly Hyperfinite C^*-algebras, abbreviated as UHF-algebras), and continued in the late 1960's and early 1970's with the classification by Bratteli and Elliott of the larger class of AF-algebras by their ordered K_0-group. These important results were back then considered to be very special and unlikely to admit any significant generalization, until Elliott around 1990 proved that a much larger class of simple nuclear C^*-algebras (the class of simple real rank zero $A\mathbb{T}$-algebras) admits classification by the Elliott invariant, now also involving K_1. Elliott suggested that a classification should be possible for all simple nuclear separable C^*-algebras, thus formally initiating the Elliott program. This problem has over the last 25 years been investigated by dozens of mathematicians, and has resulted in the publication of several hundreds articles occuping thousands of pages.

In the mid 1990's to the early 2000's it was shown by Villadsen, Rørdam and Toms that the classification conjecture, in its first optimistic form, cannot be true, as certain C^*-algebras of "infinite dimensional" type refuse classification. The complement of the class of "infinite dimensional" C^*-algebras was singled out in the conjecture by Toms and Winter, stating that three seemingly very different properties of a simple, separable, nuclear C^*-algebras are equivalent, one of them being "finite nuclear dimension" introduced by Winter and Zacharias. The conjecture by Toms and Winter is today confirmed (under some mild assumptions on the trace simplex of the C^*-algebras). The (partial) confirmation of this conjecture alone has been the topic of several articles, many of which have been published in top journals including Inventiones Math. and Acta Math., and by several authors.

Last year, in 2015, Tikuisis, White and Winter proved that faithful traces on separable nuclear C^*-algebras satisfying the so-called Universal Coefficient Theorem (UCT) (a K-theoretical condition) are automatically quasidiagonal, and hence that all stably finite, nuclear C^*-algebras in the UCT class are quasidiagonal. Among many other things, this also settled in the positive the conjecture by Rosenberg that a group is amenable if and only if its group C^*-algebra is quasidiagonal. This remarkable breakthrough in combination with a monumental work by Gong, Lin and Niu (a 283 pages long paper from 2014) followed by a much shorter paper by the same three authors and George Elliott from 2015, resulted in the complete classification of all unital simple nuclear separable C^*-algebras in the UCT class with finite nuclear dimension by the Elliott invariant. All these conditions are also necessary, and only the assumption that it belongs to the UCT class may automatically follow from the other conditions. This remains an important open problem.

These results were the main topics of our workshop, and were explained in several lectures, including ones given by Huaxin Lin, Wilhelm Winter and Stuart White.

Another recent major breakthrough within the theory of C^*-algebras, which was highligted at our workshop by a talk by Matt Kennedy, concerns the structure of C^*-algebras associated with groups. To each (countable, discrete) group one can associate a C^*-algebra arising from the left-regular representation of the group. It was shown by Powers in 1975 that the C^*-algebra associated with the free group on $n \geq 2$ generators is simple and has unique trace. (This was the first step towards verifying a conjecture by Kadison that this C^*-algebra is a simple C^*-algebra with no non-trivial projections.) It also spurred the question as to which groups give rise to simple C^*-algebras, respectively, C^*-algebras with a unique trace (e.g., to C^*-simple groups and groups with the unique trace property, respectively).

In the following decades it was shown that large classes of (non-amenable) groups are C^*-simple and have the unique trace property. In all the cases considered it was shown that C^*-simplicity was equivalent to the unique trace property, which again was equivalent to a property of the group: that it has no non-trivial normal amenable subgroups. It is fairly easy to see that both of the former conditions imply the latter, but no other relations between these three properties of a group were known to hold in general. In 2013, Kalantar and Kennedy gave a remarkable reformulation of the Furstenberg boundary of a group in terms of injective envelopes, and they were able to show that a group is C^*-simple if and only if it admits a (topologically) free boundary action (or, equivalently, that the action of the group on its Furstenberg boundary is free). Using these results, Breuillard, Kalantar, Kennedy and Ozawa proved that the unique trace property is equivalent to the absence of non-trivial normal amenable subgroups, and that C^*-simplicity implies the unique trace property. Le Boudec shortly after gave examples showing that the reverse implication does not hold: there are groups with the unique trace property that are not C^*-simple. Further examples of such groups were subsequently given by Ivanov and Omland.

These results relate to the famous problem if the Thompson group F is amenable. Indeed, as shown by Haagerup and Olesen, non-amenability of F is equivalent to C^*-simplicity of the Thompson group T. It is known that the group T has the unique trace property, but we do not know if it is also C^*-simple.

Among the many other highlights of the workshop, we would like to mention the lecture of Dimitri Shlyakhtenko who proved, using methods of Voiculescu's free probability theory, that the group von Neumann algebra of a sofic group with vanishing first L^2-Betti number is never isomorphic to a free group factor. Other highlights were the lecture of Cyril Houdayer on the classification of free Araki–Woods factors and the one of Hiroshi Ando who gave a counterexample to Popa's question on the characterization of Polish groups of finite type. Finally, a series of very short lectures on Thursday afternoon by young researchers demonstrated how diverse and in good shape the current research in Operator Algebras is.

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Workshop: C*-Algebras

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Structure and Classification: Where next?

Wilhelm Winter

The classification programme for nuclear C*-algebras has seen dramatic progress in the simple and unital case. In this talk I review the current state of affairs, highlight some loose ends that still need to be dealt with, and speculate about where to go from here.

Let $\mathcal{A}$ denote the class of separable, unital, simple, nuclear and infinite dimensional C*-algebras. Let us denote subclasses with certain properties by appropriate superscripts, e.g. $\mathcal{A}^{\dim_{\mathrm{nuc}}}$ stands for those C*-algebras in $\mathcal{A}$ which have finite nuclear dimension, $\mathcal{A}^{\mathcal{Z}}$ for $\mathcal{Z}$-stable ones, $\mathcal{A}^{\mathrm{comparison}}$ for those with strict comparison of positive elements and so on.

The regularity conjecture due to Andrew Toms and myself then says that the latter classes agree,

$$\mathcal{A}^{\dim_{\mathrm{nuc}}} = \mathcal{A}^{\mathcal{Z}} = \mathcal{A}^{\mathrm{comparison}}.$$

Moreover, for (stably) finite C*-algebras we expect finite nuclear dimension to coincide with finite decomposition rank,

$$\mathcal{A}^{\mathrm{dr}} = \mathcal{A}^{\mathrm{finite}, \dim_{\mathrm{nuc}}}.$$

Much about the conjecture is known by now, in particular the implications

$$\mathcal{A}^{\mathrm{dr}} \subset \mathcal{A}^{\dim_{\mathrm{nuc}}} \subset \mathcal{A}^{\mathcal{Z}} \subset \mathcal{A}^{\mathrm{comparison}}.$$

(of which only the first is trivial) hold in full generality. The reverse implications have been verified in many cases, but the picture is not yet complete. In particular, for the statements

$$\mathcal{A}^{\dim_{\mathrm{nuc}}} \supset \mathcal{A}^{\mathcal{Z}} \supset \mathcal{A}^{\mathrm{comparison}}$$

little is known beyond the case where the trace simplex has finite dimensional and/or compact extreme boundary. The all-important test case to deal with seems to be the Poulsen simplex. An intriguing partial result would be the statement

$$\mathcal{A}^{\mathrm{comparison}} = \mathcal{A}^{\mathrm{almost\,divisible}}$$

or even just a forward or reverse inclusion. It seems very worthwhile to also approach the problem from the opposite side: a major issue at this point is that we still have relatively little information on "non-regular" nuclear C*-algebras. The known examples are essentially based on Villadsen's ideas, which were very successfully exploited further by Rørdam, Toms, and Kerr–Giol. However, in all of these results nonregularity essentially originates from the same source—but there might be entirely different ways of building non-$\mathcal{Z}$-stable simple nuclear C*-algebras. (Of course, what I am really asking for here are systematic range results for the Cuntz semigroup!)

Let us return to the relation between $\mathcal{A}^{\mathrm{dr}}$ and $\mathcal{A}^{\mathrm{finite},\dim_{\mathrm{nuc}}}$. The issue is now completely settled for C*-algebras satisfying the UCT: By last year's result of Tikuisis–White–Winter, faithful traces on nuclear UCT C*-algebras are quasidiagonal, so in particular

$$\mathcal{A}^{\mathrm{finite},\dim_{\mathrm{nuc}},\mathrm{UCT}} = \mathcal{A}^{\mathrm{finite},\dim_{\mathrm{nuc}},\mathrm{UCT},\mathrm{T}=\mathrm{T}_{\mathrm{qd}}}.$$

And by last year's work of Elliott–Gong–Lin–Niu, building on Gong–Lin–Niu, this class is classified by the Elliott invariant. (Of course this is novel only in the presence of traces; the infinite case is just Kirchberg–Phillips classification.)

This is clearly a very general and extremely satisfactory result—but at the same time it opens up all sorts of follow-up questions:

Is the UCT really necessary in the quasidiagonality result of Tikuisis–White–Winter?

Is there a chance to achieve classification in terms of the Elliott invariant in conjunction with the Cuntz semigroup in the non-$\mathcal{Z}$-stable case?

What about the non-simple case?

Or the non-unital case?

This last question essentially boils down to handling stably projectionless C*-algebras and for now seems the most tractable of this block of questions. Surprisingly, it is also relevant to the unital case!

Let us start with a stably finite version of Kirchberg's $\mathcal{O}_2$-absorption theorem. Let $\mathcal{W}$ denote the simple, monotracial, K-trivial C*-algebra first studied by Kishimoto and Kumjian, later by Razak and then by Jacelon. $\mathcal{W}$ is KK-equivalent to the zero algebra; it can be described as a "deunitized" version of $\mathcal{Z}$ (as carried out by Jacelon and myself), or as an inductive limit of nonunital dimension drop intervals (due to Razak) or, as done originally by Kishimoto–Kumjian, as a corner of a crossed product of $\mathcal{O}_2$ by an action of the reals.

One intriguing question around $\mathcal{W}$ is whether it is self-absorbing, i.e., whether $\mathcal{W} \cong \mathcal{W} \otimes \mathcal{W}$.

Moreover, classification predicts that if $A \in \mathcal{A}^{\mathrm{monotracial}}$, then $A \otimes \mathcal{W} \cong \mathcal{W}$, and this statement would be a stably finite analogue of Kirchberg's $\mathcal{O}_2$-absorption theorem. With the results above at hand, this statement looks like a safe bet (although by no means trivial) under the extra hypothesis of A satisfying the UCT. Without the UCT, to me the problem seems to be essentially as hard as the quasidiagonality problem for simple nuclear C*-algebras. This point of view is supported by the following, which is the only new (and not yet publicly available) result of the talk:

Theorem. Let $\mathcal{D}$ be a strongly self-absorbing C*-algebra. Then $\mathcal{D} \otimes \mathcal{W} \cong \mathcal{W}$ if and only if $\mathcal{D}$ is quasidiagonal.

Quasidiagonality and amenability

Stuart White
(joint work with Aaron Tikuisis and Wilhelm Winter)

In recent years the analogies between the work of Connes, Haagerup and Popa on injective factors, and progress on the structure and classification of simple nuclear C*-algebras have become apparent, and so looking again at others aspects of this work, to find appropriate C*-versions seems increasingly worthwhile. In particular, a key step in Connes' proof of injectivity implies hyperfiniteness is the observation that a (separably acting) injective II_1 factor $\mathcal{M}$ embedds into $\mathcal{R}^\omega$ (the ultraproduct of the hyperfinite II_1 factor). It is from such an embedding that Connes obtained the finite dimensional algebras that are eventually used to witness hyperfiniteness of an injective II_1 factor $\mathcal{M}$. The analogous condition for nuclear C*-algebras is quasidiagonality.

Definition. *A* C*-*algebra A is* quasidiagonal *if there exists a net of completely positive and contractive*[1] *(c.p.c.) maps* $\phi_i : A \to M_{k_i}$ *such that:*

- $\|\phi_i(ab) - \phi_i(a)\phi_i(b)\| \to 0$ *for all* $a, b \in A$*;*
- $\|\phi_i(a)\| \to \|a\|$ *for all* $a \in A$.

When A is separable and unital, quasidiagonality gives rise to a unital embedding of A into the ultraproduct[2] $\mathcal{Q}_\omega$ of the universal UHF algebra $\mathcal{Q}$ as a sequence of maps witnessing quasidiagonality will induce an embedding $A \hookrightarrow \mathcal{Q}_\omega$, which is *liftable* to a u.c.p. map $A \to \ell^\infty(\mathcal{Q})$. By the Choi-Effros lifiting theorem liftability is automatic when A is nuclear, so for separable nuclear C*-algebras quasidiagonlity is equivalent to $\mathcal{Q}_\omega$-embeddibility.

Just as Connes used an $\mathcal{R}^\omega$ embedding to obtain local structure on an injective II_1 factor, quasidiagonality can be used to obtain powerful approximations in the C*-setting. This was first done by Popa in [9], and this appears again in the recent striking developments of Elliott, Gong, Lin and Niu (described in Huaxin's talk during this meeting). They use Winter's classification by embeddings technique [12] to obtain classifiability of simple algebras of finite nuclear dimension in the presence of enough quasidiagonality ('quasidiagonality of all traces' — see below) [5]. That is they obtain good rational tracial approximations from quasidiagonality, so that they can appeal to [7].

Here I will discuss when we can obtain this quasidiagonality. There are two fundamental obstructions. The first is essentially immediate: quasidiagonal C*-algebras must be stably finite (as $\mathcal{Q}_\omega$ contains no infinite projections). The second is more subtle: quasidiagonality entails some kind of amenability as first observed by Rosenberg (in the appendix to [8]) in the context of groups.

[1] When A is unital, these maps can be taken unital; when A is separable one can of course use a sequence.

[2] defined to be the quotient $\ell^\infty(\mathcal{Q})/\{(x_n) : \lim_{n\to\omega} \|x_n\| = 0\}$, for some fixed free ultrafilter ω on $\mathbb{N}$

Proposition (Rosenberg)**.** *Let G be a countable discrete group such that $C^*_r(G)$ is quasidiagonal. Then G is amenable.*

Sketch proof. [3] Let $(\phi_n)_n : C^*_r(G) \to M_{k_n}$ witness the quasidiagonality of $C^*_r(G)$ and embedd each M_{k_n} unitally into $\mathcal{Q}$, so that the sequence $(\phi_n)_n$ induces a unital embedding $\Phi : C^*_r(G) \to \mathcal{Q}_\omega$. Consider $C^*_r(G) \subset \mathcal{B}(\ell^2(G))$ in the canonical way. Then the map Φ extends to a unital completely positive map $\widetilde{\Phi} : \mathcal{B}(\ell^2(G)) \to \mathcal{Q}_\omega$.[4] Define a state μ on $\mathcal{B}(\ell^2(G))$ by $\mu = \tau_{\mathcal{Q}_\omega} \circ \widetilde{\Phi}$.

For $T \in \mathcal{B}(\ell^2(G))$, and the canonical unitary $u_g \in C^*_r(G)$ coming from the group element $g \in G$, a multiplicative domain argument (essentially Steinespring) gives $\widetilde{\Phi}(u_g T u_g^*) = \Phi(u_g)\widetilde{\Phi}(T)\Phi(u_g)^*$. Thus

$$\mu(u_g T u_g^*) = \tau_{\mathcal{Q}_\omega}(\Phi(u_g)\widetilde{\Phi}(T)\Phi(u_g)^*) = \tau_{\mathcal{Q}_\omega}(\widetilde{\Phi}(T)) = \mu(T).$$

Thus this state μ restricts to an invariant mean on $\ell^\infty(G) \subset \mathcal{B}(\ell^2(G))$. □

This argument goes through more generally and it shows that any unital quasidiagonal C*-algebra has an *amenable trace* τ: i.e. when A is faithfully represented on $\mathcal{H}$, then there is a state μ on $\mathcal{B}(\mathcal{H})$ extending τ, with $\mu(uTu^*) = \mu(T)$ for all $T \in \mathcal{B}(H)$ and unitaries $u \in A$.[5] Equivalently[6] one can characterise amenable traces through an approximation property: τ is *amenable* if and only if there exists completely positive and contractive maps $\phi_i : A \to M_{k_i}$ such that

- $\|\phi_i(ab) - \phi_i(a)\phi_i(b)\|_{2,\mathrm{tr}_{M_{k_i}}} \to 0$ for all $a, b \in A$;
- $\mathrm{tr}_{M_{k_i}}(\phi_i(a)) \to \tau(a)$ for all $a \in A$,

where $\mathrm{tr}_{M_{k_i}}$ is the normalised trace on M_{k_i} inducing the 2-norm $\|x\|_{2,\mathrm{tr}_{M_{k_i}}} = \mathrm{tr}_{M_{k_i}}(x^*x)^{1/2}$. The presence of this amount of amenability is the second fundamental obstruction to quasidiagonality.

Given this characterisation of amenable traces, it is natural to cast quasidiagonality at the tracial level. This was done by Brown in his excellent monograph [1]: a trace τ on A is *quasidiagonal* if and only if there exists completely positive and contractive maps $\phi_i : A \to M_{k_i}$ such that

- $\|\phi_i(ab) - \phi_i(a)\phi_i(b)\| \to 0$ for all $a, b \in A$;
- $\mathrm{tr}_{M_{k_i}}(\phi_i(a)) \to \tau(a)$ for all $a \in A$.

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[3] not Rosenberg's original proof, and not due to me either. This is the folklore proof of this fact that has evolved from Voiculescu's observation that unital quasdiagonal C*-algebras have traces. I extracted it from Nate and Taka's book.

[4] Here liftability is crucial; one uses Arveson's extension theorem to extend each ϕ_n to $\tilde{\phi}_n : \mathcal{B}(\ell^2(G)) \to M_{k_n}$, then obtain $\widetilde{\Phi}$ as the map induced by $(\tilde{\phi})_n$. Indeed liftability *must* be crucial to this argument as a fabulous result of Haagerup and Thorbjørnsen shows that $C^*_r(\mathbb{F}_2)$ is MF, so also embedds in $\mathcal{Q}_\omega$. Of course this embedding is not liftable.

[5] This concept is independent of the choice of representation. Note too that these are essentially the *hypertraces* appearing in Connes work (now set at the C*-level).

[6] via the circle of ideas in Connes' work with hypertraces; see the account in chapter 6 of Nate and Taka's book.

Rosenberg's proposition then holds at the level of traces (quasidiagonality $\Longrightarrow$ amenability) and from the approximation view point it is immediate from the inequality $\|\cdot\|_2 \leq \|\cdot\|$.

Questions:

(1) (Rosenberg's Conjecture) If G is discrete and amenable, must $\mathrm{C}^*_r(G)$ be quasidiagonal?
(2) (The Blackadar-Kirchberg Problem) Are all stably finite nuclear C*-algebras quasidiagonal?
(3) (Brown) Are all amenable traces quasidiagonal?

Progress was recently made on Rosenberg's conjecture by Ozawa, Sato and Rørdam, who resolved it positively for elementary amenable groups via classification techniques. Conceptually this is a bit of a miracle; one has no right a-priori to prove things about group C*-algebras of amenable groups (which are never simple) in this way, but it turns out to be possible.

Theorem ([11])**.** *Let A be a separable* C**-algebra in the UCT class. Then all faithful traces on A are quasidiagonal.*

This result has subsequently been extended by Jamie Gabe to faithful amenable traces on separable exact C*-algebras in the UCT class ([6]); to my mind this is the right general framework for this method of obtaining quasidiagonality.

As a consequence of the main theorem we resolve Rosenberg's conjecture positively (with thanks to Tu, Higson and Kasparov, for their work showing that $\mathrm{C}^*_r(G)$ satisfies the UCT). Via Haagerup's profound result that quasitraces are traces on exact C*-algebras, we obtain a positive answer to the Blackadar-Kirchberg question for simple C*-algebras with the UCT. Brown's question also has a positive answer for separable exact C*-algebras in the UCT class.[7] For the general version of Brown's question the key test case is the trace on the hyperfinite II_1 factor $\mathcal{R}$: is $\mathcal{R}$ quasidiagonal as a C*-algebra?

Some ingredients of the proof

A more extensive expository account of the proof can be found in [13] or the introduction to [11].

Voiculescu famously showed that quasidiagonality is invariant under homotopy. In particular for any C*-algebra A the cone $C_0((0,1]) \otimes A$ is quasidiagonal as it is homotopic to the zero C*-algebra, which is quasidiagonal.[8] With hindsight, our starting point is an answer to Brown's question on cones.

Proposition (Kirchberg-Rørdam, Sato-W-Winter, TWW, Gabe)**.** *All amenable traces on a cone $C_0((0,1]) \otimes A$ are quasidiagonal.*[9]

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[7] Even in the nuclear case this isn't immediate from the theorem, it is obtained from it using additional techniques from Brown's monograph.

[8] Surprisingly, the zero C*-algebra and discussions of its properties appeared quite frequently during this workshop!

[9] This precise statement can be found in [3], but it really comes from successive improvements of statements by the authors listed.

This result is quite specific to cones, using the duality between order zero maps and cones over *-homomorphisms. Can one answer Brown's question for more general contractible C*-algebras?

Given a trace τ_A on a separable unital and nuclear C*-algebra A, we consider the trace $\mathrm{leb}\otimes\tau_A$ on $C_0((0,1]\otimes A$ (where leb denotes Lebesgue measure). The proposition then gives a *-homomorphism $\acute{\Phi} : C_0((0,1])\otimes A \to \mathcal{Q}_\omega$ inducing $\mathrm{leb}\otimes\tau_A$, i.e. $\tau_{\mathcal{Q}_\omega}\circ\acute{\Phi} = \mathrm{leb}\otimes\tau_A$. We can also flip the interval around and consider the cone $C_0([0,1])\otimes A$. Again there is a *-homomorhism $\grave{\Phi} : C_0([0,1))\otimes A\to\mathcal{Q}_\omega$ inducing $\mathrm{leb}\otimes\tau_A$.[10] Of course if we just use the proposition twice, we can't expect any form of compatibility between $\acute{\Phi}$ and $\grave{\Phi}$. Instead Cuntz semigroup techniques can be used to carefully construct $\grave{\Phi}$ from $\acute{\Phi}$ so that they agree on the scalars (i.e. they restrict the same map on $C_0((0,1))\otimes 1_A$) and

$$\acute{\Phi}(\{\mathrm{id}_{[0,1]}\otimes 1_A) + \grave{\Phi}((1-\mathrm{id}_{[0,1]})\otimes 1_A) = 1_{\mathcal{Q}_\omega}.$$

I view this as a kind of "2-coloured quasidiagonality", as translating $\grave{\Phi}$ and $\acute{\Phi}$ back to order zero maps we would have two order zero maps $A\to\mathcal{Q}_\omega$ whose sum is unital. Indeed, this is an ingredient in the arguments of [10] and [2] obtaining finite nuclear dimension from $\mathcal{Z}$-stability.

