Mathematisches Forschungsinstitut Oberwolfach

Mini-Workshop: Surreal Numbers, Surreal Analysis, Hahn Fields and Derivations

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ABSTRACT. New striking analogies between H. Hahn's fields of generalised series with real coefficients, G. H. Hardy's field of germs of real valued functions, and J. H. Conway's field **No** of surreal numbers, have been lately discovered and exploited. The aim of the workshop was to bring quickly together experts and young researchers, to articulate and investigate current key questions and conjectures regarding these fields, and to explore emerging applications of this recent discovery.

Introduction by the Organisers

The field **No** of surreal numbers is a real closed field which simultaneously contains the real numbers and the ordinal numbers. Since its discovery, it has ever been the object of intense research. It was established by various authors (e.g. Alling, Costin, v. d. Dries, Ehrlich, Gonshor, Kruskal, to cite a few) that **No** is a *universal domain for real algebra* (in the sense that every real closed field whose domain is a set can be embedded in **No**), that it admits an exponential function, and an interpretation of the restricted real analytic functions, making it into a model of the elementary theory of the field of real numbers endowed with the exponential function and all the restricted real analytic functions.

In the last decade, it has been conjectured that **No** is a *universal domain for real differential algebra*. To this end, an immediate goal is to equip **No** with a derivation compatible with the exponential and with its natural structure as a

Hahn field. Moreover such a derivation should formally behave like the natural derivation on the germs at infinity of univariate real valued functions belonging to a *Hardy field*. A related conjecture is that **No** can also be viewed as a *universal domain for generalized series fields* equipped with an exponential function, such as the *logarithmic-exponential series* of v. d. Dries, Macintyre and Marker, the *exponential-logarithmic series* of F.- V. Kuhlmann and S. Kuhlmann and the field of *transseries* of v. d. Hoeven. This lead to the explicit conjecture by S. Kuhlmann and Matusinski (2011 & 2012) that **No** is a field of exponential-logarithmic transseries and can be equipped with a *Hardy-type series derivation*. Progress in this direction was achieved by Berarducci and Mantova (2015 & 2016), showing that the surreal numbers have a natural transseries structure and admit a compatible Hardy-type derivation.

The purpose of this mini-workshop was to bring together quickly a small team of senior and junior mathematicians, who felt the time was ripe to mine the theory of surreal numbers for insights into topics such as real algebra, asymptotic analysis, model theory, set theory and the foundations of mathematics. We concentrated our efforts mainly in three research directions: **(A)** study derivation, integration and composition operators on those fields, **(B)** investigate their algebraic and arithmetic properties, **(C)** explore their applications to computable analysis. The structure of the meeting was organised around these goals. Leading experts gave keynote lectures (tutorials/surveys) on the first and second day, in order to promptly introduce the background, and bring in the cutting- edge research. The special session organised by Mantova and Matusinki was held already on the afternoon of the second day, thereby intensifying the impact. The last day was dedicated to the talks of the doctoral students, who in the meanwhile had time to absorb the lively presentation and discussion style, and make it into their own. We now describe in detail the course of events.

1. Special lectures (tutorials) by distinguished experts

A four parts introductory tutorial on the field of surreal numbers (Matusinski), surreal exponentiation (Mantova), initial embeddings in the surreals, in particular the work of Ehrlich on that topic (Fornasiero) and generalised series as germs of surreal functions (Berarducci) occupied most of the day on Monday. This was followed on Tuesday morning by a two-part introductory tutorial on transseries (Point) and κ-bounded series (S. Kuhlmann). Wednesday morning was dedicated to further survey talks on quasi-analytic classes (Speissegger), closed ordered differential fields (Tressl) and integer parts of real closed fields (S. Kuhlmann).

2. Special session on derivations induced by right shifts

One of the key steps in the recent paper by Berarducci and Mantova in order to construct a derivation on **No**, consists in identifying the class of log-atomic monomials and defining suitable derivatives of these numbers, hence addressing the above conjecture raised by S. Kuhlmann and Matusinski. The latter authors (2011) gave a criterion relating pre-logarithms and derivations for formal series, via

the action of a right-shift automorphism on the chain of fundamental monomials. The aim of the session was to clarify the connection between the two approaches. This special session lasted half of a day. The first hour and a half were dedicated to two informal talks about right-shift automorphisms and surreal derivations. A two hour problem session followed (see paragraph on open problems below).

3. Research talks and talks of doctoral students

L'Innocente and Mantova reported in two consecutive talks on their recent joint work related to the arithmetic and algebraic properties of **Oz**. Berarducci presented the work of Costin-Ehrlich-Friedman (who did not attend the meeting) on integration, and Kaiser presented a slides talk on Lebesgue measure theory over the surreals. All the graduate students gave exciting talks. Galeotti explained the interest of surreals in descriptive set theory and computable analysis. Lehericy presented his work on asymptotic couples, and Krapp on o-minimal exponential fields. Kaplan lectured on his joint work with Ehrlich and Müller on quasi-ordered fields.

4. Open problems and questions:

Question 1. The field $\mathbb{T}$ of logarithmic exponential series has a natural derivation ∂ and an exponential map exp. It is known that $(\mathbb{T}, \exp)$ is an elementary extension of $(\mathbb{R}, \exp)$ [36] and $(\mathbb{T}, \partial)$ is an elementary substructure of $(\mathbf{No}, \partial)$ with the derivation of [10].

   (1) Is $(\mathbb{T}, \partial, \exp) \preceq (\mathbf{No}, \partial, \exp)$?
   (2) Every Hardy field can be embedded in $(\mathbf{No}, \partial)$ as a differential field [8]. Now, let $(K, \partial, \exp)$ be a Hardy field closed under exp. Can we embed it in $(\mathbf{No}, \partial, \exp)$?

Question 2. Describe the field operations of **No** using the sign sequence representation.

Question 3. Let $i = \sqrt{-1}$. Is there a good way to introduce sin and cos on **No** and an exponential map on $\mathbf{No}[i]$? Is there a surreal version of the p-adic numbers?

Question 4. The field $\mathbb{R}\langle\langle\omega\rangle\rangle$ of omega-series is the smallest subfield of the surreal numbers containing $\mathbb{R}(\omega)$ and closed under log, exp and sums of arbitrary summable sequences. In [11] it is shown that this field has a unique natural derivation and composition operator and contains, as differential subfields, the various variants of transseries fields (LE and EL-series). Each $f \in \mathbb{R}\langle\langle\omega\rangle\rangle$ (hence in particular any transseries) determines a function $\hat{f} : \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$ on positive infinite surreal numbers.

   (1) Does the structure $(\mathbf{No}, +, \times, \exp, \hat{f})_{f \in \mathbb{R}\langle\langle\omega\rangle\rangle}$ have good model theoretic properties? It may be conjectured that, restricting the various $\hat{f}$ to some infinite half-line $(a, +\infty)$, one obtains an o-minimal structure. A preliminary question is whether the intermediate value theorem holds for $\hat{f}$.

(2) Can we find a good composition operator on the whole of **No**?

(3) Are the EL-series elementary equivalent to $\mathbb{R}\langle\langle\omega\rangle\rangle$ as a differential field?

Question 5. Let κ be an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$ and $\mathbb{R}_\kappa$ be the Cauchy completion of **No**$_\kappa$. As shown in [86], one can code elements of $\mathbb{R}_\kappa$ by binary sequences of length κ and define a notion of computability over $\mathbb{R}_\kappa$ using Turing machines running for κ many steps. Let $f : \mathbb{R}_\kappa \to \mathbb{R}_\kappa$ be κ-computable (hence continuous). Does f satisfy the intermediate value theorem?

Question 6. Consider a field of κ-bounded generalized power series $\mathbb{R}((G))_\kappa$ as in [77]. Is there a Kaplansky differential embedding theorem for $\mathbb{R}((G))_\kappa$?

Question 7. Let $\kappa_x \in$ **No** be the κ-number indexed by $x \in$ **No** and let $\lambda_x \in$ **No** be the log-atomic number indexed by $x \in$ **No**. For every ordinal α we know that $\lambda_{\omega\alpha} = \kappa_\alpha$ (see Cor. 2.10 in [8]). What is the function f such that $\lambda_{f(x)} = \kappa_x$ for every $x \in$ **No**?

Question 8. The surreal numbers **No** contain a largest exponential subfield $\mathbb{R}\langle\langle\mathbb{L}\rangle\rangle$ satisfying axiom ELT4 of [75]. Are there distinct surreal derivations on **No** with the same restriction to $\mathbb{R}\langle\langle\mathbb{L}\rangle\rangle$? See [10] for background.

Question 9. Let $\kappa > \omega$ be a regular cardinal. It is known that the Hahn fields $\mathbb{R}((G))$ do not admit an exponential map, but for suitable G the κ-bounded subfields $\mathbb{R}((G))_\kappa$ do admit an exp [77]. Despite the fact that **No** is sometimes loosely described as a Hahn field $\mathbb{R}((G))$, the correct analogy is rather with the κ-bounded version $\mathbb{R}((G))_\kappa$. A general question is to explore these analogies and find ways of introducing derivations on κ-bounded series compatible with an exponential function. In particular one would like to find ∂ and exp such that $(\mathbb{R}((G))_\kappa, \exp, \partial)$ is isomorphic to a fragment of **No** with some surreal derivation (see [10]). One such fragment could be **No**$_\kappa$, the subfield of **No** consist of the surreal numbers of length $< \kappa$. Another suitable fragment could be the intersection of **No**$_\kappa$ with the field $\mathbb{R}\langle\langle\mathbb{L}\rangle\rangle$ defined in [10]) (the largest subfield of **No** satisfying axiom ELT4 of [75]).

Question 10. Given a real closed field K admitting an integer part which is a model of PA, does K admit a total exponential function? Kuhlmann proved that it admits a left-exponential function.

Question 11. Is every exponential group the value group of an exponential field? See the abstract of S. Kuhlmann on Integer parts.

Question 12. Does every RCF admit a normal integer part? See the abstract of S. Kuhlmann on Integer Parts.

Question 13. Every real closed field F admits a truncation closed embedding into a field of Hahn series and a corresponding truncation closed integer part $Z \subseteq F$ (which is a model of Open Induction). Is the class of all such truncation closed integer parts an elementary class? See the abstract of S. Kuhlmann on Integer Parts.

Question 14. In the work of L'Innocente and Mantova (see their abstracts), irreducibility in rings of generalized power series is studied with the help of a degree function deg. Can we extend deg to a field valuation?

Question 15. Can one obtain an integration theory for semialgebraic differential forms on semialgebraic **No**-submanifolds, including Stokes theorem? Can one extend the measure and integration theory on **No** beyond the semialgebraic and globally subanalytic category? See the abstract of T. Kaiser for references.

Question 16. Can one characterise the subset $\mathbb{Q}$ of **No** in terms of sign-sequences?

Question 17. Can one extend the simplicity order of **No** to functions? In which sense $+$ is the simplest function increasing in both arguments? Is exp the simplest homomorphism from $(\mathbf{No}, +)$ to $(\mathbf{No}^{>0}, \times)$ such that for all $n \in \mathbb{N}$ and positive infinite $x \in \mathbf{No}$ we have $\exp(x) > x^n$?

Question 18. Can one describe an integer part of **No** which is a model of true arithmetic? The existence of such an integer part should follow by the saturation properties of **No**, but can one construct such an integer part explicitly? (without the axiom of choice, say).

5. Conclusion and outlook

In the opinion of the organizers, this mini-workshop was a great success, because it achieved exactly the desired impact on the subject. We think that the following facts were fundamental to this very special event:

   (1) one third of the participants were doctoral students working in the general area of research, and eager to learn the background quickly and collect research problems,
   (2) we organized 20 talks as well as an afternoon dedicated to a special discussion and problems session,
   (3) since we had 20 talks distributed among 15 participants on the one hand all the graduate students were given opportunity to speak about their ongoing research, on the other hand we could dedicate more than one talk to some topics, thus allowing in-depth treatment,
   (4) we highly recommended black-board talks and encouraged audience to ask questions, speakers to include open problems, conjectures, or simply challenging exercises in their talks,
   (5) the special session was an opportunity to draw a research road map for the upcoming year on the topic of the mini-workshop.

All this produced a tremendous synergy; participants junior and senior were discussing intensely during the breaks and well into the evenings. Doctoral students collected new ideas and inspirations for their dissertations. Several collaborations were initiated during this short meeting. Our reporter Lorenzo Galeotti invested a lot of effort in keeping track and recording the open questions and exercises (see paragraph above). We intend to apply for a follow-up mini-workshop at MFO very soon, to keep the acquired research momentum and collaborations.

Mini-Workshop: Surreal Numbers, Surreal Analysis, Hahn Fields and Derivations

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Abstracts

Introduction to surreal numbers.
MICKAEL MATUSINSKI

This talk and the next one by V. Mantova, being the two firsts of the workshop, are aimed as an introduction to the whole week. We gather the fundamental definitions and results about surreal numbers, with a view toward the recent achievments that will be discussed. Our aim is also to provide the various participants with the common related basic notions and notations. The classical references are the seminal works [29] and [57].

Surreal numbers – denote their proper class by **No** – can be viewed in three different manners, each of which having been continuously studied, and having witnessed remarkable achievements that motivate this workshop:

(1) As numbers, they form a proper class containing simultaneously the set of real numbers and the proper class of ordinal numbers. Analysis on surreal numbers has been developped [1], with a still ongoing hot topic with partial answers: is there a well-defined integral on surreal numbers ? [53, 30];

(2) **No** is a real closed field extending the field of real numbers and the semiring of ordinal numbers (for the so-called natural operations). Also, Conway defined the omnific integers which form an integer part of **No** in the sense of non Archimedean real closed fields. An important topic is the description of irreducibles and prime numbers in this context, which has witnessed important progress recently [9, 13, 81];

(3) Surreal numbers can be construed as generalized series with real coefficients and surreal exponents, a formal analogue to non oscillating real functions. After [57], **No** carries a total exponential and restricted analytic functions as $\mathbb{R}$ does [36]. More recently, in [10] the authors prove that **No** is a field of transseries in the sense of [101] and define a well-behaved derivation on **No**. Building on these results and on [7], in [8] the authors show that **No** is a universal domain for Hardy fields.

1. **No** AS NUMBERS

We introduce the surreal numbers as [57]:

Definition 1.1. .
$$a \in \mathbf{No} :\Leftrightarrow a: \ \alpha \to \{\ominus, \oplus\} \text{ for some } \alpha \in \mathbf{On}$$
$$\Leftrightarrow \ a := \oplus \ominus \ominus \oplus \ominus \cdots, \text{ a well-ordered sequence of } \ominus\text{'s and } \oplus \text{ of length } \alpha.$$

The total ordering $\leq$ on **No** is lexicographical, with the following elementary rule for symbols:
$$\ominus < \text{ no symbol } < \oplus.$$

There is also a partial well-founded ordering $\leq_s$ called simplicity:

$$a \leq_s b :\Leftrightarrow a \text{ is an initial subsequence of } b.$$

E.g.

$$\oplus \oplus \ominus < \oplus \oplus \ominus \oplus \quad \text{and} \quad \oplus \oplus \ominus <_s \oplus \oplus \ominus \oplus$$

which is construed as;

$$3/2 < 7/8 \quad \text{and} \quad 3/2 <_s 7/8$$

This leads to the following binary tree representation for **No**: see Figure 1. In particular, $\mathbb{R}$ (viewed in terms of dyadic sequences) and **On** (identified with the sequences of $\oplus$'s of arbitrary length) are shown to be naturally included as classes in **No**.

Finally, we introduce the representation of a surreal number as a cut between an ordered pair of sets of surreal numbers, as was originally defined in [29]:

$$
\begin{aligned}
a &= \{L \,|\, R\} \text{ for sets of surreal numbers} L > R; \\
&= \text{ the } simplest \text{ number between } L \text{ and } R.
\end{aligned}
$$

We underline that such a representation of a surreal number as a cut is not unique. Then we focus on the so-called canonical representation, which allows to define functions and operations on **No** recursively along the simplicity ordering.

2. The field of surreal numbers.

The algebraic operations on **No** are defined recursively using the canonical representation.

Addition: $\qquad a + b \quad := \quad \{a^L + b, \, a + b^L \,|\, a^R + b, \, a + b^R\}$

Multiplication: $\quad a \cdot b \quad :=$
$$
\begin{aligned}
\{a^L \cdot b + a \cdot b^L - a^L \cdot b^L, \quad & a^R \cdot b + a \cdot b^R - a^R \cdot b^R \,| \\
a^L \cdot b + a \cdot b^R - a^L \cdot b^R, \quad & a^R \cdot b + a \cdot b^L - a^R \cdot b^L\}
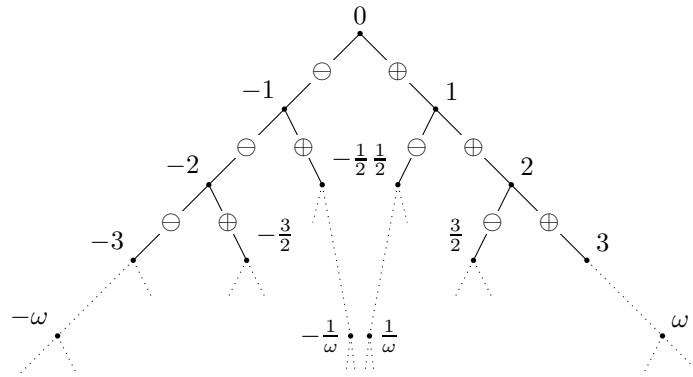\end{aligned}
$$

FIGURE 1. The tree of surreal numbers.

E.g. we show that

$$2 + (-1/2) = \{1 \mid \emptyset\} + \{-1 \mid 0\} = \{1 \mid 2\} = 3/2$$

which can be viewed as $\oplus \oplus + \ominus \oplus = \oplus \oplus \ominus$. Also:

$$\omega \cdot (1/\omega) = \{n \in \mathbb{Z}_{>0} \mid \emptyset\} \cdot \{0 \mid 1/2^n,\, n \in \mathbb{Z}_{>0}\} = \{0 \mid \emptyset\} = 1$$

which can be viewed as: $\underbrace{\oplus \oplus \cdots}_{\omega \text{ times}} \cdot \underbrace{\oplus \ominus \ominus \cdots}_{\omega \text{ times}} = \oplus$.

The key results concerning the algebraic structure of **No** and its universality are:

Theorem 2.1 (Universal real closed field)**.**

   (1) [29, Ch. 5] [57, Ch. 5, Sect. D] *The proper class* **No** *is a real closed Field.*
   (2) [29, Theorems 28 and 29] [47, Theorems 9 and 19] *Any divisible ordered Abelian group, respectively any real closed field, is isomorphic to an initial subgroup of* (**No**, +)*, respectively an initial subfield of* (**No**, +, ·)*.*

Introduction to surreal exponentiation
Vincenzo Mantova

We continue the introduction started by M. Matusinski about the basics of the class of Conway's surreal numbers **No**, with the aim of presenting the tools needed to understand the Kruskal-Gonshor surreal exponentiation. We focus on the presentation of surreal numbers as a Hahn field.

Given two surreal numbers $a, b \in$ **No**, we define the following relations:

   - $a \preceq b$ if $|a| \leq k \cdot |b|$ for some $k \in \mathbb{N}$; we say that a is *dominated* by b;
   - $a \prec b$ if $k \cdot |a| \leq |b|$ for all $k \in \mathbb{N}$; we say that a is *infinitesimal* w.r.t. b;
   - $a \asymp b$ if $a \preceq b$ and $b \preceq a$; we say that a and b are in the same *Archimedean class*.

Conway showed that the Archimedean equivalence relation has a complete class of canonical positive representatives, which can parametrised by the surreal numbers themselves through the *omega map*:

$$\omega^a := \left\{ 0, k \cdot \omega^{a^L} \mid \frac{1}{k} \cdot \omega^{a^R} \right\}$$

for k running over the positive natural numbers. It is fairly easy to verify that if $a \neq b$, then $\omega^a \not\asymp \omega^b$. Moreover, $a \leq_s b$ if and only if $\omega^a \leq_s \omega^b$.

