Mathematisches Forschungsinstitut Oberwolfach

Report No. 56/2016

DOI: 10.4171/OWR/2016/56

Surface Bundles

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4 December – 10 December 2016

Abstract. This workshop brought together specialists in algebraic topology, low dimensional topology, geometric group theory, algebraic geometry and neighboring fields. It provided a good overview of the current developments, and highlighted significant progress in the field. Furthermore it showed an increasing amount of interaction between specialists in different fields who are interested in the different facets of the rich theory of surface bundles.

Mathematics Subject Classification (2010): 57M50, 55R40, 20F65.

Introduction by the Organisers

The workshop *Surface Bundles* was held December 5 – December 9, 2016. The participants were specialists in algebraic topology, low dimensional topology, geometric group theory, algebraic geometry and neighboring fields, covering a broad spectrum of areas which are in the focus of current developments.

The lectures during the five days of the meeting were roughly organized according to different thematic themes.

In the mornings of the first two days of the meeting, longer survey lectures were presented whose aim was to give an introduction to one of the four different themes of the conference.

On Monday morning, there was a survey lecture on some of the algebraic geometric viewpoints on surface bundles as well as a lecture introducing homological stability of the mapping class group and the classification of the stable cohomology classes.

On Tuesday morning, there was a survey lecture on the differential geometry of surface bundles and a lecture presenting the viewpoint of geometric group theory.

The afternoon lectures at these days were more traditional reports on new research that picked up aspects introduced in the morning.

On Wednesday (with only three lecture due to the traditional hike), the talks focused on new advances on the interplay between algebraic geometric and algebraic topological invariants of surfaces bundles.

The lectures of the last two days were mainly devoted to results on various ways a manifold can be represented as a surface bundle, including some aspects related to rigidity, also exploring new types of constructions of examples and new ways to understand algebraic invariants.

On Thursday evening, we organized a problem session. The list of problems are included in this report.

The meeting gave a good overview of the current developments, and highlighted significant progress in the field. It also showed an increasing amount of interaction between specialists in different fields who are interested in the different facets of the rich theory of surface bundles. The workshop was attended by researchers from around the world, ranging from graduate students to scientific leaders in their respective areas.

The atmosphere during the meeting was lively and open, and greatly benefited from the ideal environment at Oberwolfach.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, “US Junior Oberwolfach Fellows”.

Workshop: Surface Bundles

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Abstracts

Signature of surface bundles mod 8 and cohomology of finite groups

David Benson

The purpose of this talk is to describe work in progress with Catarina Campagnolo, Andrew Ranicki and Carmen Rovi, giving a recipe for computing the signature of a surface bundle over a surface modulo eight, using the cohomology and representation theory of finite groups.

Let $\Sigma_g \to M \to \Sigma_h$ be an oriented surface bundle with base of genus h and fibre of genus g. Such a bundle is determined up to homeomorphism by a map $\pi_1(\Sigma_h) \to \Gamma_g$, the mapping class group of Σ_g. The action of this mapping class group on $H^1(\Sigma_g, \mathbb{Z})$ gives a map $\Gamma_g \to \mathsf{Sp}(2g, \mathbb{Z})$. We write $\chi\colon \pi_1(\Sigma_h) \to \mathsf{Sp}(2g, \mathbb{Z})$ for the composite of these maps, namely the monodromy action of $\pi_1(\Sigma_h)$ on $H^1(\Sigma_g, \mathbb{Z})$. In 1973, Meyer defined a 2-cocycle

$$\tau\colon \mathsf{Sp}(2g,\mathbb{Z}) \times \mathsf{Sp}(2g,\mathbb{Z}) \to \mathbb{Z}$$

and proved that the signature of M is given by

$$\sigma(M) = \langle \chi^*[\tau], [\Sigma_h] \rangle$$

and that this is divisible by four.

Recall that the maximal compact subgroup of $\mathsf{Sp}(2g, \mathbb{R})$ is the unitary group $\mathsf{U}(2g)$, so $\pi_1(\mathsf{Sp}(2g, \mathbb{R})) \cong \pi_1(\mathsf{U}(2g)) \cong \mathbb{Z}$. Writing $\widetilde{\mathsf{Sp}(2g, \mathbb{R})}$ for the universal cover, we pull back to $\mathsf{Sp}(2g, \mathbb{Z})$ to give a group which we call $\widetilde{\mathsf{Sp}(2g, \mathbb{Z})}$:

$$\begin{CD}
1 @>>> \mathbb{Z} @>>> \widetilde{\mathsf{Sp}(2g,\mathbb{Z})} @>>> \mathsf{Sp}(2g,\mathbb{Z}) @>>> 1 \\
@. @| @VVV @VVV @. \\
1 @>>> \mathbb{Z} @>>> \widetilde{\mathsf{Sp}(2g,\mathbb{R})} @>>> \mathsf{Sp}(2g,\mathbb{R}) @>>> 1
\end{CD}$$

As long as $g \geq 4$, this is the universal central extension of $\mathsf{Sp}(2g, \mathbb{Z})$; for smaller g there are further extensions inflated from the exceptional central extensions of $\mathsf{Sp}(2g, \mathbb{Z}/2)$. Now $\sigma(M)$ is divisible by four since $[\tau] \in H^2(\mathsf{Sp}(2g, \mathbb{Z}), \mathbb{Z})$ is four times the class of this extension.

In 1978, Deligne proved that $\widetilde{\mathsf{Sp}(2g, \mathbb{Z})}$ is not residually finite. Every subgroup of finite index contains $2\mathbb{Z}$. As a consequence, if we use inflations of 2-cocycles on finite quotients of $\mathsf{Sp}(2g, \mathbb{Z})$, the best we can hope for is to compute the signature modulo eight. This is what we shall do.

For notation, let J be the matrix $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, where I is a $g \times g$ identity matrix. Then $\mathsf{Sp}(2g, \mathbb{Z})$ consists of the matrices $X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ such that $X^t J X = J$. We write $\Gamma(n)$ for the congruence subgroup consisting of those matrices in $\mathsf{Sp}(2g, \mathbb{Z})$ which are congruent to the identity modulo n.

Definition 1. Let $\mathcal{K}$ be the subgroup consisting of the matrices

$$\begin{pmatrix} I+2A & 2B \\ 2C & I+2D \end{pmatrix} \in \mathsf{Sp}(2g,\mathbb{Z}) \quad \text{such that}$$

(i) the diagonal entries of B and C are even, and
(ii) the trace of A is even.

Thus $\Gamma(4) \le \mathcal{K} \le \Gamma(2)$ and $|\Gamma(2) : \mathcal{K}| = 2^{2g+1}$.

Theorem 2 (BCRR 2016). (i) $\mathcal{K}$ is a normal subgroup of $\mathsf{Sp}(2g,\mathbb{Z})$.

(ii) $\Gamma(2)/\mathcal{K} \cong (\mathbb{Z}/2)^{2g+1}$ is an elementary abelian 2-group of rank $2g+1$.
(iii) The extension $1 \to (\mathbb{Z}/2)^{2g+1} \to \mathsf{Sp}(2g,\mathbb{Z})/\mathcal{K} \to \mathsf{Sp}(2g,\mathbb{Z}/2) \to 1$ does not split.
(iv) The conjugation action of $\mathsf{Sp}(2g,\mathbb{Z}/2)$ on $(\mathbb{Z}/2)^{2g+1}$ gives the exceptional isomorphism $\mathsf{Sp}(2g,\mathbb{Z}/2) \cong \mathsf{O}(2g+1,\mathbb{Z}/2)$, the orthogonal group for the form on $\Gamma(2)/\mathcal{K}$ given by $\mathsf{Tr}(A) + \langle \mathsf{Diag}(B), \mathsf{Diag}(C)\rangle$ (the trace of A plus the inner product of the diagonal elements of B and C).
(v) For $g \ge 4$, we have $H^2(\mathsf{Sp}(2g,\mathbb{Z})/\mathcal{K},\mathbb{Z}/2) \cong \mathbb{Z}/2$.
(vi) For $g \ge 2$ there is a non-zero element of $H^2(\mathsf{Sp}(2g,\mathbb{Z})/\mathcal{K}),\mathbb{Z}/2)$ which inflates to $\overline{\tfrac{1}{4}[\tau]} \in H^2(\mathsf{Sp}(2g,\mathbb{Z}),\mathbb{Z}/2)$.
(vii) Restricting the extension of $\mathsf{Sp}(2g,\mathbb{Z})/\mathcal{K}$ by $\mathbb{Z}/2$ to $(\mathbb{Z}/2)^{2g+1}$, we get an almost extraspecial group of order 2^{2g+2}.

To explain part (vii), we have $H^*((\mathbb{Z}/2)^n,\mathbb{Z}/2) = \mathbb{Z}/2[z_1,\dots,z_n]$ with

$$z_i \in H^1((\mathbb{Z}/2)^n,\mathbb{Z}/2) \cong \mathsf{Hom}((\mathbb{Z}/2)^n,\mathbb{Z}/2).$$

So $H^2((\mathbb{Z}/2)^n,\mathbb{Z})$ is the space of quadratic forms on $(\mathbb{Z}/2)^n$, namely functions $q\colon (\mathbb{Z}/2)^n \to \mathbb{Z}/2$ satisfying $q(x+y) = q(x) + q(y) + b(x,y)$ with b symmetric bilinear. The interpretation of this is that given an extension

$$1 \to \mathbb{Z}/2 \to E \to (\mathbb{Z}/2)^n \to 1$$

and elements $x,y \in (\mathbb{Z}/2)^n$, choose inverse images $\hat{x},\hat{y} \in E$. Then $q(x) = \hat{x}^2$ and $b(x,y) = [\hat{x},\hat{y}]$. We say that q is *non-degenerate* if $q^{-1}(0) \cap b^\perp = \{0\}$, and *non-singular* if $b^\perp = \{0\}$.

The classification of quadratic forms over $\mathbb{Z}/2$ is as follows. For non-singularity, we have $n = 2g$ even. In this case, there are two isomorphism classes of quadratic forms, distinguished by their Arf invariant, which is 0 or 1. The corresponding extensions are called *extraspecial groups*, 2_+^{1+2g} and 2_-^{1+2g} respectively.

For singular but non-degenerate forms, we have $n = 2g+1$ odd. There is only one isomorphism class, and the corresponding extension is the *almost extraspecial group* of order 2^{2g+2}, namely the central product $\mathbb{Z}/4 \circ 2_+^{1+2g} \cong \mathbb{Z}/4 \circ 2_-^{1+2g}$.

The automorphism group of the extraspecial group is an extension of the corresponding orthogonal group by $E/Z(E) = (\mathbb{Z}/2)^{2g}$, which is non-split for g large. The automorphism group of the almost extraspecial group sits in an extension

$$1 \to (\mathbb{Z}/2)^{2g} \to \mathsf{Aut}(E) \to \mathsf{Sp}(2g,\mathbb{Z}/2) \times \mathbb{Z}/2 \to 1.$$

The extra copy of $\mathbb{Z}/2$ acts by inverting a central element of order four in E. So the subgroup $\mathsf{Aut}(E)'$ of $\mathsf{Aut}(E)$ fixing $Z(E)$ has index two. For $g \geq 3$ the extension

$$1 \to (\mathbb{Z}/2)^{2g} \to \mathsf{Aut}(E)' \to \mathsf{Sp}(2g, \mathbb{Z}/2) \to 1$$

is non-split. By a theorem of Dempwolff (1974), there is a unique non-split extension up to isomorphism.

Theorem 3 (Griess 1973). For $g \geq 3$ there is a group extension

$$1 \to E \to G \to \mathsf{Sp}(2g, \mathbb{Z}/2) \to 1$$

such that $G/Z(G) \cong \mathsf{Aut}(E)'$ and $O_2(G) = E$.

By Dempwolff's theorem, we have $G/(\mathbb{Z}/2) \cong \mathsf{Sp}(2g, \mathbb{Z})/\mathcal{K}$, and hence a central extension

$$1 \to \mathbb{Z}/2 \to G \to \mathsf{Sp}(2g, \mathbb{Z})/\mathcal{K} \to 1.$$

Theorem 4. For $g \geq 2$, the inflation map

$$H^2(\mathsf{Sp}(2g, \mathbb{Z})/\mathcal{K}, \mathbb{Z}/2) \to H^2(\mathsf{Sp}(2g, \mathbb{Z}), \mathbb{Z}/2)$$

is injective.

Proof. Use the five term sequence, and compute

$$H^0(\mathsf{Sp}(2g, \mathbb{Z})/\mathcal{K}, H^1(\mathcal{K}, \mathbb{Z}/2)) = 0. \qquad \square$$

An explicit cohomology class representing the extension described by Griess' theorem can be obtained by the action on a suitable space of theta functions. This gives a projective representation $\mathsf{Sp}(2g, \mathbb{Z})/\mathcal{K} \to \mathsf{U}(2^g)/\{\pm 1\}$ as follows. We have 2^g basis elements e_w indexed by $w \in (\mathbb{Z}/2)^g$:

$$\begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \mapsto \left(e_w \mapsto i^{w^t B w} e_w\right)$$

$$\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \mapsto \left(e_w \mapsto \sqrt{\det(A)}\, e_{(A^t)^{-1} w}\right)$$

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \mapsto \frac{1}{(1+i)^g} \sum_{w'} (-1)^{w^t w'} e_{w'}.$$

Note that here, B is interpreted as a quadratic form on $(\mathbb{Z}/2)^g$ with values in $\mathbb{Z}/4$, so that $i^{w^t B w}$ is well defined in $\mathbb{C}$.