If it was possible to arrange for $\grave{\Phi}$ and $\acute{\Phi}$ to agree on all of $C_0((0,1))\otimes A$ we would be done, as we could then use them to define a map from $C([0,1])\otimes A\to\mathcal{Q}_\omega$ inducing $\mathrm{leb}\otimes\tau$, from which quasidiagonality of τ would follow. Obtaining this directly seems highly implausible, so we must look for weaker conditions. In fact one can prove:

$$\tau \text{ is quasidiagonal} \iff \grave{\Phi} \text{ and } \acute{\Phi} \text{ are unitarily equivalent.}$$

The implication from left to right can be deduced from classification results for cones, but isn't needed to prove the theorem. The argument from right to left is a 'patching trick' working in the additional space of $M_2(\mathcal{Q}_\omega)$ to give additional room.

Proving the unitary equivalence of $\grave{\Phi}$ and $\acute{\Phi}$ directly also looks highly implausible. But one can use *stable uniqueness* theorems appearing in the work of Dadarlat, Eilers, and Lin. The exact version we use is a modification of a result from [4] which allows one to deduce approximate unitarily equivalence of these maps after the addition of large summands of themselves. Precisely for a given finite subset $\mathcal{F}$ of $C_0((0,1))\otimes A$ and $\varepsilon>0$, there exists $n\in\mathbb{N}$ such that $\grave{\Phi}\oplus\acute{\Phi}^{\oplus n}$ is ε-approximately unitarily conjugate to $\acute{\Phi}\oplus\acute{\Phi}^{\oplus n}$ on $\mathcal{F}$ (similarly for $\grave{\Phi}\oplus\grave{\Phi}^{\oplus n}$ and $\acute{\Phi}\oplus\grave{\Phi}^{\oplus n}$). With considerable care, and the UCT, one can have the same replacing $\grave{\Phi}$ and $\acute{\Phi}$ with their restrictions to smaller subintervals.[11]

[10] The point of this strange notation is that the accents correspond to the appearance of the canonical generating functions $\mathrm{id}_{[0,1]}$ and $1-\mathrm{id}_{[0,1]}$ of the cones $C_0((0,1])$ and $C_0([0,1])$ respectively.

[11] The resulting n is independent of the size of the subinterval (which is vital to the argument). To make sense of this, one has to fix scalings of the subinterval back to $[0,1]$ so that one can set this up with a fixed finite set $\mathcal{F}$.

This enables one to run the patching argument repeatedly along the interval $[0,1]$, patching $\acute{\Phi}^{\oplus(2n-i)} \oplus \grave{\Phi}^{\oplus i}$ to $\acute{\Phi}^{\oplus(2n-(i+1))} \oplus \grave{\Phi}^{\oplus(i+1)}$ across subintervals to connect $\acute{\Phi}^{\oplus 2n}$ at the left hand end to $\grave{\Phi}^{\oplus 2n}$ at the right hand end. The 'patched' maps then then be glued together to witness quasidiagonality of τ.

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Non-amenable von Neumann algebras of groups acting on trees

Sven Raum

(joint work with Cyril Houdayer)

Groups acting on locally finite trees form a particularly interesting class of locally compact groups. Bruhat-Tits theory naturally links them to algebraic groups, while Bass-Serre theory gives strong tools to reduce questions about general groups acting on trees to amalgamated free products of groups. The following folklore type I conjecture in topological group theory draws a parallel between the representation theory of reductive algebraic groups over non-Archimedean fields and a large class of groups acting on trees.

Conjecture. *Let T be a locally finite tree and $G \leq \mathrm{Aut}(T)$ a closed subgroup acting transitively on the boundary ∂T. Then G is a type* I *group.*

A locally compact group G is of type I if every unitary representation π of G generates a type I von Neumann algebra $\pi(G)''$. This notion steams from representation theory, were it reduces questions about unitary representations to the irreducible ones. Loosely speaking a group is type I if and only if all its unitary representations can be uniquely written as an integral of irreducible unitary representations

There are few instances in which the type I conjecture for groups acting on trees could be verified, such as rank one reductive algebraic groups over non-Archimedean fields or Burger-Mozes groups $\mathrm{U}(F)$ introduced in [2]. In an attempt to isolate the relevant structure for the type I conjecture, it is natural to ask whether its formulation is sharp or not. However, the picture looks even less complete and there are no results on non-type I groups available in the literature, although recently some effort was undertaken to approach this question by classical means [4].

It is well-known that a non-compact closed subgroup $G \leq \mathrm{Aut}(T)$ is boundary transitive if and only if for every $n \in \mathbb{N}_{\geq 1}$ and every $v \in \mathrm{V}(T)$ the vertex stabilisers $G_v \leq G$ act transitively on geodesic paths of length n emerging from v. We say that $G \leq \mathrm{Aut}(T)$ is locally 2-transitive, if this statement holds for $n = 2$. Clearly, a boundary transitive group is locally 2-transitive. Further, the group von Neumann algebra of every type I group is amenable, so that the following theorem gives a converse to the type I conjecture for a large class of groups acting on trees.

Theorem. *Let T be a locally finite tree and $G \leq \mathrm{Aut}(T)$ a closed subgroup acting minimally on T. Assume that G is not virtually abelian. If G does not act locally 2-transitively on T, then the group von Neumann algbera $\mathrm{L}(G)$ is non-amenable.*

While in general local 2-transitivity is strictly weaker than boundary transitivity, in the class of Burger–Mozes groups, these are equivalent. In fact, $\mathrm{U}(F)$ is boundary transitive if and only if $\mathrm{U}(F)$ is locally 2-transitive if and only if F is 2-transitive. Combining our work with the previously known type I result for Burger–Mozes groups [1, 3] and the previously mentioned characterisation of boundary transitivity, we hence obtain the following characterisation of Burger–Mozes groups of type I.

Theorem. *Let $F \leq \mathrm{S}_n$ be a permutation group and $\mathrm{U}(F)$ the associated Burger-Mozes group. The following statements are equivalent.*

- $\mathrm{U}(F)$ *is a type* I *group.*
- $\mathrm{U}(F)$ *is boundary transitive.*
- F *is 2-transitive.*

It is interesting to observe that in the case of Burger–Mozes groups our proof of non-amenability for $\mathrm{L}(\mathrm{U}(F))$ does not make use of Bass-Serre theory but can directly use an amalgamated free product decomposition, thereby making a particularly short proof possible.

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A brief visit to classification of simple C^*-algebras of finite rank

Huaxin Lin

We begin with the a short discussion of the following theorem:

Theorem 1 (....., 2015–[6], [5], [23], ..., 2000—[7] and [19])**.** *Let A and B be two unital separable simple C^*-algebras with finite nuclear dimension which satisfy the UCT. Then $A \cong B$ if and only if* $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

This is a result of decades work by many people. We only discuss the finite case.

Definition 1. *Let F_1 and F_2 be two finite dimensional C^*-algebras let $\phi_0, \phi_1 : F_1 \to F_2$ be two unital homomorphisms. Define*

$$C(F_1, F_2, \phi_0, \phi_1) = \{f \in C([0,1], F_2) \oplus F_1 : f(0) = \phi_0(a) \text{ and } f(1) = \phi_1(a),\ \ a \in F_1\}.$$

These are called Elliott-Thomsen building blocks. These are also known as one-dimensional non-commutative CW compleces (Eliers-Loring-Pedersen). Denote by $\mathcal{C}_0$ the class of all such C^*-algebras.

Definition 2. *Let A be a unital simple C^*-algebra. We say that A has generalized tracial rank at most one, if the following holds. For any $\epsilon > 0$ and finite subset $\mathcal{F} \subset A$ and any $a \in A_+ \setminus \{0\}$, there exists a projection $p \in A$ and a C^*-subalgebra $B \in \mathcal{C}_0$ with $1_B = p$ such that*

$$\|xp - px\| < \epsilon \text{ and } pxp \in_\epsilon B \text{ for all } x \in \mathcal{F}$$
$$1 - p \lesssim a.$$

If A has generalized tracial rank at most one, we write $GTR(A) \leq 1$.

This is a modification of previous notions of tracial rank one (and zero, TAF, TAI). Motivated by some earlier work as well as a Popa condition ([8], [9] , [20], [10], [11]).

Theorem 2 (Gong–L –Niu [6])**.** *Let A and B be two unital separable amenable simple C^*-algebras with $gTR(A), gTA(B) \leq 1$ whcih satisfy the UCT.Then $A \cong B$ if and only if*

$$Ell(A) \cong Ell(B).$$

Definition 3. *Let $\mathcal{N}_1$ denote the class of all unital separable simple amenable $\mathcal{Z}$-stable C^*-algebras A with $gTR(A \otimes U) \leq 1$ which satisfies the UCT for some UHF-algebra U of infinite type.*

Theorem 3 ([6])**.** *Let A and B be in $\mathcal{N}_1$. Then $A \cong B$ if and only if $Ell(A) \cong Ell(B)$.*

How large is $\mathcal{N}_1$? Which C^*-algebras are in $\mathcal{N}_1$?

Theorem 4 (Gong–L–Niu [5] and also [1])**.** *Let (G, G_+, u) be a countable abelian weakly unperforated ordered group with order unit u, let F be a countable abelian group, let Δ be a Choquet simplex and let $r : \Delta \to S_u(G)$ be a surjective affine map. Then there exists a unital C^*-algebra $A \in \mathcal{N}_1$ such that*

$$Ell(A) = ((G, G_+, u), F, \Delta, r).$$

Corollary 1. *Let B be a unital separable amenable simple stably finite C^*-algebra. Then there exists a unital simple C^*-algebra $A \in \mathcal{N}_1$ such that* $\mathrm{Ell}(B \otimes \mathcal{Z}) = \mathrm{Ell}(A)$.

Theorem 5 ([16])**.** *Let X be an infinite compact metric space and let $\sigma : X \to X$ be a minimal homeomorphism. Suppose that (X, α) has mean dimension zero. Then $C(X) \times_\alpha \mathbb{Z} \in \mathcal{N}_1$.*

Note if X has finite covering dimension, then any minimal dynamical system (X, α) has mean dimension zero. For infinite dimension cases, the proof also uses an important result of Elliott and Niu that such crossed products are $\mathcal{Z}$-stable.

Theorem 6 (Elliott–Gong–L–Niu [5])**.** *Let A be a unital separable amenable C^*-algebra which satisfies the UCT. Suppose that A has finite decomposition rank. Then $A \in \mathcal{N}_1$.*

What is actually proved is the following:

Theorem 7 ([5])**.** *Let A be a unital separable C^*-algebra which has finite nuclear dimension and satisfies the UCT. Suppose that every tracial state is quasidiagonal, then $gTR(A \otimes Q) \leq 1$.*

Then there is an amazing result:

Theorem 8 (Tikuisis, Whiter, and Winter–[23])**.** *Every tracial state of a unital simple separable amenable C^*-algebra which satisfies the UCT is quasi-diagonal.*

Combining four of the above mentioned results, one has the following:

Theorem 9. *Let A and B be two finite unital separable simple C^*-algebras with finite nuclear dimension which satisfy the UCT. Then $A \cong B$ if and only if* $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

We will glance at the path from Theorem 2 to Theorem 3.

Definition 4. *(Winter-*[24]*) Let A and B be two unital C^*-algebras. A $C([0,1])$-homomorphism*

$$\phi : A \otimes Z_{\mathfrak{p},\mathfrak{q}} \to B \otimes Z_{\mathfrak{p},\mathfrak{q}}.$$

is said to be unitarily suspended, if there are $0 < t_0 < t_1 < 1$, *a continuous path of unitaries* $\{u_t : t \in [t_0, t_1]\}$ *in* $B \otimes M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$ *and homomorphisms*

$$\phi_{\mathfrak{p}} : A \otimes M_{\mathfrak{p}} \to B \otimes M_{\mathfrak{p}} \text{ and } \psi_{\mathfrak{q}} : A \otimes M_{\mathfrak{q}} \to B \otimes M_{\mathfrak{q}}$$

such that $\phi^{(0)} = \phi_{\mathfrak{p}}$, $\phi^{(t)} = \phi_{\mathfrak{p}} \otimes \mathrm{id}_{M_{\mathfrak{q}}}$ for all $t \in (0, t_0], \phi^{(t)} = \mathrm{Ad}\, u_t \circ (\phi_{\mathfrak{p}} \otimes \mathrm{id}_{\mathfrak{q}})$ for all $t \in [t_0, t_1]$ *and* $\phi^{(t)} = \psi_{\mathfrak{q}}^{[1,3]} \otimes \mathrm{id}_{M_{\mathfrak{p}}}^{[2]}$ for all $t \in [t_1, 1)$, $\pi_1 \circ \phi = \psi_{\mathfrak{q}}$.

Theorem 10 (Winter [24])**.** *Suppose that* A *and* B *are two unital separable* C^**-algebras. If there is an isomorphism* $\phi : A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \to B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ *which is* $C([0,1])$*-isomorphism and is a unitarily suspended, then* $A \otimes \mathcal{Z} \cong B \otimes \mathcal{Z}$.

Questions:
1. Suppose that $\mathrm{Ell}(A) = \mathrm{Ell}(B)$. How to construct ϕ?
2. Suppose that there are isomorphisms $\phi_{\mathfrak{p}} : A \otimes M_{\mathfrak{p}} \to B \otimes M_{\mathfrak{p}}$ and $\psi_{\mathfrak{q}} : A \otimes M_{\mathfrak{q}} \to B \otimes M_{\mathfrak{q}}$. How to obtain ϕ?

We need the following theorems to answer both questions. Then applying Theorem 2 to obtain 3 (see also [14] and [15]).

Theorem 11 ([6] and see also [13])**.** *Let* A *be a unital separable amenable simple* C^**-algebra with* $GTR(A) \le 1$ *which satisfies the UCT and* B *be a unital* C^**-algebra. Suppose that* $\phi_1, \phi_2 : A \to B$ *are two unital homomorphisms. Then there exists a continuous path of unitaries* $\{u(t) : t \in [0, \infty)\} \subset B$ *such that*

$$\phi_1(a) = \lim_{t \to \infty} u(t)^* \phi_2(a) u(t) \text{ for all } a \in A$$

if and only if

$$(\phi_1)_T = (\phi_2)_T, \ \phi_1^{\ddagger} = \phi_2^{\ddagger}, \ [\phi_1] = [\phi_2] \text{ in } KK(A,B) \text{ and } \overline{R}_{\phi_1,\phi_2} = 0.$$

Theorem 12 ([6])**.** *Let* A *and* B *be two unital separable amenable simple* C^**-algebra with* $gTR(A), gTR(B) \le 1$ *and* A *satisfies the UCT. Suppose that* $x \in KK_e(A,B)^{++}$, $\gamma : T(B) \to T(A)$ *is an affine continuous map and* $\lambda : U(A)/CU(A) \to U(B)/CU(B)$ *is a continuous homomorphism so that* $(\kappa, \gamma, \lambda)$ *is compatible. Then there exists a unital monomorphism* $\phi : A \to B$ *such that*

$$[\phi] = x, \ (\phi)_T = \gamma \text{ and } \phi^{\ddagger} = \lambda.$$

It should be aware that there are examples that $x \in KK_e(A,B)^{++}$ but no homomorphism ϕ exists so that $[\phi] = x$ (see [12]).

Theorem 13 ([6])**.** *Let* A *and* B *be two unital separable amenable simple* C^**-algebras with* $gTR(A), gTR(B) \le 1$, A *satisfies the UCT and let* $\phi : A \to B$ *be a unital monomorphism. Suppose that* $\psi : A \to B$ *is another unital monomorphsm such that*

$$([\phi], (\phi)_T, \phi^{\ddagger}) = ([\psi], (\psi)_T, \psi^{\ddagger}). \tag{1}$$

Then $\overline{R}_{\phi,\psi} \in \mathrm{Hom}(K_1(A), \overline{\rho_B(K_0(B))})/\mathcal{R}_0$. *Conversely, for any* $\eta \in \mathrm{Hom}(K_1(A), \overline{\rho_B(K_0(B))})/\mathcal{R}_0$, *there exists a unital monomorphism* $\psi : A \to B$ *such that* (1) *holds and* $\overline{R}_{\phi,\psi} = \eta$.

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Constructing MASAs with prescribed properties

Sorin Popa

Given a separable II_1 factor M, one can construct a *maximal abelian* **-subalgebra* (abbreviated hereafter as *MASA*) A in M as an inductive limit of finer and finer partitions of 1 by projections in M. This iterative procedure pairs well with properties of MASAs that can be characterized locally, allowing the construction of A in a manner that makes "more and more" of the desired property be satisfied.

This technique has been initiated in [P81, P82] where it was used to show that any separable II_1 factor M contains a MASA $A \subset M$ whose *normalizer* $\mathcal{N}_M(A) := \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ generates a factor (A is *semiregular* in M; see [P81]), as well as a MASA $A \subset M$ whose normalizer is trivial, i.e. $\mathcal{N}_M(A) = \mathcal{U}(A)$ (A is *singular* in M; see [P82]).

In this talk, we discuss more refined applications of this method, by combining it with two additional ingredients: the intertwining by bimodule technique ([P01, P03]) and local properties of the ambient II_1 factor M, such as existence of non-trivial central sequences (i.e., property Gamma of [MvN43]) and *s-thin* approximation, which we will define below.

Recall in this respect that if Q, P are von Neumann subalgebras in a II_1 factor M, then we write $Q \prec_M P$ if there exists a Hilbert $Q-P$ sub-bimodule $\mathcal{H} \subset L^2M$ such that $\dim\mathcal{H}_P < \infty$. In certain cases (notably if Q, P are MASAs) this condition is equivalent to the existence of a non-zero partial isometry $v \in M$ such that $v^*v \in Q$ and $vQv^* \subset P$.

Our first result shows that any separable II_1 factor M contains an uncountable family of singular (respectively semiregular) MASAs $\{A_i\}_i$ such that $A_i \not\prec_M A_j$, $\forall i \neq j$, with A containing non-trivial central sequences of M whenever M does. This will in fact follow from the following stronger result.

Theorem 1. *Let M be a separable II_1 factor M and $N \subset M$ a subfactor with trivial relative commutant, $N' \cap M = \mathbb{C}$. Let $P_n \subset M$ be a sequence of von Neumann subalgebras such that $N \not\prec_M P_n$, $\forall n$. Then N contains a singular (respectively semiregular) maximal abelian *-subalgebra A of M such that $A \not\prec_M P_n$, $\forall n$. Moreover, if $N \simeq R$, then one can take A so that to satisfy $\mathcal{N}_M(A)'' = N$, and if N contains non-trivial central sequences of M, then A can be taken so that to contain non-trivial central sequences of M as well.*

We then consider the class of II_1 factors M which have an *s-MASA*, i.e., a MASA $A \subset M$ such that the von Neumann algebra $A \vee JAJ \subset \mathcal{B}(L^2M)$, generated by left and right multiplication by elements in A on the Hilbert space L^2M, is a MASA in $\mathcal{B}(L^2M)$. We obtain a local characterization of factors in this class, by proving that M has an s-MASA if and only if it satisfies the following approximation property, that we call *s-thin*: for any finite partition $\{p_i\}_i \subset M$, any finite set $F \subset M$ and any $\varepsilon > 0$, there exist a partition $\{q_j\}_j \subset M$ refining $\{p_i\}_i$ and an element $\xi \in M$ such that any $x \in F$ can be ε-approximated in the norm-$\| \ \|_2$ by linear combinations of elements of the form $q_j \xi q_k$. We show that factors with s-MASAs are closed under amplifications and inductive limits and combine their local characterization with the iterative procedure to prove the following:

Theorem 2. *If M has an s-MASA, then there exist uncountably many non-intertwinable s-MASAs in M, which in addition can be chosen singular* (*resp. semiregular*).

The typical example of s-MASAs in II_1 factors are the *Cartan* (or *regular*) MASAs, i.e., MASAs $A \subset M$ for which $\mathcal{N}_M(A)'' = M$ (cf. [FM77]). Any group measure space II_1 factor $M = L^\infty(X) \rtimes \Gamma$, obtained from a free ergodic measure preserving action $\Gamma \curvearrowright X$ of a countable group Γ on a probability measure space (X, μ), has $A = L^\infty(X)$ as a Cartan subalgebra, which is thus also an s-MASA. The above result shows that such factors necessarily have singular s-MASAs as well. Note that when M is hyperfinite, this fact was already known since ([D54, Pu60]), where the first concrete examples of singular s-MASAs were given.

By [OP07, PV11, PV12], there are large classes of group measure space II_1 factors that have unique (up to unitary conjugacy) Cartan subalgebras (= regular MASAs), while by Theorem 2 above, such a factor always has "many" non conjugate semiregular s-MASAs.

There are by now many classes of II_1 factors known to have no Cartan subalgebras, obtained first by using free probability theory ([V96]), then by using deformation-rigidity theory ([OP07, PV11, CS11, PV12, I12]). It is interesting to note that in each case when one could prove absence of Cartan MASAs by using free probability, absence of s-MASAs followed as well (notably for the free group factors $L(\mathbb{F}_n)$, cf. [G97]).

It would be important to obtain an intrinsic, local characterization of II_1 factors having Cartan subalgebras. Such characterization should allow to prove, for instance, that the class of factors with Cartan MASAs is close to inductive limits. Another problem that we leave open is to produce examples of II_1 factors that have s-MASAs but do not have Cartan subalgebras.