Theorem 1 (Conway [29])**.** *For every non-zero $a \in$* **No** *there is a $b \in$* **No** *such that $a \asymp \omega^b$. For all $a, b \in$* **No***, $\omega^{a+b} = \omega^a \cdot \omega^b$. The function $\alpha \mapsto \omega^\alpha$ coincides with ordinal exponentiation for $\alpha \in$* **On***.*

Note however that the omega map is not a good notion of surreal exponentiation: it is highly discontinuous and it has infinitely many fixed points (e.g. $\omega^{\epsilon_0} = \epsilon_0$). In particular, it shouldn't be thought of as 'exponentiation in base ω'.

As a corollary, $\omega^{\mathbf{No}}$ is a multiplicative group of representatives for $\asymp$. By the general theory of ordered fields, one can then embed $\mathbf{No}$ into the Hahn field $\mathbb{R}((\omega^{\mathbf{No}}))$, where

$$\mathbb{R}((\omega^{\mathbf{No}})) = \left\{ \sum_{i<\alpha} r_i \omega^{a_i} \ : \ \alpha \in \mathbf{On}, r_i \in \mathbb{R}^*, a_i \in \mathbf{No}, \forall i,j : (i<j) \to a_i > a_j \right\}.^1$$

Thanks to the simplicity relation, one can show that there is in fact a canonical *isomorphism* $\iota : \mathbb{R}((\omega^{\mathbf{No}})) \to \mathbf{No}$. Such isomorphism can be defined explicitly by induction on $\alpha \in \mathbf{On}$:

- if $\alpha = 0$, define $\iota(0) := 0$;
- if $\alpha = \beta + 1$, $\iota\left(\sum_{i<\beta+1} r_i \omega^{a_i}\right) := \iota\left(\sum_{i<\beta} r_i \omega^{a_i}\right) + r_\beta \omega^{a_\beta}$;
- if α is limit,

$$\iota\left(\sum_{i<\alpha} r_i \omega^{a_i}\right) := \left\{ \iota\left(\sum_{i<\beta} r_i \omega^{a_i}\right) + r_\beta^L \omega^{a_\beta} \ \middle| \ \iota\left(\sum_{i<\beta} r_i \omega^{a_i}\right) + r_\beta^R \omega^{a_\beta} \right\}.$$

Theorem 2 (Conway [29, 57]). *The map ι is an isomorphism between $\mathbf{No}$ and $\mathbb{R}((\omega^{\mathbf{No}}))$.*

In particular, this shows that $\mathbf{No}$ is a *real closed field*.

If we identify $\mathbf{No}$ and $\mathbb{R}((\omega^{\mathbf{No}}))$, every surreal number $b \in \mathbf{No}$ can be written in the so called *Conway's normal form*

$$b = \sum_{i<\alpha} r_i \omega^{a_i}.$$

When $b \in \mathbf{On}$, such expression becomes the usual Cantor's normal form for ordinal numbers; in particular, the sum is finite, the exponents a_i are in $\mathbf{On}$ and the coefficients r_i are in $\mathbb{N}$. A detailed description of how to translate between normal forms and sign sequences can be found in [48].

The Conway normal form can then be used to define the surreal exponential function by Kruskal and Gonshor [57] as done in the following talk by A. Fornasiero.

Surreal Ordered Exponential Fields
Philip Ehrlich

In his monograph *On Numbers and Games* (1972) [29], J. H. Conway introduced a real-closed field containing the reals and the ordinals as well as a great many less familiar numbers, including $-\omega$, $\omega/2$, $\omega/2$, $1/\omega$, $\sqrt{\omega}$ and $\omega - \pi$ to name only a few. Indeed, this particular real-closed field, which Conway calls $\mathbf{No}$, is so remarkably inclusive that, subject to the proviso that numbers—construed here as members

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[1] Note that $\omega^{\mathbf{No}}$ is a proper class, while the support of each formal sum is a *set*, so $\mathbb{R}((\omega^{\mathbf{No}}))$ is in fact a slight abuse of notation.

of ordered fields—be individually definable in terms of sets of NBG, it may be said to contain "All Numbers Great and Small" ([45], [47], [49])

In addition to its inclusive structure as an ordered field, **No** has a rich algebraico-tree-theoretic structure that emerges from the recursive clauses in terms of which it is defined. This *simplicity hierarchical* (or *s-hierarchical*) *structure*, as we call it, depends upon **No**'s structure as a lexicographically ordered binary tree and arises from the fact that the sums and products of any two members of the tree are the simplest possible elements of the tree consistent with **No**'s structure as an ordered group and an ordered field, respectively, it being understood that x is *simpler than y just in case x is a predecessor of y in the tree* [47].

Among the striking *s*-hierarchical features of **No** is that much as the surreal numbers emerge from the empty set of surreal numbers by means of a transfinite recursion that provides an unfolding of the entire spectrum of numbers great and small (modulo the aforementioned provisos), the recursive process of defining **No**'s arithmetic in turn provides an unfolding of the entire spectrum of ordered fields in such a way that an isomorphic copy of every such system either emerges as an initial subtree of **No**, or is contained in a theoretically distinguished instance of such a system that does. In particular:

Every real-closed ordered field is isomorphic to an initial subfield of **No** (Ehrlich 2001) [47].

In 2003, we began to suspect that if we supplement **No** with the exponential function exp on **No** (introduced by Martin Kruskal and investigated by Harry Gonshor (1986) [57]) the analog of the above result holds for models of $Th\,(\mathbb{R}, e^x)$. That is, we began to suspect that:

Every model of $Th\,(\mathbb{R}, e^x)$ is isomorphic to an initial substructure of (**No**, exp).

The purpose of this talk is to point out that this is indeed the case. In particular, we will point out that an extension of our earlier proof of the aforementioned result on real-closed fields leads to a proof of its exponential counterpart. The ingredients of the proof consist of results of Kruskal and Gonshor on surreal exponentiation, results of Ressarye (1993) [91] on models of $Th\,(\mathbb{R}, e^x)$, and some earlier results of ours on initial subspaces and subfields of **No**.

Initial embeddings in the Surreal Numbers

Antongiulio Fornasiero

The exponential function on $\mathbb{R}$ can be extended canonically to the the field of Surreal Numbers **No**; similarly, each real analytic function defined on $[-1,1]^n$ can be extended to the corresponding closed box in **No**^{n} (see [57] and [29]).

In [36], they proved that **No** is an elementary superstructure of $\mathbb{R}$ in the language $\mathcal{L}_{an,exp}$ of ordered fields with all restricted analytic functions and the global

exponential function. Moreover, **No** is a saturated $\mathcal{L}_{an,exp}$-structure (but remember that **No** is a proper class, not a set), hence every model (whose universe is a set) of $Th(\mathbb{R}_{an,exp})$ can be elementarily embedded in **No**.

Definition 1. *A subset X of **No** is* initial *if:*
$\forall x \in X \ \forall y \in \textbf{No}$ *if $x \in X$ and y is simpler than x, then $y \in X$.*
   A function $f : S \to \textbf{No}$ is initial *if its image is an initial subset of **No**.*

We show that every $\mathcal{L}_{an}$-structure elementarily equivalent to $\mathbb{R}_{an}$ can be elementarily embedded in an initial way in **No**, and similarly every $\mathcal{L}_{an,exp}$-structure elementarily equivalent to $\mathbb{R}_{an,exp}$ can be elementarily embedded in an initial way in **No**.

It was already known that every real-closed field can be embedded initially in **No** (see [47]).

Our proof mimics a similar results by Mourgues and Ressayre that every real closed field admits a truncation-closed embedding into a field of generealized power series (see [85]). The main ingredients are:

(1) $\mathbb{R}_{an,exp}$ is o-minimal (see [38]), and therefore the set of realizations of a 1-type is a convex set.
(2) The following fact (see [38]),

> **Fact 1.** *Let S be a subset of **No**. The $\mathcal{L}_{an}$-submodel of **No** generated by S is the smallest real closed subfield of **No** closed under restricted analytic fucntions. The $\mathcal{L}_{an,exp}$-submodel **No** generated by S is the smallest real closed subfield of **No** closed under restricted analytic fucntions,* exp, *and* log.

(3) The following lemma

> **Lemma 1.** *Let S be an initial subset of **No**. Then, the following are also initial subsets of **No**:*
> (a) *the real closure of the field generated by S;*
> (b) *the $\mathcal{L}_{an}$-submodel of **No** generated by S;*
> (c) *the $\mathcal{L}_{an,exp}$-submodel of **No** generated by S.*

Transseries as germs of surreal functions

ALESSANDRO BERARDUCCI

(joint work with Vincenzo Mantova)

Transseries fields are an important tool in asymptotic analysis and play a crucial role in Ècalle's approach to the problem of Dulac [42]. They appear in various versions, see for instance [32, 39, 59, 72, 40, 101, 77, 60, 61] and the bibliography therein. In [10] we proved that Conway's field **No** of surreal numbers [29] admits the structure of a field of transseries (in the sense of [101]) and a compatible derivation $\partial : \textbf{No} \to \textbf{No}$ (in fact more than one). We also proved the existence of "integrals", in the sense of antiderivatives, for the "simplest" transserial derivation

on **No**. This makes **No** into a Liouville closed H-field in the sense of [7]. The notion of H-field arises as an attempt to axiomatize some of the properties of Hardy-fields, where a Hardy field is a field of germs at $+\infty$ of eventually C^1-functions $f : \mathbb{R} \to \mathbb{R}$ closed under derivation. Such fields have been studied since the early 80's, see for instance [97, 96, 98]. Any o-minimal structure on the reals gives rise to an H-field, namely the field of germs at $+\infty$ of its definable unary functions. In [8] van den Dries, Aschenbrenner and van der Hoeven proved that, with the "simplest" derivation ∂ introduced in [10], the surreals are a universal H-field. More precisely, every H-field with "small derivations" and constant field $\mathbb{R}$ embeds in **No** as a differential field. Moreover $(\mathbf{No}, \partial)$ satisfies the complete first order theory of the logarithmic-exponential series of [39, 40] and therefore, again by [8], it admits solutions to as many differential equations as one can possibly hope for in the setting of H-fields.

Another approach to derivation and integration on the surrreal numbers was taken by Costin, Ehrlich and Friedman [30] in a more analytic vein, possibly suitable for asymptotic analysis, namely they consider derivatives and definite integrals of functions, rather than derivatives of "numbers" (elements of **No**).

Here we try to reconcile the algebraic and the analytic approach to surreal derivation and integration through a notion of composition [11]. The special session on surreal numbers at the joint AMS-MAA meeting in Seattle (6-9 Jan. 2016) was a timely occasion to discuss these developments and some of the results of this contribution were presented during that meeting.

We need some definitions. We recall that in **No**, as in any Hahn field, there is a formal notion of summability, and one can associate to each summable sequence $(x_i)_{i \in I}$ its "sum" $\sum_{i \in I} x_i \in \mathbf{No}$. We define the field of **omega-series** $\mathbb{R}\langle\omega\rangle$ as the smallest subfield of **No** containing $\mathbb{R}(\omega)$ and closed under exp, log and sums of summable sequences. Here ω is the first infinite ordinal and plays the role of a formal variable with derivative 1. It turns out that $\mathbb{R}\langle\omega\rangle$ is a very big exponential field (in fact a proper class) properly containing an isomorphic copy of the logarithmic-exponential series of [39, 40] (LE-series) and their variants, such as the exponential-logarithmic series of [72, 78] (EL-series). More precisely, we can isolate two subfields $\mathbb{R}((\omega))^{LE} \subset \mathbb{R}((\omega))^{EL}$ of $\mathbb{R}\langle\langle\omega\rangle\rangle$ which are isomorphic to the LE and EL-series respetively. The field $\mathbb{R}((\omega))^{LE}$ is a countable union $\bigcup_{n \in \mathbb{N}} X_n \subseteq$ **No**, where $X_0 := \mathbb{R}(\omega)$ and X_{n+1} is the set of all sums of summable sequences of elements in $X_n \cup \exp(X_n) \cup \log(X_n)$. In other words, a surreal number is a LE-series if it can be obtained from $\mathbb{R}(\omega)$ by finitely many applications of $\sum, \exp, \log$. This remarkably simple characterization of the LE-series, which should be compared with the original definition, is made possible by working inside the surreals, with its notion of summability and exponential structure. The EL-series admit a similar characterization.

We show that each omega-series $f \in \mathbb{R}\langle\langle\omega\rangle\rangle$, hence in particular each LE or EL-series, can be interpreted as a function from positive infinite surreal numbers to surreal numbers. The idea is simply to substitute ω with a positive infinite surreal and evaluate the resulting expression, but the proof of summability is rather long

and technical. Similar problems were tackled in [101]. This gives rise to a natural composition operator $\circ : \mathbb{R}\langle\langle\omega\rangle\rangle \times \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$ which restricts to a composition $\circ : \mathbb{R}\langle\langle\omega\rangle\rangle \times \mathbb{R}\langle\langle\omega\rangle\rangle^{>\mathbb{R}} \to \mathbb{R}\langle\langle\omega\rangle\rangle$ analogous to the composition of ordinary power series.

Finally we consider the "simplest" derivation $\partial : \mathbf{No} \to \mathbf{No}$ introduced in [10] and its compatibility with the composition. Let us recall that [10] we proved the existence of several "surreal derivations" $\partial : \mathbf{No} \to \mathbf{No}$ and we studied in detail the "simplest" one [10, Def. 6.21]. It easy to see that all surreal derivations coincide on the subfield $\mathbb{R}\langle\langle\omega\rangle\rangle$, so the latter admits a unique natural derivation $\partial : \mathbb{R}\langle\langle\omega\rangle\rangle \to \mathbb{R}\langle\langle\omega\rangle\rangle$. We show that the formal derivative ∂f of the "number" $f \in \mathbb{R}\langle\langle\omega\rangle\rangle$ can be interpreted as the derivative of the function $\hat{f} : \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$ defined by $\hat{f}(x) = f \circ x$, namely we have

$$\partial f \circ x = \lim_{\epsilon \to 0} \frac{f \circ (x+\epsilon) - f \circ x}{\epsilon},$$

where x and ϵ range in $\mathbf{No}$. Since $\partial f \circ \omega = \partial f$, this shows in particular that the derivative can be defined in terms of the composition: $\partial f = \lim_{\epsilon \to 0} \frac{f \circ (\omega + \epsilon) - f \circ \omega}{\epsilon}$. Other compatibility conditions then follow, such as the chain rule $\partial(f \circ g) = (\partial f \circ g) \cdot \partial g$.

These results tells us that any omega-series $f \in \mathbb{R}\langle\langle\omega\rangle\rangle$, hence in particular every logarithmic-exponential series, can be interpreted as a differentiable function $\hat{f} : \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$ from positive infinite surreal numers to surreal numbers. We shall prove that all such functions are surreal-analytic, namely for every $f \in \mathbb{R}\langle\langle\omega\rangle\rangle$ and $x \in \mathbf{No}^{>\mathbb{R}}$ we have

$$f \circ (x+\epsilon) = \sum_{n \in \mathbb{N}} \frac{1}{n!}(f^{(n)} \circ x) \cdot \epsilon^n$$

for every sufficiently small $\epsilon \in \mathbf{No}$. At the moment of writing we still do not know how small the ϵ should be, and in particular we do not know whether we can take ϵ in $\mathbb{R}\langle\omega\rangle \setminus \{0\}$ if x is in $\mathbb{R}\langle\langle\omega\rangle\rangle$.

It is tempting to raise the conjecture that the exponential field $\mathbf{No}$, enriched with all the functions $\hat{f} : \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$ for $f \in \mathbb{R}\langle\omega\rangle$ (possibly restricted to some interval $(a, +\infty)$) has a good model theory. For instance the restricted version could yield an o-minimal structure on $\mathbf{No}$. Notice that the family of all functions $\hat{f} : \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$ (for $f \in \mathbb{R}\langle\langle\omega\rangle\rangle$) yields a sort of non-standard Hardy field on $\mathbf{No}$, namely a field of functions closed under differentiation (it is also closed under exp, log and composition).

The derivation ∂ on $\mathbb{R}\langle\omega\rangle$ makes it into a H-field, which however it is not Liouville closed because $\partial : \mathbb{R}\langle\langle\omega\rangle\rangle \to \mathbb{R}\langle\langle\omega\rangle\rangle$ is not surjective. There are however many subfields of $\mathbb{R}\langle\langle\omega\rangle\rangle$ which are Liouville closed, among them $\mathbb{R}((\omega))^{LE}$.

We do not know up to what extent the above results can be extended beyond $\mathbb{R}\langle\langle\omega\rangle\rangle$, namely whether we can introduce a composition operator on the whole of $\mathbf{No}$, thus giving a functional interpretation to all surreal numbers. We finish with a negative result: the simplest derivation $\partial : \mathbf{No} \to \mathbf{No}$ defined in [10] cannot be compatible with a composition on the whole of $\mathbf{No}$.

Quasianalytic Ilyashenko Algebras

Patrick Speissegger

I construct a quasianalytic algebra of functions with simple logarithmic transseries as asymptotic expansions; see [106] for details. The construction is based on Ilyashenko's class of *almost regular* functions, as introduced in his book [62] on Dulac's problem [63]. This class forms a group under composition, but it is not closed under addition or multiplication; to obtain a ring, we need to allow finite iterates of the logarithm in the asymptotic expansions. Among other things, this leads to dealing with asymptotic series that have order type larger than ω, so I first need to define what I mean by "asymptotic expansion".

Let G be a multiplicative subgroup of some Hardy field of C^∞-germs at $+\infty$, and let $\mathbb{R}(\!(G)\!)$ denote the corresponding generalized series field, see van den Dries et al. [40]. (The support of such a series is a *reverse well-ordered* subset of G.) Let K be an $\mathbb{R}$-algebra of C^∞-germs at $+\infty$, and let $T : K \longrightarrow \mathbb{R}(\!(G)\!)$ be an $\mathbb{R}$-algebra homomorphism. For $F \in \mathbb{R}(\!(G)\!)$ and $g \in G$, we denote by F_g the truncation of F above g. Below, I use the dominance relation $\preceq$ on Hardy fields as introduced by Aschenbrenner and van den Dries in [4].

Definition. We say that (K, G, T) is a **quasianalytic asymptotic** (or **qaa** for short) algebra if

(1) T is injective;
(2) $T(K)$ is truncation closed;
(3) for every $f \in K$ and every $g \in G$, we have

$$f - T^{-1}((Tf)_g) \prec g.$$

My aim is to construct a qaa field $(\mathcal{K}, \mathcal{L}, T)$ such that $\mathcal{K}$ contains Ilyashenko's class of almost regular mappings and $\mathcal{L}$ is the group of monomials of the form $\log_{-1}^{\alpha_{-1}} \log_0^{\alpha_0} \cdots \log_k^{\alpha_k}$, where $k \in \{-1\} \cup \mathbb{N}$, $\alpha = (\alpha_{-1}, \ldots, \alpha_k) \in \mathbb{R}^{2+k}$ and $\log_i$ denotes the ith compositional iterate of log (so that $\log_0 = x$ and $\log_{-1} = \exp$).

The construction is based on the following principle: for $C > 0$, we define the **standard quadratic domain**

$$\Omega = \Omega_C := \left\{ z + C\sqrt{1+z} : \ z \in H(0) \right\},$$

where $H(0)$ denotes the right half-plane of $\mathbb{C}$.

Uniqueness Principle. (See Theorem 1 on p. 23 of [62].) *Let $\Omega \subseteq \mathbb{C}$ be a standard quadratic domain and $f : \bar{\Omega} \longrightarrow \mathbb{C}$ be holomorphic. If f is bounded and*

$$f \prec \exp^{-n} \quad on \ \mathbb{R}, \ for \ each \ n \in \mathbb{N},$$

then $f = 0$.

