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Mini-Workshop: Perspectives in High-Dimensional Probability and Convexity

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Abstract. Understanding the geometric structure of systems involving a huge amount of parameters is a central problem in mathematics and applied sciences today. Here, geometric and analytical ideas meet in a non-trivial way and powerful probabilistic tools play a key role in many discoveries. Two essentially independent areas of mathematics concerned with high-dimensional problems are asymptotic geometric analysis and information-based complexity. In this Mini-Workshop we brought together researchers from both fields to explore the connections and form synergies to develop new perspectives.

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Introduction by the Organisers

The mini-workshop: *Perspectives in high-dimensional probability and convexity* was organized by Joscha Prochno (Hull, UK), Christoph Thäle (Bochum, Germany) and Elisabeth M. Werner (Cleveland, USA). In total, 16 participants attended the workshop.

High-dimensional systems are frequent in mathematics and applied sciences, and understanding of high-dimensional phenomena has become an increasingly important topic. Mathematical disciplines that are most strongly related to such phenomena are functional analysis, convex geometry and probability theory. In fact, a new area emerged, now called asymptotic geometric analysis, which is at the very core of the crossroads of these disciplines and bears connection to mathematical physics and theoretical computer science as well. The last two decades have seen a tremendous growth in this area. Far reaching results were obtained and various powerful techniques, mainly of a probabilistic flavor, have been developed. A major stimulus and impulse for the theory is the famous hyperplane conjecture which stays unsolved till the present day. Against this background, we explored new perspectives during this workshop and brought together three different groups of researchers working on problems involving high-dimensional set-ups, and thus contributing different angles on high-dimensional phenomena.

To enable participants with different background to form synergies, we kept the number of talks at a minimum and restricted ourselves to 6 survey talks, each ending with a set of possible open problems for group work. Three of the lectures were given on Monday and three on Tuesday. The speakers were:

- Olivier Guédon (Paris):
  Perspectives on the Kannan-Lovász-Simonovits conjecture
- Aicke Hinrichs (Linz):
  Discrepancy and dispersion of point distributions
- Carsten Schütt (Kiel):
  Random polytopes and approximation
- David Alonso-Gutiérrez (Zaragoza):
  Random convex sets verifying the hyperplane and variance conjectures
- Jan Vybiral (Prague):
  IBC: Approximation problems and lower bounds
- Matthias Reitzner (Osnabrück):
  High-dimensional random polytopes

Monday and Tuesday afternoon ended with an open problem session. In those two sessions, the already mentioned potential problems from the survey talks were discussed in more detail and new problems were added. After agreeing on a final set of problems, the participants selected a particular question of their interest and we split up in essentially two groups. For the remainder of the week, the groups worked on and discussed these respective problems (a detailed description of the problems and the outcome of the group discussion is given below). This resulted in additional short talks in the smaller working groups to clarify certain aspects of the theory or to present some important results related to the question. On Thursday morning, each of the groups gave a 15 minutes talk, addressing the progress they had made, stating results and/or presenting the (technical) problems they had run into as well as the different approaches they had tried. On Friday before lunch, the groups gathered for a final update on their work. These presentations also formed the closure of the mini-workshop.

One special focus in this workshop was on the connections between asymptotic geometric analysis and information-based complexity (IBC), two young and essentially independent areas dealing with high-dimensional problems. A group of participants from both areas decided to work on questions in this direction. At the center of attention was the problem of the minimal dispersion of point sets. This is a classical problem in computational geometry, which is related to the notion of discrepancy and the approximation of rank-one tensors. The goal is to find the largest empty axis-parallel box amidst a point set $\mathcal{P}_n = \{t_1, \dots, t_n\}$ inside the unit cube $[0,1]^d$ (in high dimensions). If we denote by $\mathcal{B}_{\mathrm{ax}}$ the set of axis-parallel boxes inside $[0,1]^d$, then the minimal dispersion of this set is defined to be

$$\mathrm{disp}_{\mathcal{B}_{\mathrm{ax}}}(n,d) = \inf_{\substack{\mathcal{P}_n \subseteq [0,1]^d \\ |\mathcal{P}_n| = n}} \sup_{\substack{B \in \mathcal{B}_{\mathrm{ax}} \\ B \cap \mathcal{P}_n = \emptyset}} \mathrm{vol}_d(B) \,.$$

There are two very recent notable results providing upper and lower bounds in this setting and, as part of the group work and discussions, two members prepared short talks of 30 minutes to explain their proofs to the other participants. In fact, the two bounds are of a different order in d and n and the gap is quite huge. For instance, the lower bound is logarithmic in d, but is expected to display a linear behavior in d (in fact, this linear term can be seen in the upper bound up to a logarithmic factor). The participants of the workshop who focused on that particular problem are still working on improving both best known lower and upper bound. For more details we refer to the group work report below.

Another aspect that has been addressed during the week was the central limit problem for the volume of random simplices in high dimensions. For $1 \le r \le d$ let $X_0, X_1, \dots, X_r$ be independent random points that are uniformly distributed in the normalized d-dimensional cube $[-\sqrt{3}, \sqrt{3}]^d$ and denote by V_r the r-volume of their convex hull, which is almost surely a simplex of dimension r. It is known from the literature that for fixed r, the random variables V_r satisfy a central limit theorem, as $d \to \infty$. One of the open problems in this area is to show asymptotic normality for V_r also in the high-dimensional regime, where $r = r(d) \to \infty$. During the workshop we considered the particularly attractive and extremal full-dimensional case $r = d$. While in this situation there is no central limit theorem for V_d itself, it turns out that the random variables $\log V_d$ are asymptotically Gaussian. To show this new central limit theorem, the participants working on this problem have split up into further subgroups to work on the details of the proof. In particular, after having connected the question to already existing result in random matrix theory (more precisely, random determinants), it has become necessary to understand certain details in the literature. These were afterwards presented in short talks in order to put together all pieces for the proof. Currently, the participants from this working group are writing down their result and try to extend it in different directions. We expect that this will eventually lead to a joint publication.
More details on this aspect of the mini-workshop will be explained in the group work report below.