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Amenability VS Amalgamated free products

Remi Boutonnet

(joint work with C. Houdayer)

We investigate the position of certain amenable subalgebras of arbitrary amalgamated free product von Neumann algebras. Our main result roughly says that any amenable subalgebra of $A *_C B$ with large enough intersection with A is actually contained in A. This generalizes previous results of Houdayer-Ueda and Leary.

Free entropy dimension and the first L^2 Betti number

Dimitri Shlyakhtenko

In [Voi94, Voi96] Voiculescu introduced the notion of *free entropy dimension.* If $X = (X_1, \ldots, X_n)$ is an n-tuple of self-adjoint elements in a tracial von Neumann algebra, the free entropy dimension is computed as

$$\delta_0(X) = n - \limsup_{t \downarrow 0} \chi(X + tS : S)$$

where $S = (S_1, \ldots, S_n)$ is a free semicircular n-tuple, free from X, and χ is Voiculescu's (microstates) free entropy.

In [Jun07], Jung introduced the notion of a *strongly* 1*-bounded* n*-tuple* X. His definition involves strengthening the inequality $\delta_0(X) \leq 1$ (which entails $\chi(X + tS) = (n-1)\log t + O(\log t)$) to requiring that

$$\chi(X + tS) = (n-1)\log t + O(1).$$

A remarkable theorem due to Jung (see also recent work of Hayes [Hay15]) states that under certain additional assumptions on X (such as that it contains an element with finite entropy, or is a non-amenability set, or $W^*(X)$ is amenable), if for some $Y = (Y_1, \ldots, Y_m)$, $W^*(X) = W^*(Y)$, then Y is again strongly 1-bounded. In other words, strong 1-boundedness is a property of a finite von Neumann algebra which can be checked on a generating set.

Let Γ be a finitely generated discrete group and let X be a generating set for the group algebra $\mathbb{C}\Gamma$. We have previously shown [CS05] that the inequality

$$\delta_0(X) \leq \beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1$$

always holds. Here $\beta_j^{(2)}(\Gamma)$ are the L^2-Betti numbers of Γ (cf. [Lüc02]). More concretely, the quaitity $\beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1$ is precisely $\dim_{L\Gamma} C(\Gamma; \ell^2\Gamma)$, where dim stands for Murray-von Neumann dimension and $C(\Gamma; \ell^2\Gamma)$ is the set of group cocycles, i.e. maps $c : \Gamma \to \ell^2\Gamma$ satisfying $c(gh) = c(g) + \rho(g)c(h)$, ρ being the right regular representation.

Equality $\delta_0(X) = \beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1$ holds for certain classes of groups [Shl09, BDJ08].

In this talk we prove that in the case that Γ is a finitely presented infinite finitely generated group which satisfies the determinant conjecture (cf. [Lüc02]) and $\beta_1^{(2)}(\Gamma) = 0$, then there exists a set of generators for the group algebra of Γ which is strongly 1-bounded. It follows that if $L\Gamma = W^*(Y)$ for some m-tuple Y, then Y must also be strongly 1-bounded. This implies, for example, that $L\Gamma$ cannot be a free product of two diffuse von Neumann algebras, in particular $L\Gamma \not\cong L\mathbb{F}_n$ for any $n \geq 2$.

The proof of our result relies on estimating non-microstates free Fisher information for semicircular perturbations of generating sets and using the results of [BCG03] to deduce the corresponding inequality for microstates free entropy dimension (cf. [Shl16]).

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Connective C*-algebras

Marius Dadarlat
(joint work with Ulrich Pennig)

Connectivity is a homotopy invariant property of a separable C^*-algebra A which has three important consequences: absence of nonzero projections, quasidiagonality and realization of the Kasparov group $KK(A, B)$ as homotopy classes of asymptotic morphisms from A to $B \otimes \mathcal{K}$ if A is nuclear.

We denote by $CB = C_0[0, 1) \otimes B$ the cone over a C*-algebra B. Let $L(\mathcal{H})$ denote the bounded operators on a separable infinite dimensional Hilbert space and let $\mathcal{K} = K(\mathcal{H})$ denote the compact operators.

Definition. *A C*-algebra A is* connective *if there is a $*$-monomorphism*

$$\Phi \colon A \to \prod_n CL(\mathcal{H}) / \bigoplus_n CL(\mathcal{H})$$

which is liftable to a completely positive and contractive map $\varphi \colon A \to \prod_n CL(\mathcal{H})$

Let G be a countable discrete group and let $I(G)$ be the augmentation ideal defined as the kernel of the trivial representation $\iota : C^*(G) \to \mathbb{C}$.

Definition. *A countable discrete group G is connective if the ideal $I(G)$ is a connective C*-algebra.*

Connes and Higson [1] gave a realization of E-theory, the universal half-exact C*-stable homotopy functor on separable C*-algebras in terms of homotopy classes of asymptotic homomorphisms: $E(A, B) = [[SA, SB \otimes \mathcal{K}]]$. An asymptotic morphism at the level of unsuspended C*-algebras $A \to B \otimes \mathcal{K}$ contains in principle more geometric information. A full answer to the question of unsuspending in E-theory was found in [3]. The suspension map $[[A, B \otimes \mathcal{K}]] \to E(A, B)$ is an isomorphism for all separable C^*-algebras B if and only if A is *homotopy symmetric*, which means that $[[\mathrm{id}_A]] \in [[A, A \otimes \mathcal{K}]]$ has an additive inverse or equivalently that $[[A, A \otimes \mathcal{K}]]$ is a group. Unfortunately, it can be quite hard to check homotopy symmetry in practice. Addressing this point we showed that homotopy symmetry is equivalent to connectivity, a property which is significantly easier to verify:

Theorem. [4] *If A is separable and nuclear, the following conditions are equivalent:*
(i) A is homotopy-symmetric
(ii) A is connective.
(iii) $KK(A, B) \cong [[A, B \otimes \mathcal{K}]]$ for any separable C-algebra B.*

The proof of this theorem relies crucially on results of Thomsen [9]. Since connectivity is an embeddability condition, it passes to subalgebras and hence we obtain a somewhat unexpected corollary:

Theorem. [4] *Homotopy symmetry is inherited by separable nuclear subalgebras*

Connectivity has a key permanence property which does not hold for quasidiagonality:

Theorem. [5] *The class of connective separable nuclear C*-algebras is closed under extensions.*

Using this and other permanence properties such as passage to inductive limits and crossed products by compact groups, we were able to exhibit new classes of homotopy symmetric C^*-algebras [4], [5].

Theorem. [4] *If G is a countable torsion free nilpotent group, then G is connective.*

We have also shown that the class of discrete amenable groups is closed under generalized wreath products.

Let G and H be countable discrete groups and let J be a set with a left action of H. The wreath product $G \wr_J H$ is defined as the following semidirect product: $G \wr_J H = (\bigoplus_J G) \rtimes H$.

Theorem. [6] *If G and H are connective, so is the wreath product $G \wr_J H$.*

The Hantzsche-Wendt manifold is the only closed flat 3-dimensional manifold with finite homology [2]. Its fundamental group $G = \langle x, y \colon x^2yx^2 = y, \quad y^2xy^2 = x\rangle$ is a torsion free crystallographic group (Bieberbach group) that fits into an exact sequence

$$1 \to \mathbb{Z}^3 \to G \longrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1.$$

We showed that $\widehat{G} \setminus \{\iota\}$ is compact-open and use a result from [10] to prove that

Theorem. [5] *The Hantzsche-Wendt group G is not connective.*

In contrast we have proved that

Theorem. [5] *Any Bieberbach group with cyclic holonomy is connective.*

In other words we showed that if $1 \to \mathbb{Z}^n \to H \to \mathbb{Z}/m\mathbb{Z} \to 1$ is exact, then H connective if and only if H is torsion free.

Reduced C-algebras of Lie groups*
Based on classic results of representation theory, [7], [8], [11], [12] we arrived the following picture concerning connectivity for the reduced C*-algebras of Lie groups [5]:

(a) Let G be a (real or complex) linear connected nilpotent Lie group. Then $C^*(G)$ is connective if and only if G is not compact.

(b) If G is a linear connected complex semisimple Lie group, then $C^*_r(G)$ is connective if and only if G is not compact.

(c) Let G be a linear connected real reductive Lie group. The following assertions are equivalent

- (i) $C^*_r(G)$ is connective
- (ii) G does not have a discrete series representations
- (iii) G does not have a compact Cartan subgroup
- (iv) There are no nonzero projections in $C^*_r(G)$

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Actions of amenable groups on the Cantor set: $\mathcal{Z}$-stability and classifiability

David Kerr

As a consequence of recent remarkable work of Elliott, Gong, Lin, and Niu [5, 3] and Tikuisis, White, and Winter [10], we now know that the class of simple separable unital C*-algebras satisfying the UCT (universal coefficient theorem) and having finite nuclear dimension is classified by ordered K-theory paired with traces. For C*-crossed products of actions of amenable groups the UCT is automatic by a result of Tu [11], and so in this case classifiability (i.e., falling within the scope of the above classification theorem) boils down to the problem of determining when the nuclear dimension is finite. While there are very effective methods for tackling this problem that reimagine nuclear dimension in dynamical terms, as in the work of Szabo [8], Szabo-Wu-Zacharias [9], and Guentner-Willett-Yu [6], for free actions these methods all require that the group have finite asymptotic dimension, which does not always hold in the amenable case.

Here we take a different approach by developing dynamical arguments for verifying $\mathcal{Z}$-stability, which, for simple separable unital infinite-dimensional nuclear C*-algebras, is conjectured to be equivalent to finite nuclear dimension (Toms-Winter) and is known to be so equivalent when the extreme boundary of the trace simplex is compact (Bosa-Brown-Sato-Tikuisis-White-Winter [1]), a fact which was first proved in the unique trace case, under the hypothesis of quasidiagonality, by Matui and Sato [7].

We begin with the following definition modeled on the notion of strict comparison for C*-algebras.

Definition 1. *An action $G \curvearrowright X$ on the Cantor set is said to have clopen strict comparison if for all clopen sets $A, B \subseteq X$ satisfying $\mu(A) < \mu(B)$ for all G-invariant Borel probability measures μ on X there exist a clopen partition $\{A_1, \dots, A_n\}$ of A and $s_1, \dots, s_n \in G$ such that the sets $s_1A_1, \dots, s_nA_n$ are pairwise disjoint and contained in B.*

The following was observed by Glasner in Weiss in [4] using Kakutani-Rokhlin towers. It is an open question whether the statement holds when $\mathbb{Z}$ is replaced by an arbitrary countably infinite amenable group (or even just $\mathbb{Z}^2$).

Proposition 1. *A minimal $\mathbb{Z}$-action on the Cantor set has clopen strict comparison.*

An action on a compact metrizable space is said to be *strictly ergodic* if it is minimal and has a unique invariant Borel probability measure. Strict ergodicity and freeness together imply that the crossed product is simple and has unique trace. An application of the Ornstein-Weiss quasitower theorem and a characterization of $\mathcal{Z}$-stability observed by Winter yield:

Theorem 1. *Let $G \curvearrowright X$ be a free strictly ergodic action of a countably infinite amenable group on the Cantor set. Suppose that the action has clopen strict comparison. Then $C(X) \rtimes G$ is $\mathcal{Z}$-stable.*

The following is a refinement of the Jewett-Krieger theorem [12]. The proof relies on a recent tiling result of Downarowicz, Huczek, and Zhang for amenable groups [2].

Theorem 2. *Let $G \curvearrowright (Y, \mu)$ be a free probability-measure-preserving action of a countable amenable group and let H be a subgroup of G isomorphic to $\mathbb{Z}$ such that the restriction $H \curvearrowright (Y, \mu)$ is ergodic. Then there exists a free topological model $G \curvearrowright X$ for the action $G \curvearrowright (Y, \mu)$ such that the restriction $H \curvearrowright X$ is strictly ergodic.*

Theorems 1 and 2 and Proposition 1, along with the results in classification theory discussed at the outset, combine to produce many classifiable crossed products among actions of a given nontorsion countable amenable group on the Cantor set:

Theorem 3. *Let G be a nontorsion countable amenable group. Then there exist uncountably many pairwise nonconjugate strictly ergodic free minimal actions $G \curvearrowright X$ on the Cantor set such that $C(X) \rtimes G$ is $\mathcal{Z}$-stable and hence classifiable.*

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Equivariant Kirchberg-Phillips-type absorption for amenable group actions

Gábor Szabó

As we have seen in earlier talks, an important C*-algebraic regularity property is given by the tensorial absorption of some strongly self-absorbing C*-algebra $\mathcal{D}$ [11]. This ties into the Toms-Winter conjecture [1, 12, 14]. In a very influential paper [13], the term of '*localizing the Elliott conjecture at a strongly self-absorbing* C*-*algebra* $\mathcal{D}$' was coined by Winter. The most general case concerns $\mathcal{D} = \mathcal{Z}$. The earliest and perhaps most prominent case is Kirchberg-Phillips' classification [3, 8] of purely infinite C*-algebras, where the Cuntz algebra $\mathcal{O}_\infty$ played this role [4]. Together with $\mathcal{O}_2$, which plays a reverse role to $\mathcal{O}_\infty$, these two objects can be regarded as the cornerstones of that classification theory.

The notion of a strongly self-absorbing C*-algebra was generalized to the equivariant context and studied in [9, 10].

Definition. *Let $\mathcal{D}$ be a separable, unital* C*-*algebra and G a locally compact group. An action $\gamma : G \curvearrowright \mathcal{D}$ is called strongly self-absorbing, if there exists an equivariant isomorphism $\varphi : (\mathcal{D}, \gamma) \to (\mathcal{D} \otimes \mathcal{D}, \gamma \otimes \gamma)$ and unitaries $v_n \in \mathcal{U}(\mathcal{D} \otimes \mathcal{D})$ satisfying*

$$\varphi(x) = \lim_{n\to\infty} v_n(x \otimes 1)v_n^*, \quad x \in \mathcal{D}$$

and

$$\max_{g\in K} \|v_n - (\gamma\otimes\gamma)_g(v_n)\| \stackrel{n\to\infty}{\longrightarrow} 0, \quad K\subseteq G \text{ compact.}$$

In analogy to the classical theory, the most important feature of strongly self-absorbing actions is that their absorption can be characterized by a McDuff-type condition. The variant cited below is a special case that has been folklore long before the work in [9] was initiated.

Theorem 1 (generalizing Rørdam)**.** *Let G be a countable, discrete group. Let $\alpha : G \curvearrowright A$ be an action on a separable, unital* C^**-algebra. Let $\gamma : G \curvearrowright \mathcal{D}$ be a strongly self-absorbing action. Then α is (strongly) cocycle conjugate to $\alpha\otimes\gamma$ if and only if there exists an equivariant and unital $*$-homomorphism from $(\mathcal{D},\gamma)$ to $\big(A_\infty\cap A', \alpha_\infty\big)$. (If this holds, we say that α is γ-absorbing.)*

The basic question treated in this talk is whether the classical Kirchberg-Phillips absorption theorem [4] has an equivariant analog, interpreted in the above sense, for outer actions of amenable groups. Results of this kind are already known for special classes of groups due to work of Izumi [5], Goldstein-Izumi [2], Izumi-Matui [7, 6] and Phillips (in progress).

Example. *Let G be discrete and exact. By Kirchberg's $\mathcal{O}_2$-embedding theorem, we find a faithful unitary representation $v : G\to\mathcal{U}(\mathcal{O}_2)$ via some inclusion $\mathrm{C}^*_r(G)\subset\mathcal{O}_2$. Choose some embedding $\iota:\mathcal{O}_2\to\mathcal{O}_\infty$, and obtain $u: G\to\mathcal{U}(\mathcal{O}_\infty)$ via $u_g=\iota(v_g)+1-\iota(1)$. Consider*

$$\delta = \bigotimes_{\mathbb{N}} \mathrm{Ad}(v) : G \curvearrowright \bigotimes_{\mathbb{N}} \mathcal{O}_2 \cong \mathcal{O}_2\ ,\ \ \gamma = \bigotimes_{\mathbb{N}} \mathrm{Ad}(u) : G \curvearrowright \bigotimes_{\mathbb{N}} \mathcal{O}_\infty \cong \mathcal{O}_\infty .$$

Then both of these actions are faithful and strongly self-absorbing.

The main result of this talk is that an equivariant Kirchberg-Phillips-type absorption theorem holds for outer actions of all amenable group, with the model actions above forming the common framework:

Theorem 2. *Let G be a discrete, amenable group. Then up to (strong) cocycle conjugacy, δ is the unique outer, equivariantly $\mathcal{O}_2$-absorbing G-action on $\mathcal{O}_2$. In particular, $\alpha\otimes\delta$ is (strongly) cocycle conjugate to δ for any action $\alpha: G\curvearrowright A$ on a unital Kirchberg algebra.*

Theorem 3. *Let G be a discrete, amenable group. Then every outer action $\alpha : G\curvearrowright A$ on a unital Kirchberg algebra is γ-absorbing.*

Theorem 4. *Let G be a discrete, amenable group. Let $\beta: G\curvearrowright\mathcal{O}_\infty$ be an outer action. Then β is strongly cocycle conjugate to γ if and only if β is approximately representable and the inclusion $\mathrm{C}^*(G)\subset\mathcal{O}_\infty\rtimes_\beta G$ is a KK-equivalence.*

A suitable variant of this holds for every strongly self-absorbing Kirchberg algebra $\mathcal{D}$ in place of $\mathcal{O}_\infty$.

We comment that the last condition must generally be assumed for groups with torsion, as it may fail for $G=\mathbb{Z}_2$. It remains unclear whether approximate representability is always redundant in this context. For the torsion-free case, we obtain the following more satisfactory uniqueness result:

Theorem 5. *Let G be a discrete, amenable, torsion-free group and $\mathcal{D}$ a strongly self-absorbing Kirchberg algebra. Then up to (strong) cocycle conjugacy, $\gamma \otimes \mathrm{id}_{\mathcal{D}}$ is the unique outer, approximately representable G-action on $\mathcal{O}_\infty \otimes \mathcal{D} \cong \mathcal{D}$.*

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C*-algebras of free minimal actions of amenable groups: a survey of the nonclassifiable case

N. Christopher Phillips

Throughout, Γ will be an infinite countable amenable group.

We first contrast the von Neumann algebra and C*-algebra situations. In von Neumann algebras, we take (X, μ) to be a standard probability space, with an action of Γ on X which preserves μ and is free and ergodic. Then, by Connes, the group measure space construction always gives the same algebra, namely the (unique) hyperfinite factor of type II_1, regardless of the group or the action.

For the C*-algebra situation, we instead let X be a compact metric space, we assume that Γ acts freely on X (essential freeness should be good enough, but we don't address this issue here), and we assume that the action is minimal. The C*-algebra crossed product $C^*(\Gamma, X)$ is then a simple separable nuclear stably finite C*-algebra. Such algebras might be considered to be the C*-algebraic analogs of the hyperfinite factor of type II_1, but they are very far from unique. Here are some of the complications which can arise, in roughly increasing order of distance from what one might hope for by comparison with the hyperfinite factor of type II_1:

(1) There might be inequivalent projections with the same trace.
(2) There might be more than one tracial state, and, given a projection, the tracial states need not all take the same value on it.
(3) The algebra might not be approximable by finite dimensional subalgebras. (It need not be AF.)
(4) The unitary group can fail to be connected.
(5) The algebra might have few or no nontrivial projections.
(6) Comparison can fail: there might be projections p and q such that $\tau(p) < \tau(q)$ for every tracial state τ, but p is not Murray-von Neumann equivalent to a subprojection of q.
(7) The algebra can fail to have stable rank one, that is, the invertible elements need not be (norm) dense.

All of these phenomena are known by example to actually occur in simple separable nuclear stably finite C*-algebras. The phenomena (1)–(5) all occur in C*-algebras which are classifiable in the sense of the Elliott program, while (6) and (7) rule out classifiability, at least according to current knowledge.

Since we want to discuss comparison in a C*-algebra A, but there may be no nontrivial projections, we compare positive elements in $M_\infty(A)$. We therefore very briefly summarize Cuntz comparison. See [1] for an extensive survey, and Sections 1 and 2 of [7] for some additional properties needed for the work reported here.

Definition 1. *Let A be a C*-algebra, and let $a, b \in M_\infty(A)_+$. We say that a* is Cuntz subequivalent to b, *written $a \precsim b$, if there is a sequence $(v_n)_{n\in\mathbb{N}}$ in $M_\infty(A)_+$ such that $\lim_{n\to\infty} v_n^* b v_n = a$.*

Definition 2. *Let A be a unital C*-algebra. We write $\mathrm{T}(A)$ for the set of tracial states on A. For $\tau \in \mathrm{T}(A)$, we define a function $d_\tau \colon M_\infty(A)_+ \to [0, \infty)$ by $d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$.*

We think of the relation $a \precsim b$ as saying that "the open support of a is dominated by the open support of b". (This is actually correct if $A = C(X)$ and a and b are in A.) We think of $d_\tau(a)$ as a measure of "the size of the open support of a".

Definition 3 (Definition 6.1 of [10]). *Let A be a unital C*-algebra and let $r \in [0, \infty)$. We say that A* has r-comparison *if whenever $a, b \in M_\infty(A)_+$ satisfy $d_\tau(a) + r < d_\tau(b)$ for all $\tau \in \mathrm{T}(A)$, then $a \precsim b$. We further define the* radius of comparison $\mathrm{rc}(A)$ *to be*

$$\mathrm{rc}(A) = \inf\big(\{r \in [0,\infty) \colon A \text{ has } r\text{-comparison}\}\big).$$

We have $\mathrm{rc}(A) = 0$ if and only if A has strict comparison of positive elements, which is the analog of comparablity of projections in factors of type II_1. For simple separable nuclear stably finite C*-algebras A, this condition is conjectured to be equivalent to classifiability in the sense of the Elliott program.

We now describe mean dimension, introduced in [6]. We first recall some definitions related to covering dimension.

