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Representation Theory of Quivers and Finite Dimensional Algebras

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Abstract. Methods and results from the representation theory of quivers and finite dimensional algebras have led to many interactions with other areas of mathematics. Such areas include the theory of Lie algebras and quantum groups, commutative algebra, algebraic geometry and topology, and in particular the theory of cluster algebras. The aim of this workshop was to further develop such interactions and to stimulate progress in the representation theory of algebras.

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Introduction by the Organisers

The representation theory of quivers is probably one of the most fruitful parts of modern representation theory because of its various links to other mathematical subjects. This has been the reason for devoting a substantial part of this Oberwolfach meeting to problems that can be formulated and solved involving quivers and their representations. The interaction with neighbouring mathematical subjects like geometry, topology, and combinatorics is one of the traditions of such Oberwolfach meetings; it continues to be a source of inspiration. There were 26 lectures given at the meeting, and what follows is a quick survey of their main themes.

Representation varieties and moduli. The representation theory of quivers is a natural setting in which to formulate many problems in linear algebra, and by extension also in geometry. For example configurations of m subspaces in a vector space correspond to representations of a quiver with a central vertex having arrows from m other vertices (and representations of the quiver in which these arrows are injective). In case the dimension vector is d at the central vertex and 1 at the other vertices, a representation is given by m points in projective space of dimension $d-1$. In addition to trying to classify indecomposable representations, one can study moduli spaces of representations, and also problems of enumerative geometry. In his talk, Reineke considered the number of irreducible degree d curves in the complex projective plane passing through $2d-1$ points in general position and tangent to order d at a given point on a given line and also the Euler characteristic of the moduli space of $2d-1$ points in projective space of dimension $d-1$. Reineke explained that there is no direct connection between these numbers, but that they can be computed by the same recursive formula, so they are equal.

The link to geometry was also pursued by Hille, who used inverse limits of moduli spaces of representations of quivers to construct various moduli spaces of pointed curves of genus zero. One can also consider moduli spaces of representations of algebras, or equivalently of quivers with relations, and Kinser described a general decomposition theorem for such moduli spaces and used it to show that any irreducible component of a moduli space for a special biserial algebra is isomorphic to a product of projective spaces. Sauter, motivated by geometric considerations, such as desingularizations of quiver Grassmannians and orbit closures in representation varieties, described a homological construction for algebras of positive dominant dimension.

In his fundamental work on representations of quivers, Kac used the Weil conjectures to prove that the number of absolutely indecomposable representations of a quiver over a finite field is polynomial in the number of elements of the field. In important recent work, Schiffmann has adapted this to quasi-parabolic vector bundles on curves, where the polynomial now also depends on the Weil number of the curve. In particular it holds for weighted projective lines in the sense of Geigle and Lenzing. Plamondon spoke about his joint work with Schiffmann, applying this to show the existence of polynomials for representations of canonical algebras.

Links to differential equations. One of the features of this meeting was a small attempt to strengthen the links between the representation theory of algebras and quivers and the area of integrable systems and related differential equations. In this regard, Boalch and Hiroe both spoke on different aspects of the use of quivers and graphs in the study of moduli spaces of meromorphic connections on Riemann surfaces, especially in the case of irregular' connections.

Two other talks were also related to this link. Burban spoke about Cohen-Macaulay modules for a certain algebra of dihedral quasi-invariants associated to a rational Calogero-Moser operator, and Stroppel spoke about a version of Schur-Weyl duality between, on the one hand, algebras arising from the Yang-Baxter equation, and on the other hand, equivariant cohomology algebras of Grassmannians.

Algebras related to Lie theory. There is a strong connection relating representations of quivers to the representation theory of Kac-Moody algebras. This includes the realization of the enveloping algebra of the positive part of a symmetric Kac-Moody algebra and the construction of a semicanonical basis due to Lusztig. Schröer reported on an ongoing project to extend this from the case of symmetric Kac-Moody algebras to the symmetrizable case, involving the representations of quivers with relations for generalized Cartan matrices. Another connection arises from Ringel-Hall algebras and their relation to quantum groups. In his talk Xiao explained a categorification of Green's formula for the comultiplication of the Ringel-Hall algebra, by relating this to the comultiplication via Lusztig's restriction functors.

Quasihereditary algebras form another class of algebras arising from Lie theory via the study of highest weight categories. A talk of Conde focused on strongly quasihereditary algebras and the Ringel duality for these algebras.

Algebras given by surfaces. The talk of Erdmann extended the classical connection between blocks of group algebras that are of finite type and Brauer tree algebras. This involves the new concept of a weighted surface algebra and covers then algebras of generalized quaternion and dihedral type. Algebras of similar flavour came up in King's talk, again with origin in the theory of cluster algebras. The talk explained a formula of Marsh and Scott for twisted minors as an instance of the Caldero-Chapoton formula for cluster characters.

Algebraic groups and group schemes. A stratification result for modular representations of finite groups schemes was discussed in Pevtsova's talk. It is an analogue of Quillen's stratification for the cohomology of a finite group, and it amounts to a classification of localising and colocalising subcategories of the stable module category via the cohomology of the group scheme.

The commutative algebraic groups form an abelian category which becomes a length category after inverting the isogenies. In his talk Brion explained the structure of this quotient category, pointing out in particular the connection with representations of finite dimensional hereditary algebras.

Higher Auslander-Reiten theory. Auslander's bijection between representation-finite algebras and Auslander algebras is classical. Its higher dimensional analogue is a bijection between d-cluster tilting modules and d-Auslander algebras, a non-commutative analog of regular local rings in Krull dimension $d+1$.

The translation quivers $\mathbb{Z}A_\infty$ and $\mathbb{Z}A_\infty/\tau^n$ are typical structures in Auslander-Reiten quivers. As their higher dimensional analogues, Külshammer and Jasso introduced higher Nakayama algebras $A_\infty^{(d)}$ and $\widetilde{A}_{n-1}^{(d)}$. They are Nakayama algebras for $d=1$, and the mesh categories of $\mathbb{Z}A_\infty$ and $\mathbb{Z}A_\infty/\tau^n$ for $d=2$. Their module categories admit d-cluster tilting subcategories, which are equivalent to $A_\infty^{(d+1)}$ and $\widetilde{A}_{n-1}^{(d+1)}$ respectively. Moreover $A_\infty^{(d)}$ gives an md-Calabi-Yau triangulated category, which admits a weakly d-cluster tilting subcategory and an md-spherical object.

For a finite dimensional algebra A, there exists a bijection between functorially finite thick (wide) subcategories of mod$A$ and ring epimorphisms $A \to B$ such that B is finite dimensional and $\mathrm{Tor}_1^A(B, B) = 0$. Herschend introduced the notion of thick subcategories of a d-cluster tilting subcategory $\mathcal{M}$ of mod$A$, and explained a bijection between functorially finite thick subcategories of $\mathcal{M}$ and d-rigid ring epimorphisms from A.

Cohen-Macaulay modules. Graded Cohen-Macaulay modules over a (not necessarily commutative) graded Gorenstein ring R form a Frobenius category, whose stable category is triangle equivalent to the singularity category of Buchweitz and Orlov. A powerful approach to study Cohen-Macaulay modules is to find a tilting object T in the stable category of R since it gives a triangle equivalence with the derived category of the endomorphism algebra of T. There are many known examples of graded Gorenstein rings whose stable categories have tilting objects, e.g. trivial extension algebras, selfinjective algebras, simple singularities, etc.

In Minamoto's talk, it is shown that the stable category of a finite dimensional graded Gorenstein algebra R can be realized in the derived category of graded R-modules in an explicit way. This improves a result by Orlov. In Buchweitz's talk, it is shown that the stable category of a commutative reduced graded Gorenstein ring R in Krull dimension one always has a tilting object T. An explicit description of the endomorphism algebra of T is given for some important examples.

Tilting theory and stability. Tilting theory is fundamental to study derived and triangulated categories since it enables us to control their equivalences.

Tilted algebras are classical in representation theory, and they are characterized by the existence of slices in their module categories. Cluster-tilted algebras were introduced in cluster tilting theory in this century. In Schiffler's talk, he introduced local slices of cluster-tilted algebras, and explained their role in the study of cluster-tilted algebras.

Silting objects are an important generalization of tilting objects. For a finite dimensional algebra, silting objects correspond bijectively with algebraic t-structures (König-Yang). Angeleri Hügel spoke about a bijection between pure injective cosilting objects in a compactly generated triangulated category $\mathcal{T}$ and smashing non-degenerate t-structures in $\mathcal{T}$ whose hearts are Grothendieck categories.

For a finite dimensional algebra A, two-term silting objects form a distinguished class from the point of view of mutations. They correspond bijectively with support τ-tilting A-modules. Taking g-vectors, they give rise to a simplicial fan in the Grothendieck group $K_0(\mathrm{proj}A)$. This gives a connection with geometric invariant theory for A-modules. In independent talks by Brüstle and Thomas, they described the wide subcategory of θ-semistable modules in terms of τ-tilting theory for a given stability θ in $K_0(\mathrm{proj}A) \otimes_{\mathbb{Z}} \mathbb{R}$.

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Workshop: Representation Theory of Quivers and Finite Dimensional Algebras

Table of Contents

Abstracts

Quivers with relations for generalized Cartan matrices: Crystal graphs

Jan Schröer
(joint work with Christof Geiß, Bernard Leclerc)

1. Introduction

There is a remarkable geometric universe relating the representation theory of quivers and preprojective algebras with the representation theory of symmetric Kac-Moody algebras. This includes the realization of the enveloping algebra $U(\mathfrak{n})$ of the positive part $\mathfrak{n}$ of a symmetric Kac-Moody algebra $\mathfrak{g}$ as an algebra of constructible functions on varieties of modules over path algebras [9] and over preprojective algebras [6, 7]. The latter leads to the construction of a semicanonical basis $\mathcal{S}$ of $U(\mathfrak{n})$ due to Lusztig [7]. The elements of $\mathcal{S}$ are parametrized by the irreducible components of varieties of modules over preprojective algebras. Furthermore, closely linked with varieties of modules over preprojective algebras, there is a geometric realization of the crystal graph $B(-\infty)$ of the quantized enveloping algebra $U_q(\mathfrak{n})$ due to Kashiwara and Saito [5]. This crystal graph controls the decompositions of tensor products of irreducible integrable highest weight $\mathfrak{g}$-modules.

Many geometric constructions for symmetric Kac-Moody algebras, especially the construction of Lusztig's semicanonical basis, do not exist for non-symmetric Kac-Moody algebras. Nandakumar and Tingley [8] recently realized $B(-\infty)$ in the symmetrizable case via varieties of modules over preprojective algebras associated with species. In the non-symmetric cases, their construction cannot be carried out over algebraically closed fields, especially not over $\mathbb{C}$. There exists also a folding technique, which sometimes allows to transfer results from the symmetric cases to the non-symmetric ones.

In our setting, symmetric and symmetrizable cases are dealt with uniformly. For example, in [3] (based on results in [1] and [2]) we proved that in all Dynkin types, the enveloping algebras $U(\mathfrak{n})$ are isomorphic to algebras of constructible functions on varieties of modules over a class of Iwanaga-Gorenstein algebras defined over $\mathbb{C}$.

2. Generalized preprojective algebras

Let $C = (c_{ij}) \in M_n(\mathbb{Z})$ be a *symmetrizable generalized Cartan matrix*, i.e. the following hold:

- (C1) $c_{ii} = 2$ for all i;
- (C2) $c_{ij} \leq 0$ for all $i \neq j$;
- (C3) There is a diagonal integer matrix $D = \mathrm{diag}(c_1, \ldots, c_n)$ with $c_i \geq 1$ for all i such that DC is symmetric.

The matrix D is a *symmetrizer* of C.

For each (i,j) with $c_{ij} < 0$ we choose some $\operatorname{sgn}(i,j) \in \{\pm 1\}$ such that $\operatorname{sgn}(i,j) = -\operatorname{sgn}(j,i)$. Define $g_{ij} := |\gcd(c_{ij}, c_{ji})|$ and $f_{ij} := |c_{ij}|/g_{ij}$. Let $Q := Q(C) := (I, \mathbb{Q}_1)$ be the quiver with the set of vertices $I := \{1, \dots, n\}$ and with the set of arrows

$$Q_1 := \{\alpha_{ij}^{(g)} : j \to i \mid c_{ij} < 0, 1 \leq g \leq g_{ij}\} \cup \{\varepsilon_i \colon i \to i \mid i \in I\}.$$

If $g_{ij} = 1$, we also write α_{ij} instead of $\alpha_{ij}^{(1)}$.

For $Q = Q(C)$ and a symmetrizer $D = \operatorname{diag}(c_1, \dots, c_n)$ of C, we define an algebra

$$\Pi := \Pi(C, D) := KQ/J$$

where KQ is the path algebra of Q and J is the ideal defined by the following relations:

(P1) For each i we have

$$\varepsilon_i^{c_i} = 0.$$

(P2) For $c_{ij} < 0$ and $1 \leq g \leq g_{ij}$ we have

$$\varepsilon_i^{f_{ji}} \alpha_{ij}^{(g)} = \alpha_{ij}^{(g)} \varepsilon_j^{f_{ij}}.$$

(P3) For each i we have

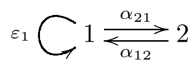
$$\sum_{j : c_{ji} < 0} \sum_{g=1}^{g_{ji}} \sum_{f=0}^{f_{ji}-1} \operatorname{sgn}(i,j) \varepsilon_i^f \alpha_{ij}^{(g)} \alpha_{ji}^{(g)} \varepsilon_i^{f_{ji}-1-f} = 0.$$

We call Π a *(generalized) preprojective algebra* of type C. These algebras generalize the classical preprojective algebras associated with quivers. Up to isomorphism, the algebra Π does not depend on the choice the sgn map.

As an example, let

$$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

and $\operatorname{sgn}(1,2) = 1$. Thus C is a Cartan matrix of Dynkin type B_2. We have $\Pi = KQ/J$ where $Q = Q(C)$ is the quiver



and J is generated by the set $\{\varepsilon_1^2,\ \alpha_{12}\alpha_{21}\varepsilon_1 + \varepsilon_1\alpha_{12}\alpha_{21},\ -\alpha_{21}\alpha_{12}\}$.

3. Main results

As before, let $\Pi = \Pi(C, D)$ be a generalized preprojective algebra. We assume that our ground field K is algebraically closed. For $d \in \mathbb{N}^n$, let $\operatorname{nil}_E(\Pi, d)$ be the variety of E-filtered Π-modules with dimension vector d. (For each vertex $i \in I$ let E_i be $K[\varepsilon_i]/(\varepsilon_i^{c_i})$ seen as a Π-module. Then a Π-module is *E-filtered* if it has a filtration by these modules E_i.)

Let $G(d)$ be the product of linear groups, which acts on $\operatorname{nil}_E(\Pi, d)$ by conjugation.

For $d = (d_1, \ldots, d_n)$ and $D = \operatorname{diag}(c_1, \ldots, c_n)$ define $d/D := (d_1/c_1, \ldots, d_n/c_n)$. Let q_{DC} be the quadratic form associated with $1/2DC$.

Theorem 3.1. *For each irreducible component Z of $\operatorname{nil}_E(\Pi, d)$ we have*

$$\dim(Z) \leq \dim G(d) - q_{DC}(d/D).$$

Let $\operatorname{Irr}(\operatorname{nil}_E(\Pi, d))^{\max}$ be the set of irreducible components of $\operatorname{nil}_E(\Pi, d)$ of maximal dimension $\dim G(d) - q_{DC}(d/D)$.

Assume that C is symmetric and D is the identity matrix. Then Π is a classical preprojective algebra associated with an acyclic quiver, the $\operatorname{nil}_E(\Pi, d)$ are Lusztig's nilpotent varieties, $\dim G(d) - q_{DC}(d/D)$ is the dimension of the affine space of representations of the acyclic quiver with dimension vector d, and all irreducible components of $\operatorname{nil}_E(\Pi, d)$ are maximal.

Let $\mathfrak{n}(C)$ be the positive part of the symmetrizable Kac-Moody algebra $\mathfrak{g}(C)$ associated with C. Let $B(-\infty)$ be the crystal graph of the quantized enveloping algebra $U_q(\mathfrak{n}(C))$.

The following theorem is our first main result.

Theorem 3.2 ([4])**.** *Let $\Pi = \Pi(C, D)$, and set*

$$\mathcal{B} := \bigsqcup_{d \in \mathbb{N}^n} \operatorname{Irr}(\operatorname{nil}_E(\Pi, d))^{\max}.$$

Then there are isomorphisms of crystals

$$(\mathcal{B}, \operatorname{wt}, \tilde{e}_i, \tilde{f}_i, \varphi_i, \varepsilon_i) \cong (\mathcal{B}, \operatorname{wt}, \tilde{e}_i^*, \tilde{f}_i^*, \varphi_i^*, \varepsilon_i^*) \cong B(-\infty).$$

The operators and maps $\operatorname{wt}, \tilde{e}_i, \tilde{f}_i, \varphi_i, \varepsilon_i$ (and their $*$-versions) appearing in Theorem 3.2 are defined in a module theoretic way in the fashion of Kashiwara and Saito [5], see also Nandakumar and Tingley [8]. Kashiwara and Saito only work with symmetric Kac-Moody algebras (C symmetric and D the identity matrix), and Nandakumar and Tingley need to work over fields which are not algebraically closed in case C is non-symmetric. For C symmetric and D the identity matrix, Theorem 3.2 coincides with Kashiwara and Saito's result.

As an example, we display in Figure 1 part of the geometric crystal graph $B(-\infty)$ of type B_2. (Each box in the figure contains a Π-module. The orbit closure of this Π-module is a maximal irreducible component.)

4. Convolution algebras and semicanonical functions

For $K = \mathbb{C}$ the field of complex numbers, let $\widetilde{\mathcal{F}}(\Pi)$ be the convolution algebra of constructible functions on the representation varieties $\operatorname{rep}(\Pi, d)$, and let

$$\widetilde{\mathcal{M}}(\Pi) = \bigoplus_{d \in \mathbb{N}^n} \widetilde{\mathcal{M}}(\Pi)_d$$

be the subalgebra generated by the characteristic functions $\{\widetilde{\theta}_i := 1_{E_i} \mid 1 \leq i \leq n\}$. We assume that all constructible functions are invariant on isomorphism classes of modules. The elements in $\widetilde{\mathcal{M}}(\Pi)_d$ are constructible functions $\operatorname{nil}_E(\Pi, d) \to \mathbb{C}$.

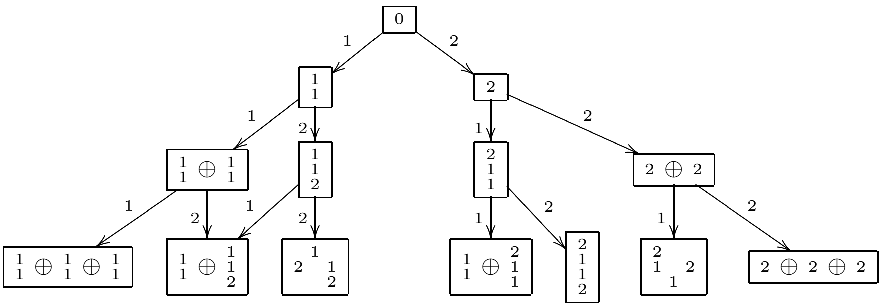


FIGURE 1. The first four layers of the geometric crystal graph $B(-\infty)$ of type B_2.

For a constructible function $f\colon \mathrm{nil}_E(\Pi, d) \to \mathbb{C}$ and an irreducible component Z of $\mathrm{nil}_E(\Pi, d)$ let $\rho_Z(f)$ be the generic values of f on Z.

Theorem 4.1 ([4]). *For $K = \mathbb{C}$ and $\Pi = \Pi(C, D)$, the convolution algebra $\widetilde{\mathcal{M}}(\Pi)$ contains a set*

$$\widetilde{\mathcal{S}} := \{\widetilde{f}_Z \mid Z \in \mathcal{B}\}$$

of constructible functions such that for each $Z' \in \mathcal{B}$ we have

$$\rho_{Z'}(\widetilde{f}_Z) = \begin{cases} 1 & \text{if } Z = Z', \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$\mathcal{M}(\Pi) := \widetilde{\mathcal{M}}(\Pi)/\mathcal{I}$$

where $\mathcal{I}$ is the ideal generated by the Serre relations $\{\widetilde{\theta}_{ij} \mid 1 \leq i, j \leq n \text{ with } c_{ij} \leq 0\}$ where

$$\widetilde{\theta}_{ij} := \mathrm{ad}(\widetilde{\theta}_i)^{1-c_{ij}}(\widetilde{\theta}_j).$$

Let

$$\theta_i := \widetilde{\theta}_i + \mathcal{I} \quad \text{and} \quad f_Z := \widetilde{f}_Z + \mathcal{I}$$

be the residue classes of $\widetilde{\theta}_i$ and $\widetilde{f}_Z$ in $\mathcal{M}(\Pi)$.

For a constructible function $f\colon \mathrm{nil}_E(\Pi, d) \to \mathbb{C}$ let

$$\mathrm{supp}(f) := \{M \in \mathrm{nil}_E(\Pi, d) \mid f(M) \neq 0\}$$

be the *support* of f. By Theorem 3.1 we have $\dim \mathrm{supp}(f) \leq \dim G(d) - q_{DC}(d/D)$.

Conjecture 4.2 (Support Conjecture). *The direct sum of the subspaces*

$$\{f \in \widetilde{\mathcal{M}}(\Pi)_d \mid \mathrm{supp}(f) < \dim G(d) - q_{DC}(d/D)\}$$

is an ideal in $\widetilde{\mathcal{M}}(\Pi)$.

The next theorem is our second main result.

Theorem 4.3 ([4]). *Suppose Conjecture 4.2 is true. For* $K = \mathbb{C}$, $\Pi = \Pi(C, D)$ *and* $\mathfrak{n} = \mathfrak{n}(C)$ *the following hold:*

(i) *There is a Hopf algebra isomorphism*

$$\eta_\Pi \colon U(\mathfrak{n}) \to \mathcal{M}(\Pi)$$

defined by $e_i \mapsto \theta_i$.

(ii) *Via the isomorphism* η_Π, *the set*

$$\mathcal{S} := \{f_Z \mid Z \in \mathcal{B}\}$$

is a $\mathbb{C}$*-basis of* $U(\mathfrak{n})$.

(iii) *For* $0 \neq f \in \widetilde{\mathcal{M}}(\Pi)_d$ *the following are equivalent:*
   (a) $f \in \mathcal{I}$;
   (b) $\dim \operatorname{supp}(f) < \dim G(d) - q_{DC}(d/D)$.

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Cluster-tilted and quasi-tilted algebras

Ralf Schiffler

(joint work with Ibrahim Assem, Khrystyna Serhiyenko)

Cluster-tilted algebras were introduced by Buan, Marsh and Reiten [BMR] and, independently in [CCS] for type $\mathbb{A}$ as a byproduct of the now extensive theory of cluster algebras of Fomin and Zelevinsky [FZ]. Since then, cluster-tilted algebras have been the subject of several investigations.

In particular, in [ABS] is given a construction procedure for cluster-tilted algebras: let C be a triangular algebra of global dimension two over an algebraically closed field k, and consider the C-C-bimodule $\operatorname{Ext}^2_C(DC, C)$, where $D = \operatorname{Hom}_k(-, k)$ is the standard duality, with its natural left and right C-actions. The trivial extension of C by this bimodule is called the *relation-extension* $\widetilde{C}$ of C. It is shown there that, if C is tilted, then its relation-extension is cluster-tilted, and every cluster-tilted algebra occurs in this way.

In this talk we report on two recent papers [ASS1, ASS2]. In [ASS1], we study the relation-extensions of a wider class of triangular algebras of global dimension two, namely the class of quasi-tilted algebras, introduced by Happel, Reiten and Smalø in [HRS]. In general, the relation-extension of a quasi-tilted algebra is not cluster-tilted, however it is 2-Calabi-Yau tilted.

Theorem 1. *Let C be a quasi-tilted algebra. Then its relation-extension $\widetilde{C}$ is cluster-tilted or it is 2-Calabi-Yau tilted of canonical type.*

We then look more closely at those cluster-tilted algebras which are tame and representation-infinite. According to [BMR], these coincide exactly with the cluster-tilted algebras of euclidean type. We ask then the following question: Given a cluster-tilted algebra B of euclidean type, find all quasi-tilted algebras C such that $B = \widetilde{C}$.

For this purpose, we generalize the notion of reflections of [ABS4]. A local slice Σ in $\operatorname{mod} B$ is called rightmost, if all its sources are injective. If $I(x)$ is a source in a rightmost local slice Σ, we define the *completion H_x of x* by the following three conditions.

(a) $I(x) \in H_x$.
(b) H_x is closed under predecessors in Σ.
(c) If $L \to M$ is an arrow in Σ with $L \in H_x$ having an injective successor in H_x then $M \in H_x$.

We can decompose H_x as the disjoint union of three sets as follows. Let $\mathcal{J}$ denote the set of injectives in H_x, let $\mathcal{J}^-$ be the set of non-injectives in H_x which have an injective successor in H_x, and let $\mathcal{E} = H_x \setminus (\mathcal{J} \cup \mathcal{J}^-)$ denote the complement of $(\mathcal{J} \cup \mathcal{J}^-)$ in H_x. Thus

$$H_x = \mathcal{J} \sqcup \mathcal{J}^- \sqcup \mathcal{E}$$

is a disjoint union. Then the reflection of the slice Σ in x is

$$\sigma_x^+ \Sigma = \tau^{-2}(\mathcal{J} \cup \mathcal{J}^-) \cup \tau^{-1}\mathcal{E} \cup (\Sigma \setminus H_x),$$

where $\tau^{-2} I(x) = P(x)$. We obtain an induced notion of reflections on the level of tilted algebras. Indeed, if $C = B/\mathrm{Ann}\Sigma$ then let $\sigma_x^+ C = B/\mathrm{Ann}\sigma_x^+\Sigma$.

We prove that this operation allows to produce all tilted algebras C such that $B = \widetilde{C}$.

Theorem 2. *Let C_1 and C_2 be two tilted algebras that have the same relation-extension. Then there exists a sequence of reflections and coreflections σ such that $\sigma C_1 \cong C_2$.*

In [ABS4] this result was shown only for cluster-tilted algebras of tree type. We also prove that, unlike those of [ABS4], reflections in the sense of [ASS1] are always defined, that the reflection of a tilted algebra is also tilted of the same type, and that they have the same relation-extension. Because all tilted algebras having a given cluster-tilted algebra as relation-extension are given by iterated reflections, this gives an algorithmic answer to our question above.

Returning to quasi-tilted algebras, we define a particular two-sided ideal of a cluster-tilted algebra, which we call the partition ideal. We show that the quasi-tilted algebras which are not tilted but have a given cluster-tilted algebra B of euclidean type as relation-extension are the quotients of B by a partition ideal.

Theorem 3. *Let C be a quasi-tilted algebra whose relation-extension B is cluster-tilted of euclidean type. Then C is one of the following.*

(a) *$C = B/\operatorname{Ann}\Sigma$ for some local slice Σ in $\Gamma(\operatorname{mod} B)$, and then C is tilted.*
(b) *$C = B/K$ for some partition ideal K.*

In the second paper [ASS2], we characterize the indecomposable transjective B-modules that do not lie on a local slice.

Theorem 4. *Let B be a cluster-tilted algebra and M an indecomposable transjective B-module. Then the following are equivalent.*

(a) *M does not lie on a local slice.*
(b) *There exist a rightmost slice Σ and a strong sink x with $I(x)$ a source in Σ such that the completion H_x contains a sectional path*

$$\omega : I(i) \to \cdots \to \tau M \to \cdots \to I(j)$$

with $I(i)$ and $I(j)$ injective.

As a consequence we obtain the following sharp upper bound.

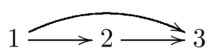
Corollary 5. *Let B be a representation-infinite cluster-tilted algebra. The number of indecomposable transjective B-modules that do not lie on a local slice is at most*

$$(2^{t-1} - 1)(n - 2)$$

where t is the number of indecomposable transjective projective B-modules and n is the number of all indecomposable projective B-modules.

Remark 6. The bound in Corollary 5 is sharp. Consider the following example.

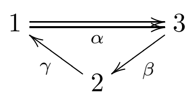
Let $A = \widetilde{\mathbb{A}}_{2,1}$ be the path algebra of the quiver



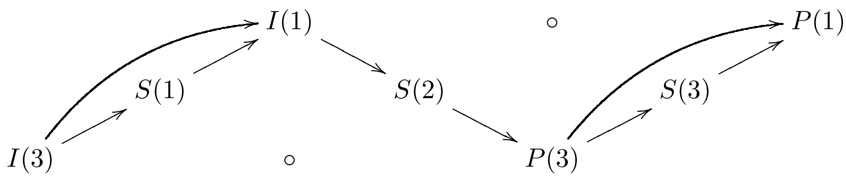
Let $\mathcal{C}_A$ be the corresponding cluster category and let $T \in \mathcal{C}_A$ be the cluster-tilting object

$$T = P(1) \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus P(3)$$

The cluster-tilted algebra $B = \operatorname{End}_{\mathcal{C}_A}(T)$ is given by the following quiver with relations $\alpha\beta = \beta\gamma = \gamma\alpha = 0$.