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Mini-Workshop: Perspectives in High-Dimensional Probability and Convexity

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Abstracts

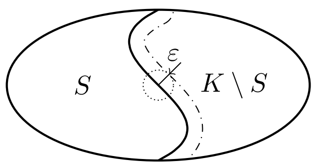
Perspectives on the Kannan-Lovász-Simonovits conjecture

Olivier Guédon

After recalling the problem of the computational complexity of the volume of a convex body (with randomized algorithms), I have presented a conjecture due to Kannan, Lovász and Simonovits about isoperimeters of the measure uniformly distributed on a convex body. This is related to the problem of evaluating the conductance of a convex body and we refer to [20] for a survey of these problems.

The isoperimetric problem for convex bodies is the following. Let K be a convex body in $\mathbb{R}^n$ and μ be the uniform measure on K. Let S be a subset of K and define the boundary measure of S as

$$\mu^+(S) = \liminf_{\varepsilon \to 0} \frac{\mu(S + \varepsilon B_2^n) - \mu(S)}{\varepsilon}.$$



This definition is also valid for any measure on $\mathbb{R}^n$ with log-concave density. The question is to find the largest possible h such that

$$\forall S \subset K: \quad \mu^+(S) \geq h\, \mu(S)\big(1 - \mu(S)\big). \tag{1}$$

Without any assumption on the measure, we can easily imagine a situation where h may be as close to 0 as we wish. In our situation, we made the assumption that the measure is isotropic and log-concave. This avoids many non regular situations. Kannan, Lovász and Simonovits [10] conjectured that there is a universal constant such that inequality (1) holds, and the extremal set S should be a half space of the same measure as $K \setminus S$.

Conjecture 1 (KLS Conjecture)**.** *There exists $c > 0$ such that for any dimension n and any isotropic log-concave probability measure μ on $\mathbb{R}^n$,*

$$\forall S \subset \mathbb{R}^n: \quad \mu^+(S) \geq c\, \mu(S)\big(1 - \mu(S)\big).$$

We refer to [1, 8] for a detailed survey of various conjectures related to this problem. In this talk, I presented results due to E. Milman [19] who proposed several functional versions of this problem, which lead to equivalent questions. In particular he proved the following

Theorem 1. *Let X be the random vector distributed according to a log-concave probability measure μ and let D_∞ be the largest constant such that for every 1-Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}$ we have $\mathrm{Var}\ F(X) \leq \frac{1}{D_\infty}$. Then $h^2 \approx D_\infty$.*

Even the case of F being the Euclidean norm is not known. There were several attempts due to Klartag [13], Fleury, Guédon, Paouris [6], Fleury [5] and Guédon-Milman [9] to prove this inequality for the Euclidean norm. We refer to Chapter 3 in [16] and to [7] for a more detailed description of the links between the

Poincaré inequality and concentration of measure and we just emphasize the fact that Conjecture 1 implies a very strong concentration inequality of the Euclidean norm.

Conjecture 2 (Thin Shell Conjecture). *There exists $c > 0$ such that for any random vector X, distributed according to a log-concave isotropic probability, we have*

$$\forall t > 0 : \quad \mathrm{Prob}\left(\left||X|_2 - \sqrt{n}\right| \geq t\sqrt{n}\right) \leq 2e^{-ct\sqrt{n}}.$$

Such concentration inequalities are related with reverse Hölder (or Khintchine) inequalities for the Euclidean norm, which we even don't know in the following weak form.

Conjecture 3 (Weak Thin Shell Conjecture). *There exists $c > 0$ such that for every log-concave random vector X, we have*

$$\forall p \geq 1 : \quad \left(\mathbb{E}|X|_2^p\right)^{1/p} \leq \mathbb{E}|X|_2 + c\,\sigma_p(X).$$

Two months ago, Lee and Vempala posted on arXiv.org a paper [18] where, developing the approach of Eldan [3], they prove the following

Theorem 2. *Let X be the random vector distributed according to an isotropic log-concave probability μ, then for every 1-Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}$,*

$$\mathrm{Var}\, F(X) \leq C\sqrt{n}, \tag{2}$$

where C is a universal constant.

Up to now this is the best known upper bound in this Poincaré type inequality. To emphasize the depth of this result, it should be noted, that using a result of Eldan and Klartag [4], it is known that (2) implies that the isotropic constant of a convex body is bounded above by $Cn^{1/4}$. Up to now, this is the best upper bound for the isotropic constant due to Klartag [12], which he obtained with a completely different approach.

In another direction, it is interesting in probability to understand which family of random vectors X satisfy for any norm $\|\cdot\|$

$$\left(\mathbb{E}\|X\|^p\right)^{1/p} \leq C\,\mathbb{E}\|X\| + c \sup_{\|z\|_\star \leq 1} \left(\mathbb{E}\langle z, X\rangle^p\right)^{1/p},$$

where $\|\cdot\|_\star$ is the dual norm of $\|\cdot\|$ and $C, c > 0$ are numerical constants. It is known to be true with $C = 1$ for Gaussian [2] or Rademacher (see [17, Thm. 4.7]) random vectors. We refer to [15] and [14] where such questions are discussed. In the area of log-concave measures, Latała and Wojtaszczyk [15] asked the following.

Conjecture 4 (Weak and Strong Moments Conjecture). *There exists $c > 0$ such that for any log-concave random vector X and any norm $\|\cdot\|$ we have for all $p \geq 2$,*

$$\left(\mathbb{E}\left(\|X\| - (\mathbb{E}\|X\|^2)^{1/2}\right)^p\right)^{1/p} \leq c \sup_{\|z\|_\star \leq 1} \left(\mathbb{E}\langle z, X\rangle^p\right)^{1/p},$$

where $\|\cdot\|_\star$ *is the dual norm of* $\|\cdot\|$.

The approach suggested in [15] to tackle such inequality is to prove a strong concentration inequality CI(β) for log-concave measures

Conjecture 5 (Concentration Inequality)**.** *There exist* $\beta > 0$ *such that for any random vector* X *distributed according to a log-concave probability* μ*, we have, for every* $p \geq 2$ *and every set* $A \subset \mathbb{R}^n$ *with* $\mu(A) > 1/2$,

$$\mu\big(A + \beta Z_p(\mu)\big) \geq \min\big\{e^p \mu(A), 1/2\big\},$$

where $Z_p(\mu)$ *is the convex body defined by its support function*

$$h_{Z_p(\mu)}(\theta) = \big(\mathbb{E}|\langle X, \theta\rangle|^p\big)^{1/p}.$$

In my talk, I have tried to explain the connections between these different problems.