In view of the Uniqueness Principle, I define $\mathcal{A}_0$ to be the set of all germs at $+\infty$ of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ that have a bounded, holomorphic extension $\mathbf{f} : \Omega \longrightarrow \mathbb{C}$

to (the closure of) some standard quadratic domain Ω and for which there exist real numbers $0 \leq \nu_0 < \nu_1 < \cdots$ and $a_0, a_1, \ldots$ such that $\lim_{n \to \infty} \nu_n = +\infty$ and

$$(1) \qquad \mathbf{f}(z) - \sum_{n=0}^{N} a_n e^{-\nu_n z} = o\left(e^{-Nz}\right) \quad \text{as} \quad |z| \to \infty \text{ in } \Omega, \quad \text{for each } N \in \mathbb{N}.$$

For convenience, for holomorphic $\phi, \psi : \Omega \longrightarrow \mathbb{C}$, I write $\phi \prec_\Omega \psi$ if $\phi(z) = o(\psi(z))$ as $|z| \to \infty$ in Ω. Thus, writing $\mathbf{exp}$ for the holomorphic extension of $\exp$ to Ω, I can write (1) as

$$(2) \qquad \mathbf{f} - \sum_{n=0}^{N} a_n \, \mathbf{exp}^{-\nu_n} \prec_\Omega \mathbf{exp}^{-N} \quad \text{for each } N \in \mathbb{N}.$$

In this situation, I set $T_0 f := \sum_{n=0}^{\infty} a_n \exp^{-n} \in \mathbb{T}$; by the Uniqueness Principle, the triple $(\mathcal{A}_0, \mathcal{L}, T_0)$ is a qaa algebra.

Remark. The algebra $\mathcal{A}_0 \circ (-\log)$ is the algebra $\mathcal{A}_1$ considered in [65].

Next, I let $\mathcal{F}_0$ be the fraction field of $\mathcal{A}_0$ and extend T_0 to $\mathcal{F}_0$ in the obvious way. Note that the germs in $\mathcal{F}_0$ do not all have *bounded* holomorphic extensions to standard quadratic domains; hence the need for first defining $\mathcal{A}_0$.

I now construct qaa fields $(\mathcal{F}_k, \mathcal{L}, T_k)$, for $k \in \mathbb{N}$, such that $\mathcal{F}_k$ is a subfield of $\mathcal{F}_{k+1}$ and T_{k+1} extends T_k, as follows: assuming $(\mathcal{F}_k, \mathcal{L}, T_k)$ has been constructed, I set

$$\mathcal{F}'_{k+1} := \mathcal{F}_k \circ \log$$

and define $T'_{k+1} : \mathcal{F}'_{k+1} \longrightarrow \mathbb{T}$ by

$$T'_{k+1}(f \circ \log) := (T_k f) \circ \log.$$

Note, in particular, that every $f \in \mathcal{F}'_{k+1}$ has a holomorphic extension $\mathbf{f} : \Omega \longrightarrow \mathbb{C}$ on some standard quadratic domain Ω depending on f. Then $\left(\mathcal{F}'_{k+1}, \mathcal{L}, T'_{k+1}\right)$ is a qaa field, and I let $\mathcal{A}_{k+1}$ be the set of all germs at $+\infty$ of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ that have a bounded, holomorphic extension $\mathbf{f} : \Omega \longrightarrow \mathbb{C}$ to some standard quadratic domain Ω and for which there exist real numbers $0 \leq \nu_0 < \nu_1 < \cdots$ and germs $a_0, a_1, \ldots$ in $\mathcal{F}'_{k+1}$ such that $\lim_{n \to \infty} \nu_n = +\infty$ and

$$\mathbf{f} - \sum_{n=0}^{N} \mathbf{a}_n \, \mathbf{exp}^{-\nu_n} \prec_\Omega \mathbf{exp}^{-\nu_N} \quad \text{for each } N \in \mathbb{N}.$$

In this situation, I set $T_{k+1} f := \sum_{n=0}^{\infty} \left(T'_{k+1} a_n\right) \cdot \exp^{-n} \in \mathbb{T}$; by the Uniqueness Principle, the triple $(\mathcal{A}_{k+1}, \mathcal{L}, T_{k+1})$ is again a qaa algebra. Finally, I let $\mathcal{F}_{k+1}$ be the fraction field of $\mathcal{A}_{k+1}$ and extend T_{k+1} correspondingly.

Remarks.
   (1) $\mathcal{A}_1 \circ (-\log)$ contains all transition maps near hyperbolic singularities of planar real analytic vector fields, see Theorem 3 on p. 24 of [62].
   (2) One shows, by induction on k, that both $\mathcal{F}_k$ and $\mathcal{F}'_{k+1}$ are subalgebras of $\mathcal{F}_{k+1}$, and that the restrictions of T_{k+1} to $\mathcal{F}_k$ and $\mathcal{F}'_{k+1}$ are T_k and T'_{k+1}, respectively.

In view of Remark 2 above, I set $\mathcal{F} := \bigcup_k \mathcal{F}_k$ and $T := \bigcup_k T_k$; it follows that $(\mathcal{F}, \mathcal{L}, T)$ is a qaa field.

What other closure properties does $\mathcal{F}$ have? I say that a field H of germs $+\infty$ of real functions is **closed under** log**-composition** if, for $f, g \in H$ such that $g \succ 1$, the composition $f \circ \log \circ g$ belongs again to H.

Proposition. *The field $\mathcal{F}$ is a Hardy field closed under* log*-composition.*

It follows that the field $\mathcal{F} \circ (-\log)$ of germs at 0^+—which contains all transition maps near hyperbolic singularities of planar real analytic vector fields—is a Hardy field closed under composition.

On the field of transseries-construction and first-order properties.
Françoise Point

In this talk, we recalled the construction of the field of transseries $\mathbb{T}$ and tried to give an idea of the main result of the monograph of M. Aschenbrenner, L. van den Dries and J. van der Hoeven [7]. The construction of $\mathbb{T}$ goes back to the works of B. Dahn and P. Göring who investigated the question of A. Tarski on the decidability of the field of real numbers with the exponential function (see the introduction of [40]). Let us outline the construction (see Appendix A in [7]). Let F be an ordered field endowed with a partial exponential function exp and with the following direct sum decomposition: $F = A \oplus B$ where B is a convex additive subgroup of F and the exponential function exp is a strictly increasing group morphism from B to the multiplicative group of F (of strictly positive elements). Let $exp^*(A)$ be an order preserving multiplicative copy of the ordered additive group A and consider the ordered Hahn field $F^* := F((exp^*(A)))$, with the convention that the supports of the elements are reverse well-ordered. The key step of the construction is to go from the partial exponential ordered field (F, A, B, exp) to (F^*, A^*, B^*, exp^*), where $F^* = A^* \oplus B^*$, $A^* := F((exp^*(A)^{>1}))$ and $exp^*(a + b + \epsilon)$ is defined as $exp^*(a).exp(b).\sum_{n \geq 0} \frac{\epsilon^n}{n!}$, where $a \in A$, $b \in B$ with ϵ is an infinitesimal element of F^*. Note that F is now included in the domain of exponential function exp^*.

The field $\mathbb{T}$ is defined as $\bigcup_{n \geq 0} \mathbb{T}_n^E$, where $\mathbb{T}^E := \bigcup_{n \geq 0} F_n$, $F_0 := \mathbb{R}((x^{\mathbb{R}}))$, $x > \mathbb{R}$ and $F_{n+1} := F_n^*$. On F_0, one defines $exp(r + \epsilon) := e(x).\sum_{n \geq 0} \frac{\epsilon^n}{n!}$, where $e(x)$ is the exponential function on $\mathbb{R}$ and $\epsilon \in \mathbb{R}((x^{\mathbb{R}^{<0}}))$. Then, one introduces formal indeterminates ℓ_n, $n \in \mathbb{N}$ with $\ell_0 = x$. One repeats the previous construction with $\mathbb{R}((\ell_1^{\mathbb{R}}))$, $\ell_1 > \mathbb{R}$, and denotes the corresponding exponential field $\mathbb{T}_1^E$. Then one embeds $\mathbb{T}^E$ in $\mathbb{T}_1^E$, sending x^r to $exp(r.\ell_1)$ and define $log(u) = v$ iff $exp(v) = u$. Then one verifies that $\mathbb{T}$ is an ordered field with an exponential function everywhere defined and a logarithmic function on strictly positive elements. Finally one introduces a derivation by setting $\delta(x) = 1$, $\delta(r) = 0$ for every $r \in \mathbb{R}$, and asking that δ is an E-derivation, namely $\delta(exp(u)) = \delta(u).exp(u)$ and that it is strongly additive.

Let $\mathcal{L} := \{+, -, ., 0, 1, \leq, \prec, \delta\}$; from now on $\mathbb{T}$ will be considered as an $\mathcal{L}$-structure, with $a \prec b$ if $v_a(a) \geq v_a(b)$, for v_a the archimedean valuation on $\mathbb{T}$. Let $\mathcal{L}_{an} := \{+, -, ., 0, 1, f; \ f$ a restricted analytic function on $[-1\ 1]\}$ and $\mathcal{L}_{an,exp} := \mathcal{L}_{an} \cup \{exp, log\}$.

The $\mathcal{L}_{an,exp}$-theory of $\mathbb{T}$ admits quantifier-elimination [38, Corollary 4.5, 4.6], is o-minimal [38, Corollary 5.13] and the field $\mathbb{R}$ as an $\mathcal{L}_{an,exp}$-structure is an elementary substructure [38, Corollary 4.6]. The theory of $\mathbb{T}$ considered now as a differential valued ordered field is certainly not weakly o-minimal (the subfield of constants is definable but not convex). However the $\mathcal{L}$-theory of $\mathbb{T}$ is NIP, o-minimal at ∞ and the intersection of any definable set with the subfield of constants is semi-algebraic.

To make a comparison with other theories of topological fields endowed with a derivation, one can note the following. Late seventies, M. Singer showed that the class of existentially closed ordered differential fields with no a priori interaction between the derivation and the order, was axiomatizable and admits quantifier elimination in the language of ordered differential fields [105]. The theory of this class is denoted by $CODF$, it is NIP, the subfield of constants is dense, it has no prime model and one has a good notion of dimension. Moreover if K is a model of $CODF$, then $K(i)$ is a differentially closed field [104]. (Later, T. Scanlon [100] described an elementary class of existentially closed differential valued fields (of characteristic 0) assuming that $v(\delta(x)) \geq v(x)$. In models of that class the subfield of constants is dense.)

Based on the example of Hardy fields, M. Rosenlicht investigated a class of valued differential fields where there was a strong interaction between the valuation and the derivative [94]. Let us recall the precise setting. Let (K, v, δ) be a differential valued field, where v denotes a valuation on K and δ a derivation. We denote by $\mathcal{O}$ the corresponding valuation ring, $\mathfrak{o}$ the maximal ideal and by $\Gamma := v(K^{\times})$ the value group. Denote by $\Gamma^{\times}$ the set of non-zero elements of Γ. An *asymptotic field* is a differential valued field where for all f, $g \in \mathfrak{o}^{\times}$, we have $v(f) < v(g) \leftrightarrow v(\delta(f)) < v(\delta(g))$. Denote by $a^{\dagger}$, the logarithmic derivative of a. Then $\Gamma^{\times}$, one can define two maps $\delta_v : v(a) \to v(\delta(a))$ and $\psi : v(a) \to v(a^{\dagger})$ [95]. One gets that $\psi(\Gamma^{\times}) < \delta_v(\Gamma^{>0})$ and the pair (Γ, ψ) is called an *asymptotic couple*. An asymptotic field is an H-asymptotic field if for all f, $g \in \mathfrak{o}^{\times}$, $v(f) < v(g) \to v(f^{\dagger}) \geq v(g^{\dagger})$. The asymptotic couple encodes in particular the behaviour of Liouville extensions of K. Assuming that K is an H-asymptotic field, there are three mutually excluding possibilities: there exists $\beta \in \Gamma$ such that $\psi(\Gamma^{\times}) < \beta < \delta(\Gamma^{>0})$, $\psi(\Gamma^{\times})$ has a maximum (K is *grounded*) or $\Gamma = \delta_v(\Gamma^{\times})$ (K has asymptotic integration).

In an H-asymptotic field K with asymptotic integration, one can build a sequence of elements as follows: pick an element ℓ_0 with $v(\ell_0) < 0$, if ℓ_ρ is defined, choose $\ell_{\rho+1}$ such that $v(\delta(\ell_{\rho+1})) = v(\ell_\rho^{\dagger})$ and then if σ a limit ordinal and for all $\rho < \sigma$, ℓ_ρ is defined, choose ℓ_σ such that $0 > v(\ell_\sigma) > v(\ell_\rho)$. Then define $\gamma_\rho = \ell_\rho^{\dagger}$, $\lambda_\rho = -\gamma_\rho^{\dagger}$ and $\omega_\rho = -2\delta(\lambda_\rho) - \lambda_\rho^2$. Then one can check that these sequences of

elements of K are pseudo-Cauchy sequences; they have a pseudo-limit in an extension of K which will be denoted by respectively $\gamma_K, \lambda_K, \omega_K$. If λ_K (respectively ω_K) belongs to a proper extension of K, one says that K is λ-free (respectively ω-free). One can also link the existence of such pseudo-limit in K itself, to the asymptotic behaviour of certain second order equations.

From now on, we will assume that the H-asymptotic field K is in addition totally ordered with the following compatibility relation: $\forall a\ (a > C \to \delta(a) > C)$ and $\mathcal{O} = C + \mathfrak{o}$, where C is the subfield of constants (such fields are called H-fields).

One easily observes that in any H-field K, the equation $\delta^{(2)}(x) + x = 0$ has no non-trivial solutions (and so it doesn't have a solution in $K(i)$ and so $K(i)$ is never a model of DCF_0).

Coming back to the field of transseries $\mathbb{T}$, we note that the derivation that we have defined on $\mathbb{T}$ (making $\mathbb{R}$ the field of constants) is surjective and so because we have an exponential function, $\mathbb{T}$ is Liouville closed with a small derivation, namely $\delta(\mathfrak{o}) \subseteq \mathfrak{o}$.

Concerning the problem of determining which second-order differential equations have a solution in $\mathbb{T}$, one can show that the equation $\delta(x) + f.x = 0$ has a solution in $\mathbb{T}$ iff $f < \frac{1}{4x^2} + \frac{1}{4x^2 log^2(x)} + \frac{1}{4x^2 log^2(x)(loglog(x))^2} + \cdots$. Note that this element does not belong to $\mathbb{T}$ ($\mathbb{T}$ is ω-free). Based on an earlier work of van der Hoeven who devised in the subfield $\mathbb{T}_g \subset \mathbb{T}$ of grid-based series, algorithms to solve differential equations, they defined the notion of being *newtonian* ([7, chapter 13, page 519]. That last property is related to the property of d-henseliality in the following way. Given a differential polynomial $P(Y) \in \mathcal{O}\{Y\}$, one can decompose into its homogeneous parts : $P_0 := P(0)$, $P_1 = \sum_{n=0}^{r} \frac{\partial P}{\partial Y^{(n)}}(0)Y^{(n)}, \cdots$ and one puts on $K\{Y\}$ the Gaussian valuation. Recall that K is d-henselian if its residue field is linearly surjective and if for any differential polynomial $P(Y) \in \mathcal{O}\{Y\}$ with $v(P_0) > 0$, $v(P_1) = 0$ there exists y with $v(y) > 0$ such that $P(y) = 0$. Let $a \in K$ be such that $a^{-1}\delta$ small, consider the differential valued field: $(K, a^{-1}\delta, v^\sharp)$, where $v^\sharp$ is the coarsening of v (in $(K, a^{-1}\delta)$) obtained by quotienting out Γ by $\Gamma^\sharp := \{\gamma :\ \psi(\gamma) > 0\} \cup \{0\}$. Then, suppose K is H-asymptotic, non-trivially valued and λ-free, then K newtonian is equivalent to: for all $a \in K$ such that $a^{-1}\delta$ is small, the field $(K, a^{-1}\delta, v^\sharp)$ is d-henselian. They show that if K is an H-field where the derivative is surjective, which is a directed union of H-fields, grounded and spherically complete, then K is ω-free and newtonian. As a corollary, $\mathbb{T}$ is newtonian.

Finally, the theory T of $\mathbb{T}$ is equal to the $\mathcal{L}$-theory of newtonian H-fields, ω-free, Liouville closed with a small derivation. All these properties are recursively axiomatizable and so T is decidable. The theory T is model-complete and $\mathbb{T}_g \preceq \mathbb{T}$. The theory T admits quantifier elimination if one adds to the language $\mathcal{L}$ two additional unary predicates Λ, Ω, defined as follows: $\Lambda(a)$ iff $\exists y\ (1 \prec y\ \&\ a = -y^{\dagger\dagger})$ and $\Omega(a)$ iff $\exists y\ (y \neq 0\ \&\ 4\delta^{(2)}(y) + ay = 0)$.

Using asymptotic couples to define derivations on fields of generalized power series.

Gabriel Lehéricy

Fields of generalized power series play an important role in real algebraic geometry. Given a field k and an ordered abelian group $(G, \leq)$, one defines the field $k((G))$ of generalized power series as the set of all maps from G to k whose support is well-ordered, where sums and products of series are defined as for usual analytic series. The field $k((G))$ is naturally endowed with a valuation, which we always denote by v, given by the minimum of the support of an element. If k happens to be an ordered field then there is a natural way of extending this order to $k((G))$: we say that an element of $k((G))$ is positive if and only if its leading coefficient is positive as an element of k. Fields of generalized power series are a useful way of constructing valued fields of given value group and residue field. Moreover, we know from Kaplansky (see [67]) that fields of generalized power series are universal domains for valued fields and thus suitable domains for the study of real algebra. This naturally leads to the following question: Are fields of generalized power series suitable domains for the study of differential valued fields? In other words, we are concerned with the following question:

Question 1: Can one define a derivation D on $k((G))$ so that (k, v, D) is a differential valued field? A H-field?

By differential valued field we mean a differential field endowed with a differential valuation in the sense of Rosenlicht (see [94]). Question 1 was already addressed in [74] but not fully answered. In this talk we go further than in [74], giving a general method to define a derivation on any field of generalized power series. Our approach uses asymptotic couples. Asymptotic couples were introduced by L.v.d.Dries and M.Aschenbrenner in [2] and play a central role in the study of H-fields and transseries (see [3],[5] and [6]). An asymptotic couple is a pair (G, ψ) where G is an ordered abelian group and $\psi : G^{\neq 0} \to G$ a map such that $\psi(g + h) \geq \min(\psi(g), \psi(h))$, $\psi(ng) = \psi(g)$ and $\psi(g) < \psi(h) + |h|$ for any $g, h \in G^{\neq 0}$ and $n \in \mathbb{Z} \setminus \{0\}$. If moreover $g \leq h < 0 \Rightarrow \psi(g) \leq \psi(h)$ for any g, h, we say that (G, ψ) is an H-type asymptotic couple. Asymptotic couples appear naturally as the value group of differential valued fields. Indeed, if (K, v, D) is a differential valued field with value group G, then D induces a map $\psi : G^{\neq 0} \to G$ defined by $\psi(v(a)) := v(\frac{D(a)}{a})$ making (G, ψ) an asymptotic couple. If moreover (K, v, D) is an H-field, then (G, ψ) is H-type. Given an ordered abelian group G, we can always define a map ψ such that (G, ψ) is an H-type asymptotic couple, which is why we can answer Question 1 by answering the following question:

Question 2: Given an H-type asymptotic couple (G, ψ) and a field k, can we define a derivation D on $K := k((G))$ such that (K, v, D) is a differential valued field of asymptotic couple (G, ψ)? If k is an ordered field, can we do this so that (K, v, D) is an H-field?