We can now give the required recipe for computing the signature of a surface bundle modulo eight. Denote by ϕ the composite

$$\pi_1(\Sigma_h) \to \Gamma_g \to \mathsf{Sp}(2g, \mathbb{Z})/\mathcal{K} \to \mathsf{U}(2^g)/\{\pm 1\}.$$

We have $\pi_1(\Sigma_h) = \langle a_1, b_1, \ldots, a_h, b_h \rangle / r$ where r is the single relation

$$r = [a_1, b_1] \ldots [a_h, b_h].$$

For each i, the commutator $[\phi(a_i), \phi(b_i)]$ is well defined in $\mathsf{U}(2^g)$, since negating $\phi(a_i)$ or $\phi(b_i)$ does not affect the commutator. The product

$$[\phi(a_1), \phi(b_1)] ... [\phi(a_h), \phi(b_h)]$$

is plus or minus the identity matrix. If the answer is the identity matrix, the signature is zero modulo eight. If the answer is minus the identity matrix, the signature is four modulo eight.

The simplicial volume of surface bundles

Caterina Campagnolo
(joint work with Michelle Bucher)

Let $F \to X \to B$ denote a surface bundle over a surface (everything being closed, connected and oriented).

We recall important numerical invariants of surface bundles over surfaces: the Euler characteristic, the signature and the simplicial volume. We focus on simplicial volume.

Two results of M. Bucher [3, 4] show that

$$\|X\| \geq 6\chi(X),$$

with equality when $X = F \times B$. By a simple pullback argument, equality also holds if the image of the monodromy representation is finite. We then ask the natural question:

Do there exist surface bundles over surfaces X with $\|X\| > 6\chi(X)$?

In order to answer this question, we first briefly introduce bounded cohomology and the duality relation to simplicial volume [2, 6]. We then present the orientation cocycle on the circle, that can be used to find a representative of the real Euler class $e \in H^2(X, \mathbb{R})$ of the bundle X [1, 7]. We also explain the technique of alternation of cocycles. We then show bounds on the Gromov norm of e and $e \cup e$. We use them to prove the following results:

Proposition 1. Let $[N] \in H_2(X, \mathbb{Z})$ be the Poincaré dual of e. Then

$$\|X\| \geq 3\|[N]\|.$$

Theorem 2.

$$\|X\| \geq 36|\sigma(X)|.$$

Theorem 3. For bundles coming from Morita's construction [8],

$$\|X\| \geq 6\chi(E) + 6|\chi(\Sigma')|(d-1).$$

The latter answers the above question in the positive.

Eduard Looijenga asked whether one can prove that if the monodromy contains a pseudo-Anosov element, then $\|X\| > 6\chi(E)$.

Christopher Leininger asked whether the inequality in the Proposition is expected to have a big gap in general. Note that in the case of the trivial bundle this is an equality.

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Homology stability for spaces of embedded surfaces

Federico Cantero Moran

In this talk I presented a joint work with Oscar Randal-Williams [CRW13], where we study the homology of the space of oriented embedded surfaces in a background manifold using the methods of Harer, Galatius, Madsen, Tillmann and Weiss [Har85, GMTW09]. More concretely, let Σ_g be an compact, connected oriented surface of genus g, and let M be a manifold. The set $E_g(M)$ of oriented submanifolds of M which are diffeomorphic to Σ_g receives a surjective map from the space of all embeddings of Σ_g into M, that takes an embedding to its image. We endow the former set of submanifolds with the quotient topology.

Madsen and Tillmann defined a generalisation of the Pontryagin-Thom map whose source is the space of submanifolds of $\mathbb{R}^\infty$ and whose target is a certain infinite loop space denoted $\Omega^\infty\mathbf{MTSO}(2)$. This map can be further generalised to a map from the space of submanifolds of any manifold M to the space $\Gamma_c(S_2(M))$ of compactly supported sections of a certain fibre bundle $S_2(M) := \mathrm{Th}^{\mathrm{fib}}(\gamma_2^\perp(TM)) \to M$. A point in this fibre bundle is a pair (p, L), where p is a point in M and L is a point in the one-point compactification of the affine Grassmannian of oriented 2-dimensional planes in T_pM.

If M is simply connected and of dimension at least 5, then the connected components of this space of sections are in bijection with $2\mathbb{Z} \times H_2(M)$, and we write $\Gamma_c(S_2(M))_g$ for the union of the components $\{2-2g\} \times H_2(M)$. The image of the generalised Pontryagin-Thom map lands in $\Gamma_c(S_2(M))_g$. Our main result states that, under these assumptions on M, the generalised Pontryagin-Thom map

$$E_g(M) \longrightarrow \Gamma_c(S_2(M))_g$$

induces a homology isomorphism in degrees $* \leq \frac{2g-2}{3}$. Additionally, if M is non-compact, the homotopy type of the right hand-side is independent of g. Therefore, the homology if the left hand-side is independent of g in the forementioned range.

This theorem generalises the Madsen–Weiss theorem in the same way that McDuff [McD75] generalises the Barrat–Priddy–Quillen theorem [BP72].

Martin Palmer has announced a similar result for the homology of spaces of disconnected submanifolds: he fixes d-manifolds P, W and an n-manifold M, and considers the space of all submanifolds of M diffeomorphic to the disjoint union of W and k copies of P. He proves that, under certain hypotheses, the homology of this space is independent of k, provided that k is big enough. The case of the space of disjoint unions of unlinked circles in $\mathbb{R}^3$, which does not satisfy the hypotheses of Palmer's theorem, is addressed in [Kup13].

The rational cohomology of the infinite loop space $\Omega^\infty \mathbf{MTSO}(2)$ is a free polynomial algebra with one generator κ_i in each even degree $2i$, which is called the i-th Miller–Morita–Mumford class. When $M = \mathbb{R}^n$, we can use the rational homotopy theory of Sullivan and Quillen to give a description of the rational cohomology of the space $\Gamma_c(S_2(\mathbb{R}^n))_g$ in terms of these classes. The result is as follows:

Let V_n be the dual of the vector space generated by $\kappa_{-1}, \kappa_0, \ldots, \kappa_{n-3}$, and let $\mathbb{L}_{n-1}(V_n)$ be the free graded Lie algebra on V_n, with a bracket of degree $n-1$, i.e., $\deg([a,b]) = \deg(a) + \deg(b) + n - 1$. If n is odd, then $H^*(\Gamma_c(S_2(\mathbb{R}^n)_g; \mathbb{Q})$ is the free polynomial algebra on the dual of the graded vector space $\mathbb{L}_{n-1}(V_n)$.

Let W_n be the dual of the vector space generated by $\kappa_{-1}, \kappa_0, \ldots, \kappa_{n-4}$ and a class ξ of degree $3n-9$. Let $\mathbb{L}_{n-1}(W_n)$ be the free graded Lie algebra on W_n, with a bracket of degree $n-1$. If n is even, then $H^*(\Gamma_c(S_2(\mathbb{R}^n))_g; \mathbb{Q})$ is the free polynomial algebra on the dual of the graded vector space $\mathbb{L}_{n-1}(W_n)$.

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The sections problem

Lei Chen

The classifying space $\mathrm{BDiff}(S_{g,n})$ of the orientation-preserving diffeomorphism group of the surface $S_{g,n}$ of genus $g > 1$ with n ordered marked points has a universal bundle

$$S_{g,n} \to \mathrm{UDiff}(S_{g,n}) \to \mathrm{BDiff}(S_{g,n}).$$

The n marked points provide n sections of the right hand side projection π which are called tautological.

This talk explains a proof of a conjecture of R. Hain: Any section of π is homotopic to one of the tautological sections.

The proof uses the following result of independent interest: Let $\mathrm{PConf}_n(S_g)$ be the set of ordered n-tuples of distinct points in S_g. Then any surjective homomorphism $\pi_1(\mathrm{PConf}_n(S_g)) \to \pi_1(S_g)$ is equal to one of the n forgetful maps

$$\rho_i : \pi_1(\mathrm{PConf}_n(S_g)) \to \pi_1(S_g),$$

which erases all but the i-th marked point, possibly post-composed with an automorphism of $\pi_1(S_g)$. Using similar arguments, one can show that the universal surface bundle does not have any section which respects the n marked points as a set.

Free-by-cyclic groups via fibered 3–manifolds

Spencer Dowdall

(joint work with Ilya Kapovich and Christopher J. Leininger)

It is well-known that a fibered 3–manifold M with first Betti-number at least 2 fibers as a surface bundle over the circle in infinitely many ways. While this is often discussed in the context of the Thurston norm on $H_2(M;\mathbb{R})$, which was introduced in 1986, the existence of infinitely many fiberings in fact follows from basic differential topology and goes back to the work of Tischler [9] in 1970. To see this consider a fibration of M

$$\Sigma \hookrightarrow M \xrightarrow{\sigma} S^1$$

realized by a smooth map $\sigma\colon M \to S^1$. Letting $d\theta \in \Omega^1(S^1)$ be the volume form on S^1, the pull-back $\omega = \sigma^*(d\theta)$ is a closed, nowhere vanishing 1–form whose kernel $\ker(\omega)$ is a 2–plane bundle tangent to the foliation of M by the fibers $\sigma^{-1}(x) \cong \Sigma$ for $x \in S^1$. Since nowhere-vanishing is an open and scale-invariant condition, one can find a neighborhood $U \subset H^1(M;\mathbb{R})$ of $[w]$ with the property that every cohomology class $[\alpha] \in (\mathbb{R}_+U)$ in the positive cone on U may be represented by a nowhere-vanishing closed 1–form α. For every such *integral* class $[\alpha] \in H^1(M;\mathbb{Z}) \cap (\mathbb{R}_+U)$, of which there are clearly infinitely many, integration against α defines a map $\sigma_\alpha\colon M \to \mathbb{R}/\mathbb{Z} = S^1$ which is in fact a fibration of M. In general, we say that a class $\beta \in H^1(M;\mathbb{Z}) \cong [M, S^1]$ is *fibered* if, as in this case, it is represented by a fibration under which $d\theta$ pulls back to β.

Letting $\epsilon_\omega \in H^2(M;\mathbb{Z})$ denote the Euler class of the 2–plane bundle $\ker(w)$, the Euler characteristic of the fiber Σ is given by the pairing $\langle \epsilon_\omega, [\Sigma]\rangle$, where $[\Sigma] \in H_2(M;\mathbb{Z})$ is the homology class of Σ. For a different fibering $\Sigma_\alpha \to M \overset{\alpha}{\to} S^1$, the Euler characteristic $\chi(\Sigma_\alpha)$ is similarly given by pairing $\langle \epsilon_\alpha, [\Sigma_\alpha]\rangle$ with the Euler class ϵ_α of the bundle $\ker(\alpha)$. However, since $\alpha \in \mathbb{R}_+ \cdot U$ is just a small perturbation of ω, the bundles $\ker(\alpha)$ and $\ker(\omega)$ are homotopic and thus have the same Euler class $\epsilon_\omega = \epsilon_\alpha$! This shows that $\chi(\Sigma_\alpha) = \langle \epsilon_\omega, [\Sigma_\alpha]\rangle$; that is, for *every* integral class in $\mathbb{R}_+ \cdot U$, the Euler characteristic of the fiber is given by pairing with ϵ_ω.

Let us now relate this to the Thurston norm: One may define a function ξ on $H^1(M;\mathbb{Z})$ by declaring $\xi(\alpha)$ to be $\min_S \overline{\chi}(S)$, where the minimum is taken over all surfaces $S \subset M$ representing the Poincaré dual of α and $\overline{\chi}(S)$ is the sum of the negative Euler characteristics $-\chi(S_i)$ of the non-spherical components S_i of S. Thurston [8] proved that ξ extends to a pseudo-norm on $H^1(M;\mathbb{R})$, meaning $\xi(\lambda\alpha) = |\lambda|\xi(\alpha)$ and $\xi(\alpha + \beta) \le \xi(\alpha) + \xi(\beta)$; this is the *Thurston norm.*

It is easy to see that for a fibration $\Sigma \to M \overset{\sigma}{\to} S^1$, the value of ξ on the fibered class $\omega = \sigma^*(d\theta)$ is the negative Euler characteristic $\xi(\omega) = -\chi(\Sigma)$ of the fiber. Therefore, our above discussion shows that ξ is *linear* in a cone neighborhood $\mathbb{R}_+ \cdot U$ of any fibered class ω, as it is given by pairing with the (Poincaré dual) of the Euler class of $\ker(\omega)$. This proves that the unit ball $B_\xi \subset H^1(M;\mathbb{R})$ of the norm ξ has flat "faces" below any fibered class. In fact, Thurston proved that B_ξ is a polyhedron with finitely many faces, and that every fibered class lives over the interior of a top dimensional face. What's more, Thurston proved the following dichotomy: for each top-dimensional face F of B_ξ, either *every* integral class in the cone $\mathbb{R}_+ \cdot F$ is fibered, or *none* of the classes in $\mathbb{R}_+ \cdot F$ are fibered. Thus the Thurston norm provides a complete classification of the fibrations of M: there is a finite set $\mathcal{F}$ of top-dimensional faces of B_ξ such that a class $\alpha \in H^1(M;\mathbb{Z})$ is fibered if and only if it lies in $\mathbb{R}_+ \cdot F$ for some face $F \in \mathcal{F}$.