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Discrepancy & dispersion of point distributions

Aicke Hinrichs

The topic of this survey lecture is the study of uniform distribution of finite point sets and its connections to complexity and tractability problems for integration of functions from Sobolev spaces of dominating mixed smoothness. We concentrate here on the geometric notions of discrepancy and dispersion of finite point sets in the unit cube $[0,1]^d$ and, in particular, present important open problems.

Uniform distribution theory has its historical origins in Weyl's equidistribution theory presented in [9]. That it is still a vibrant area of research has two main reasons. First, there are still some basic open questions with historical roots. Second, the close connection to the complexity of numerical integration revived and extended the interest of researchers to high dimensional problems not present in the classical literature.

The classical discrepancy function (in the unit cube, for anchored axis parallel boxes) of a point set $\mathcal{P}_n$ of n points in the unit cube $[0,1]^d$ is defined as

$$\mathrm{disc}(\mathcal{P}_n, x) = \frac{\#\mathcal{P}_n \cap [0,x]}{n} - x_1 \cdots x_n$$

for $x \in [0,1]^d$. The bridge between the geometric notion of uniform distribution and discrepancy of point sets to worst case integration errors for functions from spaces with dominating mixed smoothness is provided by Koksma-Hlawka type inequalities.

Different norms of the discrepancy function describe the uniform distribution properties of the point set and in turn the quality of the associated quasi-Monte Carlo integration rule for numerical integration of functions from related Sobolev spaces of dominating mixed smoothness. Most of the research until about 15 years ago was focused on fixed dimension d and on classical norms like the L_p-norms of the discrepancy function. Here the picture is fairly complete for the range $1 < p < \infty$. The celebrated result of Roth [8] for the L_2-discrepancy and the subsequent adaption of harmonic analysis methods like Littlewood-Paley inequalities and Haar series lead to a unified approach for the correct lower bounds in this case. We present the proof of Roth in a modern harmonic analysis language.

For $p = 1$ and $p = \infty$, known lower and upper bounds, for a survey see [3], are still far apart and give rise to the two grand old open problems of discrepancy theory.

Problem 1. *For fixed dimension d, what is the optimal order of the L_1- and L_∞-norms of the discrepancy function?*

Equally important problems arise for tractability questions about the L_∞-norm of the discrepancy, the star discrepancy. Let us denote by

$$\mathrm{disc}^*(n,d) = \inf_{\mathcal{P}_n \subset [0,1]^d} \sup_{x \in [0,1]^d} \mathrm{disc}(\mathcal{P}_n, x)$$

the minimal L_∞-norm of the discrepancy of an n-point subset $\mathcal{P}_n \subset [0,1]^d$.

The best known upper bound was first shown in [5]:

$$\mathrm{disc}^*(n,d) \leq c\sqrt{\frac{d}{n}}. \tag{1}$$

The point sets achieving this are independent uniformly distributed points, it is a probabilistic argument. The best known lower bound is

$$\mathrm{disc}^*(n,d) \geq c\frac{d}{n} \tag{2}$$

for n at least proportional to d, see [6]. To close the gap between upper and lower bound is a more recent, but nevertheless very exciting and important problem.

Problem 2. *Improve either the upper bound* (1) *or the lower bound* (2)*.*

Even if the upper bound (1) already turns out to be the right one, it is highly important to find explicit constructions of point sets satisfying this estimate.

Problem 3. *Find explicit constructions satisfying the upper bound* (1) *for the star discrepancy or, at least, find random constructions involving "less" randomness than the known constructions.*

A quantity closely related to the discrepancy of a point set is the dispersion of the point set. The dispersion deals with the problem of finding the largest empty axis-parallel box, i.e. the largest box not containing any point of the set in the interior. In dimension two, this is a standard problem in computational geometry and computational complexity theory. Here the emphasis is on the word finding, that is, researchers are actually interested in the complexity of algorithms whose output is the largest empty rectangle.

The problem has probably been introduced by Naamad, Lee and Hsu [7], and generalizes in a natural way to the multi-dimensional case, where one has to find the largest empty axis-parallel box amidst a point configuration in the d-dimensional unit cube. Given the prominence of the problem, it is quite surprising that, until recently, very little was known about the size of the largest empty box. Again, there are actually two problems, a "lower bound problem" and an "upper bound problem": one asking for the minimal size of the largest empty box for *any* point configuration, and one asking for the maximal size of the largest empty box for an optimal point configuration. Obviously, the dispersion is also a lower bound for the star discrepancy.

For a point set $\mathcal{P}_n$ of n points in the unit cube $[0,1]^d$, let $\mathrm{disp}(\mathcal{P}_n)$ denote its dispersion. Naturally, we are in particular interested in the minimal dispersion of point sets; thus we set

$$\operatorname{disp}^*(n,d) = \inf_{\substack{\mathcal{P}_n \subset [0,1]^d \\ |\mathcal{P}_n| = n}} \operatorname{disp}(\mathcal{P}_n).$$

It is known that this minimal dispersion is of asymptotic order $1/n$ as a function of n. So the remaining interesting problems here are tractability problems concerning the dependence on the dimension d.

Problem 4. *Improve the known upper bound*

$$\operatorname{disp}^*(n,d) \leq \min\left(\frac{c_1^d}{n}, \frac{c_2 d \log(n/d)}{n}\right)$$

for the dispersion on the unit cube.

We expect that an upper bound of the form

$$\operatorname{disp}^*(n,d) \leq c\,\frac{d}{n} \quad \text{for } n \geq d$$

holds. An approach to remove the logarithmic term in the known estimate can be oriented on the similar successful approach for the discrepancy [5, 1]. This approach is based on the consideration of random point sets and on a chaining argument, or the corresponding estimates for suprema of empirical processes based on chaining. Hence this would again give a probabilistic proof of the existence of such point sets. It is not immediately clear how this can be adapted for the dispersion, since the dispersion is not directly visible as an empirical process. Nevertheless, the geometry of the problem admits different chaining type approaches. It would already be interesting to compute the discrepancy of more explicit low discrepancy point sets such as lattices. The known upper bound $2^{7d}/n$ due to G. Larcher, see [2], is based on digital nets with a t value linear in the dimension.

Problem 5. *Improve the known lower bound*

$$\operatorname{disp}^*(n,d) \geq \frac{\log_2 d}{4(n + \log_2 d)}.$$

for the dispersion on the cube.