Definition 4. *Let X be a compact Hausdorff space.*

(1) *Let $\mathcal{U}$ be a finite open cover of X. The* order *of $\mathcal{U}$ is the least number n such that the intersection of any $n+2$ distinct elements of $\mathcal{U}$ is empty.*
(2) *Let $\mathcal{U}$ and $\mathcal{V}$ be finite open covers of X. Then $\mathcal{V}$* refines *$\mathcal{U}$ (written $\mathcal{V} \prec \mathcal{U}$) if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subset U$.*
(3) *Let $\mathcal{U}$ be a finite open cover of X. We let $\mathcal{D}(\mathcal{U})$ be the least possible order of a finite open cover which refines $\mathcal{U}$.*
(4) *The* covering dimension $\dim(X)$ *is the supremum of $\mathcal{D}(\mathcal{U})$ over all finite open covers $\mathcal{U}$ of X.*
(5) *If $\mathcal{U}$ and $\mathcal{V}$ are finite open covers of X, their* join *is $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}$.*

Definition 5 ([6])**.** *Let X be a compact metric space and let $h\colon X \to X$ be a homeomorphism. Then the* mean dimension *of h is*

$$\operatorname{mdim}(h) = \sup_{\mathcal{U}} \lim_{n \to \infty} \frac{\mathcal{D}\big(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})\big)}{n}.$$

The supremum is over all finite open covers of X (as for $\dim(X)$).

The mean dimension of an action of Γ is defined using Følner sets in Γ in place of intervals in $\mathbb{Z}$. See [6]. We write $\operatorname{mdim}(\Gamma, X)$. The definition is designed so that if K is sufficiently nice (for example, a finite complex), then the shift on K^{Γ} has mean dimension $\dim(K)$.

Based on thin evidence, we hope for the following:

- $\operatorname{rc}\big(C^*(\Gamma, X)\big) = \frac{1}{2}\operatorname{mdim}(\Gamma, X)$.
- $C^*(\Gamma, X)$ should always have stable rank 1.

When $\Gamma = \mathbb{Z}$, the first is known as the "Phillips-Toms conjecture", and the second is an explicit conjecture in [2], of the authors of that paper and Niu. In particular, among algebras of the form $C^*(\Gamma, X)$, phenomenon (6), the first on the list above which is incompatible with classification, is supposed to occur (and does in examples, as was first shown in [4]), but phenomenon (7) is not supposed to occur.

The following results are known. (For the last two, not all details have yet been written, but they seem essentially certain to work.)

- For a class of actions of $\mathbb{Z}$ slightly generalizing the Giol-Kerr examples (see [4]), and thus including actions with arbitrarily large mean dimension, $\operatorname{rc}\big(C^*(\mathbb{Z}, X)\big) = \frac{1}{2}\operatorname{mdim}(\mathbb{Z}, X)$. (Joint work with Hines and Toms; see [5].)
- For $\Gamma = \mathbb{Z}$ and if X has infinitely many connected components, we have $\operatorname{rc}\big(C^*(\mathbb{Z}, X)\big) \leq \frac{1}{2}\operatorname{mdim}(\mathbb{Z}, X)$. (Joint work with Hines and Toms; see [5].)
- For $\Gamma = \mathbb{Z}$ and X arbitrary, if $\operatorname{mdim}(\mathbb{Z}, X) = 0$ then $\operatorname{rc}\big(C^*(\mathbb{Z}, X)\big) = 0$. (Elliott and Niu; see [3].)

- For $\Gamma = \mathbb{Z}$ and X arbitrary, $\mathrm{rc}\big(C^*(\mathbb{Z}, X)\big) \leq 1 + 2\,\mathrm{mdim}(\mathbb{Z}, X)$. (See [8]. The current version says $\mathrm{rc}\big(C^*(\mathbb{Z}, X)\big) \leq 1 + 36\,\mathrm{mdim}(\mathbb{Z}, X)$; a revision in preparation will give the improved bound.)
- For $\Gamma = \mathbb{Z}$ and if X has infinitely many connected components, $C^*(\mathbb{Z}, X)$ has stable rank 1. (Joint work with Archey; see [2].)
- For $\Gamma = \mathbb{Z}^d$ and if the action has a factor system which is a free minimal action on the Cantor set, $\mathrm{rc}\big(C^*(\mathbb{Z}^d, X)\big) \leq \frac{1}{2}\mathrm{mdim}(\mathbb{Z}^d, X)$. (See [9].)
- Under the same hypotheses as in the previous item, $C^*(\mathbb{Z}^d, X)$ has stable rank 1. (See [9].)

In the last two items, there are free minimal actions of many other countable amenable groups Γ on the Cantor set such that, if an action of Γ has one of these systems as a factor, then $\mathrm{rc}\big(C^*(\Gamma, X)\big) \leq \frac{1}{2}\mathrm{mdim}(\Gamma, X)$ and $C^*(\Gamma, X)$ has stable rank 1.

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Classification of a family of non almost periodic free Araki-Woods factors

Cyril Houdayer

(joint work with Dimitri Shlyakhtenko and Stefaan Vaes)

Following [7], to any finite symmetric Borel measure on $\mathbf{R}$ and any symmetric Borel multiplicity function $m : \mathbf{R} \to \mathbf{N} \cup \{+\infty\}$ (that we always assume to satisfy $m \geq 1$ μ-almost everywhere), one associates the free Araki–Woods von Neumann algebra $\Gamma(\mu, m)''$ which comes equipped with the free quasi-free state $\varphi_{\mu,m}$. When μ is an atomic measure that is not concentrated on $\{0\}$, the free quasi-free state is almost periodic and the von Neumann algebras $\Gamma(\mu, m)''$ were completely classified in [7]:

$\Gamma(\mu_1, m_1)'' \cong \Gamma(\mu_2, m_2)''$ if and only if the sets of atoms of μ_1 and μ_2 generate the same subgroup of $(\mathbf{R}, +)$.

It is a very intriguing open problem to classify the free Araki–Woods factors $\Gamma(\mu, m)''$ beyond the almost periodic case and there is not even a conjectural classification statement. So far, one could only distinguish between families of non almost periodic free Araki–Woods factors by computing their invariants, like Connes' τ-invariant (see [8, 9]), or by structural properties of their continuous core (see [9, 3, 2]). In this paper, we fully classify the free Araki–Woods factors in the case where the atomic part μ_a is nonzero and not concentrated on $\{0\}$ and where the continuous part μ_c satisfies $\mu_c * \mu_c \prec \mu_c$. We find in particular that in that case, the free Araki–Woods factor does not depend on the multiplicity function m.

In order to state our main results, we first introduce some terminology. Let $(X, \mathcal{X})$ be any standard Borel space endowed with its σ-algebra of Borel sets. A *measure class* on X is a subset $\mathcal{C} \subset \mathcal{X}$ that contains the empty set $\emptyset$ and that is closed under taking subsets and countable unions. For any (positive) σ-finite Borel measure μ on X, denote by $\mathcal{C}(\mu)$ the measure class consisting of all μ-null Borel subsets of X and write $\mu = \mu_c + \mu_a$ where μ_c (resp. μ_a) is the continuous (resp. atomic) part of μ. For any sequence $(\mu_k)_{k\in\mathbf{N}}$ of finite Borel measures on X, we define the *joint* measure class of $(\mu_k)_{k\in\mathbf{N}}$ by $\mathcal{C}(\bigvee_{k\in\mathbf{N}} \mu_k) := \bigcap_{k\in\mathbf{N}} \mathcal{C}(\mu_k)$.

We show that free Araki–Woods factors $\Gamma(\mu, m)''$ arising from finite symmetric Borel measures μ on $\mathbf{R}$ whose atomic part μ_a is nonzero and not concentrated on $\{0\}$ have the joint measure class $\mathcal{C}(\bigvee_{k\geq 1} \mu^{*k})$ as an invariant. More precisely, we obtain the following result.

Theorem 1. *Let μ, ν be finite symmetric Borel measures on $\mathbf{R}$ and $m, n : \mathbf{R} \to \mathbf{N} \cup \{+\infty\}$ any symmetric Borel multiplicity functions. Assume moreover that $\nu_a \neq 0$ and either $\operatorname{supp}(\nu_a) \neq \{0\}$ or $\operatorname{supp}(\nu_a) = \{0\}$ and $n(0) \geq 2$.*
If the free Araki–Woods factors $\Gamma(\mu, m)''$ and $\Gamma(\nu, n)''$ are $*$-isomorphic then there exist nonzero projections $p \in (\Gamma(\mu, m)'')^{\varphi_{\mu,m}}$ and $q \in (\Gamma(\nu, n)'')^{\varphi_{\nu,n}}$ and a state-preserving surjective $*$-isomorphism

$$(p\,\Gamma(\mu, m)''\,p, (\varphi_{\mu,m})_p) \cong (q\,\Gamma(\nu, n)''\,q, (\varphi_{\nu,n})_q)$$

where $(\varphi_{\mu,m})_p = \frac{\varphi_{\mu,m}(p\,\cdot\,p)}{\varphi_{\mu,m}(p)}$ and $(\varphi_{\nu,n})_q = \frac{\varphi_{\nu,n}(q\,\cdot\,q)}{\varphi_{\nu,n}(q)}$.
In particular, the joint measure classes $\mathcal{C}(\bigvee_{k\geq 1} \mu^{*k})$ and $\mathcal{C}(\bigvee_{k\geq 1} \nu^{*k})$ are equal.

Denote by $\mathcal{S}(\mathbf{R})$ the set of all finite symmetric Borel measures $\mu = \mu_c + \mu_a$ on $\mathbf{R}$ satisfying the following two properties:

(i) $\mu_c * \mu_c \prec \mu_c$ and
(ii) $\mu_a \neq 0$ and $\operatorname{supp}(\mu_a) \neq \{0\}$.

Denote by $\Lambda(\mu_a)$ the countable subgroup of $\mathbf{R}$ generated by the atoms of μ_a and by $\delta_{\Lambda(\mu_a)}$ a finite atomic measure on $\mathbf{R}$ whose support is $\Lambda(\mu_a)$.

We obtain a complete classification of the free Araki–Woods factors arising from measures in $\mathcal{S}(\mathbf{R})$.

Corollary 1. *The set of free Araki–Woods factors* $\Gamma(\mu, m)''$ *where*

- $\mu \in \mathcal{S}(\mathbf{R})$ *and*
- $m : \mathbf{R} \to \mathbf{N} \cup \{+\infty\}$ *is a symmetric Borel multiplicity function*

is exactly classified, up to ∗-isomorphism, by the countable subgroup $\Lambda(\mu_a)$ *and the measure class* $\mathcal{C}(\mu_c * \delta_{\Lambda(\mu_a)})$.

The family $\mathcal{S}(\mathbf{R})$ is large and provides many nonisomorphic free Araki–Woods factors having the same Connes invariants and in particular the same τ-invariant. Note that so far, only two non almost periodic free Araki–Woods factors having the same τ-invariant could be distinguished, see [9, Theorem 5.6].

We then show that free Araki–Woods factors $\Gamma(\mu, m)''$ arising from continuous finite symmetric Borel measures μ on $\mathbf{R}$ have all their centralizers amenable, i.e. the centralizer of any faithful normal state is amenable. More generally, we obtain the following result.

Corollary 2. *Let* μ *be any finite symmetric Borel measure on* $\mathbf{R}$ *and* $m : \mathbf{R} \to \mathbf{N} \cup \{+\infty\}$ *any symmetric Borel multiplicity function. The free Araki–Woods factor* $\Gamma(\mu, m)''$ *has all its centralizers amenable if and only if the atomic part* μ_a *of* μ *is either zero or concentrated on* $\{0\}$ *with* $m(0) = 1$.

In the setting of Corollary 2 and under the additional assumption that the Fourier transform of the continuous finite symmetric Borel measure μ_c vanishes at infinity, it was shown in [3, Theorem 1.2] that the continuous core of the corresponding free Araki–Woods factor $\Gamma(\mu, m)''$ is *solid* (see [4]), meaning that the relative commutant of any diffuse subalgebra that is the range of a faithful normal conditional expectation is amenable. Any type III_1 factor whose continuous core is solid has all its centralizers amenable. Observe that there are many free Araki–Woods factors arising in Corollary 2 whose Connes τ-invariant (see [1]) is not the usual topology on $\mathbf{R}$. In particular, these free Araki–Woods factors have a continuous core that is not *full* (see [1, 9]) and hence not solid (see [4, Proposition 7] with $\mathcal{N}_0 = \mathcal{M}$). Therefore, Corollary 2 provides many new examples of type III_1 factors whose centralizers are all amenable.

Our main technical tool to prove the results mentioned so far is a deformation/rigidity criterion for the unitary conjugacy of two faithful normal states on a von Neumann algebra M. We prove that a corner of the state ψ is unitarily conjugate with a corner of the state φ if and only if in the continuous core $\mathrm{c}(M)$, there is a Popa intertwining bimodule (in the sense of [5, 6]) between the canonical subalgebras $\mathrm{L}_\psi(\mathbf{R})$ and $\mathrm{L}_\varphi(\mathbf{R})$ of $\mathrm{c}(M)$.

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A free monotone transport result for $L(\mathbb{F}_\infty)$

Brent Nelson
(joint work with Qiang Zeng)

Originally defined by Bożejko and Speicher [1], mixed q-Gaussian algebras are the von Neumann algebras generated by self-adjoint operators $x_i = \ell_i + \ell_i^*$, $i \in I$, where $\{\ell_i\}_{i\in I}$ are the Fock space representations of the commutation relations

$$\ell_i^*\ell_j - q_{ij}\ell_j\ell_i^* = \delta_{i,j} \qquad i,j \in I,$$

corresponding to a symmetric array $Q = (q_{ij})_{i,j\in I} \subset (-1,1)$. These von Neumann algebras, denoted Γ_Q, come equipped with a faithful normal trace, namely the vector state associated to the vacuum vector in the Fock space. In the case that $q_{ij} = q$ for all $i, j \in I$ and some fixed parameter $q \in (-1,1)$, these von Neumann algebras are simply the q-deformed free group factors Γ_q. Furthermore, when $q = 0$ one has $\Gamma_0 \cong L(\mathbb{F}_{|I|})$.

When I is a finite set, Guionnet and Shlyakhtenko [3] showed that for sufficiently small $|q|$ (depending on $|I|$), $\Gamma_q \cong L(\mathbb{F}_{|I|})$. Moreover, this isomorphism is trace-preserving and even holds at the level of C^*-algebras. Their proof, which relied on estimates of Dabrowski [2], was the first application of their free monotone transport result. Free transport is the non-commutative analogue of classical transport between probability spaces; that is, a measurable way to transform one probability measure into another via a push-forward.

In the case of non-constant array $Q = (q_{ij})_{i,j\in I}$, in joint work with Zeng we proved the analogous result: for I finite and sufficiently small $\max_{i,j} |q_{ij}|$ (depending on $|I|$) one has $\Gamma_Q \cong L(\mathbb{F}_{|I|})$ [5]. Later, by adapting Guionnet and Shlyakhtenko's free monotone transport result to handle an infinite number of generators, we proved that a result in the I infinite case [4]. Namely, if $I = \mathbb{N}$ and loosely speaking Q satisfies :

(i) $\sup_{i,j} |q_{ij}|$ sufficiently small; and
(ii) $|q_{ij}| \to 0$ sufficiently rapidly as $i + j \to \infty$,

then $\Gamma_Q \cong L(\mathbb{F}_\infty)$. Furthermore, these isomorphisms are all trace-preserving and hold at the level of C^*-algebras.

The key observation is that the joint law of the generating variables $\{x_n\}_{n\in\mathbb{N}}$ (i.e. the values of monomials under the trace) is a solution to a non-commutative partial differential equation. In fact, the PDE belongs to a class parametrized by *potentials*: formal power series $V \in \mathbb{C}\langle t_n : n \in \mathbb{N}\rangle$. For $q_{ij} = 0$ for all $i, j \in \mathbb{N}$ (when the generators are freely independent semicircular variables) the associated potential is quadratic:

$$V_0 = \frac{1}{2}\sum_{n=1}^{\infty} t_n^2.$$

For Q satisfying (i) and (ii) above, the generators of Γ_Q have a joint law with associated potential V_Q that is "close" to V_0 as elements of a certain Banach algebra. One consequence of this is that the solution to the PDE associated to V_Q has a unique solution. This closeness also allows one to construct invertible, convergent power series in freely independent semicircular variables whose joint law is also a solution to the PDE associated to V_Q. The uniqueness of the solution combined with the invertibility of the convergent power series yields the isomorphism result.

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Group actions on Banach spaces and a duality spaces/operators

Mikael de la Salle

The polar of a subset C of a locally convex topological vector space E is the set

$$C^\circ = \{x^* \in E^* | \langle x^*, x\rangle \geq -1\}.$$

The very classical bipolar theorem [1, II §6] states that the bipolar of C (*i.e.* the polar of C° in E^* endowed with the weak-* topology) coincides with the closed convex hull of $C \cup \{0\}$. The inclusion of the closed convex hull of $C \cup \{0\}$ in the bipolar of C is obvious; the content of the theorem is the other inclusion, which follows from the Hahn-Banach theorem.

The aim of my talk was to present an analogous of the bipolar theorem, when one replaces E (respectively E^*) by the class **Ban** of Banach spaces and E^* (respectively E) by the class $\mathbf{Op_p}$ of bounded linear maps between subspaces of L_p spaces. The first half of my talk was devoted to motivations for this question, mainly from representation theory and operator algebras [8]. In this extended abstract I proceed the other way around: I start by presenting the objects and results involved in the polarity, and then I explain by some examples why I care about such questions.

Here p is a fixed number in $[1, \infty)$, for example $p = 2$.

Of course, neither **Ban** nor $\mathbf{Op_p}$ has a linear structure, so we are not at all in the setting of a pair of vector spaces in duality. The little structure that remains is a way to assign, to each pair (X, T) in $\mathbf{Ban} \times \mathbf{Op_p}$, a number $\|T_X\| \in [0, \infty]$, the (possibly infinite) norm of $T \otimes \mathrm{id}_X$ between the subspaces $\mathrm{dom}(T) \otimes X$ and $\mathrm{ran}(T) \otimes X$ of $L_p(\Omega_i, m_i; X)$. This is enough to define the notion of polar, analogously to the linear setting.

Definition. *If $A \subset$ **Ban** is a class of Banach spaces, we define its* polar *A° as the class of operators $T \in \mathbf{Op_p}$ such that $\|T_X\| \leq 1$ for every X in A.*

Definition. *If $B \subset \mathbf{Op_p}$, we define its polar B° as the class of Banach spaces $X \in$ **Ban** such that $\|T_X\| \leq 1$ for every T in B.*

Actually most natural geometric properties of Banach spaces (uniform convexity with a given modulus, UMD with a given UMD constant, type/cotype with a given type/cotype constant etc) are *defined* as the polar of some set of operators B between sub L_p spaces for the correct p. And so any question of the form "does property P_1 imply property P_2?" can be reformulated as "is B_1° contained in B_2°?", or equivalently as "is B_2 contained in the bipolar of B_1?". This is a strong motivation for obtaining a desctiption of the bipolar of a set of operators.

This duality is a variant of the one considered in [6], where Pisier restricts to operators between L_p spaces (and not subspaces of L_p spaces). There is a form of the bipolar theorem in this setting, which again takes informally the form of "the bipolar of C contains no more elements than the *obvious* elements", for the correct notion of *obvious*. For the bipolar of a class of Banach spaces, this is due to Hernandez.

Theorem. *([2]) The bipolar $A^{\circ\circ}$ of a class of Banach spaces $A \subset$ **Ban** is the class of Banach spaces finitely representable in the class of all finite ℓ_p-direct sums of elements in A.*

To state our main result, the bipolar theorem for sets of operators, we have to introduce some definition.

Definition. *A* spacial isometry *between finite dimensional subspaces of L_p spaces is a composition of isometries of the form:*

- *(Change of phase and measure) Restriction to a subspace of $L_p(\Omega, m)$ of the multiplication by a nonvanishing measurable function $h \colon \Omega \to \mathbf{C}^*$,* i.e. *$f \in L_p(\Omega, m) \mapsto hf \in L_p(\Omega, |h|^{-p} m)$.*
- *(Equimeasurability outside of 0) Maps of the form $T \colon \mathrm{dom}(T) \subset L_p(\Omega, m) \to L_p(\Omega', m')$ such that for every finite family $f_1, \ldots, f_n \in \mathrm{dom}(T)$ and every Borel subset $E \subset \mathbf{C}^n \setminus \{0\}$,*

$$m(\{x, (f_1(x), \ldots, f_n(x)) \in E\}) = m'(\{x, (Tf_1(x), \ldots, Tf_n(x)) \in E\}).$$

It is important that we require $0 \notin E$, as we want for example that $f \in L_p([0,1]) \mapsto f \oplus 0 \in L_p([0,1]) \oplus_p L_p([0,1])$ to be a spacial isometry.

If p is not an even integer, it is known that every isometry between (separable) susbpaces of L_p spaces is a spacial isometry ([5]).

We can now state the version of the bipolar theorem for sets of operators, which gives an answer to [6, Problem 4.1].

Theorem. *The bipolar $B^{\circ\circ}$ of a class $B \subset \mathbf{Op_p}$ is the smallest class $B' \subset \mathbf{Op_p}$ containing B and satisfying the following properties :*

1. *B' contains the convex combinations of spacial isometries.*
2. *B' is stable under finite ℓ_p-direct sums.*
3. *Let $T \in B'$ such that $T\colon \mathrm{dom}(T) \subset L_p(\Omega_1, m_1) \oplus L_p(\Omega, m) \to L_p(\Omega_1, m_1) \oplus L_p(\Omega, m)$ is of the form $(f, g) \mapsto (Sf, g)$ for some $S \in \mathbf{Op_p}$ with domain equal to the image of $\mathrm{dom}(T)$ by the first coordinate projection. Then $S \in B'$.*
4. *If $T \in B'$ and U, V are spacial isometries then $U \circ T \circ V \in B'$.*
5. *If $T \in \mathbf{Op_p}$ is an operator between subspaces of $L_p(\Omega, m)$ and $L_p(\Omega', m')$ and if, for every finite family $f_1, \ldots, f_n$ in the domain of T and every $\varepsilon > 0$, there is $S \in B'$ with domain contained in $L_p(\Omega, m)$ and range contained in $L_p(\Omega', m')$ and elements $g_1, \ldots, g_n \in \mathrm{dom}(S)$ such that $\|f_i - g_i\| \leq \varepsilon$ and $\|Tf_i - Sg_i\| \leq \varepsilon$, then $T \in B'$.*

We regard (3) as more subtle than the other properties defining B', and as responsible for the fact that in many concrete situations, the bipolar of a set of operators is not explicitely understood.