Each fibration $\Sigma_\alpha \to M \overset{\alpha}{\to} S^1$ of M induces a well-defined (up to isotopy) *monodromy* homeomorphism $f_\alpha \colon \Sigma_\alpha \to \Sigma_\alpha$ of the fiber, and there is now a well-developed theory describing how the different fibrations are related to each other. For example, if the monodromy f_α of one fibration is pseudo-Anosov, then so is the monodromy of *every* fibration. What's more, the pseudo-Anosov dilatations of these monodromies vary, in an appropriate sense, real-analytically in $H^1(M;\mathbb{R})$.

These striking results motivate the study of whether similar phenomena hold in other settings, such as that of free-by-cyclic groups $F_n \rtimes \mathbb{Z}$. Indeed, if the fiber Σ is a punctured surface, then the fibration $\Sigma \hookrightarrow M \to S^1$ gives a splitting of $\pi_1(M)$ as a free-by-cyclic group. However, there are many free-by-cyclic groups that do not arise in this fashion. In joint work with Ilya Kapovich and Christopher Leininger [5, 3, 4, 6], we have begun to study such free-by-cyclic groups and to explore how their various splittings are related to each other.

Our results are simplest to state for the case of a free-by-cyclic group $G = F_n \rtimes_\phi \mathbb{Z}$ for which $\phi \colon F_n \to F_n$ is a fully irreducible and atoroidal automorphism of the rank n free group F_n. This means that no power ϕ^k, $k \ge 1$, preserves the conjugacy class of any proper free factor or nontrivial conjugacy class of F_n

and implies (by work of Brinkmann [2]) that G is hyperbolic. It is known that whenever $\mathrm{rank}(H^1(G;\mathbb{R})) \geq 2$, the group G splits as a free-by-cyclic group in infinitely many ways. More precisely, there is an open, $\mathbb{R}_+$–invariant set $\Sigma(G) \subset \mathrm{Hom}(G,R) = H^1(G;R)$ (namely, the Bieri–Neumann–Strebel invariant [1]) such that a nontrivial homomorphism $u \in \mathrm{Hom}(G,\mathbb{Z})$ has $\ker(u)$ finitely-generated if and only if $u \in \Sigma(G) \cap (-\Sigma(G))$. Further, for each such u, $\ker(u)$ is a finite-rank free group, and the short exact sequence

$$1 \longrightarrow \ker(u) \longrightarrow G \xrightarrow{u} \mathbb{Z} \cong u(G) \longrightarrow 1$$

induces a monodromy (outer) automorphism $\phi_u \in \mathrm{Out}(\ker(u))$ that expresses G as the free-by-cyclic group $G = \ker(u) \rtimes_{\phi_u} \mathbb{Z}$. In order to have rich theory as in the case of fibered 3-manifolds, one would like to have a means of explicitly calculating the set $\Sigma(G)$ and of studying and relating the various monodromies ϕ_u.

To this end, we constructed a polynomial $\mathfrak{m} \in \mathbb{Z}[H_1(G;\mathbb{Z})/\mathrm{torsion}]$ that is modeled on McMullen's Teichmüller polynomial for fibered hyperbolic 3–manifolds [7]. This polynomial has the form $\mathfrak{m} = a_1h_1 + \ldots + a_kh_k$ for some $a_i \in \mathbb{Z}$ and $h_i \in H_1(G;\mathbb{Z})/\mathrm{torsion}$ and is computed in terms of a train-track graph map $f\colon \Gamma \to \Gamma$ representing $\phi\colon F_n \to F_n$. Further, $\mathfrak{m}$ explicitly determines 3 pieces of information:

1. an open, convex, finite-sided polyhedral cone $\mathcal{C}_{\mathfrak{m}} \subset H^1(G;\mathbb{R})$,
2. a specialized Laurent polynomial $\mathfrak{m}_u(t) = a_1t^{u(h_1)} + \ldots + a_kt^{u(h_k)}$ in $\mathbb{Z}[t,t^{-1}]$ for each $u \in \mathcal{C}_{\mathfrak{m}} \cap H^1(G;\mathbb{Z})$, and
3. a convex, real-analytic, homogeneous of degree -1 function $\mathfrak{h}_{\mathfrak{m}}\colon \mathcal{C}_{\mathfrak{m}} \to \mathbb{R}$ that tends to ∞ at $\partial\mathcal{C}_{\mathfrak{m}}$ and is defined on integral classes $u \in \mathcal{C}_{\mathfrak{m}} \cap H^1(G;\mathbb{Z})$ by $\mathfrak{h}_{\mathfrak{m}}(u) = \log(\text{largest root of } \mathfrak{m}_u)$.

The train track representative f of ϕ allows us to build a "folded mapping torus" X, which is a $K(G,1)$-space that comes equipped with a semiflow ψ_t. Through a careful analysis of the dynamical system (X,ψ), we prove the following:

1. The cone $\mathcal{C}_{\mathfrak{m}}$ is a component of the BNS-invariant $\Sigma(G)$.
2. For every integral class $u \in \mathcal{C}_{\mathfrak{m}} \cap H^1(G;\mathbb{Z})$, there is exists a cross section Θ_u of ψ (meaning a finite, embedded graph $\Theta_u \hookrightarrow X$ that is transverse to the flow and has the property that every flowline $\{\psi_t(\xi) : t \in \mathbb{R}\}$ intersects Θ_u infinitely often) with the following properties:
   - Θ_u is "Poincaré dual" to the cohomology class u.
   - The first return map $f_u\colon \Theta_u \to \Theta_u$ of ψ is a train-track graph map representing the monodromy automorphism $\phi_u\colon \ker(u) \to \ker(u)$ of the splitting $1 \to \ker(u) \to G \xrightarrow{u} \mathbb{Z} \to 1$.
   - The transition matrix of f_u has characteristic polynomial given by $\mathfrak{m}_u(t)$ (up to a factor of t^k for some $k \in \mathbb{Z}$).
   - Consequently, $\mathfrak{h}_{\mathfrak{m}}(u)$ is the topological entropy of the graph map f_u and $e^{\mathfrak{h}_{\mathfrak{m}}(u)}$ is the algebraic stretch factor of the automorphism ϕ_u.
3. For every $u \in \mathcal{C}_{\mathfrak{m}} \cap H^1(G;\mathbb{Z})$ with $\ker(u)$ finitely generated, the monodromy ϕ_u is a fully irreducible and atoroidal automorphism of $\ker(u)$.

Thus our polynomial both identifies an interesting family of splittings of G, by calculating a component of $\Sigma(G)$, and gives detailed dynamical information about all of the splittings in this family. In particular, it shows that the algebraic stretch factors of these monodromies give rise to a convex, real-analytic function on this component of $\Sigma(G)$.

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Survey talk: Surface bundles and homotopy theory

Johannes Ebert

In this talk, an outline of the Madsen-Weiss theorem [15] was given, with the aim to make the statement comprehensible to the part of the audience without homotopy-theoretic background. I reported on the following results.

- The Earle-Eells-Schatz theorem [3] [2]: The topological group $\mathrm{Diff}^+_\partial(F_{g,r})$ of orientation-preserving diffeomorphisms of a surface of genus g with r boundary components which fix the boundary pointwise has contractible components, unless $g = r = 0$ or $r = 0, g = 1$. A topological proof of this result can be found in [8], and this is reproduced in [11].
- The Harer stability theorem [10], with the improved bounds due to Ivanov [12] and Boldsen [1]. A highly recommended source for this result is [21].
- The definition of the κ-classes [19], [18], [17], using a homotopy theoretic definition of the Gysin map.
- The geometric model for the classifying space $B\mathrm{Diff}(M)$ in terms of submanifolds of $\mathbb{R}^\infty$ diffeomorphic to M.
- The definition of a spectrum in general, the spectrum $\mathrm{MTSO}(d)$.
- The construction of a map $\alpha : B\mathrm{Diff}^+(M) \to \Omega^\infty\mathrm{MTSO}(d)$ using a version of the Pontrjagin-Thom construction and a more general version of it for manifolds with boundary [14].

- The statement of the Madsen-Weiss theorem [15]: Consider the space
$$\mathcal{N}(S^1) := \coprod_{g\geq 0} B\mathrm{Diff}^+_\partial(F_{g,1}) \simeq \coprod_{g\geq 0} B\Gamma_{g,1}$$
(this is the space of all connected oriented surfaces with one boundary component). Adding a torus with 2 boundary components induces a stabilization map $S : \mathcal{N}(S^1) \to \mathcal{N}(S^1)$ and one forms the (homotopy) colimit
$$\mathcal{N}(S^1)_\infty := \mathrm{hocolim}(\mathcal{N}(S^1) \stackrel{S}{\to} \ldots) \simeq \mathbb{Z} \times B\Gamma_\infty.$$
The previously introduced maps α induce a map
$$\alpha : \mathcal{N}(S^1)_\infty \to \Omega^\infty \mathrm{MTSO}(2),$$
and the Madsen-Weiss theorem states that it is a homology equivalence. The first proof [15] is a 90 page tour de force. A simplified argument appeared in [7] and a further simplification in [5]. This simplified argument is almost elementary, and [11] offers a good outline of the main ideas. These simplifications led to generalizations to the high-dimensional case, proven in [6] and [4]. It is also worth mentioning that Tillmann proved earlier [20] that $\mathcal{N}(S^1)_\infty$ has the homology of an infinite loop space, and this insight led to the development of the whole theory.
- Some computations of homotopy and homology groups of $\Omega^\infty \mathrm{MTSO}(2)$. The computation have been carried out in [13]. Among other things, they show that the Madsen-Weiss theorem implies the Mumford conjecture [19].
- The target space $\Omega^\infty \mathrm{MTSO}(2)$ has simply connected components. Thus α can be identified with a Quillen plus construction, which leads to a strong analogy with algebraic K-theory. One might view the homotopy groups $\pi_k(\Omega^\infty \mathrm{MTSO}(2))$ as the "higher algebraic K-groups of surfaces".

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Stories about surface bundles

Benson Farb

In this talk I discussed the context, background, and statements of various open problems on surface bundles. Such problems included the conjecture that there exist surface bundles over surfaces not admitting any flat connection, as well as the problem of finding explicit, nontrivial, unstable and odd-dimensional rational cohomology classes of mapping class groups in genus $g > 6$, as none are known [Note: the night before my lecture someone had actually posted a paper constructing such a class for some g!] One of the reasons we care about this cohomology is that every characteristic class of surface bundles must be of this form ...

Regularity and group actions on one-manifolds

Thomas Koberda
(joint work with Hyungryul Baik, Sang-hyun Kim, Yash Lodha)

In this talk, we discuss the relationship between the algebraic structure of a group G and the possible degrees of regularity of faithful action of G on a compact one–manifold, concentrating on mapping class groups of surfaces, right-angled Artin groups, and chain groups. We generally follow the results of [2] and [6].

It is a classical result of Nielsen that if S is a surface of genus at least two with one marked point, then the mapping class group $\mathrm{Mod}(S)$ acts faithfully by orientation-preserving homeomorphisms on the circle. This action has intrinsic non–differentiability, and Farb–Franks [4] (and independently Ghys) showed that sufficiently complicated mapping class groups admit no faithful C^2 actions on the circle or on the interval (in fact, they show that all such actions are trivial). Parwani [8] showed that sufficiently complicated mapping class groups admit no faithful C^1 actions on the circle.

Because of general analogies between mapping class groups and lattices in semi–simple Lie groups, it is natural to ask whether finite index subgroups of mapping class groups can act faithfully by diffeomorphisms on a compact one–manifold (cf. [3, 9]). The main result of [2] is that no finite index subgroup of a mapping class group acts faithfully by C^2 diffeomorphisms on a compact one–manifold, provided the mapping class group is not virtually a direct product of a free group and a cyclic group.

The tool which allows us to study finite index subgroups of mapping class groups is their right-angled Artin subgroups, as expounded by the author in [7]. This is because once a right-angled Artin group occurs in a mapping class group, it persists inside of all finite index subgroups of that mapping class group. In [2], it is proved that the right-angled Artin group on the graph P_4, the path on four vertices, admits no faithful C^2 action on a compact one-manifold. This result characterizes the mapping class groups which admit faithful C^2 virtual actions on a compact one-manifold, since these are precisely the ones which do not contain the right-angled Artin group on P_4. Moreover, this result proves that braid groups, Torelli groups, and many other natural examples of groups cannot admit faithful C^2 virtual actions on a compact one-manifold.

It should be noted that the compactness of the manifold and the regularity of the action are both essential hypotheses. Indeed, Farb–Franks [5] prove that every right-angled Artin group admits a faithful C^1 action on every one–manifold. Morever, in [1] it is proved that every right-angled Artin group admits a faithful C^∞ action on $\mathbb{R}$.

Right-angled Artin group actions on compact one-manifolds naturally give rise to the notion of chain groups, as introduced in [6]. A chain of intervals is a collection $\{J_1, \ldots, J_k\}$ of open subintervals of the real line, with $J_i \cap J_{i+1} \neq \emptyset$, and with all other intersections empty. We then take homeomorphisms $\{f_1, \ldots, f_k\}$ of $\mathbb{R}$ such that the support of f_i is exactly J_i. The homeomorphisms $\{f_1, \ldots, f_k\}$ generate a chain group provided that $\langle f_i, f_{i+1}\rangle$ is isomorphic to Thompson's group F for each i. This condition, while strange at first glance, is quite natural and dynamically stable. Indeed for arbitrary choices of f_i and f_{i+1}, these homeomorphisms will always generate a copy of F, provided that they are replaced by sufficiently high powers if necessary.