We use the theory of valued groups presented in chapter 0 of [72]. We recall that if G is an abelian group, then a $\mathbb{Z}$-module valuation on G is a surjective map $v : G \to \Gamma \cup \{\infty\}$ where Γ is a totally ordered set and such that $v(g) = \infty \Leftrightarrow g = 0$,

$v(g+h) \geq \min(v(g), v(h))$ and $v(ng) = v(g)$ for any $g, h \in G$ and $n \in \mathbb{Z}\backslash\{0\}$. Note in particular that if (G, ψ) is an asymptotic couple, then ψ can be seen as a $\mathbb{Z}$-module valuation on G. The skeleton of a valuation v is by definition the pair $(\Gamma, (G^\gamma/G_\gamma)_{\gamma \in \Gamma})$, where $G^\gamma = \{g \in G \mid v(g) \geq g\}$ and $G_\gamma = \{g \in G \mid v(g) > g\}$. If v is a $\mathbb{Z}$-module valuation on G with skeleton $(\Gamma, (B_\gamma)_\gamma)$, then we define the Hahn product of the B_γ's, denoted by $\mathrm{H}_{\gamma \in \Gamma} B_\gamma$, as the subgroup of the direct product $\Pi_{\gamma \in \Gamma} B_\gamma$ consisting of all elements with well-ordered support, endowed with the valuation given by the minimum of the support of an element. Elements of $\mathrm{H}_{\gamma \in \Gamma} B_\gamma$ can be thought of as formal sums $\sum_{\gamma \in \Gamma} g_\gamma \tau^\gamma$ with well-ordered support and $g_\gamma \in B_\gamma$. We recall the following result due to Hahn: If v is a $\mathbb{Z}$-module valuation on an abelian group G with skeleton $(\Gamma, (B_\gamma)_\gamma)$, then (G, v) is isomorphic as a valued $\mathbb{Z}$-module to a subgroup of $\mathrm{H}_{\gamma \in \Gamma} B_\gamma$.

We now give a method to answer Question 2. We fix an asymptotic couple (G, ψ) and a field k. We denote by v_G the natural valuation on G, i.e the $\mathbb{Z}$-module valuation defined by $v_G(g) \leq v_G(h) \Leftrightarrow \exists n \in \mathbb{N}, |h| \leq n|g|$. Now take a valuation w which is a coarsening of v_G and such that ψ (seen as a valuation) is a coarsening of w. We denote by $(\Gamma, (B_\gamma)_\gamma)$ the skeleton of (G, w). Thanks to Hahn's theorem, we can see (G, w) as a subgroup of $H := \mathrm{H}_{\gamma \in \Gamma} B_\gamma$, and we can extend ψ to H in a unique way. We shall define a derivation on $k((G))$ by first defining it on $K := k((H))$. Note that since ψ is a coarsening of w, ψ induces a map $\Gamma \to G$, which we also denote ψ, defined by $\psi(w(g)) = w(\psi(g))$. We assume that we are given a family $(\epsilon_\gamma)_{\gamma \in \Gamma}$ of group homomorphisms from B_γ to $(k, +)$. We also assume that for any $\gamma \in \Gamma$, $\{\delta \in \Gamma \mid \psi(\delta) = \psi(\gamma)\}$ is finite.

To define a derivation on K we proceed in three steps: Step 1: define D on the fundamental monomials, i.e define $D(t^{g_\gamma})$ for each $g_\gamma \in B_\gamma$ for every $\gamma \in \Gamma$. Step 2: extend D to all monomials by using a strong Leibniz rule. Step 3: extend D to K by strong linearity. This idea is inspired by the work in [74]. In [74], the authors assumed that the map D was already given on the fundamental monomials and gave conditions for this map to be extendable to the whole field. They only do this in the particular case where G is a Hahn product of copies of $\mathbb{R}$. Here we do it in a more general setting since we do not make any assumption on (G, ψ) except that it is an H-type asymptotic couple; moreover, we define D explicitly on the fundamental monomials. The idea for step 1 comes from the following remark: if (K, v, D) is a differential valued field, then for any $a \in K$ we have $v(D(a)) = v(a) + \psi(v(a))$. We thus naturally want to define $D(t^{g_\gamma}) = \epsilon_\gamma(g_\gamma) t^{g_\gamma + \psi(\gamma)}$. Note that a similar idea was already used in [5], but only in the case where G is divisible and admits a valuation basis, which is a strong restriction. For a $g = \sum_\gamma g_\gamma \tau^\gamma \in H$, we use strong Leibniz rule to define $D(t^g) = \sum_{\gamma \in \Gamma} \epsilon(g_\gamma) t^{g+\psi(\gamma)}$. With our assumptions, we can easily check that the family $(\epsilon(g_\gamma) t^{g+\psi(\gamma)})_{\gamma \in supp(g)}$ is summable, so that D is well-defined on the set of monomials of K. For an $a = \sum_{g \in G} a_g t^g \in K$, we apply strong linearity and get $D(a) = \sum_{g \in G} a_g \sum_{\gamma \in \Gamma} \epsilon(g_\gamma) t^{g+\psi(\gamma)}$. Again, one can check that the family $(a_g \epsilon(g_\gamma) t^{g+\psi(\gamma)})_{g \in supp(a), \gamma \in supp(g)}$ is summable, so that our definition

makes sense. The key to summability is the fact that $w(g - h) < w(\psi(g) - \psi(h))$ for any $g \neq h$; this was already proved in [3] for $w = v_G$.

Thus, we have defined a series derivation on K and we see from the definition of D that $k((G))$ is stable under D, so D defines a derivation on $k((G))$. It remains to see if (K, v, D) is a differential valued field. One can show that the leading term of $D(a)$ is $(\sum_\delta a_g \epsilon(g_\delta)) t^{g + \psi(\gamma)}$, where $g = v(a)$, $\gamma = w(g)$ and δ ranges over elements of Γ such that $\psi(\delta) = \psi(\gamma)$. This implies in particular that $v(D(a)) \geq v(a) + \psi(v(a))$. Unfortunately, equality is not verified in general, and K endowed with the derivation D is not a differential valued field in the sense of Rosenlicht; in fact, we have no control over the field of constants. However, if we choose $w := \psi$, then the leading term of $D(a)$ is $a_g \epsilon(g_\gamma) t^{g + \psi(\gamma)}$; if we assume moreover that each ϵ_γ is injective, this implies $v(D(a)) = v(a) + \psi(v(a))$. It follows that (K, v, D) is a differential valued field with associated asymptotic couple (G, ψ). Moreover, if k is an ordered field, and if we assume that each ϵ_γ is order-reversing, then (K, v, D) is an H-field, which is what we wanted. Note that the condition of injectivity of ϵ_γ is not surprising: if (K, v, D) is a differential valued field, we can define $\epsilon_\gamma(g_\gamma)$ as the leading coefficient of $D(t^{g_\gamma})$, and one can check that this defines an embedding of groups from B_γ to $(k, +)$.

We can thus give the following answer to Question 2: *given a field (respectively, an ordered field) k and an asymptotic couple (G, ψ), if $(\Gamma, (B_\gamma)_{\gamma \in \Gamma})$ is the skeleton of the valuation ψ, then we can define a derivation D on $k((G))$ making $(k((G)), v, D)$ a differential valued field (respectively, a H-field) if and only if each B_γ is embeddable into $(k, +)$ as a group (respectively, as an ordered group). In that case, D can be defined by the method given above.* The following questions are still open:

1)Let k, k' be two elementary equivalent fields and $(G, \psi), (G', \psi')$ two elementary equivalent asymptotic couples. Are $k((G))$ and $k'((G))$ elementary equivalent as differential valued fields (with the derivation given above)?

2)Can we extend D to an exponential derivation on $k((G))^{LE}$?(with $k = \mathbb{R}$)

Exponential-Logarithmic fields of κ-bounded series
Salma Kuhlmann

In this talk we presented the construction of power series models of real exponentiation given in our paper [K-S (2005)] Kuhlmann, S. - Shelah, S.: κ–bounded Exponential Logarithmic Power Series Fields, APAL, **136**, 284-296. We aim to endow these models with derivation and composition operators, and conjecture that they are appropriate candidates for a differential version of Kaplansky embedding theorem. A monomial $m := t^g$ is *log-atomic* if the n-th iterate of its logarithm $\log_n(m)$ is a monomial for all n. We explain below how to generalise the construction given in [K-S (2005)] to kill from the beginning all log-atomic monomials.

For $(K, +, \cdot, 0, 1, <)$ an ordered field we denote by v its natural valuation, by $v(K) := \{v(x) \,|\, x \in K, x \neq 0\}$ its value group, by $R_v := \{x \,|\, x \in K \text{ and } v(x) \geq 0\}$

its valuation ring, by $U_v^{>0} := \{x \mid x \in R_v, x > 0, v(x) = 0\}$ its group of positive units and by $I_v := \{x \mid x \in K \text{ and } v(x) > 0\}$ its valuation ideal. K is an *exponential field* if there exists an isomorphism of ordered groups (an *exponential* exp) $\exp : (K, +, 0, <) \longrightarrow (K^{>0}, \cdot, 1, <)$. A *logarithm* on K is the compositional inverse $\log = \exp^{-1}$ of an exponential. We require the exponentials (logarithms) to be *v-compatible*: $\exp(R_v) = U_v^{>0}$ or $\log(U_v^{>0}) = R_v$. A logarithm log is a (GA)-logarithm if it satisfies

$$(3) \qquad\qquad v(x) < v(\log(x)) \text{ for all } x \in K^{>0} \setminus R_v .$$

Hahn Groups and Fields. for Γ any totally ordered set and R any ordered abelian group, we let R^Γ be the set of all maps g from Γ to R such that the *support* $\{\gamma \in \Gamma \mid g(\gamma) \neq 0\}$ of g is well-ordered in Γ. Endowed with the lexicographic order and pointwise addition, R^Γ is an ordered abelian group, called the *Hahn group*. Fix a strictly positive element $\mathbf{1} \in R$. For every $\gamma \in \Gamma$, denote by $\mathbf{1}_\gamma$ the characteristic function of the singleton $\{\gamma\}$. For $g \in R^\Gamma$ write $g = \sum_{\gamma \in \Gamma} g_\gamma \mathbf{1}_\gamma$. For $G \neq 0$ an ordered abelian group, k an archimedean ordered field, $k((G))$ is the (generalized) *power series field* with coefficients in k and exponents in G. A series $s \in k((G))$ is written $s = \sum_{g \in G} s_g t^g$ with $s_g \in k$ and well-ordered support $\{g \in G \mid s_g \neq 0\}$. The natural valuation on $k((G))$ is $v(s) = \min \operatorname{support} s$ for any series $s \in k((G))$. The value group is G and the residue field is k. The valuation ring $k((G^{\geq 0}))$ consists of the series with non-negative exponents, and the valuation ideal $k((G^{>0}))$ of the series with positive exponents. The *constant term* of a series s is the coefficient s_0. The units of $k((G^{\geq 0}))$ are the series in $k((G^{\geq 0}))$ with a non-zero constant term.

Additive Decomposition Given $s \in k((G))$, we can truncate it at its constant term and write it as the sum of two series, one with strictly negative exponents, and the other with non-negative exponents. Thus a complement in $(k((G)), +)$ to the valuation ring is the Hahn group $k((G^{<0}))$.

Multiplicative Decomposition Given $s \in k((G))^{>0}$, we can factor out the monomial of smallest exponent $g \in G$ and write $s = t^g u$ with u a unit with a positive constant term. Thus a complement in $(k((G))^{>0}, \cdot)$ to the subgroup $U_v^{>0}$ of positive units is the group consisting of the (monic) monomials t^g, denoted by **Mon** $k((G))$.

Now fix a regular uncountable cardinal κ. The *κ-bounded Hahn group* $(R^\Gamma)_\kappa \subseteq R^\Gamma$ consists of all maps of which support has cardinality $< \kappa$. The *κ-bounded power series field* $k((G))_\kappa \subseteq k((G))$ consists of all series of which support has cardinality $< \kappa$. It is a truncation closed subfield of $k((G))$. We denote by $k((G^{\geq 0}))_\kappa$ its valuation ring. Note that $k((G))_\kappa$ contains the monic monomials. We denote by $k((G^{<0}))_\kappa$ the complement to $k((G^{\geq 0}))_\kappa$. We have proved

Proposition 1. Set $K = k((G))_\kappa$. Then $(K, +, 0, <)$ decomposes lexicographically as the sum:

$$(4) \qquad\qquad (K, +, 0, <) = k((G^{<0}))_\kappa \oplus k((G^{\geq 0}))_\kappa .$$

$(K^{>0}, \cdot, 1, <)$ decomposes lexicographically as the product:

$$(5) \qquad\qquad (K^{>0}, \cdot, 1, <) = \mathrm{Mon}\,(K) \times U_v^{>0}$$

$\mathrm{Mon}\,(K)$ is order isomorphic to G through the isomorphism $t^g \mapsto -g$.

Proposition 1 allows us to define an exponential (logarithm) on $k((G))_\kappa$ by defining the logarithm on $\mathrm{Mon}\,(K)$ and on $U_v^{>0}$:

Theorem 2. *Let Γ be a chain, $G = (\mathbb{R}^\Gamma)_\kappa$ and $K = \mathbb{R}((G))_\kappa$. Assume that*

$$l : \Gamma \to G^{<0}$$

is an embedding of chains. Then l induces an embedding of ordered groups (a prelogarithm)

$$log \ : (K^{>0}, \cdot, 1, <) \longrightarrow (K, +, 0, <)$$

as follows: given $a \in K^{>0}$, write $a = t^g r(1 + \varepsilon)$ (with $g = \sum_{\gamma \in \Gamma} g_\gamma \mathbf{1}_\gamma$, $r \in \mathbb{R}^{>0}$, ε infinitesimal), and set

$$(6) \qquad\qquad \log(a) := -\sum_{\gamma \in \Gamma} g_\gamma t^{l(\gamma)} + \log r + \sum_{i=1}^\infty (-1)^{(i-1)} \frac{\varepsilon^i}{i}$$

This prelogarithm satisfies

$$(7) \qquad\qquad v(\log t^g) = l(\min \mathrm{support}\, g)$$

Moreover, $\log$ is surjective (a logarithm) if and only if l is surjective, and $\log$ satisfies (GA) if and only if

$$(8) \qquad\qquad l(\min \mathrm{support}\, g) > g \qquad \textit{for all } g \in G^{<0}\ .$$

Finally, to get a pair admitting such a **surjective** l as in the theorem, we start with any pair consisting of a chain Γ and a section ι (i.e. an embedding) of Γ into $G^{<0}$, and then construct by transfinite induction over the pair (Γ, ι) the κ-th iterated lexicographic closure $(\Gamma_\kappa, \iota_\kappa)$, thus forcing surjectivity of $l := \iota_\kappa$. In [K-S (2005)] Section 4, this is performed starting with the basic section $\iota : \Gamma \to G^{<0}$ defined by $\gamma \mapsto -\mathbf{1}_\gamma$. We note that since the elements in the image of the section have singleton support, all the initial fundamental monomials $\gamma \in \Gamma$ become log-atomic in the construction. To avoid producing log-atomic monomials, it suffices to start instead with a section for which the support of the images are not singleton. The details will appear in our joint work in preparation (initiated during the workshop) with A. Berarducci, V. Mantova and M. Matusinski.

The transserial structure of surreal numbers

Vincenzo Mantova

(joint work with Alessandro Berarducci)

In this Special Session, we discuss the problem of constructing and classifying derivations of Hardy type on fields of transseries, of generalised power series, and surreal numbers. The motivation for this stems from the theory of Hardy fields and their abstract counterpart, the H-fields, its connection with transseries, as presented in the talk by A. Berarducci.

In this talk, we focus on the case of surreal numbers. We call *surreal derivation* an $\mathbb{R}$-linear function $D : \mathbf{No} \to \mathbf{No}$ satisfying the fundamental equation

$$D\left(\sum_{i<\alpha} r_i e^{\gamma_i}\right) = \sum_{i<\alpha} r_i e^{\gamma_i} D(\gamma_i)$$

and moreover such that $D(a) > 0$ when $a > \mathbb{R}$. One can verify that a function satisfying these conditions is a derivation making $\mathbf{No}$ into an H-field. Several such functions do indeed exist, among which a "simplest" one $\partial : \mathbf{No} \to \mathbf{No}$ [10]. Moreover, $\mathbf{No}$ equipped with the simplest ∂ turns out to be a universal H-field with small derivation and standard kernel [8].

It is tempting to use the fundamental equation as an inductive definition to construct surreal derivations. However, such induction is not well-founded on $\mathbf{No}$. To overcome this obstacle, it is fundamental to understand the *transserial structure* of $\mathbf{No}$, by which we mean the interaction between the function exp and the structure of Hahn field. The nature of this interaction was conjectured in [60], and later confirmed in [10]. We shall present some detail about this, with an eye towards surreal derivations.

1. Log-atomic numbers

Call a positive infinite surreal number a *log-atomic* when a and all its iterated logarithms are monomials, i.e., of the form ω^b. The existence of such numbers is already non-obvious, but it was already observed by Gonshor that ω and all ϵ-numbers (the solutions of $\epsilon = \omega^\epsilon$) are log-atomic [57]. The fundamental equation does not give information about the value of D on log-atomic numbers, so it is necessary to first find all log-atomic surreal numbers.

The approach to this was started in [75] and completed in [10]. We introduce two equivalence relations, both coarsening of the Archimedean equivalence. Given $a, b \in \mathbf{No}^{>\mathbb{R}}$, we say that:

- a and b are in the same exp-log *class*, and write $a \sim_{\exp} b$, if $\log_k(b) \leq a \leq \exp_k(b)$ for some $k \in \mathbb{N}$ [75];
- a and b are in the same *level*, and write $a \sim_\ell b$, if $\log_k(a) \sim \log_k(b)$ for some $k \in \mathbb{N}$ [98, 10].

One can check that level equivalence is a refinement of exp-log equivalence.

Just as Conway defined the omega map to describe canonical representatives for Archimedean equivalence, one can give canonical representatives for the above equivalence relations:

$$\kappa_a := \left\{ k, \exp_k(a^L) \mid \log_k(a^R) \right\},$$

$$\lambda_a := \left\{ k, \exp_k(h \cdot \log_k(a^L)) \mid \exp_k\left(\frac{1}{h} \cdot \log_k(a^R) \right) \right\},$$

for k, h running over the positive natural numbers. One can easily verify that $\kappa_{\mathbf{No}}$ and $\lambda_{\mathbf{No}}$ are complete classes of representatives for respectively exp-log equivalence and level equivalence. Some of these number can be made explicit: for instance, $\kappa_0 = \lambda_0 = \omega$, while $\kappa_1 = \epsilon_0$, $\lambda_1 = \exp(\omega)$. Moreover, for all $a \in \mathbf{No}$, $\lambda_{a+1} = \exp(\lambda_a)$ [8].

By carefully calculating the sign sequences of the κ numbers, S. Kuhlmann and M. Matusinski proved that all such numbers are log-atomic.

Theorem 1.1 (Kuhlmann-Matusinski [75]). *For all $a \in \mathbf{No}$, κ_a is log-atomic.*

Moreover, they strictly contain the ϵ-numbers (e.g., $\kappa_0 = \omega$, $\kappa_1 = \epsilon_0$).

Theorem 1.2 (Berarducci-M. [10]). *For all $a \in \mathbf{No}$, λ_a is log-atomic, and all log-atomic numbers are of this form.*

In particular, all κ numbers are of the form λ_b for some $b \in \mathbf{No}$. One can verify that λ_b is a κ number if and only if b is purely infinite. It would be interesting to have a description of the map $f : \mathbf{No} \to \mathbf{No}$ such that $\kappa_a = \lambda_{f(a)}$.

We give some details about the proof of the latter theorem. Start with any $a_0 \in \mathbf{No}^{\mathbb{N}}$. First, truncate all the terms in the normal form of a_0 except for the leading term and replace its coefficient by 1, obtaining a new number a_1. Then do the same truncation on the exponent of the remaining monomial, obtaining a_2, and so on. We thus obtain the sequence

$$a_0 = r_0 e^{\gamma_0} + \cdots \to a_1 = e^{\gamma_0} = e^{s_0 e^{\delta_0} + \cdots} \to a_2 = e^{e^{\delta_0}} = e^{e^{t_0 e^{\varepsilon_0} + \cdots}} \to a_3 = e^{e^{e^{\varepsilon_0}}} \ldots.$$

By studying the interaction between exp and simplicity, one can prove that a_1 is simpler than a_0, then a_2 is simpler than a_1, and so on. In particular, since simplicity is *well-founded*, the sequence becomes constant, and its eventual value must be some log-atomic number λ. Moreover, one can prove that $\lambda \sim_\ell a$. If $a = \lambda_b$, since λ_b is by construction the simplest number in its level, we must have $\lambda = \lambda_b$, so λ_b is log-atomic (see [10]).