Motivation. Given a group G and a Banach space X, we denote by $C^*_X(G)$ the completion of $C_c(G)$ for the norm

$$\sup\{\|\pi(f)\|_{B(X)} | \pi \text{isometric representation of } G \text{ on } X\}.$$

This is a Banach algebra that encodes the isometric representation theory of G. There is also a reduced version, $C^*_{\lambda,X}(G)$ the completion of $C_c(G)$ for the norm of the convolution by f on the space $L_2(G; X)$, *i.e.*, with the previous notation, for the norm $\|(\lambda(f))_X\|$. The relevance of this Banach algebra for the study of general Banach space representations of G comes from the following generalization of the fact that the full and reduced C^*-algebras of an amenable group coincide : if G is amenable, then the formal identity extends to a contraction $C^*_X(G) \to C^*_{\lambda,X}(G)$.

Let G be a locally compact group with a compact symmetric generating set S. We say that G has (F_X) if every action of G by affine isometries on X has a fixed point. The case when X is a Hilbert space corresponds to Kazhdan's property (T), and in general it is a challenging problem to understand for which Banach spaces a given group has property (F_X). For example, despite several partial results the following conjecture by Bader, Furman, Gelander and Monod is still open (the analougous for groups over nonarchimedean local fields is known thanks to the work of Lafforgue and Liao) : *higher rank connected simple real Lie groups have (F_X) for every superreflexive Banach space X*. The same should be true for spaces of type > 1.

A consequence of [8] is that "property (F_X) for every superreflexive space X" can be expressed purely in terms of some variants of the Banach algebras $C^*_X(G)$ (formally it is equivalent to robust property (T_X) for every superreflexive X).

Moreover, the ideas introduced by Lafforgue and further developped by Liao, myself, de Laat and Mimura for the study of representations of higher rank connected simple Lie groups G and their nonarchimedean counterparts rely on the maximal compact subgroup K of G. In particular to obtain a good understanding of Banach space representations of G and to prove the conjecture above, all is needed is to have a good understanding of $C^*_{\lambda,X}(K)$. The question is really to understand whether some very explicit convolution operators (see [7, 4, 3]) on $L_2(K)$ belong to the bipolar of B, for B the class of operators defining the class of Banach spaces we are interested in (eg the spaces of Banach space with a given type p constant).

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The C^*-algebras of lamplighter groups over finite groups

Alain Valette

A celebrated theorem of Higson-Kasparov (1997) states that the Baum-Connes conjecture (BC) holds for every countable, amenable group G. The general machinery used in the proof establishes that the assembly map $\mu_G : K_*(\underline{EG}) \to K_*(C^*_r G)$ is an isomorphism, without identifying any of the objects. It seems interesting to try to establish BC by hand for specific classes of groups.

One class where this can be achieved is the class of lamplighter groups $G = F \wr \mathbb{Z}$, with F a finite group. The existence of a 2-dimensional $\underline{EG}$ (resp. the Pimsner-Voiculescu sequence) allows for computations on the left-hand side (resp. right-hand side). As a result, K_1 is infinite cyclic (and corresponds to the inclusion $\mathbb{Z} \hookrightarrow G$), while K_0 is free abelian on countably many generators, with explicit generators set into correspondence under μ_G. As an application, we get new proofs of some results concerning full shifts in topological dynamics.

Varying the finite group F, we see that the C^*-algebras C^*L cannot be distinguished by K-theory; this raises the question of classifying the C^*L's up to isomorphism. We show that, if F_1, F_2 are finite groups with F_1 abelian, $C^*(F_1 \wr \mathbb{Z}) \simeq$

$C^*(F_2 \wr \mathbb{Z})$ if and only if F_2 is abelian and $|F_1| = |F_2|$. This is a joint project with Ramon FLORES (Sevilla) and my PhD student Sanaz POOYA.

An intrinsic algebraic characterization of C*-simplicity

Matthew Kennedy

A group G is said to be C*-simple if its reduced C*-algebra $\mathrm{C}^*_\lambda(G)$ is simple, i.e. has no proper non-trivial two-sided closed ideals. There has recently been a great deal of progress in our understanding of groups with this property.

A dynamical characterization of discrete C*-simple groups was established in [8]. Recall that a compact G-space is said to be a G-boundary if the G-action on X is minimal and strongly proximal.

Theorem 1. *A discrete group G is C*-simple if and only if it has a free boundary action.*

The following averaging property was introduced by Powers' [9] in his proof that the free group $\mathbb{F}_2$ on 2 generators is C*-simple.

Definition 1. *A discrete group G is said to have* Powers' averaging property *if for every element a in the reduced C*-algebra $\mathrm{C}^*_\lambda(G)$ and $\epsilon > 0$ there are $g_1, \ldots, g_n \in G$ such that*

$$\left\| \frac{1}{n} \sum_{i=1}^{n} \lambda_{g_i} a \lambda_{g_i}^{-1} - \tau_\lambda(a) 1 \right\| < \epsilon,$$

where τ_λ denotes the canonical tracial state on $\mathrm{C}^*_\lambda(G)$.

It is straightforward to prove that a discrete group with Powers' averaging property is necessarily C*-simple. Recently, Haagerup [3] and the present author [7] independently proved that the converse holds.

Theorem 2. *A discrete group G is C*-simple if and only if it has Powers' averaging property.*

For a large class of discrete groups, it is known that C*-simplicity is equivalent to the intrinsically algebraic property of having no non-trivial normal amenable subgroups. For some time it was thought that this might be true for all discrete groups, however Le Boudec [5] and Ivanov and Omland [4] recently constructed examples of non-C*-simple groups with trivial amenable radical.

In [7], we established an algebraic characterization of C*-simplicity. Before stating this result, we state a characterization of C*-simplicity based on the notion of a uniformly recurrent subgroup, introduced by Glasner and Weiss [2].

Let G be a discrete group and let $\mathcal{S}(G)$ denote the compact space of subgroups of G equipped with the Chabauty topology, which coincides with the product topology on $\{0,1\}^G$. Convergence in the Chabauty topology can be described in the following way: a net of subgroups $(H_i)_{i \in I} < G$ converges in the Chabauty topology to a subgroup $H < G$ if

(1) every $h \in H$ eventually belongs to H_i and

(2) for every subnet $(H_j)_{j \in J}$, $\cap_j H_j \subset H$.

The space $\mathcal{S}(G)$ is a G-space with respect to the conjugation action of G. Let $\mathcal{S}_a(G)$ denote the (closed) G-invariant subspace of amenable subgroups of G.

Definition 2. *A compact G-subspace $X \subset \mathcal{S}(G)$ is said to be a* uniformly recurrent subgroup *of G if it is minimal, i.e. if $\{gHg^{-1} \mid g \in G\}$ is dense in X for every $H \in X$. If $X \subset \mathcal{S}_a(G)$, then X is said to be* amenable. *If $X \neq \{\{e\}\}$, where $\{e\}$ denotes the trivial subgroup of G, then X is said to be* non-trivial.

Theorem 3. *A discrete group G is C*-simple if and only if it has no non-trivial amenable uniformly recurrent subgroups.*

The characterization in Theorem 3 has a dynamical flavour. The algebraic characterization of C*-simplicity is obtained by unwinding the definition of a uniformly recurrent subgroup. Before stating this result, we require the notion of a recurrent subgroup.

Definition 3. *Let G be a countable discrete group. A subgroup $H < G$ is said to be* recurrent *if for every sequence $(g_n) \in G$ there is a subsequence (g_{n_k}) such that*

$$\bigcap_k g_{n_k} H g_{n_k}^{-1} \neq \{e\}.$$

Theorem 4. *A countable discrete group is C*-simple if and only if it has no amenable recurrent subgroups.*

The results stated above were recently utilized by Le Boudec and Matte Bon [6] to prove that Thompson's group F is non-amenable if and only if Thompson's group T is C*-simple.

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Roe algebras as relative commutants

Aaron Tikuisis
(joint work with J. Špakula)

Roe algebras are certain C*-algebras constructed from (proper) metric spaces, which encode coarse information about the spaces. A metric space is *proper* if all its closed balls are compact. For concreteness, this talk focused on *uniformly discrete* metric spaces, i.e., metric spaces (X, d) for which $d(x, y) \geq 1$ whenever $x \neq y$.

An extremely important example, which drives the study of metric spaces from a coarse perspective, is a finitely generated group with the word metric.

Definition. *Let (X, d) be a uniformly discrete metric space, let $a \in \mathcal{B}(l^2(X))$, and let $R, \epsilon > 0$.*

(i) *a has* propogation at most $R > 0$ *if $faf' = 0$ whenever $f, f' \in l^\infty(X)$ such that $d(\mathrm{supp} f, \mathrm{supp} f') > R$.*

(ii) *The* uniform Roe algebra *of X is*

$$\mathrm{C}^*_u(X) := \overline{\{b \in \mathcal{B}(l^2(X)) : b \text{ has finite propogation}\}}^{\|\cdot\|}.$$

(This is an algebra.)

(iii) *a has* ϵ-propogation at most $R > 0$ *if $\|faf'\| = 0$ whenever $f, f' \in l^\infty(X)$ such that $d(\mathrm{supp} f, \mathrm{supp} f') > R$ and $\|f\|, \|f'\| \leq 1$.*

Straightforward analysis shows that if $a \in \mathrm{C}^*_u(X)$ then a has finite ϵ-propogation for all $\epsilon > 0$. Finite propogation operators have also been called *band operators*, a name which is easily motivated by the picture of such an operator in the case $X = \mathbb{Z}$. When X is a finitely generated group G with the word metric, G acts on $l^\infty(G)$ by translation, and

$$\mathrm{C}^*_u(G) \cong l^\infty(G) \rtimes_r G.$$

Interestingly, the uniform Roe algebra captures a lot of information about the space X.

Theorem (Špakula–Willett [3])**.** *Let X, Y be metric spaces with property (A). Then X and Y are coarsely equivalent if and only if $\mathrm{C}^*_u(X)$ and $\mathrm{C}^*_u(Y)$ are Morita equivalent.*

Definition. *Let (X, d) be a uniformly discrete metric space. A function $f \in l^\infty(X)$ is a* Higson function *if for every $L > 0$, there exists a finite set $K \subset X$ such that $f|_{X \setminus K}$ is L-Lipschitz. The set (C*-algebra) of all Higson functions is denoted $C_h(X)$.*

For example, when X is a finitely generated group G with the word metric, $C_h(X)$ is the preimage of $(l^\infty(G)/c_0(G))^G$ (the fixed point algebra of the action induced by translation) under the quotient map $l^\infty(G) \to l^\infty(G)/c_0(G)$. When X has bounded geometry, it is easy to see that for any $a \in \mathrm{C}^*_u(X)$ and $f \in C_h(X)$,

$$[a, f] \in \mathcal{K}(l^2(X)).$$

Definition. *Let* (X, d) *be a uniformly discrete metric space. A bounded sequence* $(f_n)_{n=1}^{\infty} \subset l^{\infty}(X)$ *is* very Lipschitz *if for every* $L > 0$*, there exists* n_0 *such that, if* $n \geq n_0$ *then* f_n *is L-Lipschitz. The set (*C*-algebra) of all very Lipschitz bounded sequences is denoted* $\mathrm{VL}(X)$*, and* $\mathrm{VL}_{\infty}(X)$ *denotes its quotient in the sequence algebra:*

$$\mathrm{VL}_{\infty}(X) := \mathrm{VL}(X)/c_0(\mathbb{N}, l^{\infty}(X)) \subset l^{\infty}(X)_{\infty}.$$

For example, when X is a finitely generated group G with the word metric, $\mathrm{VL}_{\infty}(G) = (l^{\infty}(G)_{\infty})^G$ (the fixed point algebra of the action induced by translation). When X has bounded geometry, it is easy to see that for any $a \in \mathrm{C}^*_u(X)$ and $f = (f_n)_{n=1}^{\infty} \in \mathrm{VL}_{\infty}(X)$,

$$[a, f] = 0, \quad \text{i.e.,} \quad \lim_{n\to\infty} \|[a, f_n]\| = 0.$$

The main result presented in this talk is the following.

Theorem (Špakula-T)**.** *Let* (X, d) *be a uniformly discrete metric space and let* $a \in \mathcal{B}(l^2(X))$*. The following are equivalent:*

- (i) *a has finite* ϵ*-propogation for every* $\epsilon > 0$*;*
- (ii) $[a, f] = 0$ *for every* $f \in \mathrm{VL}_{\infty}(X)$*;*
- (iii) $[a, g] \in \mathcal{K}(l^2(X))$ *for every* $g \in C_h(X)$*.*

When X has finite asymptotic dimension (or even finite decomposition complexity, as defined by Guentner, Tessera, and Yu [1]*), these are also equivalent to:*

- (iv) $a \in \mathrm{C}^*_u(X)$*.*

The implication from the first three conditions to (iv) is the most important, as it provides new ways of showing that operators are in the uniform Roe algebra. In the case that $X = \mathbb{Z}^d$, the result (ii) $\Rightarrow$ (iv) was proven by Lange and Rabinovich using harmonic analysis [2].

Finite decomposition complexity is a property known to hold widely. It implies property (A), and the converse (does property (A) imply finite decomposition complexity?) is open.

It is an interesting question whether (i)–(iii) imply (iv) in the absence of finite decomposition complexity.

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Controlled K-theory and dynamic asymptotic dimension

Rufus Willett
(joint work with Erik Guentner, Guoliang Yu)

Probably the most important tools to compute C^*-algebra K-theory are the six term exact sequences arising from ideals in various ways. For example, if A is a C^*-algebra and I and J are ideals of A such that $A = I + J$, then there is a Mayer-Vietoris sequence

$$\begin{array}{ccccc} K_0(I\cap J) & \longrightarrow & K_0(I)\oplus K_0(J) & \longrightarrow & K_0(A) \\ \uparrow & & & & \downarrow \\ K_1(A) & \longleftarrow & K_1(I)\oplus K_1(J) & \longleftarrow & K_1(I\cap J) \end{array}$$

that allows information about $K_*(A)$ to be deduced from information and $K_*(I)$ and $K_*(J)$. When A is simple, such tools are of no use. The aim of my talk was to describe techniques that give analogues of the above six-term exact sequence even in the absence of non-trivial ideals, and discuss applications to the K-theory of crossed product C^*-algebras. This is based on joint work with Erik Guentner and Guoliang Yu: see [1].

To make precise the sort of 'weak Mayer-Vietoris sequence' one can construct, we need controlled K-theory groups. Let A be a non-unital C^*-algebra, and let $E \subseteq A$ be a self-adjoint subspace. Let $\widetilde{A}$ be the unitization of A, and let $\widetilde{E}$ be the subspace of $\widetilde{A}$ spanned by E and the unit. For each n, let $M_n(\widetilde{E})$ denote those $n \times n$ matrices over $\widetilde{A}$ that have entries coming from A. Fix $\epsilon \in (0, 1/4)$ and say that $p \in M_n(\widetilde{E})$ is an *ϵ-quasi-projection* if $p = p^*$ and $\|p^2 - p\| < \epsilon$. Note that the spectrum of p misses $1/2$, so that if $\kappa = \chi_{\{x>1/2\}}$, then there is a projection $\kappa(p) \in M_n(\widetilde{A})$ for any ϵ-quasi-projection $p \in M_n(\widetilde{E})$. Define the *rank* of p to be the rank of the image of $\kappa(p)$ in $M_n(\mathbb{C})$.

Set $M_\infty(\widetilde{E}) = \bigcup_{n=1}^\infty M_n(\widetilde{E})$ where the union is defined via the usual inclusions $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Define

$$K_0^\epsilon(E) := \{(p, \mathrm{rank}(p)) \in M_n(\widetilde{E}) \times \mathbb{N} \mid p \text{ an } \epsilon\text{-quasi-projection}\}/\sim,$$

where $(p, \mathrm{rank}(p)) \sim (q, \mathrm{rank}(q))$ if there is $k \in \mathbb{N}$ so that the

$$\begin{pmatrix} p & 0 \\ 0 & 1_{k+\mathrm{rank}(q)} \end{pmatrix} \text{ is homotopic to } \begin{pmatrix} q & 0 \\ 0 & 1_{k+\mathrm{rank}(p)} \end{pmatrix}$$

through quasi-projections in $M_\infty(\widetilde{E})$.

One can similarly define $K_1^\epsilon(E)$ by using 'quasi-unitaries': elements satisfying $\|uu^* - 1\| < \epsilon$ and $\|u^*u - 1\| < \epsilon$. Straightforward adaptations of the usual proofs show that $K_0^\epsilon(E)$ is an abelian group, $K_1^\epsilon(E)$ is an abelian semigroup, and that $K_i^\epsilon(E)$ is isomorphic to $K_i^\epsilon(A)$ whenever E is dense in A. See [3] for a full development of closely related controlled K-theory groups.

Now we discuss a Mayer-Vietoris sequence. Let A be a C^*-algebra, and let E, F be arbitrary self-adjoint subspaces of A (these will play the role of ideals). Assume that A is generated by self-adjoint subspace S, and let $\widetilde{S}$ denote the subspace of $\widetilde{A}$ spanned by S and the unit as before. For each $n \in \mathbb{N}$, define

$$E^{(n)} := \operatorname{span}(S^n \cdot E \cdot S^n),$$

and similarly for $F^{(n)}$. The pair E, F is *uniformly excisive* if for all $\epsilon > 0$ and $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that

$$E^{(n)} \cap F^{(n)} \subseteq_\epsilon (E \cap F)^{(m)},$$

where '$\subseteq_\epsilon$' means 'containment of the unit balls, up to ϵ error'. Then for any $\epsilon, r > 0$ there exists $s > 0$ and a sequence

$$\cdots \to K_i^\epsilon(E^{(r)}) \oplus K_i^\epsilon(F^{(r)}) \to K_i^\epsilon(E^{(r)} + F^{(r)}) \xrightarrow{\partial} K_{i-1}^\epsilon(E^{(s)} \cap E^{(s)}) \to \cdots .$$

It has exactness properties of the sort: for any r, ϵ there exists $t \geq r$ such that if $x \in K_i^\epsilon(E^{(r)} + F^{(r)})$ goes to zero under ∂, then there are $y \in K_i^\epsilon(E^{(t)})$ and $z \in K_i^\epsilon(F^{(t)})$ such that x is equal to $y + z$ in $K_i^\epsilon(E^{(t)} + F^{(t)})$. Similar statements hold at the other positions in the sequence. For applications, it is crucial that the various constants appearing depend only on the constants appearing in the definition of uniform excisiveness.

Now, in order to get any use out of this, one needs some interesting examples. Let Γ be a finitely generated group equipped with the associated work length $|\cdot|$, and say Γ acts on some compact space X. The *dynamic asymptotic dimension* (defined by the three current authors in earlier work [2]) is the smallest $d \in \mathbb{N}$ such that for all $r > 0$ there exists $s > 0$ and an open cover $\{U_0, ..., U_d\}$ of X such that if

$$x,\ g_1x,\ g_2g_1x, ...,\ g_n \cdots g_1 x$$

are all elements of the same U_i with all $|g_k| \leq r$ for all k, then $|g_n \cdots g_1| \leq s$. There are many interesting actions with finite dynamic asymptotic dimension: perhaps the most basic are minimal (free) $\mathbb{Z}$ actions, but there are many others: see [2].

Say now A is the (stabilized) crossed product associated to some action of a finitely generated group Γ on a compact space X. Let S be the subspace of A spanned by some finite symmetric generating set for Γ together with $C(X)$ (suitably stabilized). The above dynamic asymptotic dimension condition, plus an induction on d is enough to use the controlled Mayer-Vietoris sequence discussed above to get information on the K-theory of A. For example, we can use these techniques to prove that if the action of Γ on X has finite dynamic asymptotic dimension, then the Baum-Connes conjecture is true for Γ with coefficients in $C(X)$.

We should remark that this result is known: it follows from work of Tu on the Baum-Connes conjecture for amenable groupoids [5]. Nonetheless, our proof is rather more direct and elementary: in particular, it uses a new and concrete model for the Baum-Connes conjecture with coefficients that uses no equivariant or bivariant K-theory. It proceeds in quite a different way to Tu's proof, amounting in principle to a computation of $K_*(A)$ that uses only the internal structure, and does not replace A with a simpler KK-equivalent C^*-algebra.

As well as the fact that the proof is more elementary, there are two other motivations for our new proof: first, the techniques are likely to adapt to enable us to say something about K-theory of more general classes of C^*-algebras (compare for example [4]; second, the proofs avoid any of the inherently analytic material in Tu's proof, and thus adapt to the setting of the Farrell-Jones conjecture in algebraic K-theory.

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Unitarizability, Maurey–Nikishin factorization and Polish groups of finite type

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(joint work with Yasumichi Matsuzawa, Andreas Thom and Asger Törnquist)

I report our recent work [AMTT16] on the characterization problem of Sorin Popa's class $\mathscr{U}_{\text{fin}}$ of finite type Polish groups. Here, a Polish group G is said to be of *finite type* if it is embeddable into the the unitary group $\mathcal{U}(M)$ of a II_1 factor equipped with the strong operator topology. In [Po07] Popa showed the celebrated cocycle superrigidity Theorem. The theorem treats measurable cocycles for a certain class of probability measure preserving (pmp) actions with target groups in the class $\mathscr{U}_{\text{fin}}$. The theorem has many applications in the study of II_1 factors and ergodic theory. It is therefore of interest to understand the class $\mathscr{U}_{\text{fin}}$. It is clear that the class $\mathscr{U}_{\text{fin}}$ contains all countable discrete groups and all compact Polish groups [1]. But are there more groups that fall into this class? Interestingly, very little has been known about the class $\mathscr{U}_{\text{fin}}$. In particular, there is no abstract characterization of finite type groups. In this situation, Popa pointed out that there are two necessary conditions for a Polish group G to be of finite type. Namely,

(i) G must be unitarily representable. That is, G must be embeddable into the closed subgroup of $\mathcal{U}(\ell^2)$.