Chain groups exhibit a combination of uniformity and diversity. On the one hand, their centers are always trivial. Moreover, they either have simple commutator subgroups, or they admit certain canonical quotients which are themselves naturally isomorphic to chain groups with simple commutator subgroups.

On the other hand, every finitely generated subgroup of the group of homeomorphisms of the interval embeds in some chain group. This shows that there are uncountably many isomorphism types of chain groups on three or more intervals, and in particular there exist chain groups which are not finitely presented. With some more work, one can show that chain groups give rise to uncountably many different isomorphism types of simple subgroups of the homeomorphism group of the interval.

Returning to regularity, one can use certain examples due to B. Neumann to show that there exist uncountably many isomorphism types of chain groups which can be realized by homeomorphisms, but not by C^2 diffeomorphisms.

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Fundamental Groups of Kodaira Fibrations of Genus 3

Laure Flapan

A Kodaira fibration is an algebraic surface X equipped with a non-isotrivial fibration $f\colon X \to B$ over a smooth algebraic curve B such that for every $b \in B$ the fiber F_b is a smooth algebraic curve. Such a fibration yields a monodromy representation $\rho\colon \pi_1(B) \to Sp(V)$, where $V = H^1(F_b, \mathbb{Q})$. The connected monodromy group is the connected component of the image of the identity element in the $\mathbb{Q}$-Zariski closure of the image of ρ.

In this talk, we determine the possibilities for connected monodromy groups realized by Kodaira fibrations whose fibers have genus 3, which is the minimal possible fiber genus.

Surface bundles and low-dimensional geometry

Dieter Kotschick

In many ways, three-dimensional surface bundles, i. e. mapping tori of surface diffeomorphisms, are well understood. For example, their geometrization was described long ago by Thurston in terms of the classification of isotopy classes of surface diffeomorphisms. This description in particular settles questions of existence and uniqueness of Einstein metrics, and it also leads to an exact calculation of the simplicial volume, which turns out to be a universal multiple of the hyperbolic volume (of any hyperbolic pieces contained in the geometric decomposition).

In this survey talk I discussed some geometric structures – symplectic, complex, or Einstein – on four-dimensional surface bundles, i. e. surface bundles over surfaces, and their relations to inequalities between the numerical invariants of such bundles.

Let X be the total space of an oriented surface bundle with closed oriented fiber F over a closed oriented surface B. For simplicity we assume that the genera of both F and B are ≥ 2.

1. Geometric structures

1.1. **Symplectic structures.** By the so-called Thurston construction, every such total space X admits symplectic structures inducing both orientations, the given one, and the opposite one. Therefore, work of Taubes implies that the Seiberg–Witten invariants are non-trivial for both orientations. This is quite a rare situation, since for most other four-manifolds the non-triviality of gauge theoretic invariants is destroyed by a reversal of orientation.

1.2. **Complex structures.** While symplectic structures always exist, complex structures on surface bundles are very rare. The reason is that if such an X admits a complex structure, then by the Kodaira classification it is a minimal surface of general type, cf. [5]. In particular, it is Kähler and even projective algebraic. Since for fixed F and B the Chern numbers (equivalently, the signature and Euler characteristic) of the surface X are bounded (see 2.1 below), boundedness results for the moduli space of surfaces of general type imply that for fixed F and B there are at most finitely many surface bundles admitting a complex structure at all. One can also prove that any complex structure on the total space X in fact comes from a holomorphic family of complex curves, see for example [6], so that finiteness also follows from the results of Parshin and Arakelov about algebraic families.

Since almost all surface bundles cannot be complex, it is perhaps surprising that so many concrete or explicit constructions actually give complex examples.

This is true for the first non-trivial constructions of Atiyah and Kodaira, and also for many later constructions up until [3].

1.3. **Einstein metrics.** Concerning the existence of Einstein metrics on surface bundles X, note that any such metric has to have negative scalar curvature, or Einstein constant. Since the complex examples are not only minimal algebraic, but have ample canonical bundles, the work of Aubin and Yau implies the existence of (negative) Kähler–Einstein metrics on them. For non-complex surface bundles the existence of Einstein metrics is wide open.

2. Inequalities between numerical invariants

The most important numerical invariants of an oriented surface bundle X are its Euler characteristic $\chi(X)$, its signature $\sigma(X)$, and its simplicial volume $||X||$. They all behave multiplicatively in finite coverings.

The Euler characteristic is also multiplicative in fiber bundles, and therefore satisfies

$$\chi(X) = \chi(F) \cdot \chi(B) = 4(g(F) - 1)(g(B) - 1) \ .$$

The signature and the simplicial volume are not multiplicative in fiber bundles, and are not determined by F and B, but really depend on the bundle under consideration. The most convenient way to understand their possible ranges is to compare them to the Euler characteristic.

2.1. **Signature versus Euler characteristic.** Since the signature changes sign under orientation reversal, one considers its absolute value. There are positive constants c, such that for all surface bundles X one has

$$0 \leq c \cdot |\sigma(X)| \leq \chi(X) \ . \tag{1}$$

Now one would like to find the largest possible c (that works for all X).

Exploiting the existence of symplectic structures to apply results from Seiberg–Witten theory, I proved a long time ago that one can take $c = 2$, and I conjectured that $c = 3$ should be possible, cf. [5]. A proof of this conjecture was announced several years ago, but has not appeared in print.

Under the additional assumption that X admits an Einstein metric, I did prove that (1) holds with $c = 3$, and in fact the inequality is then strict, cf. [5, 6]. The argument for this uses the interaction between an Einstein metric and solutions to the Seiberg–Witten equations with respect to this metric. As mentioned above, any complex surface bundle does admit an Einstein metric.

Examples of surface bundles with large signature (compared to the Euler characteristic) provide upper bounds on the largest possible constant c in (1). As far as I am aware, the best bound comes from some examples of Catanese and Rollenske [3], showing that $c \leq 9/2$.

2.2. **Simplicial volume versus Euler characteristic.** It is known that the simplicial volume of surface bundles X as above is always positive [4] and can be bounded from below in terms of the Euler characteristic:

$$d \cdot \chi(X) \leq ||X|| \ . \tag{2}$$

In this case one needs to find the largest possible constant d that works for all X. With Hoster [4] we proved that one can take $d = 4$, and this was improved to $d = 6$ by Bucher [1]. This is the optimal constant, since product bundles $X = F \times B$ satisfy $6\chi(X) = ||X||$.

It is an interesting problem to understand the possible values of the simplicial volume $||X||$ for fixed F and B. For example, can $||X||$ be bounded from above in terms of the Euler characteristic? Are there finitely or infinitely many possible values? These questions are open for arbitrary surface bundles, but for those admitting an Einstein metric one can say more. For a four-dimensional Einstein manifold the simplicial volume is always bounded in terms of the Euler characteristic, by what I called the Gromov–Hitchin–Thorpe inequality. Compare [4] and the stronger inequalities in [7].

2.3. **Simplicial volume versus signature.** Combining the inequalitites (1) and (2), one trivially obtains

$$e \cdot |\sigma(X)| \leq ||X|| \tag{3}$$

with the constant $e = c \cdot d$. The known values $c = 2$ and $d = 6$ give $e = 12$. (In [4] we had already proved that (3) holds with $e = 12$, although at the time it was not yet known that one can take $d = 6$.) The conjectured $c = 3$ would imply $e = 18$. In any case, since $d = 6$ is best possible, and $c \leq 9/2$, the largest constant e one can hope to obtain in (3) by combining (1) and (2) is $e = 27$. Interestingly, the best possible constant e in (3) is not obtained by combining (1) and (2), since recently Bucher and Campagnolo [2] proved that (3) holds with $e = 36 > 27$.

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Virtual constructions in closed hyperbolic 3-manifolds

Yi Liu

In this talk, we discuss how to build an essentially immersed subsurface of odd Euler characteristic in a closed hyperbolic 3-manifold. A solution can be obtained by developing the good pants constructions invented by J. Kahn and V. Markovic. A few further applications of the improved technques are discussed. In particular, it is shown that closed hyperbolic 3-manifolds admit exhausting towers of irregular finite covers with exponential homological torsion growth.

Strata of abelian differentials and Miller-Morita-Mumford classes

Mark Pedron

The aim of this talk was to introduce the cycle classes of strata of abelian differentials in a differential-topological setting as follows.

Let us suppose we are given

- a closed oriented surface bundle $F \to E \to M$ over a closed manifold M, with fibre a closed surface F of genus $g \geq 2$
- a smooth complex structure J of the vertical tangent bundle T^vE of E
- a smooth fibrewise holomorphic section ω of the fibrewise holomorphic cotangent bundle of E
- a partition P of $2g-2$ as $2g-2 = m_1 + \cdots + m_k$.

Then one can consider the locus

$$M(P) = \{p \in M \mid \omega|F_p \text{ has zero partition } m_1, \ldots, m_k\}.$$

We obtain the following result.

Theorem 1. The Poincaré dual class $h(P) \in H^*(M; \mathbb{Q})$ is a stable characteristic class of surface bundle. These classes form a basis for the rational stable characteristic classes of surface bundles.

Now let $\mathcal{M}_g$ be the the moduli space of surfaces of genus g. The above result is a geometric consequence of the following more structural result.

There is an associative commutative algebra structure on the free vector space $\mathbb{Q}[\Pi_\Sigma(2g-2)]$ on the partitions of $2g-2$, and there is a morphism of algebras

$$\mathbb{Q}[\Pi_\Sigma(2g-2)] \to H^*(\mathcal{M}_g; \mathbb{Q}).$$

A chase for invariants of integral homology spheres

Wolfgang Pitsch

Let $\Sigma_{g,1}$ denote a surface of genus $g \geq 3$ with one boundary component and let $\pi = \pi_1(\Sigma_{g,1})$ denote its fundamental group. Let $\mathcal{M}_{g,1}$ denote the mapping class group of $\Sigma_{g,1}$, relative to its boundary. The group $\mathcal{M}_{g,1}$ acts on π, and therefore on the lower central series of π, given by $\Gamma_0 = \pi$ and for $k \geq 0$ $\Gamma_{k+1} = [\Gamma_k, \pi]$; we have then an induced action on the nilpotent quotients of π given by $N_k = \pi/\Gamma_k$. Denote by $\mathcal{M}_{g,1}(k)$ the kernel of the map $\mathcal{M}_{g,1} \to \mathrm{Aut} N_k$.

Let $\mathbb{S}^3$ denote the oriented standard 3-sphere. The canonical embedding of $\Sigma_{g,1} \hookrightarrow \mathbb{S}^3$ splits the sphere into two oriented handlebodies, we denote by $\mathcal{H}_g$ the inner one and by $-\mathcal{H}_g$ the outer one. Then commuting diagram expressing $\mathbb{S}^3$ as the union of the two handlebodies glued along their common boundary:

$$\begin{array}{ccc} \Sigma_{g,1} & \longrightarrow & \mathcal{H}^g \\ \downarrow & & \downarrow \\ -\mathcal{H}_g & \longrightarrow & \mathbb{S}^3 \end{array}$$

induces a commutative diagram of mapping class groups:

$$\begin{array}{ccc} \mathcal{M}_{g,1} & \longleftarrow & \mathcal{B}_{g,1} \\ \uparrow & & \uparrow \\ \mathcal{A}_{g,1} & \longleftarrow & \mathcal{AB}_{g,1} \end{array}$$

where $\mathcal{A}_{g,1}$ (resp. $\mathcal{B}_{g,1}$) denotes the subgroup of those mapping classes that extend over the handlebody $-\mathcal{H}_g$ (resp. $\mathcal{H}_g$) and $\mathcal{AB}_{g,1} = \mathcal{A}_{g,1} \cap \mathcal{B}_{g,1}$ is the subgroup of those mapping classes that extend over both handlebodies, and hence by a result of Waldhausen that extend simultaneously over both handlebodies. It is easy to check that these subgroups are compatible under the stabilization morphism $\mathcal{M}_{g,1} \to \mathcal{M}_{g+1,1}$, and we can use them to parametrize the set $\mathcal{V}(3)$ of oriented diffeomorphism classes of closed 3-manifolds. Given the above spitting of $\mathbb{S}^3 = \mathcal{H}_g \bigcup_{\iota_g} -\mathcal{H}_g$, where the two boundaries of the handlebodies are identified through a map ι_g , we denote by $\mathbb{S}^3_\phi$ the manifold $\mathcal{H}_g \bigcup_{\iota_g \phi} -\mathcal{H}_g$, where we glue back the boundaries of the handlebodies through the map $\iota_g \phi$.