2. Nested truncation and T4

Let $\mathbb{L}$ be the class of log-atomic numbers. If one chooses appropriately the values of D on $\mathbb{L}$, then the fundamental equation gives an inductive construction of the unique extension of D to "$\mathbb{R}\langle\langle \mathbb{L} \rangle\rangle$", the smallest subfield of $\mathbf{No}$ containing all of $\mathbb{L}$ and closed under exp and infinite sum ([101, 10]). This subfield is the largest one satisfying condition ELT4 of [75].

This is still not sufficient to construct a surreal derivation, as $\mathbb{R}\langle\langle\mathbb{L}\rangle\rangle \neq \mathbf{No}$. However, the described procedure to prove that λ-numbers are log-atomic generalises to the notion of *nested truncation* (omitted). One observes that a nested truncation of a number is necessarily *simpler* than the original number; therefore, any sequence of numbers, each a nested truncation of the previous one, must be eventually constant.

This behaviour is equivalent to the following conditions, isolated by van der Hoeven and Schmeling [101], and called "T4": for any sequence $(\gamma_n : n \in \mathbb{N})$ of surreal numbers whose normal form can be written as

$$\gamma_n = \sum_{i<\alpha_n} r_{n,i} e^{\gamma_{n,i}} + r_n e^{\gamma_{n+1}} + \underbrace{\sum_{\alpha_n<i<\beta_n} r_{n,i} e^{\gamma_{n,i}}}_{\delta_n},$$

we have $\delta_n = 0$ and $r_n = \pm 1$ for all n sufficiently large.

Theorem 2.1 (Berarducci-M. [10]). *T4 holds in* **No**.

In turn, T4 can be used to show that the derivations on $\mathbb{R}\langle\langle\mathbb{L}\rangle\rangle$ can be extended in a canonical way to **No**. Therefore, provided we fix a starting partial derivation on $\mathbb{L}$ satisfying some suitable compatibility conditions, such partial derivation extends to a full surreal derivation.

Theorem 2.2 (Berarducci-M. [10]). *There exist several surreal derivations, among which a "simplest one"* $\partial : \mathbf{No} \to \mathbf{No}$.

A precise formulation of the compatibility conditions for which a partial derivation on $\mathbb{L}$ extends to **No** can possibly be given by combining the results of [101], [73] and [10] (part of which is discussed in the following talk of this Special Session by M. Matusinski).

A given (compatible) choice of a partial derivation on $\mathbb{L}$ determines a unique derivation on $\mathbb{R}\langle\langle\mathbb{L}\rangle\rangle$. It is an open problem whether there can be different extensions to a surreal derivations to **No**. A possible answer could be given by assigning non-zero values to the 'right-most paths' and checking that this still yields a surreal derivation.

Derivations on generalized series fields and Exp-Log fields.

Mickael Matusinski

The aim of this second talk of the Special Session is to provide a quick survey on the relevant results from [74, 73] in connection with the problem presently studied.

1. Derivations on generalized series fields.

In [74], we study the question of endowing generalized series fields with well-behaved derivations. This setting for our work was motivated by the following classical facts. By Hahn's Embedding Theorem [58], Hahn groups are universal domains for ordered abelian groups, and by Kaplansky's Embedding Theorem [67] generalized series fields are universal domains for valued fields.

The (multiplicative) Hahn group Γ over a given totally ordered set (or possibly proper class) Φ is by definition the lexicographically ordered group of formal products

$$\alpha = \prod_{\phi \in \Phi} \phi^{\alpha_\phi}$$

where $\alpha_\phi \in \mathbb{R}$ and the support of this product if anti-well-ordered. Thus Φ – the fundamental monomials – are naturally included in the so-called group of monomials Γ. It consists in canonical representatives of the multiplicative equivalence classes, the values of the natural valuation on Γ that we denote by LF (for leading fundamental monomial). E.g. in **No**, one can take canonically $\Phi = \omega^{\omega^{\mathbf{No}}}$. Also, we will use the class of log-atomic $\mathbb{L}$ as fundamental monomials.

Given an ordered abelian group Γ – e.g. the Hahn group over some chain Φ – the field of generalized series $\mathbb{R}((\Gamma))$ with real coefficients and monomials in Γ consists in formal power series $a = \sum_{\alpha \in \Gamma} r_\alpha \, \alpha$. The ordering $\preccurlyeq$ on Γ extends on $\mathbb{R}((\Gamma))$ to a dominance relation. The latter is equivalent to the natural valuation, which we call the leading monomial LM(a) of a nonzero series a.

We are interested in so-called series derivations, i.e. derivations d such that:

(D0): $\ker d = \mathbb{R}$;

(D1) Strong Leibniz rule: $\forall \alpha \in \Gamma, \; d(\alpha) = \alpha \cdot \sum_{\phi \in \Phi} \alpha_\phi \dfrac{d(\phi)}{\phi}$;

(D2) Strong additivity: $\forall a \in \mathbb{R}((\Gamma)), \; d(a) = \sum_{\alpha \in \Gamma} r_\alpha d(\alpha)$.

We want also the derivation to be similar to the derivation in a Hardy field:

(HD0): $\ker d$ is isomorphic to the residue field of the natural valuation;

(HD1) Strong L'Hospital's rule: $\forall a, b \not\asymp 1, \; a \succcurlyeq b \Leftrightarrow d(a) \succcurlyeq d(b)$;

(HD2) Rule for logarithmic derivatives: $\forall |a| \succ |b| \succ 1, \; \dfrac{d(a)}{a} \succcurlyeq \dfrac{d(b)}{b}$.

Our main results in [74] consist in giving an explicit criterion on a map $d : \Phi \to \mathbb{R}((\Gamma)) \setminus \{0\}$ so that it extends to a well-defined derivation d on $\mathbb{R}((\Gamma))$ via (D1) and (D2), and such that d is of Hardy type. More specifically, the corresponding version for a derivation of monomial type is as follows:

Proposition 1.1. *A map* $d : \Phi \to \mathbb{R} \cdot \Gamma \setminus \{0\}$ *is extends to a well-defined derivation of Hardy type on* $\mathbb{R}((\Gamma))$ *if and only if:*

$$\forall \phi \prec \psi, \mathrm{LM}\left(\frac{d(\phi)}{\phi}\right) \prec \mathrm{LM}\left(\frac{d(\psi)}{\psi}\right) \;\; and \;\; \mathrm{LF}\left(\mathrm{LM}\left(\frac{d(\psi)}{\psi}\right) \Big/ \mathrm{LM}\left(\frac{d(\phi)}{\phi}\right)\right) \prec \psi.$$

As an example, given a map $\sigma : \Phi \to \Phi$ order preserving and such that $\sigma(\phi) \prec \phi$ for any ϕ, one can choose for each $\phi \in \Phi$:

$$d(\phi) := \phi \prod_{n \in \mathbb{Z}_{>0}} \left(\sigma^n(\phi)\right)^{\theta_{\phi,n}},$$

for some $\theta_{\phi,n} \in \mathbb{R}$, $\theta_{\phi,1} > 0$. Note that the derivation $\partial'_{\mathbb{L}}$ in [10], for $\Phi = \mathbb{L}$ and $\sigma = \log$, can be obtained via these formulas.

2. Derivations on Exp-Log series fields.

In [73], we investigate how to endow prelogarithmic generalized series fields and Exp-Log series fields in the sense of [72] with compatible derivations of Hardy type as above, i.e. such that $d(\log(a)) = d(a)/a$. Conversely, starting with a generalized series field with derivation, we give an explicit criterion to obtain a compatible prelogarithm.

By a prelogarithm on $\mathbb{R}((\Gamma))$, we mean a map $\log$ extending $\log$ on $\mathbb{R}_{>0}$ such that:

$$\begin{aligned} \log: \quad (\mathbb{R}((\Gamma)))_{>0} \quad &\to \quad \mathbb{R}((\Gamma)) \\ a = \alpha \cdot r_\alpha \cdot (1+\varepsilon) \quad &\mapsto \quad \log(\alpha) + \log(r_\alpha) + \sum_{n \in \mathbb{Z}_{>0}} \frac{(-1)^{n+1}\varepsilon^n}{n}, \end{aligned}$$

with $\log(\alpha) = \sum_{\phi \in \Phi} \alpha_\phi \log(\phi)$. As a key particular case, take the prelogarithm induced by $\log = \sigma$ as above on Φ. For any $\psi \in \Phi$, let us denote its convex orbit:

$$\mathcal{C}_\psi := \{\phi \,/\, \exists k, \; \sigma^k(\psi) \preccurlyeq \phi \preccurlyeq \sigma^{-k}(\psi)\},$$

and the corresponding truncation of a monomial:

$$\mathrm{Trunc}_{\mathcal{C}_\psi}(\alpha) := \prod_{\phi \in \mathcal{C}_\psi} \phi^{\alpha_\phi}.$$

We proved that:

Proposition 2.1. *There exists a compatible monomial derivation of Hardy type on $\mathbb{R}((\Gamma))$ if and only if for any $\phi \prec \psi$:*

$$\mathrm{Trunc}_{\mathcal{C}_\psi}\left(\frac{\mathrm{LM}(d(\phi)/\phi)}{\mathrm{LM}(d(\psi)/\psi)}\right) = \mathrm{Trunc}_{\mathcal{C}_\psi}\left(\prod_{k \in \mathbb{Z}_{>0}} \frac{\sigma^k(\psi)}{\sigma^k(\phi)}\right).$$

In particular, $\mathrm{LM}(d(\sigma(\phi))/\sigma(\phi)) = \dfrac{\mathrm{LM}(d(\phi)/\phi)}{\sigma(\phi)}.$

This proposition, combined with the other results in [73] (extension of the derivation to the corresponding Exp-Log field $\mathbb{R}((\Gamma))^{\mathrm{EL}}$) and with other results in [10] (extension of the derivation from $\mathbb{R}((\Gamma))^{\mathrm{EL}}$ to **No** where the Hahn group Γ is taken over $\Phi = \mathbb{L}$, the class of log-atomic numbers) might lead to many new examples of derivations on **No** and perharps to a description of the space of compatible monomial derivations on **No**.

Closed ordered differential fields

Marcus Tressl

An ordered differential field (ODF) is a an ordered field $(K, \leq)$ equipped with a derivation $\delta : K \longrightarrow K$; no interaction of $\leq$ and δ is assumed. A closed ordered differential field (CODF) is an existentially closed ODF. In [105], Singer has shown that CODFs are axiomatizable and constitute the model completion with quantifier elimination of ODFs in the first order language $\{+, -, \cdot, \leq, 0, 1, \delta\}$.

We give a brief overview of know results, produce a geometric axiomatisation and prove an intermediate value theorem in the CODF context.

1. Geometric axiomatisation.   The original axioms for CODF can be found in [105]. Geometrically, an ordered differential field K is a CODF if and only if K (as a field) is real closed and for all $n \in \mathbb{N}$ and every irreducible variety $V \subseteq R^n \times R$ of dimension n that is not of the form $W \times R$ for a subvariety $W \subseteq R^n$, the set of regular points of V of the form $(a, \delta(a), \ldots, \delta^n(a))$, is dense in the regular points of V; the topology being the euclidean topology.

2. Summary of properties of CODFs.

(1) CODF has NIP as follows easily from Singer's quantifier elimination result.

(2) If M is a CODF, then $M[i]$ is a differentially closed field [104].

(3) There is an embedding theorem for differentially finitely generated ODFs into germs of real meromorphic functions, similar to Seidenberg's embedding theorem for differential fields of characteristic 0. However, unlike in Seidenberg's theorem, there is no relative version of the embedding theorem for ODFs and it is not known if there is a model of CODF within germs of real meromorphic functions. See [105].

(4) There is a cell decomposition theorem and a notion of geometric differential dimension which agrees with transcendence degree on types, see [18],[93].

(5) CODF has an o-minimal open core, see [89] who deduces this from (4). In fact, using the geometric axiomatisation given above one can show directly that every continuous definable function $M^n \to M$, $M \models CODF$ is semi-algebraic and so every closed definable subset of M^n is semi-algebraic.

(6) Every model of CODF is definably complete (hence every bounded definable subset has a supremum). Further, CODF eliminates $\exists^\infty$, i.e., for every definable family in a CODF, there is some $N \in \mathbb{N}$ such that each member of the family of size at least N, is infinite. Both items follow easily from (5): For the first one, consider the closure of the set in question. For the second one, use compactness and observe that a definable set is finite just if its closure is discrete.

(7) CODF has elimination of imaginaries, see [89].

(8) By an unpublished note of A. Onshuus, CODF is rosy.

(9) The positive solution to Hilbert's 17th problem for the real field transfers to CODFs, see [19].

(10) There is a model of CODF whose underlying field is the field of real numbers. This is shown in Q. Brouette's PhD thesis based on earlier work of

C. Michaux. However, the proof is a pure existence statement. There are no known natural models of CODF.

(11) Existence and uniqueness of Picard-Vessiot extensions of CODFs have been shown in [31] and generalised in [66].

There is also a model completion with quantifier elimination of ordered fields equipped with several commuting derivatives, see [107] and [92] for an explicit axiomatisation. Again, these structures have the NIP, but apart from that much less is known compared to the ordinary case.

3. The Intermediate Value Theorem in CODF

Let $M \models$ CODF and let $s : M^{n+1} \to M$ be continuous and semi-algebraic. Let $\bar{a}, \bar{b} \in M^{n+1}$ with $a_0 < b_0$ and suppose $s(\bar{a}) < 0 < s(\bar{b})$. Then there is some $c \in R$ with $a_0 < c < b_0$ such that $s(c, c', \ldots, c^{(n)}) = 0$.

When s is a polynomial in $n+1$ variables, considered as a differential polynomial in 1 variable, of order $\leq n$, then it follows that for all $a, b \in M$ with $s(a) < 0 < s(b)$, there is a some $c \in M$ between a and b such that $s(c) = 0$.

Integer Parts of Real Closed Fields

Salma Kuhlmann

In this talk we survey the results of our papers on this topic. An **integer part (IP)** Z of an ordered field K is a discretely ordered subring (1 is least positive element) such that $\forall x \in K \, \exists z \in Z : \ z \leq x < z + 1$. Here $z := \lfloor x \rfloor$ is the Gauß bracket. **Peano arithmetic (PA)** is the first-order theory, in the language $L := \{+, \cdot, <, 0, 1\}$, of discretely ordered commutative rings with 1 whose set of non-negative elements satisfies, for each formula $\Phi(x, \overline{y})$, the associated induction axiom: $\forall \overline{y} \, [\Phi(0, \overline{y}) \& \forall x \, [\Phi(x, \overline{y}) \to \Phi(x + 1, \overline{y})] \to \forall x \Phi(x, \overline{y})]$. An ordered field K is **real closed** if every positive element has a square root in K, and every polynomial in $K[x]$ of odd degree has a root. **Open Induction (OI)** is the fragment of PA obtained by taking the induction axioms associated to open formulas only. Shepherdson proved that IP's of real closed fields are precisely the models of OI. **Remark:** $\mathbb{Z}$ is an IP of K iff K is archimedean ([Hölder] iff K is isomorphic to a subfield of $\mathbb{R}$). We will only consider **non-archimedean** fields. An ordered field K need not admit an IP, see [71]. In general, different IP's need not be isomorphic, not even elementarily equivalent.

Q: *Does every real closed field admit an IP? If yes, how to construct such?*

First construct divisible ordered abelian groups (DOAG): Let Γ be any ordered set, $\{A_\gamma \, ; \, \gamma \in \Gamma\}$ a family of divisible archimedean groups (subgroups of $\mathbb{R}$). For $g \in \prod_\Gamma A_\gamma$, set support$g := \{\gamma \in \Gamma \, ; \, g_\gamma \neq 0\}$ The **Hahn group** is the subgroup of $\prod_\Gamma A_\gamma \ \mathbf{H}_\Gamma A_\gamma := \{g \, ; \, \text{support}g \text{ is well-ordered in } \Gamma\}$ ordered lexicographically by "first differences". The **Hahn sum** is the subgroup $\oplus_\Gamma A_\gamma := \{g \, ; \, \text{support}g \text{ is finite}\}$. **Hahn's embedding Theorem** states that a divisible ordered abelian group with rank Γ and archimedean components $\{A_\gamma \, ; \, \gamma \in \Gamma\}$ is (isomorphic to) a subgroup of $\mathbf{H}_\Gamma A_\gamma$.

Next construct real closed fields (RCF): Let G be any divisible ordered abelian group, k a real closed archimedean field (a real closed subfield of $\mathbb{R}$). The **Hahn field** is $k((G)) = \{s = \sum_{g \in G} s_g t^g$; support s is w.o. in $G\}$ the field of generalized power series, with convolution multiplication (Cauchy product) and lexicographic order. The Hahn field $\mathbb{K}$ is a **valued** field: the map $v : \mathbb{K} \to G \cup \infty$ $v(s) := \min$ support s is a **valuation** with **valuation ring** $\mathcal{O} := k((G^{\geq 0}))$, **group of units** $\mathcal{O}^\times$, **valuation ideal** $\mathcal{M} := k((G^+))$, **residue field** k, **value group** G. **Kaplansky Embedding's Theorem** states that if K is a real closed field with residue field k and value group G, then K is (analytically isomorphic to) a subfield of a field of $k((G))$.

So we know how to construct all DOAG and all RCF, now we want to construct IP of RCF. To this end the **direct sum (respectively product) decompositions** are useful:

$$\begin{array}{ccccccc} k((G)) & = & k((G^-)) & \oplus & k & \oplus & k((G^+)) \\ k((G))^{>0} & = & t^G & \times & k^+ & \times & [\, 1 + k((G^+)) \,] \end{array}$$

Proposition: $Z := k((G^-)) \oplus \mathbb{Z}$ is an IP of $\mathbb{K}$.
Observation: If F is a **truncation closed** subfield of $\mathbb{K}$ ($\forall s : s \in F$ implies $s_{<0} \in F$), then $Z_F := [k((G^-)) \cap F] \oplus \mathbb{Z}$ is an IP of F.

Mourgues-Ressayre or Kaplansky revisited: Let K be real closed field with residue field k and value group G. Then K is (isomorphic to) a truncation closed subfield of a field of $k((G))$, thus K has an IP (which is a model of OI).
Remark: A valuation theoretic interpretation of truncation closed embeddings was given in [54]. A Truncation Integer Part (TIP) is never normal and is never a model of PA, see [21] and [22]. Their prime and irreducible elements were studied in [13], establishing a criterion for primality in Corollary 4.14 and 4.15, in showing in particular the primality of the series $\sum t^{g/n} + 1$. We now address the following **Q**: *Does a RCF admit an IP that is a model of PA?*
To this end the following is useful. **Observation:** The graph of the exponential function $2^y = z$ on $\mathbb{N}$ is definable by an L-formula, and PA proves the basic properties of exponentiation. Thus any model of PA is endowed with an **exponential function** exp. This provides a key connection to real closed exponential fields which we shall now explain and exploit. Indeed, the direct sum (respectively product) decompositions hold for any RCF K with valuation ring $\mathcal{O}$, value group G and residue field $\overline{K}$: $(K, +) = \mathbb{A} \oplus \mathcal{O}$ and $(K^+, \cdot) = \mathbb{B} \times \mathcal{O}_+^\times$, where $\mathbb{A}$ and $\mathbb{B}$ are unique up to isomorphism, the rank of $\mathbb{A}$ is (isomorphic to) G^-, its archimedean components are (isomorphic to) $\overline{K}$ and $\mathbb{B} \simeq G$. A RCF K has **left exponentiation** iff there is an isomorphism from a group complement $\mathbb{A}$ of $\mathcal{O}$ in $(K, +, 0, <)$ onto a group complement $\mathbb{B}$ of $\mathcal{O}_+^\times$ in $(K^+, \cdot, 1, <)$. Let G be a DOAG with rank Γ and archimedean components $\{A_\gamma : \gamma \in \Gamma\}$. We say that G is an **exponential group** (in C) if Γ is isomorphic (as linear order) to the negative cone G^-, and each A_γ is isomorphic (as ordered group) to C, for some archimedean group C. A Characterization of countable exponential groups is given in [72]. This is closely related to recursively saturated RCF, see [34]. It was further observed in [72] that

if K admits a left exponential, then the value group G of K is an exponential group in $\overline{K}$. In [23] we show that if K admits an IPA (i.e K is an IPA real closed field), then K admits a left exponential, therefore the value group of K is an exponential group in $\overline{K}$.