Although this bound may be far from the truth, we do not immediately see a way for improvement using the combinatorial approach from [2]. The method was already refined in the recent work of Dumitrescu and Jiang [4] without a further improvement in the rate. Only the constant is slightly better. It is conceivable that some kind of discrete or continuous volume argument should be involved.

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Random polytopes and approximation

Carsten Schütt

This is the first part of a survey on random polytopes and approximation of convex bodies by polytopes. The second part was given by M. Reitzner.

An important invariant of convex bodies that is relevant here is the affine surface area of a convex body K in $\mathbb{R}^n$

$$\mathrm{as}(K) = \int_{\partial K} \kappa^{\frac{1}{n+1}}(x)\, d\mu_{\partial K}(x)$$

where κ is the generalized Gauss-Kronecker curvature and $\mu_{\partial K}$ the Lebesgue measure on the boundary of K. We discuss its properties and its relation to the convex floating body. The convex floating body K_t of a convex body K in $\mathbb{R}^n$ is the intersection of all halfspaces whose defining hyperplane cuts off a set of volume t from K, i.e.,

$$K_t = \bigcap_{\mathrm{vol}_n(K\cap H^-)=t} H^+.$$

The convex floating body and the affine surface area are related in the following way

$$\lim_{t\to 0} \frac{\mathrm{vol}_n(K) - \mathrm{vol}_n(K_t)}{t^{\frac{2}{n+1}}} = \frac{1}{2}\left(\frac{n+1}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{2}{n+1}} \int_{\partial K} \kappa(x)^{\frac{1}{n+1}}\, d\mu_{\partial K}(x).$$

This formula was first shown by Blaschke for convex bodies with smooth boundaries and later for all convex bodies by Schütt and Werner [17]. Other extensions of the affine surface area to general convex bodies were given by Leichtweiss [8] and Lutwak [9].

A random polytope in a convex body K is the convex hull of finitely many points in K that are chosen at random with respect to a probability measure $\mathbb{P}$ on K or ∂K. For a fixed number N of points we are interested in the expectation of the volume of that part of K that is not contained in the convex hull $[x_1, \dots, x_N]$ of the chosen points

$$\mathbb{E}(K, N, \mathbb{P}) = \int_{K\times\cdots\times K} \mathrm{vol}_n([x_1, \dots, x_N])\, d\mathbb{P}(x_1)\dots d\mathbb{P}(x_N).$$

We have the following asymptotic formula

$$c(n)\lim_{N\to\infty}\frac{\mathrm{vol}_n(K)-\mathbb{E}(K,N,\mu_K)}{\left(\frac{\mathrm{vol}_n(K)}{N}\right)^{\frac{2}{n+1}}}=\int_{\partial K}\kappa(x)^{\frac{1}{n+1}}\,d\mu_K(x)$$

where μ_K is the normalized Lebesgue measure on K and κ is the generalized Gauß-Kronecker curvature and

$$c(n)=2\left(\frac{\mathrm{vol}_{n-1}(B_2^{n-1})}{n+1}\right)^{\frac{2}{n+1}}\frac{(n+3)(n+1)!}{(n^2+n+2)(n^2+1)\Gamma\left(\frac{n^2+1}{n+1}\right)}.$$

For dimension 2 this formula was proved by Rényi and Sulanke [13, 14]. This was actually the starting point for the research on random polytopes. For general dimensions the formula was first proved by Bárány for bodies with C^2-boundaries [1] and later for arbitrary convex bodies by Schütt [15].

Related to these problems and formulae is the economic cap covering of Bárány and Larman [2]: let $\epsilon>0$ and let K be a convex body with $\mathrm{vol}_n(K)>0$. Then there are caps $C_1,\dots,C_m$ of K and pairwise disjoint subsets $W_1,\dots,W_m$ such that

(i) $\forall i=1,\dots,m:\quad W_i\subseteq C_i$,
(ii) $\quad\bigcup_{i=1}^m W_i\subseteq K\setminus K_\epsilon\subseteq\bigcup_{i=1}^m C_i$,
(iii) $\forall i=1,\dots,m:\quad \mathrm{vol}_n(C_i)\le 6^n\epsilon$,
(iv) $\forall i=1,\dots,m:\quad \mathrm{vol}_n(W_i)\ge(6n)^{-n}\epsilon$.

For random polytopes whose vertices are chosen from the boundary of the convex body K with respect to a probabilty measure with density f we have the following asymptotic formula

$$\lim_{N\to\infty}\frac{\mathrm{vol}_n(K)-\mathbb{E}(f,N)}{\left(\frac{1}{N}\right)^{\frac{2}{n-1}}}=c_n\int_{\partial K}\frac{\kappa(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}}\,d\mu_{\partial K}(x),$$

where

$$c_n=\frac{(n-1)^{\frac{n+1}{n-1}}\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!\left(\mathrm{vol}_{n-2}(\partial B_2^{n-1})\right)^{\frac{2}{n-1}}}.$$

This formula was shown by Reitzner for bodies with a C^2-boundary [12] and, independently, for a more general class of bodies by Schütt and Werner [18, 19].

Gruber studied best approximation of smooth convex bodies by polytopes [4, 5, 6, 7]. Among other things he showed for the symmetric difference metric d_S that the quantity

$$\inf\left\{d_S(K,P_N):P_N\subseteq K\text{ and }P_N\text{ has at most }N\text{ vertices}\right\}$$

is asymptotically equivalent to

$$\frac{1}{2}\,\mathrm{del}_{n-1}\left(\int_{\partial K}\kappa^{\frac{1}{n+1}}\,d\mu_{\partial K}\right)^{\frac{n+1}{n-1}}N^{-\frac{2}{n-1}}$$

Using the formulae for best and random approximation we can show that random approximation of convex bodies by polytopes is almost as good as best approximation. To accomplish this we need precise formulae for the approximation of the Euclidean ball by polytopes [3, 10, 11].

Finally, we study the floating body algorithm for approximation of convex bodies by polytopes [16].

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Random convex sets verifying the hyperplane and variance conjectures

David Alonso-Gutiérrez

The hyperplane conjecture poses the question of the existence of a positive absolute constant c such that any convex body $K \subseteq \mathbb{R}$ of any dimension n verifies that it has a hyperplane section $K \cap H$ whose $(n-1)$-dimensional volume verifies

$$|K \cap H| \geq c|K|^{\frac{n-1}{n}}.$$

This question was posed by Bourgain in 1990 and has been answered in the positive when K is restricted to several classes of convex bodies. However the best general bound known up to now gives a constant of the order $n^{-1/4}$ which is not absolute. This estimate was proved by Klartag in [8], where he improved by a logarithmic factor the estimate shown by Bourgain [4] and has been recently obtained using different methods by Lee and Vempala. [11]. This problem can be formulated in terms of the boundedness of the isotropic constant of any convex body.