Theorem 1 (Singer, 1953). The following map is well defined and is a bijection:

$$\begin{array}{rcl} \lim_{g\to\infty} \mathcal{M}_{g,1}/\sim & \longrightarrow & \mathcal{V}(3) \\ \phi & \longmapsto & \mathbb{S}^3_\phi \end{array}$$

where $\sim$ is the equivalence relation given by taking double cosets

$$\mathcal{M}_{g,1}/\sim = \mathcal{A}_{g,1} \backslash \mathcal{M}_{g,1} / \mathcal{B}_{g,1}.$$

We have produced a decreasing filtration of the mapping class group, known as the Johnson filtration, which is also compatible with the stabilization map

$$\mathcal{M}_{g,1} \supset \mathcal{M}_{g,1}(1) \supset \mathcal{M}_{g,1}(2) \supset \cdots \mathcal{M}_{g,1}(k) \supset \cdots$$

and by letting $\mathcal{V}_k(3) \subset \mathcal{V}(3)$ denote the subset that is the image of the group $\lim_{g\to\infty} \mathcal{M}_{g,1}(k)$ under the composition:

$$\lim_{g\to\infty} \mathcal{M}_{g,1}(k) \to \lim_{g\to\infty} \mathcal{M}_{g,1} \to \mathcal{V}_k(3) \subset \mathcal{V}(3)$$

then we get a parallel decreasing filtration of $\mathcal{V}(3)$:

$$\mathcal{V}(3) \supset \mathcal{V}_1(3) \supset \mathcal{V}_2(3) \supset \cdots \mathcal{V}_k(3) \supset \cdots$$

It is an exercise in the Mayer-Viettoris sequence to check that $\mathcal{V}_1(3)$ is the set of integral homology 3-spheres. More generally we ask the following

Question 1. Give a description of the subset $\mathcal{V}_k(3)$.

The case of the first steps in the filtration where addressed by S. Morita [2] in his seminal work o the Casson invariant; the known answers are:

- $\mathcal{V}_2(3) = \mathcal{V}_1(3)$ (Morita [2])
- $V_3(3) = \mathcal{V}_2(3)$ (Pitsch [3], Massuyeau-Meylan [1])

One may then wonder if the following is true:

Question 2. Is $\mathcal{V}_k(3) = \mathcal{V}_1(3)$ for all $k \geq 1$?

This seems to be a very hard question, mainly because the combinatorics in the Johnsons filtration are rapidly very complicated and largely unknown.

There is a convenient way describe integral homology spheres via their Heegaard splittings and the subgroup $\mathcal{M}_{g,1}(1)$. For any $k \geq 1$ set $\mathcal{A}_{g,1}(k) = \mathcal{A}_{g,1} \cap \mathcal{M}_{g,1}(k)$, and $\mathcal{B}_{g,1}(k) = \mathcal{B}_{g,1} \cap \mathcal{M}_{g,1}(k)$

Definition 2. Consider on $\mathcal{M}_{g,1}(1)$ the following equivalence relation:

$$\phi \approx \psi \Leftrightarrow \exists(\mu, \xi_a, \xi_b) \in \mathcal{AB}_{g,1} \times \mathcal{A}_{g,1}(1) \times \mathcal{B}_{g,1}(1) \text{ such that } \phi = \mu\xi_a\psi\xi_b\mu^{-1}$$

The analgous statement for the subgroups $\mathcal{M}_{g,1}(k)$, $k \geq 2$ is unknown, the difficulty rests in the following problem:

Question 3. Let $\phi \in \mathcal{A}_{g,1}(k)$ and $\psi \in \mathcal{B}_{g,1}$. Assume that $\forall x \in \pi$

$$\phi(x) = \psi(x)i \bmod \Gamma_{k+1}.$$

Does there exist $\mu \in \mathcal{AB}_{g,1}$ such that $\forall x \in \pi$

$$\mu(x) = \phi(x) = \psi(x) \bmod \Gamma_{k+1}?$$

The equivalence relation $\approx$ on $\mathcal{M}_{g,1}(1)$ is in fact the same as the one induced from $\mathcal{M}_{g,1}$ by the double coset relation:

Lemma 1. Let $\phi, \psi \in \mathcal{M}_{g,1}(1)$, then ϕ and ψ belong to the same double coset in $\mathcal{A}_{g,1}\backslash\mathcal{M}_{g,1}/\mathcal{B}_{g,1}$ if and only if $\phi \approx \psi$

As a consequence we have an analogue for Singer's theorem for integral homology spheres:

Theorem 3. [3] The following map is well defined and is a bijection:

$$\begin{array}{rcl} \lim_{g\to\infty} \mathcal{M}_{g,1}(1)/\approx & \longrightarrow & \mathcal{V}_1(3) \\ \phi & \longmapsto & \mathbb{S}^3_\phi \end{array}$$

We may use this to understand group-valued invariants of integral homology spheres. Given any such invariant with values in an abelian group $F : \mathcal{V}_1(3) \to A$, then we may consider the associated maps $F_g : \mathcal{M}_{g,1}(1) \to A$ given by the composites:

$$\mathcal{M}_{g,1}(1) \to \lim_{g\to\infty} \mathcal{M}_{g,1}(1)/\approx \to \mathcal{V}_1(3) \xrightarrow{F} A$$

Then we consider the associated trivialized cocycle on $\mathcal{M}_{g,1}(1)$ with values in A:

$$C_g(\phi, \psi) = F_g(\phi\psi) - F_g(\phi) - F_g(\psi)$$

which can be thought of as a kind of generalized surgery formula for the invariant F. It turns out that from the cocycle C_g one can reconstruct the invariant F:

Theorem 4. [3] Let a familly of 2-cocycles $(C_g)_{g\geq 3}$ on the group $\mathcal{M}_{g,1}(1)$ with values in an abelian group without 2-torsion A be given. Assume that the cocycles satisfy the following conditions:

(1) $C_{g+1}|_{\mathcal{M}_{g,1}(1)} = C_g$
(2) C_g is trivial on $\mathcal{A}_{g,1}(1) \times \mathcal{M}_{g,1}(1) \bigcup \mathcal{M}_{g,1}(1) \times \mathcal{B}_{g,1}(1)$
(3) C_g is $\mathcal{AB}_{g,1}$-invariant, $\forall \mu \in \mathcal{AB}_{g,1}$ $C_g(\mu - \mu^{-1}, \mu - \mu^{-1}) = C_g(-,-)$.
(4) The cohomology class in $H^2(\mathcal{M}_{g,1}(1); A)$ defined by each C_g is trivial
(5) The torsor of the induced action of $AB_{g,1}$ on the set of trivializations of the cocycle $C_g$is trivial:

$$[\rho_{C_g}] = 0 \in H^1(\mathcal{AB}_{g,1}; Hom(\mathcal{M}_{g,1}(1), A))$$

Then for each g the cocycle C_g admits a unique $\mathcal{AB}_g$-invariant F_g, they are compatible with stabilization, and the induced map

$$\lim_g F_g : \lim_g \mathcal{M}_{g,1}(1) \to A$$

is constant on the equivalence classes of the equivalence relation $\approx$, and hence induces an invariant of integral homology spheres.

As an application, we can construct in an elementary way the Casson invariant, for details see [3]. Recall that $N_1 = H_1(\Sigma_{g,1}; \mathbb{Z})$. Let $\omega : \Lambda^2 N_1 \to \mathbb{Z}$ denote the symplectic intersection form given by transverse intersection of oriented paths on $\Sigma_{g,1}$. By a classical result of Johnson, there is a canonical surjective homomorphism $\mathcal{M}_{g,1} \to \Lambda^3 N_1$ that gives the abelianization of $\mathcal{M}_{g,1}(1)$ up to 2-torsion. Let $B \subset N_1$ be the kernel of the map $H_1(\Sigma_{g,1}; \mathbb{Z}) \to H_1(\mathcal{H}_g; \mathbb{Z})$ (resp. A the kernel of $H_1(\Sigma_{g,1}; \mathbb{Z}) \to H_1(-\mathcal{H}_g; \mathbb{Z})$). Then $A \oplus B = N_1$, and these are two transverse lagrangians for the intersection form ω. From the decomposition $A \oplus B = N_1$, we have an induced decomposition $\Lambda^3 N_1 = \Lambda^3 A \oplus W_{AB} \oplus \Lambda^3 B$, and in turn ω induces a pairing:

$$\Lambda^3\omega : \Lambda^3 B \times \Lambda^3 A \to \mathbb{Z}$$

Consider the bilinear form J_g on $\Lambda^3 N_1 = \Lambda^3 A \oplus W_{AB} \oplus \Lambda^3 B$, given by the blocks matrix:

$$\begin{pmatrix} 0 & 0 & \Lambda^3\omega \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

then

Theorem 5. (1) The cocycles $2J_g$ satisfy the conditions of the above theorem.
(2) The associated invariant is the Casson invariant.

The proof of 2 comes rests on the key fact that the Casson invariant is determined by surgery properties, i.e. by how the value of the invariant varies when one performs a surgery along a knot in a homology sphere and one checks that the invariant given by the first part of the theorem satisfies the same surgery properties.

In recent work R. Riba [4] extended the above theorems to mod-p homology spheres, for p a prime. In this case one has to replace the group $\mathcal{M}_{g,1}(1) = \ker(\mathcal{M}_{g,1} \to Sp_{2g}(\mathbb{Z}))$ by $\mathcal{M}_{g,1}[p] = \ker(\mathcal{M}_{g,1} \to Sp_{2g}(\mathbb{Z}/p\mathbb{Z}))$. Then the obvious generalizations of Theorems 3 and 4 hold true, with the following slight modification in Theorem 4: $A = \mathbb{Z}/p\mathbb{Z}$ and there are then exactly p trivializations that glue back to form p invariants of mod-p homology spheres.

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The high-dimensional cohomology of the moduli space of curves with level structures

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(joint work with Neil Fullarton)

Acknowledgement: Neil Fullarton: Supported in part by an AMS-Simons grant and an MSRI postdoctoral fellowship and Andrew Putman: Supported in part by NSF grant DMS-1255350 and the Alfred P. Sloan Foundation

Let Σ_g be a closed oriented genus g surface. The *mapping class group* of Σ_g, denoted Mod_g, is the group of isotopy classes of orientation-preserving diffeomorphisms of Σ_g. The group Mod_g lies at the crossroads of many areas of mathematics. One fundamental reason for this is that Mod_g is the orbifold fundamental group of the moduli space $\mathcal{M}_g$ of genus g Riemann surfaces. In fact, even more is true: as an orbifold, $\mathcal{M}_g$ is an Eilenberg–MacLane space for Mod_g, which implies in particular that

$$\text{H}^*(\text{Mod}_g; \mathbb{Q}) = \text{H}^*(\mathcal{M}_g; \mathbb{Q}).$$

See [FarMar] for a survey of Mod_g and $\mathcal{M}_g$.

Stable cohomology. Let $\kappa_i \in \mathrm{H}^{2i}(\mathrm{Mod}_g;\mathbb{Q})$ be the i^{th} Miller–Mumford–Morita class. We then have a graded ring homomorphism $\mathbb{Q}[\kappa_1,\kappa_2,\ldots] \to \mathrm{H}^*(\mathrm{Mod}_g;\mathbb{Q})$, and the Mumford conjecture (proved by Madsen–Weiss [MadW]) says that this graded ring homomorphism is an isomorphism in degrees less than or equal to $\frac{2}{3}(g-1)$. Aside from some low-genus computations, no nontrivial elements of $\mathrm{H}^*(\mathrm{Mod}_g;\mathbb{Q})$ have been found outside this stable range. However, Harer–Zagier [HareZ] proved that the Euler characteristic of Mod_g is enormous, so there must exist vast amounts of unstable rational cohomology.

Level structures. The cohomology of finite-index subgroups of Mod_g (or, equivalently, finite covers of $\mathcal{M}_g$) is also of interest. For $\ell \geq 2$, the *level ℓ congruence subgroup* of Mod_g, denoted $\mathrm{Mod}_g(\ell)$, is the kernel of the action of Mod_g on $\mathrm{H}_1(\Sigma_g;\mathbb{Z}/\ell)$. It fits into a short exact sequence

$$1 \longrightarrow \mathrm{Mod}_g(\ell) \longrightarrow \mathrm{Mod}_g \longrightarrow \mathsf{Sp}_{2g}(\mathbb{Z}/\ell) \longrightarrow 1.$$

The symplectic group appears here because the action of Mod_g on $\mathrm{H}_1(\Sigma_g;\mathbb{Z}/\ell)$ preserves the algebraic intersection pairing. The associated finite cover of $\mathcal{M}_g$ is the moduli space $\mathcal{M}_g(\ell)$ of genus g curves equipped with a full level ℓ structure (i.e. a basis for the ℓ-torsion in their Jacobian). A conjecture of the second author (see [Pu, §1] for a discussion) asserts that

$$\mathrm{H}_k(\mathrm{Mod}_g(\ell);\mathbb{Q}) \cong \mathrm{H}_k(\mathrm{Mod}_g;\mathbb{Q}) \qquad (g \gg k).$$

This holds for $k=1$ by work of Hain [Hai] and for $k=2$ by work of the second author [Pu].

Cohomological dimension. The main topic of this report is what happens outside the stable range. Harer [Hare2] proved that the virtual cohomological dimension (vcd) of Mod_g is $4g-5$, so $\mathrm{H}^i(\mathrm{Mod}_g;\mathbb{Q}) = 0$ for $i > 4g-5$. The first place where one might hope to find some unstable cohomology is thus in degree $4g-5$. However, a theorem proved independently by Morita–Sakasai–Suzuki [MorSaSu] and by Church–Farb–Putman [ChuFarPu] says that $\mathrm{H}^{4g-5}(\mathrm{Mod}_g;\mathbb{Q}) = 0$. This might seem to contradict the fact that the vcd of Mod_g is $4g-5$. However, the definition of the vcd of a group makes use not only of ordinary cohomology, but also cohomology with respect to arbitrary twisted coefficient systems. Harer's theorem thus only asserts that there exists some $\mathbb{Q}[\mathrm{Mod}_g]$-module M (necessarily nontrivial, in light of [MorSaSu, ChuFarPu]) such that $\mathrm{H}^{4g-5}(\mathrm{Mod}_g;M) \neq 0$.