In the Auslander-Reiten quiver of $\operatorname{mod} B$ we have the following local configuration. Note that the projective B-modules $P(1)$ and $P(3)$ lie in the transjective component of $\Gamma(\operatorname{mod} B)$ while the projective $P(2)$ lies in a tube.



The only transjective B-module not lying on a local slice is $S(2)$. On the other hand the formula gives

$$(2^{t-1} - 1)(n - 2) = (2^{2-1} - 1)(3 - 2) = 1.$$

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Torsion pairs in silting theory

Lidia Angeleri Hügel
(joint work with Frederik Marks, Jorge Vitória)

Le $\mathcal{T}$ be a compactly generated triangulated category. Recall that a torsion pair $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ in $\mathcal{T}$ is said to be

- *nondegenerate* if $\cap_{n\in\mathbb{Z}}\mathcal{U}[n] = 0 = \cap_{n\in\mathbb{Z}}\mathcal{V}[n]$;
- a *t-structure* if $\mathcal{U}[1] \subseteq \mathcal{U}$,
- a *co-t-structure* if $\mathcal{U}[-1] \subseteq \mathcal{U}$.

It was shown in [3] that the *heart* $\mathcal{H}_{\mathbb{T}} = \mathcal{U}[-1] \cap \mathcal{V}$ of a t-structure $\mathbb{T}$ is always an abelian category. We are going to investigate t-structures whose heart is even a Grothendieck category, relating them to an interesting class of objects.

According to [9], an object C in $\mathcal{T}$ is called *cosilting* if the classes $\mathcal{U} = {}^{\perp_{\leq 0}}C$ and $\mathcal{V} = {}^{\perp_{>0}}C$ form a t-structure $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ in $\mathcal{T}$ and $C \in \mathcal{V} = {}^{\perp_{>0}}C$. Two cosilting objects are *equivalent* if they give rise to the same t-structure in $\mathcal{T}$.

For example, it is shown in [10] that every cotilting module over a ring R is a cosilting object in the derived category D(Mod-R). More generally, using [11] one can show that a bounded complex of injective modules $C \in K^b$(Inj-R) is a cosilting object in D(Mod-R) if and only if it is a *cosilting complex*, i. e.

(i) $\mathrm{Hom}_{\mathrm{D(Mod\text{-}R)}}(C^I, C[i]) = 0$ for all $i > 0$ and all sets I, and
(ii) K^b(Inj-R) is the smallest triangulated subcategory of D(Mod-R) containing all direct summands of direct products of copies of C.

Every t-structure $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ arising as above from a cosilting object C is nondegenerate and *smashing*, i. e. $\mathcal{V}$ is closed under coproducts. Moreover, C is a cogenerator of $\mathcal{T}$, and the cohomological functor $H^0_{\mathbb{T}} : \mathcal{T} \to \mathcal{H}_{\mathbb{T}}$ associated to $\mathbb{T}$ maps C to an injective cogenerator of the heart $\mathcal{H}_{\mathbb{T}}$, see [9].

In fact, by dualising arguments from [8], one can show that these properties characterise the t-structures given by cosilting objects: there is a bijection

$$\left\{ \begin{array}{c} \text{equivalence classes of} \\ \text{cosilting objects} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{smashing nondegenerate t-structures} \\ \text{whose heart has an} \\ \text{injective cogenerator} \end{array} \right\}$$

In order to determine the cosilting objects which give rise to t-structures with Grothendieck heart, we need the theory of purity for triangulated categories developed in [4, 6]. We consider the category Mod-$\mathcal{T}^c$ of contravariant functors $\mathcal{T}^c \to$ Mod-$\mathbb{Z}$ where $\mathcal{T}^c$ denotes the class of compact objects in $\mathcal{T}$, together with the Yoneda functor $y : \mathcal{T} \longrightarrow$ Mod-$\mathcal{T}^c$ assigning to an object X the functor $\mathrm{Hom}_T(-, X) \mid_{\mathcal{T}^c}$. Using that Mod-$\mathcal{T}^c$ is a Grothendieck category, one can carry the classical notions of purity in abelian categories to $\mathcal{T}$. For example, pure triangles are triangles that correspond to short exact sequences under the Yoneda functor, and pure-injective objects correspond to injectives in Mod-$\mathcal{T}^c$. Similarly, one can introduce the notion of a definable subcategory of $\mathcal{T}$, cf. [7] for details.

It turns out in [1] that the bijection above restricts to a bijection

$$\left\{ \begin{array}{c} \text{equivalence classes of} \\ \text{pure-injective} \\ \text{cosilting objects} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{smashing nondegenerate t-structures} \\ \text{whose heart is a} \\ \text{Grothendieck category} \end{array} \right\}$$

Moreover, if $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ is a triple of classes in $\mathcal{T}$ where $(\mathcal{U}, \mathcal{V})$ is a non-degenerate t-structure and $(\mathcal{V}, \mathcal{W})$ is an adjacent co-t-structure, then $\mathcal{V}$ is a definable subcategory of $\mathcal{T}$ if and only if the t-structure $(\mathcal{U}, \mathcal{V})$ arises as above from some pure-injective cosilting object C.

Combining this with a recent result in [5], it follows that the heart of any nondegenerate compactly generated t-structure in $\mathcal{T}$ is a Grothendieck category. But in general there are more t-structures with this property. Indeed, it is shown in [2] that all cosilting complexes in K^b(Inj-R) over a ring R are pure-injective in D(Mod-R) and thus give rise to t-structures whose heart is a Grothendieck category, although they need not be compactly generated. For example, over a valuation domain R with a non-zero maximal idempotent ideal $\mathfrak{m}$, the simple module $R/\mathfrak{m}$ yields a two-term cosilting complex in D(Mod-R) whose t-structure is not compactly generated.

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On finite dimensional graded Iwanaga-Gorenstein algebras and their stable category of Cohen-Macaulay modules

Hiroyuki Minamoto

(joint work with Kota Yamaura)

We discuss finite dimensional graded Iwanaga-Gorenstein (IG) algebras $A = \bigoplus_{i=0}^{\ell} A_i$ and their stable categories $\underline{\mathsf{CM}}^{\mathbb{Z}} A$ of graded Cohen-Macaulay (CM) modules.

Via quasi-Veronese algebra construction, a finite dimensional graded algebra $A = \bigoplus_{i=0}^{\ell} A_i$ is graded Morita equivalent to a trivial extension algebra $\Lambda \oplus C$ with the grading $\deg \Lambda = 0$ and $\deg C = 1$. Hence, in principle, we may reduce our study to the case where A is a trivial extension algebra $A = \Lambda \bigoplus C$ with the grading $\deg \Lambda = 0$ and $\deg C = 1$.

In this situation, we regard the derived category $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$ as a full subcategory of $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A)$ by regarding a complex M of Λ-modules as a complex of graded A-modules concentrated at 0-th degree. We denote by π the canonical projection from the derived category of A to the singular derived category of A and denote by ϖ the restriction to $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$

$$\pi : \mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A) \to \mathrm{Sing}^{\mathbb{Z}}A := \mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A)/\mathsf{K}^{\mathrm{b}}(\mathrm{proj}^{\mathbb{Z}}A)$$
$$\varpi : \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda) \to \mathrm{Sing}^{\mathbb{Z}}A$$

First, we give a criterion that a complex $M \in \mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A)$ belongs to the perfect derived category $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}^{\mathbb{Z}}A)$ of graded A-modules in terms of Λ and C. Here we don't assume that A is IG. The following three Corollaries play important role in the main subject of this talk.

(1) We give a description of the kernel of the canonical functor ϖ.
(2) We give a necessary and sufficient condition that A is of finite global dimension in terms of Λ and C.
(3) We give a necessary and sufficient condition that A is IG. This condition is given by using derived tensor products and derived Homs.

Next, we assume that Λ is of finite global dimension. Then, we show that the condition (3) has a triangulated categorical interpretation. More precisely, we give a condition for A to be IG in terms of the action of C on the perfect derived category of Λ by the derived tensor product.

Third, we show that if A is IG, then their stable category $\underline{\mathsf{CM}}^{\mathbb{Z}}A$ is realized as an admissible subcategory T of the perfect derived category of Λ. (Recall that an admissible subcategory is a such subcategory that the inclusion functor has a left and right adjoint functors.) We point that the admissible subcategory T plays an important role in the above categorical characterization. As a corollary, we obtain a result applicable for any finite dimensional graded IG algebra A which is not necessary a trivial extension algebra. Assume that the degree 0 part A_0 is of finite global dimension. Then the Grothendieck group $K_0(\underline{\mathsf{CM}}^{\mathbb{Z}}A)$ is free of finite rank. An explicit bound of the rank is given. We note that in the talk we will deal with the case where Λ is IG. Even in this case we have similar results.

Finally, we give several applications.

(1) Let Λ be an iterated tilted algebra of Dynkin type and C is a Λ-Λ-bimodule. If the trivial extension algebra $A = \Lambda \oplus C$ is IG, then A is of finite CM type.
(2) Assume that Λ is of finite global dimension and $C = M \otimes_K N$ is the tensor product of a left Λ-module M and a right Λ-module N over the base fields K. Then the trivial extension algebra $A = \Lambda \oplus C$ is IG and not of finite global dimension if and only if $\mathbb{R}\mathrm{Hom}_\Lambda(N, N) \cong K$ and $\mathbb{R}\mathrm{Hom}_\Lambda(N, \Lambda) = M[-p]$ for some $p \in \mathbb{Z}$. If this is the case, p is the projective dimension of N and the number of non-projective indecomposable CM-modules is $p+1$.

(3) Let $\Lambda := K[1 \to 2 \to 3]$ be the path algebra of the directed A_3-quiver. Using the categorical characterization, we classify Λ-Λ-bimodules C such that $\Lambda \oplus C$ is IG (and of infinite global dimension).

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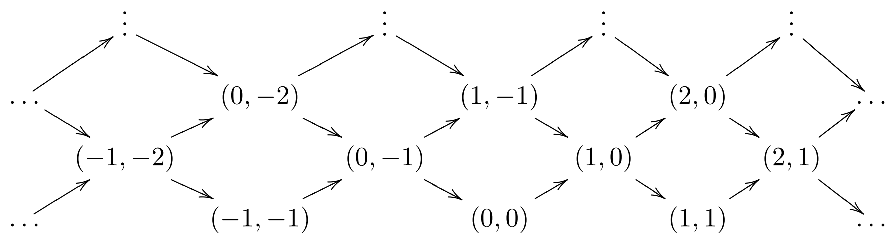
Higher Nakayama algebras

Julian Külshammer
(joint work with Gustavo Jasso)

In 'classical' Auslander–Reiten theory, the shape of the connected components of the Auslander–Reiten quiver was a particularly important question. Many important results have been obtained for particular classes of algebras. For example, for a path algebra kQ of a quiver Q not of Dynkin or Euclidean type, Ringel [9] proved that every connected component of the Auslander–Reiten quiver not containing a projective or an injective module is of the form $\mathbb{Z}\mathbb{A}_\infty$:



For group algebras, Webb [10] showed that the components of the stable Auslander–Reiten quiver which do not contain a τ-periodic module are of the form $\mathbb{Z}\Delta$ for Δ a Euclidean diagram or an infinite Dynkin diagram. Erdmann [4] showed an analogue of Ringel's result for group algebras, proving that every stable Auslander–Reiten component for the block of wild representation type of a group algebra is of the form $\mathbb{Z}\mathbb{A}_\infty$. For tame algebras, Crawley-Boevey showed that all but finitely many modules in each dimension lie in homogeneous tubes, i.e. components of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau)$. (In the above picture, τ acts on vertices by subtracting $(1,1)$.) Recently, it has also been proven that the Auslander–Reiten components of the derived category of an algebra of finite global dimension of radical square zero are of the form $\mathbb{Z}\tilde{Q}^{op}$ where $\tilde{Q}$ is the minimal gradable covering of the Gabriel quiver Q of the algebra or the form $\mathbb{Z}\mathbb{A}_\infty$, see [1].

Higher Auslander–Reiten theory started with Iyama's definition of a d-cluster-tilting subcategory of a module category [5], therein called maximal $(d-1)$-orthogonal subcategory. For an abelian category $\mathcal{A}$, a subcategory $\mathcal{M} \subseteq \mathcal{A}$ is called *d-cluster-tilting* if it is generating–cogenerating and functorially finite and satisfies

$$\begin{aligned}\mathcal{M} &= \{X \in \mathcal{A} | \mathrm{Ext}^i(X, \mathcal{M}) = 0 \,\forall i = 1, \dots, d-1\} \\ &= \{Y \in \mathcal{A} | \mathrm{Ext}^i(\mathcal{M}, Y) = 0 \,\forall i = 1, \dots, d-1\}\end{aligned}$$

The category $\mathcal{M}$ can be seen as a 'replacement' of an abelian category, where the shortest exact sequences are no longer short exact sequences, but sequences of length $d+2$. There is an analogous notion of a d-almost split sequence in such subcategories. An exact sequence

$$0 \to X_0 \to X_1 \to \cdots \to X_d \to X_{d+1} \to 0$$

with terms in $\mathcal{M}$ is called *d-almost split sequence* if $X_i \to X_{i+1}$ is in the (Jacobson) radical of $\mathcal{A}$ and each map $W \to X_{d+1}$ which is not a split epimorphism factors through X_d. One can of course ask about the shape of the components of the quiver of irreducible morphisms (which one could call 'higher Auslander–Reiten quiver'). In particular, it would be of interest to describe analogues of the components $\mathbb{Z}\mathbb{A}_\infty$ and $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ which seem to be the most typical shapes of components for many particular classes of algebras. A problem is, that at present, there is no known example of a finite dimensional algebra A having a d-cluster-tilting subcategory with infinitely many isomorphism classes of indecomposable modules.

There is another way in which an Auslander–Reiten component of type $\mathbb{Z}\mathbb{A}_\infty$ arises. In fact, $\mathbb{Z}\mathbb{A}_\infty$ is the Auslander–Reiten quiver of the category of finite dimensional representations of the quiver

$$\mathbb{A}_\infty^\infty = (\cdots \to -1 \to 0 \to 1 \to \dots).$$

In the figure on the previous page, the coordinate (i, j) stands for an 'interval module' that is a representation which has the ground field on each of the vertices in the interval $[j, i]$, connected by identities and zeros elsewhere. Closely related are the *Nakayama algebras*, which are the algebras such that every projective and every injective indecomposable module has a unique composition series. The connected Nakayama algebras are precisely the admissible quotients of the linearly oriented type $\mathbb{A}_n$-quiver, i.e. the full subquiver of $\mathbb{A}_\infty^\infty$ on the vertices $0, \dots, n-1$, and the linearly oriented type $\tilde{\mathbb{A}}_{n-1}$-quiver, i.e. the quiver obtained from $\mathbb{A}_\infty^\infty$ by identifying vertices which are the same modulo n.

In joint work with Gustavo Jasso [7], we were able to define higher analogues of $\mathbb{Z}\mathbb{A}_\infty$, $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, and the Nakayama algebras. We define $A_\infty^{(d)}$ as the idempotent quotient of the incidence category of $\mathbb{Z}^d$ with the product order by the idempotent ideal generated by the objects not in the set of ordered sequences

$$\mathbf{os}^{(d)} = \{(\lambda_1, \dots, \lambda_d) \in \mathbb{Z}^d | \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d\}.$$

Then, the category $\operatorname{mod} A_\infty^{(d)}$ of finite dimensional modules over $A_\infty^{(d)}$ has a d-cluster-tilting subcategory $\mathcal{M}_\infty^{(d)}$ whose indecomposable modules are interval modules. We see $A_\infty^{(d)}$ as a higher analogue of the path algebra of $\mathbb{A}_\infty^\infty$ and the quiver of irreducible morphisms of $\mathcal{M}_\infty^{(d)}$ as a higher analogue of $\mathbb{Z}\mathbb{A}_\infty$. Higher analogues of the path algebra of $\mathbb{A}_n$ have already been defined by Iyama in [6] and studied extensively in [8]. They are the idempotent quotients of $A_\infty^{(d)}$ by the idempotents corresponding to the vertices with $n-1 \geq \lambda_1, \lambda_d \geq 0$. In [7], we also defined higher analogues $\tilde{A}_{n-1}^{(d)}$ of the path algebra of $\tilde{\mathbb{A}}_{n-1}$ as quotients $A_\infty^{(d)}/(\varphi_d^n)$ where φ_d is the automorphism given on objects by subtracting $(1,1,\dots,1)$. Then, the category of finite dimensional nilpotent representations of $\tilde{A}_{n-1}^{(d)}$ has a d-cluster-tilting subcategory which is the image of the subcategory of $\operatorname{mod} A_\infty^{(d)}$ under the push-down functor.

To define analogues of the Nakayama algebras, recall that these can be parametrised by means of their *Kupisch series*, i.e. the sequence $\underline{\ell} = (\dim P_i)_{i=0,\dots,n-1}$. For quotients of the path algebra of $\mathbb{A}_n$, the corresponding interval modules are the modules corresponding to the vertices (i,j) with $i-j+1 \leq \ell_i$. Similarly, for the higher Nakayama algebras, define $A_{\underline{\ell}}^{(d)}$ to be the idempotent quotient of $A_n^{(d)}$ by the idempotents not in the set

$$\{(\lambda_1,\dots,\lambda_d) \in \mathbf{os}_n^{(d)} | \lambda_1 - \lambda_d + 1 \leq \ell_{\lambda_1}\}.$$

Then, the algebra $A_{\underline{\ell}}^{(d)}$ has a d-cluster-tilting subcategory in its module category. Similarly, higher Nakayama algebras of type $\tilde{\mathbb{A}}$ can be constructed from $\tilde{A}_{n-1}^{(d)}$ using results of Darpö and Iyama [3].

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Quiver moduli and Gromov-Witten invariants

Markus Reineke
(joint work with Thorsten Weist)

1. Problems of enumerative geometry

Enumerative geometry asks for counts of geometric objects subject to incidences. Classical examples are, for example, the number of lines through two general points (one, Euklid), the number of circles tangent to three general circles (eight, Apollonius), the number of lines in a general cubic surface (27, Cayley),. . .

A famous modern instance is M. Kontsevich's formula [3] for the numbers N_d of irreducible rational degree d curves in the projective plane through $3d-1$ general points, which is recursively given by $N_1 = 1$ and

$$N_d = \sum_{k+l=d} N_k N_l k^2 l \left(l \binom{3d-4}{3k-2} - k \binom{3d-4}{3k-1} \right).$$

Here we consider the problem of calculating the numbers Z_d of irreducible rational degree d curves in the projective plane through $2d-1$ general points, and tangent of order d to a fixed line in a fixed point. A recursive formula for Z_d is derived in [2] (see below).

2. Point configurations

We would like to study point configurations in projective space up to symmetries, which amounts to studying the action of $\mathrm{PGL}_d(\mathbf{C})$ on $(\mathbf{C}P^{d-1})^m$ for given $m > d \geq 1$. We call a point configuration $(p_1, \ldots, p_m)$ stable if

$$\dim\langle p_i \,:\, i \in I\rangle_{\mathbf{C}^d} > \frac{d}{m}|I|$$

for every proper subset $I \subset \{1, \ldots, m\}$, and equality for $I = \{1, \ldots, m\}$.

By Mumford's Geometric Invariant Theory, the quotient space

$$M_{d,m} = (\mathbf{C}P^{d-1})^m_{\mathrm{stable}}/\mathrm{PGL}_d(\mathbf{C})$$

of the stable locus admits a canonical structure of a connected complex algebraic manifold, which is compact in case $\gcd(d, m) = 1$.

Typical examples include $M_{d,d+1}$, which is a single point, $M_{2,4}$, which is a three-punctured projective line, and $M_{2,5}$, which is a degree five del Pezzo surface.

3. Main result

The main result which is reported here is:

Theorem: For all $d \geq 1$, we have

$$Z_d = \chi(M_{d,2d-1}),$$

the topological Euler characteristic of the moduli space of stable configurations of $2d-1$ points in $\mathbf{C}P^{d-1}$.

From a geometric point of view, there is no a priori reason for this equality, the left hand side being a rather delicate intersection theoretic invariant of a certain moduli space of curves in the projective plane, whereas the right hand side is a purely topological invariant of a much simpler moduli space. Moreover, there exists a conjectural refinement of the equality, the right hand side refining to the Poincaré polynomial, the left hand side refining to Block-Göttsche invariants. Note also that another instance of such an equality is derived in [5], but the present identity seems to be of a rather different nature.

4. Idea of the proof

The left hand side of the equality in the theorem is computed in [2] using a passage to a tropical curve count via a correspondence theorem, a discretization to a count of combinatorial objects called labelled floor diagrams, an a direct combinatorial enumeration of the latter, leading to the recursive formula $Z_1 = 1$ and

$$Z_{d+1} = \sum_{d=d_1+\ldots+d_s} \frac{1}{s!} \frac{(2d)!}{\prod_i (2d_i)!} \prod_i d_i^2 Z_{d_i}.$$

The same recursion can be established for the right hand side using an interpretation as a quiver moduli space and methods from the theory of Donaldson-Thomas invariants of quivers:

$M_{d,m}$ is identified with the moduli space $M_{\mathbf{d}}^{\Theta-\mathrm{st}}(Q)$ of stable representations of the m-subspace quiver Q (with m arrows pointing from vertices $i_1, \ldots, i_m$ to a unique sink j) of dimension vector $\mathbf{d} = \sum_{k=1}^m i_k + dj$, for the stability function $\Theta = \sum_{k=1}^m di_k^* - mj^*$. We also consider a framed version $M_{\mathbf{d},\mathbf{n}}^{\Theta}(Q)$ as in [1]. Then Donaldson-Thomas invariants $\mathrm{DT}_{\mathbf{d}}^{\Theta}(Q)$ of Q with respect to Θ can be defined [4] by the Euler product factorization

$$1 + \sum_{\Theta(\mathbf{d})=0} \chi(M_{\mathbf{d},\mathbf{n}}^{\Theta}(Q)) x^{\mathbf{d}} = \prod_{\Theta(\mathbf{d})=0} (1 - x^{\mathbf{d}})^{-(\mathbf{n}\cdot\mathbf{d})\cdot \mathrm{DT}_{\mathbf{d}}^{\Theta}(Q)}.$$

Specialization of this identity to the present situation yields

$$\chi(M_{d,2d+1}) = \sum_{d=d_1+\ldots+d_s} \frac{1}{s!} \frac{(2d)!}{\prod_i (2d_i)!} \prod_i d_i \mathrm{DT}_{d_i,2d_i}.$$

Using a detailed analysis of the geometric effect of small deformations of the stability function, one finds

$$\chi(M_{d,2d+1}) = \chi(M_{d+1,2d+1}) \text{ and } \mathrm{DT}_{d,2d} = d \cdot \chi(M_{d,2d-1}),$$

which recovers the recursion above.

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Beyond the stars

Philip Boalch

In the late 90's the author worked on a project to extend some of the technology related to connections on curves to the case of irregular connections (often called the "wild" case, although this tame/wild dichotomy is different to that in the quiver world). The basic idea is that tame (regular singular) connections are classified by fundamental group representations, whereas irregular connections are classified by more general topological data: Stokes local systems or Stokes structures Thus, for example, one can generalise the Atiyah-Bott-Goldman symplectic structure to this setting [B99, B01a], and the non-abelian Hodge hyperkähler metrics ([BB04], with O. Biquard). Even the simplest wild examples seem to be significant, for example the Poisson variety underlying the Drinfeld–Jimbo quantum group appears immediately as a space of Stokes data for connections with an order two pole on a disk ([B01b] for GL_n, [B02a] for any reductive group). The notion of a Stokes structure involves recording the filtrations (coming from the exponential growth rates of solutions) on sectors at each pole, and in general it is tricky to classify such flags (involving the Malgrange–Sibuya nonabelian cohomology space [BV89]), whereas the Stokes local systems involve unipotent Stokes multipliers, and are easy to classify. See for example [B15b] for an exposition of a simple class of examples, and the relation between these two algebraically equivalent approaches to Stokes data.

Somehow all of this was done with the stubborn refusal to use methods that only work for general linear groups. However in 2007 the author finally understood that many of the (additive) spaces $\mathcal{M}^*$ of connections considered in [B01a] were in fact (Nakajima) quiver varieties, and that this was a good way to understand the isomorphisms between such spaces ([B07] Exercise 3, [B08, B12, HY14]). In the tame case this had been understood earlier by Crawley-Boevey [CB03] (although even in that case one can better understand the reflections via the isomorphism between tame and wild character varieties coming from the Fourier–Laplace transform). This tame case corresponds to star-shaped quivers, so the upshot is that we can now play the same game with a larger class of quivers, now called "supernova quivers", generalising the stars. For example all the complete graphs and all the complete k-partite graphs are supernova. In brief this amounts to a new modular interpretation of the Nakajima quiver varieties for such quivers (as moduli spaces of meromorphic connections on trivial vector bundles on the Riemann sphere, or, better, as presentations of modules for the first Weyl algebra [B12]).

Then we can go back to the original viewpoint and look at the spaces of Stokes data (the wild character varieties) in these cases. In fact they can be viewed as multiplicative analogues of quiver varieties, but they are *not* in general isomorphic to the previous "classical" definition of multiplicative quiver varieties, suggested by the work of Crawley-Boevey and Shaw on deformed multiplicative preprojective algebras (see [CBS06, VdB08b, VdB08a, Yam08]). Rather, these space of Stokes data yield a generalisation of the classical multiplicative quiver varieties [B15a]. In particular this leads to some new noncommutative algebras (fission algebras) generalising the deformed multiplicative preprojective algebras. In this talk I will discuss some simple examples of wild character varieties/multiplicative quiver varieties and show how, in work with R. Paluba and D. Yamakawa, it leads to a new approach to the fact that many cluster algebras are functions on certain wild character varieties (for example the FST cluster algebras from surfaces correspond to taking $\mathfrak{sl}_2$ in the Stokes story).

Appendix—simple examples and more detailed review.

The classical multiplicative quiver varieties are built out of the following pieces, for each edge of a graph:

$$\mathcal{B}(V_1, V_2) = \{(a, b) \in \mathrm{Hom}(V_2, V_1) \oplus \mathrm{Hom}(V_1, V_2) \bigm| 1 + ab \text{ invertible}\}.$$

As explained in [Yam08] they were shown to be quasi-Hamiltonian spaces, with group valued moment map $(1 + ab, (1 + ba)^{-1})$ by Van den Bergh [VdB08b, VdB08a].

Given any graph Γ with nodes I, and an I-graded vector space V, the vector space $\mathrm{Rep}(\Gamma, V)$ of representations of the graph Γ on V is well defined (we view Γ as a doubled quiver—each edge denotes two oppositely oriented quiver edges). Then using the Van den Bergh edges one can define an open subset $\mathrm{Rep}^*(\Gamma, V) \subset \mathrm{Rep}(\Gamma, V)$ which is a quasi-Hamiltonian space for the group $H = \prod_I \mathrm{GL}(V_i)$. Performing the multiplicative symplectic reduction (at a central value of the moment map) yields the classical multiplicative quiver varieties [CBS06, Yam08]. However there are other possible choices of open subsets $\mathrm{Rep}^*(\Gamma, V) \subset \mathrm{Rep}(\Gamma, V)$ which are quasi-Hamiltonian H-spaces and the resulting "new multiplicative quiver varieties" often appear in wild nonabelian Hodge theory.

For example let $V_1 = \mathbb{C} = V_2$, $V = V_1 \oplus V_2 = \mathbb{C}^2$, $G = \mathrm{GL}(V)$ and consider the following groups

$$U_+ = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \; U_- = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}, \; H = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset G.$$

In [B02b] it was shown that the affine variety

$$\mathcal{A} = \mathcal{A}^k = G \times (U_+ \times U_-)^k \times H$$

is a quasi-Hamiltonian $G \times H$-space[1]. The reduction of $\mathcal{A}$ by G at the identity value of the moment map is

$$\mathcal{B} = \mathcal{B}^k = \{(\mathbf{S}, h) \in (U_+ \times U_-)^k \times H \mid hS_{2k} \cdots S_2 S_1 = 1\}$$

which is a quasi-Hamiltonian H-space with moment map h^{-1}. Here $\mathbf{S} = (S_1, \ldots, S_{2k})$ with $S_{odd/even} \in U_{+/-}$ respectively. Now let Γ be the graph with two nodes and $k-1$ edges connecting them. Using the Gauss decomposition it is easy to see:

Lemma 1. *$\mathcal{B}$ is a Zariski open subset of* $\mathrm{Rep}(\Gamma, V)$.