A convex body K is said to be isotropic if it has volume 1, it is centered at the origin and its covariance matrix is a multiple of the identity, i.e., $|K| = 1$, and for every unit vector $\theta \in S^{n-1}$ we have

$$\int_K \langle x, \theta \rangle \, dx = 0 \quad \text{and} \quad \int_K \langle x, \theta \rangle^2 \, dx = L_K^2,$$

where L_K is a constant independent of θ which is called the isotropic constant of K. It is known that every convex body has a unique (up to orthogonal transformations) affine image which is isotropic and thus the isotropic constant of any convex body K can be defined as the isotropic constant of such an affine image. This affine image appears as the solution of a minimization problem and hence L_K can be defined by the equation

$$nL_K^2 = \min \left\{ \frac{1}{|K|^{\frac{2}{n}}} \frac{1}{|TK|} \int_{a+TK} |x|^2 \, dx : a \in \mathbb{R}^n, T \in GL(n) \right\}.$$

The hyperplane conjecture is equivalent to the existence of an absolute constant $C > 0$, such that $L_K \leq C$ for every convex body K in any dimension. Another important open problem in Asymptotic Geometry is the variance conjecture, which appeared in the context of the central limit problem. The central limit problem posed the question of the existence of directions θ such that if X is a random vector uniformly distributed on an isotropic convex body K then the one-dimensional marginal $\langle X, \theta \rangle$ is almost Gaussian. This was shown to be true under some assumptions on the concentration of the Euclidean ball on a thin shell of radius $\sqrt{n}L_K$ (cf. [3]) and this concentration was proved by Klartag in [9]. Nevertheless, the width of this thin shell is not yet completely understood and it is not yet known whether the variance of the square of the Euclidean norm of a random vector uniformly distributed on a convex body verifies the estimate

$$\mathrm{Var}|X|^2 \leq C \sup_{\theta \in S^{n-1}} \mathbb{E}\langle X, \theta \rangle^2 \, \mathbb{E}|X|^2,$$

where C is an absolute constant. This is known as the variance conjecture and is a stronger conjecture than the hyperplane conjecture (cf. [6]). The best constant known up to now is of the order $n^{1/4}$, which was proved in [11].

In [10], Klartag and Kozma showed that Gaussian random polytopes verify the hyperplane conjecture with an overwhelming probability, i.e., a probability that grows to 1 exponentially fast with the dimension. They also considered other models of randomness in which the random polytopes were obtained as the convex hull of independent random vectors with independent coordinates. In this survey talk we will explain their aproach and how it has been adapted to cover other models of randomness, which include the case of random polytopes generated by independent random vectors uniformly distributed on the sphere [1], on an isotropic convex body [5], or an arbitrary isotropic convex body [2, 7].

Regarding the variance conjecture, there are not so many families of random convex bodies that verify it. Nonetheless, we will present some of the random results which are known.

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IBC: Approximation problems and lower bounds

Jan Vybiral
(joint work with A. Hinrichs, A. Kolleck and J. Prochno)

High-dimensional algorithms are very often randomized. Their analysis is therefore usually connected with probability in high dimensions and geometry of underlying objects. This talk presents two aspects of such analysis. The first part gives a general setting of approximation problems in high dimension and the second part concerns with the analysis of optimality of proposed algorithms.

If $f : \Omega \subset \mathbb{R}^d \to \mathbb{R}$ is a function of many ($d \gg 1$) variables, we may want to

- approximate f using only function values $f(x_1), \ldots, f(x_n)$
- approximate f using only values of linear functionals $L_1(f), \ldots, L_n(f)$.

The first setting is called *standard information*, the second one *linear information.* In both cases we want to achieve the smallest possible error using only a prescribed number n of function values or linear functionals.

We show, how to formalize the notation in the setting of standard information. Let $\mathcal{F}_d$ be a given class of d-variate functions on $\Omega_d \subset \mathbb{R}^d$ and let $Y(\Omega_d)$ be a function space on Ω_d. To make the evaluations of functions meaningful, we will assume that $\mathcal{F}_d \subset C(\Omega_d)$. A mapping $N : \mathcal{F}_d \to \mathbb{R}^n$ given by $N(f) = (f(x_1), \ldots, f(x_n)) \in \mathbb{R}^n$ will be called *information map.* For a continuous *recovery map* $\phi : \mathbb{R}^n \to Y(\Omega_d)$ we further consider their composition, namely the *sampling operator* $S_n = \phi \circ N : \mathcal{F}_d \to Y(\Omega_d)$. Its *approximation error* in the worst-case setting is then just $e(S_n) := \sup_{f \in \mathcal{F}_d} \|f - S_n(f)\|_Y$. Finally, the search for the best sampling algorithm is expressed in the definition of *sampling numbers:* $g_{n,d}(\mathcal{F}_d, Y) := \inf_{S_n} e(S_n)$. It is the error of the "best algorithm" when using only n function values. Its inverse function is then $n(\varepsilon, d) = \min\{n \in \mathbb{N} : g_{n,d}(\mathcal{F}_d, Y) \leq \varepsilon\}$, i.e. the minimal number of sampling points needed to achieve an approximation of the error at most $\varepsilon > 0$.

We present a series of results from classical approximation theory as well as from the area of *Information Based Complexity*, which is especially interested in the influence of high dimension $d \gg 1$ on the quality of approximation. We give a survey of a couple of results by Hinrichs, Novak, and Woźniakowski which show, that uniform approximation of analytic, monotone, or convex functions all suffer from the curse of high dimension.