Main theorem. This brings us to our main theorem, which says that in contrast to what conjecturally happens in the stable range, the group $\mathrm{Mod}_g(\ell)$ has an enormous amount of cohomology in its vcd.

Theorem A. *Fix $g,\ell \geq 2$ and let p be a prime dividing ℓ. Then*

$$\dim_{\mathbb{Q}} \mathrm{H}^{4g-5}(Mod_g(\ell);\mathbb{Q}) \geq \frac{|\mathsf{Sp}_{2g}(\mathbb{F}_p)|}{g(p^{2g}-1)} = \frac{1}{g} p^{2g-1} \prod_{k=1}^{g-1} (p^{2k}-1) p^{2k-1}.$$

Remark 1. Our lower bound is super-exponential; its leading term is $\frac{1}{g} p^{\binom{2g}{2}}$. To give an idea of how quickly it grows, the following are some special cases:

$$\dim_{\mathbb{Q}} \mathrm{H}^3(\mathrm{Mod}_2(2);\mathbb{Q}) \geq 24 \qquad \dim_{\mathbb{Q}} \mathrm{H}^3(\mathrm{Mod}_2(3);\mathbb{Q}) \geq 216$$
$$\dim_{\mathbb{Q}} \mathrm{H}^7(\mathrm{Mod}_3(2);\mathbb{Q}) \geq 11520 \qquad \dim_{\mathbb{Q}} \mathrm{H}^7(\mathrm{Mod}_3(3);\mathbb{Q}) \geq 4199040$$
$$\dim_{\mathbb{Q}} \mathrm{H}^{11}(\mathrm{Mod}_4(2);\mathbb{Q}) \geq 92897280 \qquad \dim_{\mathbb{Q}} \mathrm{H}^{11}(\mathrm{Mod}_4(3);\mathbb{Q}) \geq 6685442749440.$$

Remark 2. Our lower bound is almost certainly not sharp. One difficulty with improving it is that as described below, we exploit a connection to the Tits building for the group $\mathrm{SL}_n(\mathbb{F}_p)$. Presumably better results could be obtained by studying the obvious analogue of this building for the finite group $\mathrm{SL}_n(\mathbb{Z}/\ell)$, but for ℓ not prime this is poorly understood.

Remark 3. In his 1986 paper [Hare2] (see p. 175), Harer asserts that "it is possible to show" that $\mathrm{H}^{4g-5}(\mathrm{Mod}_g(\ell);\mathbb{Q}) \neq 0$. However, he never published a proof of this.

Application to algebraic geometry. Theorem A has an interesting application to the algebraic geometry of $\mathcal{M}_g$, which recall is the (coarse) moduli space of genus g Riemann surfaces. We begin with the following conjecture of Looijenga [FabLo].

Conjecture 1 (Looijenga). For $g \geq 2$, the quasiprojective variety $\mathcal{M}_g$ can be covered by $(g-1)$ open affine subsets.

For example, this conjecture asserts that $\mathcal{M}_2$ is itself affine, which is a consequence of the fact that every genus 2 Riemann surface is hyperelliptic. More generally, Fontanari–Pascolutti [FoPa] proved that Conjecture 1 holds for $2 \leq g \leq 5$.

Conjecture 1 would imply a bound on the coherent cohomological dimension of $\mathcal{M}_g$, which is defined as follows. If X is a variety, then the *coherent cohomological dimension* of X, denoted $\mathrm{CohCD}(X)$, is the maximum value of k such that there exists some coherent sheaf $\mathcal{F}$ on X with $\mathrm{H}^k(X;\mathcal{F}) \neq 0$. The coherent cohomological dimension of a variety reflects interesting geometric properties of the variety. For example, Serre [Se] proved that $\mathrm{CohCD}(X) = 0$ if and only if X is an affine variety. See [Hart] for more information on coherent cohomological dimension.

Since $\mathcal{M}_g$ is separated, the intersection of two affine open subsets of $\mathcal{M}_g$ is itself an affine open subset. Thus if Conjecture 1 were true and $\mathcal{F}$ were a coherent sheaf on $\mathcal{M}_g$, then we could apply the Mayer–Vietoris spectral sequence to the cover of $\mathcal{M}_g$ given by Conjecture 1 to deduce that $\mathrm{H}^k(\mathcal{M}_g;\mathcal{F}) = 0$ for $k > g-2$. In other words, Conjecture 1 would imply that $\mathrm{CohCD}(\mathcal{M}) \leq g-2$.

Using our main theorem (Theorem A), we will prove the following, which asserts that this conjectural upper bound is sharp.

Theorem B. *For $g \geq 2$, we have $\mathrm{CohCD}(\mathcal{M}_g) \geq g-2$ with equality for $2 \leq g \leq 5$.*

Remark 4. This implies that $\mathcal{M}_g$ cannot be covered with fewer than $(g-1)$ open affine subsets. This was already known. Indeed, it follows from work of Chaudhuri [Cha], who gave a lower bound on the cohomological excess of $\mathcal{M}_g$ (which is defined using constructible sheaves).

Remark 5. The only paper we are aware of concerning upper bounds for things related to $\mathrm{CohCD}(\mathcal{M}_g)$ is recent work of Mondello [Mon], who proved that the

Dolbeault cohomological dimension of $\mathcal{M}_g$ is at most $2g-2$. The Dolbeault cohomological dimension of a complex analytic variety X is the maximal k such that there exists a holomorphic vector bundle $\mathcal{B}$ on X such that $\mathrm{H}^k(\mathcal{M}_g;\mathcal{B}) \neq 0$. As will be clear from our argument below, our work also establishes a lower bound of $g-2$ on the Dolbeault cohomological dimension of $\mathcal{M}_g$.

Proof of Theorem B. The derivation of Theorem B from Theorem A is so simple that we give it here. We wish to thank Eduard Looijenga for explaining this argument to us. Fix some $\ell \geq 3$, so $\mathcal{M}_g(\ell)$ is smooth. The projection $\mathcal{M}_g(\ell) \to \mathcal{M}_g$ is a finite surjective map, so $\mathrm{CohCD}(\mathcal{M}_g(\ell)) = \mathrm{CohCD}(\mathcal{M}_g)$ (see [Hart, Proposition 1.1]). It is thus enough to prove that $\mathrm{CohCD}(\mathcal{M}_g(\ell)) \geq g-2$. Assume for the sake of contradiction that this is false. The Hodge–de Rham spectral sequence for $\mathcal{M}_g(\ell)$ converges to $\mathrm{H}^*(\mathcal{M}_g(\ell);\mathbb{C})$ and has

$$E_1^{pq} = \mathrm{H}^p(\mathcal{M}_g(\ell);\Omega^q).$$

Since the complex dimension of $\mathcal{M}_g(\ell)$ is $3g-3$, we have $\Omega^q = 0$ for $q \geq 3g-2$, and thus

$$E_1^{pq} = 0 \qquad (q \geq 3g-2). \tag{2}$$

Since Ω^q is a coherent sheaf on $\mathcal{M}_g(\ell)$, our assumption that $\mathrm{CohCD}(\mathcal{M}_g(\ell)) < g-2$ implies that

$$E_1^{pq} = \mathrm{H}^p(\mathcal{M}_g(\ell);\Omega^q) = 0 \qquad (p \geq g-2). \tag{3}$$

From (2) and (3), we deduce that $E_1^{pq} = 0$ whenever $p+q = 4g-5$. This implies that $\mathrm{H}^{4g-5}(\mathcal{M}_g(\ell);\mathbb{C}) = 0$, contradicting Theorem A. The fact that we have equality for $2 \leq g \leq 5$ follows from the aforementioned theorem of Fontanari–Pascolutti [FoPa] asserting that Conjecture 1 holds for $2 \leq g \leq 5$.

Proof outline for Theorem A. Our proof of Theorem A has four steps.

(1) First, we use the fact that the mapping class group satisfies Bieri–Eckmann duality [Hare2] to translate the theorem into an assertion about the action of $\mathrm{Mod}_g(\ell)$ on the Steinberg module for the mapping class group, i.e. the unique nonzero homology group of the curve complex.

(2) Next, we study this action by constructing a novel surjective homomorphism from the Steinberg module for the mapping class group to a vector space $\mathrm{St}_{2g}^{\mathrm{ns}}(\mathbb{F}_p)$ that is a quotient of the Steinberg module $\mathrm{St}_{2g}(\mathbb{F}_p)$ for the finite group $\mathrm{SL}_{2g}(\mathbb{F}_p)$, i.e. the unique nonzero homology group of this finite group's Tits building.

It follows from the previous two steps that the dimension of $\mathrm{H}^{4g-5}(\mathrm{Mod}_g(\ell);\mathbb{Q})$ is at least $\dim_{\mathbb{Q}} \mathrm{St}_{2g}^{\mathrm{ns}}(\mathbb{F}_p)$. At this point, one might think that we have at least proved that $\mathrm{H}^{4g-5}(\mathrm{Mod}_g(\ell);\mathbb{Q}) \neq 0$. However, there is a problem – from its definition, it is not clear that $\mathrm{St}_{2g}^{\mathrm{ns}}(\mathbb{F}_p) \neq 0$. The next two steps analyze this vector space.

3. The third step is representation-theoretic. As a prelude to analyzing $\mathrm{St}_{2g}^{\mathrm{ns}}(\mathbb{F}_p)$, we show how to decompose the restriction of the $\mathrm{SL}_{2g}(\mathbb{F}_p)$-representation $\mathrm{St}_{2g}(\mathbb{F}_p)$ to the subgroup $\mathsf{Sp}_{2g}(\mathbb{F}_p)$, i.e. we construct a branching rule between these two different classical groups.

4. Finally, to use the third step to show that $\mathrm{St}^{\mathrm{ns}}_{2g}(\mathbb{F}_p)$ is nonzero and satisfies the bound in Theorem A, we apply classical theorems concerning the partition function and exponential generating functions (some of which go back to Euler).

Remark 6. One might expect that the Steinberg module for the finite group $\mathsf{Sp}_{2g}(\mathbb{F}_p)$ would appear here rather than the Steinberg module for $\mathrm{SL}_{2g}(\mathbb{F}_p)$. We tried to do this initially, but were unsuccessful. Indeed, every map from the Steinberg module for the mapping class group to the Steinberg module for $\mathsf{Sp}_{2g}(\mathbb{F}_p)$ that we were able to concoct ended up being the zero map. What is more, our lower bound on the dimension of $\mathrm{H}^{4g-5}(\mathrm{Mod}_g(\ell);\mathbb{Q})$ is significantly larger than the dimension of the Steinberg module for $\mathsf{Sp}_{2g}(\mathbb{F}_p)$, namely p^{g^2}, so it seems unlikely that one could use the Steinberg module for $\mathsf{Sp}_{2g}(\mathbb{F}_p)$ to prove a theorem as strong as Theorem A.

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Mapping class groups and plane curves

Nick Salter

1. Algebro-geometric preliminaries

The central notion of this talk is that of a *smooth complex projective plane curve of degree d*. A complex projective plane curve of degree d is the vanishing locus $V(f) \subset \mathbb{C}P^2$ of some homogeneous polynomial f of degree d in three variables. Such a $V(f)$ is *smooth* if 0 is a regular value of f when viewed as a mapping $\mathbb{C}^3 \to \mathbb{C}$; in such a case, $V(f) \cong \Sigma_g$ is diffeomorphic to some closed topological surface of genus g. The genus is related to the degree via the formula $g = \binom{d-1}{2}$. In particular, most genera of Riemann surfaces do not contain any plane curves at all.

Plane curves of a fixed degree d are in correspondence with the projectivization of the vector space of homogeneous polynomials of degree d; this will be written as $\mathbb{C}P^N$. The *discriminant polynomial* $\mathcal{D}_d$ is a homogeneous polynomial of the coefficients of an element $f \in \mathbb{C}P^N$, and $\mathcal{D}_d(f) \neq 0$ if and only if $V(f)$ is smooth. Consequently, we define

$$\mathcal{P}_d = \mathbb{C}P^N \setminus V(\mathcal{D}_d)$$

to be the *parameter space of smooth plane curves of degree d*. By construction it is a hypersurface complement.

$\mathcal{P}_d$ is the base space for the "universal smooth plane curve". This is a topological fiber bundle

$$\Sigma_g \to \mathfrak{X}_d \to \mathcal{P}_d \tag{1}$$

with fiber Σ_g, where the fiber over $f \in \mathcal{P}_d$ is the associated plane curve $V(f)$.

2. Monodromy and mapping class groups

Let $\Sigma_g \to E \to B$ be a Σ_g bundle ($g \geq 2$). The *monodromy representation* is a homomorphism

$$\rho : \pi_1(B) \to \mathrm{Mod}_g,$$

where Mod_g denotes the mapping class group of Σ_g. We define

$$\rho_d : \pi_1(\mathcal{P}_d) \to \mathrm{Mod}_g$$

to be the monodromy representation associated to the universal plane curve bundle (1), and we define $\Gamma_d = \mathrm{im}(\rho_d)$.