Remark: There are plenty of DOAG that are not exponential groups in $\overline{K}$. For example, take the Hahn group $G = \mathbf{H}_{\gamma \in \Gamma} A_\gamma$ where the archimedean components A_γ are divisible but not all isomorphic and/or Γ is not a dense linear order without endpoints (say, a finite Γ). Alternatively, we could choose all archimedean components to be divisible and all isomorphic, say to C, and Γ to be a dense linear order without endpoints, but choose the residue field so that $\overline{K}$ not isomorphic to C.

A class of not IPA real closed fields: Let k be any real closed subfield of $\mathbb{R}$. Let $G \neq \{0\}$ be any DOAG which is *not* an exponential group in k. Consider the Hahn field $k((G))$ and its subfield $k(G)$ generated by k and $\{t^g : g \in G\}$. Let K be any real closed field satisfying $k(G)^{rc} \subseteq K \subseteq k((G))$ where $k(G)^{rc}$ is the real closure of $k(G)$. Any such K has G as value group and k as residue field. By Corollary above, K does not admit an IPA.

We now summarize important feedback that we got during or after the talk:

• What do IPA RCF look like? Models of PA define numerous other fast growing functions, of particular interest would be to investigate a definition of the Ackermann function in the context of RCF.

• We conjectured that the field of surreal numbers No admits an integer part which is a model of true arithmetic. It was pointed out by some participants, in response to this question, that No should behave like a saturated, hence resplendent, real closed field, and therefore such an IP Z can be constructed. Note that Z will not be TIP **Oz** of **No**.

• Fornasiero asked whether an IPA real closed field actually admits a total exponential function (not just a left-exponential function as we proved). During the workshop already, Berarducci and Fornasiero started working on these issues. They aim to show that any model of true arithmetic is an IP of a model of $T_{\exp}$. This is closely related to results of [20].

• We asked whether every real closed field admits a normal integer part. It turns out that this is closely related to the existence of dense subfields and the results of [71], which we will further investigate.

Irreducible generalized power series, Part I

Sonia L'Innocente

(joint work with Vincenzo Mantova)

If K is a field and G an additive abelian ordered group, a *formal series* with *coefficients* in K and *exponents* in G is a formal sum $a = \sum_\gamma a_\gamma t^\gamma$, where $a_\gamma \in K$ and $\gamma \in G$. We call *support* of a the set $S_a := \{\gamma \in G : a_\gamma \neq 0\}$. A formal series a is said to be a *generalised power series* if its support S_a is well-ordered. The collection of all generalised power series, denoted by $K((G))$, is a field with respect to the obvious operations $+$ and $\cdot$ defined for ordinary power series (see [58]).

When K is ordered, then $K((G))$ has a natural order as well, obtained by stipulating that $0 < t^\gamma < a$ for any $\gamma \in G^{>0}$ and for any positive element a of the field K. The field $K((G))$ is a valuable tool for the study of real closed fields. One can use them to prove, for instance, that every real closed field R has an integer part (i.e., a subring Z such that for all $x \in R$ there exists a unique integer part $\lfloor x \rfloor \in Z$ of x such that $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$) [85]. For example, $\mathbb{Z} + K((G^{<0}))$ is an integer part of $K((G))$, where $K((G^{<0}))$ is the subring of the series with the support contained in the negative part $G^{<0}$ of the group G.

The ring $\mathbb{Z} + K((G^{<0}))$ has a non-trivial arithmetic behaviour, some of which is already visible in $K + K((G^{<0})) = K((G^{\le 0}))$. Assuming the divisibility of G, $K((G^{\le 0}))$ is non-noetherian, as for instance we have $t^{-1} = t^{-\frac{1}{2}} \cdot t^{-\frac{1}{2}} = t^{-\frac{1}{4}} \cdot t^{-\frac{1}{4}} \cdot t^{-\frac{1}{4}} \cdot t^{-\frac{1}{4}} = \ldots$. The general aim is to understand how the elements can be factorized. We provide here a new class of irreducibles and prove some further cases of uniqueness of the factorisation.

Berarducci [9] proved that $K((G^{\le 0}))$, when $\mathbb{Q} \subseteq G$, contains irreducible series, such as $1 + \sum_n t^{-\frac{1}{n}}$, answering a question of Conway [29]; in fact, the result implies that $1 + \sum_n t^{-\frac{1}{n}}$ is irreducible in the ring of omnific integers, which are the natural integer part of surreal numbers.

In order to state Berarducci's result, let the *order type* $\mathrm{ot}(a)$ of a power series $a \in K((G^{\le 0}))$ be the ordinal number representing the order type of its support S_a. Moreover, let J be the ideal of the series that are divisible by t^γ for some $\gamma \in G^{<0}$ (as noted before for $\gamma = -1$, such series cannot be factored into irreducibles, since $t^\gamma = t^{\frac{\gamma}{2}} t^{\frac{\gamma}{2}} = \ldots$).

Theorem 1 ([9, Thm. 10.5]). *If $a \in K((\mathbb{R}^{\le 0})) \setminus J$ (equivalently, $a \in K((\mathbb{R}^{\le 0}))$ not divisible by t^γ for any $\gamma < 0$) has order type ω^{ω^α} for some ordinal α, then both a and $a + 1$ are irreducible.*

This result was obtained by constructing a function resembling a valuation but taking values into ordinal numbers.

Definition 2 ([9, Def. 5.2]). *For $a \in K((G^{\le 0}))$, the* order-value $v_J(a)$ *of a is:*

(1) *if $a \in J$, then $v_J(a) := 0$;*
(2) *if $a \in J + K$ and $a \notin J$, then $v_J(a) := 1$;*
(3) *if $a \notin J + K$, then $v_J(a) := \min\{\mathrm{ot}(a') : a - a' \in J + K\}$.*

The difficult key result of [9] is that for $G = \mathbb{R}$ the function v_J is *multiplicative*.

Theorem 3 ([9, Thm. 9.7]). *For all $a, b \in K((\mathbb{R}^{\le 0}))$ we have $v_J(ab) = v_J(a) \odot v_J(b)$ (where $\odot$ is Hessenberg's natural product on ordinal numbers).*

This immediately implies, for instance, that the ideal J is prime, so the quotient ring of *germs* $K((\mathbb{R}^{\le 0}))/J$ is an integral domain (in fact, J is prime for arbitrary choices of G, see [88]), and also each elements admitis a factorization into irreducibles.

The above comments and theorems support and motivate the following conjectures. If $a = b_1 \cdot \ldots \cdot b_n$ is a factorisation of a series a, possibly with some reducible

factors, a *refinement* is another factorisation of a obtained by replacing each b_i with a further factorisation of b_i. More formally, a refinement is a factorisation $a = c_1 \cdot \ldots \cdot c_m$ such that, up to reordering $c_1, \ldots, c_m$, $b_i = k_i \cdot c_{m_i+1} \cdot \ldots \cdot c_{m_{i+1}}$ for some constants $k_i \in K^*$ and some natural numbers $0 = m_1 \leq \cdots \leq m_{n+1} = m$.

Conjecture 4 (Conway [29]). For every non-zero series $a \in K((\mathbb{R}^{\leq 0}))$, any two factorisations of a admit common refinements.

For instance, it is easy to verify that for all $\gamma < 0$, any two factorisations of t^γ admit a common refinement.

Conjecture 5 (Berarducci [9]). Every non-zero germ in $K((\mathbb{R}^{\leq 0}))/J$ admits a unique factorisation into irreducibles.

Berarducci's work was partially strengthened by Pitteloud [87], who proved that the a(n irreducible) series in $K((\mathbb{R}^{\leq 0}))$ of order type ω or $\omega + 1$ with $v_j(a) = \omega$ are actually prime.

Adapting Pitteloud's technique, we shall prove that the germs of order-value ω are prime in $K((\mathbb{R}^{\leq 0}))/J$; in particular, the germs of order-value at most ω^3 admit a unique factorisation into irreducibles, supporting Berarducci's conjecture.

Theorem 6. *All germs in $K((\mathbb{R}^{\leq 0}))/J$ of order-value ω are prime. Every non-zero germ in $K((\mathbb{R}^{\leq 0}))/J$ of order-value $\leq \omega^3$ admits a unique factorisation into irreducibles.*

Moreover, we shall isolate the notion of *germ-like* series: we say that $a \in K((\mathbb{R}^{\leq 0}))$ is germ-like if either $\mathrm{ot}(a) = v_J(a)$ or $v_J(a) > 1$ and $\mathrm{ot}(a) = v_J(a) + 1$. The main result of [9] can be rephrased as saying that germ-like series of order-value ω^{ω^α} are irreducible, while the main result of [87] is that germ-like series of order-value ω are prime. Moreover, Pommersheim and Shahriari [90] proved that germ-like series of order-value ω^2 have a unique factorisation, and that some of them are irreducible.

By generalising an argument in [9], we shall see that germ-like series always have factorisations into irreducibles. Together with Pitteloud's result, we shall be able to prove that the factorisation into irreducibles of germ-like series of order-value at most ω^3 must be unique.

Theorem 7. *All non-zero germ-like series in $K((\mathbb{R}^{\leq 0}))$ admit factorisations into irreducibles. Every non-zero germ-like series in $K((\mathbb{R}^{\leq 0}))$ of order-value $\leq \omega^3$ admits a unique factorisation into irreducibles.*

For completeness, we shall also verify that irreducible germs and series of order-value ω^3 do exist.

Theorem 8. *There exist irreducible germs in $K((\mathbb{R}^{\leq 0}))/J$ and irreducible series in $K((\mathbb{R}^{\leq 0}))$ of order-value ω^3.*

To prove this result, we follow a strategy similar to the one of [90]. In particular, in order to find a sufficient criterion for irreducibility of series of order-value ω^3, we picture a series $a \in K((\mathbb{R}^{\leq 0}))$ of order-value $\omega^{\alpha+1}$ as if it were a series of

order-value ω with coefficients that are themselves series of order-value ω^α. In other words, we describe a as the sum of ω series of order-value ω^α.

We propose the following conjecture, which seems to be a reasonable intermediate statement between Conway's conjecture and Berarducci's conjecture.

Conjecture 9. Every non-zero germ-like series in $K((\mathbb{R}^{\leq 0}))$ admits a unique factorisation into irreducibles.

The definition of germ-like in terms of critical point suggests an alternative multiplicative order-value map whose value is the *first* term of the Cantor normal form of the order type, rather than the last infinite one. By using this order value, we prove in a second part of the work that the series in $K((\mathbb{R}^{\leq 0}))$ admit factorisation into irreducibles and a "small part".

Irreducible generalised power series, part II

Vincenzo Mantova

(joint work with Sonia L'Innocente)

We now report on a new, unpublished work in progress on the factorisation of generalised power series. We shall introduce a new valuation on the ring $K((\mathbb{R}^{\leq 0}))$, the degree, which in a sense refines Berarducci's valuation v_J, and we shall give some valuation-theoretic constructions which in turn give a best-possible result about the existence of a factorisation into irreducibles. We shall then lift the result to omnific integers.

1. A new valuation with ordinal values

Recall that for $b \in K((\mathbb{R}^{\leq 0}))$, Berarducci's order-value $v_J(b)$ can be 0, 1, or the last infinite term of the Cantor normal form of $\mathrm{ot}(b)$. We define a new valuation by looking instead at the first term of such Cantor normal form, based on an easy observation on critical points [81].

Definition 1.1. Given $b \in K((\mathbb{R}^{\leq 0}))$, the *degree* of b, denoted by $\deg(b)$, is the ordinal α such that ω^α is the first term of the Cantor normal form of $\mathrm{ot}(b)$ (with the convention that $0 = \omega^{-\infty}$).

Theorem 1.2. *For all $b, c \in K((\mathbb{R}^{\leq 0}))$,*

- $\deg(b + c) \leq \max\{\deg(b), \deg(c)\}$;
- $\deg(b \cdot c) = \deg(b) \oplus \deg(c)$.

The proof of this statement is rather short, and it relies heavily on the hard results in [9]. In fact, even the argument itself is inspired by a short lemma in [9] which already contains all the necessary ingredients.

2. A factorisation theorem

A series b has degree 0 if and only if its support is finite. We shall denote the subring of series with finite support by $K(\mathbb{R}^{\leq 0})$. It follows at once from the above theorem that any series $b \in K((\mathbb{R}^{\leq 0}))^*$ can be factored as $b = b_1 \cdot \ldots \cdot b_n$, where each b_i has the following property: if $b_i = cd$, then $\deg(c) = 0$ or $\deg(d) = 0$ (hence $c \in K(\mathbb{R}^{\leq 0})$ or $d \in K(\mathbb{R}^{\leq 0})$). We shall prove that each b_i has a maximal divisor in $K(\mathbb{R}^{\leq 0})$, and so it can be factor as a series of degree 0 and an actually irreducible series.

Theorem 2.1 (L'Innocente-M.). *For all $b \in K((\mathbb{R}^{\leq 0}))^*$, we can write*

$$b = p \cdot b_1 \cdot \ldots \cdot b_n$$

where $p \in K(\mathbb{R}^{\leq 0})$ and each b_i is irreducible with infinite support. Moreover, p is unique up to multiplication by an element of K^*.

Note moreover that the problem of factoring series in $K(\mathbb{R}^{\leq 0})$ is well-known (see e.g. [51]), and in particular that $K(\mathbb{R}^{\leq 0})$ is a GCD-domain. Therefore, the conjecture of common refinement reduces to asserting that all irreducible series are prime.

The proof of this theorem goes through a construction of valuation-theoretic flavor. We write $b \sim c$ when $\deg(b - c) < \deg(b)$. This is an equivalence relation, and we denote the equivalence class of b by $[b] := \{c : c \sim b\}$. We then define the ring $\mathcal{H}$ as the free ring generated by the classes $[b]$ modulo the following relations:

- $[b] \cdot [c] = [bc]$;
- if $\deg(b) = \deg(c)$ and $\deg(b + c) = \deg(b)$, then $[b] + [c] = [b + c]$;
- if $\deg(b) = \deg(c)$ and $\deg(b + c) < \deg(b)$, then $[b] + [c] = [0]$.

The map $b \mapsto [b]$ from $K((\mathbb{R}^{\leq 0}))$ to $\mathcal{H}$ is *multiplicative*, but in general it does not preserve sums, so it is not a ring homomorphism. On the other hand, $p \mapsto [p]$ is an embedding of rings when restricted to $p \in K(\mathbb{R}^{\leq 0})$. We shall identify $K(\mathbb{R}^{\leq 0})$ with its isomorphic copy in $\mathcal{H}$.

Let $\mathcal{P}$ the ring generated by the equivalence classes $[b]$ of the series $b \in K((\mathbb{R}^{\leq 0}))$ such that $\mathrm{ot}(b) = \omega^\alpha$ for some $\alpha \in \mathbf{On}$ and $\sup(\mathrm{supp}(b)) = 0$. We can then prove the following factorisation of the ring $\mathcal{H}$.

Theorem 2.2 (L'Innocente-M.). $\mathcal{H} = \mathcal{P} \otimes_K K(\mathbb{R}^{\leq 0})$.

This technical observation is now crucial for proving the theorem. To illustrate this, let us verify that each element $B \in \mathcal{H}$ has a maximal divisor in $K(\mathbb{R}^{\leq 0})$. Choose any K-linear basis $\{C_i\}$ of $\mathcal{P}$. Then there is a unique expression of B as

$$B = \sum_i p_i C_i$$

with $p_i \in K(\mathbb{R}^{\leq 0})$. Moreover, only finitely many such series p_i are non-zero. It is now easy to verify that $p \mid B$ if and only if $p \mid p_i$ for all i. Since $K(\mathbb{R}^{\leq 0})$ is a GCD-domain, we deduce immediately that the maximal divisor of B in $K(\mathbb{R}^{\leq 0})$ is the greatest common divisor of all the series p_i. Such observations can be transferred

back to the ring $K((\mathbb{R}^{\leq 0}))$ with some straightforward arguments by induction on the degree, and ultimately lead to the proof of the factorisation theorem.

3. Factorisation of omnific integers

Our factorisation theorem does not extend as stated to the ring $K((G^{\leq 0}))$ for an arbitrary divisible ordered group G. For instance, when G is not Archimedean, there are series b with infinite support which are divisible by all series whose support is "negligible" with respect to b (see [13] for examples in this sense). Using techniques inspired by [9] and [13], we can still extend the theorem, provided with relax appropriately the notion of irreducibility. We now do this in the case of the ring of Conway's *omnific integers* **Oz** [29]:

$$\mathbf{Oz} := \{x \in \mathbf{No} : x = \{x - 1 \mid x + 1\}\} = \mathbb{R}((\omega^{\mathbf{No}^{>0}})) + \mathbb{Z}.$$

For a surreal number $b \in \mathbf{No}$, let $\mathrm{ord}(b)$ be the maximum surreal number in its support; in other words, it is the unique surreal number such that $b \asymp \omega^{\mathrm{ord}(b)}$. We first relax irreducibility as follows.

Definition 3.1. Given $b \in \mathbf{Oz}$, we say that b is **pseudo-irreducible** if whenever $b = cd$ for some $c, d \in \mathbf{Oz}$ we have $\mathrm{ord}(c) \prec \mathrm{ord}(b)$ or $\mathrm{ord}(d) \prec \mathrm{ord}(b)$.

Moreover, we note that when $b \in \mathbf{Oz}$, for any $x \in \mathrm{supp}(b)$, $\frac{x}{\mathrm{ord}(b)}$ is a finite surreal number, so it has a standard part in $\mathbb{R}$. We denote the standard part by st.

Definition 3.2. Given $b \in \mathbf{Oz}$, we call the **coarse support** of b the set

$$\overline{\mathrm{supp}}(b) := \left\{ \mathrm{st}\left(\frac{x}{\mathrm{ord}(b)}\right) \in \mathbb{R} : x \in \mathrm{supp}(b) \right\}.$$

We say that b is **monic** if $0 \notin \overline{\mathrm{supp}}(b - 1)$.

Theorem 3.3 (L'Innocente-M.)**.** *For all $b \in \mathbf{On}$, we can write*

$$b = p \cdot c \cdot \omega^x \cdot b_1 \cdot \dots \cdot b_n$$

where:

- *p has finite coarse support, is monic, and either $p = 1$ or $\mathrm{ord}(p) \asymp \mathrm{ord}(b)$;*
- *$b_1, \dots, b_n$ are pseudo-irreducible with infinite coarse support;*
- *$\mathrm{ord}(c) \prec \mathrm{ord}(b)$.*

Moreover, p is unique up to multiplication by a non-zero real number, and the convex class $x + o(\mathrm{ord}(b))$ is unique.

We can also say something more depending on whether $\overline{\mathrm{supp}}(b)$ has a maximum:

- if $\overline{\mathrm{supp}}(b)$ has a maximum, then we may assume that $b_1, \dots, b_n$ are monic irreducible, in which case c and x are unique;
- if $\overline{\mathrm{supp}}(b)$ has no maximum, then c can be any series such that $\mathrm{ord}(c) \prec \mathrm{ord}(b)$ and x can be any value in $x + o(\mathrm{ord}(b))$.

Surreal Numbers and generalised computable analysis

Lorenzo Galeotti

(joint work with Merlin Carl, Benedikt Löwe and Hugo Nobrega)

In classical computability theory one studies the computational properties of functions over natural numbers and transfers these properties to arbitrary countable spaces via codings. A similar approach is taken in computable analysis. By using codings, in fact, one can transfer computational results from the Baire space ω^ω to sets of cardinality $2^{\aleph_0}$. In particular, by encoding the real numbers, one can use the Baire space to study computability in the context of real analysis.