Further, it is slightly more involved to check that:

Proposition 2 ([B14])**.** *If $k = 2$ then $\mathcal{B}^k \cong \mathcal{B}(V_1, V_2)$ as quasi-Hamiltonian H-spaces.*

On the other hand if $k = 3$ it is not true that $\mathcal{B}^k \cong \mathcal{B}(V_1, V_2) \circledast \mathcal{B}(V_1, V_2)$ (cf. [B15a] Rmk 6.10), and similarly for higher k. If $k = 3$ then Γ has two edges, and we thus have two possible definitions of $\mathrm{Rep}^*(\Gamma, V)$. To distinguish them we colour the edges of Γ. If they are coloured differently then we define

$$\mathrm{Rep}^*(\Gamma, V) = \mathcal{B}(V_1, V_2) \circledast \mathcal{B}(V_1, V_2)$$

as in the classical theory, fusing two of the Van den Bergh edges. Otherwise, if they are coloured the same (so Γ is monochromatic), we set

$$\mathrm{Rep}^*(\Gamma, V) \cong \mathcal{B}^3.$$

The reductions of this are not isomorphic to the usual/classical multiplicative quiver varieties. If $k = 3$ then Γ is the affine A_1 graph, and the corresponding additive/Nakajima quiver varieties have real dimension four, and were the first known examples of complete hyperkähler manifolds (due to Eguchi–Hanson [EH78]). (They are $T^*\mathbb{P}^1$ in one complex structure and generic coadjoint orbits of $\mathrm{SL}_2(\mathbb{C})$ in another.) In the multiplicative case it is not known if the classical multiplicative quiver varieties admit complete hyperkähler metrics beyond the star-shaped case, but it follows from [BB04] that the monochromatic ones do:

Theorem 3. *For generic $q \in H$ the reduction $\mathcal{B}^k /\!\!/_q H$ is a complete hyperkähler manifold.*

Many more general results are known (see [B15a, B15b]). To understand where these results come from one needs to study the Stokes phenomenon. However it should be no surprise that Stokes data play a central role in algebraic Lie theory—indeed the study of algebraic groups originated in the work of Kolchin on differential Galois theory (see [Kol48] and its MathReview by Chevalley), and Ramis

[1] In fact [B02b] proves this for arbitrary complex reductive groups G (with $U_\pm$ the unipotent radicals of opposite Borels $B_\pm$). The spaces $\mathcal{A}$ are denoted $\widetilde{\mathcal{C}}/L$ in [B02b] Rmk 4 p.6. The non-abelian extension appears in [B09, B14] (with $B_\pm$ replaced by arbitrary opposite parabolics, and H by their common Levi subgroup).

has shown how the Stokes multipliers (plus the monodromy and exponential tori) generate the differential Galois group [Ram85].

To end let us briefly discuss the larger class of graphs that occur in more general examples, and mention the additive analogues of the spaces $\mathcal{B}$. Let $G = \mathrm{GL}_n(\mathbb{C})$ with maximal torus T, and Lie algebras $\mathfrak{t} \subset \mathfrak{g}$. The simplest spaces of Stokes data are associated to connections on the Riemann sphere, which locally (in some trivialisation at each pole) take the form

$$\nabla = d - A, \qquad A = dQ + \Lambda\frac{dz}{z} + \Theta(z)dz$$

where $Q = \sum_1^k A_i/z^i$ with $A_i \in \mathfrak{t}$, $\Lambda \in \mathfrak{h}$, Θ is a holomorphic map from a disk to $\mathfrak{g}$, and z is a local coordinate vanishing at the marked point. The diagonal element Q is the "irregular type" of the connection, and $\mathfrak{h} = \mathrm{Lie}(H)$, where $H \subset G$ is the centraliser of Q (the subgroup commuting with each A_i). Solutions of such connections involve the essentially singular term $\exp(Q)$ (note that the growth/decay of the entries of $\exp(Q)$ will vary dependent on the direction one goes towards $z = 0$—this dependence on the direction is central to the Stokes phenomenon). The examples $\mathcal{B} = \mathcal{B}^k$ considered above had $n = 2$ and just one marked point on the Riemann sphere, and irregular type Q with just one term $Q = A_k/z^k$ (with A_k having distinct eigenvalues and k the same k as above).

The definition of the graph Γ above generalises to arbitrary Q as follows: Let I be the set of common eigenspaces of all the A_i, and let $\bigoplus_I V_i$ be the resulting grading of $V = \mathbb{C}^n$. Let $q_i \in \mathbb{C}[z^{-1}]$ be the "eigenvalue" of Q on V_i. For $i, j \in I$ let d_{ij} be the degree of the polynomial $q_i(z^{-1}) - q_j(z^{-1})$. Then let $\Gamma(Q)$ be the graph with nodes I and $d_{ij} - 1$ edges between nodes $i, j \in I$. We call this the "fission graph" of Q (it was defined in an equivalent way, involving splaying/fission, in [B08] Appx. C). For example if $k = 2$ then $\Gamma(Q)$ is a complete s-partite graph (where s is the number of eigenvalues of A_k), and all complete s-partite graphs arise in this way, and in particular all the complete graphs.

Attached to Q there is a fission space $\mathcal{A}(Q)$ (see [B14] Thm 7.6, generalising [B02b, B09]) which is a quasi-Hamiltonian $G \times H$-space, and a reduced space $\mathcal{B}(Q) = \mathcal{A}(Q)/\!\!/G$ which is a quasi-Hamiltonian H-space and parameterises Stokes data for connections on $\mathbb{P}^1$ with just one pole, with irregular type Q. Then it is straightforward to check (cf. [B15a]):

Lemma 4. *$\mathcal{B}(Q)$ is isomorphic to a Zariski open subset of* $\mathrm{Rep}(\Gamma(Q), V)$.

So we can colour $\Gamma(Q)$ monochromatically and define the invertible representations $\mathrm{Rep}^*(\Gamma(Q), V) \subset \mathrm{Rep}(\Gamma(Q), V)$ in this way, and fuse such pieces together etc.

For the additive analogues of these spaces (in the usual world of Hamiltonian spaces) we consider connections on the trivial vector bundle over the Riemann sphere with the given irregular type. In general one can get symplectic moduli spaces of such connections by fixing the whole principal part of the connection at each pole (upto local holomorphic isomorphism), as in [B01a] §2. In brief we fix an irregular type Q and a formal residue Λ at each pole (as above), and identify $dQ + \Lambda dz/z$ with a point of the dual of the Lie algebra of the group

$G_{k+1} = \mathrm{GL}_n(\mathbb{C}[z]/z^{k+1})$ of jets of bundle automorphisms at each pole (see op. cit.). Then the coadjoint orbit $\mathcal{O}$ through $dQ + \Lambda dz/z$ is a symplectic manifold, and it has a Hamiltonian action of G via the inclusion $G \subset G_{k+1}$ and the coadjoint action. Repeating at each pole gives a symplectic description:

$$\mathcal{M}^* = (\mathcal{O}_1 \times \cdots \times \mathcal{O}_m) /\!\!/ G$$

of the moduli space. The notation $\mathcal{M}^*$ indicates this space is an open part of the full (de Rham) moduli space one can consider, allowing non-trivial vector bundles.

Now each orbit $\mathcal{O}$ may be decoupled as follows. Let $B_{k+1} \subset G_{k+1}$ be the kernel of the evaluation map $G_{k+1} \twoheadrightarrow G$. Then dQ may be viewed as a point of the dual of the Lie algebra of B_{k+1}. Let $\mathcal{O}_B \subset \mathrm{Lie}(B_{k+1})^*$ be its coadjoint orbit. Let $\widetilde{\mathcal{O}} = \mathcal{O}_B \times T^*G$. We call this space $\widetilde{\mathcal{O}}$ the "extended orbit"; it is a Hamiltonian $G \times H$-space and is the additive analogue of the fission space $\mathcal{A}$. It arises by allowing the formal residue Λ to vary and adding a compatible framing (cf. [B01a] §2)—the term "extended" is by analogy with the set-up of Jeffrey [Jef94]. The orbit $\mathcal{O}$ arises as the reduction $\widetilde{\mathcal{O}} /\!\!/_{\Lambda} H$ at the value Λ of the moment map. The additive analogue of $\mathcal{B}$ is thus the reduction $\widetilde{\mathcal{O}} /\!\!/ G$. However the action decouples:

Lemma 5. *The reduction $\widetilde{\mathcal{O}} /\!\!/ G$ is isomorphic to $\mathcal{O}_B$.*

Now $\mathcal{O}_B$ is a coadjoint orbit of a unipotent group, so has global Darboux coordinates (as noted in [B01a]), but to make the link to Nakajima quiver varieties we need a stronger result:

Theorem 6 ([B08, HY14])**.** *$\mathcal{O}_B$ is isomorphic to $\mathrm{Rep}(\Gamma(Q), V)$ as a Hamiltonian H-space.*

Consequently in the case when there is just one pole this identifies $\mathcal{M}^*$ with a Nakajima quiver variety. The reduction by H at Λ corresponds to gluing a leg (type A Dynkin graph) onto each node of $\Gamma(Q)$ to obtain a larger graph—we call such graphs "supernova graphs" as they generalise the stars (cf. [B12] Defn 9.1). The fact that one can get the Eguchi–Hanson space in this way first appeared in [B07] Ex.3, and this set the ball rolling. More generally one can allow a finite number of simple poles and still obtain that $\mathcal{M}^*$ is a quiver variety (as in [B08]), giving more general "modular" interpretations of certain Nakajima quiver varieties (as moduli spaces of connections), making contact with the viewpoint of [CB03] in the tame case.

However as soon as there is more than one irregular singularity in this set-up, one will retain some factors of T^*G in this description of $\mathcal{M}^*$. Such spaces can still be embedded in quiver varieties though, via the inclusion $T^*G \subset T^*\mathfrak{g}$, as has been pursued by Hiroe [Hir13].

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Spherical objects in higher Auslander–Reiten theory

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(joint work with Julian Külshammer)

Let k be a field and δ an integer number. Consider the graded algebra $B_\delta := k[\varepsilon]$ of dual numbers with ε placed in cohomological degree δ. We view B_δ as a differential graded algebra with vanishing differential. It is known [3] that the perfect derived category $\operatorname{perf} B_\delta$ of B_δ is a δ-Calabi–Yau triangulated category in the sense that $[\delta]\colon \operatorname{perf} B_\delta \to \operatorname{perf} B_\delta$ is a Serre functor. Moreover, an elementary application of derived Morita theory of A_∞-algebras shows that, up to equivalence of triangulated categories, this is the only (algebraic) triangulated category classically generated by an object whose graded endomorphism algebra is isomorphic to B_δ as a graded algebra [7].

Suppose now that m is an integer greater than or equal to 2. Then, $\operatorname{perf} B_m$ can be realised as the m-cluster category $\mathcal{C}_{\infty,m}$ of the category of finite dimensional representations of the path category

$$A_\infty := k(\cdots \to -1 \to 0 \to 1 \to \cdots),$$

Following [1] and [6], this category is defined as the orbit quotient of the bounded derived category $D^b(\operatorname{mod} A_\infty)$ which trivialises the action of the autoequivalence

$$\nu[-m]\colon D^b(\operatorname{mod} A_\infty) \to D^b(\operatorname{mod} A_\infty),$$

where $\nu := - \otimes^{\mathbb{L}}_{A_\infty} D(A_\infty)$ is the Serre functor of $D^b(\operatorname{mod} A_\infty)$. An elementary calculation shows that the image of any simple A_∞-module under the canonical projection functor $D^b(\operatorname{mod} A_\infty) \to \mathcal{C}_{\infty,m}$ is an m-spherical object which classically generates $\mathcal{C}_{\infty,m}$.

Let d be a positive integer and m an integer greater than or equal to 2. In our work in progress we apply a modified version of the above construction to the category $A^{(d)}_\infty$, a suitable d-dimensional analogue of A_∞ we introduced in [5]. This yields an md-Calabi–Yau triangulated category $\mathcal{C}^{(d)}_{\infty,m}$ with the following properties:

- If $d = 1$, then $\mathcal{C}^{(1)}_{\infty,m} = \mathcal{C}_{\infty,m}$.
- There exists a canonically defined weakly $d\mathbb{Z}$-cluster-tilting subcategory $\mathcal{O}^{(d)}_{\infty,m}$ of $\mathcal{C}^{(1)}_{\infty,m}$ (we expect that $\mathcal{O}^{(d)}_{\infty,m}$ is functorially finite in $\mathcal{C}^{(d)}_{\infty,m}$ so that the adjective 'weakly' can be removed).
- There exists an objects $S \in \mathcal{O}^{(d)}_{\infty,m}$ which is md-spherical when viewed as an object of $\mathcal{C}^{(d)}_{\infty,m}$ and which generates $\mathcal{O}^{(d)}_{\infty,m}$ in a suitable sense.
- In the case $m = 2$, the mutation combinatorics of $2d$-cluster-tilting objects in $\mathcal{C}^{(d)}_{\infty,2}$ which are contained in $\mathcal{O}^{(d)}_{\infty,2}$ is controlled by the triangulations of the cyclic apeirogon of dimension $2d$ (a higher dimensional analogue of the ∞-gon considered in [2] where this result is established in the case $d = 1$).

Our results build on earlier work of Oppermann and Thomas [8]. From the viewpoint of Iyama's higher Auslander–Reiten theory [4] the above properties suggest that $\mathcal{C}^{(d)}_{\infty,m}$ should be considered as a 'd-dimensional analogue' of $\operatorname{perf} B_m$.

Note that the differential graded algebra B_m can be interpreted as the graded singular cohomology algebra of the m-dimensional sphere with coefficients in k (we assume that k has characteristic zero for this). We conclude with the following question.

Question. *Is there a 'nice' differential graded category* $\mathcal{B}_m^{(d)}$ *such that* $\mathcal{B}_m^{(1)} = B_m$ *(viewing* B_m *as a differential graded category having a single object) and* $\operatorname{perf}\mathcal{B}_m^{(d)} = \mathcal{C}_{\infty,m}^{(d)}$*?*

Ideally, such a 'nice' differential graded category $\mathcal{B}_m^{(d)}$ would admit a topological interpretation analogous to that of B_m.

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A decomposition theorem for moduli of representations of algebras, with application to moduli of special biserial algebras

Ryan Kinser

1. Background

This is a report on two very recent projects concerning moduli spaces of representations of finite-dimensional associative algebras. We assume throughout that $\Bbbk$ is an algebraically closed field of characteristic 0. All algebras are assumed to be basic, finite-dimensional, and associative, thus of the form $A = \Bbbk Q/I$ for a quiver Q and (admissible) ideal I.

Given a quiver Q and *dimension vector* $\mathbf{d}\colon Q_0 \to \mathbb{Z}_{\geq 0}$, we study the *representation variety*

$$\operatorname{rep}_Q(\mathbf{d}) = \prod_{a \in Q_1} \operatorname{Mat}(\mathbf{d}(ta), \mathbf{d}(ha)), \tag{1}$$

where $\mathrm{Mat}(m,n)$ denotes the variety of matrices with m rows, n columns, and entries in $\Bbbk$. We consider the left action of the *base change group* and its action on $\mathrm{rep}_Q(\mathbf{d})$

$$(2) \qquad GL(\mathbf{d}) = \prod_{z\in Q_0} GL(\mathbf{d}(z)) \qquad g\cdot M = (g_{ha}M_a g_{ta}^{-1})_{a\in Q_1}.$$

where $g = (g_z)_{z\in Q_0} \in GL(\mathbf{d})$ and $M = (M_a)_{a\in Q_1} \in \mathrm{rep}_Q(\mathbf{d})$. Inside $\mathrm{rep}_Q(\mathbf{d})$ we have the $GL(\mathbf{d})$-invariant closed subvariety

$$\mathrm{rep}_A(\mathbf{d}) = \{M \in \mathrm{rep}_Q(\mathbf{d}) \mid M(r) = 0 \quad \forall r \in I\}$$

which, in general, may have many irreducible components.

We now recall the definitions of (semi)stability and moduli spaces of representations from [1]. First we must fix a *weight* $\theta\colon Q_0 \to \mathbb{Z}$. A representation M of A is said to be *θ-semistable* if $\theta\cdot\mathbf{dim}M = 0$ and $\theta\cdot\mathbf{dim}N \le 0$ for any subrepresentation $N \le M$, and *θ-stable* if the inequality is strict whenever N is nonzero and proper. Every semistable representation M has a well-defined collection of stable composition factors; the direct sum of these is called the associated polystable representation $\widetilde{M}$.

For any $GL(\mathbf{d})$-invariant closed subvariety $C \subseteq \mathrm{rep}_A(\mathbf{d})$, we have the *space of semi-invariants*

$$\mathrm{SI}(C)_\theta = \{f \in K[C] \mid g\cdot f = \left(\prod_{z\in Q_0} \det(g_z)^{\theta(z)}\right) f\}.$$

Summing over nonnegative multiples of θ gives a graded ring, and then the *moduli space of semistable representations in C* (with respect to θ) is defined as

$$\mathcal{M}(C)^{ss}_\theta = \mathrm{Proj}\left(\bigoplus_{m\ge 0} \mathrm{SI}(C)_{m\theta}\right).$$

2. The decomposition theorem

In joint work with Calin Chindris [3], we prove a general decomposition theorem which expresses moduli spaces of semistable representations in terms of smaller moduli spaces. To make this precise, we need one more definition, which we introduce in our new work. It can be viewed as an analogue of the Jordan-Holder theorem for varieties of semistable representations.

Definition 3. Let C be a $GL(\mathbf{d})$-invariant, irreducible, closed subvariety of $\mathrm{rep}(A,\mathbf{d})$. Consider a collection $(C_i \subseteq \mathrm{rep}(A,\mathbf{d_i}))_i$ of $GL(\mathbf{d}_i)$-invariant, irreducible, and closed subvarieties such that $C_i \ne C_j$ for $i \ne j$, along with a collection of multiplicities $(m_i \in \mathbb{Z}_{>0})_i$. We say that $(C_i, m_i)_i$ is a *θ-stable decomposition of C* if the following conditions hold:

(TS1) For a general representation $M \in C^{ss}_\theta$, its corresponding θ-polystable representation $\widetilde{M}$ is isomorphic to a representation in $(C^s_{1,\theta})^{m_1}\times\ldots\times(C^s_{r,\theta})^{m_r}$.

(TS2) If $(C'_i \subseteq \mathrm{rep}(A, \mathbf{d}_i))_i$ is any collection of subvarieties and $(m'_i \in \mathbb{Z}_{>0})$ a list of multiplicities which together also satisfy (TS1), then (after perhaps relabeling) we have that $(C^s_{1,\theta})^{m_1} \times \ldots \times (C^s_{r,\theta})^{m_r} \subseteq (C'^s_{1,\theta})^{m'_1} \times \ldots \times (C'^s_{r,\theta})^{m'_r}$.

We use the expression

$$C = m_1 C_1 \dotplus \ldots \dotplus m_r C_r \qquad m_i \in \mathbb{Z}_{\geq 0}. \tag{4}$$

to indicate a θ-stable decomposition of C.

Chindris and I first prove that every such C with $\mathrm{SI}(C)_\theta \neq 0$ admits a unique θ-stable decomposition. Then the main theorem of our work is the following result.

Theorem 5. Let A be a finite-dimensional algebra and $C \subseteq \mathrm{rep}(A, \mathbf{d})$ a $GL(\mathbf{d})$-invariant, irreducible, closed subvariety of $\mathrm{rep}(A, \mathbf{d})$ with θ-stable decomposition $C = m_1 C_1 \dotplus \ldots \dotplus m_r C_r$.

(a) Suppose that C_1 is an orbit closure. Then we have that

$$\mathcal{M}(C)^{ss}_\theta \simeq \mathcal{M}(\overline{C_2^{\oplus m_2} \oplus \cdots \oplus C_r^{\oplus m_r}})^{ss}_\theta.$$

(b) Assume now that none of the C_i are orbit closures. There is a natural map

$$\Psi \colon S^{m_1}(\mathcal{M}(C_1)^{ss}_\theta) \times \ldots \times S^{m_r}(\mathcal{M}(C_r)^{ss}_\theta) \to \mathcal{M}(C)^{ss}_\theta$$

which is surjective, finite, and birational. Furthermore, if C^{ss} is a normal variety, then Ψ is an isomorphism.

While it is clear that stable composition factors which are orbit closures do not affect the dimensions of families of representations, part (a) says that in fact they do not affect the geometry of moduli spaces in any way at all.

It was asked during the talk whether we could hope for a stronger result, such as the map Ψ being bijective or even an isomorphism. In general, it is not even bijective; roughly, positive-dimensional loci of stable representations which lie in the intersection of more than one stable irreducible component can give rise to disconnected fibers. We do, however, get the following corollary by combining our theorem with a result of Carroll and Chindris [2, Prop. 12].

Corollary 6. Any moduli space of semistable representations of a tame (or even Schur-tame) algebra is a rational variety.

3. Application to special biserial algebras

Although arbitrary projective varieties can arise as moduli spaces spaces of representations of finite-dimensional algebras, one may ask how representation theoretic properties of a given algebra restrict the geometry of its moduli spaces.

In joint work with Chindris, Andrew Carroll, Amelie Schreiber, and Jerzy Weyman [4], we study moduli spaces of representations of special biserial algebras. Our main result is the following theorem.

Theorem 7. Let A be a special biserial algebra. Then any irreducible component of a moduli space of θ-semistable A-modules is a product of projective spaces.

The proof involves studying the θ-stable decompositions of irreducible components and establishing a connection with varieties of circular complexes. As the irreducible components of the latter varieties are known to be normal, a study of moduli spaces of stable components combined with Theorem 5 above gives the desired isomorphism.

A couple of natural problems are being investigated for future work.

Problem 8. Determine whether or not Ψ is an isomorphism for all tame algebras.

Problem 9. Determine whether or not moduli spaces of tame algebras are always smooth.

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Weighted surface algebras, algebras of generalized quaternion type, and of generalized dihedral type

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(joint work with Andrzej Skowroński)

One main motivation for this work is to place blocks of group algebras of finite groups into a wider context. This has been done for blocks of finite type, they are special cases of Brauer tree algebras. In order to study blocks of tame representation type, algebras of dihedral type, of semidihedral type, and quaternion type were introduced (see [3]). Work of [1] is partly motivated by certain blocks of symmetric groups.

Another motivation is the problem of understanding periodicity. We consider finite-dimensional algebras A over some algebraically closed field. The algebra A is said to be periodic if it is periodic with respect to the syzygy operator Ω_{A^e} in the category $A, A-$bimodules. If A has period d then all one-sided A-modules are Ω_A-periodic of period dividing d. Periodic algebras are selfinjective, and such algebras occur in various contexts. In particular, blocks with quaternion defect groups are periodic, of period four.

Recently, there has been new input coming from the theory of cluster algebras, and this has led to new algebras which place blocks of tame type into a wider context. (For more background, see the introduction of [4].)

1. Weighted surface algebra

Definition 1.1. *A triangulation quiver is a pair (Q, f) where Q is a 2-regular quiver together with a permutation f of the arrows such that for each arrow α of Q, $t(\alpha) = s(f(\alpha))$ and $f^3 = 1$. (A quiver is 2-regular if at each vertex two arrows start and two arrows end).*

In [4] it is shown that triangulation quivers correspond to quivers defined from triangulated surfaces where triangles may have arbitrary orientation, and where a boundary component corresponds to a fixed point of the permutation f.

To define weighted surface algebras, we need more data. Since Q is 2-regular there are two other permutations of the arrows of Q. First, if α is an arrow starting at i, define $\bar{\alpha}$ to be the other arrow starting at i. Next, for any arrow α, define $g(\alpha) := \overline{f(\alpha)}$. If α is an arrow, let $\mathcal{O}(\alpha)$ be the g-orbit containing α, and let $n_\alpha = |\mathcal{O}(\alpha)|$. To each α we associates a multiplicity, $m_\alpha \geq 1$ and a weight $c_\alpha \in K \setminus \{0\}$, where the functions $m_\bullet$ and $c_\bullet$ are constant on g-orbits. In the path algebra KQ, we need elements which are products along a cycle of g. If α is an arrow, set $B_\alpha := (\alpha g(\alpha) \ldots q^{n_\alpha - 1}(\alpha))^{m_\alpha}$ of length $n_\alpha m_\alpha$, and let A_α be the subpath of B_α such that $A_\alpha g^{n_\alpha - 1}(\alpha) = B_\alpha$.

Definition 1.2. *A weighted surface algebra is an algebra $A = KQ/I$ where (Q, f) is a triangulation quiver, and $I = I(f, m_\bullet, c_\bullet)$ is generated by:*
(1) $\alpha f(\alpha) - c_{\bar{\alpha}} A_{\bar{\alpha}}$ for each arrow α, and
(2) $\alpha f(\alpha) g(f(\alpha))$ for each arrow α.

A similar definition is given in [7]. It is natural to consider degenerations of such algebras. This leads to the following:

Definition 1.3. *A biserial weighted surface algebra is an algebra $B = KQ/J$ where (Q, f) is a triangulation quiver and $J = J(f, m_\bullet)$ is generated by:*
(1) $\alpha f(\alpha)$ for each arrow α, and
(2) $B_\alpha - B_{\bar{\alpha}}$ for each arrow α.

Biserial weighted surface algebras are special biserial, hence tame. A weighted surface algebra degenerates to a biserial weighted surface algebra, hence it is also tame. The following is proved in [4].

Theorem 1.4. *Assume A is a weighted surface algebra, or is socle equivalent to a weighted surface algebra. Then*
(1) A is tame and symmetric.
(2) A is periodic as a bimodule, of period four, unless A is a singular tetrahedral algebra.

The singular tetrahedral algebra has quiver obtained from a tetrahedron, with the natural orientation, where all multiplicities m_α are equal to 1, and where the product of all c_α is equal to 1, for details see Section 6 of [4].

2. Generalized quaternion type

We say that an algebra A is of generalized quaternion type if A is tame symmetric, and all simple A-modules have Ω_A-period four. Recall from [3] that A is an algebra of quaternion type if in addition the Cartan matrix of A is non-singular. An indecomposable algebra of quaternion type has at most three simple modules (see [3]). On the other hand, without the condition on the Cartan matrix, the number of simple modules can be arbitrary. The following is proved in [5].

Theorem 2.1. *Assume Q is a 2-regular quiver and $A = KQ/I$ for some admissible ideal I. Then A is of generalized quaternion type if and only if A is a weighted surface algebra (but not the singular tetrahedral algebra).*

3. Generalized dihedral type

Algebras of dihedral type as defined in [3] generalize blocks of group algebras with dihedral defect groups. We generalize further, and modify slightly.

Definition 3.1. *Assume $D = KQ/I$ where Q is a quiver and I is an admissible ideal of KQ. Then D is of generalized dihedral type if*
(1) D is symmetric and tame.
(2) The stable AR-quiver of D consists of
(i) tubes of rank $1, 3$, and
(ii) nonperiodic components isomorphic to $\mathbb{Z}A_\infty^\infty$ or $\mathbb{Z}\widetilde{A}_n$.
(3) Ω_D fixes each tube of rank 3.

Originally, algebras of dihedral type were defined to satisfy (1) and (2) and in addition, the Cartan matrix should be non-singular, and the number of 3-tubes should be at most two.

Every biserial weighted surface algebra is of generalized dihedral type. The converse is not quite true. An algebra of generalized dihedral type can have pairs of arrows $\alpha : i \to j$ and $\beta : j \to i$ which are not loops and satisfy $\alpha\beta = 0$ and $\beta\alpha = 0$, then the modules αD and βD belong to two different tubes of rank 1. We have an explicit construction relating D to a biserial weighted surface algebra (see [6]):

Theorem 3.2. *Assume $D = KQ/I$ where I is an admissible ideal of KQ. Then D is of generalized dihedral type if and only if D is a certain idempotent algebra of a biserial weighted surface algebra.*

4. Idempotent algebras

One may ask what type of algebras one gets as idempotent algebras of biserial weighted surface algebras. We have the following [6]:

Theorem 4.1. *The following are equivalent for an algebra A:*
(1) A is a Brauer graph algebra;
(2) A is symmetric and special biserial;
(3) A is isomorphic to eBe where B is a biserial weighted surface algebra.

A different realisation of Brauer graph algebras was obtained by L. Demonet [2].

5. Partial degenerations

By degenerating all relations of type (1) in 1.2 of a weighted surface algebra one obtains a biserial weighted surface algebra. If one only degenerates the type (1) relations for some of the cycles of f, then one obtains algebras which have some components isomorphic to $\mathbb{Z}D_\infty$, and which may have periodic simple modules of period four, and also may have simple modules in components isomorphic to $\mathbb{Z}A_\infty^\infty$. These properties are natural generalizations of 'algebras of semidihedral type', as defined in [3]. The study of these algebras remains to be done.

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On additive Deligne-Simpson problem

Kazuki Hiroe

The additive Deligne-Simpson problem asks the existence of an irreducible Fuchsian differential equation on the Riemann sphere with prescribed local data. A system of first order linear differential equations is called *Fuchsian* if it is of the form

$$\frac{d}{dx}Y = \sum_{i=1}^{p} \frac{A_i}{x-a_i}Y \quad (A_i \in M(n,\mathbb{C}),\, i = 1,\ldots,p).$$

Here we call each A_i the *residue matrix* at the singular point a_i. Also $A_0 = -\sum_{i=1}^{p} A_i$ is called the residue matrix at ∞. We say that $\frac{d}{dx}Y = \sum_{i=1}^{p}\frac{A_i}{x-a_i}Y$ is *irreducible* if $A_0,\ldots,A_p$ have no nontrivial simultaneous invariant subspace of $\mathbb{C}^n$, i.e., if there exists $W \subsetneqq \mathbb{C}^n$ such that $A_iW \subset W$ for all $i = 0,\ldots,p$, then $W = \{0\}$.