We can now state the fundamental question of the talk. *What is Γ_d? When is it finite-index? When does $\Gamma_d = Mod_g$?*

3. An approximate answer

A fundamental tool in studying the mapping class group is the *symplectic representation*. This is encapsulated in the following short exact sequence:

$$1 \longrightarrow \mathcal{I}_g \longrightarrow \mathrm{Mod}_g \xrightarrow{\Psi} \mathsf{Sp}(2g, \mathbb{Z}) \longrightarrow 1 \tag{2}$$

Here, $\Psi : \mathrm{Mod}_g \to \mathsf{Sp}(2g, \mathbb{Z})$ is the *symplectic representation* given by

$$\Psi(f) = f_* \in \mathsf{Aut}(H_1(\Sigma_g, \mathbb{Z})).$$

The kernel $\mathcal{I}_g$ is known as the *Torelli group*.

Beauville determined an "approximation" to Γ_d; namely, he computed $\Psi(\Gamma_d)$.

Theorem 3 (Beauville). For $d \geq 4$, the group $\Psi(\Gamma_d)$ is given by

$$\Psi(\Gamma_d) = \begin{cases} \mathsf{Sp}(2g, \mathbb{Z}) & (d \text{ even}) \\ \mathsf{Sp}(2g, \mathbb{Z})[q] & (d \text{ odd}) \end{cases}$$

Here, $\mathsf{Sp}(2g, \mathbb{Z})[q] \leq \mathsf{Sp}(2g, \mathbb{Z})$ is finite-index. It is defined as the stabilizer of a "spin structure". These will be discussed further in Section 5 below.

The question remains - is there an equality

$$\Gamma_d = \Psi^{-1}(\Psi(\Gamma_d))?$$

This is answered by the following theorem. The first statement is folklore, while the second follows from work of Sipe.

Theorem 4. There is an invariant ϕ_d known as a "$(d-3)$-spin structure" that is preserved by Γ_d. Letting $\mathrm{Mod}_g[\phi_d]$ denote the stabilizer (a finite-index subgroup of Mod_g), it follows that there is a containment

$$\Gamma_d \subseteq \mathrm{Mod}_g[\phi_d].$$

For $d \geq 6$, the containment $\mathrm{Mod}_g[\phi_d] \subsetneqq \Psi^{-1}(\Psi(\Gamma_d))$ is *strict*.

This allows us to refine the central question of the talk: *Is the containment* $\Gamma_d \subseteq \mathit{Mod}_g[\phi_d]$ *an equality?*

4. Low-degree cases

The answer to this question is well-understood for very small d. The case $d = 3$ corresponds to the case of elliptic curves. Every elliptic curve is planar; it follows that there is an equality $\Gamma_3 = \mathrm{Mod}(\Sigma_1) = \mathrm{SL}(2, \mathbb{Z})$. The case $d = 4$ corresponds to genus 3 curves. If X is a non-hyperelliptic curve of genus 3, then the canonical embedding is planar. This implies the equality $\Gamma_4 = \mathrm{Mod}_3$.

For $d = 5$, the corresponding genus is $g = 6$. In this case, the generic curve of genus 6 is not planar. Nevertheless, the monodromy Γ_5 is as large as possible given the constraint of Theorem 4.

Theorem 5 (S-). There is an equality

$$\Gamma_5 = \mathrm{Mod}_6[\phi_5].$$

Here ϕ_5 is a (classical) spin structure.

We conjecture that the monodromy groups should continue to be as large as possible.

Conjecture 6. For all $d \geq 4$, there is an equality

$$\Gamma_d = \mathrm{Mod}_g[\phi_d]$$

5. (Higher) spin structures

Let X be a Riemann surface with associated unit tangent bundle T^1X. Topologically, a *spin structure* on X is a 2-sheeted cover $\widetilde{T^1X} \to T^1X$ such that the restriction to the S^1-fiber of T^1X is the connected covering of S^1. This is encoded by the data of a cohomology class $\phi \in H^1(T^1X; \mathbb{Z}/2\mathbb{Z})$. Algebro-geometrically, a spin structure, also known as a *theta characteristic*, is a line bundle L satisfying $L^{\otimes 2} \cong K_X$, where K_X denotes the canonical bundle. The connection between the two definitions arises from the fact that the underlying real bundle associated to K_X is the cotangent bundle.

For $n > 2$, an n-spin structure is either an n-sheeted covering of T^1X, or else a cohomology class $\phi \in H^1(T^1X; \mathbb{Z}/n\mathbb{Z})$, or else a line bundle L satisfying $L^{\otimes n} \cong K_X$. As before, all of these notions are equivalent.

The $(d-3)$-spin structure of Theorem 4 is constructed from the adjunction formula: the line bundle $\mathcal{O}(d-3)$ on $\mathbb{C}P^2$ restricts to K_X for any X a smooth plane curve of degree d. Then the corresponding spin structure on a plane curve X is the restriction of the line bundle $\mathcal{O}(1)$.

6. On the proof of Theorem 5

The starting point for the proof of Theorem 5 is the following theorem due to Lönne:

Theorem 7 (Lönne). There is an explicit presentation for the group $\pi_1(\mathcal{P}_d)$. Under the monodromy representation ρ_d, the generators are sent to Dehn twists $T_c \in \mathrm{Mod}_g$ about nonseparating simple closed curves.

Theorem 5 is proved by explicitly determining the configuration of the curves c arising from Lönne's presentation. This is accomplished by understanding how the relations in Lönne's presentation constrain the configuration. For instance, many of Lönne's generators commute; this forces the associated curves c and d to be disjoint. Others satisfy a braid relation $aba = bab$; this forces the associated curves c and d to intersect exactly once.

The crux of the argument is to exhibit the Torelli group $\mathcal{I}_6$ as a subgroup of Γ_5. This is in fact sufficient, in light of Theorem 3 and the sequence (2). To do this, we make use of Johnson's finite generating set for $\mathcal{I}_g$. Johnson's generating set is quite large (it contains 4470 elements for $g = 6$), but it is highly structured and we are able to reduce to only 8 cases arranged in 2 families. The proof is facilitated by the development of a new relation among mapping classes that we call the "genus-g star relation".

Number theoretic aspects of surface homeomorphisms

Balázs Strenner

Let S be a finite type surface. A surface homeomorphism $\psi : S \to S$ is *pseudo-Anosov* if there are $\lambda > 1$ and transverse singular measured foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ such that $\psi(\mathcal{F}^u) = \lambda\mathcal{F}^u$ and $\psi(\mathcal{F}^s) = \lambda^{-1}\mathcal{F}^s$. The number λ has a purely topological interpretation: $\log(\lambda)$ is the topological entropy of ψ. On the other hand, λ is an algebraic integer so it also has a number theoretic aspect to it. The goal of the talk is to explore connections between number theory and topology in the context of surface homeomorphisms.

1. Five problems relating number theory and topology

1.1. **Characterizing pseudo-Anosov stretch factors.** The number λ is an algebraic unit. It is also a bi-Perron number, meaning that all Galois conjugates of λ (except λ^{-1} if it is a Galois conjugate) lie in the open annulus $\lambda^{-1} < |z| < \lambda$. The folklore conjecture (originally suggested by Fried in a slightly different form) is that conversely, every bi-Perron algebraic unit is a pseudo-Anosov stretch factor.

1.2. **Algebraic degrees.** Denote by $\deg(\lambda)$ the algebraic degree of λ (the degree of its minimal polynomial). Thurston [1] showed that on the closed orientable surface S_g of genus g, we have $2 \le \deg(\lambda) \le 6g-6$. He claimed that his construction of pseudo-Anosov maps can be used to prove that the upper bound is sharp, but he did not give a proof, nor did anyone else since then. Suprisingly it is not true that all degrees between 2 and $6g-6$ occur: Long [7] showed that if $\deg(\lambda)$ is odd, then $\deg(\lambda) \le 3g-3$. The following theorem shows that these are the only constraints on the possible degrees, and in particular proves Thurston's claim.

Theorem 1 (S.). [3] The possible algebraic degrees of S_g are the even numbers in $[2, 6g-6]$ and the odd numbers in $[3, 3g-3]$.

In the second part of the talk, we sketch of the proof of this theorem.

1.3. **Degrees and covers.** Franks and Rykken [6] proved that if $\mathcal{F}^u$ and $\mathcal{F}^s$ are orientable, then the following are equivalent:

(1) $\deg(\lambda) = 2$,
(2) ψ is a lift of an Anosov map of the torus by a branched covering.

Farb conjectured that for all $d \ge 2$ there is a positive integer $h(d)$ such that every pseudo-Anosov map with a degree d stretch factor is a lift of a pseudo-Anosov map on a genus at most $h(d)$ surface.

1.4. **Surface bundles.** The mapping torus M_ψ of ψ is a hyperbolic 3-manifold.

Question 4. How are the number theoretic properties of λ reflected in the topology and geometry of M_ψ?

1.5. **Penner's construction.** Let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_m\}$ be a pair of multicurves on a surface S. Suppose that A and B are filling, that is, A and B are in minimal position and the complement of $A \cup B$ is a union of disks and once punctured disks. Penner [5] showed that any product of positive Dehn twists about a_j and negative Dehn twists about b_k is pseudo-Anosov provided that all $n + m$ Dehn twists appear in the product at least once.

One may wonder which pseudo-Anosov maps arise from the construction. The construction is very general: there is a large freedom in choosing the pair of multicurves and also the product of Dehn twists. In fact, Penner conjectured the following.

Conjecture 2. Every pseudo-Anosov mapping class has a power that arises from Penner's construction.

It turns out that the conjecture is false, and this fact has number theoretic reasons.

Theorem 3 (Shin-S.). [2] If λ has a Galois conjugate on the unit circle, then it does not arise from Penner's construction.

The fact that the Galois conjugates of Penner stretch factors cannot lie on the unit circle motivated us to study them further.

Theorem 4 (S.). [4] If $g \geq 2$, then there are multicurves A and B such that the Galois conjugates of stretch factors arising from Penner's construction using the pair (A, B) are dense in $\mathbb{C}$.

2. Sketch of the proof of Theorem 1

One of the difficulties in proving such a result is the lack of irreducibility criteria that work for minimal polynomials of pseudo-Anosov stretch factors. We get around this by using an asymptotic irreducibility criterion that works for a sequence of polynomials, not for a single polynomial. The proof is short and completely elementary.

Lemma 2 (S.). Let $p_n(x) \in \mathbb{Z}[x]$ be a sequence of monic degree d polynomials whose constant coefficients are ± 1. Suppose there is a sequence of real numbers $\lambda_n \to \infty$ such that $p_n(\lambda_n) = 0$ for all n, and suppose that $p_n(1) \neq 0$ for all n. If

$$\lim_{n \to \infty} \frac{p_n(x)}{x - \lambda_n} = x(x-1)^{d-2},$$

then $p_n(x)$ is irreducible for all but finitely many n.

In the rest of the talk, we sketch the ideas about how to construct such sequences of polynomials using Penner's construction.

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Surface bundles with fiberwise group action

Bena Tshishiku

For a surface S and a finite subgroup $G < \text{Diff}(S)$, we consider S bundles $X \to B$ with structure group the centralizer $C_{\text{Diff}(S)}(G)$. We call these (S, G) *bundles.* For such a bundle G acts on X by bundle maps covering the identity. Important examples are the Atiyah–Kodaira bundles.

Basic invariants for an (S, G) bundle $X \to B$ are the Chern classes of the associated Hodge eigenbundles: the Hodge bundle $H^1(S; \mathbb{R}) \to E \to B$ (a complex vector bundle after choosing a fiberwise almost complex structure on $X \to B$) admits a G action and decomposes into eigenbundles $E \simeq \bigoplus_q E_q$, corresponding to decomposition of $H_1(S; \mathbb{R})$ as a G-representation. The Chern classes $c_i(E_q) \in H^{2i}(B)$ can be expressed in terms of the MMM and Euler classes of $X \to B$ by the G-index theorem for families for the signature operator. In this talk we discussed some consequences of this fact. We list two below (see [1] for details).

(1) For an (S, G) bundle $X \to \Sigma$ over a surface Σ, the integers $\langle c_1(E_q), [\Sigma] \rangle$ depend only on the G-bordism class of X.
(2) Using the Atiyah–Kodaira construction of (S, G) bundles $X \to \Sigma$, one can produce surface group representations $\alpha : \pi_1(\Sigma) \to \prod_j \text{SU}(p_j, q_j)$ with image contained in a lattice, whose Toledo number is proportional to $\text{sig}(X)$ and in particular is nonzero.

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Problem session

Notes taken by Johannes Ebert

1. Proposed by Christopher Leininger

Fact. For each $R > 0$, there exists an injective homomorphism $\rho_R : \pi_1(\Sigma_2) \to \mathrm{Mod}_5$ such that the associated Σ_5-bundle $E_R \to \Sigma_2$ has the following property: for each closed curve $\gamma : S^1 \to \Sigma_2$, the total space of the pull-back bundle $\gamma^* E_R \to S^1$ has simplicial volume $\|\gamma^* E_R\| \geq R$. The same is true for larger genera. Reference: [3, Corollary 1.3].

Question 5. Does $\lim_{R\to\infty} \|E_R\| = \infty$ hold?

Remarks. Suggested strategy: Consider the lift to the universal cover

$$\begin{array}{ccc} \tilde{E} & \longrightarrow & \mathcal{T}_{5,1} \\ \downarrow & & \downarrow \\ \tilde{\Sigma}_2 & \longrightarrow & \mathcal{T}_5 \end{array}$$

Giving the Teichmüller spaces the Weil-Petersson metrics, the map along the bottom is a quasi-isometric embedding. Try to straighten simplicies and apply Gromov's strategy for hyperbolic manifolds.