The second part of the talk concerns with Carl's inequality and its application in optimality of algorithms. Let us first define the necessary geometrical quantities, usually called *s-numbers.* Let X, Y be two Banach spaces and let $T : X \to Y$ be a bounded linear operator. Then we define its

- **Approximation numbers**

$$a_n(T) := \inf\{\|T - L\| : L\colon X \to Y, \operatorname{rank}(L) < n\},$$

- **Entropy numbers**

$$e_n(T) = \inf\left\{\varepsilon > 0 : T(B_X) \subset \bigcup_{j=1}^{2^{n-1}} (y_j + \varepsilon B_Y)\right\},$$

- **Gelfand numbers**

$$c_n(T) = \inf_{\substack{M \subset\subset X \\ \operatorname{codim} M < n}} \sup_{\substack{x \in M \\ \|x\|_X \le 1}} \|Tx\|_Y,$$

- **Kolmogorov numbers**

$$d_n(T) = \inf_{\substack{N \subset\subset Y \\ \dim N < n}} \sup_{\|x\|_X \le 1} \inf_{z \in N} \|Tx - z\|_Y.$$

Carl's inequality then states, that the lower bound of entropy numbers imply corresponding lower bounds also for approximation, Gelfand, and Kolmogorov numbers. To be more specific, we formulate it as follows.

Theorem. *Let $T : X \to Y$ be a bounded linear operator between Banach spaces X and Y. Let $\alpha > 0$. Then there is $c_\alpha > 0$, such that for every natural number $n \in \mathbb{N}$*

$$\sup_{1 \le k \le n} k^\alpha e_k(T) \le c_\alpha \sup_{1 \le k \le n} k^\alpha s_k(T).$$

Here, $s_k(T)$ stands for approximation, Gelfand, or Kolmogorov numbers, respectively. The main message of Carl's theorem is then, that the polynomial order of decay of approximation, Gelfand, respectively Kolmogorov numbers, is smaller than that of the entropy numbers.

Further, we discuss how lower bounds on Gelfand numbers imply lower bounds for optimality of approximation algorithms. We treat in detail the setting of Compressed Sensing, where this approach was used already in the original paper of Donoho. It turns out that the behavior of Gelfand numbers of $\mathrm{id} : \ell_p^N \to \ell_2^N$ is of interest. They are very well known and the result reads as follows.

Theorem. *For $0 < p \le 2$*

$$c_n(\mathrm{id} : \ell_p^N \to \ell_2^N) \approx_p \min\left\{1, \frac{1 + \log(N/n)}{n}\right\}^{1/p-1/2}.$$

The proof of this statement goes back to the works of Kashin ('77), Garnaev & Gluskin ('84), Donoho ('06), Foucart, Pajor, Rauhut & Ullrich ('10).

There is unfortunately one gap in this argument. Carl's inequality was proved by Bernd Carl [1] only for Banach spaces. Furthermore, he made a heavy use of Hahn-Banach theorem, which is known to fail for quasi-Banach spaces ℓ_p^N for $0 < p < 1$. We fill this gap by showing the following:

Theorem ([2])**.** *Carl's inequality is true for quasi-Banach spaces and Gelfand, Kolmogorov, and approximation numbers.*

Together with classical results on $e_n(\mathrm{id} : \ell_p^N \to \ell_2^N)$ (Schütt, Kühn, Guédon & Litvak) this gives an alternative proof of the lower bounds of Gelfand numbers and, therefore, of the lower bound for the optimality in compressed sensing.

We further elaborate on this approach in the setting of recovery of low-rank matrices. The corresponding analogue of ℓ_p^N-unit balls are the unit balls of Schatten classes, which are defined as follows: Let $0 < p \leq \infty$. Then S_p^N is the N^2-dimensional space of all real $N \times N$ matrices with

$$\|A\|_{S_p^N} = \left(\sum_{j=1}^{N} \sigma_j(A)^p\right)^{1/p},$$

where $\sigma_j(A)$, $j = 1, \ldots, N$ are the singular values of A. The entropy numbers of embeddings of Schatten classes were recently calculated.

Theorem ([3]). *Let $n, N \in \mathbb{N}$. If $0 < p \leq q \leq \infty$, then*

$$e_n\big(\mathrm{id} : S_p^N \to S_q^N\big) \asymp_{p,q} \begin{cases} 1, & 1 \leq n \leq N, \\ \big(\frac{N}{n}\big)^{1/p-1/q}, & N \leq n \leq N^2, \\ 2^{-\frac{n}{N^2}} \cdot N^{1/q-1/p}, & N^2 \leq n. \end{cases}$$

If $0 < q < p \leq \infty$, then

$$e_n\left(\mathrm{id} : S_p^N \to S_q^N\right) \asymp_{p,q} 2^{-\frac{n}{N^2}} \cdot N^{1/q-1/p}.$$

We close the talk by showing how to use the Carl's inequality again to prove lower bounds for corresponding Gelfand numbers, and therefore also for the problem of approximation of low-rank matrices.

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High-dimensional random polytopes

Matthias Reitzner

This talk surveys some of the results on random polytopes $P_n \subset \mathbb{R}^d$ which concern highdimensional aspects. Hence in most cases we are interested in limits of functionals $f(P_n)$ where the dimension of the underlying space $\mathbb{R}^d$ tends to infinity.

The starting point is the connection between the slicing problem and random polytopes. The slicing problem asks whether there exists a constant C independent of the dimension d such that for all convex bodies K with $V_d(K) = 1$

$$\max_H V_{d-1}(K \cap H) \geq C\,?$$

One possible way to prove this was pointed out by Mark Meckes: Choose n random points in a convex body $K \subset \mathbb{R}^d$ according to the uniform distribution. The convex hull of these random points is a random polytope P_n^K.

> Is the expected volume of a random simplex a monotone function in K? More precisely, if $\mathbb{E}V_d(P_{d+1}^K) \leq \mathbb{E}V_d(P_{d+1}^L)$ for all $K \subset L$, then this implies a positive solution of the slicing problem.
>
> (Meckes [8])

This monotonicity was proved by Rademacher [13] in dimension $d = 2$, but disproved for $d \geq 4$! The case $d = 3$ was settled recently by Kunis, Reichenwallner and Reitzner [7], where also monotonicity does not hold.

The original question by Meckes was the more general problem:

> Does there exist $c(d)$ with $\lim_{d\to\infty} c(d)^{\frac{1}{d}} < \infty$, such that for all $K \subset L$ one has $\mathbb{E}V_d(P_{d+1}^K) \leq c(d)\, \mathbb{E}V_d(P_{d+1}^L)$.
>
> (Meckes [8])

Now, this is equivalent to a positive solution to the slicing problem. It would be of interest to investigate the counterexamples to monotonicity given by Rademacher and computing the lower bound for $c(d)$ given by them.