[1] take the left regular representation. Note that any tracial von Neumann algebra with separable predual embeds into some II_1 factor

(ii) G is SIN. That is, G must admit a two-sided invariant metric d compatible with the topology.

Indeed, if $G \subset \mathcal{U}(M)$ for a II_1 factor M with the tracial state τ, then clearly $G \hookrightarrow \mathcal{U}(\ell^2)$ and G admits a two-sided invariant metric $d(u,v) := \|u-v\|_2$, where $\|x\|_2 := \tau(x^*x)^{\frac{1}{2}}$. Popa asked whether the above two conditions are sufficient. I.e., whether the class $\mathscr{U}_{\mathrm{SIN}}$ of all unitarily representable SIN Polish groups coincides with $\mathscr{U}_{\mathrm{fin}}$. In [AM12-2], I and Matsuzawa showed a partial positive answer that if $G \in \mathscr{U}_{\mathrm{SIN}}$ is either (a) locally compact or (b) amenable (there exists a left-invariant mean on the space $\mathrm{RUCB}(G)$ of all right-uniformly continuous bounded functions on G), then $G \in \mathscr{U}_{\mathrm{fin}}$. However, the general case was left open. In [AMTT16], we showed that $\mathscr{U}_{\mathrm{SIN}} \neq \mathscr{U}_{\mathrm{fin}}$. There we found an unexpected connection about Popa's question on $\mathscr{U}_{\mathrm{fin}}$ and the theory of uniformly bounded representations of countable discrete groups. The main result of [AMTT16] is as follows.

Theorem 1. *Let Γ be a countable discrete group, and let $\pi\colon \Gamma \to \mathrm{GL}(H)$ be a group homomorphism of Γ into the group of all invertible operators on a separable Hilbert space H.*

(i) *G is SIN, if and only if π is uniformly bounded: $\sup_{s\in\Gamma}\|\pi(s)\| < \infty$.*
(ii) *G is of finite type, if and only if π is unitarizable: there exists $V \in \mathrm{GL}(H)$ such that $V^{-1}\pi(\,\cdot\,)V$ is a unitary representation of Γ.*

A countable grouop Γ is said to be *unitarizable* if all its uniformly bounded representations are unitarizable, and many non-amenable groups are known to be non-unitarizable. There is even a question by Dixmier asking whether unitarizable groups are actually amenable. We will not go into the details of the history of unitarizability question. For more details, see e.g., [Dix50, Oz06, Pi01, Pi05]. The point here is that there are many non-unitarizable uniformly bounded representations and they give a family of unitarily representable Polish SIN groups which fail to be in the class $\mathscr{U}_{\mathrm{fin}}$. As a byproduct of the main theorem, we also showed:

Theorem 2. *Let π be a uniformly bounded representation of a countable discrete group Γ on H. Then the following three conditions are equivalent.*

(i) *$G = H \rtimes_\pi \Gamma$ is of finite type.*
(ii) *π is unitarizable.*
(iii) *There exists a continuous positive definite function f on H (regarded as an additive group) which generates a neighborhood basis of $0 \in H$, such that $f(\pi(s)\xi) = f(\xi)$ $(s \in \Gamma,\ \xi \in H)$ holds.*

Here, a positive definite function $f\colon H \to \mathbb{C}$ is said to generate a neighborhood basis of 0, if for every closed set $A \subset H$ not containing 0, $\sup_{\xi\in A}|f(\xi)| < f(0)$ holds. The crucial point in the main theorem is to show that if $G = H \rtimes_\pi \Gamma$ is in $\mathscr{U}_{\mathrm{fin}}$, then π is unitarizable. The outline of the proof is as follows: by assumption we have an embedding $G \overset{\alpha}{\hookrightarrow} \mathcal{U}(M)$ for some II_1 factor M. Then we use the Lie algebra approach as in [AM12-1, AM12-2]: set $\alpha(s) := \alpha(0,s)$ and $\alpha(\xi) := \alpha(\xi,1)$ $(s \in \Gamma,\ \xi \in H)$ Then for each $\xi \in H$, $\mathbb{R} \ni t \mapsto \alpha(t\xi) \in \mathcal{U}(M)$ is a strongly continuous one-parameter subgroup of $\mathcal{U}(M)$, whence there exists a (possibly unbounded) self-adjoint operator $T(\xi)$ affiliated with M such that $\alpha(t\xi) = e^{iT(\xi)}$ $(t \in \mathbb{R})$. Since $\alpha(s)\alpha(\xi)\alpha(s)^* = \alpha(\pi(s)\xi)$, one has

$$\alpha(s)T(\xi)\alpha(s)^* = T(\pi(s)\xi), \quad s \in \Gamma, \, \xi \in H. \tag{1}$$

Denote by $L^0(M,\tau)$ the space of all closed densely defined operators affiliated with M equipped with the so-called τ-measure topology. One can show that $T\colon H \to L^0(M,\tau)$ is a real-linear homeomorphism onto its range. Then since the von Neumann subalgebra $\{\alpha(\xi); \xi \in H\}''$ is commutative and (1) holds, the group Γ acts on $\mathcal{A}$ by $\mathrm{Ad}(\alpha(\,\cdot\,)$ in a τ-preserving way. Therefore there exists a compact metric space X with a Borel probability measure m, and an m-preseriving Γ action such that

$$\Gamma \overset{\mathrm{Ad}(\alpha(\,\cdot\,))}{\curvearrowright} \mathcal{A} \cong \Gamma \curvearrowright L^\infty(X,m).$$

Therefore we get a map $T\colon H \to L^0(X,m)$ (the space of all measurable maps on X). We then use a convexity argument to improve Maurey–Nikishin factorization Theorem [Ni70, Ma72] (see also [GR85]) to show that there exists a m-a.e. positive Γ-*invariant* function $\varphi \in L^\infty(X,m)$ such that

$$\int_X \varphi(x)[T\xi](x)\,\mathrm{d}m(x) < \infty, \quad \xi \in H.$$

Then $\tilde{T}\colon H \to \xi \mapsto \varphi^{\frac{1}{2}} \cdot T\xi \in L^2(X,m)$ satisfies $\alpha(s)\tilde{T}(\xi)\alpha(s)^* = \tilde{T}(\pi(s)\xi)$ $(s \in \Gamma, \, \xi \in H)$, so that the complexification (note that T is only real-linear) of

$$\langle \xi, \, \eta \rangle_\pi := \langle \tilde{T}(\xi), \, \tilde{T}(\eta) \rangle_{L^2(M)}$$

gives a Γ-invariant inner product on H. With an extra argument on the topology of L^2 and the τ-measure topology, one can show that the resulting inner-product is equivalent to the original inner-product, which then shows that π is unitarizable. The omitted details can be found in [AMTT16].

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A new bicommutant theorem

Ilijas Farah

Ultrapowers[1] $A^{\mathcal{U}}$ of separable operator algebras are, being subject to well-developed model-theoretic methods, reasonably well-understood. Ultraproducts of Banach spaces, C*-algebras, tracial von Neumann algebras, representations of C*-algebras, as well as ultrapowers of 'metric structures' can be construed as special cases of the general ultraproduct construction ([1]; see also [6, Theorem 1.2] and [4]). Since the early 1970s and the influential work of McDuff and Connes central sequence algebras $A' \cap A^{\mathcal{U}}$ play an even more important role than ultrapowers in classification of II_1 factors and (more recently) C*-algebras. While they do not have a well-studied abstract analogue, in [6, Theorem 1] it was shown that the central sequence algebra of a strongly self-absorbing algebra ([9]) is isomorphic to its ultrapower (this applies to both C*-algebras and II_1-factors; note that the hyperfinite II_1 factor is the only strongly self-absorbing II_1 factor). Relative commutants $B' \cap D^{\mathcal{U}}$ of separable subalgebras of ultrapowers of strongly self-absorbing C*-algebras play an increasingly important role in classification program for separable C*-algebras ([7, §3], [3]; see also [8], [11]).

C*-algebra B is *primitive* if it has representation that is both faithful and irreducible. Since $\mathcal{B}(H)^{\mathcal{U}}$ can be naturally identified with a subalgebra of $\mathcal{B}(H^{\mathcal{U}})$ We prove an analogue of the well-known consequence of Voiculescu's theorem ([10, Corollary 1.9]) and von Neumann's bicommutant theorem ([2, §I.9.1.2]). $\overline{A}^{\mathrm{WOT}}$ denotes the closure of A in the weak operator topology.

Theorem 1. *Assume $\prod_{\mathcal{U}} B_j$ is an ultraproduct of unital, primitive* C*-*algebras and A is a separable unital* C*-*subalgebra. Then (with the weak operator closure $\overline{A}^{\mathrm{WOT}}$ computed in the ultraproduct of faithful irreducible representations of B_js)*

$$A = \Big(A' \cap \prod_{\mathcal{U}} B_j\Big)' = \overline{A}^{\mathrm{WOT}} \cap \prod_{\mathcal{U}} B_j.$$

A slightly weaker version of the following corollary to Theorem 1 (stated here with Aaron Tikuisis's kind permission) was originally proved by using very different methods ($Z(A)$ denotes the center of A).

Corollary 1 (Farah–Tikuisis, 2015)**.** *Assume $\prod_{\mathcal{U}} B_j$ is an ultraproduct of simple unital* C*-*algebras and A is a separable unital subalgebra. Then $Z(A' \cap \prod_{\mathcal{U}} B_j) = Z(A)$.* □

[1] Throughout $\mathcal{U}$ denotes a nonprincipal ultrafilter on $\mathbb{N}$.

The proof of Theorem 1 combines model-theoretic methods with the application of Hahn–Banach theorem known as the 'method of Day' used to transfer some first-order statements between C*-algebra B and its double dual B^{**} (see [5] for the details).

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Free subgroups of amenable (Polish) groups

Alessandro Carderi
(joint work with Andreas Thom)

The free group F_2 is the prototipe of non-amenable group: any countable group which contains F_2 is not amenable and on the contrary it requires some work to construct a non amenable group without free subgroups. Similarly any locally compact group which contains a discrete free subgroup is not amenable. For Polish groups this is no longer true and the situation is much more complicated. Andreas Thom conjectured the following.

Conjecture (A. Thom). *The unitary group of the hyperfinite* II_1 *factor has no uniformly discrete free non abelian subgroup (uniformly discrete with respect to the* 2*-norm).*

In order to understand better this conjecture, we want to study Polish groups which have similar properties to the above group. We will focus on the following important facts.

- The unitary group of the hyperfinite II_1 factor is Polish and SIN with respect to the 2-norm.
- The quotient by its center is simple, see [3].
- It is extremely amenable, see [2].

Our work is focused in describing a group which satisfies similar properties and which is not one of its subgroups. For this we need to fix a finite field $\mathbb{F}_q$ with $q = p^h$ elements and we let $\mathrm{S}L_n(q)$ be the special linear group over $\mathbb{F}_q$. We denote by $r(k)$ the *rank* of a matrix $k \in M_n(\mathbb{F}_q)$. We equip the groups $\mathrm{S}L_n(q)$ with the (normalized) *rank-distance*, $d_r(g,h) := \frac{1}{n} r(g-h) \in [0,1]$. Note that d_r is a bi-invariant metric, which means that it is a metric such that $d_r(gh, gk) = d_r(h,k) = d_r(hg, kg)$ for every $g,h,k \in \mathrm{S}L_n(q)$. For every $n \in \mathbb{N}$, we consider the diagonal embedding

$$\varphi_n : \mathrm{S}L_{2^n}(q) \to \mathrm{S}L_{2^{n+1}}(q), \qquad \text{defined by} \qquad \varphi_n(g) := \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}.$$

Observe that for every n, φ_n is an isometric homomorphism. We denote by $A_0(q)$ the countable group arising as the inductive limit of the family $\{(\mathrm{S}L_{2^n}(q), \varphi_n)\}_n$ and observe that we can extend the rank-metric d_r canonically to $A_0(q)$. Let $A(q)$ be the metric-completion of $A_0(q)$ with respect to d_r, i.e., $A(q)$ is a Polish group and the natural extension of the rank-metric is complete and bi-invariant.

Our main result is the following theorem.

Theorem ([1]). *The Polish group $A(q)$ has the following properties:*

- *every strongly continuous unitary representation of $A(q)$ on a Hilbert space is trivial,*
- *the group $A(q)$ is extremely amenable,*
- *the center of $A(q)$ is isomorphic to $\mathbb{F}_q^\times$ and the quotient by its center is topologically simple,*
- *$A(q)$ contains every countable amenable group and, in case q is odd, the free group on two generators as discrete subgroups.*

The above theorem shows that if the conjecture is true, then it has to be true for very specific reasons. Moreover we would like to add that the free subgroup constructed in the above theorem is maximally discrete, that is the distance between any two distinct elements is 1. On the contrary it is well known that the unitary group of the hyperfinite II_1 factor cannot have maximally discrete free subgroups.

It would be also very interesting to understand further properties of the group $A(q)$, for example it is unknown whether it is contractible, generated by involution or if it acts by isometries on a reflexive Banach space.

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Cartan subalgebras in C*-algebras

Xin Li
(joint work with J. Renault)

In the first part of my talk, I explained the motivation to study Cartan subalgebras in C*-algebras and continuous orbit equivalence for topological dynamics. The goal is to develop a topological version of measured group theory [3].

Definition 1. *A Cartan subalgebra B of a C*-algebra A is a maximal abelian selfadjoint subalgebra of A which contains an approximate unit for A, satisfies the condition that $N_A(B) := \{n \in A : n^* \subseteq B, n^* B n \subseteq B\}$ generates A as a C*-algebra, and with the property that there is a faithful conditional expectation $A \twoheadrightarrow B$.*

Two Cartan pairs (A_1, B_1) and (A_2, B_2) (a Cartan pair is a C*-algebra together with a Cartan subalgebra) are equivalent if there exists a C*-algebra isomorphism $\varphi : A_1 \to A_2$ with $\varphi(B_1) = B_2$. Kumjian and Renault showed that every Cartan subalgebra comes from a twisted groupoid [4, 8]. This justifies why we can think of Definition 1 as the exact C*-algebraic analogue of Cartan subalgebras in von Neumann algebras [1, 2].

At the moment, very little is known about C*-algebraic Cartan pairs, in particular concerning existence and uniqueness. We say that a C*-algebra A has unique Cartan if any Cartan pairs (A, B_1) and (A, B_2) must be equivalent. A weaker notion is given by distinguished Cartans within a certain class of C*-algebras. This means that we can choose Cartan subalgebras for certain C*-algebras so that whenever two C*-algebras from that class are isomorphic, the corresponding Cartan pairs will be equivalent. For instance, AF algebras have distinguished Cartan subalgebras.

To explain the role Cartan subalgebras play for connections between C*-algebras and topological dynamics, let us introduce the notion of continuous orbit equivalence.

Definition 2. *$G \curvearrowright X$ and $H \curvearrowright Y$ are continuously orbit equivalent if there exists a homeomorphism $\varphi : X \xrightarrow{\cong} Y$ together with continuous maps $a : G \times X \to H$ and $b : H \times Y \to G$ such that $\varphi(g.x) = a(g,x).\varphi(x)$ and $\varphi^{-1}(h.y) = b(h,y).\varphi^{-1}(y)$ for all $g \in G$, $x \in X$, $h \in H$ and $y \in Y$. Here $G \times X \to X$, $(g,x) \mapsto g.x$ denotes the G-action on X.*

The connection between continuous orbit equivalence and Cartan pairs is provided by the following result (see [5] for details).

Theorem 1. *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free systems. The following are equivalent:*

- *$G \curvearrowright X$ and $H \curvearrowright Y$ are continuously orbit equivalent.*
- *The transformation groupoids $G \ltimes X$ and $H \ltimes Y$ are isomorphic as topological groupoids.*
- *$(C_0(X) \rtimes_r G, C_0(X))$ and $(C_0(Y) \rtimes_r H, C_0(Y))$ are equivalent as Cartan pairs.*

Cartan subalgebras and continuous orbit equivalence turn out to be closely related to quasi-isometry, the fundamental notion of geometric group theory. The following dynamical characterizations of quasi-isometry and bilipschitz equivalence have been established in [7] and [6].

Theorem 2. *Let G and H be finitely generated groups.*

- *G and H are bilipschitz equivalent if and only if there exist continuously orbit equivalent topologically free dynamical systems $G \curvearrowright X$ and $H \curvearrowright Y$ on totally disconnected compact spaces X and Y.*
- *G and H are quasi-isometric if and only if there exist stably continuously orbit equivalent topologically free dynamical systems $G \curvearrowright X$ and $H \curvearrowright Y$ on totally disconnected compact spaces X and Y.*

Here, $G \curvearrowright X$ and $H \curvearrowright Y$ are called stably COE if $\mathbb{Z} \times G \curvearrowright \mathbb{Z} \times X$ and $\mathbb{Z} \times H \curvearrowright \mathbb{Z} \times Y$ are continuously orbit equivalent. This notion corresponds to stable isomorphism for the Cartan pairs, hence the name.

Theorems 1 and 2 show that the picture in the topological setting is very much analogous to the measurable framework [3], and this forms the starting point for our goal to develop a topological version of measured group theory.

Inspired by [9], I recently found the following alternative dynamical characterization of quasi-isometry and bilipschitz equivalence, which in contrast to Theorem 2 provides concrete models of dynamical systems.

Theorem 3. *Let G and H be finitely generated groups.*

- *G and H are bilipschitz equivalent if and only if $G \curvearrowright \beta G$ and $H \curvearrowright \beta H$ are continuously orbit equivalent.*
- *G and H are quasi-isometric if and only if $G \curvearrowright \beta G$ and $H \curvearrowright \beta H$ are stably continuously orbit equivalent.*

Here $G \curvearrowright \beta G$ is the canonical action of G on its Stone-Cech compactification.

I then went on to present ongoing work with Jean Renault on Cartan subalgebras in C*-algebras.

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Howe-Moore type theorems for quantum groups and rigid C^*-tensor categories

Tim de Laat
(joint work with Yuki Arano and Jonas Wahl)

The Howe-Moore property is a property for locally compact groups that plays a crucial role in the proofs of several important rigidity results. A locally compact group is said to have the Howe-Moore property if for every unitary representation without invariant vectors, the matrix coefficients vanish at infinity. This property was first established by Howe and Moore [3] and Zimmer [8] for connected non-compact simple Lie groups with finite center. Howe and Moore also proved the property for certain subgroups of algebraic groups over non-Archimedean local fields.

For Lie groups and algebraic groups over non-Archimedean local fields, much more can be said about the asymptotic behaviour of matrix coefficients. A powerful result of Veech [7] asserts that every weakly almost periodic function on a connected non-compact simple Lie group with finite center has a limit at infinity, and this limit is equal to the (unique invariant) mean of the weakly almost periodic function. As a consequence, it follows that for every uniformly bounded representation of such a Lie group on a reflexive Banach space that does not have any invariant vectors, the matrix coefficients vanish at infinity. In a recent work of Bader and Gelander [2], this was also shown to hold for connected simple algebraic groups over non-Archimedean local fields.

In a recent article with Yuki Arano and Jonas Wahl [1], we initiated the study of Howe-Moore type phenomena in the setting of quantum groups and rigid C^*-tensor categories. The unitary representation theory for quantum groups has been studied extensively. Recently, Popa and Vaes developed a theory of unitary representations (called admissible $*$-representations) for "subfactor related group-like objects" [6] (see also [5]). This representation theory is formulated in the setting of rigid C^*-tensor categories. Two important notions in this theory are the notions of completely positive and completely bounded multiplier. The completely positive multipliers span an algebra that is the analogue in the setting of rigid C^*-tensor categories of the Fourier-Stieltjes algebra. Indeed, this algebra can alternatively be defined as the algebra of matrix coefficients of admissible $*$-representations of the fusion algebra of the category. This naturally leads to the following formulation of the Howe-Moore property in the setting of C^*-tensor categories.

Definition 1. *Let $\mathcal{C}$ be a rigid C^*-tensor category, and let* $\mathrm{Irr}(\mathcal{C})$ *denote the set of equivalence classes of irreducible objects in $\mathcal{C}$. Then $\mathcal{C}$ is said to have the Howe-Moore property if every completely positive multiplier $\omega : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ has a limit at infinity.*

Our main result is a Howe-Moore type theorem for the representation categories of q-deformations of compact simple Lie groups, which are ubiquitous and motivating examples of compact quantum groups. Recall that the representation category $\mathrm{Rep}(\mathbb{G})$ of a compact quantum group $\mathbb{G}$ is the rigid C^*-tensor category of finite-dimensional unitary representations of $\mathbb{G}$.

Theorem 1. *Let $q \in (0, 1]$, and let K_q be a q-deformation of a connected compact simple Lie group K with trivial center. Then every completely bounded multiplier on* $\mathrm{Rep}(K_q)$ *has a limit at infinity. In particular, the representation category* $\mathrm{Rep}(K_q)$ *has the Howe-Moore property.*

Another important source of rigid C^*-tensor categories comes from Jones's theory of subfactors. From an inclusion $N \subset M$ of II_1-factors with finite index $[M : N] < \infty$, we can construct its Jones tower $M_{-1} \subset M_0 \subset M_1 \subset M_2 \subset \dots$ of II_1-factors, where $M_{-1} = N$ and $M_0 = M$ (see [4] for details). The standard invariant of $N \subset M$ is the lattice of relative commutants $M_i' \cap M_j$, with $i \leq j$. A crucial example of a standard invariant is the Temperley-Lieb-Jones standard invariant $\mathrm{TLJ}(\lambda)$, which is an initial object for the category of standard invariants. Given the Jones tower of a subfactor $N \subset M$ with standard invariant $\mathrm{TLJ}(\lambda)$ with $\lambda^{-1} \geq 4$, we can consider the rigid C^*-tensor category $\mathcal{C}_M$ consisting of all M-bimodules that are isomorphic to a finite direct sum of M-subbimodules of ${}_M L^2(M_i)_M$, with $i \geq 0$. Such a category $\mathcal{C}_M$ is equivalent to the representation category of the compact quantum group $\mathrm{PSU}_q(2)$, where q is the unique number $0 < q \leq 1$ such that $q + \frac{1}{q} = \lambda^{-\frac{1}{2}}$. Hence, the following theorem is a direct consequence of Theorem 1.