2. Proposed by Jonathan Hillman

The following questions are (speculative) attempts to put the coherence of 3-manifold groups into a wider context.

Setting. Let E be the total space of a bundle with base B and fibre F closed surfaces, and let $\theta : \pi_1(B) \to \mathrm{Mod}(F)$ be the associated action.

Question 6.
(1) Is $\pi = \pi_1(E)$ coherent? In other words, are finitely generated subgroups of π finitely presented?
(2) If $G \subset \pi$ is a finitely presented subgroup, does there exist a finite $K(G,1)$-space?
(3) More generally: if G is an $FP_{[n/2]}$-subgroup of a PD_n-group is it FP?

Remark 7. Recall that a group G is of type FP_n, $n \in \mathrm{N}$, if there is a partial projective resolution

$$P_n \to P_{n-1} \to \ldots \to P_0 \to \mathbb{Z}$$

of $\mathbb{Z}$ by finitely generated $\mathbb{Z}[G]$-modules. A group G is a PD_n-group if it satisfies Poincaré duality in dimension n [11].

It is easy to see that if $\chi(F) = 0$ then π is coherent, so we may assume that $\chi(F) < 0$. If the action θ is *not* injective and $\chi(E) > 0$ then π contains a copy of $F(2) \times F(2)$, and so cannot be coherent.

One could extend (2) to higher dimensions by asking whether a finitely presentable $FP_{[n/2]}$ subgroup G of the fundamental group of an aspherical n-manifold has a finite $K(G,1)$-complex. Part (3) is a homological version of this extension.

Parts (2) and (3) hold if G is a normal subgroup with quotient a PD_r-group, for some $r < n$, by an argument using duality and Shapiro's Lemma. The condition that G be $FP_{[n/2]}$ probably cannot be weakened further. (This is clear when $n = 4$, by the remark above on (1).)

3. Proposed by Nick Salter

Setting. Observe that the Kodaira fibrations obtained by the Kodaira-Atiyah construction fibre in two different ways. Also, there are known examples of 4-manifolds E_n which admit $\geq n$ fiberings over surfaces [19]. These examples are known to be not algebraic, hence not Kodaira fibrations.

Question 7. Are there complex compact surfaces which admit three or more Kodaira fiberings over surfaces?

Remark 8. This question has also been asked by Catanese [1, Question 10]. One can try to use "resonance" in $H^*(E;\mathbb{R})$, compare with [18, Lemmas 7.8 - 7.10]. To show that every Kodaira fibration admits at most two fiberings, it is sufficient to prove that for any E that admits two Kodaira fiberings $p_i : E \to B_i$, one has $H^1(E;\mathbb{R}) \cong p_1^* H^1(B_1;\mathbb{R}) \oplus p_2^* H^1(B_2;\mathbb{R})$.

4. Proposed by Martin Möller

Consider a fibred complex surface $f : X \to B$, but with singular fibres, allowing nodal singularities. Assume that X is Kähler. The *irregularity* of E is $q := H^1(X;\mathcal{O}) = \frac{1}{2} b_1(X)$. The *relative irregularity* $q_f := q - g(B)$ is non-negative and bounded above by $g(F)$. The upper bounded is attained precisely for isotrivial fibrations. For non-isotrivial fibrations, Xiao proved that $q_f \leq \frac{5g(F)+1}{6}$ [21] and conjectured that $q_f \leq \frac{g(F)+1}{2}$. Pirola [17] gave examples with $q_f = 3$ and $g(F) = 4$.

Problem 1. *Can one give better (or asymptotically sharp) bounds for large values of $g(F)$? What can be said about this question if f is moreover a Kodaira fibration.*

5. Proposed by Benson Farb

Problem 2. *Does there exist a non-flat surface bundle $\Sigma_g \to E \to \Sigma_h$ over a surface? In other words, is there an example such that the lifting problem*

$$\begin{array}{ccc} & & \mathit{Diff}^+(F_g) \\ & \nearrow & \downarrow \\ \pi_1(\Sigma_h) & \longrightarrow & \mathit{Mod}_g \end{array}$$

(in the category of groups!) cannot be solved? The Atiyah-Kodaira constructions might give examples; the hard part is to provide obstructions. The same question can be asked for $\mathrm{Homeo}^+(F_g)$ *instead of* $\mathit{Diff}^+(F_g)$.

Remark 9 (Looijenga). The case $h = 1$ might be interesting to look at. Here the question comes down to: do two commuting mapping classes lift to two commuting diffeomorphisms?

6. Proposed by Benson Farb

A classical result of Harer and Zagier [8] computes the Euler characteristic of Mod_g. It is asymptotically equal to the orbifold Euler characteristic, which by the main result of the same paper is

$$\frac{B_{2g}}{4g(g-1)} = \frac{\zeta(1-2g)}{2-2g} \sim (-1)^{g+1}\frac{(2g-1)!}{2^{2g-2}\pi^{2g}(g-1)}.$$

Problem 3 (The dark matter problem 1)**.** *For $g \geq 6$, find elements in the rational cohomology ring $H^*(Mod_g;\mathbb{Q})$ which are not polynomials in the κ_i-classes.*

Problem 4 (The dark matter problem 2)**.** *For even g, find odd degree classes in $H^*(Mod_g;\mathbb{Q})$. These must exist by* [8].

Remark 10 (By E. Looijenga). Since the kappa class κ_d is in degree $2d$ and of Hodge bidegree (d,d), any cohomology class of odd degree or of even degree $2d$, but not of Hodge bidegree (d,d) is not the subalgebra generated by the kappa classes.

For example, whenever you have a nonzero holomorphic p-form α on $\mathfrak{M}_g$ (in the orbifold sense) which has poles of order at most one along the Deligne-Mumford boundary, then mixed Hodge theory tells us that it defines a nonzero element of $H^p(\mathfrak{M}_g,\mathbb{C})$; to be precise, when a k-fold iterated residue of α along a k-fold boundary crossing is nonzero and k is maximal for that property, then it will map to a nonzero element of $F^pH^p(\mathfrak{M}_g,\mathbb{C}) \cap W_{p+k}H^p(\mathfrak{M}_g,\mathbb{C})$. Such an example occurs for $g=3$ with $p=k=6$ [13], and it is likely that there are many more such examples in higher genus.

The other known examples identify in the cohomology of $\mathfrak{M}_{g,n}$ a Tate twist of the *Ramanujan motive*. This means the following: the weight 12 cusp form Δ for $\mathrm{SL}(2,\mathbb{Z})$ can be understood as spanning a one-dimensional piece H_Δ of bidegree $(11,0)$ of a polarizable Hodge structure ($H_\Delta \oplus \overline{H}_\Delta$ is the complexification of a symplectic vector space defined over $\mathbb{Q}$). This Hodge structure appears in the cohomology of the of $\overline{\mathfrak{M}}_{1,11}$. Pikaart [16] had proved that for g large, $H^*(\overline{\mathfrak{M}}_{g,n})$ contains up to a Tate twist a copy of H_Δ. This was used by Graber-Pandharipande [5] to prove that this is also true for some $\mathfrak{M}_g$. Van Zelm recently posted a paper [20] showing that this is in fact holds for any $\mathfrak{M}_{g,n}$ when g is large enough.

7. Proposed by Benson Farb

Problem 5. *Find asymptotics for $\sharp\mathfrak{M}_g(\mathbb{F}_q)$, the set of $\mathbb{F}_q$-valued points of the coarse moduli space, using the Grothendieck-Lefschetz trace formula. More specifically: is*

$$0 < \lim_{g\to\infty}\frac{\sharp\mathfrak{M}_g(\mathbb{F}_q)}{q^{3g-3}} < \infty?$$

8. Proposed by Dawei Chen

Question 8. Does $\mathfrak{M}_4$ contain a complete complex 2-dimensional subvariety?

Remarks. There are complete complex curves in $\mathfrak{M}_g$, $g \geq 3$, as can be shown using the Satake compactification.

A Theorem of Diaz [4] states that any complete complex subvariety of $\mathfrak{M}_g$ has dimension at most $g-2$ (this also follows from vanishing results in cohomology proven by Ionel [10], Looijenga [14], Graber-Vakil [6]; also reproven by Grushevsky-Krichever [7])

The papers [23] and [22] are also relevant to this problem.

9. Proposed by Samuel Grushevsky

Question 9. What is the maximal dimension of a complete algebraic subvariety of $\mathfrak{A}_4$ (the moduli space of principally polarized abelian fourfolds), over the field of complex numbers?

Remark 11. From the existence of the Satake compactification of $\mathcal{A}_g$, which is a projective variety with boundary of codimension g, it follows that $\mathcal{A}_g$ has a complete subvariety of *dimension* $g-1$, for any g. It is also known that over $\mathbb{F}_p$, the locus of abelian varieties with no non-zero p-torsion point is a complete *codimension* g (and thus dimension $g(g-1)/2$) subvariety of $\mathcal{A}_g$) [15]. Over the field of complex numbers, the theorem of Keel and Sadun [12] states that there does not exist a complete complex subvariety of $\mathcal{A}_g$ of *codimension* g. Thus over the complex numbers, $\mathcal{A}_3$ contains a complete surface, but not a complete threefold, while the maximal dimension of a complete subvariety of $\mathcal{A}_4$ must be ≥ 3 and ≤ 5.

10. Proposed by Jarek Kedra

Question 10. Are the κ-classes bounded cohomology classes? It is known that κ_{2i-1} is bounded, since it is induced from $\mathrm{Sp}_{2g}(\mathbb{Z})$.

11. Proposed by Grigori Avramidi

Let $\Gamma \subset \mathrm{Mod}_g$ be torsionfree and of finite index.

Question 11. Is the cohomology group $H^{2g-1}(\Gamma; H_{2g-2}(\mathfrak{C}_g))$ infinitely generated?

This is motivated by an obstruction problem.

12. Proposed by Alan Reid

Among the finitely generated subgroups of $\mathrm{SL}_3(\mathbb{Z})$, there are plenty examples of solvable and free groups. There are surface groups of every genus, even Zariski dense ones. Some few 3-manifold groups are also contained in $\mathrm{SL}_3(\mathbb{Z})$ (e.g. the Heisenberg group).

Question 12. Does there exist a finite volume hyperbolic M^3 and an injective $\pi_1(M) \to \mathrm{SL}_3(\mathbb{Z})$?

If the answer is yes, then $\mathrm{SL}_3(\mathbb{Z})$ does not have the finitely generated intersection property. Similar question (Serre): Is $\mathrm{SL}_3(\mathbb{Z})$ coherent? Or does $\mathrm{SL}_3(\mathbb{Z})$ contain $(\mathbb{Z}\times\mathbb{Z})*\mathbb{Z}$?

One might replace $\mathrm{SL}_3(\mathbb{Z})$ by groups such as $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{2}])$,

13. Proposed by Andrew Putman

It is known that the virtual cohomological dimension of Mod_g is equal to $4g-5$ [9]

Conjecture 1 ([2]). $H^{(4g-5)-i}(\mathrm{Mod}_g;\mathbb{Q}) = 0$ for $g >> i$.

14. Proposed by Mustafa Korkmaz

Question 13. Consider the forgetful map $\mathrm{Mod}_{g,1} \to \mathrm{Mod}_g$. Let us by given a relation $t_{a_1}\cdots t_{a_n} = 1$ of Dehn twists in Mod_g. Can you move the curves away from the puncture and obtain curves $\tilde{a_i}$ in the punctured surface such that $t_{\tilde{a_1}}\cdots t_{\tilde{a_n}} = 1 \in \mathrm{Mod}_{g,1}$?

15. Proposed by Eduard Looijenga after the workshop

Question 14. Does there exist a function $f : \mathfrak{M}_g \to \mathbb{R}$ which is C^∞ (in an orbifold sense), proper, bounded below and $(g-2)$-convex (this means that $\sqrt{-1}\partial\bar{\partial}f$ defines on each tangent space of $\mathfrak{M}_g$ a Hermitian form which is not negative definite on any subspace of dimension $> g-2$).

Such a function can be approximated by a Morse function with Morse indices $\leq \dim_{\mathbb{C}} \mathfrak{M}_g + (g-2) = 4g-5$ and so this would recover Harer's theorem [9] which says that $\mathfrak{M}_g$ has the homotopy type of a CW-complex of dimension $\leq 4g-5$ (but we would get in fact the stronger assertion that every constructible sheaf on $\mathfrak{M}_g$ has no cohomology in degree $> 4g-5$). At the same time this would imply that for every coherent analytic sheaf $\mathcal{F}$ on $\mathfrak{M}_g$, $H^k(\mathfrak{M}_g,\mathcal{F}) = 0$ for $k > g-2$. This implies the theorem of Diaz (if $i : Y \subset \mathfrak{M}_g$ is a complete subvariety of dimension d, then Y supports a coherent sheaf $\mathcal{F}$ on $\mathfrak{M}_g$ with $H^d(\mathfrak{M}_g,\mathcal{F}) \neq 0$ and so $d \leq g-2$), but is in fact considerable stronger. The theorem Andy Putman discussed in his talk shows that the bound $g-2$ is sharp.

It would be just as good if you can prove the existence of a function $f_1 : \mathfrak{M}_{g,1} \to \mathbb{R}$ which is C^∞, proper, bounded below and $(g-1)$-convex, for this will have all the consequences mentioned above.

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