Open Problem: *Define*

$$c(d) = \min_{K \subset L} \frac{\mathbb{E}V_d(P_{d+1}^K)}{\mathbb{E}V_d(P_{d+1}^L)}.$$

Compute a lower bound for $c(d)$!

A possible alternative approach to the slicing conjecture via random polytopes goes back to a paper by Blaschke [3] and a result by Milman and Pajor [11]. Blaschke proved that among all convex bodies the triangle has the property that the expected area of an inscribed random triangle is maximized. A generalization of his proof to higher dimensions seems to be out of reach. Milman and Pajor showed that a solution of this problem would be equivalent to the slicing problem:

> Let T^d be the d-dimension regular simplex. Then the inequality $\mathbb{E}V_d(P_{d+1}^K) \leq \mathbb{E}V_d(P_{d+1}^{T^d})$ for all convex bodies $K \subset \mathbb{R}^d$ with $V_d(K) = V_d(T^d)$ implies a positive solution to the slicing problem.

Since the expected volume of a random simplex in a simplex plays an important role, it would be of interest to determine the occurring values. The planar case $d = 2$ is classical, see e.g. [1, 12]), and the value for $d = 3$ (Buchta and Reitzner [4]) is also known explicitly, but for higher dimensions nothing seems to be known.

Open Problem: *Can one compute the asymptotic behaviour of* $\mathbb{E}V_d(P_{d+1}^{T^d})$ *as* $d \to \infty$*?*

In a more general setting it would be of interest to determine the expected volume of a random simplex for 'nice' convex bodies as T^d, the unit cube C^d or the cross polytope C_1^d yet explicit formulae are out of reach at the moment.

Open Problem: *Can one compute the asymptotic behaviour of* $\mathbb{E}V_d(P_{d+1}^{C^d})$ *and* $\mathbb{E}V_d(P_{d+1}^{C_1^d})$ *as* $d \to \infty$*?*

The same question concerning random polytopes in a ball has been settled by Miles [10]. From a stochastic point of view a more detailed analysis going beyond expectations would be highly interesting. The problem of determining the variances and proving limit distributions seems to be hard. For the ball a weak solution was given by Ruben [14] who used Miles' method to compute the characteristic function.

> **Open Problem:** *Can one prove CLTs for* $V_d(P_{d+1}^{B^d})$, $V_d(P_{d+1}^{T^d})$, $V_d(P_{d+1}^{C^d})$ *and* $V_d(P_{d+1}^{C_1^d})$ *as* $d \to \infty$*?*

The last part of the talk dealt with Gaussian polytopes and 0-1-polytopes. Let P_n^d be a random 0-1-polytope, i.e. each coordinate of the random points is an i.i.d. Bernoulli random variable and thus a vertex of the cube $C^d = [0,1]^d$ and P_n^d is the convex hull of n independent random 0-1-points. The geometry of 0-1-polytopes [9] and the asymptotics of the expected number of facets $\mathbb{E} f_{d-1}(P_n^d)$ as $d \to \infty$ is well investigated, [2, 5, 6], but in general $\mathbb{E} f_\ell(P_n^d)$, $2 \le \ell \le d-2$ is unknown.

> **Open Problem:** *Determine the asymptotic behaviour of* $\mathbb{E} f_\ell(P_n^d)$ *for* $2 \le \ell \le d-2$.

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Bounds for the minimal dispersion

Workgroup report

A classical problem in computational geometry is to find the largest empty axis-parallel box amidst a point set $\mathcal{P}_n = \{t_1, \dots, t_n\}$ inside the unit cube $[0,1]^d$. This question is related to the notion of (minimal) dispersion. Let $\mathcal{P}_n$ be a set of n points in $[0,1]^d$ and $\mathcal{B} \subseteq \mathfrak{B}([0,1]^d)$, where $\mathfrak{B}([0,1]^d)$ denotes the Borel σ-algebra on $[0,1]^d$. Then we define the dispersion to be

$$\mathrm{disp}(\mathcal{P}_n, \mathcal{B}) := \sup_{\substack{B \in \mathcal{B} \\ B \cap \mathcal{P}_n = \emptyset}} \mathrm{vol}_d(B)\,.$$

The minimal dispersion of the set $\mathcal{B}$ is the defined as

$$\mathrm{disp}_{\mathcal{B}}(n,d) := \inf_{\substack{\mathcal{P}_n \subseteq [0,1]^d \\ |\mathcal{P}_n| = n}} \mathrm{disp}(\mathcal{P}_n, \mathcal{B}).$$

An example of particular importance is the case where $\mathcal{B}$ is the set $\mathcal{B}_{\mathrm{ax}}$ of axis-parallel boxes inside $[0,1]^d$. There are two very recent results providing upper and lower bounds in this setting.

The best known lower bound was proved by Aistleitner, Hinrichs and Rudolf in 2015 [1], improving upon earlier work of Dumitrescu and Jiang [2]:

Theorem 1. *For all $n, d \in \mathbb{N}$, we have*

$$\mathrm{disp}_{\mathcal{B}_{\mathrm{ax}}}(n,d) \geq \frac{1}{4} \cdot \frac{\log_2 d}{n + \log_2 d}\,.$$

The best known lower bound for moderate n was proved by Rudolf [3] in 2017 and gives the following estimate.

Theorem 2. *Let $n, d \in \mathbb{N}$ and assume that $n > 2d$. Then,*

$$\mathrm{disp}_{\mathcal{B}_{\mathrm{ax}}}(n,d) \leq 4 \cdot \frac{d}{n} \log_2 \frac{9n}{d}\,.$$

Moreover, for very large n there is a better bound with the optimal order in n also proved in [1].

Theorem 3. *Let $n, d \in \mathbb{N}$. Then,*

$$\mathrm{disp}_{\mathcal{B}_{\mathrm{ax}}}(n,d) \leq \frac{2^{7d}}{n}\,.$$

This bound obviously has much worse dependence on the dimension d. The large gap between upper and lower bounds was the starting point for the group work within the Mini-Workshop 1706c and is currently work in progress.

We are trying to improve upon both upper and lower bound. However, it is not clear what the optimal order of the minimal dispersion with respect to the number of points n and the dimension d should be.

The proof by Rudolf for the upper bound uses points that are distributed uniformly at random inside $[0,1]^d$ and requires the construction of a suitably small delta cover. Currently we are working on improving the cardinality of such a net, by a kind of chaining argument that should allow us to better balance the probability and cardinality estimates involved.