Of particular interest in computable analysis is the study of the computational content of theorems from classical analysis. The idea is that of formalizing the complexity of theorems by means similar to those used in computability theory to classify functions over the natural numbers. In this context, the Weihrauch theory of reducibility plays an important role. For an introduction to the theory of Weihrauch reductions see [17]. Weihrauch reductions can be used to classify functions over the Baire space ω^ω. By using this concept it is possible to arrange many theorems from classical real analysis in a complexity hierarchy called the Weihrauch hierarchy. A study of the Weihrauch degrees of some of the most important theorems from real analysis can be found in [16, 15].

Recently, the study of the descriptive set theory of the generalised Baire spaces κ^κ for cardinals $\kappa > \omega$ has been catching the interest of set theorists (see [55] for an overview on the subject). This fact is also witnessed by the increasing number of workshops dedicated to generalised Baire spaces that have been very successfully organized in the last three years[2].

In this talk, we are exploring a version of computable analysis for generalised Baire space.

The first step in this generalisation is that of finding a generalisation of $\mathbb{R}$ on which we can prove a version of theorems from classical analysis. The problem of generalising the real line is not new in mathematics. Different approaches have been tried for very different purposes. A good introduction to these number systems can be found in [46]. Among the most influential contributions to this field particularly important are the works of Sikorski [103] and Klaua [68] on the *real ordinal numbers* and that of Conway [29] on the *surreal numbers*. Sikorski's idea was to repeat the classical Dedekind construction of the real numbers starting from an ordinal equipped with the Hessenberg operations (i.e., commutative operations over the ordinal numbers). Unfortunately, one can prove that these fields do not have the density properties that, as we will see, have a central role in the context of real analysis. The surreal numbers were introduced by Conway in order to generalise both the Dedekind construction of real numbers and the Cantor construction of ordinal numbers. In his introduction to surreal numbers, Conway

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[2]The Bonn Set Theory Workshop on Generalized Baire spaces in 2016, the Hamburg Workshop on Set Theory (HST) in 2015 and the Amsterdam Workshop on Set Theory (AST) in 2014.

proved that they form a (class) real closed field. Later, Ehrlich [44] proved that every real closed field is isomorphic to a subfield of the surreal numbers, showing that they behave like a universal (class) model for real closed fields. It is then natural for us to use this framework in the developing of generalisation of $\mathbb{R}$ to uncountable cardinals.

The first part of the talk will be devoted to the presentation of the construction of the real closed field $\mathbb{R}_\kappa$, an extension of $\mathbb{R}$ suitable for generalising computable analysis to uncountable cardinals κ (see [56]).

In the second part of the talk we will show how to do analysis over $\mathbb{R}_\kappa$. In particular we will present results from [24] on generalisations of the Bolzano-Weierstraß Theorem (BWT) for real closed field extensions of $\mathbb{R}$. We will show that due to its incompleteness $\mathbb{R}_\kappa$ can not satisfy the classical version of the BWT. Moreover we will show that $\mathbb{R}_\kappa$ does not even satisfy the BWT restricted to κ-sequences which was proved true by Sikorski on real ordinal numbers for regular κ. Finally we will introduce a new generalised version of the BWT and we will show that it holds on $\mathbb{R}_\kappa$ if and only if κ is a weakly-compact cardinal.

In the last part of the talk we will present results from [86]. We will use Ordinal Turing Machine, introduced by P. Koepke in [70], to extend the theory of type two computability to uncountable cardinals. In particular, given an uncountable cardinal κ such that $\kappa^{<\kappa} = \kappa$, we will define a notion of type two κ-computability over the generalised Cantor space 2^κ. Then we will follow the classical theory to induce a notion of computability over spaces of cardinality 2^κ. In particular we will show that, as in the classical case, under suitable codings, the field operations over $\mathbb{R}_\kappa$ are computable.

Number systems with simplicity hierarchies
Elliot Kaplan
(joint work with Philip Ehrlich)

As a full lexicographically ordered binary tree, Conway's ordered field **No** of surreal numbers has a rich simplicity hierarchical structure in which sums and products are the simplest elements consistent with **No**'s ordered field structure, it being understood that x is simpler than y just in case x is a predecessor of y in the tree. Many familiar structures and classes of structures, such as the ordinals, real numbers, all divisible ordered abelian groups, and all real-closed ordered fields emerge as initial substructures of **No**, that is, as substructures A of **No** where $y \in A$ whenever $x \in A$ and y is simpler than x.

In [47], the algebraico-tree-theoretic simplicity hierarchical structure of **No** was brought to the fore and employed to provide necessary and sufficient conditions for an ordered field to be isomorphic to an initial subfield of **No**. In this talk, we establish corresponding results for ordered abelian groups and ordered domains. These results are employed to characterize the convex subgroups and convex subdomains of initial subfields of **No** that are themselves initial, and we further show that an initial subdomain of **No** is discrete if and only if it is a subdomain of **No**'s

canonical integer part $\mathbf{Oz}$ of omnific integers. We end by extending the results of [47] to show that the theories of nontrivial divisible ordered abelian groups and real-closed ordered fields are the sole theories of nontrivial densely ordered abelian groups and ordered fields all of whose models are isomorphic to initial subgroups and initial subfields of $\mathbf{No}$.

Let $\mathbb{R}((t^\Gamma))_{\mathbf{On}}$ be the ordered group (ordered domain; ordered field) of power series consisting of all formal power series of the form $\sum_{\alpha < \beta} r_\alpha t^{y_\alpha}$ where $(y_\alpha)_{\alpha < \beta \in \mathbf{On}}$ is a possibly empty descending sequence of elements of an ordered class (ordered commutative monoid; ordered abelian group) Γ and $r_\alpha \in \mathbb{R} - \{0\}$ for each $\alpha < \beta$. A subclass $A \subseteq \mathbb{R}((t^\Gamma))_{\mathbf{On}}$ is said to be *truncation closed* if whenever $\sum_{\alpha < \beta} r_\alpha t^{y_\alpha} \in A$ and $\sigma \le \beta$, we also have $\sum_{\alpha < \sigma} r_\alpha t^{y_\alpha} \in A$. The subclass A is said to be *cross sectional* if $\{t^y : y \in \Gamma\} \subseteq A$. For $y \in \Gamma$ we let $\mathbb{R}_y$ denote the *y-coefficient group* $\{r \in \mathbb{R} : rt^y \in A\}$. Making use of the fact (due to Conway [29]) that every surreal number can be uniquely written in the form $\sum_{\alpha < \beta} \omega^{y_\alpha} \cdot r_\alpha$ where $(y_\alpha)_{\alpha < \beta}$ is a descending sequence of surreal numbers and each r_α is a nonzero real number, we see that there is a canonical isomorphism of ordered fields from $\mathbf{No}$ onto $\mathbb{R}((t^{\mathbf{No}}))_{\mathbf{On}}$ that sends each surreal number $\sum_{\alpha < \beta} \omega^{y_\alpha} \cdot r_\alpha$ to $\sum_{\alpha < \beta} r_\alpha t^{y_\alpha}$. In [47], it was shown that:

A subfield of $\mathbf{No}$ is initial if and only if it is isomorphic (via the canonical isomorphism) to a truncation closed, cross sectional subfield of a power series field $\mathbb{R}((t^\Gamma))_{\mathbf{On}}$, where Γ is an initial subgroup of $\mathbf{No}$.

Let $\mathbb{D}$ denote the ring of dyadic rationals: $\{m/2^n : m \in \mathbb{Z}, n \in \mathbb{N}\}$. We have the analogous theorems for groups and domains:

Theorem 1. *A subgroup of $\mathbf{No}$ is initial if and only if it is isomorphic (via the canonical isomorphism) to a truncation closed, cross sectional subgroup G of a power series group $\mathbb{R}((t^\Gamma))_{\mathbf{On}}$, where (i) Γ is an initial ordered subclass of $\mathbf{No}$, (ii) every y-coefficient group $\mathbb{R}_y$ of G is an initial subgroup of $\mathbb{R}$, and (iii) $\mathbb{D} \subseteq \mathbb{R}_y$ whenever $x, y \in \Gamma$, y is greater than x, and y is simpler than x.*

Theorem 2. *A subdomain of $\mathbf{No}$ is initial if and only if it is isomorphic (via the canonical isomorphism) to a truncation closed, cross sectional subdomain K of a power series domain $\mathbb{R}((t^\Gamma))_{\mathbf{On}}$, where (i) Γ is an initial submonoid of $\mathbf{No}$, (ii) every y-coefficient group $\mathbb{R}_y$ of K is an initial subgroup of $\mathbb{R}$, and (iii) $\mathbb{D} \subseteq \mathbb{R}_y$ whenever $x, y \in \Gamma$, y is greater than x, and y is simpler than x.*

In the case that the subdomain of $\mathbf{No}$ in Theorem 2 is densely ordered, conditions *(ii)* and *(iii)* can be replaced with the condition that $\mathbb{D}$ is a subdomain of K. Let $\mathbf{Oz}$ denote the omnific integers: a canonical integer part of $\mathbf{No}$ consisting of the surreal numbers of the form $x = \{x - 1 \mid x + 1\}$. Then for discretely ordered domains, we have the additional result:

Theorem 3. *An initial subdomain of $\mathbf{No}$ is discrete if and only if it is a subdomain of the omnific integers $\mathbf{Oz}$.*

It is easy to find an example of a discrete initial subgroup of $\mathbf{No}$ which is not contained in $\mathbf{Oz}$, but the following question is open:

Question 1. Is every discrete initial subgroup of **No** isomorphic to an initial subgroup of **Oz**?

Using Theorems 1 and 2, we can identify the convex subgroups of initial subgroups of **No** that are themselves initial, as well as the convex subdomains of initial subfields of **No** that are likewise initial. A nontrivial initial subgroup A of **No** is said to be α-*Archimedean* if α is the height of $\mathbf{On} \cap A$ considered as a subtree of A. Every nontrivial initial subgroup of **No** is ω^ϕ-Archimedean for some nonzero ordinal ϕ and every nontrivial initial subdomain of **No** is ω^{ω^ϕ}-Archimedean for some ordinal ϕ.

If A is α-Archimedean initial subgroup of **No**, then for each infinite ordinal $\beta < \alpha$, let

$$A[\beta] =: \{x \in A : -\rho < x < \rho \text{ for some } \rho < \beta\}.$$

When β is of the form ω^τ, then $A[\beta]$ is a subgroup of A. When A is a domain and β is of the form ω^{ω^τ}, then $A[\beta]$ is a subdomain of A. We have the following results:

Theorem 4. *Let A be an ω^ϕ-Archimedean initial subgroup of* **No**. *Then K is a nontrivial initial convex subgroup of A if and only if $K = A[\omega^\tau]$ for some nonzero $\tau \leq \phi$.*

Theorem 5. *Let A be an ω^{ω^ϕ}-Archimedean initial subfield of* **No**. *Then K is an initial convex subdomain of A if and only if $K = A\left[\omega^{\omega^\tau}\right]$ for some $\tau \leq \phi$.*

We end the talk by showing that the results in regarding divisible ordered abelian groups and real-closed ordered fields are optimal in the following sense: let T^D and T^{DIV} be the theories of nontrivial densely ordered abelian groups and nontrivial divisible ordered abelian groups in the language $\{\leq, +, 0\}$ of ordered additive groups, and let T^{OF} and T^{RCF} be the theories of ordered fields and real-closed ordered fields in the language $\{\leq, +, \cdot, 0, 1\}$ of ordered fields.

Theorem 6. *(i) If T is a theory in $\{\leq, +, 0\}$ containing T^D, then every model of T is isomorphic to an initial subgroup of* **No** *if and only if $T = T^{DIV}$.*

(ii) If T is a theory in $\{\leq, +, \cdot, 0, 1\}$ containing T^{OF}, then every model of T is isomorphic to an initial subfield of **No** *if and only if $T = T^{RCF}$.*

The "if" portions of these statements are proved in [47]. The "only if" portions make critical use of class models, which raises the question:

Question 2. Can the "only if" portions of Theorem 6 be established in NBG appealing solely to models whose universes are sets?

O-minimal exponential fields and their residue fields
Lothar Sebastian Krapp

This extended abstract shall give an overview of some results and questions under investigation in my doctoral reasearch project *Algebraic and Model Theoretic Properties of O-minimal Exponential Fields*[3]. Its aim is the study of o-minimal exponential fields, which exhibit strong connections to the decidability problem of the real exponential field as well as Schanuel's Conjecture.

1. O-MINIMAL EXPONENTIAL FIELDS

Definition 1.1. Let $(K, +, \cdot, 0, 1, <)$ be an ordered field. An *exponential* exp on K is an order-preserving isomorphism from the ordered additive group $(K, +, 0, <)$ to the ordered multiplicative group $(K^{>0}, \cdot, 1, <)$. The structure $\mathcal{K}_{\exp} = (K, +, \cdot, 0, 1, <, \exp)$ is called an *ordered exponential field*. The inverse log of exp is called the *logarithm* on $\mathcal{K}_{\exp}$.

Definition 1.2. An ordered structure $(M, <, \dots)$ is called *o-minimal* if every parametrically definable subset of M is a finite union of intervals and points.

Example 1.3. 1. Let $\exp_{\mathbb{R}}$ denote the standard exponential function $x \mapsto e^x$ on $\mathbb{R}$. The most prominent ordered exponential field is the real exponential field $\mathcal{R}_{\exp} = (\mathbb{R}, +, \cdot, 0, 1, <, \exp_{\mathbb{R}})$. It was shown in [109] that $T_{\exp}$, the theory of $\mathcal{R}_{\exp}$, is model complete and o-minimal.

2. Let $\mathcal{K}_{\exp}$ be an ordered exponential field. Set $F_0 := \mathbb{Q}$ and inductively $F_{i+1} := F_i \left(\exp(F_i), \log \left(F_i^{>0} \right) \right)$. Then the exponential-logarithmic closure of $\mathbb{Q}$ given by $\mathbb{Q}^{\exp} := \bigcup_{i=0}^{\infty} F_i$ is the domain of an ordered exponential field with exponential exp. Note that it is possible for $\mathbb{Q}^{\exp}$ to be a non-archimedean field.

Schanuel's Conjecture (SC) [real case]. *Let $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ be $\mathbb{Q}$-linearly independent. Then $\mathrm{td}_{\mathbb{Q}} \mathbb{Q} \left(\alpha_1, \dots, \alpha_n, e^{\alpha_1}, \dots, e^{\alpha_n} \right) \geq n$.*

Based on the model completeness of $T_{\exp}$, it was shown in [83] that, under the assumption of (SC), $T_{\exp}$ is decidable.

Theorem 1.4. ([83]) *Assume (SC). Then $T_{\exp}$ is decidable.*

In fact, the decidability of $T_{\exp}$ is equivalent to the following weaker form of (SC).

First Root Conjecture. *Let $n \in \mathbb{N} \setminus \{0\}$ and $f \in \mathbb{Z}[x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}]$. Then one can effectively find $\eta(n, f) \in \mathbb{N}$ such that if f has a zero in $\mathbb{R}^n$, then it also has a zero $\alpha \in \mathbb{R}^n$ with $\|\alpha\| < \eta(n, f)$.*

Proposition 1.5. *Let $\mathcal{K}_{\exp}$ be an o-minimal exponential field. Then exp is differentiable with derivative $\exp' = \exp'(0) \exp$.*

Let EXP be the conjunction of the axioms for ordered exponential fields together with an axiom stating $\exp' = \exp$. The main aim of this project is to investigate whether (SC) implies the following conjecture.

Conjecture 1.6. *Let* $\mathcal{K}_{\exp}$ *be o-minimal and* EXP*. Then* $\mathcal{K}_{\exp} \equiv \mathcal{R}_{\exp}$*.*

Theorem 1.7. ([12]) *Assume Conjecture 1.6. Then* $T_{\exp}$ *is decidable.*

Since the decidability of $T_{\exp}$ is equivalent to a weaker version of (SC), Theorem 1.7 justifies why it might be necessary to assume (SC) for a proof of Conjecture 1.6.

Theorem 1.8. ([80]) *Let* $\mathcal{K}_{\exp}$ *be o-minimal,* EXP *and archimedean. Then* $\mathcal{K}_{\exp} \preccurlyeq \mathcal{R}_{\exp}$*.*

Theorem 1.8 implies that Conjecture 1.6 is true if the underlying ordered exponential field is archimedean. An approach towards our aim is therefore an investigation of the relation of an o-minimal exponential field with its archimedean residue exponential field.

2. Residue Exponential Fields

Let K be an ordered field, v the natural valuation on K, $\mathcal{O}_v$ the valuation ring, $\mathcal{I}_v$ the valuation ideal, $\mathcal{U}_v^{>0} = \{x \in K^{>0} \mid v(x) = 0\}$ the multiplicative group of positive units, and $\overline{K} = \mathcal{O}_v/\mathcal{I}_v$ the residue field of K.

Defintion 2.1. Let exp be an exponential on K. Then exp is called *v-compatible* if $\exp(\mathcal{I}_v) = 1 + \mathcal{I}_v$ and $\exp(\mathcal{O}_v) = \mathcal{U}_v^{>0}$.

Theorem 2.2. ([72]) *Let* $\mathcal{K}_{\exp}$ *be an ordered exponential field. Then the following are equivalent:*
  1. exp *is v-compatible.*
  2. $v(\exp(1) - 1) = 0$*.*
  3. $\overline{\exp} : \overline{K} \to \overline{K}^{>0}, \overline{a} \mapsto \overline{\exp(a)}$ *defines an exponential on* $\overline{K}$*.*

Definition 2.3. If exp is *v-compatible*, we call $\overline{\mathcal{K}_{\overline{\exp}}} = \left(\overline{K}, +, \cdot, \overline{0}, \overline{1}, <, \overline{\exp}\right)$ the *residue exponential field* of $\mathcal{K}_{\exp}$.

The following proposition is a consequence of Theorem 2.2 and the existence of Taylor expansions in o-minimal structures.

Proposition 2.4. *Let* $\mathcal{K}_{\exp}$ *be o-minimal and* EXP*. Then* exp *is v-compatible and* $\overline{\mathcal{K}_{\overline{\exp}}}$ *is* EXP*. Moreover,* $\overline{\mathcal{K}_{\overline{\exp}}}$ *is a substructure of* $\mathcal{R}_{\exp}$*.*

3. Open Questions

In the following, we will state some conjectures which are possible approaches to our main conjecture.

Conjecture 3.1. *Let* $\mathcal{K}_{\exp}$ *be o-minimal and* EXP*. Then* $\overline{\mathcal{K}_{\overline{\exp}}} \equiv \mathcal{K}_{\exp}$*.*

Conjecture 3.2. *Let* $\mathcal{K}_{\exp}$ *be o-minimal and* EXP*. Then* $\overline{\mathcal{K}_{\overline{\exp}}}$ *is o-minimal.*

Conjecture 3.3. *Let* $\mathcal{K}_{\exp}$ *be o-minimal and* EXP*. Then* $\overline{\mathcal{K}_{\overline{\exp}}} \preccurlyeq \mathcal{K}_{\exp}$*.*

Conjecture 3.1 implies Conjecture 1.6 by Theorem 1.8. Moreover, Conjecture 3.1 directly implies Conjecture 3.2. Finally, Conjecture 3.3 implies Conjecture 3.1 and is thus the strongest conjecture. A proof of any of these might be based on the assumption of (SC).

Progress during the workshop: It was pointed out by Antongiulio Fornasiero that Conjecture 3.2 might be implied by general results on o-minimal structures without the assumption of (SC) in [14, 108]. During the workshop I finished the proof that (SC) implies that for any $\mathcal{K}_{\exp}$ which is o-minimal and EXP its exponential residue field $\overline{\mathcal{K}_{\exp}}$ is embeddable as a structure into $\mathcal{K}_{\exp}$. This is a necessary condition for Conjecture 3.3. Tobias Kaiser mentioned that it might also be possible to derive this result from [37, 35]. A result from [36] presented during the workshop is that $(\mathbf{No}, \exp)$, the surreal numbers with its exponential, elementarily extends $\mathcal{R}_{\exp}$. Thus, a new approach to Conjecture 1.6. would be showing that any o-minimal and EXP exponential field is elementarily embeddable into $(\mathbf{No}, \exp)$.