Definition 1 (additive Deligne-Simpson problem)**.** An *additive Deligne-Simpson problem for equations* consists of a collection of points $a_1,\ldots,a_p$ in $\mathbb{C}$ and of conjugacy classes $C_0, C_1,\ldots,C_p$ in $M(n,\mathbb{C})$. A *solution* of the additive Deligne-Simpson problem is an *irreducible* Fuchsian differential equation

$$\frac{dY}{dx} = \sum_{i=1}^{p} \frac{A_i}{x-a_i}Y \quad (A_i \in M(n,\mathbb{C}),\, i = 1,\ldots,p)$$

whose residue matrices $A_i \in C_i$ for $i = 0,\ldots,p$.

This problem is developed by V. Kostov as an analogy of the problem studied by P. Deligne and C. Simpson (see [5]), so called multiplicative Deligne-Simpson problem. After Kostov's several works, a complete necessary and sufficient condition of the existence of a solution is given by W. Crawley-Boevey [2].

As a generalization of this problem, it seems to be natural to consider similar problems for non-Fuchsian equations which might be first considered by Boalch [1]. In our generalization along with Boalch's way, conjugacy classes, i.e., $\mathrm{GL}(n,\mathbb{C})$-orbits of Jordan normal forms, are replaced by the following orbits of (unramified) Hukuhara-Turrittin-Levelt normal forms.

Definition 2 (Hukuhara-Turrittin-Levelt normal form)**.** If $B \in M(n, \mathbb{C}(\!(x)\!))$ is of the form

$$B = \mathrm{diag}(q_1(x^{-1})I_{n_1} + R_1x^{-1}, \dots, q_m(x^{-1})I_{n_m} + R_mx^{-1})$$

with $q_i(s) \in s^2\mathbb{C}[s]$ satisfying $q_i \neq q_j$ if $i \neq j$ and $R_i \in M(n_i, \mathbb{C})$, then B is called the *Hukuhara-Turrittin-Levelt normal form* or the *HTL normal form* shortly. Here I_m is the identity matrix of $M(m, \mathbb{C})$.

For $k \geq 1$, let us define $G_k = \mathrm{GL}(n, \mathbb{C}[\![x]\!]/x^k\mathbb{C}[\![x]\!])$ and the vector space

$$\mathfrak{g}_k^* := M(n, x^{-k}\mathbb{C}[\![x]\!]/\mathbb{C}[\![x]\!]) = \left\{ \frac{A_k}{x^k} + \cdots + \frac{A_1}{x} \;\middle|\; A_i \in M(n, \mathbb{C}) \right\}.$$

Then an HTL normal form B can be seen as an element in $\mathfrak{g}_k^*$ for some k and consider the G_k-orbit $\mathcal{O}_B = \left\{ gBg^{-1} \in \mathfrak{g}_k^* \mid g \in G_k \right\}$ of B in $\mathfrak{g}_k^*$.

Definition 3 (generalized additive Deligne-Simpson problem for equations)**.** A *generalized additive Deligne-Simpson problem for equations* consists of a collection of points $a_1, \dots, a_p$ in $\mathbb{C}$, of nonzero positive integers $k_0, \dots, k_p$ and of HTL normal forms $B^{(i)} \in \mathfrak{g}_{k_i}^* \subset M(n, \mathbb{C}(\!(x)\!))$ for $i = 0, \dots, p$. A *solution* of the generalized additive Deligne-Simpson problem for equations is an *irreducible* differential equation

$$\frac{d}{dx}Y = \left(\sum_{i=1}^{p} \sum_{j=1}^{k_i} \frac{A_{i,j}}{(x-a_i)^j} + \sum_{2 \leq j \leq k_0} A_{0,j}x^{j-2} \right) Y$$

satisfying that $A^{(i)}(x) \in \mathcal{O}_{B^{(i)}}$ for $i = 0, \dots, p$. Here $A^{(i)}(x) = \sum_{j=1}^{k_i} A_{i,j}x^{-j}$ for $i = 0, \dots, p$ and $A_{0,1} = -\sum_{i=1}^{p} A_{i,1}$. We say

$$\frac{d}{dx}Y = \left(\sum_{i=1}^{p} \sum_{j=1}^{k_i} \frac{A_{i,j}}{(x-a_i)^j} + \sum_{2 \leq j \leq k_0} A_{0,j}x^{j-2} \right) Y$$

is *irreducible* if the collection $(A_{i,j})_{\substack{0 \leq i \leq p \\ 1 \leq j \leq k_i}}$ of coefficient matrices is irreducible.

This can be seen as a natural generalization of additive Deligne-Simpson problems which contains the original problems for Fuchsian equations as the special case $k_0 = \cdots = k_p = 1$.

In order to determine the existence condition of solutions of Deligne-Simpson problems, Crawley-Boevey [2] shows that Fuchsian systems can be realized as representations of quivers, and applies the existence theorem of irreducible representations of deformed preprojective algebras associated with the quivers to the existence of solutions of additive Deligne-Simpson problems. Our generalization of his theory is as follows. Let us suppose that $B^{(0)}, \ldots, B^{(p)}$ are written by

$$B^{(i)} = \operatorname{diag}\left(q_1^{(i)}(x^{-1})I_{n_1^{(i)}} + R_1^{(i)}x^{-1}, \ldots, q_{m^{(i)}}^{(i)}(x^{-1})I_{n_{m^{(i)}}^{(i)}} + R_{m^{(i)}}^{(i)}x^{-1}\right)$$

and choose complex numbers $\xi_1^{[i,j]}, \ldots, \xi_{e_{[i,j]}}^{[i,j]}$ so that

$$\prod_{k=1}^{e_{[i,j]}} (R_j^{(i)} - \xi_k^{[i,j]}) = 0$$

for $i = 0, \ldots, p$ and $j = 1, \ldots, m^{(i)}$. Set $I_{\mathrm{irr}} = \{i \in \{0, \ldots, p\} \mid m^{(i)} > 1\} \cup \{0\}$ and $I_{\mathrm{reg}} = \{0, \ldots, p\} \backslash I_{\mathrm{irr}}$. If $m^{(i)} = 1$, then all coefficients of x^{-j} for $j > 2$ in $B^{(i)}$ are scalar matrices.

Then let Q be the quiver with the set of vertices

$$\mathsf{Q}_0 = \left\{ [i,j] \;\middle|\; \begin{array}{l} i \in I_{\mathrm{irr}}, \\ j = 1, \ldots, m^{(i)} \end{array} \right\} \cup \left\{ [i,j,k] \;\middle|\; \begin{array}{l} i = 0, \ldots, p, \\ j = 1, \ldots, m^{(i)}, \\ k = 1, \ldots, e_{[i,j]} - 1 \end{array} \right\}$$

and the set of arrows

$$\begin{aligned} \mathsf{Q}_1 = & \left\{ \rho_{[i,j']}^{[0,j]} \colon [0,j] \to [i,j'] \;\middle|\; \begin{array}{l} j = 1, \ldots, m^{(0)}, \\ i \in I_{\mathrm{irr}} \backslash \{0\}, \\ j' = 1, \ldots, m^{(i)} \end{array} \right\} \\ & \cup \left\{ \rho_{[i,j],[i,j']}^{[k]} \colon [i,j] \to [i,j'] \;\middle|\; \begin{array}{l} i \in I_{\mathrm{irr}},\ 1 \leq j < j' \leq m^{(i)}, \\ 1 \leq k \leq d_i(j,j') \end{array} \right\} \\ & \cup \left\{ \rho_{[i,j,1]} \colon [i,j,1] \to [i,j] \mid i \in I_{\mathrm{irr}},\ j = 1, \ldots, m^{(i)} \right\} \\ & \cup \left\{ \rho_{[0,j]}^{[i,1,1]} \colon [i,1,1] \to [0,j] \mid i \in I_{\mathrm{reg}},\ j = 1, \ldots, m^{(0)} \right\} \\ & \cup \left\{ \rho_{[i,j,k]} \colon [i,j,k] \to [i,j,k-1] \;\middle|\; \begin{array}{l} i = 0, \ldots, p, \\ j = 1, \ldots, m^{(i)}, \\ k = 2, \ldots, e_{[i,j]} - 1 \end{array} \right\}. \end{aligned}$$

Here $d_i(j,j') = \deg_{\mathbb{C}[x]}(q_j^{(i)}(x) - q_{j'}^{(i)}(x)) - 2$. To each vector $\beta \in \mathbb{Z}^{\mathsf{Q}_0}$, we associate integers $q(\beta) = \sum_{a \in \mathsf{Q}_0} \beta_a^2 - \sum_{\rho \in \mathsf{Q}_1} \beta_{s(\rho)}\beta_{t(\rho)}, p(\beta) = 1 - q(\beta)$. Here $s(\rho)$ and $t(\rho)$ are the source and target of the arrow ρ respectively. The following sublattice of $\mathbb{Z}^{\mathsf{Q}_0}$ plays an essential role,

$$\mathcal{L} = \left\{ \beta \in \mathbb{Z}^{\mathsf{Q}_0} \;\middle|\; \sum_{j=1}^{m^{(0)}} \beta_{[0,j]} = \sum_{j=1}^{m^{(i)}} \beta_{[i,j]} \text{ for all } i \in I_{\mathrm{irr}} \backslash \{0\} \right\}.$$

Then the following is the main theorem.

Theorem 4 ([3], cf. Crawley-Boevey [2] and Boalch [1]). *There exist* $\alpha \in (\mathbb{Z}_{\geq 0})^{\mathsf{Q}_0}$ *and* $\lambda \in \mathbb{C}^{\mathsf{Q}_0}$ *such that the generalized additive Deligne-Simpson problem has a solution if and only if the following are satisfied,*

(1) α *is a positive root of* Q *and* $\alpha \cdot \lambda = \sum_{a \in \mathsf{Q}_0} \alpha_a \lambda_a = 0$*,*
(2) for any decomposition $\alpha = \beta_1 + \cdots + \beta_r$ *where* $\beta_i \in \mathcal{L}$ *are positive roots of* Q *satisfying* $\beta_i \cdot \lambda = 0$*, we have*

$$p(\alpha) > p(\beta_1) + \cdots + p(\beta_r).$$

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Stability conditions and torsion classes

Thomas Brüstle
(joint work with David Smith, Hipolito Treffinger)

The notion of (semi-)stability has been introduced in representation theory of quivers by Schofield [9] and King [6], and it was formalised in the context of abelian categories by Rudakov [8]. Since then, the study of rings of quiver semi-invariants by Derksen and Weyman has been expanded to the context of cluster algebras in [4] and [5]. The work of Igusa, Orr, Todorov and Weyman shows that walls in the semi-invariant picture correspond to the c-vectors in cluster theory. These vectors are also studied in quantum field theory, where they are interpreted as charges of BPS particles. It turns out that maximal green sequences, that is, maximal paths in the semi-invariant picture which are oriented in positive direction, give rise to a "complete" sequence of charges, called spectrum of a BPS particle.

The semi-invariant picture of quiver representations has re-appeared in mathematical physics and mirror symmetry as scattering diagrams such as in Kontsevich and Soibelman's study of wall crossing in the context of Donaldson-Thomas invariants in integrable systems and mirror symmetry [7]. Its wall and chamber structure is also studied by Bridgeland in [2].

It seems very natural to join two recent developments, the scattering diagrams and their wall-and-chamber structure as described in [2], with the combinatorial structure of the fan associated with τ-tilting modules as given in [1, 3]. In fact, we learned at the Oberwolfach conference that David Speyer and Hugh Thomas were independently following similar ideas, we refer to the report of Hugh Thomas [10] in this volume for more details.

We recall from King [6] the notion of stability: Let $\mathcal{A}$ be an abelian category and $\theta : K_0(\mathcal{A}) \to \mathbb{R}$ an additive function on the Grothendieck group of $\mathcal{A}$. An object $M \in \mathcal{A}$ is called *θ-semi-stable* if $\theta(M) = 0$ and $\theta(L) \leq 0$ for every subobject L of M.

Suppose now that $\mathcal{A}$ has t simple objects, hence $K_0(\mathcal{A})$ is isomorphic to $\mathbb{Z}^t$. We denote by $(\mathbb{R}^t)^*$ the dual vector space of $\mathbb{R}^t$. Then, for an object M of $\mathcal{A}$, we define the *stability space of* M to be

$$\mathfrak{D}(M) = \{\theta \in (\mathbb{R}^t)^* : M \text{ is } \theta\text{-semi-stable}\}.$$

The stability space $\mathfrak{D}(M)$ of M is contained in the hyperplane defined by the linear form θ, but it could have smaller dimension. We say that $\mathfrak{D}(M)$ is a *wall* when $\mathfrak{D}(M)$ has codimension one. Outside the walls, there are only linear functions θ having no θ-semi-stable modules other that the zero object. Removing the closure of all walls we obtain a space

$$\mathfrak{R} = (\mathbb{R}^t)^* \setminus \overline{\bigcup_{M \in \mathcal{A}} \mathcal{D}(M)}.$$

A connected component $\mathfrak{C}$ of dimension t of $\mathfrak{R}$ is called a *chamber*.

From now on, we consider the case when $\mathcal{A}$ is the category of finitely generated modules mod A over a finite dimensional algebra A over an algebraically closed field k. We recall from Adachi, Iyama and Reiten [1] the definition of τ-rigid modules and τ-tilting pairs:

(a) Let M be an A-module. Then M is *τ-rigid* if $\mathrm{Hom}_A(M, \tau M) = 0$;
(b) A pair $(M, P) \in (\text{mod } A) \times (\text{proj } A)$ is *τ-rigid* if M is a τ-rigid module and $\mathrm{Hom}_A(P, M) = 0$;
(c) A τ-rigid pair (M, P) is *support τ-tilting* if $|M| + |P| = |A|$;

We explain in the talk how the $\tau-$tilting fan introduced by Demonet, Iyama and Jasso in [3] can be embedded into King's stability manifold: Each support τ-tilting pair (M, P) yields a chamber $\mathfrak{C}_{(M,P)}$, and one can give a complete description of the walls bordering this chamber $\mathfrak{C}_{(M,P)}$. Finally, we associate to each chamber $\mathfrak{C}$ a torsion class $\mathcal{T}_{\mathfrak{C}}$. For similar results, compare also with [10]:

Proposition 1. *Every support τ-tilting pair (M, P) defines a chamber $\mathfrak{C}_{(M,P)}$.*

Proposition 2. *Let A be a finite-dimensional algebra over an algebraically closed field. Then the function $\mathfrak{C}$ mapping a support τ-tilting pair (M, P) to its corresponding chamber $\mathfrak{C}_{(M,P)}$ is injective. Moreover, if A is τ-tilting finite then $\mathfrak{C}$ is also surjective.*

Consider now a continuous function $\gamma : [0, 1] \to \mathbb{R}^t$ such that $\gamma(0) = (1, \ldots, 1)$ and $\gamma(1) = (-1, \ldots, -1)$. If we fix an A-module M, $\gamma(t)$ induces a continuous function $\rho_M : [0, 1] \to \mathbb{R}$ defined as $\rho_M(t) = \theta_{\gamma(t)}([M])$, where θ_y denotes the linear function $x \mapsto x \cdot y$. Note that $\rho_M(0) > 0$ and $\rho_M(1) < 0$. Therefore, for every module there is at least one $t \in (0,1)$ such that $\theta_{\gamma(t)}([M]) = 0$. This easy remark leads us to the definition of green paths as follows:

A *green path in* $\mathbb{R}^t$ is a continuous function $\gamma : [0,1] \to \mathbb{R}^t$ such that $\gamma(0) = (1,\dots,1)$, $\gamma(1) = (-1,\dots,-1)$ and such that for every $M \in \operatorname{mod} A$ there is exactly one $t_M \in [0,1]$ such that $\theta_{\gamma(t_M)}([M]) = 0$.

Note that we allow green paths to pass through the intersection of walls. It is easy to see that every green path $\gamma : [0,1] \to \mathbb{R}^t$ induces a stability function in the sense of Rudakov $\phi_\gamma : \text{modA} \to [0,1]$ defined by $\phi_\gamma(M) = t_M$, where t_M is the unique element in $[0,1]$ such that $\theta_{\gamma(t_M)}(M) = 0$. Moreover M is ϕ_γ-semi-stable if and only if M is $\theta_{\gamma(t_M)}$-semi-stable.

Suppose now that there is a finite set $\{M_1,\dots,M_n\}$ of ϕ_γ-stable modules. Then γ induces a green sequence which is maximal if $t_{M_i} \neq t_{M_j}$ for all $i \neq j$. Moreover, every maximal green sequence is obtained in this way.

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Applications of derived categories to Brauer algebras and to Schur algebras

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(joint work with Yiping Chen, Ming Fang, Wei Hu)

Classical Schur-Weyl duality relates group algebras of general linear groups over infinite fields with group algebras of finite symmetric groups, by a double centraliser property on tensor space. The action of the general linear group on tensor space factors through a finite dimensional quotient algebra, the classical Schur algebra. Similarly, quantised Schur algebras are related with Hecke algebras. In orthogonal and symplectic Schur-Weyl duality, Brauer algebras take the role of group algebras of symmetric groups.

We address two questions about the ring structure and homological dimensions of such algebras by using derived categories as main tools.

(1) **Brauer algebras** $B_r(\delta)$ are finite dimensional algebras over a field k, depending on an integer r and a parameter $\delta \in k$. By definition, $B_r(\delta)$ has a basis consisting of diagrams, where r edges connect $2r$ vertices. Multiplication is by deformed concatenation.

In a series of articles starting in the 1950s with Brown's work and continued by Hanlon and Wales, by Wenzl and, obtaining final results, by Rui and Si [7] in 2006, it has been clarified when $B_r(\delta)$ is semisimple; this depends on the characteristic of k relative to r and on the choice of δ.

The global dimension of $B_r(\delta)$ is finite if and only the characteristic of k is zero or bigger than r and this is equivalent to $B_r(\delta)$ being quasi-hereditary, and also to its Cartan determinant being one, by [6].

When $\delta \neq 0$, which is assumed from now on, $B_r(\delta)$ is cellularly stratified in the sense of [5]. In particular, it has a chain of two-sided ideals, each generated by a particular idempotent. These ideals are stratifying ideals, which means there are homological epimorphisms from the algebra to each quotient modulo such an ideal.

When is $B_r(\delta)$ symmetric or self-injective or Frobenius? A class of examples, which have been classified by Rui and Si [8], consists of those Brauer algebras whose stratifying chain splits, in particular the semisimple Brauer algebras. Such a Brauer algebra is Morita equivalent to a direct sum of group algebras of symmetric groups, of varying size. It turns out that these examples are the only ones: A Brauer algebra is self-injective or Frobenius if and only if it is symmetric, and this happens if and only the Brauer algebra is Morita equivalent to a direct sum of group algebras of symmetric groups, by a splitting of the cellularly stratified chain. The key to proving in [1] that there are no other examples is a property of the derived category of self-injective algebras:

Theorem. [1] *An indecomposable self-injective algebra Λ is derived simple with respect to the bounded derived module category. That is, in any recollement of $D^b(\Lambda - mod)$ by derived module categories, one of the outer terms must be zero.*

(2) **Schur algebras.** Denote by p the characteristic of the ground field k. Totaro [9] determined the global dimension of (non-semisimple) classical Schur algebras $S(n,r)$ (with $n \geq r$) as $2(r-s)$, where s is the sum of the digits in the p-adic expansion of r. The dominant dimension of $S(n,r)$ (when non-semisimple) has been computed as $2(p-1)$, see [4]. As Schur algebras are, in general, not indecomposable as algebras, it remains to prove similar formulae for their blocks, the indecomposable algebra direct summands, which are parametrised by two combinatorial data, p-core and weight. By results of Chuang and Rouquier [2], who classified blocks of group algebras of symmetric groups up to derived equivalence, blocks of Schur algebras are derived equivalent when they have the same weight.

Homological dimensions are, however, not in general invariant under derived equivalences. But Schur algebras and their blocks satisfy additional properties and thus are covered by the following invariance results:

Theorem. [3] *Let A and B be two derived equivalent algebras over a splitting field k and suppose both A and B admit anti-automorphisms that preserve simple modules up to isomorphisms. Then:*
(a) A and B have the same global dimension.
(b) Suppose in addition that both A and B have dominant dimension at least one. Then A and B have the same dominant dimension.

These results also can be applied to quantised Schur algebras.

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(Strongly) quasihereditary algebras and Ringel duality

Teresa Conde

Quasihereditary algebras are abundant in mathematics. They often arise as endomorphism algebras of modules endowed with some sort of 'stratification', and in many cases they possess a module with a double centralizer property. Examples coming from the classic realm of semisimple Lie algebras and algebraic groups include the Schur algebras, blocks of the category $\mathcal{O}$ and extensions and generalisations of these. There are many more examples of quasihereditary algebras arising from the most diverse contexts.

1. Strongly quasihereditary algebras

It has been noticed that certain endomorphism algebras emerging naturally in representation theory are particularly well behaved: they are strongly quasihereditary in the sense of Ringel ([12]).

Definition ([12])**.** A quasihereditary algebra $(B, \Phi, \sqsubseteq)$ is said to be *right strongly quasihereditary* if $\operatorname{Rad} \Delta(i) \in \mathcal{F}(\Delta)$ for all $i \in \Phi$. Dually, B is called *left strongly quasihereditary* if $(B\mathrm{op}, \Phi, \sqsubseteq)$ is a right strongly quasihereditary algebra. The generic term *strongly quasihereditary algebra* is used for left or right strongly quasihereditary algebras.

The main motivation to study strongly quasihereditary algebras is twofold. On the one hand, these are not so complicated among the class of all quasihereditary algebras, so they provide a good starting point to test conjectures. On the other hand, they are quite prevalent in module theoretical contexts.

Examples of strongly quasihereditary algebras include: the Auslander algebras, associated to algebras of finite type; the endomorphism algebras constructed by Iyama, used in his famous proof of the finiteness of the representation dimension of Artin algebras ([10]); certain cluster-tilted algebras investigated by Geiß–Leclerc–Schröer ([9]), Buan–Iyama–Reiten–Scott ([1]) and Iyama–Reiten ([11]). The Auslander–Dlab–Ringel algebra (ADR algebra) R_A of a finite-dimensional algebra A, defined as the basic version of

$$\operatorname{End}_A \left(\bigoplus_{i:\, i \geq 0} A/\operatorname{Rad}^i A \right)^{op},$$

is yet another example of a strongly quasihereditary endomorphism algebra ([3], [4]).

It turns out that all the algebras mentioned in the previous paragraph can be seen as instances of a general construction which produces strongly quasihereditary endomorphism algebras and gives natural bounds for their global dimension (see [2, Chapter 5]).

1.1. **RUSQ algebras.** The ADR algebra, the cluster-tilted algebras in [11] and some instances of the algebras in [10] are all examples of strongly quasihereditary algebras satisfying the following additional condition, up to dualisation:

(A): the injective hull of every simple standard module is a tilting module.

We call these algebras *RUSQ algebras*. The acronym RUSQ stands for "right ultra strongly quasihereditary". RUSQ algebras have a dual counterpart.

The structure of a RUSQ algebra is specially nice. It was shown in [3] that the standard modules are uniserial and that the simple modules can be labelled in a natural way by pairs (i, j) so that $\Delta(i, j)$ has radical $\Delta(i, j+1)$ for $1 \leq j < l_i$ and $\Delta(i, l_i)$ is simple. The next result summarises some of our findings.

Theorem ([3],[2])**.** *Let B be a RUSQ algebra. The injective hull Q_{i,l_i} of the simple B-module with label (i, l_i) lies in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ (that is, Q_{i,l_i} is a tilting module). Moreover, the chain of inclusions*

$$0 \subset T(i, l_i) \subset \cdots \subset T(i, j) \subset \cdots \subset T(i, 1) = Q_{i,l_i},$$

where $T(i,j)$ is the tilting module corresponding to the label (i,j), is the unique filtration of Q_{i,l_i} by costandard modules. For $1 \leq j < l_i$, the injective hull $Q_{i,j}$ of the simple module with label (i,j) is isomorphic to $Q_{i,l_i}/T(i,j+1)$.

In addition, the Ringel dual $\mathcal{R}(B)$ of B is such that $\mathcal{R}(B)\,\mathrm{op}$ is still a RUSQ algebra.

1.2. **Ringel (self)duality for RUSQ algebras.** It seems that strongly quasihereditary algebras (and, in particular, RUSQ algebras) are 'rarely' isomorphic to their own Ringel dual, contrary to what happens often for quasihereditary algebras arising from classic contexts.

One should be clear about the meaning of 'isomorphism' or 'equivalence' of quasihereditary algebras. We shall say that two quasihereditary algebras $(B, \Phi, \sqsubseteq)$ and $(C, \Lambda, \trianglelefteq)$ are *equivalent* if there is an equivalence between the respective categories of standardly filtered modules $\mathcal{F}(\Delta)$ and $\mathcal{F}(\Delta')$. Such correspondences preserve the quasihereditary data. *Ringel selfduality* occurs when $(B, \Phi, \sqsubseteq)$ is equivalent to its Ringel dual $(\mathcal{R}(B), \Phi, \sqsubseteq \mathrm{op})$.

In joint work (in progress) with Karin Erdmann (see also [2, Chapter 3]), it was shown that the Ringel dual of the ADR algebra R_A is in general 'similar' to the algebra $(R_{A^{\mathrm{op}}})\mathrm{op}$. The following was proved.

Theorem. *Suppose that all indecomposable projective and injective A-modules have the same radical length, and are also rigid. Then the Ringel dual $\mathcal{R}(R_A)$ of R_A is equivalent to $(R_{A^{\mathrm{op}}})\mathrm{op}$.*

As pointed out in [12, Section A.2], one should not expect strongly quasihereditary algebras to be Ringel selfdual (for instance, because every such algebra must have global dimension at most 2). This is corroborated by joint work (in progress) with Karin Erdmann: we have shown that Ringel selfduality occurs for the ADR algebra R_A if and only if the underlying algebra A is a selfinjective Nakayama algebra.

2. Ringel selfduality in general

It is not unusual for a quasihereditary algebra $(B, \Phi, \sqsubseteq)$ to be Ringel selfdual. This phenomenon is frequently observed in quasihereditary algebras and highest weight categories arising from the theory of semisimple Lie algebras and algebraic groups. Donkin ([7]) and Erdmann–Henke ([8]) have proved that, under certain conditions, a (q-)Schur algebra of type A is its own Ringel dual. Furthermore, Soergel ([13]) has shown that the category $\mathcal{O}$ is Ringel selfdual. Therefore, it is natural to ask which quasihereditary algebras are Ringel selfdual.

To the best of our knowledge, most of the Ringel selfdual quasihereditary algebras investigated in the literature satisfy the following properties: they possess a simple preserving duality, have dominant dimension at least 2 (and thus have a module satisfying the double centraliser property – see [14]), and satisfy $\mathrm{gl.dim}\, B = 2\,\mathrm{proj.dim}\, T = 2\,\mathrm{inj.dim}\, T$ (here T denotes the associated characteristic tilting module). It seems then reasonable to inquire whether all connected

Ringel selfdual algebras satisfy such properties and, more generally, to look for homological criteria for an algebra to be Ringel selfdual.

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Shifted modules and desingularizations

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(joint work with Matthew Pressland)

Let K be an algebraically closed field. All modules and algebras considered here are finite-dimensional and basic. Modules are always left modules. The functor $\mathrm{D} = \mathrm{Hom}_K(-, K)$ converts right modules into left modules. Varieties are by definition reduced schemes which are of finite type over K.

For an algebra B and an idempotent $e \in B$ we set $A = eBe$ and get a recollement, i.e. a diagram of six functors

$$\Gamma/\Gamma e\Gamma\text{-mod} \;\underset{p}{\overset{q}{\leftleftarrows}}\!\!\xrightarrow{\;i\;}\; \Gamma\text{-mod} \;\underset{r}{\overset{\ell}{\leftleftarrows}}\!\!\xrightarrow{\;e\;}\; A\text{-mod}.$$

fulfilling certain conditions. We also get a seventh functor $c = \mathrm{im}(\ell \to r)$. Furthermore, we get a TTF-triple associated to the recollement

$$\mathrm{TTF}(e) = (\ker q = \mathrm{gen}(Be), \ker e, \ker p = \mathrm{cogen}(\mathrm{D}(eB)))$$

where $\mathrm{gen}(Be)$ are all modules whose projective cover is in $\mathrm{add}(Be)$ and $\mathrm{cogen}(\mathrm{D}eB)$ are all modules whose injective hull lies in $\mathrm{D}(eB)$. We refer to $\mathrm{gen}(Be)$ as the *e-stable* modules and to $\mathrm{cogen}(\mathrm{D}(eB))$ as *e-costable* modules. Furthermore, we have $\mathrm{im}\, c = \ker q \cap \ker p$.

Associated varieties and regular maps. Assume that $A = KQ/I$ for a finite quiver Q and admissible relations I. For $\underline{d} \in \mathbb{N}_0^{Q_0}$ we define the *representation space* as

$$\mathrm{R}_A(\underline{d}) := \{M \in \prod_{(i\to j)\in Q_1} \mathrm{Hom}_K(K^{d_i}, K^{d_j}) \mid I \cdot M = 0\}.$$

This carries a natural operation of the group $\mathbf{Gl}_{\underline{d}} := \prod_{i\in Q_0} \mathbf{Gl}_{d_i}$. For an A-module M we define the *quiver Grassmannian* as

$$\mathrm{Gr}_A \binom{M}{\underline{d}} := \{U \subset M \mid U \text{ submodule of } M, \underline{\dim}\, U = \underline{d}\}.$$

From the idempotent $e \in B$ we obtain maps of varieties.