Theorem 2. *Let $N \subset M$ be an inclusion of II_1 factors with index $[M : N] = \lambda^{-1} \geq 4$ and Temperley-Lieb-Jones standard invariant* $\mathrm{TLJ}(\lambda)$ *(and hence principal graph A_∞). The rigid C^*-tensor category $\mathcal{C}_M$ of M-bimodules associated with the Jones tower of $N \subset M$ has the Howe-Moore property.*

Theorem 1 and Theorem 2 follow from a more general result on the convergence of completely bounded multipliers on certain rigid C^*-tensor categories. This more general result also holds for the representation categories of the free orthogonal quantum groups and for the Kazhdan-Wenzl categories. We refer to [1] for the details.

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Bounded Normal Generation

Philip Dowerk
(joint work with Andreas Thom)

Let G be a group and for $g \in G$ denote the conjugacy class of g by $g^G := \{hgh^{-1} \mid h \in G\}$, similarly $g^{-G} := (g^{-1})^G$. We say that G has the bounded normal generation property (BNG) if for every nontrivial element $g \in G$, the conjugacy classes of g and g^{-1} generate the whole group in finitely many steps, i.e., there exists $k \in \mathbb{N}$ such that

$$G = (g^G \cup g^{-G})^k.$$

A function $f : G \setminus \{1_G\} \to \mathbb{R}$ giving an upper bound on the number of steps that are required is called normal generation function for G.

Obviously, any group with property (BNG) is simple, however the converse is not true in general – even though many naturally occuring simple groups do have property (BNG). For example, a Baire category argument (see [2, Proposition 2.2]) implies that any compact simple group has property (BNG). Already in this case it is much harder to provide explicit normal generating functions. Informally, we say that a normal generating function is optimal if it is best possible up to a multiplicative constant for some family of groups.

For finite simple groups an optimal normal generation function can be found in seminal work of Liebeck and Shalev [6, Theorem 1.1] – leading to many fruitful applications. For compact connected simple Lie groups, an explicit (non-optimal) normal generation function was given in work of Nikolov-Segal [7, Proposition 5.11].

We show that the projective unitary group $\mathrm{PU}(\mathcal{M})$ has property (BNG) whenever $\mathcal{M}$ is a factor of type I_n, II_1 or III. Our proof does not use simplicity, which was proven by de la Harpe [5]. However, in the II_1 case, we need a modified version of a result by Broise [1]. Note that the projective unitary group of a type I_∞ or II_∞ factor does not have property (BNG), because it contains finite and infinite rank perturbations of the identity.

We present optimal normal generation functions for PU(n) [2, Corollary 5.11] and for the projective unitary group of a type III factor [3, Theorem 1.3], given by

$$f(g) = c / \inf_{\lambda \in S^1} \|1 - \lambda g\|, \qquad g \neq 1,$$

for some universal constant $c \in \mathbb{N}$, where $\|\cdot\|$ denotes the operator norm.

In the case of the projective unitary group of the connected component of the identity of the Calkin algebra, we prove property (BNG) and obtain as a normal generation function

$$f(g) = c / \inf_{\lambda \in S^1} \|1 - \lambda g\|_{\mathrm{ess}}, \qquad g \neq 1,$$

where $\|\cdot\|_{\mathrm{ess}}$ denotes the essential operator norm and $c \in \mathbb{N}$ a universal constant, see [3, Theorem 1.2].

For the projective unitary group of a type II_1 factor with trace τ we provide a concrete (close to optimal) normal generation function in [2, Theorem 1.3]: let $\ell(g) = \inf_{\lambda \in S^1} \|1 - \lambda g\|_1$, where $\|\cdot\| = \tau(|\cdot|)$ denotes the 1-norm. Then

$$f(g) = c \cdot |\log \ell(g)| / \ell(g)$$

defines a normal generation function for some universal constant $c \in \mathbb{N}$.

As an application of our results on bounded normal generation, we show that every homomorphism from the projective unitary group $\mathrm{PU}(\mathcal{M})$ of a type I_n or II_1 factor $\mathcal{M}$, equipped with the strong operator topology, into any separable SIN group is continuous. This requires some modification of the setting in [8].

Another application of our techniques combined with a result from Gartside and Pejić [4] is the uniqueness of the Polish group topology of $\mathrm{PU}(\mathcal{M})$.

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ε-representations

Marcus De Chiffre
(joint work with Andreas Thom)

Let $\varepsilon > 0$, let G be a group and let $\mathcal{M}$ be a von Neumann algebra with unitary group $\mathcal{U}(\mathcal{M})$. An ε-representation of G (with respect to $\|\cdot\|$) is a map $\varphi\colon G \to \mathcal{U}(\mathcal{M})$ such that $\|\varphi(gh) - \varphi(g)\varphi(h)\| < \varepsilon$ for all $g, h \in G$. Here, $\|\cdot\|$ is some norm on $\mathcal{M}$, e.g. the operator norm or, if $\mathcal{M}$ has a tracial state, the 2-norm.

A natural question, which is the topic of this 15 minutes talk, is the following: Given an ε-representation as above, is there an honest representation $\pi\colon G \to \mathcal{U}(\mathcal{M})$ such that $\|\varphi(g) - \pi(g)\|$ is small for all $g \in G$ (depending on ε)?

Kazhdan proved in [2] that in the case where G is amenable and $\mathcal{M}$ is equipped with the operator norm $\|\cdot\|_{op}$ the question has a very satisfying answer: For every $0 < \varepsilon < \frac{1}{200}$ and for every ε-representation $\varphi\colon G \to \mathcal{U}(\mathcal{M})$ there exists a representation $\pi\colon G \to \mathcal{U}(\mathcal{M})$ such that $\|\varphi(g) - \pi(g)\|_{op} < 2\varepsilon$ for every $g \in G$.

In the tracial case, where $\mathcal{M}$ is equipped with the 2-norm, the situation becomes more subtle, even in finite dimensions if $\mathcal{M} = \mathcal{M}_n$ is a I_n-factor. Of course, since $\|\cdot\|_2 \leq \|\cdot\|_{op} \leq \sqrt{n}\|\cdot\|_2$, Kaszdan's result implies that if φ is an ε-representation into $\mathcal{U}_n$ the unitary group on an n-dimensional Hilbert space, then there is a representation $\pi\colon G \to \mathcal{U}_n$ such that $\|\varphi(g) - \pi(g)\|_2 < 2\sqrt{n}\varepsilon$, but this estimate depends on the dimension n.

Recently, Gowers and Hatami [1] managed to avoid the dimension dependence in a certain sense, at least when G is finite. They proved that if $0 < \varepsilon < \frac{1}{16}$ and G is finite and $\mathcal{M} = \mathcal{M}_n$ for some natural number n, then for any $\varphi\colon G \to \mathcal{U}_n$ there exists $m \in \{0, \ldots, \lfloor 3\varepsilon^2 n \rfloor\}$ and a representation $\pi\colon G \to \mathcal{U}_{n+m}$ such that $\|\varphi(g) \oplus I_m - \pi(g)\|_2 < 31\varepsilon$. As it turns out, the addition of the m extra dimensions is necessary in order to get a dimension independent result, even if G is finite.

In a work in progress, we generalize the result of Gowers and Hatami to the case where G is amenable and $\mathcal{M}$ is a finite von Neumann algebra equipped with the 2-norm coming from a trace.

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Dynamical characterizations of paradoxicality for groups

Eduardo Scarparo

Consider the following conditions on a group G:

(1) G is not equidecomposable with two disjoint subsets;
(2) No non-empty subset of G is equidecomposable with two disjoint subsets of itself;

(3) G is not equidecomposable with a proper subset. Equivalently, no subset of G is equidecomposable with a proper subset of itself.

By Tarski's theorem, the first condition is equivalent to amenability, and the second one to supramenability.

If a group has subexponential growth, then it is supramenable. It is not known if the converse holds. Also, given G and H supramenable groups, it is unknown if $G \times H$ is supramenable, even if one of the groups is $\mathbb{Z}$.

In [3], Kellerhals, Monod and Rrdam showed that a group G is supramenable if and only if $\ell^\infty(G) \rtimes_{\mathrm{r}} G$ has no properly infinite projections, if and only if every co-compact action of G on a locally compact, Hausdorff space admits a non-zero, invariant, regular measure.

In [5], we showed that the class of supramenable groups, in a certain way, plays the role of amenable groups in the context of partial actions:

Theorem 1. *A group G is supramenable if and only if whenever it partially acts on a unital* C^**-algebra A which has a tracial state, then $A \rtimes G$ has a tracial state.*

Kellerhals, Monod and Rrdam also showed in [3] that if a group is locally finally, then $\ell^\infty(G) \rtimes_{\mathrm{r}} G$ is finite, and asked if the converse holds. In [6], we gave an affirmative answer for their question:

Theorem 2 (Kellerhals-Monod-Rrdam, Scarparo)**.** *Let G be a group. The following conditions are equivalent:*

(1) *G is locally finite;*
(2) *$\ell^\infty(G) \rtimes_{\mathrm{r}} G$ is finite;*
(3) *G is not equidecomposable with a proper subset of itself;*

If G is a countable, locally finite group, then $\mathrm{C}^*(G)$ is clearly AF. It is not known if the converse holds. For nilpotent groups, it does, by a result of Kaniuth in [2].

We presented a proof that, for finitely generated, elementary amenable groups, also $\mathrm{C}^*(G)$ being AF implies that G is (locally) finite.

Lemma 1. *If G is an infinite, finitely generated, elementary amenable group, then there is a subgroup of finite index of G which admits a homomorphism onto $\mathbb{Z}$.*

Proof. Let $\mathcal{A}$ be the class of all finite groups, all non-finitely generated groups, and all groups containing a finite index subgroup which maps onto $\mathbb{Z}$.

We claim that $\mathcal{A}$ contains the class of elementary amenable groups. Obviously, $\mathcal{A}$ contains all finite groups, it contains $\mathbb{Z}$, and it is closed under taking inductive limits (with injective connecting maps), and extensions by $\mathbb{Z}$.

Let us check that $\mathcal{A}$ is also closed under taking extensions by finite groups. Let $H \in \mathcal{A}$, F be a finite group, and G a group which fits into the short exact sequence

$$1 \to H \to G \to F \to 1.$$

If G is infinite and finitely generated, then also H is infinite and finitely generated. Hence, H contains a finite index subgroup H' which maps onto $\mathbb{Z}$. Since F is finite, also H' has finite index in G. Therefore, $G \in \mathcal{A}$.

By [4, Corollary 2.1], it follows that $\mathcal{A}$ contains the elementary amenable groups. □

Theorem 3. *If G is a finitely generated, elementary amenable group and $\mathrm{C}^*(G)$ is AF, then G is finite.*

Proof. Suppose G is infinite. By Lemma 1, there is a subgroup H of G with finite index n, and which admits a homomorphism onto $\mathbb{Z}$. This gives rise to a $*$-homomorphism $\varphi\colon \mathrm{C}^*(G) \to M_n(\mathrm{C}^*(H))$, and a surjective $*$-homomorphism $\psi\colon M_n(\mathrm{C}^*(H)) \to M_n(\mathrm{C}^*(\mathbb{Z}))$, such that the image of $\psi \circ \varphi$ is infinite-dimensional.

Since $M_n(\mathrm{C}^*(\mathbb{Z})) \simeq M_n(C(S^1))$ does not contain any infinite-dimensional AF algebra, we get a contradiction. Hence, G is finite. □

If there exists an elementary amenable, non-locally finite group G such that $\mathrm{C}^*(G)$ is AF, then, by Theorem 3, $\mathrm{C}^*(G)$ can be written as an inductive limit of group C^*-algebras which are not AF. There are many examples in the literature of AF algebras which are naturally given by inductive limits of non-AF algebras (see [1] for various references).

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Bernoulli crossed products of type III

Peter Verraedt

(joint work with Stefaan Vaes)

Crossed products with noncommutative Bernoulli actions were introduced by Connes as the first examples of full factors of type III. In this talk, we provide a complete classification of the factors $(P,\phi)^{\mathbb{F}_n} \rtimes \mathbb{F}_n$, where $\mathbb{F}_n$ is the free group and P is an amenable factor with a normal faithful state ϕ that either is almost periodic, or has a weakly mixing modular automorphism group. We show that the family of factors $(P,\phi)^{\mathbb{F}_n} \rtimes \mathbb{F}_n$ with ϕ almost periodic, is completely classified by the rank n of the free group $\mathbb{F}_n$ and Connes's Sd-invariant; and that the family of factors $(P,\phi)^{\mathbb{F}_n} \rtimes \mathbb{F}_n$ with ϕ a weakly mixing state, is classified by n and the action $\mathbb{F}_n \curvearrowright (P,\phi)^{\mathbb{F}_n}$, up to state-preserving conjugation of the action.

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The complete classification of unital graph C^*-algebras

Søren Eilers

(joint work with Gunnar Restorff, Efren Ruiz, Adam Sørensen)

We report on a classification result for all unital graph C^*-algebras – prominently containing the complete family of Cuntz-Krieger algebras – by K-theoretical invariants. The invariant used, the so-called *filtered K-theory*, has its origins in classification of symbolic dynamical systems ([3], [15]) and was conjectured to be complete in the real rank zero case for quite some time, but the generality of the result as well as the method of proof contains a number of surprises.

Indeed, the classification result applies also to graph C^*-algebras of real rank one, including those Cuntz-Krieger algebras that come about as iterated extensions of (stabilized) circle algebras and hence are postliminal. That the Elliott program is successful there is rather astonishing, and has interesting applications to a class of quantum lens spaces ([13]) which may be thus presented. And the method of proof allows the classification to be interpreted geometrically, in the sense that when two graphs E and F yield the same stabilized graph C^*-algebra, one may transform E into F by a finite number of moves resembling the role of Reidemeister moves in knot theory. Indeed, a key component of the proof is the definition of a new such move and the proof that it leaves the graph C^*-algebras invariant up to Morita equivalence. This combines with earlier work of the authors ([11], [10]) to allow the conclusion that the list of such moves is now complete in the sense described above.

Using [5], the classification results in full generality allow versions that are *strong* in the sense that any given isomorphism at the level of the invariant lifts to a $*$-isomorphism. This in turn leads to exact isomorphism by augmenting the K-theoretical invariant by the class of the unit.

As noted already by Cuntz and Krieger ([7]), the classical moves originating essentially in symbolic dynamics induce isomorphisms that preserve the canonical diagonal subalgebra, and combining recent work of Matsumoto and Matui with an observation of Sørensen ([14], [17]) one may in fact prove in the simple case that one graph may be transformed into another using only these moves (avoiding the *Cuntz splice* ([6], [16]) and the new move described above) if and only if the graph C^*-algebras are stably isomorphic in a diagonal-reserving way. We conjecture that this is true for all unital graph C^*-algebras and have confirmed this in the case of Cuntz-Krieger algebras. This last observation draws on forthcoming work with Arklint, Carlsen and Ortega ([1], [4]).

In conclusion, let us briefly discuss the status of the classification problem for general (not necessarily unital) graph C^*-algebras with finitely many ideals. Since we see no obvious way to reduce to the unital case solved here, nor to mimic the geometric approach which is essential for our proof, it stands to reason that entirely different methods are going to be necessary. It is worth noting that complete classification results exist in the case where all simple subquotients are of the same type — either AF (solved in [12]) or purely infinite (solved in [2]) — and the early general results obtained by three of the authors ([8], [9]) similarly require restrictions on the amount of “mixing”, so we predict that the key will be to resolve how such issues influence the general KK-theoretic machinery which we in the unital case may replace with a geometric aproach.

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C^*-completions of Hecke algebras and property (T)

Nadia S. Larsen
(joint work with Rui Palma)

The problem of identifying C^*-algebra completions of the $*$-algebra $H(\Gamma, \Gamma_0)$ associated to a Hecke pair (Γ, Γ_0) received a good deal of attention starting with the construction by Bost and Connes of a quantum statistical dynamical system whose underlying C^*-algebra $\mathcal{C}_{\mathbb{Q}}$ is a reduced C^*-algebra associated to a Hecke pair coming from the inclusion of $\mathbb{Z}$ in $\mathbb{Q}$, [1].

A group-subgroup pair (Γ, Γ_0) is a Hecke pair if each double coset $\Gamma_0 g \Gamma_0$ contains finitely many left cosets as $g \in \Gamma$. The space of complex-valued functions on $\Gamma_0 \backslash \Gamma / \Gamma_0$ whose support is finite admits a convolution product and an involution operation that turn it into a $*$-algebra $H(\Gamma, \Gamma_0)$. The particular example giving rise to $\mathcal{C}_{\mathbb{Q}}$ was a discrete pair, but not long afterwards a systematic study was undertaken by Hall with the goal of identifying whether the category of nondegenerate $*$-representations of the Hecke algebra was equivalent with the category of unitary representations of Γ generated by their Γ_0-fixed vectors [2]. In her approach, an important role was played by a pair formed of a compact open subgroup of a locally compact (totally disconnected) group. Hall proposed a positivity condition for a certain bilinear form under which the above equivalence would be valid. She proved, however, that this positivity condition failed for the pair $(SL_2(\mathbb{Q}_p), SL_2(\mathbb{Z}_p))$. Closely related to this, she also proved that the $*$-algebra $H\big(SL_2(\mathbb{Q}_p), SL_2(\mathbb{Z}_p)\big)$ did not admit a universal C^*-completion due to the fact that certain generators are unbounded in every $*$-representation.

The study of C^*-completions of a (discrete) Hecke pair (Γ, Γ_0) was put firmly into the framework of harmonic analysis by Tzanev, [6], who used work of Schlichting to associate an essentially unique topological pair consisting of a totally disconnected group G with a compact open subgroup H containing dense embeddings of the original pair. In this setting, $H(\Gamma, \Gamma_0)$ becomes the corner of $C_c(G)$ determined by the self-adjoint projection p_0 equal to the characteristic function of H. Thus, immediately, two new C^*-completions are available, with $p_0 C^*(G) p_0$ being a natural quotient of $C^*(L^1(G, H))$. Kaliszewski, Landstad and Quigg studied these completions further and gave a new and streamlined proof of Hall's equivalence using the theory of Fell-Rieffel imprimitivity bimodules for $*$-algebras [3]. In [3], the question was raised whether the canonical surjection from $C^*(G, H)$ onto $p_0 C^*(G) p_0$ was injective for the pair $(SL_n(\mathbb{Q}_p), SL_n(\mathbb{Z}_p))$ at $n = 2$. They also acknowledged a private communication from Tzanev asserting that the map was not injective at $n = 3$.

In [5], Palma answered the question raised by Kaliszewski, Landstad and Quigg by showing that the canonical surjection was not injective for $n = 2$. We then looked to find a proof for the non-injectivity claim in case $n = 3$. We establish that the canonical surjection is not injective by showing that the trivial representation of $p_0 C^*(G) p_0$ is isolated in its natural hull-kernel topology, due to property (T) of $SL_n(\mathbb{Q}_p)$, while the trivial representation of $C^*(L^1(G, H))$ is not isolated in the hull-kernel topology, for all $n \geq 3$. Indeed, the same result holds for a larger class of simple algebraic groups of rank at least 2, taken with a suitable compact open subgroup [4]. We also show that for Gelfand pairs, injectivity of the canonical surjection and existence of a universal C^*-completion can be characterised by positive definiteness and/or boundedness of so-called $*$-spherical functions [4].

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Classification results for C*-algebras associated to Smale spaces

Karen R. Strung

(joint work with Robin J. Deeley)

A Smale space is a dynamical system (X, φ) where X is a compact metric space and φ is a homeomorphism with a particularly tractable local structure: at each point $x \in X$ there is a small neighbourhood which splits into two sets, which we think of as local coordinates. Along one coordinate, the systems is expanding; along the other it is contracting. Ruelle defined Smale spaces in [3] to model the restriction of an Axiom A diffeomorphism to its nonwandering set or one of its basic sets. As such this class of dynamical systems is quite diverse: it includes subshifts of finite type, William's attractors (for example solenoids), Anosov diffeomorphisms (for example hyperbolic toral automorphisms), among many others.

When considering a Smale space, we are interested in the asymptotic behaviour of expansion and contraction. From this point of view, there are three naturally associated groupoids called the stable, unstable, and homoclinic groupoids. Suitably defined these are all amenable étale groupoids and, using the groupoid C*-algebra construction, results in separable, nuclear C*-algebras. Each of these algebras is stably finite and nuclear, and the homoclinic C*-algebra is unital.

In this talk, I reported on the results in [1] as well as forthcoming work (also joint with Deeley) that examines group actions on Smale spaces and properties of the resulting C*-algebraic crossed products [2]. Our main result in [1] is that the homoclinic algebras of mixing Smale spaces can be classified by their Elliott invariant which consists of K-theory data paired with tracial states. This required using Guentner, Willet and Yu's dynamic asymptotic dimension for groupoids [4] so that we might determine finiteness of the nuclear dimension of the Smale space C*-algebras.

Let (X, φ) be a Smale space. For a finite set of φ-invariant periodic points, the groupoid $G^S(P)$ consists of stable equivalence classes of points in X that are unstably equivalent to some point in P. Similarly, $G^U(P)$ will be the groupoid consisting of unstable equivalence classes of points in X that are stably equivalent to a point in P. The homoclinic groupoid, G^H, consists of all points $(x, y) \in X \times X$ such that x and y are both stably and unstably equivalent.

Theorem 1. *Let (X, φ) be a mixing Smale space and P a finite set of φ-invariant periodic points. Then $G^H(P)$ and $G^U(P)$ have finite dynamic dimension.*

As a consequence of hyperbolicity, one cannot define a Smale space structure unless X has finite covering dimension. Because of this, we can pass to finite nuclear dimension of the C*-algebras.