Quasi-ordered fields: A uniform approach to orderings and valuations
SIMON MÜLLER

A **quasi-order** (q.o.) $\leq$ on a set S is a binary reflexive and transitive relation on S. It defines an equivalence relation $\sim$ on S by setting $a \sim b :\Leftrightarrow a \leq b \wedge b \leq a$. In his note [52], S.M. Fakhruddin introduces the notion of **quasi-ordered fields** $(K, \leq)$ by demanding that $\leq$ is a total quasi-order on K satisfying

Q1 $x \sim 0 \Rightarrow x = 0$,
Q2 $0 \leq x \wedge y \leq z \Rightarrow xy \leq xz$,
Q3 $x \leq y \wedge z \not\sim y \Rightarrow x + z \leq y + z$.

The subject of [52] is to prove that if $(K, \leq)$ is a quasi-ordered field, then it is either an ordered field or else there is a valuation v on K such that

$$\forall a, b \in K : a \leq b \Leftrightarrow v(b) \leq' v(a),$$

where $\leq'$ denotes the order of the value group $v(K^*)$. In this case we denote the quasi-order $\leq$ also by $\leq_v$ and call it a **proper quasi-order** (p.q.o).
Fakhruddin obtains this dichotomy by distinguishing whether the equivalence class of 1 with respect to $\sim$, denoted by E_1, is trivial or not. If $E_1 = \{1\}$, then $\leq$ is antisymmetric and Fakhruddin shows that $(K, \leq)$ is either an ordered field or else the prime field of characteristic 2, whose unique quasi-order $0 < 1$ is induced by the trivial valuation. If $E_1 \neq \{1\}$, the quasi-order is induced by the valuation

$$v : K \to K^*/E_1 \cup \{\infty\}, \ a \mapsto \left\{ \begin{array}{ll} \infty, & a = 0 \\ aE_1, & a \neq 0 \end{array} \right.$$

Thus, quasi-ordered fields are a natural way to unify the theories of ordered and valued fields. We demonstrate this on the basis of two concrete theorems, namely the characterization of compatible valuations and the Baer-Krull Theorem.

If $(K, \leq)$ is a quasi-ordered field, a valuation v on K is **compatible** with $\leq$ iff $\forall a, b \in K : 0 \leq a \leq b \Rightarrow a \leq_v b$. If v and w are valuations on a field K, then v is a **coarsening** of w (or w is a **refinement** of v) iff $\forall a, b \in K : a \leq_w b \Rightarrow a \leq_v b$. Hence, if $(K, \leq)$ is a quasi-ordered field such that $\leq$ is induced by some valuation w, then a valuation v on K is compatible with $\leq$ iff v is a coarsening of w, i.e. the compatible valuations are precisely the coarser valuations.
If $\leq$ is an order, this is the usual notion of compatible valuations.

> **Theorem:** Let $(K, \leq)$ be a quasi-ordered field. For a valuation v on K with order $\leq'$ on $v(K^*)$, the following conditions are equivalent:
> (1) v is compatible with $\leq$,
> (2) the valuation ring $A := \{a \in K : 0 \leq' v(a)\}$ is convex w.r.t. $\leq$,
> (3) the maximal ideal $I := \{a \in K : 0 <' v(a)\}$ is convex w.r.t. $\leq$,
> (4) $I < 1$,
> (5) the quasi-order $\leq$ induces canonically via the residue map a quasi-order $\preceq$ on the residue field A/I, and $\leq$ is an order iff $\preceq$ is an order.

If the quasi-order $\leq$ is an order, this is a well-known result and can for instance be found in [79, Theorem 2.3 and Proposition 2.9]. In 2013, Kuhlmann, Matusinski and Point proved this characterization of compatible valuations in the case where $\leq$ is induced by some valuation [76, Theorem 2.2], and thereby, by exploiting Fakhruddin's dichotomy, for quasi-ordered fields in general. We give a uniform proof for this theorem, i.e. without using the dichotomy.

The equivalence of the first four conditions is proved similarly as in the ordered case. Interesting is the implication (3) $\Rightarrow$ (5). While it is convenient to work with positive cones in the ordered case (see the proof in [79]), there can be no such unary description for quasi-orders. One reason for this is that if the quasi-order is induced by some valuation, then any element in K is non-negative, i.e. the positive cone would coincide with the whole field K.
Hence, in order to adapt the proofs from the ordered case, we have to translate them from positive cones to orderings in the original binary sense. By doing so, we eventually get that the quasi-order induced by the residue map is given by

$$\overline{x} \preceq \overline{y} :\Leftrightarrow \exists c_1, c_2 \in I : x + c_1 \leq y + c_2.$$

To complete the proof, it remains to verify all the axioms of a quasi-ordered field. For this purpose, one needs the following two lemmas exploiting convexity.

> **Lemma:** Suppose I is convex, $a \in I$ and $b \in A \backslash I$. Then $a \not\sim b$.

> **Lemma:** Suppose I is convex. For any valuation unit $u \in A \backslash I$, we have
> $$0 \leq u \Rightarrow \forall c \in I : 0 \leq u + c$$
> $$u < 0 \Rightarrow \forall c \in I : u + c < 0.$$

With these lemmas in place, the rest of the proof is quite routine.

Note that above we fixed the quasi-order and let the valuation run. So it is natural to ask what happens the other way round, i.e. if we fix the valuation and

let the quasi-order run. The answer is given by the Baer-Krull Theorem. In the ordered case, this theorem states the following (see [50, Theorem 2.2.5]):

Theorem: Let K be a field, $v : K \to \Gamma \cup \{\infty\}$ a valuation on K with valuation ring A and maximal ideal I, and $\{\pi_j : j \in J\} \subseteq K^*$ such that $\{\overline{v(\pi_j)} : j \in J\}$ is a $\mathbb{F}_2$-basis of $\Gamma/2\Gamma$. There is a bijective correspondence

$$\{v\text{-compatible orders on } K\} \longleftrightarrow \{-1,1\}^J \times \{\text{orders on } A/I\}$$

More precisely: if $\leq$ is a v-compatible order on K, define $\eta_\leq : J \to \{-1,1\}$ by $\eta_\leq(j) = 1$ iff $\pi_j \geq 0$. Then $\leq \mapsto (\eta_\leq, \preceq)$ is the above bijection. Here $\preceq$ denotes the order on the residue field induced by $\leq$.

When we translated the theorem about compatible valuations from the ordered to the quasi-ordered setting, we just replaced "order" with "quasi-order" everywhere. Translating the Baer-Krull Theorem is more complicated. As already mentioned, if $(K, \leq)$ is a proper quasi-ordered field, any element is non-negative. But this means that the map η is trivial, i.e. $\eta = 1$. Thus, in the case quasi-ordered case, the bijective correspondence states as

$$\{v\text{-compatible q.o. on } K\} \longleftrightarrow \{\text{orders on } A/I \times \{-1,1\}^J\} \sqcup \{\text{p.q.o. on A}/I\}.$$

Recall that for the characterization of compatible valuations we fixed the quasi-order and let the valuation run. Then, if $\leq = \leq_w$ for some valuation w, the theorem characterizes the coarsenings w. In the Baer-Krull Theorem we fixed some valuation v on K and let the v-compatible quasi-orders run. If we restrict our attention to v-compatible proper quasi-orders, this characterizes the refinements of v.

The difficult part of the proof of the Baer-Krull Theorem is to construct the v-compatible quasi-order $\leq$ on K, given a map $\eta : J \to \{-1,1\}$ and a quasi-order $\preceq$ on the residue field A/I. Let us give the translation from the unary ordered case done in [50] to the binary quasi-ordered case that we need for our purposes. So let $x, y \in K$, not both zero (we define $0 \leq 0$ separately). Define $\gamma := \gamma_{x,y} := \max\{-v(x), -v(y)\} \in \Gamma$. Then there is some $a \in K^*$ such that

$$\gamma = \sum \gamma_j + 2v(a) = v\left(\prod \pi_j a^2\right).$$

We consider $x \prod \pi_j a^2$ and $y \prod \pi_j a^2$. By the choice of γ both of them are in the valuation ring, so we can take residues. Moreover, it is easy to verify that $x \prod \pi_j a^2$ is a unit iff $v(x) \leq v(y)$. This observation will be frequently used in the proof of the axioms of a quasi-order. We may now define $\leq$ by

$$x \leq y :\Leftrightarrow \left(\overline{x \prod \pi_j a^2} \preceq \overline{y \prod \pi_j a^2} \text{ and } \prod \eta(j) = 1\right) \text{ or}$$
$$\left(\overline{y \prod \pi_j a^2} \preceq \overline{x \prod \pi_j a^2} \text{ and } \prod \eta(j) = -1\right).$$

The hardest axioms to prove are transitivity and the axiom Q3, because there we have to deal with different $\gamma's$ and not only one. So we conclude this abstract by sketching how transitivity is shown.

Let $x \leq y$ and $y \leq z$. In the case where $\gamma_{x,y} = \gamma_{y,z} = \gamma_{x,z}$, the transitivity of $\leq$ follows immediately from the transitivity of $\preceq$. This occurs when two of the values $v(x), v(y)$ and $v(z)$ coincide and the third one is greater or equal than the other ones. So we may reduce to the case that there is a unique smallest value. In these three cases, still two of the three $\gamma's$ coincide. If for instance $v(x)$ is the uniquely smallest value, then $\gamma_{x,y} = \gamma_{x,z}$. By the observation stated above, both $y \prod \pi_j a^2$ and $z \prod \pi_j a^2$ are non-units (where the π_j and a^2 are used in the representation of $\gamma_{x,y}$, respectively $\gamma_{x,z}$), i.e. their residues equal zero. From there it is easy to verify that $x \leq z$. The case $v(z) < v(x), v(y)$ is dealth with analogously. Finally, the same kind of arguments shows that the assumption $v(y) < v(x), v(z)$ leads to a contradiction. Hence, $\leq$ is transitive.

Lebesgue measure and integration theory for the semialgebraic category over the field of surreal numbers
Tobias Kaiser

Costin, Ehrlich and Friedman have developed in [30] an integration theory on the field of surreal numbers for unary functions. It is also shown that from a set theoretic point of view the restriction to a tame setting is essential. We show how one can establish a full Lebesgue measure and integration theory for the category of semialgebraic classes and functions in arbitrary dimension on the field of surral numbers such that the main properties of the real Lebesgue measure and integration theory hold. The construction relies on model theoretic arguments (compare with [64]). By **No** we denote the field of surreal numbers.

1. Starting point

One starting point is the following consequence of the seminal work of Comte, Lion and Rolin [28, 82] on integration of globally subanalytic sets and functions over the reals (i.e. definable in the field $\mathbb{R}_{an}$ of real numbers with restricted analytic functions):

Fact 1: Let $n \in \mathbb{N}$ and let $p \in \mathbb{N}_0$.

(A) Let $A \subset \mathbb{R}^{p+n}$ be semialgebraic. The set $\mathrm{Fin}(A) := \{t \in \mathbb{R}^p \mid \lambda_n(A_t) < \infty\}$ is semialgebraic and the function $\mathrm{Fin}(A) \to \mathbb{R}_{\geq 0}, t \mapsto \lambda_n(A_t)$, is definable in the field $\mathbb{R}_{an,exp}$ of real numbers with restricted analytic functions and real exponentiation.

(B) Let $f : \mathbb{R}^{p+n} \to \mathbb{R}$ be semialgebraic. The set $\mathrm{Fin}(f) := \{t \in \mathbb{R}^p \mid f_t \text{ integrable}\}$ is semialgebraic and the function $\mathrm{Fin}(f) \to \mathbb{R}, t \mapsto \int f_t(x)\,dx$, is definable in $\mathbb{R}_{an,exp}$.

Here λ_n denotes the usual Lebesgue measure on $\mathbb{R}^n$. □

The other starting point is the following result by Van den Dries and Ehrlich [36]:

Fact 2: **No** equipped with restricted analytic functions and exponentiation is an elementary extension of the field $\mathbb{R}_{\text{an,exp}}$ of real numbers with restricted analytic functions and real exponentiation. □

2. Construction of the measure and integral

Let ∞ be an element which is bigger than every element of **No**.

Construction of the measure: Let $n \in \mathbb{N}$ and let $A \subset \mathbf{No}^n$ be semialgebraic. We define its measure $\lambda_n(A) \in \mathbf{No}_{\geq 0} \cup \{\infty\}$ as follows:

- Take a formula $\phi(x, y)$ in the language of ordered rings, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_q)$, and a point $a \in \mathbf{No}^q$ such that $A = \phi(\mathbf{No}^n, a)$.
- Then the graph of the function $F : \mathbb{R}^q \to \mathbb{R}$ given by

$$F(c) := \begin{cases} \lambda_n\big(\phi(\mathbb{R}^n, c)\big), & \lambda_n\big(\phi(\mathbb{R}^n, c)\big) < \infty, \\ & \text{if} \\ -1, & \lambda_n\big(\phi(\mathbb{R}^n, c)\big) = \infty. \end{cases}$$

  is by Fact 1 defined in $\mathbb{R}_{\text{an,exp}}$ by an $\mathcal{L}_{\text{an,exp}}$-formula $\psi(y, z)$ where $\mathcal{L}_{\text{an,exp}}$ denotes the canonical language of $\mathbb{R}_{\text{an,exp}}$.
- By Fact 2, the formula $\psi(y, z)$ defines in **No** the graph of a function $F_{\mathbf{No}} : \mathbf{No}^q \to \mathbf{No}$.
- A routine model theoretic argument shows that $F_{\mathbf{No}}(a)$ does not depend on the choices of ϕ, a and ψ.
- This allows us to define $\lambda_n(A) := F_{\mathbf{No}}(a)$ if $F_{\mathbf{No}}(a) \geq 0$, and $\lambda_n(A) = \infty$ otherwise (that is, $F_{\mathbf{No}}(a) = -1$). □

By a classical model theoretic transfer argument we obtain the elementary properties of the measure: finite additivity, monotonicity, translation invariance, product formula. Moreover, the measure reflects elementary geometry (for example, the measure of an interval is its length).

In the same way we construct the integral $\int f(x)dx = \int f \, d\lambda_n \in \mathbf{No}_{\geq 0} \cup \{\infty\}$ of a nonnegative semialgebraic function $f : \mathbf{No}^n \to \mathbf{No}_{\geq 0}$ and extend integration to general semialgebraic functions as usually by consisting the positive and the negative part. We also obtain the usual elementary properties: linearity, monotonicity.

3. Main results of integration

By the usual transfer argument we obtain semialgebraic versions of the transformation formula and of Lebesgue's theorem on dominated convergence. We present the latter:

Lebesgue's theorem on dominated convergence: Let $f : \mathbf{No}^{n+1} \to \mathbf{No}$, $(t, x) \mapsto f(t, x) = f_t(x)$, be semialgebraic. Assume that there is some integrable semialgebraic function $h : \mathbf{No}^n \to \mathbf{No}$ such that $|f_t| \leq |h|$ for all sufficiently large $t \in \mathbf{No}$. Then the semialgebraic function $\lim_{t \to \infty} f_t$ is integrable and

$$\int \lim_{t \to \infty} f_t \, d\lambda_n = \lim_{t \to \infty} \int f_t \, d\lambda_n.$$

$\square$

For the fundamental theorem of calculus and Fubini's theorem we use the results of Cluckers and D. Miller [25, 26, 27] who have extended the work of Comte, Lion and Rolin. We formulate the latter result.

Definition: A function $f : \mathbf{No}^n \to \mathbf{No}$ is called **constructible** if it is a finite sum of finite products of globally subanalytic functions and logarithms of positive globally subanalytic functions.

Fubini's theorem: Let $f : \mathbf{No}^{m+n} \to \mathbf{No}$ be a constructible function that is integrable. There is a constructible function $g : \mathbf{No}^m \to \mathbf{No}$ such that $g(x) = \int_{\mathbf{No}^n} f(x,y) dy$ for all $x \in \mathbf{No}^m$ such that $f_x : \mathbf{No}^n \to \mathbf{No}$ is integrable. Then g is integrable and

$$\int_{\mathbf{No}^{m+n}} f(x,y)\, d\lambda(x,y) = \int_{\mathbf{No}^m} g(x) dx.$$

$\square$

4. Outlook and open questions

Outlook: One can extend the above construction to obtain an integration theory for semialgebraic differential forms on semialgebraic **No**-submanifolds, including Stokes' theorem.

Open questions: Can one extend the measure and integration theory on the field of surreal numbers beyond the semialgebraic and globally subanalytic category? How about classes and functions defined by restricted analytic functions and exponentiation? Going back to the reals: Are parametrized integrals of functions definable in $\mathbb{R}_{an,exp}$ definable in an o-minimal extension?

Integration on the surreals: A Conjecture of Conway, Kruskal and Norton

Ovidiu Costin, Philip Ehrlich
(joint work with Harvey Friedman)

In his seminal work *On Numbers and Games* [29, 29], J. H. Conway introduced the system **No** of *surreal numbers*, a strikingly inclusive real-closed field containing the reals and the ordinals. An important subsequent advance was the extension from the reals to **No** of simple functions, including the log and the exponential by Bach, Norton, Conway, Kruskal, Gonshor and others (e.g [29, 57]). The definitions of these functions, like Conway's definitions of **No**'s field operations, are inductive and are based on the *simplicity hierarchical structure* central to **No** [47].

There has been a longstanding program, initiated by Conway, Kruskal and Norton, to develop analysis on **No**, starting with a consistent definition of integration. It was motivated in part by the broader goal of providing a new foundation for asymptotic analysis which would include new and more general tools for resumming divergent series and for solving complicated differential equations.

In real analysis and mathematical physics, the asymptotic series expansions at infinity of solutions to many problems have zero radius of convergence. A prototypical example of such a divergent series is that of the exponential integral: $e^{-x}\mathrm{Ei}(x) \sim \sum_{k=0}^{\infty} k! x^{-k-1}$, $x \to \infty$. In the 1980's Ècalle discovered the analyzable functions, a vast generalization of the analytic functions which allow for divergent expansions (possibly followed by series of exponential corrections multiplied by other divergent expansions). These expansions are called *transseries*. Écalle also devised the powerful tools of *accelero-summation* for resumming the divergent expansions that arise in most applications [41], [43]. For example, the resummation of $\sum_{k=0}^{\infty} k! x^{-k-1}$ is $e^{-x}\mathrm{Ei}(x)$.

In **No**, on the other hand, for all surreal $x > \infty$, $\sum_{k=0}^{\infty} k! x^{-k-1}$ is convergent in the sense of Conway. Accordingly, since the exponential integral, as well as most functions arising in applications are analytic for large $x \in \mathbb{R}$, the question naturally arises as to whether we can link the finite and infinite domains. In particular, building on convergence in the sense of Conway, can we find a way of extending functions and their integrals past ∞ or, more generally, past a singularity at which asymptotic expansions do not exist or are divergent?

The initial attempts at defining integration, in particular the schema proposed by Norton [29, page 227], turned out as Kruskal discovered to have fundamental flaws [29, page 228]. Despite this disappointment, the search for a theory of surreal integration has continued [53], [99] and remains largely open. Indeed, in his recent survey [102, page 438], Siegel characterizes the question of the existence of a reasonable definition of surreal integration as "perhaps the most important open problem in the theory of surreal numbers".

In this paper, we address the extension and integration problems with both positive and negative results.

In the positive direction, we show that extensions to **No**, and thereby integrals, exist for most functions covered by Écalle's theory, and surreal-based summation coincides with Écalle summation. In this direction, we are working on various ways of substantially simplifying resummation methods.

In the negative direction, however, we show that the existence of nice extensions and integrals of more general types of functions (e.g. smooth functions) is obstructed by considerations from the foundations of mathematics. In particular, we show that, in a sense made precise, there is no description which, provably in NBG, defines extensions or integrals (inductive or otherwise) from the finite to the infinite domain, even on spaces of entire functions that rapidly decay towards ∞. The fact that the obstructions to integration of general classes of functions originate in the foundations of mathematics, and that such obstructions are often hard to detect, explains why the question remained open for such a long time.

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