(1) (a) $e\colon \mathrm{R}_B(\underline{d}, \underline{r}) \to \mathrm{R}_A(\underline{d}),\ X \mapsto eX.$
Assume $char K = 0$. By [1, Lemma 6.3], we have that the affine quotient variety

$$\mathrm{R}_B(\underline{d}, \underline{r}) /\!\!/ \mathbf{Gl}_{\underline{r}} \cong \mathrm{im}\, e$$

is isomorphic to the image of this map.

(b) Assume $char K = 0$. The e-stable modules coincide with stable modules (and with semi-stable) with respect to a specific stability notion cf. [1, Section 7]. Following King [4] we get a projective map

$$\mathrm{R}_B(\underline{d}, \underline{r})^{st}/\mathbf{Gl}_{\underline{r}} \to \mathrm{im}\, e$$

(2) $e\colon \mathrm{Gr}_B \binom{c(M)}{\underline{d},\underline{r}} \to \mathrm{Gr}_A \binom{M}{\underline{d}},\ U \mapsto eU$ is a projective map (since quiver Grassmannians are projective varieties).

Question: Given an algebra A and an A-module M, can we find $B, e, \underline{r}$ such that these maps give rise to desingularizations of orbits closures and of (subvarieties of) quiver Grassmannians? This has a very nice answer when A is representation-finite using the projective quotient algebra as B, cf. [1]. Our aim is to relax this condition to *finiteness* conditions on the module M.

1. Shifted algebras

Let Γ be an algebra and Π a maximal projective-injective summand. For a natural number k, we say that Γ has dominant dimension at least k if there is an exact sequence

$$0 \to \Gamma \to \Pi_0 \to \cdots \to \Pi_{k-1}$$

with $\Pi_i \in \mathrm{add}(\Pi)$. By [5] this is equivalent to the existence of an exact sequence

$$\Pi^{k-1} \to \cdots \to \Pi^0 \to \mathrm{D}\Gamma \to 0$$

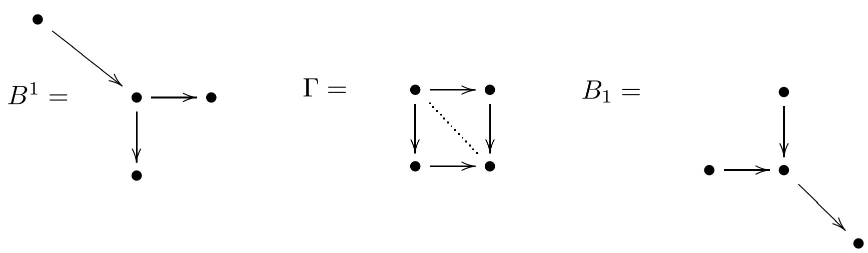
with $\Pi^i \in \mathrm{add}(\Pi)$. From now on we assume $\mathrm{domdim}\,\Gamma \geq k \geq 1$.

In this case, we define T_k to be the basic module with $\mathrm{add}T_k = \mathrm{add}(\Pi \oplus \Omega^{-k}\Gamma)$ and call this the *k-shifted* module and its endomorphism ring $B_k = \mathrm{End}_\Gamma(T_k)^{\mathrm{op}}$ the *k-shifted algebra*.
Analogously, we define C_k to be the basic module with $\mathrm{add}C_k = \mathrm{add}(\Pi \oplus \Omega^k \mathrm{D}\Gamma)$ and call this the *k-coshifted* module and $B^k = \mathrm{End}_\Gamma(C_k)^{\mathrm{op}}$ the *k-coshifted algebra*.

Lemma 1.1. *The module T_k is a k-tilting module (i.e. a tilting module with* $\mathrm{pd} \leq k$*) and C_k is k-cotilting module (i.e. cotilting with* $\mathrm{id} \leq k$*).*

These modules are studied independently in upcoming work by I. Reiten, G. Todorov and coworkers. They also already appear in [3].

Example 1.2. (1) $\mathrm{domdim}\,\Gamma \geq 1$ is equivalent to the property QF3. An example for $\mathrm{domdim} = 1$:



(2) $\mathrm{domdim}\,\Gamma \geq 2$ is by the so-called Morita-Tachikawa correspondence equivalent to $\Gamma = \mathrm{End}_A(E)^{\mathrm{op}}$ for an algebra A and an A-module E with $A \oplus \mathrm{D}A \in \mathrm{add}E$. To find (A, E) from Γ choose idempotents $e_0, \epsilon_0 \in \Gamma$ such that $\Pi = \Gamma\epsilon_0 = \mathrm{D}(e_0\Gamma)$. Then, one has $A = e_0\Gamma e_0$ and $E = e_0\Gamma$.
The algebra Γ is d-Auslander Gorenstein (i.e. $\mathrm{id}\,{}_\Gamma\Gamma \leq d+1 \leq \mathrm{domdim}\,\Gamma$, cf. [2]) if and only if there is an exact sequence

$$0 \to \Gamma \to \Pi_0 \to \cdots \to \Pi_d \to \mathrm{D}\Gamma \to 0$$

with $\Pi_i \in \mathrm{add}(\Pi)$. Therefore, in this case one has $T_k = C_{d+1-k}$ and $B^k = B_{d+1-k}$ for all k.

The following properties are the generalizations of the analogous properties for projective quotient algebras.

Proposition 1.3. (1) *We have* $\mathrm{gldim}\,B_k \leq \mathrm{gldim}\,\Gamma, \mathrm{gldim}\,B^k \leq \mathrm{gldim}\,\Gamma$.

(2) *For $0 < k <$ $\mathrm{domdim}\,\Gamma$ we have $\Gamma = \mathrm{End}_A(E)^{\mathrm{op}}$ with $E = X \oplus A = Y \oplus \mathrm{D}A$. We denote by $\mathcal{H}(A)$ the homotopy category of finite-dimensional A-modules and consider a minimal injective corepresentation $X \to I_\bullet$ and*

a minimal projective presentation $P_\bullet \to Y$. We define objects in $\mathcal{H}(A)$

$$\begin{array}{ll} deg & \quad 0 \qquad 1 \qquad\qquad\qquad k \qquad\qquad\qquad\qquad k \\ E_k = & (\,Y \to I_0 \to I_1 \to \cdots \to I_{k-1}) \oplus (0 \to \cdots \to 0 \to \mathrm{D}A), \end{array}$$

$$\begin{array}{ll} deg & \quad -k \qquad\qquad -1 \qquad 0 \qquad -k \\ E^k = & (\,P_{k-1} \to \cdots \to P_0 \to X) \oplus (\,A \to 0 \to \cdots \to 0) \end{array}$$

then one has $B_k \cong \mathrm{End}_{\mathcal{H}(A)}(E_k)^{\mathrm{op}}$ and $B^k \cong \mathrm{End}_{\mathcal{H}(A)}(E^k)^{\mathrm{op}}$.

From every (co)tilting module we get an idempotent as follows: Let (Γ, T, B) consist of an algebra Γ a (co)tilting module T and $B = \mathrm{End}_\Gamma(T)^{op}$. Let Π be the maximal projective injective summand of Γ. Then we get an idempotent $e \in B$ by projecting onto the summand Π of T, i.e. $Be = \mathrm{Hom}_\Gamma(T, \Pi)$.

So, for (Γ, T_k, B_k) and (Γ, C_k, B^k) we get idempotents $e_k \in B^k$ and $\epsilon_k \in B^k$ by projecting onto the summand Π. We observe that $\Pi = \Gamma\epsilon_0 = \mathrm{D}(e_0\Gamma)$.

Theorem 1.4. (1) *We have isomorphisms of algebras $B^k/(e_k) \cong \Gamma/(e_0)$ and $B_k/(\epsilon_k) \cong \Gamma/(\epsilon_0)$.*

(2) *For $0 < k < \mathrm{domdim}\,\Gamma$ let $\Gamma = \mathrm{End}_A(E)^{op}$ with $E = e_0\Gamma = \mathrm{D}(\epsilon_0\Gamma)$. We denote by c_k (resp. c^k) the intermediate extension associated to ϵ_k (resp. e_k). Then we have isomorphisms $c_k(E) \cong \mathrm{D}T_k$ of B_k-Γ-bimodules and $c^k(E) \cong \mathrm{D}C_k$ of B^k-Γ-bimodules.*

2. Geometric applications

(1) (a) Let $E = \bigoplus_{i=1}^t M_i$ and $m = (m_1, \ldots, m_t) \in \mathbb{N}_0^t$, then we define the associated *rank variety* as

$$\mathcal{C}_m^E := \{N \in \mathrm{R}_A(\underline{d}) \mid \dim_K \mathrm{Hom}_A(M_i, N) \geq m_i \forall i\}.$$

Wlog: We assume $A \oplus \mathrm{D}A \in \mathrm{add}(E)$, else add summands and extend m by zero entries. Let $E = \bigoplus_{i=1}^t M_i \oplus A$ with M_i indecomposable non-projective and $m = (m_1, \ldots, m_t, 0) \in \mathbb{N}_0^{t+1}$. We take $B = B^1$ for $\Gamma = \mathrm{End}_A(E)^{\mathrm{op}}$ and $e = e_1$. Furthermore, we set $r_i = d^{(M_i)} - m_i$ with $d^{(M_i)} := \dim_K \mathrm{Hom}_A(P_{M_i}, X)$ where $P_{M_i} \to M_i$ is a minimal projective cover and $X \in \mathrm{R}_A(\underline{d})$ arbitrary, $1 \leq i \leq t$. Then we have an isomorphism of varieties

$$\mathrm{R}_B(\underline{d}, \underline{r}) /\!\!/ \mathbf{Gl}_{\underline{r}} \cong \mathcal{C}_m^E.$$

(b) Assume $char K = 0$ and $(\underline{d}, \underline{r}) = \underline{\dim}\, c(M)$. For $M \in \mathrm{R}_A(\underline{d})$ we denote the $\mathbf{Gl}_{\underline{d}}$-orbit of M by $\mathcal{O}_M$. By using (a) and a suitable restriction of the maps from the beginning we get a projective and birational map

$$\overline{\mathcal{O}_{c(M)}}^{st} / \mathbf{Gl}_{\underline{r}} \to \overline{\mathcal{O}_M}$$

in which we may assume $c(M)$ is rigid. The missing property is the smoothness. We can prove the following (which is analogue to a result of Zwara [6])

Proposition 2.1. *If M is quotient-finite (i.e. only finitely many A-module isomorphism classes of quotients), then one can choose B such that $\overline{\mathcal{O}_{c(M)}}^{st}$ is smooth.*

(2) Let M be an A-module and $N \in \mathrm{Gr}_A\binom{M}{\underline{d}}$. We denote by $\mathcal{S}_{[N]} := \{U \in \mathrm{Gr}_A\binom{M}{\underline{d}} \mid U \cong N\}$. Then the map from the beginning (for the dimension vector $\underline{\dim}\, c(N)$) restricts to projective and birational map

$$\overline{\mathcal{S}_{[c(N)]}} \to \overline{\mathcal{S}_{[N]}}.$$

Proposition 2.2. *If N is quotient-finite, then one can choose B such that $\overline{\mathcal{S}_{[c(N)]}}$ is smooth.*

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Cluster variables and perfect matchings

Alastair King

(joint work with Ilke Canakci and Matthew Pressland)

The objective of this talk is to interpret the Marsh-Scott formula for twisted minors as an instance of the Caldero-Chapoton formula for cluster characters.

§1. Background. The homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}_k^n]$ of the Grassmannian, e.g. the ring of $SL_k(\mathbb{C})$ invariants of $k \times n$ matrices, is generated in degree 1 by (maximal) minors ϕ_J for $J \in \binom{n}{k}$, the k-subsets of $\{1, \dots, n\}$.

Fomin-Zelevinsky and Scott showed that $\mathbb{C}[\mathrm{Gr}_k^n]$ has a cluster structure in which the clusters of minors are given by maximal non-crossing sets $\mathcal{C} \subseteq \binom{n}{k}$. In the case $k = 2$, these correspond to triangulations of an n-gon, while for $k \geq 3$ they can be generated by Postnikov's alternating strand diagrams. As described in [1], such a diagram is equivalent to a dimer model $D(\mathcal{C})$ on a disc, that is, either a bipartite graph with faces labelled by the $J \in \mathcal{C}$, or its dual quiver with potential, with vertices so labelled (see Figure 2 for an example).

In [3], an (additive) categorification of $\mathbb{C}[\mathrm{Gr}_k^n]$ is given by the category $\mathrm{CM}(B)$ of Cohen-Macauly modules for a certain Gorenstein order B, the path algebra of the double quiver of an n-cycle (see Figure 1 for an example), subject to relations of the form $xy = yx$ and $x^k = y^{n-k}$, where x and y label the clockwise and anti-clockwise arrows, respectively. The algebra B has centre $Z = \mathbb{C}[t]$, for $t = xy$, and $\mathrm{CM}(B)$ is more simply the category of B-modules that are free over Z. Such a module has rank m when, as a representation of the quiver, it has Z^m at each vertex. The (necessarily rigid) rank 1 modules are M_J, for each $J \in \binom{n}{k}$, where $y_j = t$ for $j \in J$ and $x_j = t$ for $j \notin J$. It is possible to define a cluster character $\Phi\colon \mathrm{CM}(B) \to \mathbb{C}\big[\mathrm{Gr}_k^n\big]$, so that $\Phi(M_J) = \phi_J$.

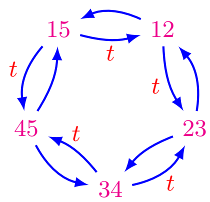

FIGURE 1. Quiver of B with the rank 1 module M_{24}.

The set $\mathcal{C}$ also determines a cluster tilting object $T = \bigoplus_{J\in\mathcal{C}} M_J$ and the main result of [1] is that $A = \mathrm{End}_B(T)$ is the 'dimer algebra' associated to $D(\mathcal{C})$, i.e. the frozen Jacobi algebra of the dual quiver with potential. Note that B is recovered from A as the 'boundary algebra' eAe for e with $eT \cong B$, the projective summand of T. This summand corresponds to the frozen variables in the cluster, that is, the minors labelled by cyclic intervals, which are then also the natural labels for the vertices of the quiver of B (as in Figure 1).

§2. Marsh-Scott. For the initial cluster $\{\phi_J : J \in \mathcal{C}\}$ and any $w \in \mathbb{Z}^{\mathcal{C}}$, we write a formal monomial $\phi^w = \prod_{J\in\mathcal{C}} \phi_J^{w_J}$. We write $\mathcal{C}^\circ$ for the labels of unfrozen variables and

$$\mathrm{PM}(J) = \big\{\text{perfect matchings on dimer model } D(\mathcal{C}) \text{ with boundary value } J\big\}$$

In [4], Marsh-Scott write any twisted minor, which is (up to a frozen factor) a particular Grassmannian cluster variable, as a dimer model partition function

$$\widetilde{\phi_J} = \left(\prod_{I\in\mathcal{C}^\circ} \phi_I\right)^{-1} \sum_{\mu\in\mathrm{PM}(J)} \phi^{w(\mu)} \tag{10}$$

where $w(\mu) = \sum_{e\in\mu} w(e)$ and $w(e)$ is the sum of the face labels opposite (through the white node) the edge e. For example, for the matching shown in Figure 2, we have $w(\mu) = 2\delta_{13} + \delta_{45}$. In that case, for the same boundary value 24, there is one other matching μ' with $w(\mu') = \delta_{13} + \delta_{34} + \delta_{15}$ and thus

$$\widetilde{\phi_{24}} = \frac{\phi_{13}\phi_{13}\phi_{45} + \phi_{13}\phi_{34}\phi_{15}}{\phi_{13}\phi_{14}} = \phi_{35}$$

We can rewrite (10) as

$$\widetilde{\phi_J} = \sum_{\mu \in \mathrm{PM}(J)} \phi^{\gamma_{MS}(\mu)} \tag{11}$$

where $\gamma_{MS}(\mu) = \sum_{e\in\mu} w(e) - \sum_{I\in\mathcal{C}^\circ} \delta_I$.

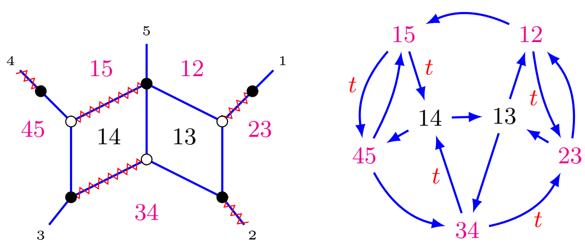

Figure 2. Bipartite graph with matching $\mu \in \mathrm{PM}(24)$ and dual quiver with the CM-module N_μ.

§3. Caldero-Chapoton. In this context (Fu-Keller [2] and Palu [5]), this is a formula for the cluster character $\Phi(M)$ of a module $M \in \mathrm{CM}(B)$, expressed in terms of the same initial cluster $\{\phi_J : J \in \mathcal{C}\}$ as above, corresponding to the cluster tilting object $T = \bigoplus_{J\in\mathcal{C}} M_J$ with $\mathrm{End}_B(T) = A$.

Consider $FM = \mathrm{Hom}(T, M)$ and $GM = \mathrm{Ext}^1(T, M)$ as A-modules. Then

$$\Phi(M) = \phi^{\pi(FM)} \sum_{\overline{N}\leq GM} \phi^{-\pi(\overline{N})} \tag{12}$$

where $\pi\colon K(A) \to \mathbb{Z}^{\mathcal{C}}$ gives the coordinates in the basis of indecomposable projective A-modules $\{FM_J : J \in \mathcal{C}\}$. Note, in case the sum is infinite, it should be interpreted using the Euler characteristic of the variety of submodules of GM, but this will not occur in the cases we are interested in.

Now, for $M \in \mathrm{CM}(B)$, we have $FM \in \mathrm{CM}(A)$ and there is a short exact sequence

$$0 \to F'M \to FM \to G\Omega M \to 0$$

where $F'M = (Ae \otimes_B M)/\mathrm{Tors}$ in also in $\mathrm{CM}(A)$ and ΩM is the syzygy of M, i.e. the kernel of a projective cover $PM \to M$. Thus, applying (12) to ΩM, we get

$$\Phi(\Omega M) = \sum_{F'M\leq N\leq FM} \phi^{\gamma_{CC}(N)} \tag{13}$$

where $\gamma_{CC}(N) = \pi(F\Omega M) - \pi(N/F'M) = \pi(FPM) - \pi(N)$.

Our goal is now clear: to show that the Marsh-Scott formula (11) can be interpreted as a special case of the Caldero-Chapoton formula (13). In other words, we show that $\widetilde{\phi_J} = \Phi(\Omega M_J)$ by showing that the sums in the two formulae coincide. Note that this requires a particular choice for the syzygy ΩM, i.e. a particular choice of (non-minimal) projective cover PM.

The key observation is that the ranges of summation match up, i.e. there is a natural one-to-one correspondence between the perfect matchings $\mu \in \mathrm{PM}(J)$ and the modules N in the range $F'M_J \leq N \leq FM_J$. This follows because F' and F are the left and right adjoints to the restriction functor $\mathrm{CM}(A) \to CM(B)\colon N \mapsto eN$ and so such N are precisely those with $eN = M_J$. It is then straightforward to associate to each $\mu \in \mathrm{PM}(J)$ a module N_μ with $eN_\mu = M_J$, and vice-versa. An example of such a module N_μ is shown on the right of Figure 2 corresponding to the perfect matching μ shown on the left of the same figure.

To complete the proof one must show that $\gamma_{CC}(N_\mu) = \gamma_{MS}(\mu)$, which is done by finding an explicit projective resolution of N_μ.

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Ringel-Hall algebras beyond their quantum group

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(joint work with Fan Xu, Minghui Zhao)

1. Introduction

The Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ of a (small) abelian category $\mathcal{A}$ was introduced by Ringel as a model to realize the quantum group. When $\mathcal{A}$ is the category $\mathrm{Rep}_{\mathbb{F}_q}Q$ of finite dimensional representations for a simply-laced Dynkin quiver Q over a finite field $\mathbb{F}_q$, the Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ is isomorphic to the positive/negative part of the corresponding quantum group. For any acyclic quiver Q and $\mathcal{A} = \mathrm{Rep}_{\mathbb{F}_q}Q$, the composition subalgebra of $\mathcal{H}(\mathcal{A})$ generated by the elements corresponding to simple representations is isomorphic to the positive/negative part of the quantum group of type Q. This gives the algebraic realization of the positive/negative part of a (Kac-Moody type) quantum group. This realization was achieved by Green, through solving a natural question whether there is a comultiplication on $\mathcal{H}(\mathcal{A})$ compatible with the corresponding multiplication so that the above isomorphism is an isomorphism between bialgebras. Now it is well-known that Green's comultiplication depends on a remarkable homological formula, which is called the Green formula in the following.

Lusztig gave the geometric realization of the positive/negative part of a quantum group and then constructed the canonical basis for it. Let $Q = (Q_0, Q_1, s, t)$ be a quiver and

$$\mathbb{E}_\alpha := \bigoplus_{h \in Q_1} \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^{\alpha_{s(h)}}, \mathbb{K}^{\alpha_{t(h)}})$$

be the variety with the natural action of the algebraic group

$$G_\alpha := \prod_{i \in Q_0} GL(\alpha_i, \mathbb{K})$$

for a given dimension vector $\alpha = \sum_{i \in Q_0} \alpha_i i \in \mathbb{N}Q_0$. For any $\mathbf{i} = (i_1, i_2, \cdots, i_s)$, $i_l \in Q_0$ and $\mathbf{a} = (a_1, a_2, \cdots, a_s)$, $a_l \in \mathbb{N}$ such that $\sum_{l=1}^{s} a_l i_l = \alpha$, Lusztig defined the flag variety $F_{\mathbf{i},\mathbf{a}}$ and the subvariety $\tilde{F}_{\mathbf{i},\mathbf{a}} \subseteq \mathbb{E}_\alpha \times F_{\mathbf{i},\mathbf{a}}$. Fix any type $(\mathbf{i}, \mathbf{a})$, consider the canonical proper morphism $\pi_{\mathbf{i},\mathbf{a}} : \tilde{F}_{\mathbf{i},\mathbf{a}} \to \mathbb{E}_\alpha$. By the decomposition theorem of Beilinson, Bernstein and Deligne, the complex $\pi_{\mathbf{i},\mathbf{a}!}\mathbf{1}$ is semisimple, where $\mathbf{1}$ is the constant perverse sheaf on $\tilde{F}_{\mathbf{i},\mathbf{a}}$. Let $\mathcal{Q}_\alpha$ be the category of complexes isomorphic to sums of shifts of simple perverse sheaves appearing in $\pi_{\mathbf{i},\mathbf{a}!}\mathbf{1}$, K_α the Grothendieck group of $\mathcal{Q}_\alpha$ and

$$K(\mathcal{Q}) = \bigoplus_{\alpha \in \mathbb{N}Q_0} K_\alpha.$$

Lusztig already endowed $K(\mathcal{Q})$ with the multiplication and comultiplication structures by introducing his induction and restriction functors. He proved that the comultiplication is compatible with the multiplication in $K(\mathcal{Q})$ and $K(\mathcal{Q})$ is isomorphic to the positive/negative part of the corresponding quantum group as bialgebras up to a twist. By this isomorphism, the isomorphism classes of simple perverse sheaves in $\mathcal{Q}_\alpha$ provide a basis of the positive/negative part of the corresponding quantum group, which is called the canonical basis.

For a long time, we have been asked what the explicit relation exists between Green's comultiplication and Lusztig's restriction functor. As one of the main results, the following Theorem 2.1 and the definition of the comultiplication operator Δ provide us this strong and clear link. Thanks to an embedding property (Theorem 3.6), we can lift the Green formula from finite fields to the level of sheaves. This is finally suitable to apply Lusztig's restriction functor to the larger categories of the so-called Weil complexes, whose Grothendieck groups realize the weight spaces of a generic Ringel-Hall algebra. By using the direct sum of these Grothendieck groups, we also give the categorification of Ringel-Hall algebras via Lusztig's geometric method. The simple perverse sheaves provide the canonical basis. We show that the compatibility of the induction and restriction functor holds for these perverse sheaves. Therefore the generic Ringel-Hall algebra has a structure of bialgebra.

2. Main results

Let $\mathbb{F}_q$ be a finite field with q elements and $\mathbb{K}$ the algebraic closure of $\mathbb{F}_q$. Let G be an algebraic group over $\mathbb{K}$ and X be a scheme of finite type over $\mathbb{K}$ together with an G-action. Assume that X and G have $\mathbb{F}_q$-structures. Denote by

$\mathcal{D}^b_{G,w}(X)$ the triangulated category of ι-mixed Weil complexes. Let $K_{G,w}(X)$ be the Grothendieck group of $\mathcal{D}^b_{G,w}(X)$. Denote by $\mathcal{D}^b_{im,G,w}(X)$ the subcategory of $\mathcal{D}^b_{G,w}(X)$ consisting of ι-mixed Weil complexes of integer weights. Let $I_{im,G,w}(X)$ be the corresponding Grothendieck group.

Let $Q = (Q_0, Q_1, s, t)$ be a quiver. For a given dimension vector $\alpha \in \mathbb{N}Q_0$, $\mathbb{E}_\alpha$ and G_α have been defined in Section 1. Set $\mathbf{K}_w = \bigoplus_\alpha K_{G_\alpha,w}(\mathbb{E}_\alpha)$ and $\mathbf{I}_w = \bigoplus_\alpha I_{im,G_\alpha,w}(\mathbb{E}_\alpha)$.

Consider the following diagram

$$\mathbb{E}_\alpha \times \mathbb{E}_\beta \xleftarrow{p_1} \mathbb{E}' \xrightarrow{p_2} \mathbb{E}'' \xrightarrow{p_3} \mathbb{E}_{\alpha+\beta}\ .$$

This induces a functor $\mathfrak{m} : \mathcal{D}^b_{G_\alpha\times G_\beta,w}(\mathbb{E}_\alpha \times \mathbb{E}_\beta) \to \mathcal{D}^b_{G_{\alpha+\beta},w}(\mathbb{E}_{\alpha+\beta})$ described as the composition of p_1^*, $(p_2)_\flat$ and $(p_3)_!$.

Consider the diagram

$$\mathbb{E}_\alpha \times \mathbb{E}_\beta \xleftarrow{\kappa} F_{\alpha,\beta} \xrightarrow{i} \mathbb{E}_{\alpha+\beta}\ .$$

This induces a functor $\Delta_{\alpha,\beta} : \mathcal{D}^b_{G_{\alpha+\beta},w}(\mathbb{E}_{\alpha+\beta}) \to \mathcal{D}^b_{G_\alpha\times G_\beta,w}(\mathbb{E}_\alpha \times \mathbb{E}_\beta)$ as the composition of i^* and $\kappa_!$.

Theorem 2.1. *(1) For simple perverse sheaves $\mathcal{L} \in \mathcal{D}^b_{G_\alpha\times G_\beta,w}(\mathbb{E}_\alpha \times \mathbb{E}_\beta)$, $\mathfrak{m}(\mathcal{L})$ is still semisimple in $\mathcal{D}^b_{G_{\alpha+\beta},w}(\mathbb{E}_{\alpha+\beta})$.*

(2) For simple perverse sheaves $\mathcal{L} \in \mathcal{D}^b_{G_{\alpha+\beta},w}(\mathbb{E}_{\alpha+\beta})$, $\Delta_{\alpha,\beta}(\mathcal{L})$ is still semisimple in $\mathcal{D}^b_{G_\alpha\times G_\beta,w}(\mathbb{E}_\alpha \times \mathbb{E}_\beta)$.

(3) For two simple perverse sheaves $\mathcal{L}_1 \in \mathcal{D}^b_{G_\alpha,w}(\mathbb{E}_\alpha)$ and $\mathcal{L}_2 \in \mathcal{D}^b_{G_\beta,w}(\mathbb{E}_\beta)$, we have

$$\Delta(\mathcal{L}_1 * \mathcal{L}_2) = \Delta(\mathcal{L}_1) * \Delta(\mathcal{L}_1).$$

This theorem can be viewed as the categorification of the Green formula. As a corollary, the $\mathbb{Z}$-module $\mathbf{K}_w$ and $\mathbf{I}_w$ are bialgebras with a multiplication induced by the functor $\mathfrak{m}$ and a comultiplication induced by the functor Δ.

The algebra $\mathbf{K}_w$ and $\mathbf{I}_w$ have natural $\mathbb{A} = \mathbb{Z}[v, v^{-1}]$-module structures by $v[\mathcal{L}] = [\mathcal{L}[1](\frac{1}{2})]$ and $v^{-1}[\mathcal{L}] = [\mathcal{L}[-1](-\frac{1}{2})]$ for any dimension vector α and $\mathcal{L} \in \mathcal{D}^b_{G_\alpha,w}(\mathbb{E}_\alpha)$. Let $\mathbf{B}$ be the set of all simple perverse sheaves with weight 0. Then, the bialgebra $\mathbf{I}_w$ is free $\mathbb{A}$-module with $\mathbf{B}$ as a basis, which is called the canonical basis of $\mathbf{I}_w$.

3. The sketch of proof

First, we should recall Lusztig's construction of Hall algebras via functions.