The lower bound exploits the pigeonhole principle and the fact that for sufficiently large dimensions d the coordinates of a small number of points within the cube follow a certain structure. A possible approach to improve this bound is, given a point set $\mathcal{P}_n$, we "throw" in axis-parallel boxes of conjectured (optimal) size at random, where the distribution of the center of such a box depends on the ℓ_∞-distance to the point set. However, this makes the corresponding analysis quite involved. Another possible approach with respect to the lower bound uses volume covering arguments adapted to the dispersion setting. Such arguments are successfully used in related problems for the discrepancy of point sets and complexity of integration.

We expect that a lower bound linear in d should be true. This is supported by the corresponding recent result of M. Ullrich for the periodic setting, i.e. dispersion of axis parallel boxes on the d-dimensional torus, see [4].

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CLT for the logarithmic volume of a full-dimensional random simplex in the cube

Workgroup report

By a result of Maehara [3] it can be concluded that the volume of a random simplex of fixed dimension in a high-dimensional cube is asymptotically normal distributed. To be more precise, let $r \in \mathbb{N}$ be fixed and consider uniformly i.i.d. random vectors $X_0, \dots, X_r \in [-\sqrt{3}, \sqrt{3}]^d$, $d \geq r$, and set Σ_r^d as the r-dimensional simplex spanned by $\{X_0, \dots, X_r\}$. Then, by the results obtained in [3] we can conclude that the r-dimensional volume of this random simplex, $\mathrm{vol}_r(\Sigma_r^d)$, is asymptotically normal distributed, as $d \to \infty$, with explicitly known mean and variance.

However, if the dimension r of the random simplex is not fixed, but grows with the dimension d of the ambient space, then the problem is wide open. We considered the extremal case, i.e., $r(d) = d$: Let $X_0, \dots, X_d$ be uniformly i.i.d. random vectors in the cube $C^d = [-\sqrt{3}, \sqrt{3}]^d$. By construction, the random variables X_i, $i = 0, \dots, d$, have zero mean and variance one. We denote by Σ_d^d the full-dimensional (random) simplex generated by the $(d+1)$ random vectors, i.e.,

$$\Sigma_d^d = \mathrm{conv}\,\big\{X_0, \dots, X_d\big\}.$$

The d-dimensional volume of Σ_d^d can be expressed as

$$\operatorname{vol}_d\left(\Sigma_d^d\right) = \frac{1}{d!}\left|\det\begin{pmatrix} X_0 & \cdots & X_d \\ 1 & \cdots & 1\end{pmatrix}\right|. \tag{1}$$

There is no CLT for the actual volume of Σ_d^d, but we believe that $\log \operatorname{vol}_d(\Sigma_d^d)$ is asymptotically normal distributed. This conjecture is based on the following observations: In the case that X_0 is fixed at the origin, i.e.,

$$\widetilde{\Sigma}_d^d := \operatorname{conv}\big\{0, X_1, \dots, X_d\big\},$$

we have that

$$\operatorname{vol}_d\left(\widetilde{\Sigma}_d^d\right) = \frac{1}{d!}\left|\det\left(X_1 \cdots X_d\right)\right|. \tag{2}$$

Recently, Nguyen and Vu [1] established a Central limit theorem (CLT) for the logarithm of the absolute value of the determinant of a random matrix. To be more precise, let $A_d \in \mathbb{R}^{d\times d}$ be a random matrix whose entries are independent real random variables with zero mean, variance one and sub-exponential tails. Then, Nguyen and Vu [1, Thm. 1.1] were able to show that

(3)
$$\sup_{t\in\mathbb{R}}\left|\mathbb{P}\left(\frac{\log|\det A_d| - (1/2)\log(d-1)!}{\sqrt{(1/2)\log d}} \le t\right) - \mathbb{P}\big(\mathcal{N}(0,1)\le t\big)\right| \le (\log d)^{-\frac{1}{3}+o(1)},$$

where $\mathcal{N}(0,1)$ is a standard normal distributed random variable with zero mean and variance one. Therefore, by (2), a CLT for $\log \operatorname{vol}_n(\widetilde{\Sigma}_d^d)$ holds.

We considered different approaches to establishing a CLT for $\log \operatorname{vol}_d(\Sigma_d^d)$ and believe, that the most promising approach is the following: write the rows of the matrix in (1) as vectors $Y_i \in \mathbb{R}^{d+1}$ for $i = 1, \dots, d$, that is,

$$B_{d+1} := \begin{pmatrix} X_0 & \cdots & X_d \\ 1 & \cdots & 1\end{pmatrix} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_d \\ e\end{pmatrix} \in \mathbb{R}^{(d+1)\times(d+1)},$$

where we set $e := (1, \dots, 1) \in \mathbb{R}^{d+1}$. Moreover, we set H_d as the linear hyperplane in $\mathbb{R}^{d+1}$ that is spanned by $\{Y_1, \dots, Y_d\}$ and set F_d as the d-dimensional simplex spanned by $\{0, Y_1, \dots, Y_d\}$ in H_d. This yields,

$$\begin{aligned}\operatorname{vol}_d\left(\Sigma_d^d\right) = \frac{1}{d!}\left|\det B_{d+1}\right| &= (d+1)\operatorname{vol}_{d+1}\big(\operatorname{conv}\{0, Y_1, \dots, Y_d, e\}\big) \\ &= \operatorname{dist}(e, H_d)\ \operatorname{vol}_d(F_d),\end{aligned}$$

where $\operatorname{dist}(e, H_d) \ge 0$ denotes the distance of e to the subspace H_d in $\mathbb{R}^{d+1}$. The main idea is to use the CLT of Nguyen and Vu on B_{d+1}. However, the matrix B_{d+1} does not a priori satisfy the needed conditions. To remedy this, we instead consider a matrix C_{d+1} where the last row e of B_{d+1} is replaced by a standard Gaussian random vector $Y_{d+1} \in \mathbb{R}^{d+1}$, so that (3) can be applied to C_{d+1}. To control the error generated by this replacement we have to establish asymptotic bounds for

$$\log \left|\det B_{d+1}\right| - \log \left|\det C_{d+1}\right| = \log \operatorname{dist}\big(e, H_d\big) - \log \operatorname{dist}\big(Y_{d+1}, H_d\big). \tag{4}$$

We believe that bounds for (4) are well within reach, but the details will need further investigations.

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