Theorem 2. *With (X, φ) and P as above, the groupoid* C*-*algebras $G^S(P)$ and $G^U(P)$ have finite nuclear dimension. Moreover, since* C*(G^H) *is stably isomorphic to $G^S(P) \otimes G^U(P)$, it also has finite nuclear dimension.*

As a consequence, $C^*(G^H)$ is in the Elliott class of C*-algebras, that is, those C*-algebras which can be distinguished by Elliott invariants.

Theorem 3. *With (X, φ) as above, if A is any simple separable unital* C*-*algebra in the UCT class that in addition has finite nuclear dimension, then $A \cong C^*(G^H)$ if and only if their Elliot invariants are isomorphic. We can therefore deduce that $C^*(G^H) \otimes \mathcal{U}$ is tracially approximately finite in the sense of Lin for any UHF algebra $\mathcal{U}$ and hence $C^G(H)$ is approximately subhomogeneous.*

Note in particular that the above holds within the class of homoclinic algebras of mixing Smale spaces. This raises the question: now that we can distiguish these C*-algebras by a computable invariant, can we in turn use this invariant to show some sort of equivalence of the underlying Smale spaces? This is an open question.

I finished the talk by discussing work in preparation on group actions on Smale spaces, by which we mean a group acting by φ-equivariant homeomorphisms on X. There are many interesting actions. For example, the automorphsims group of the full 2-shift contains every finite group. It is also natural to consider symmetries of aperiodic substitution tiling systems. The main theorem that appeared in the talk (further results will appear in [2]) was the following:

Theorem 4. *If Γ is a discrete elementary amenable group which acts effectively on a mixing Smale space (X, φ), then the induced action by Γ on $C * (G^H)$ is strongly outer. It follows that the crossed product by Γ with respect to this action, $C^*(G^H) \rtimes \Gamma$, is simple and has finite nuclear dimension. Since $C^*(G^H) \rtimes \Gamma$ is moreover simple and unital, it is classified by its Elliott invariant.*

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Tail algebras, amalgamated free products and KMS states

Ken Dykema

(joint work with Yoann Dabrowski, Claus Köstler, Kunal Mukherjee and John Williams)

A classical theorem of B. de Finetti [6] shows that an infinite sequence of classical random variables is exchangeable (namely, has distribution invariant under arbitrary permutations of the variables) if and only if the random variables are conditionally independent over the tail σ-algebra. Hewitt and Savage [7] used this to prove that the set of symmetric Borel probability measures on an infinite product space $\prod_1^\infty Z$, where Z is compact Hausdorff, is a Choquet simplex, whose extreme points are the product measures of the form $\prod_1^\infty \mu$, for μ a Borel probability measure on Z. Here "symmetric" means invariant under the obvious permutation action of S_∞ on the above product space.

E. Størmer [10] extended the purview of the classical de Finetti theorem to the realm of C*-algebras, showing the symmetric states on the infinite tensor product $\bigotimes_1^\infty A$ of a unital C*-algebra A with itself, form a Choquet simplex and that the extreme points of this simplex are the infinite tensor product states $\otimes_1^\infty \phi$, of states ϕ on A. Here "symmetric" means invariant under the obvious permutation action of S_∞ on the above tensor product algebra.

In the paper [4], we investigate the symmetric states on the universal unital free product $*_1^\infty A$ of a C*-algebra A with itself infinitely many times, these being those that are invariant under the obvious action of S_∞ that permutes the copies of A. We let $\mathrm{SS}(A)$ denote the set of all symmetric states on $*_1^\infty A$. For $\psi \in \mathrm{SS}(A)$, let π_ψ be the GNS representation of it, let $\mathcal{M}_\psi$ denote the von Neumann algebra generated by the image of π_ψ, and let $\hat{\psi}$ denote the normal state on $\mathcal{M}_\psi$ so that $\hat{\psi} \circ \pi_\psi = \psi$. The *tail algebra* of ψ is the von Neumann subalgebra

$$\mathcal{T}_\psi = \bigcap_{n=1}^{\infty} W^*\big(\bigcup_{j\geq n} \pi_\psi(A_j)\big),$$

of $\mathcal{M}_\psi$, where A_j is the j-th copy of A in $\mathfrak{A}$. An example of Weihua Liu [9] (described in [4]) shows that there need not be a normal, $\hat{\psi}$-preserving conditional expectation from $\mathcal{M}_\psi$ onto $\mathcal{T}_\psi$. However, we show that there is a $\hat{\psi}$-preserving, S_∞-invariant conditional expectation from the C*-algebra

$$Q_\psi := C^*(\mathcal{T}_\psi \cup \pi_\psi(\mathfrak{A}))$$

onto $\mathcal{T}_\psi$, and we define the *tail C*-algebra* D_ψ to be the the smallest unital C*-subalgebra of $\mathcal{T}_\psi$ containing $E_\psi(C^*(D_\psi \cup \pi_\psi(\mathfrak{A})))$. We ask two open questions about tail algebras of symmetric states:

Question 1. Do we always have $D_\psi \subseteq \pi_\psi(\mathfrak{A})$?

Question 2. Is $\mathcal{T}_\psi$ generated as a von Neumann algebra by D_ψ?

Theorem 3 ([4])**.** *If the restriction of $\hat{\psi}$ to D_ψ is a pure state, then ψ is an extreme point of* SS(A)*, while the converse holds if we also assume $D_\psi \subseteq \pi_\psi(\mathfrak{A})$.*

Definition 4. A state ψ on $*_1^\infty A$ is said to be *quantum symmetric* if it is invariant under the natural actions of the quantum permutation groups of S. Wang [11]. (See [4]) for more details.)

We let QSS(A) denote the set of such quantum symmetric states. It is easy to see QSS$(A) \subseteq$ SS(A).

Here is a noncommutative de Finetti theorem, whose proof is patterned after the proof Köstler and Speicher's original noncommutative de Finetti theorem [8], which is about quantum exchangeable random variables. Also, S. Curran has a version [2] that requires faithfulness of states.

Theorem 5 ([4])**.** *If $\psi \in$ QSS(A), then the family*

$$\big(C^*(\pi_\psi(A_j) \cup D_\psi)\big)_{j=1}^\infty$$

is free with amalgamation over D_ψ with respect to E_ψ. Moreover, we have

$$D_\psi \subseteq \pi_\psi(*_1^\infty A). \tag{1}$$

Thus, the C-algebra $\pi_\psi(*_1^\infty A)$ is isomorphic to the C*-algebra arising in the reduced amalgamated free product $(*_{D_\psi})_1^\infty(B, E)$, where $B = C^*(\pi_\psi(A_j) \cup D_\psi)$ and E is the restriction of E_ψ.*

The proof of (1) uses an amalgamated version of the Haagerup inequality, found in [1].

Clearly, the set QSS(A) is a convex set that is compact in the weak*-topology, but it is easy to see that it is not a Choquet simplex. However, we now consider several special sets of quantum symmetric states that are Choquet simplices.

Definition 6. Given $\psi \in$ QSS(A), we say that ψ is *central* if the tail algebra $\mathcal{T}_\psi$ lies in the center $Z(\mathcal{M}_\psi)$ of $\mathcal{M}_\psi$. We let ZQSS(A) be the set of all central quantum symmetric states.

Theorem 7 ([4])**.** ZQSS(A) *is a Choquet simplex, and ψ is in the extreme boundary of* ZQSS(A) *if and only if $\psi = *_1^\infty \phi$ is the free product of a state $\phi \in S(A)$ of A.*

From the above theorem and the one of Størmer mentioned above, we see that the Choquet simplex ZQSS(A) is the same as the Choquet simplex of all symmetric states on $\bigotimes_1^\infty A$; namely, it is the Bauer simplex with extreme boundary equal to the state space of A.

Definition 8. The set of *tracial quantum symmetric states* of A is

$$\mathrm{TQSS}(A) = \mathrm{QSS}(A) \cap T(*_1^\infty A).$$

Theorem 9 ([3])**.** *If A has a tracial state, then* TQSS(A) *is nonempty. Moreover, it is a Choquet simplex and a face of $T(*_1^\infty A)$. If A is separable, then it is the Poulsen simplex, namely, the metrizable simplex whose extreme points are dense.*

Suppose $\sigma = (\sigma_t)_{t\in\mathbf{R}}$ be a one-parameter automorphism group of A (pointwise norm continuous). Recall that the set of entire analytic elements of A is

$$\mathfrak{A} = \{a \in A \mid \exists \text{ a holomorphic extension } \mathbf{C} \ni z \mapsto \sigma_z(a) \in A\}$$

and forms a dense $*$-subalgebra of A. Recall that a state ϕ is σ-KMS (at inverse temperature -1) if for all $a \in A$ and all $b \in \mathfrak{A}$, we have

$$\phi(a\sigma_{-i}(b)) = \phi(ba).$$

Let $*_1^\infty\sigma$ denote the one-parameter automorphism group of $*_1^\infty A$ whose value at parameter t is the free product automorphism $*_1^\infty\sigma_t$ of $*_1^\infty A$.

Definition 10.

$$\mathrm{QSS}_\sigma(A) = \{\psi \in \mathrm{QSS}(A) \mid \psi \text{ is } (*_1^\infty)\text{-KMS}\}.$$

Theorem 11 ([5])**.** *If A has a σ-KMS state, then* $\mathrm{QSS}_\sigma(A)$ *is nonempty; it is, moreover, a Choquet simplex and a face of the simplex of all $(*_1^\infty\sigma)$-KMS states of $*_1^\infty A$.*

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Cantor minimal $\mathbb{Z}^d$-actions and cohomology

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(joint work with I.F. Putnam and C.F. Skau)

1. The cohomology of free minimal actions of $\mathbb{Z}^d$ on the Cantor set

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Let us review some properties of the cohomology of a free minimal action (X, φ) of $\mathbb{Z}^d$ on the Cantor set. Recall (see for example [HF]) that we consider this cohomology as the group cohomology of $\mathbb{Z}^d$ with coefficient module $C(X, \mathbb{Z})$ but with no preferred choice of projective resolution.

We then have:

1) $H^0(X, \varphi) = \{f \in C(X, \mathbb{Z})\,;\, f = f \circ \varphi\,\} = \mathbb{Z}$, as the system is minimal.

2) $H^d(X, \varphi) = C(X, \mathbb{Z})/\{f - f \circ \varphi\,;\ f \in C(X, \mathbb{Z})\,\}$ is the group of co-invariants of (X, φ). Note that $d > 1$, this group may have torsion .

3) $H^1(X, \varphi) = Z^1(X, \varphi)/B^1(X, \varphi)$, where $Z^1(X, \varphi)$ denotes the set of continuous cocycles $\Theta : X \times \mathbb{Z}^d \to \mathbb{Z}$ and the coboundaries $B^1(X, \varphi)$ given by $\Theta(x, n) = h(\varphi(n)(x) - h(x)$, for $h \in C(X, \mathbb{Z})$.

Let us state some properties of the first group of cohomology:

a) $H^1(X, \varphi)$ is a torsion free group. Moreover, for $d = 1$, $H^1(X, \varphi)$ is a simple dimension group, and as a direct consequence of [HPS], for any simple dimension group G, there is a minimal homeomorphism on the Cantor set whose first group of cohomology is G.

b) Let $\{e_i\,;\, 1 \le i \le d\,\}$ be the canonical basis of $\mathbb{Z}^d$. The group $\mathbb{Z}^d$, realized as the subgroup generated by the cocycles $\Theta_j : X \times \mathbb{Z}^d \to \mathbb{Z}\,,\ 1 \le j \le d$, given by $\Theta_j(x, e_i) = \delta_j(i)$ is canonically imbedded in $H^1(X, \varphi)$.

Definition. *For an invariant probability measure μ of a free, minimal $\mathbb{Z}^d$-action (X, φ) on the Cantor set, let $\tau^1_\mu : H^1(X, \varphi) \to Hom\,(\mathbb{Z}^d, \mathbb{R})$ and $\tau^d_\mu : H^d(X, \varphi) \to \mathbb{R}$ denote the two group homomorphisms given by:*

$$\tau^1_\mu([\Theta])(n) = \int_X \Theta(x, n) d\mu(x) \quad \textit{and} \quad \tau^d_\mu([f]) = \int_X f d\mu\,.$$

By [GPS], for any dense countable subgroup H of $\mathbb{R}$ containing $\mathbb{Z}$, there exists a uniquely ergodic, minimal homeomorphism φ of the Cantor set such that $\tau^1_\mu(H^1(X, \varphi)) = H$.

More precisely, if $H \subset \mathbb{Q}$, then (X, φ) is a $\mathbb{Z}$-odometer. Recall that a *Denjoy homeomorphism* is an aperiodic homeomorphism of the circle which is not conjugate to a pure rotation. By a *Denjoy system* we mean a Denjoy homeomorphism restricted to its unique invariant Cantor set (See [PSS]). Then there exists a Denjoy system (X, φ) such that $\tau^1_\mu(H^1(X, \varphi)) = H$ if H is not contained in $\mathbb{Q}$. Note that (see [GPS]) that any uniquely ergodic, minimal homeomorphism of the Cantor set is orbit equivalent to an odometer or a Denjoy system.

These results for a minimal homeomorphism on the Cantor set lead to the following question:

For $d \geq 1$, what is the class of countable subgroups H of $\mathbb{R}^d$ containing $\mathbb{Z}^d$ which can be realized dynamically, i.e. for which there exists a free minimal action (X, φ) of $\mathbb{Z}^d$ on the Cantor set such that $\tau^1_\mu(H^1(X, \varphi)) = H$?

Remark: In [CS], A. Clark and L. Sadun constructed recently an example of a uniquely ergodic, free and minimal action (X, φ) of $\mathbb{Z}^2$ on the Cantor set such that $\tau^1_\mu(H^1(X, \varphi)) = \mathbb{Z}^2$.

2. First case: $\mathbb{Z}^d \subset H \subset \mathbb{Q}^d$, H dense.

These subgroups will be realized using $\mathbb{Z}^d$-odometers. These $\mathbb{Z}^d$-actions were introduced by M.-I. Cortez in [C]. With S. Petite in [CP], she then extended the definition to any residually finite G. Let us recall the definition for $\mathbb{Z}^d$.

If Z is any subgroup of $\mathbb{Z}^d$, let φ_Z denote the $\mathbb{Z}^d$-action on $\mathbb{Z}^d/Z$ given by

$$\varphi_Z(k)(l + Z) = k + l + Z\ ,\ k, l \in \mathbb{Z}^d\ .$$

Let $\mathcal{G}$ denote a decreasing sequence $(Z_n)_{n\geq 1}$ of finite index subgroups of $\mathbb{Z}^d$ whose intersection is trivial, and $(X_\mathcal{G}, \varphi_\mathcal{G})$ the inverse limit of the systems

$$(\mathbb{Z}^d/Z_1, \varphi_{Z_1}) \leftarrow (\mathbb{Z}^d/Z_2, \varphi_{Z_2}) \leftarrow \cdots$$

Definition. *A $\mathbb{Z}^d$-odometer is any system $(X_\mathcal{G}, \varphi_\mathcal{G})$, where $\mathcal{G}$ is as above.*

Then $X_\mathcal{G}$ is a Cantor set and $(X_\mathcal{G}, \varphi_\mathcal{G})$ is a free, minimal action of $\mathbb{Z}^d$, which is equicontinuous and therefore uniquely ergodic.

Recall that if Z is a finite index subgroup of $\mathbb{Z}^d$, then it is a lattice in $\mathbb{R}^d$ and its dual lattice Z^* is a subgroup of $\mathbb{Q}^d$ containing $\mathbb{Z}^d$ as a finite index subgroup. Then to a decreasing sequence $(Z_n)_{n\geq 1}$ of finite index subgroups of $\mathbb{Z}^d$ we associate the increasing sequence $(Z^*_n)_{n\geq 1}$ of subgroups of $\mathbb{Q}^d$ containing $\mathbb{Z}^d$ and we denote by H its union. It is easy to verify that H is dense in $\mathbb{R}^d$ if and only if the intersection of the sequence $(Z_n)_{n\geq 1}$ is $\{0\}$. We then have

Theorem. *1) Two $\mathbb{Z}^d$-odometers are conjugate if and only if their subgroups H of $\mathbb{Q}^d$ are equal.*

2) Any subgroup H of $\mathbb{Q}^d$ containing $\mathbb{Z}^d$ and dense (in $\mathbb{R}^d$) is associated to a $\mathbb{Z}^d$-odometer.

For a subgroup H of $\mathbb{Q}^d$ containing $\mathbb{Z}^d$ and dense (in $\mathbb{R}^d$), we will denote by $\mathbb{Z}^d$*-odometer*(H), the unique, up to conjugation, $\mathbb{Z}^d$-odometer associated to H.

Before stating the next theorem, let us recall some notions of equivalences between $\mathbb{Z}^d$-dynamical systems.

Definition. *Let (X, φ) and (X', φ') be Cantor $\mathbb{Z}^d$- and $\mathbb{Z}^{d'}$-dynamical systems. Then the two systems are:*

1) conjugate if there exists a homeomorphism $F : X_1 \to X_2$ intertwining φ_1 and φ_2.

2) isomorphic if there exists an automorphism α of $\mathbb{Z}^d$ such that $(X_1, \varphi_1 \circ \alpha)$ and (X_2, φ_2) are conjugate.

3) orbit equivalent (OE) if there exists a homeomorphism $F : X_1 \to X_2$ such that $F(Orbit_{\varphi_1}(x)) = Orbit_{\varphi_2}(Fx)$ for all $x \in X_1$.

4) continuous orbit equivalent (COE) if they are OE and the associated orbit cocyles are continuous.

Remarks: i) Conjugacy and isomorphism naturally imply that $d = d'$.
ii) The notion of continuous orbit equivalence for a countable discrete group action was introduced by X. Li in [L]. For $d = 1$ and Cantor minimal systems, it is equivalent to the notion of isomorphism or flip conjugacy (see [GPS]).
iii) Conjugacy implies isomorphism, which implies COE and COE implies OE.

Then we have:

Theorem. *Let H and H' be two subgroups of $\mathbb{Q}^d$ containing $\mathbb{Z}^d$ and dense (in $\mathbb{R}^d$). Then the two corresponding $\mathbb{Z}^d$-odometers are isomorphic if and only if there exists $\alpha \in GL_d(\mathbb{Z})$ such that $\alpha(H) = H'$.*

Theorem. *Let H (resp. H') be a subgroup of $\mathbb{Q}^d$ (resp. $\mathbb{Q}^{d'}$) containing $\mathbb{Z}^d$ (resp. $\mathbb{Z}^{d'}$) and dense. Then the two corresponding odometers are COE if and only if $d = d'$ and there exists $\alpha \in GL_d(\mathbb{Q})$ with det $(\alpha^2) = 1$ and $\alpha(H) = H'$.*

Theorem. *For any dense subgroup $\mathbb{Z}^d \subset H \subset \mathbb{Q}^d$, the first cohomology group of the $\mathbb{Z}^d$-odometer (H) is isomorphic to H.*

Recall that $\mathbb{Z}^d$-odometer (H) is uniquely ergodic; let μ denote its unique invariant measure. Identifying Hom $(\mathbb{Z}^d, \mathbb{R})$ with $\mathbb{R}^d$, then τ^1_μ is a homomorphism from $H^1(\mathbb{Z}^d\text{-odometer}\,(H))$ to $\mathbb{R}^d$. We then have

Theorem. *For any dense subgroup $\mathbb{Z}^d \subset H \subset \mathbb{Q}^d$, then the map*

$$\tau^1_\mu : H^1(\mathbb{Z}^d\text{-}odometer\,(H)) \to H$$

is an isomorphism.

In the next theorem, we investigate orbit equivalence for $\mathbb{Z}^d$-odometers. Let us first introduce the following definition we will need.

Definition. *Let $\mathbb{Z}^d \subset H \subset \mathbb{Q}^d$. Then the superindex of $\mathbb{Z}^d$ in H is given by*

$$[[H : \mathbb{Z}^d]] = \{[H' : \mathbb{Z}^d]\ ;\ \mathbb{Z}^d \subset H',\ [H' : \mathbb{Z}^d] < \infty\}.$$

Recall that for a free, minimal $\mathbb{Z}^d$-Cantor system (X, φ), the cohomology group $H^d(X, \varphi)$ is the group $D(X, \varphi)$ of co-invariants of (X, φ).

Theorem. *For any dense subgroup $\mathbb{Z}^d \subset H \subset \mathbb{Q}^d$, then the map*

$$\tau^d_\mu : H^d(\mathbb{Z}^d\text{-}odometer(H)) \to \cup_{m \in [[H:\mathbb{Z}^d]]} m^{-1}\mathbb{Z}$$

is an isomorphism of ordered abelian group with order units.

Using [GMPS], we then get

Corollary. *Let H (resp. H') be a subgroup of $\mathbb{Q}^d$ (resp. $\mathbb{Q}^{d'}$) containing $\mathbb{Z}^d$ (resp. $\mathbb{Z}^{d'}$) and dense. Then the two corresponding odometers are orbit equivalent if and only if their superindex are equal.*

These results lead to the following striking results. The examples for (b) and (c) already appear in [CP] and [L].

Corollary. *a) Two $\mathbb{Z}$-odometers are conjugate if and only if they are orbit equivalent.*

b) There are pairs of $\mathbb{Z}^2$-odometers which are conjugate, without being isomorphic.

c) There are pairs of $\mathbb{Z}^2$-odometers which are isomorphic, without being COE.

d) There are pairs of $\mathbb{Z}^2$-odometers which are COE, without being OE.

3. Second case: $\mathbb{Z}^d \subset H \subset \mathbb{R}^d$, H dense.

Recall that if (X, φ) is a Cantor minimal $\mathbb{Z}$-system, then its first cohomology group is is the group $D(X, \varphi)$ of co-invariants of (X, φ). Then Hermann, Putnam and Skau's result [HPS] and Effros, Handelman and Shen's characterization of dimension groups [EHS] show that any simple dimension group is the first cohomology group of a Cantor minimal system.

Recently, we have shown (but still need to be carefully checked) that

Theorem. *For any dense subgroup H of $\mathbb{R}^2$ containing $\mathbb{Z}^2$, there exists a uniquely ergodic, free, minimal $\mathbb{Z}^2$-action (X, φ) on the Cantor set such that $H^1(X, \varphi) \cong H$.*

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