Define $\mathcal{CF}^F_\alpha$ to be the $\overline{\mathbb{Q}}_l$-space generated by G^F_α-invariant functions: $\mathbb{E}^F_\alpha \to \overline{\mathbb{Q}}_l$, where F is the Frobenius automorphism. Let $\mathcal{CF}^F(Q) = \bigoplus_\alpha \mathcal{CF}^F_\alpha$. Consider

$$\mathbb{E}^F_\alpha \times \mathbb{E}^F_\beta \xleftarrow{p_1} \mathbb{E}'^F \xrightarrow{p_2} \mathbb{E}''^F \xrightarrow{p_3} \mathbb{E}^F_{\alpha+\beta}\ .$$

There is a linear map (called the induction map)

$$\underline{m} : \mathcal{CF}_{G_\alpha\times G_\beta}(\mathbb{E}^F_\alpha \times \mathbb{E}^F_\beta) \to \mathcal{CF}_{G_{\alpha+\beta}}(\mathbb{E}^F_{\alpha+\beta}) = \mathcal{CF}^F_{\alpha+\beta}$$

sending g to $|G_\alpha^F \times G_\beta^F|^{-1}(p_3)_!(p_2)_! p_1^*(g)$.

Lemma 3.1. *Given three kQ-modules M, N and L, let $1_{\mathcal{O}_M}, 1_{\mathcal{O}_N}$ and $1_{\mathcal{O}_L}$ be the characteristic functions over orbits, respectively. Then $\underline{m}(1_{\mathcal{O}_M^F}, 1_{\mathcal{O}_N^F})(x) = F_{MN}^L$, for any $x \in \mathcal{O}_L^F$.*

Consider

$$\mathbb{E}_\alpha^F \times \mathbb{E}_\beta^F \xleftarrow{\kappa} F_{\alpha,\beta}^F \xrightarrow{i} \mathbb{E}_{\alpha+\beta}^F .$$

There is also a linear map (called the restriction map)

$$\delta_{\alpha,\beta} : \mathcal{CF}_{G_{\alpha+\beta}}(\mathbb{E}_{\alpha+\beta}^F) \to \mathcal{CF}_{G_\alpha \times G_\beta}(\mathbb{E}_\alpha^F \times \mathbb{E}_\beta^F)$$

sending $f \in \mathcal{CF}_{G_{\alpha+\beta}}(\mathbb{E}_{\alpha+\beta}^F)$ to $\kappa_! i^*(f)$. Let $\delta_{\alpha,\beta}^{tw} = q^{-\sum_{i\in Q_0}\alpha_i\beta_i}\delta_{\alpha,\beta}$ be the twist of $\delta_{\alpha.\beta}$.

Lemma 3.2. *With the notations in Lemma 3.1 and $\underline{\dim} M = \alpha, \underline{\dim} N = \beta$, we have $\delta_{\alpha,\beta}^{tw}(1_{\mathcal{O}_L^F})(x_1, x_2) = h_L^{MN}$, for any $x_1 \in \mathcal{O}_M^F$ and $x_2 \in \mathcal{O}_N^F$.*

Then, we shall consider the relation between the induction and restriction map. Fix dimension vectors $\alpha, \beta, \alpha', \beta'$ with $\alpha + \beta = \alpha' + \beta' = \alpha$. Let $\mathcal{N}$ be the set of quadruples $\lambda = (\alpha_1, \alpha_2, \beta_1, \beta_2)$ of dimension vectors such that $\alpha = \alpha_1 + \alpha_2, \beta = \beta_1 + \beta_2, \alpha' = \alpha_1 + \beta_1$ and $\beta' = \alpha_2 + \beta_2$. Consider the following diagram

$$\begin{array}{ccccccc}
\mathbb{E}_\alpha^F \times \mathbb{E}_\beta^F & \xleftarrow{p_1} & \mathbb{E}'^F_{\alpha,\beta} & \xrightarrow{p_2} & \mathbb{E}''^F_{\alpha,\beta} & \xrightarrow{p_3} & \mathbb{E}_\alpha^F \\
\uparrow{\scriptstyle i'} & & & & & & \uparrow{\scriptstyle i} \\
\coprod_{\lambda\in\mathcal{N}} F_\lambda^F & & & & & & F_{\alpha',\beta'}^F \\
\downarrow{\scriptstyle \kappa'} & & & & & & \downarrow{\scriptstyle \kappa} \\
\coprod_{\lambda\in\mathcal{N}} E^F(\lambda) & \xleftarrow{p_1'} & \coprod_{\lambda\in\mathcal{N}} \mathbb{E}'^F_\lambda & \xrightarrow{p_2'} & \coprod_{\lambda\in\mathcal{N}} \mathbb{E}''^F(\lambda) & \xrightarrow{p_3'} & \mathbb{E}_{\alpha'}^F \times \mathbb{E}_{\beta'}^F
\end{array}$$

where $\mathbb{E}^F(\lambda) = \mathbb{E}_{\alpha_1}^F \times \mathbb{E}_{\alpha_2}^F \times \mathbb{E}_{\beta_1}^F \times \mathbb{E}_{\beta_2}^F$, $\mathbb{E}'^F(\lambda) = \mathbb{E}'^F_{\alpha_1,\beta_1} \times \mathbb{E}'^F_{\alpha_2,\beta_2}$ and $\mathbb{E}''^F(\lambda) = \mathbb{E}''^F_{\alpha_1,\beta_1} \times \mathbb{E}''^F_{\alpha_2,\beta_2}$ for $\lambda = (\alpha_1, \alpha_2, \beta_1, \beta_2)$. This induces the maps between F-fixed subsets and then the maps between vector spaces of functions as follows:

$$\begin{array}{ccc}
\mathcal{CF}_\alpha^F \times \mathcal{CF}_\beta^F & \xrightarrow{\underline{m}_{\alpha,\beta}} & \mathcal{CF}_\alpha^F \\
\downarrow{\scriptstyle \delta^{tw}} & & \downarrow{\scriptstyle \delta^{tw}_{\alpha',\beta'}} \\
\mathcal{CF}^F(\prod_{\lambda\in\mathcal{N}} \mathbb{E}_\lambda) & \xrightarrow{\underline{m}} & \mathcal{CF}_{\alpha'}^F \times \mathcal{CF}_{\beta'}^F
\end{array} \tag{14}$$

Lemma 3.3. *For $M \in \mathbb{E}_\alpha^F, N \in \mathbb{E}_\beta^F, M' \in \mathbb{E}_{\alpha'}^F, N' \in \mathbb{E}_{\beta'}^F$, we have*

$$\delta_{\alpha',\beta'}^{tw}\underline{m}_{\alpha,\beta}(1_{\mathcal{O}_M^F}, 1_{\mathcal{O}_N^F})(M', N') = \sum_{[L]\in\mathbb{E}_\alpha^F/G_\alpha^F} F_{MN}^L h_L^{M'N'}.$$

Lemma 3.4. *For* $M \in \mathbb{E}^F_\alpha, N \in \mathbb{E}^F_\beta, M' \in \mathbb{E}^F_{\alpha'}, N' \in \mathbb{E}^F_{\beta'}$, *we have*

$$\underline{m}\delta^{tw}(1_{\mathcal{O}^F_M}, 1_{\mathcal{O}^F_N})(M', N') = \sum_{[X],[Y_1],[Y_2],[Z]} F^{M'}_{XY_2} F^{N'}_{Y_1 Z} h^{XY_1}_M h^{Y_2 Z}_N.$$

where $[X] \in \mathbb{E}^F_{\alpha_2}/G^F_{\alpha_2}, [Y_1] \in \mathbb{E}^F_{\alpha_1}/G^F_{\alpha_1}, [Y_2] \in \mathbb{E}^F_{\beta_2}/G^F_{\beta_2}, [Z] \in \mathbb{E}^F_{\beta_1}/G^F_{\beta_1}$.

Hence the Green formula implies the following theorem.

Theorem 3.5. *With the above notation, diagram 14 is commutative, i.e.,*

$$\delta^{tw}_{\alpha',\beta'} \underline{m}_{\alpha,\beta} = \underline{m}\delta^{tw}.$$

This theorem means that the comultiplication is compatible with the multiplication and then $\mathcal{CF}^F(Q)$ is a bialgebra,

Let G be an algebraic group over $\mathbb{K}$ and X be a scheme of finite type over $\mathbb{K}$ together with an G-action. Assume that X and G have $\mathbb{F}_q$-structures. Let $x \in X^F$ be a closed point. For any $(\mathcal{F}, j)$ in $\mathcal{D}^b_{G,w}(X)$, we get automorphisms $F_{i,x} : \mathcal{H}^i_G(\mathcal{F})_{|x} \to \mathcal{H}^i_G(\mathcal{F})_{|x}$. One can define the G-equivariant version of χ^{F^n} for $n \in \mathbb{N}$ as follows:

$$\chi^{F^n}_{\mathcal{F}}(x) = \sum_i (-1)^i tr(F^n_{i,x}, \mathcal{H}^i_G(\mathcal{F})_{|x}).$$

Let $\mathcal{CF}^{F^n}(Q) = \bigoplus_\alpha \mathcal{CF}_{G_\alpha}(\mathbb{E}^{F^n}_\alpha)$. Hence we have $\chi^{F^n} : \mathbf{K}_w \to \mathcal{CF}^{F^n}(Q)$ for any $n \in \mathbb{N}$.

Theorem 3.6. *The map*

$$\chi = \prod_{n\in\mathbb{N}} \chi^{F^n} : \mathbf{K}_w \to \prod_{n\in\mathbb{N}} \mathcal{CF}^{F^n}(Q)$$

is injective.

By the definition, the maps χ^{F^n} are homomorphism of algebras and coalgebras. Hence, Theorem 3.5 and 3.6 imply the main result Theorem 2.1.

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Inverse Limits of Moduli Spaces

Lutz Hille

(joint work with Mark Blume)

Moduli spaces of quiver representations have been introduced by King ([4]), they depend on a stability notion θ. For a quiver Q and a dimension vector d the moduli space of θ–stable representations of Q with dimension vector d is denoted by $\mathcal{M}^\theta(Q, d)$. Unfortunately, there is no canonical choice for the stability notion. Thus it is desirable (also in view of Theorem 1.1) to define a moduli space, that does not depend on this choice. Moreover, the moduli space $\mathcal{M}^\theta(Q,d)$ is defined over the integers (see [1]), so we get schemes over $\mathbb{Z}$.

The principal aim of this talk is to define the inverse limit $\overline{\mathcal{M}}(Q,d)$ of the moduli spaces $\mathcal{M}^\theta(Q,d)$ over all θ, to compute it in some small instances and to compare those with already existing ones, like the Losev-Manin moduli space $\overline{L}_t$, the Grothendieck-Knudsen moduli space $\overline{M}_{0,t}$ (of pointed curves of genus one) and the Hassett moduli space $H_{a,t}$. This, in particular, gives a common frame work to define all those moduli spaces in one strike, show they are defined over $\mathbb{Z}$ and also obtain corresponding generalizations (by using other quivers or dimension vectors).

1. Inverse Limits

Before we can define inverse limits, we need some background on walls and stability notions. Note that for any given θ in $M_{\mathbb{R}}(d)$ (the space of all real functions θ on the Grothendieck group with $\theta(d)=0$) there is a small neighbourhood, so that for all η in this small neighbourhood, we get a restriction morphism

$$\pi(\eta,\theta) : \mathcal{M}^\eta(Q,d) \longrightarrow \mathcal{M}^\theta(Q,d).$$

This says, any θ–stable representation M is also η–stable, and any η–stable representation is θ–semistable. In this case, we just write $\eta \geq \theta$. We can define a partial order on all equivalence classes of stability notions using this relation $\geq$. Note that two different weights (then called equivalent) can satisfy both inequalities. This means η is equivalent to θ precisely when the set of stable modules, as well as the set of semistable modules coincide. Using this partial order and the corresponding morphism above, we define the inverse limit via

$$\overline{\mathcal{M}}(Q,d) := \varprojlim_{\theta} \mathcal{M}^\theta(Q,d)$$

where the limit runs over all θ with non-empty moduli space. Note that there are only finitely equivalence classes of θ with respect to this partial order. This follows from the fact, that any dimension vector has only a finite number of subdimension vectors. In other words, a point in $\overline{\mathcal{M}}(Q,d)$ consists of a tuple $(M_\theta)_\theta$ of θ–stable modules M_θ (where θ runs through all possible stability notions with non-empty moduli space), subject to the following compatibility relation

$$\pi(\eta,\theta)M_\eta \simeq M_\theta \text{ for all } \eta \geq \theta.$$

In fact, one can always work with a finite tuple, since there are only finitely many equivalence classes of θ, and one can reduce even more (see [1]) for further details.

The nice fact with these moduli spaces is, they behave much better with respect to restrictions to subquivers. On the negative side, they are more complicated than the original moduli space, to compute the inverse limit one needs to compute $\mathcal{M}^\theta(Q,d)$ for all weights and in addition the compatibility condition. However, we show in the next section, those inverse limits are in some cases already intensively studied and should generalize to higher dimensions (we can replace curves by planes, or by more complicated configurations) by just changing the dimension vector or the quiver.

As a property of the inverse limit, that allows to compute the space recursively, we mention the following result, here e is a subdimension vector of d, that is unique. For example, if R is a subquiver of Q, closed under successors, the restriction $e := d\mid_R$ is such a unique subdimension vector.

Theorem 1.1. *Let Q be a quiver with dimension vector d and e a unique subdimension vector, then we have two projections (surjective morphisms)*

$$\overline{\mathcal{M}}(Q,d) \longrightarrow \overline{\mathcal{M}}(Q,e) \text{ and } \overline{\mathcal{M}}(Q,d) \longrightarrow \overline{\mathcal{M}}(Q,d-e).$$

These projections identify the larger space with a certain tautological family of objects, thus can be studied in more detail using a recursion over all possible e. For our main result, this is exactly the tautological family of objects, see details below.

2. Main Results

In this part we consider two particular quivers, Q_t (the subspace quiver with $t+1$ vertices, and arrows $0 \longrightarrow i$ for $i = 1,\dots,t$) and R_t (the complete bipartite quiver with $t+2$ vertices, and arrows $i \longrightarrow j$ for $i = 1,\dots,t$; $j = t+1, t+2$). The first quiver Q_t we consider with the dimension vector $d = (2,1,1,\dots,1)$ (one-dimensional subspaces in the plane), the second quiver R_t we consider with dimension vector $d = (1,\dots,1)$. Note that the second one defines a toric variety $\mathcal{M}^\theta(R_t,(1,\dots,1))$, consequently, the inverse limit $\overline{\mathcal{M}}(R_t,(1,\dots,1))$ is toric as well.

Next we mention already existing moduli spaces, we need three variants, for a precise definition we refer to the references.

$\overline{M}_{0,t}$ is the Grothendieck-Knudsen moduli space, it is the moduli space of stable curves of genus zero with t marked points ([5]).

$\overline{L}_t$ is the Losev-Manin moduli space of stable curves of genus zero with t marked points ([6]).

$\overline{H}_{a,t}$ is the Hassett moduli space of stable curves of genus zero with t marked points of weight a ([2]).

Note that the weight $a = (a_1,\dots,a_t)$ corresponds to a function on the vertices $i = 1,\dots,t$ of the quiver Q_t, where $d_i = 1$. It satisfies $\sum_{i=1}^t a_i \geq 2$. The weight θ satisfies (by definition) the equation $\sum_{i=1}^t \theta_i = 2$.

Theorem 2.1. *The following inverse limits of moduli spaces of quiver representations (for Q_t and R_t and the particular dimension vector) are defined over the integers and we have isomorphisms*

$$\begin{array}{rcl}
\overline{\mathcal{M}}(Q_t,(2,1,\dots,1)) \simeq \varprojlim \mathcal{M}^\theta(Q_t,(2,1,\dots,1)) & \simeq & \overline{M}_{0,t} \\
\varprojlim_{\theta\leq a} \mathcal{M}^\theta(Q_t,(2,1,\dots,1)) & \simeq & \overline{H}_{a,t} \\
\varprojlim_{\theta\sim(2,-1-1,0,\dots)} \mathcal{M}^\theta(Q_{t+2},(2,1,\dots,1)) \simeq \varprojlim \mathcal{M}^\eta(R_t,(1,1,\dots,1)) & \simeq & \overline{L}_t
\end{array}$$

where the first limit is over all θ, the second limit over all θ bounded by a, and the third limit is taken over all weights in a sufficiently small neighbourhood of the weight $(2, -1, -1, 0, \dots, 0)$. *The last limit is then taken over all weights η for the quiver R_t.*

3. Generalizations

In the last part we only mention some generalizations. First of all, the inverse limit can be seen also as a moduli space of 'compatible modules', and there are several other ways to construct it. One of the most important, non-trivial, ways is the Chow quotient of Kapranov ([3]).

The obvious generalization, to replace the 2 in the dimension vector d by $n > 2$ leads to a moduli space of projective spaces (they are in fact configurations of those spaces) with marked points and a similar stability notion. Those spaces seem to be new, the first theorem allows also for those spaces to describe them recursively. The corresponding toric version can be described using its fan, as worked out by the second author.

Using the projection morphism in Theorem 1.1 for $d = (2, 1, 1, \dots, 1)$ and $e = (2, 1, \dots, 1)$ (forget the last one) yields the universal family of stable curves. The same holds for the Losev-Manin moduli space and the Hassett moduli space.

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Labelling the τ-tilting fan

Hugh Thomas

(joint work with David Speyer)

τ-tilting theory associates to a finite-dimensional algebra a certain fan. Its full-dimensional cones are naturally associated to functorially finite torsion classes, while Asai shows that its codimension one faces are naturally labelled by bricks. We show how to recover these labellings in terms of stability conditions in the sense of King. This project turned out to have considerable overlap with the project described in the talk of Thomas Brüstle. I have tried to keep notation consistent between the two talks.

1. τ-tilting theory

We begin by recalling some key features of τ-tilting theory, as introduced by Adachi, Iyama, and Reiten [AIR]. Let A be a finite-dimensional algebra over an arbitrary ground field with t non-isomorphic simple modules.

(M, P) is called a τ-rigid pair if M is an A-module, P is a projective A-module, $\operatorname{Hom}(M, \tau M) = 0$, and $\operatorname{Hom}(P, M) = 0$. It is called τ-tilting if, in addition, the number of non-isomorphic indecomposable summands of M plus the number of non-isomorphic indecomposable summands of P equals t. For simplicity, we usually assume that M and P are basic (i.e. their decompositions into indecomposable direct summands have no summand occurring with multiplicity greater than 1).

Theorem 1 ([AIR]). *For any indecomposable summand of (M, P), there is a unique way to replace it by a different summand obtaining a new τ-tilting pair.*

Note that this is reminiscent both of classical tilting theory and of cluster-tilting.

We now associate to each indecomposable summand in a τ-tilting object a vector in $(\mathbb{R}^t)^*$, called its g-vector. We fix the standard basis $e_1^*, \ldots, e_t^*$ of $(\mathbb{R}^t)^*$. We define the g-vector of $(P_i, 0)$ to be

$$g^{(P_i,0)} = g^{P_i} = e_i^*.$$

For a general module M, we take a minimal projective presentation of M, say $P_1 \to P_0 \to M \to 0$ and define

$$g^{(M,0)} = g^M = g^{P_0} - g^{P_1}.$$

We also define

$$g^{(0,P_i)} = -e_i^*.$$

To a τ-rigid pair, we associate the cone generated by the g-vectors of its summands.

Theorem 2 ([DIJ]). *The cones corresponding τ-rigid objects form a* fan *(i.e., any two intersect in a common face of each).*

This fan is called the *τ-tilting fan.*

It was shown by Mizuno [Mi] that if A is a finite-type preprojective algebra, then the τ-tilting fan can be identified with the Coxeter fan.

2. Labelling the τ-tilting fan: chambers

A key point of τ-tilting theory is that there is a bijection between τ-tilting pairs and functorially finite torsion classes, which sends (M, P) to $\operatorname{Fac} M$. The chambers (i.e., full-dimensional cones) of the τ-tilting fan are therefore naturally labelled by functorially finite torsion classes. We now show how to recover this labelling in terms of stability.

Define a pairing between elements of $(\mathbb{R}^t)^*$ and the Grothendieck group of A by setting $e_i^*([S_j]) = \delta_{ij}$.

For $\theta \in (\mathbb{R}^t)^*$, define

$$\mathcal{T}_\theta = \{M \mid \theta([N]) \geq 0 \; \forall\, M \twoheadrightarrow N\}$$

$\mathcal{T}_\theta$ is a torsion class, as is shown, for example, in [BKT], though the result may be older.

Theorem 3 (Brüstle-Smith-Treffinger, Speyer-T.)**.** *If θ is in the interior of the chamber associated to the τ-tilting pair (M, P), then $\mathcal{T}_\theta = \operatorname{Fac} M$.*

3. Labelling the τ-tilting fan: walls

If two τ-tilting pairs (M, P) and (M', P') have $n-1$ summands in common, they are related by *mutation*. Suppose without loss of generality that $\operatorname{Fac} M$ contains $\operatorname{Fac} M'$. Then there exists a summand M_i of M such that

$$M_i \to X \to M_i' \to 0,$$

where $M_i \to X$ is a left add $M \backslash M_i$-approximation, and M' is obtained by replacing M_i by M_i'. If $M_i' = 0$, then M_i should be replaced by a suitable $(0, P_j)$ instead. (See [AIR] and the improvement by [Zh].)

This exact sequence is reminiscent of the situation in tilting or cluster-tilting. Note that there is no exactness on the left.

The codimension one cone associated to this mutation is that spanned by the g^{M_j} for $j \neq i$, and each codimension one cone in the τ-tilting fan arises like this for exactly one mutation. Asai [As] labels this cone by the quotient of M_i by the sum of the images of all the radical maps from M. This module is a brick, i.e., its endomorphism ring is a division algebra.

The same labelling, defined differently, has already studied for finite-type preprojective algebras by [BKT, IRRT].

For $\phi \in (\mathbb{R}^t)^*$, we follow [Ki] in saying that M is ϕ-semistable if $\phi([M]) = 0$ and $\phi([N]) \leq 0$ for any submodule N of M.

Theorem 4 (Speyer-T.)**.** *If θ lies in the interior of the codimension one cone associated to the mutation from (M, P) to (M', P') as above, then the unique indecomposable semistable with respect to θ is the brick which labels that wall.*

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Specialization finite group schemes with applications to (co)-stratification and local duality

Julia Pevtsova

(joint work with Dave Benson, Srikanth B. Iyengar, Henning Krause)

Let R be a commutative Noetherian ring, and consider the derived category of ModR, $T = \mathsf{D}(R)$. The compact objects of T form the bounded derived category of perfect complex over R, $T^c = \mathsf{D}^{\mathrm{perf}}(R)$. For a complex M in $\mathsf{D}(R)$, and a prime $\mathfrak{p} \in \mathsf{Spec}\, R$, we can consider the localization $M_{\mathfrak{p}}$ of M at p and the specialization $M \otimes^{\mathbb{L}}_R k(\mathfrak{p})$ at $\mathfrak{p}$, invariants which provide "local information" about the complex M at the point $\mathfrak{p}$. One invariant we can use this local information for is the support.

Definition 1.

$$\operatorname{supp} M = \{\mathfrak{p} \in \mathsf{Spec}\, R \mid M \otimes^{\mathbb{L}}_R k() \ncong 0\}$$

This geometric invariant is faithful in the sense that it detects vanishing of the object M; it also behaves nicely with respect to the standard operations in $\mathsf{D}(R)$: completing triangles, shifts, direct sums and tensor products.

One result in commutative algebra which motivates some of our considerations in modular representation theory is Neeman's classification of colocalising subcategories in $\mathsf{D}(R)$ [16]: namely, there is one-to-one correspondence

$$\left\{ \begin{array}{c} \text{Colocalising} \\ \text{subcategories of } \mathsf{D}(R) \end{array} \right\} \sim \left\{ \begin{array}{c} \text{subsets of} \\ \mathsf{Spec}\, R \end{array} \right\}$$

Combined with Neeman's classification of localizing subcategories, this gives one-to-one correspondence

$$\left\{ \begin{array}{c} \text{Colocalising} \\ \text{subcategories of } \mathsf{D}(R) \end{array} \right\} \sim \left\{ \begin{array}{c} \text{Localising} \\ \text{subcategories of } \mathsf{D}(R) \end{array} \right\}$$

given by taking a subcategory C to $C^{\perp}$

Finally, restricted to $\mathsf{D}^{\mathrm{perf}}(R)$ this gives one-to-one correspondence between the thick subcategories of $\mathsf{D}^{\mathrm{perf}}(R)$ and specialization closed subsets of $\mathsf{Spec}\, R$. The latter can be expressed in the language of *triangular geometry* introduced by P. Balmer [1]:

$$\mathsf{Spec}_{\mathrm{Bal}}\, \mathsf{D}^{\mathrm{perf}}(R) \cong \mathsf{Spec}\, R \tag{15}$$

where the left hand side is the *Balmer spectrum* of the tensor triangulated category $\mathsf{D}^{\mathrm{perf}}(R)$.

We see that for schemes the notion of a "point" can be realized on the level of categories: namely, a scheme theoretic point $\mathsf{Spec}\, K \to \mathsf{Spec}\, R$ corresponds to a ring map $R \to K$ which gives rise to a triangulated functor which is the specialization: $\mathsf{D}^{\mathrm{perf}}\, R \to \mathsf{D}^{\mathrm{perf}}\, K$. Finally, applying the Balmer spectrum, we get back the original point $\mathsf{Spec}\, K \to \mathsf{Spec}\, R$. Hence, the category $\mathsf{D}^{\mathrm{perf}}\, K$ together with the triangulated functor $\mathsf{D}^{\mathrm{perf}}\, R \to \mathsf{D}^{\mathrm{perf}}\, K$ "realizes" the point on the spectrum on the categorical level. It is this construction that we would like to mimic in modular representation theory.

Now let k be an algebraically closed field of positive characteristic, and let G be a finite group scheme defined over k. The coordinate algebra of G, $k[G]$, is a finite dimensional commutative Hopf algebra. We denote its linear dual, $\mathrm{Hom}_k(k[G], k)$, by kG. This is a finite dimensional cocommutative Hopf algebra whose category of modules is equivalent to the category of rational representations of G over k. Hence, we may identify representations of G with kG-modules; for the rest of this note we shall refer to representations of G as *G-modules*. Examples of finite group schemes include finite groups, restricted Lie algebras and Frobenius kernels of algebraic groups.

The tensor triangulated category associated to G is the stable module category $\mathsf{StMod}\, G$. Recall that the objects of $\mathsf{StMod}\, G$ are G-modules, whereas the Hom-sets are defined as follows:

$$\underline{\mathrm{Hom}}(M, N) := \frac{\mathrm{Hom}_G(M, N)}{\mathrm{PHom}_G(M, N)}$$

with $\mathrm{PHom}_G(M, N)$ being the subset of all G-maps between M and N which factor through a projective G-module. The category $\mathsf{StMod}\, G$ is a compactly generated tensor triangulated category with the compact objects being the finite dimensional G-modules. This subcategory is denoted $\mathsf{stmod}\, G$.

Let $R = H^*(G, k)$ be the cohomology algebra of G. It is finitely generated as a k-algebra by a celebrated theorem of Friedlander and Suslin [15]. We have R acting on $\underline{\mathrm{Hom}}^*(M, N) = \bigoplus_{n \in \mathbb{Z}} \underline{\mathrm{Hom}}^n(M, N)$ via Yoneda product. The question motivated by the classical construction in commutative algebra is then the following:

Let M be a G-module, and $\mathfrak{p} \in \mathsf{Proj}\, R$ be a homogenenous prime ideal strictly contained in the irrelevant ideal. How can we "specialize" or "localize" M at $\mathfrak{p}$?

We give two answers to this question: one involves a representation theoretic construction and the notion of a π-point and the other uses the local cohomology functors introduced by Benson, Iyengar and Krause.

Definition 2 ([13], [14])**.** *A* π-point *of G, defined over a field extension K of k, is a flat map of K-algebras*

$$\alpha : K[t]/(t^p) \to KG_K$$

which factors through the group algebra of a unipotent abelian subgroup scheme C of G_K.

Given a π-point $\alpha : K[t]/(t^p) \to KG_K$, we can construct a point $\mathfrak{p} = \Psi(\alpha) \in \mathsf{Proj}\, R$ as follows. Let $H^*(\alpha)$ be the map on cohomology induced by α:

$$H^*(\alpha) : H^*(G, k) \xrightarrow{-\otimes_k K} H^*(G, K) \xrightarrow{\alpha^*} H^*(K[t]/t^p, K)\ ,$$

and define $\Psi(\alpha) := \sqrt{\ker H^*(\alpha)}$. This determines a surjective correspondence:

Theorem 3 ([13], [14])**.** *For any $\mathfrak{p} \in \mathsf{Proj}\, R$ there exists a π-point α of G such that*

$$\Psi(\alpha) = \mathfrak{p}.$$

In this way, we can "realize" points on $\mathsf{Proj}\, R$ as π-points. This, in turn, gives a way to specialize G-modules at prime ideals on $\mathsf{Spec}\, R$ and to define supports (and cosupports).

For M a G-module, and $\mathfrak{p} \in \mathsf{Proj}\, R$ a homogenenous prime ideal, we consider a π-point $\alpha_{\mathfrak{p}}$ such that $\Psi(\alpha_{\mathfrak{p}}) = \mathfrak{p}$. Then the pull-back $\alpha_{\mathfrak{p}}^*(K \otimes_k M)$ which is a $K[t]/t^p$-module plays the role of a "specialization" of M at $\mathfrak{p}$.

Definition 4. *The* π-support *of M is the subset of* $\mathsf{Proj}\, H^*(G,k)$ *defined by*

$$\pi\text{-}\operatorname{supp}(M) := \{\mathfrak{p} \in \mathsf{Proj}\, H^*(G,k) \mid \alpha_{\mathfrak{p}}^*(K \otimes_k M) \textit{ is not projective}\}.$$

The π-cosupport *of M is the subset of* $\mathsf{Proj}\, H^*(G,k)$ *defined by*

$$\pi\text{-}\operatorname{cosupp}(M) := \{\mathfrak{p} \in \mathsf{Proj}\, H^*(G,k) \mid \alpha_{\mathfrak{p}}^*(\operatorname{Hom}_k(K,M)) \textit{ is not projective}\}.$$

It was proved in [13] that these invariants are well-defined (that is, independent of the choice of $\alpha_{\mathfrak{p}}$ corresponding to $\mathfrak{p}$.

The usefulness of π-support and π-cosupport is postulated in the following theorem.

Theorem 5. *i [Detection] Let M be a G-module. Then $\pi\text{-}\operatorname{supp} M = \emptyset$ if and only if M is a projective G-module.*

ii [Tensor and Hom formulae] Let M and N be G-modules. Then there are equalities

$$\pi\text{-}\operatorname{supp}(M \otimes_k N) = \pi\text{-}\operatorname{supp}(M) \cap \pi\text{-}\operatorname{supp}(N),$$
$$\pi\text{-}\operatorname{cosupp}(\operatorname{Hom}_k(M,N)) = \pi\text{-}\operatorname{supp}(M) \cap \pi\text{-}\operatorname{cosupp}(N).$$

The detection property is an ultimate generalization of the famous Dade's lemma [12]. It builds on the work of many authors, see [3], [2], [17], [18]. In this generality it is proved in [7].

From the triangular geometry point of view, a π-point α gives rise to a restriction functor

$$\mathsf{StMod}\, G \to \mathsf{StMod}\, K[t]/t^p$$

which, once we apply Balmer's Spec construction, realizes the corresponding point $\Psi(\alpha) \in \mathsf{Spec}\, H^*(G,k)$.

A different approach to localization is given by Benson-Iyngar-Krause local cohomology functors. To any homogeneous prime ideal $\mathfrak{p} \in \mathsf{Proj}\, R$ one associates a universal local cohomology module $\Gamma_{\mathfrak{p}}$ (see [4]). Then the *cohomological* support and cosupport are defined as follows:

Definition 6 ([4], [5]).

$$\operatorname{supp}(M) := \{\mathfrak{p} \in \mathsf{Proj}\, H^*(G,k) \mid \Gamma_{\mathfrak{p}}(k) \otimes_k M \textit{ is not projective}\}.$$

$$\operatorname{cosupp}(M) := \{\mathfrak{p} \in \mathsf{Proj}\, H^*(G,k) \mid \operatorname{Hom}_k(\Gamma_{\mathfrak{p}}(k), M) \textit{ is not projective}\}.$$

One important property of universal local cohomology modules developed in [7], [9] is that they satisfy the "reduction to closed points principle":

Theorem 7. *Let $\mathfrak{p}$ be a point on $\mathsf{Proj}\, H^*(G,k)$, and let $d = \dim H^*(G,k)/\mathfrak{p}$. There exists a field extension K/k of transcendence degree d, and a maximal ideal $\mathfrak{m} \in \mathsf{Proj}\, H^*(G_K, K)$ lying over $\mathfrak{p}$, such that there is an isomorphism*

$$\Gamma_{\mathfrak{p}} \cong \mathsf{Res}_G^{G_K}(\Gamma_{\mathfrak{m}} K \otimes K//b)$$

Here, $K//b$ is a *Koszul object* associated to the prime ideal $\mathfrak{p}$.

The point of this theorem is that it allows to reduce questions at prime ideal $\mathfrak{p} \in \mathsf{Proj}\, R$ to closed point, that is, to maximal homogeneous prime ideals $\mathfrak{m} \in \mathsf{Spec}\, R_K$ where they become more approachable.

Corollary 8. *In the notation of the theorem, the restriction functor*

$$\Gamma_{\mathfrak{m}}(\mathsf{StMod}\, G_K) \to \Gamma_{\mathfrak{p}}(\mathsf{StMod}\, G)$$

is full and dense.

The Detection theorem 5 is the key step in identifying the two support theories: the π-supp and the local cohomology support of Benson-Iyengar-Krause. That unified support theory, combined with the powerful reduction to closed points principle and one more new construction, that of a point module associated to a π-point α, allows us to prove the ultimate analogue of Neeman's classification for finite groups schemes:

Theorem 9 ([8])**.** *For any finite group scheme G, there is a one-to-one correspondence*

$$\left\{ \begin{array}{c} \textit{Colocalizing Hom-closed} \\ \textit{subcategories of } \mathsf{StMod}\, G \end{array} \right\} \sim \left\{ \begin{array}{c} \textit{subsets of} \\ \mathsf{Proj}\, H^*(G,k) \end{array} \right\}$$

given by cosupport.

This classification implies in the usual manner the classification for localising tensor ideal subcategories in $\mathsf{StMod}\, G$ and the tensor ideal subcategories in $\mathsf{stmod}\, G$ but it will be misleading given the historical development of the subject to state these classifications as corollaries.

Another application of the local techniques we develop, including the reduction to closed points principle, is the local Serre duality for the category $\Gamma_{\mathfrak{p}}(\mathsf{StMod}\, G)$. In the theorem below, $I_{\mathfrak{p}}$ is the universal injective cohomology object in $\mathsf{StMod}\, G$ introduced in [10], and δ_G is the one dimensional modular character of G which in a sense measures how far kG is from being symmetric. The functor $\Omega^d \delta_G \otimes_k -$ plays the role of the local Serre functor in the sense of Bondal-Kapranov [11]

Theorem 10 ([9])**.** *Let $C = (\Gamma_{\mathfrak{p}}\, \mathsf{StMod}\, G)^c$ be the category of compact objects in $\Gamma_{\mathfrak{p}}\, \mathsf{StMod}\, G$, and let $M, N \in \mathcal{C}$. There is a natural isomorphism:*

$$\mathrm{Hom}_R(\mathrm{Hom}^*_{\mathcal{C}}(M,N), I_{\mathfrak{p}}) \simeq \mathrm{Hom}_{\mathcal{C}}(N, \Omega^d \delta_G \otimes M)$$

where $d = \dim R/\mathfrak{p}$.

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Kac polynomials for canonical algebras

Pierre-Guy Plamondon
(joint work with Olivier Schiffmann)

In the 1980's, V. Kac proved the following surprising result on representations of quivers over finite fields.

Theorem 0.1 ([5]). *Let Q be a quiver, and let $d \in \mathbb{N}^{Q_0}$ be a dimension vector for Q. For any finite field k, denote by $A_{Q,d}(k)$ the number of isomorphism classes of absolutely indecomposable representations of Q of dimension vector d. Then there* *exists a polynomial* $P_{Q,d}(T) \in \mathbb{Z}[T]$ *such that*

$$A_{Q,d}(k) = P_{Q,d}(|k|)$$

for all finite fields k*, where* $|k|$ *is the number of elements of* k*.*

V. Kac conjectured that the polynomials $P_{Q,d}(T)$ had non-negative coefficients, a result recently proved in [3].

From there, one can ask about the following generalization:

Question 0.2. *Given a basic finite-dimensional algebra* B *over some finite field* k_0 *and a dimension vector* d *for this algebra, let* $A_{B,d}$ *be the number of isomorphism classes of absolutely indecomposable representations of* B *of dimension vector* d*. Is there a polynomial* $P_{B,d}(T) \in \mathbb{Z}[T]$ *such that for any finite field extension* k *of* k_0*, we have*

$$A_{B,d}(k) = P_{B,d}(|k|)?$$

The answer to this question is no in general; in fact, for any projective variety V, there is an algebra and a dimension vector such that the number of isomorphism classes of absolutely indecomposable modules with this dimension vector is equal to the number of points in V.

The aim of this talk is to present a positive answer for a certain class of algebras:

Main Theorem 0.3 ([7])**.** *Question 0.2 admits a positive answer for (almost concealed-) canonical algebras.*

1. Weighted projective lines

We define the category of coherent sheaves over a weighted projective line, following [2]. Let n be a positive integer, $p = (p_0, \dots, p_n)$ be a tuple of positive integers and $\lambda = (\lambda_0, \dots, \lambda_n)$ be a tuple of pairwise distinct points in $\mathbb{P}^1(k_0)$, normalized so that $\lambda_0 = \infty, \lambda_1 = 0$ and $\lambda_2 = 1$.

Define $S(p, \lambda) := k_0[X_0, \dots, X_n]/(X_i^{p_i} = X_1^{p_1} - \lambda_i X_0^{p_0}, i = 2, \dots, n)$. This is an $L(p)$-graded algebra, where $L(p) := \bigoplus_{i=0}^{n} \mathbb{Z}\vec{x}_i/(p_i\vec{x}_i = p_j\vec{x}_j)$, and $\deg X_i = \vec{x}_i$.

Definition 1.1 ([2])**.** *The* category of coherent sheaves over the weighted projective line $\mathbb{X}$ *defined by* p *and* λ *is the Serre quotient of abelian categories*

$$\operatorname{coh}\mathbb{X} = \operatorname{mod}_{gr}\big(S(p,\lambda)\big)/\operatorname{mod}_{gr,0}\big(S(p,\lambda)\big),$$

where $\operatorname{mod}_{gr}\big(S(p,\lambda)\big)$ *is the category of finitely generated graded* $S(p,\lambda)$*-modules, and* $\operatorname{mod}_{gr,0}\big(S(p,\lambda)\big)$ *is its full subcategory of finite length modules.*

Recall that an object T of a hereditary abelian category $\mathcal{A}$ is *tilting* if $\operatorname{Ext}^1_{\mathcal{A}}(T,T) = 0$, and T generates the bounded derived category $\mathcal{D}^b(\mathcal{A})$.

Theorem 1.2 ([2])**.** *The category* $\operatorname{coh}\mathbb{X}$ *is a hereditary abelian category, and it contains tilting objects.*

Definition 1.3 ([6])**.** *Let* T *be a tilting object in* $\operatorname{coh}\mathbb{X}$*. Then its endomorphism algebra* $C = \operatorname{End}_{\mathbb{X}}(T)$ *is an* almost concealed-canonical algebra.

Examples include Ringel's canonical algebras [8] and Brenner and Butler's squid algebras [1].

2. Ideas of the proof of the main theorem

The methods of proof are inspired from the following result of O. Schiffmann. For an abelian category $\mathcal{A}$, let $K_0(\mathcal{A})$ be its Grothendieck group, and $K_0^+(\mathcal{A}) = \{[X] \in K_0(\mathcal{A}) \mid X \in \mathcal{A}\}$ be the submonoid of classes of objects of $\mathcal{A}$.

Theorem 2.1 ([9])**.** *Let $d \in K_0^+(\operatorname{coh}\mathbb{X})$. There is a polynomial $P_{\mathbb{X},d}(T) \in \mathbb{Q}[T]$ such that for all finite field extension k of k_0,*

$$A_{\mathbb{X},d}(k) = P_{\mathbb{X},d}(|k|),$$

where $A_{\mathbb{X},d}(k)$ is the number of isomorphism classes of absolutely indecomposable objects of $\operatorname{coh}\mathbb{X}$ of class d.

In summary, the tools needed to prove the main theorem are the following. Let C be the endomorphism algebra of a tilting object T in $\operatorname{coh}\mathbb{X}$.

- Since T is tilting, the derived categories $\mathcal{D}^b(\operatorname{coh}\mathbb{X})$ and $\mathcal{D}^b(\operatorname{mod} C)$ are equivalent.
- The object T induces a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\operatorname{coh}\mathbb{X}$.
- By results of [6], the indecomposable modules over C lie, in $\mathcal{D}^b(\operatorname{coh}\mathbb{X})$, either in $\mathcal{T}$ or in $\mathcal{F}[1]$.
- There is an isomorphism $\psi : K_0(\operatorname{mod} C) \to K_0(\operatorname{coh}\mathbb{X})$.
- If $d \in K_0^+(\operatorname{mod} C)$, then $\psi(d) \in K_0^+(\mathcal{T}) \cup K_0^-(\mathcal{F})$.
- This implies that

$$A_{C,d}(k) = \begin{cases} A_{\mathcal{T},\psi(d)}(k) & \text{if } \psi(d) \in K_0^+(\mathcal{T}), \\ A_{\mathcal{F},-\psi(d)}(k) & \text{if } \psi(d) \in K_0^-(\mathcal{F}). \end{cases}$$

- The main idea borrowed from [9] is that the number of isomorphism classes of indecomposable sheaves in $\mathcal{T}$ (or $\mathcal{F}$) of a given class in the Grothendieck group and endowed with a nilpotent endomorphism is governed by the Green pairing in the spherical Hall algebra of $\operatorname{coh}\mathbb{X}$. This is the main technical point of the proof, and we omit the details in this extended abstract.
- From there, it is possible to show that $A_{\mathcal{T},\psi(d)}(k)$ and $A_{\mathcal{F},-\psi(d)}(k)$ are given by polynomials in $\mathbb{Q}[T]$.
- A lemma of N. M. Katz (see the appendix to [4]) allows us to further conclude that these polynomials are in $\mathbb{Z}[T]$.

3. A positivity conjecture

In view of the positivity conjecture of [5] and of the results of [3], it is natural to conjecture the following:

Conjecture 3.1 ([7])**.** *If C is an almost concealed-canonical algebra, then the coefficients of $P_{C,d}(T)$ are non-negative for all dimension vectors d.*

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R-matrices and convolution algebras arising from Grassmannians

Catharina Stroppel

(joint work with Vassily Gorbounov, Christian Korff)

We describe how to construct interesting convolution algebras from equivariant cohomology of Grassmannians and give a geometric construction of certain R-matrices arising in the 5-vertex models. Details appear in [GKS17].

1. Integrable systems

In the following let N be a fixed natural number. We fix as the ground field the complex numbers $\mathbb{C}$ and let $\mathbb{C}[t]$ be the polynomial ring in a variable t. For any finite dimensional vector space W denote $W[t] = W \otimes \mathbb{C}[t]$, the $\mathbb{C}[t]$-module obtained by scalar extension, and $W[t_1, t_2, \ldots, t_N] = W \otimes \mathbb{C}[t_1, t_2, \ldots t_N]$.

Let $V = \mathbb{C}^2$ with a fixed basis v_0, v_1. It induces a standard (or tensor) basis of $V \otimes V$. Then $V[t] = V \otimes \mathbb{C}[t]$ has $\mathbb{C}[t]$-basis v_0, v_1. We identify $V[t]^{\otimes N} = V \otimes [t_1, t_2, \ldots, t_N]$ as vector space (in the obvious way identifying t_i with t from the ith tensor factor). A *Lax matrix* is a 2×2-matrix

$$L = L(x,t) = \begin{pmatrix} A(x,t) & B(x,t) \\ C(x,t) & D(x,t) \end{pmatrix} \quad \in \quad \mathrm{M}(2 \times 2, \mathrm{M}(2 \times 2, \mathbb{C}[x,t]) \tag{16}$$

with entries in $\mathrm{M}(2\times 2, \mathbb{C}[x,t])$. It defines a $\mathbb{C}[x,t]$-linear endomorphism of $V\otimes V$. The *monodromy matrix* $M = M(x,t_1,t_2,\dots,t_N)$ is the endomorphism

$$L_{0,N}(x,t_N)\dots L_{0,2}(x,t_2)L_{0,1}(x,t_1) = \begin{pmatrix} A(x,t_1,t_2,\dots,t_N) & B(x,t_1,t_2,\dots,t_N) \\ C(x,t_1,t_2,\dots,t_N) & D(x,t_1,t_2,\dots,t_N)\end{pmatrix}$$

of $V[x]\otimes V^{\otimes N}[t_1,\dots,t_N]$. A pair $(R(x,y),L(x,t))$ of Lax matrices *satisfies the Yang Baxter equation* if

$$L_{2,3}(y,t)L_{1,3}(x,t)_{1,2}R(x,y) \quad = \quad R_{1,2}(x,y)L_{1,3}(x,t)R_{2,3}(y,t) \tag{17}$$

as endomorphisms of $V[t]\otimes V[x]\otimes V[y]$. One can easily verify that $(R(x,y),L(x,t))$ satisfies (17) if and only if

$$\begin{aligned} & M_2(x_2,t_1,\dots,t_N)M_1(x_1,t_1,\dots,t_N)R_{1,2}(x_1,x_2) \\ = \; & R_{1,2}(x_1,x_2)M_1(x,t_1,\dots,t_N)M_2(x_1,t_1,\dots,t_N) \end{aligned} \tag{18}$$

Important examples appearing in the physics literature are the so-called 5-vertex models which are certain (not well understood) degeneration of the (well-known) 6-vertex model and given by the following pairs $(R(x,y),L(x,y))$ with $z = x-y$.

$$\left(\begin{pmatrix}1&0&0&0\\0&z&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&z&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\right) \qquad \left(\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&-z&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}z&0&0&0\\0&1&1&0\\0&1&1&0\\0&0&0&0\end{pmatrix}\right)$$

The *Yang-Baxter algebra* is the $\mathbb{C}[t_1,\dots t_N]$-subalgebra of endomorphisms of $V^{\otimes N}$ generated by the coefficients $A_i(t_1,\dots t_N)$, $B_i(t_1,\dots t_N)$, $C_i(t_1,\dots t_N)$, and $D_i(t_1,\dots t_N)$ of the x^i of the entries of M. It is easy (although maybe not very useful) to write down an explicit presentation of this algebra for the above special choices of $(R(x,y),L(x,t))$ in which we call the algebra Y_N respectively Y'_N.

Lemma 1. *The action of Y_N and of Y'_N on $V[t]^{\otimes N}$ commutes with the action of the symmetric group S_N permuting the factors.*

2. Geometry of Grassmannians

Let $X_k = \mathrm{Gr}(k,N)$ be the Grassmannian of k-dimensional subspaces in $\mathbb{C}^N$ with its standard torus T (=diagonal matrices)-action. If we fix the standard basis $e_1,\dots,e_N$ of $\mathbb{C}^N$ then the T-fixed points are precisely the coordinate spaces $p_I = \langle e_i \mid i\in I\rangle$, where I is a k-element subset of $\mathbb{I} = \{1,2,\dots,N\}$. For each regular integral cocharacter (i.e. $\chi_{\mathbf{a}} : \mathbb{C}^* \to T$ sending $t\in\mathbb{C}^*$ to the diagonal matrix $(t_1^{a_1},\dots,t^{a_N})$ with $a_i\in\mathbb{Z}$ pairwise distinct) we have the attracting cells $C_I = \{x\in X_n \mid \lim_{t\to 0}\chi_{\mathbf{a}}(t).x = p_I\}$. Note that the cell decomposition does in fact not depend on the tuple $\mathbf{a}$, but only on the Weyl chamber containing it. Let us identify Weyl chambers with Weyl group elements by mapping the antidominant chamber to the identity element. Then we obtain geometric interpretations of integrable systems bases:

Proposition 1. *Each Weyl chamber, hence each Weyl group element w, defines a basis S_I^w, $I \subset \mathbb{I}$, $|I| = k$, of the T-equivariant cohomology ring $H_T(X_k)$.*

(1) Sending a basis vector $v_{i_1} \otimes \ldots \otimes v_{i_N}$ of $V[t]^{\otimes N} = V \otimes [t_1, t_2, \ldots, t_N]$ to S_I^e where $I = \{j \mid i_j = 1\}$ defines an isomorphism of $\mathbb{C}[t_1, t_2, \ldots, t_N]$-modules

$$\Phi : \quad V[t]^{\otimes N} \quad \cong \quad \bigoplus_{k=0}^{N} H_T(X_k)$$

after localisation at all $t_i - t_j, i \neq j$.

(2) Under this isomorphism the normalized Bethe basis vectors for the subalgebra generated by the coefficients of x^i's in $A(x)$ are mapped to the geometric fixed point basis vectors.

The first R-matrix from the 5-vertex model has a beautiful interpretation (the second can be obtained by some renormalizing):

Theorem 1 (Wall-crossing)**.** *The base change from S_I^w to $S_I^{ws_i}$ (that is the wall crossing in the ith wall) is given via Φ by $R_{a,b}(t_b, t_a)$ acting on the ath and bth tensor factor of $V[t]^{\otimes N}$, where $a = w(i)$ and $b = w(i+1)$.*

Theorem 2. *The Yang Baxter-algebras Y_N and of Y'_N can be realized via certain correspondences*

$$X_k \longleftarrow X_{k,k+1} \longrightarrow X_{k+1}$$

involving the two-steps partial flag varieties $X_{k,k+1}$.

Remark 1. *The construction can be seen as an analogue of the Maulik-Okounkov construction [MO12]. Instead of working with cotangent bundles and their symplectic structure we work instead with the base X_k itself. The Schubert varieties $\overline{C}_I$ are then classical analogues of their stable manifolds. Our assignment $p_I \mapsto [\overline{C_I}]$ from the set of T-fixed points to bases of $H_T(X_k)$ form a stabilization map.*

3. Yang-Baxter algebras, Yangians and current algebras

Consider now $V[t]$ as the natural (evaluation) module of the current algebra $\mathfrak{gl}_2[t]$. Let $H_N = \mathbb{C}[S_N]\#\mathbb{C}[t_1, t_2, \ldots t_N]$ be the smash product of the group algebra of the Weyl group S_N with the polynomial ring. It naturally acts on $V[t]^{\otimes N}$.

Theorem 3 (Schur-Weyl duality)**.** *(1) The H_N-action on $V^{\otimes N}[t_1, t_2, \ldots t_N]$ commutes with the $U(\mathfrak{gl}_2[t])$-action and they generate each others commutants.*

(2) This is still true after localization, where we take the localization of the bimodule at $(t_i - t_j)$'s as before, of the algebra H_N at its center (given by symmetric polynomials), and the Ore-localization of $U(\mathfrak{gl}_2[t])$ at the multiplicative set $\mathbf{1} \otimes t^k$, $k \geq 0$, where $\mathbf{1} \in \mathfrak{gl}_2$ denotes the identity matrix.

Since the Yang-Baxter algebra actions are $\mathbb{C}[t_1, t_2, \ldots, t_N]$-linear and commute with the S_N-action they must be subalgebras of the image of the localised $U(\mathfrak{gl}_2[t])$-action. Using the geometric interpretation we show

Theorem 4. *The subalgebra of endomorphisms generated by the two Yang-Baxter algebras Y_N and of Y'_N is isomorphic to the image of the $U(\mathfrak{gl}_2[t])$-action.*

Remark 2. Taking the limit $N \to \infty$ allows to mimic the construction of [BLM90] for $\mathfrak{gl}_n$ now for current algebras.

Remark 3. Our construction works more general for $U(\mathfrak{gl}_n[t])$ by using partial flag varieties up to n steps; although the formulas are not anymore totally explicit.

Remark 4. Geometric constructions of the current Lie algebras seem to be not available, in contrast to Yangian (a certain quantum deformation of the loop algebra). The R-matrix corresponding to the Yangian corresponds to the "generic" 6-vertex model, and via for instance [MO12] to the geometry of the cotangent bundle of partial flag varieties and Nakajima quiver varieties. Working with the base instead specialises the deformation parameter and corresponds to a non-trivial interesting degeneration on the side of R-matrices. This is the passage to the 5-vertex model. As far as we understand, these degenerations cannot be deduced easily from the "generic" model. Our stable manifolds do not arise from the construction in [MO12] by pushing their stable manifolds down to the base space of the cotangent bundle.

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Cohen–Macaulay modules over the algebra of surface quasi–invariants, Calogero–Moser systems and matrix problems

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(joint work with Alexander Zheglov)

As an input datum, let us fix any pair $(\Pi, \underline{\mu})$, where

- $\Pi \subset \mathbb{C}$ is a finite subset such that $\alpha - \beta \notin \pi\mathbb{Z}$ for any $\alpha \neq \beta \in \Pi$.
- $\Pi \xrightarrow{\underline{\mu}} \mathbb{N}_0, \ \alpha \mapsto \mu_\alpha := \underline{\mu}(\alpha)$ is any multiplicity function.

Next, for any $\alpha \in \Pi$ we denote: $l_\alpha(z_1, z_2) := -\sin(\alpha)z_1 + \cos(\alpha)z_2 \in R := \mathbb{C}[z_1, z_2]$. Then main object of our study is the following $\mathbb{C}$–algebra of $(\Pi, \underline{\mu})$–quasi–invariant polynomials:

$$A = A\big(\Pi, \underline{\mu}\big) := \Big\{ f \in R \mid l_\alpha^{2\mu_\alpha+1} \text{ divides } \big(f - s_\alpha(f)\big) \text{ for all } \alpha \in \Pi \Big\}, \tag{19}$$

where $R \xrightarrow{s_\alpha} R$ is the involution associated with the reflection $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$ keeping invariant the line $l_\alpha = 0$.

Theorem (Burban & Zheglov [3]). The following results are true.

1. The algebra A is a finitely generated Cohen–Macaulay $\mathbb{C}$–algebra of Krull dimension two, which is Gorenstein in codimension one.
2. Let $\mathsf{CM}_1^{\mathrm{lf}}(A)$ be the abelian group of Cohen–Macaulay A–modules of rank one, which are locally free in codimension one [4, 5] (it is an analogue of the divisor class group of a normal domain [2, Section 7.3]). Then there exists an isomorphism of abelian groups

$$\mathsf{CM}_1^{\mathrm{lf}}(A) \longrightarrow K\big(\Pi, \underline{\mu}\big) := \prod_{\alpha\in\Pi} \big(\mathbb{C}(\rho)[\sigma]/(\sigma^{\mu_\alpha}), \circ\big),$$

where the group law $\circ$ on $\mathbb{C}(\rho)[\sigma]/(\sigma^{\mu_\alpha})$ is given by the rule

$$\gamma_1 \circ \gamma_2 := (\gamma_1 + \gamma_2)\cdot(1 + \sigma\gamma_1\gamma_2)^{-1}.$$

3. Let M be a Cohen–Macaulay A–module of rank one, which is not locally free in the codimension one. Then there exists a multiplicity function $\Pi \xrightarrow{\underline{\mu}'} \mathbb{N}_0$ such that $\mu_\alpha \geq \mu'_\alpha$ for any $\alpha \in \Pi$ and $M' \in \mathsf{CM}_1^{\mathrm{lf}}(A')$ for $A' = A(\Pi, \underline{\mu}')$ such that M is isomorphic to M', where M' is viewed as a module over $A \subset A'$.
4. For any $\alpha \in \mathbb{C}$ and $m \in \mathbb{N}$, we construct a certain homomorphism of abelian groups $\big(\mathbb{C}[[z_1, z_2]], +\big) \xrightarrow{\Upsilon_{(\alpha,m)}} \big(\mathbb{C}[[\rho]][\sigma]/(\sigma^m), \circ\big)$, giving a homomorphism

$$\big(\mathbb{C}[[z_1, z_2]], +\big) \xrightarrow{\Upsilon} \prod_{\alpha\in\Pi} \big(\mathbb{C}[[\rho]][\sigma]/(\sigma^{\mu_\alpha}), \circ\big), \quad h \mapsto \big(\Upsilon_{(\alpha, 2\mu_\alpha)}(h)\big)_{\alpha\in\Pi}.$$

Then $\mathsf{Pic}(A) \cong \mathrm{Im}(\Upsilon) \cap K^\diamond\big(\Pi, \underline{\mu}\big)$, where $K^\diamond\big(\Pi, \underline{\mu}\big) := \prod\limits_{\alpha\in\Pi} \mathbb{C}[\rho][\sigma]/(\sigma^{\mu_\alpha})$.

More explicitly, let $\Gamma\big(\Pi, \underline{\mu}\big) := \big\{h \in \mathbb{C}[[z_1, z_2]] \big| \Upsilon(h) \in K^\diamond\big(\Pi, \underline{\mu}\big)\big\}$. Then for any such $h \in \Gamma\big(\Pi, \underline{\mu}\big)$, we have a projective A–module $P(h)$ of rank one, defined as

$$P(h) := \big\{f \in R \,\big|\, \exp(h)f \text{ is } (\Pi, \underline{\mu}) - \text{quasi–invariant}\big\}.$$

Conversely, for any $P \in \mathsf{Pic}(A)$, there exists $h \in \Gamma\big(\Pi, \underline{\mu}\big)$ such that $P \cong P(h)$. Moreover,

- $P(h_1) \cong P(h_2)$ if and only if $\Upsilon(h_1) = \Upsilon(h_2)$.
- The multiplication map $P(h_1) \otimes_A P(h_2) \longrightarrow P(h_1 + h_2)$, $f_1 \otimes f_2 \mapsto f_1 f_2$ is an isomorphism of A–modules.

A motivation to study the algebra of quasi–invariants A comes from the theory of rational Calogero–Moser systems (see [10] for an introduction). Let $\Pi = \Pi_n := \big\{0, \frac{\pi}{n}, \ldots, \frac{n-1}{n}\pi\big\}$ for some $n \in \mathbb{N}_{n\geq 2}$, $m \in \mathbb{N}$ and $(a, b) \in \mathbb{R}^2$. The rational Calogero–Moser operator of type (Π_n, m) is given by the expression

$$H = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \sum_{\alpha\in\Pi_n} \frac{m(m+1)}{l_\alpha^2(x, y)}, \tag{20}$$

where $l_\alpha(x, y) = (x - a)\sin(\alpha) - (y - b)\cos(\alpha)$.

Let $A = A\big(\Pi_n, m\big)$ be the algebra of quasi–invariants corresponding to the datum (Π_n, m) and $\mathfrak{D} = \mathbb{C}[[x, y]][\partial_x, \partial_y]$. Then the following results are known:

- There exists an injective algebra homomorphism $A \xrightarrow{L} \mathfrak{D}$ such that $L(z_1^2 + z_2^2) = H$ (Chalykh & Veselov [7]). In other words, the Calogero–Moser operator H can be included into a large family of pairwise commuting differential operators (quantum integrability of H).
- The A–module $F = \mathfrak{D}/(x,y)\mathfrak{D}$ (the spectral module of the Calogero–Moser system) is Cohen–Macaulay of rank one (Chalykh & Veselov [7]) and Kurke & Zheglov [12]).

The following result gives an answer to a question, asked in a recent paper of Feigin and Johnston [11].

Theorem (Burban & Zheglov [3]). The spectral module F of the Calogero–Moser system (20) is projective and isomorphic to $P(-az_1 - bz_2)$.

Our work is based on the following two key ingredients. Firstly, we essentially use the theory of multivariate Baker–Akhieser functions of (generalized) Calogero–Moser systems [7, 8, 1, 9, 6]. The second ingredient is the "matrix–problem method" of [4, 5] to study Cohen–Macaulay modules over singular surfaces with non–isolated singularities.

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Thick subcategories of n-cluster tilting subcategories

Martin Herschend

(joint work with Peter Jørgensen, Laertis Vaso)

Let K be a field and A a finite dimensional associative K-algebra. Denote by $\mathsf{mod}A$ the category of finitely generated right A-modules. By a subcategory $\mathcal{W} \subseteq \mathsf{mod}A$ we always mean a full subcategory closed under isomorphisms, direct sums and direct summands. We say that $\mathcal{W} \subseteq \mathsf{mod}A$ is thick if it is also closed under kernels, cokernels and extensions.

Classifying thick subcategories of $\mathsf{mod}A$ can give some insight into the representation theory of A. In particular, the combinatorics of these subcategories can be quite interesting (see for instance [4] for the case when A is hereditary).

One way to get some control over thick subcategories is to use algebra epimorphisms. More precisely, let B be another finite dimensional K-algebra and $\phi : A \to B$ an algebra morphism that is an epimorphism in the category of rings. Then ϕ induces a natural restriction of scalars functor $\phi_* : \mathsf{mod}B \to \mathsf{mod}A$, which is full and faithful. Moreover, $\phi_*(\mathsf{mod}B) \subseteq \mathsf{mod}A$ is closed under kernels and cokernels. A version of this statement formulated for the categories of all modules over arbitrary (i.e., not necessarily finite dimensional) K-algebras appeared in [2]. In [2] was also introduced a natural equivalence relation on such epimorphisms. More specifically, replacing ϕ by $\psi \circ \phi$, where $\psi : B \to B'$ is some isomorphism will not change the subcategory $\phi_*(\mathsf{mod}B)$ so it is natural consider epimorphisms ϕ up to the corresponding equivalence relation. In our setting we are dealing with finite dimensional algebras and modules and so we get in addition that $\phi_*(\mathsf{mod}B)$ is a functorially finite subcategory of $\mathsf{mod}A$ (see [5, Theorem 1.6.1]).

To make sure that $\phi_*(\mathsf{mod}B) \subseteq \mathsf{mod}A$ is closed under extensions we need to impose the condition that $\mathrm{Tor}_1^A(B,B) = 0$. Such algebra epimorphisms were studied in [1] under the name pseudoflat epimorphisms. The relation to closure under extensions can be found in [8, Theorem 4.8]

The above observations can be strengthened to provide a classification of functorially finite thick subcategories, which can be seen as a combination of results in [1], [2], [3], [5] and [8]:

Theorem 1. *Let A be a finite dimensional K-algebra. Then there is a bijection between equivalence classes of algebra epimorphisms $\phi : A \to B$, with B finite dimensional, satisfying $\mathrm{Tor}_1^A(B,B) = 0$ and functorially finite thick subcategories of $\mathsf{mod}A$, which is given by $\phi \mapsto \phi_*(\mathsf{mod}B)$.*

In Iyama's higher dimensional Auslander-Reiten theory [6] one replaces $\mathsf{mod}A$ with a n-cluster tilting subcategory $\mathcal{M} \subseteq \mathsf{mod}A$, meaning that $\mathcal{M}$ is functorially finite and satisfies

$$\begin{aligned}\mathcal{M} &= \{X \in \mathsf{mod}A \mid \mathrm{Ext}^i_A(X,\mathcal{M}) = 0 \text{ for all } 1 \le i \le n-1\},\\ &= \{X \in \mathsf{mod}A \mid \mathrm{Ext}^i_A(\mathcal{M},X) = 0 \text{ for all } 1 \le i \le n-1\}.\end{aligned}$$

In particular, $\mathsf{mod}A$ is the unique 1-cluster tilting subcategory of $\mathsf{mod}A$. For $n > 1$, there may be several or no n-cluster tilting subcategories $\mathcal{M}$. Moreover,

$\mathcal{M}$ is not abelian. Instead $\mathcal{M}$ is n-abelian in the sense of [7], meaning that the notions of kernel, cokernel and extension, are replaced by their higher dimensional counterparts: n-kernel, n-cokernel and n-extension. These are longer sequences satisfying similar conditions to their counterparts in abelian categories. Naturally, we call a subcategory $\mathcal{W} \subseteq \mathcal{M}$ thick if it is closed under these three operations.

In my talk I presented work in progress joint with Peter Jørgensen and Laertis Vaso, in which we generalize Theorem 1 to classify thick subcategories $\mathcal{W} \subseteq \mathcal{M}$, where $\mathcal{M} \subseteq \mathsf{mod} A$ is n-cluster tilting. In order to state our result we call $(A, \mathcal{M})$ an n-homological pair if A is a finite dimensional K-algebra and $\mathcal{M} \subseteq \mathsf{mod} A$ is n-cluster tilting. If $(B, \mathcal{N})$ is another n-homological pair and $\phi : A \to B$ is an algebra epimorphism such that $\phi_*(\mathcal{N}) \subseteq \mathcal{M}$ and $\mathrm{Tor}_n^A(B, B) = 0$, then we call $\phi : (A, \mathcal{M}) \to (B, \mathcal{N})$ an n-rigid epimorphism. The equivalence relation on algebra epimorphisms defined above naturally extends to n-rigid epimorphisms.

Theorem 2. *Let $(A, \mathcal{M})$ be an n-homological pair. Then there is a bijection between equivalence classes of n-rigid epimorphisms $\phi : (A, \mathcal{M}) \to (B, \mathcal{N})$ and functorially finite thick subcategories of $\mathcal{M}$, which is given by $\phi \mapsto \phi_*(\mathcal{N})$.*

A interesting special case to consider is when A has global dimension n. Then $\mathcal{M}$ is unique if it exists. Now let's specialize even more by assuming that A is a quotient of the path algebra KQ of the quiver

$$Q: \quad m \to \cdots \to 2 \to 1$$

by an ideal $0 \neq I \subsetneq \mathrm{rad}^2 KQ$. Then it is shown in [9], that A has global dimension n and admits a n-cluster tilting subcategory $\mathcal{M}$ if and only if n is even, and $I = \mathrm{rad}^l KQ$, where

$$\frac{m-1}{l} = \frac{n}{2}.$$

Then the n-cluster tilting subcategory $\mathcal{M}$ consists of all modules which have only projective and injective direct summands. The indecomposables in $\mathcal{M}$ can be uniquely labelled

$$M_1, M_2, \ldots, M_{m+l-1}$$

such that $\mathrm{Hom}_A(M_i, M_j) = 0$ for all $i > j$. As an application of Theorem 2 we can classify all thick subcategories of $\mathcal{M}$ that are not semisimple.

Theorem 3. *Let $(A, \mathcal{M})$ be the n-homological pair described above. Assume that $\mathcal{W} \subseteq \mathcal{M}$ is a subcategory that is not semisimple. Then $\mathcal{W}$ is thick if and only if $M_i \in \mathcal{W}$ implies $M_{i+kl} \in \mathcal{W}$ for all integers k such that M_{i+kl} is well defined. In particular there are $2^l - l - 1$ such subcategories.*

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The isogeny category of commutative algebraic groups

Michel Brion

The objects of the talk are the *commutative algebraic groups* over a fixed field k, that we will assume of characteristic 0 for simplicity of the presentation. These form an abelian category $\mathcal{C}$, with morphisms being the homomorphisms of algebraic groups. It is easy to show that $\mathcal{C}$ is artinian and not noetherian. Also, $\mathcal{C}$ has no non-zero injective object, and its projective objects are exactly the *unipotent groups* (i.e., the direct sums of copies of the additive group $\mathbb{G}_a$), see [2]. Thus, $\mathcal{C}$ does not have enough projectives.

When k is algebraically closed, this drawback was remedied by Serre via the construction of the category $\hat{\mathcal{C}}$ of *profinite groups*. He showed that $\hat{\mathcal{C}}$ is a hereditary abelian category having enough projectives and containing $\mathcal{C}$ as a Serre subcategory (see [10]). As a consequence, $\mathcal{C}$ is hereditary as well. For an arbitrary field k, Milne showed in [6] that $\mathrm{hd}(\mathcal{C}) = 1 + \mathrm{cd}(\Gamma)$, where $\mathrm{hd}(\mathcal{C})$ denotes the homological dimension, and $\mathrm{cd}(\Gamma)$ the cohomological dimension of the absolute Galois group of k (a profinite group). In particular, $\mathrm{hd}(\mathcal{C})$ can be arbitrarily large.

The building blocks of $\mathcal{C}$ are the *linear algebraic groups* (or equivalently, the affine ones) and the *abelian varieties* (i.e., the connected algebraic groups which are projective varieties). More specifically, every commutative algebraic group G lies in an exact sequence

$$0 \longrightarrow L \longrightarrow G \longrightarrow A \longrightarrow 0,$$

where L is linear and A is an abelian variety. Moreover, one easily checks that $\mathrm{Hom}_{\mathcal{C}}(A, L) = 0$ and every morphism $L \to A$ factors through a finite quotient group of L; hence $\mathrm{Hom}_{\mathcal{C}}(L, A)$ is finite. With an obvious notation, this defines two Serre subcategories $\mathcal{L}$, $\mathcal{A}$ of $\mathcal{C}$. By assigning to G its coordinate ring, one obtains an anti-equivalence of $\mathcal{L}$ with the category of (commutative, co-commutative) Hopf algebras over k which are finitely generated as k-algebras. Thus, linear algebraic groups may be viewed as purely algebraic objects. In contrast, dealing with abelian varieties involves some arithmetic geometry, as we will see below.

To simplify the set-up, we consider the quotient category of $\mathcal{C}$ by its Serre subcategory $\mathcal{F}$ consisting of finite algebraic groups (so that $\mathcal{F} \subset \mathcal{L}$ is anti-equivalent to the category of Hopf algebras which are finite-dimensional as k-vector spaces). The abelian category $\mathcal{C}/\mathcal{F}$ is also obtained from $\mathcal{C}$ by inverting the *isogenies*, i.e., the morphisms with finite kernel and cokernel. One easily shows that $\mathcal{C}/\mathcal{F}$ is a length category, equivalent to its full subcategory $\underline{\mathcal{C}}$ with objects being the connected algebraic groups; moreover, $\mathrm{Hom}_{\underline{\mathcal{C}}}(G,H) = \mathrm{Hom}_{\mathcal{C}}(G,H) \otimes_{\mathbb{Z}} \mathbb{Q}$ for all $G, H \in \underline{\mathcal{C}}$ (see [2]). In particular, the *isogeny category* $\underline{\mathcal{C}}$ is $\mathbb{Q}$-linear. In fact, $\mathbb{Q}$ is the largest ring with this property: indeed, $\mathrm{End}_{\mathcal{C}}(\mathbb{G}_m) = \mathbb{Z}$, where $\mathbb{G}_m$ denotes the multiplicative group, and hence $\mathrm{End}_{\underline{\mathcal{C}}}(\mathbb{G}_m) = \mathbb{Q}$.

We may also consider the isogeny categories $\underline{\mathcal{L}}$, $\underline{\mathcal{A}}$ of linear algebraic groups, resp. abelian varieties. Then $(\underline{\mathcal{L}}, \underline{\mathcal{A}})$ turns out to be a torsion pair of Serre subcategories of $\underline{\mathcal{C}}$, and $\mathrm{Hom}_{\underline{\mathcal{C}}}(A, L) = 0$ for any $A \in \underline{\mathcal{A}}$ and $L \in \underline{\mathcal{L}}$. Moreover, $\underline{\mathcal{A}}$ is known to be semi-simple and Hom-finite, with infinitely many simple objects. Also, $\underline{\mathcal{L}}$ is semi-simple; it is Hom-finite if and only if k is a number field. The simple objects of $\underline{\mathcal{L}}$ are the additive group $\mathbb{G}_a$ and the simple tori; the latter correspond to the irreducible continuous finite-dimensional representations of the Galois group Γ. Thus, when k is algebraically closed, the simple objects of $\underline{\mathcal{L}}$ are just $\mathbb{G}_a$ and $\mathbb{G}_m$.

By easy homological arguments, it follows that $\underline{\mathcal{C}}$ is hereditary and $\mathrm{Ext}^i_{\underline{\mathcal{C}}}(L, A) = 0$ for all L, A as above and all $i \geq 0$. Also, linear algebraic groups are projective in $\underline{\mathcal{C}}$, and abelian varieties are injective. To complete the picture, it remains to describe $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(A, L)$. For this, it will be convenient to consider the category $\tilde{\mathcal{L}}$ of *affine group schemes*, which is anti-equivalent to the category of Hopf algebras and contains $\mathcal{L}$ as a Serre subcategory. Also, we have the category $\tilde{\mathcal{F}}$ of *profinite group schemes* and the corresponding isogeny category $\underline{\tilde{\mathcal{F}}} := \tilde{\mathcal{L}}/\tilde{\mathcal{F}}$, which turns out to be semi-simple. Now there exists an exact functor $\mathbf{F} : \underline{\mathcal{A}} \to \underline{\tilde{\mathcal{L}}}$ such that

$$\mathrm{Ext}^1_{\underline{\mathcal{C}}}(A, L) = \mathrm{Hom}_{\underline{\tilde{\mathcal{L}}}}(\mathbf{F}(A), L)$$

for all A and L. Thus, we may consider the category $\underline{\tilde{\mathcal{C}}}$ of extensions

$$0 \longrightarrow \tilde{L} \longrightarrow \tilde{G} \longrightarrow A \longrightarrow 0,$$

where $\tilde{L} \in \underline{\tilde{\mathcal{L}}}$ and $A \in \underline{\mathcal{A}}$. Then $\underline{\tilde{\mathcal{C}}}$ is a hereditary category having enough projectives and containing $\underline{\mathcal{C}}$ as a Serre subcategory; we may view $\underline{\tilde{\mathcal{C}}}$ as the isogeny category of *quasi-compact* group schemes (see [3] for these results).

Recall that a scheme is said to be quasi-compact if every open covering admits a finite refinement. All affine schemes are quasi-compact, as well as all connected group schemes. Moreover, every connected group scheme $\tilde{G}$ lies in an extension as above, by a result of Perrin (see [7, 8]). The projective cover in $\underline{\tilde{\mathcal{C}}}$ of any abelian variety A is the universal extension

$$0 \longrightarrow \mathbf{F}(A) \longrightarrow \tilde{G} \longrightarrow A \longrightarrow 0$$

corresponding to the identity in $\mathrm{End}_{\underline{\tilde{\mathcal{L}}}}(\mathbf{F}(A)) = \mathrm{Ext}^1_{\underline{\tilde{\mathcal{C}}}}(A, \mathbf{F}(A))$.

Finally, we discuss the *indecomposable objects* of the isogeny category $\underline{\mathcal{C}}$; these are exactly the connected algebraic groups G admitting no decomposition $G = G_1 + G_2$, where $G_1, G_2 \subset G$ are connected algebraic subgroups of positive dimension, and $G_1 \cap G_2$ is finite. For this, we assume that k is a number field (so that $\underline{\mathcal{C}}$ is Hom-finite) and we consider the Serre subcategory $\underline{\mathcal{C}}_{E,F} \subset \underline{\mathcal{C}}$ generated by finite sets E of simple objects of $\underline{\mathcal{A}}$, and F of simple objects of $\underline{\mathcal{L}}$ (since $\underline{\mathcal{C}}$ has infinitely many simples). Then $\underline{\mathcal{C}}_{E,F}$ is equivalent to the category of finite-dimensional representations of a basic finite-dimensional hereditary $\mathbb{Q}$-algebra, which is uniquely determined. The associated valued graph $\Delta_{E,F}$ has vertices $E \sqcup F$ and edges joining $A \in E$ with $L \in F$ whenever $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(A, L) \neq 0$; such an edge is labeled with a pair of positive integers, the dimensions of $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(A, L)$ as a vector space over $D_A := \mathrm{End}_{\underline{\mathcal{C}}}(A)$, resp. $D_L := \mathrm{End}_{\underline{\mathcal{C}}}(L)$ (these are finite-dimensional division algebras over $\mathbb{Q}$). In particular, $\Delta_{E,F}$ is bipartite. By the main result of [4], the category $\underline{\mathcal{C}}_{E,F}$ is of finite representation type if and only if each connected component of $\Delta_{E,F}$ is a Dynkin diagram.

To fully characterize those pairs (E, F) such that $\underline{\mathcal{C}}_{E,F}$ is of finite representation type, one would need additional information on the above functor $\mathbf{F}$. This raises questions of an arithmetic nature, as seen from the example where F just consists of the additive group $\mathbb{G}_a$. Then $\mathbf{F}(A)$ is the dual of the k-vector space $H^1(A, \mathcal{O}_A)$ (of dimension $\dim(A)$), viewed as a unipotent group. In particular, every edge of the valued graph $\Delta_{E,\mathbb{G}_a}$ contains the vertex $\mathbb{G}_a$. Thus, the possible Dynkin diagrams are those having a vertex connected to all others. When $k = \mathbb{Q}$, one may check that the maximal such diagrams are exactly:

$$\mathbf{C}_3 : \quad E \;\text{———}\; \mathbb{G}_a \xrightarrow{\;(2,1)\;} A$$

$$\mathbf{D}_4 : \quad E_1 \;\text{———}\; \mathbb{G}_a \;\text{———}\; E_2, \qquad \mathbb{G}_a \;\text{———}\; E_3$$

$$\mathbf{G_2} : \quad \mathbb{G}_a \xrightarrow{\;(3,1)\;} B$$

Here the labels E, E_1, E_2, E_3 denote elliptic curves; A stands for a simple abelian surface, and B for a simple abelian threefold. In addition, we must have $\dim_{\mathbb{Q}}(D_A) = 2$ and $\dim_{\mathbb{Q}}(D_B) = 3$, i.e., A and B are *abelian varieties of* GL_2 *type* as defined by Ribet in [9]. Examples of such abelian varieties have been constructed by González, Guárdia and Rotger in dimension 2 (see [5]) and by Baran in dimension 3 (see [1]). A conjectural construction of all abelian varieties of GL_2 type has been proposed by Ribet in [9], generalizing the Shimura-Taniyama-Weil conjecture proved by Taylor and Wiles.

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Graded Maximal Cohen–Macaulay Modules over One-dimensional Graded Gorenstein Rings

Ragnar-Olaf Buchweitz

(joint work with Osamu Iyama, Kota Yamaura)

Introduction.
The key theoretical results here have already been presented in the extended abstract of the author in the Oberwolfach Report OWR 2016–46 for the workshop "*Singularities*", see https://www.mfo.de/occasion/1639 for more details and references.

Graded Gorenstein Rings of Dimension One.
Our results pertain to commutative positively graded Gorenstein rings $R = \oplus_{i\geqslant 0} R_i$ of Krull dimension 1 with $R_0 = k$ a field. The total graded ring of fractions of R is $\mathcal{K} = \mathcal{N}^{-1}R$, where $\mathcal{N} \subset R$ is the set of homogeneous non-zero-divisors of R.

Because R is Gorenstein, $\operatorname{Ext}^i_R(k, R) = 0$ for $i \neq 1$ and $\operatorname{Ext}^1_R(k, R) = k(-a)$ as graded R–modules. The occurring integer $a = a(R)$ is the *a–invariant* of R.

Example 1. For a quasi–homogeneous curve singularity $R \cong k[x, y]/(f(x, y))$, the a–invariant is $a(R) = \deg f - \deg x - \deg y$, thus, for example, for $f = xy$ with any (positive) degrees for x, y, the a–invariant is always 0.

Example 2. The ring $R = k[x, y]/(y^2)$ is Gorenstein and with arbitrarily assigned degrees $\deg x, \deg y > 0$ the ring is obviously graded. One has $a(R) = \deg y - \deg x$, whence the a–invariant can take on any integer value.

For the purpose of this abstract we restrict to the case that k is algebraically closed of characteristic zero, that R is reduced, and that $\mathcal{K}$ contains a non-zero-divisor of degree 1. In this case, we have the following precise information.

Lemma 3. *Under the assumptions made, $\mathcal{K} \cong \prod_{j=1}^{r} k[t_j, t_j^{-1}]$ is a product of graded rings of Laurent polynomials in one variable with $\deg t_j = 1$. The ring $\mathcal{K}_{\geqslant 0} \cong \prod_{j=1}^{r} k[t_j]$ is then the normalization of R.*

If $a(R) < 0$, then $R \cong k[t]$ is a polynomial ring in one variable with $\deg t = -a = 1$.

Henceforth assuming $a = a(R) \geqslant 0$, consider the following graded R–modules

- $T_i = R_{\geqslant i}(i) = R(i)_{\geqslant 0}$, for $i = 1, ..., a$, and
- $T_{i+j} = k[t_j]$, for $j = 1, ..., r$, where $\mathcal{K} \cong \prod_{j=1}^{r} k[t_j, t_j^{-1}]$ as above.

Theorem 4. *The R–module $T = \oplus_{i=1}^{a+r} T_i$ is a tilting object in the (triangulated) stable category of graded maximal Cohen–Macaulay (= torsionfree) R–modules The endomorphism algebra E of degree preserving endomorphisms of T in the stable category is the same as that in the category of all R–modules. In matrix form, recording the vector spaces of degree-preserving homomorphisms $\mathrm{Hom}_R(T_j, T_i)$, it is*

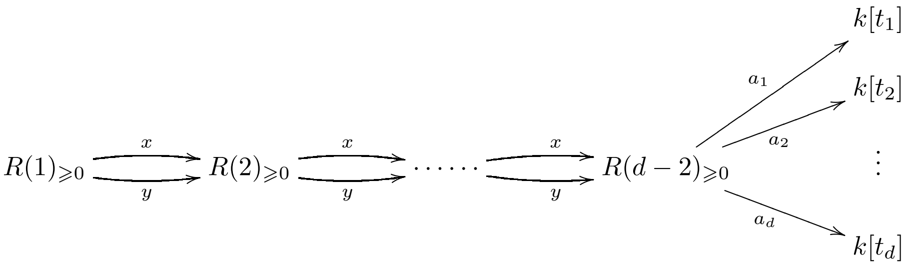
$\mathrm{End}_R(T)$	$R_{\geqslant 1}(1)$	$\cdots$	$R_{\geqslant j}(j)$	$\cdots\cdots$	$R_{\geqslant a}(a)$	$k[t_1]$	$\cdots$	$k[t_r]$
$R_{\geqslant 1}(1)$	k	$\cdots$	0	$\cdots\cdots$	0	0	$\cdots$	0
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$
$R_{\geqslant i}(i)$	R_{i-1}	$\cdots$	R_{i-j}	$\cdots\cdots$	0	0	$\cdots$	0
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$
$R_{\geqslant a}(a)$	R_{a-1}	$\cdots$	R_{a-j}	$\cdots\cdots$	k	0	$\cdots$	0
$k[t_1]$	k	$\cdots$	k	$\cdots\cdots$	k	k	$\cdots$	0
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$
$k[t_r]$	k	$\cdots$	k	$\cdots\cdots$	k	0	$\cdots$	k

The algebra thus satisfies $\dim_k E = \sum_{i=0}^{a-1}(a-i)\dim_k R_i + (a+1)r$ and, as a triangular algebra, it is of finite global dimension at most $a + r$.

Corollary 5. *The Grothendieck group of the stable category of graded maximal Cohen–Macaulay R–modules is isomorphic to $\mathbb{Z}^{a+r}$.*

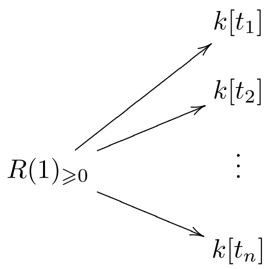
We now list three concrete examples and write down the quiver of the endomorphism algebra.

Line Arrangements in the Plane. Consider $R = k[x, y]/(f)$ with $\deg x = \deg y = 1$, and $f = \prod_{i=1}^{d} f_i$, with $d \geqslant 2$, where each $f_i = \alpha_i x + \beta_i y$ is a linear form in $k[x, y]$ for $1 \leqslant i \leqslant d$ such that $(f_i) \neq (f_j)$ for all $i \neq j$. Then $a(R) = d - 2, r = d$, and the quiver of E is given by



Note that $k[t_i] \cong R/(f_i)$. The relations in this quiver are $xy - yx$ and $a_i(\alpha_i x + \beta_i y)$ for $i = 1, ..., d$. The algebra E satisfies $\operatorname{gldim} E \leqslant 2$.

Sections of Cones over Elliptic Curves. Let C be an elliptic curve and $C \hookrightarrow \mathbb{P}^{n-1}$ an embedding of C through a line bundle of degree $n \geqslant 3$. The homogeneous coordinate ring R of a generic hyperplane section is then a Gorenstein ring of Krull dimension 1 and of a–invariant 1. For $n \geqslant 5$ it is not a complete intersection, but for $n \geqslant 4$ it is Koszul. The underlying space of R is a union of n lines through the origin in $\mathbb{A}^{n-1}$. The endomorphism ring of the tilting object is hereditary and its quiver is of the form



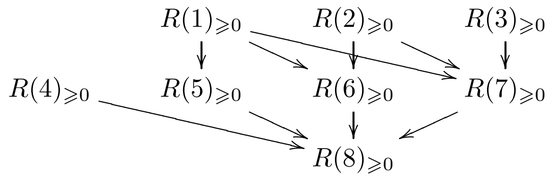
For $n = 3$, this represents a representation–finite algebra of type D_4 with source orientation. It is also the special case of the previous example for $d = 3$.
For $n = 4$, the ring R is a complete intersection of two quadrics in 3 variables and the algebra E is tame, of type $\widetilde{D}_4$ with source orientation. Gelinas [Gel17] has written down all graded indecomposable maximal Cohen–Macaulay modules over R using the well–known classification of quiver representations of $\widetilde{D}_4$.

Numerical Semigroup Rings. A numerical semigroup $\Gamma \subseteq (\mathbb{N}_{\geqslant 0}, +)$ is a submonoid of the nonnegative integers under addition such that $g := |\mathbb{N}_{\geqslant 0} \setminus \Gamma|$ is finite. The associated semigroup ring

$$R = k[\Gamma] = k[t^\gamma, \gamma \in \Gamma] \subseteq k[t] = \mathcal{K}_{\geqslant 0}$$

is of Krull dimension one. According to a classical result by Kunz [Kun70] it is Gorenstein if, and only if, $2g - 1 \notin \Gamma$, if, and only if, $a(R) = 2g - 1$. Such a numerical semigroup is called *symmetric* as an integer $x \in \mathbb{Z}$ satisfies either $x \in \Gamma$ or $2g - 1 - x \in \Gamma$. With t of degree 1, note that R_i is of k–dimension 1 or 0, according to whether i is in Γ or not. The corresponding endomorphism algebras are thus “tic tac toe” algebras over k in the terminology of [Mit70, p. 33].

As a particular example, consider the symmetric numerical semigroup generated by $4, 5, 6 \in \mathbb{N}$. The corresponding semigroup ring is Gorenstein of a–invariant 7. Indeed, it is even a complete intersection ring, $R \cong k[x, y, z]/(y^2 - xz, z^2 - x^3)$, with $(x, y, z) \mapsto (t^4, t^5, t^6)$. The quiver of the endomorphism algebra of the tilting object looks as follows



Note that $R(8)_{\geqslant 0} \cong k[t] \cong \mathcal{K}_{\geqslant 0}$. The relations on the quiver express that the "squares" contained in the quiver all commute. The global dimension of the algebra E is at most 2.

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