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O-Minimality and its Applications to Number Theory and Analysis

Organised by
Tobias Kaiser, Passau
Jonathan Pila, Oxford
Patrick Speissegger, Hamilton
Alex Wilkie, Manchester

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Abstract. The workshop brought together researchers in the areas of o-minimal structures, analysis and number theory. The latest developments in o-minimality and their applications to number theory and analysis were presented in a series of talks; one focus, in particular, was on the Pila-Wilkie Theorem and its impact on diophantine problems.

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Introduction by the Organisers

The workshop

"O-minimality and its Applications to Number Theory and Analysis"

was organized by (in alphabetical order):

- Tobias Kaiser (University of Passau)
- Jonathan Pila (University of Oxford)
- Patrick Speissegger (McMaster University)
- Alex Wilkie (University of Manchester)

There were 52 participants, many of whom are working on o-minimality with an eye on applications. A significant number of them are number theorists and analysts who use o-minimality in their work. The resulting interactions between these fields were well reflected in the lectures. Overall there were 27 talks, including 10 shorter ones by junior participants.

One main focus of the workshop was on the Pila-Wilkie Theorem, which establishes a subpolynomial bound for the number of rational points, in terms of their height, on sets definable in o-minimal structures. In recent years, this result has had a deep impact on the study of unlikely intersections in diophantine geometry. In several of the talks new aspects of the Pila-Wilkie theorem and new improvements on the respective bounds were presented. One highlight along these lines was the proof of the Wilkie conjecture (claiming poly-logarithmic bounds) in the case of the real field with the restricted exponential and sine functions. Other talks, notably by junior number theorists, described applications of the Pila-Wilkie Theorem and o-minimality to diophantine problems related to Manin's and the André-Oort conjectures, in the settings of elliptic curves or Shimura varieties.

Another main topic of the workshop was that of general o-minimal geometry. In the corresponding lectures we saw, among others, a new and very useful decomposition of sets definable in o-minimal structures into a special type of cells and a classification of definable surface singularities. For these results, combinatorial topological concepts turned out to play a major role.

How tame geometric properties of o-minimal structures lead to new insights in analysis was the third main topic of the workshop. Researchers presented their related work on integration, differentiability spaces and trajectories of vector fields.

The workshop schedule allowed for lively discussions and fruitful exchanges between participants. The organizers also took advantage of the Simons Visiting Professor program: Ta Lê Loi from the University of Dalat in Vietnam visited the University of Passau for two weeks before the workshop, and Chris Miller from The Ohio State University in Columbus, Ohio, visited the universities of Konstanz and Savoie-Mont Blanc in the two weeks following the workshop.

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Workshop: O-Minimality and its Applications to Number Theory and Analysis

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Abstracts

Expansions of the real field by real entire functions

Chris Miller
(joint work with Ovidiu Costin, Rodica Costin)

This is a preliminary report of some ongoing work. The reader is assumed to be familiar with the notions of expansions of the real field $\overline{\mathbb{R}}$ (see [2] for an introduction). Recall that an expansion of $\overline{\mathbb{R}}$ is o-minimal if every definable sets has only finitely many connected components.

We are interested in expansions of $\overline{\mathbb{R}}$ by transcendental real entire functions, that is, nonpolynomial functions $\mathbb{R} \to \mathbb{R}$ given by power series $\sum c_n x^n$ having infinite radius of convergence. We begin by considering simple infinite products.

Let $\bar{a} := (a_n)$ be an increasing and unbounded sequence of positive real numbers such that $\sum(1/a_n) < +\infty$, where the indices range over either the nonnegative integers or the positive integers according to convenience. Then

$$F := \prod \left(1 + \frac{z}{a_n}\right) : \mathbb{C} \to \mathbb{C}$$

is entire, transcendental, real on real, and positive on $(-a_0, +\infty)$. We are interested in the differential algebraic, asymptotic and model-theoretic properties of F regarded as a real function, and the interplay of these properties.
The first question: What can be said about structures of the form $(\overline{\mathbb{R}}, F{\restriction}[R, +\infty))$ for $R \in \mathbb{R}$? Currently we know of only two verified outcomes:

(O). For every $R \in \mathbb{R}$, $(\overline{\mathbb{R}}, F{\restriction}[R, +\infty))$ is o-minimal.

(PH). For every $R \in \mathbb{R}$, $(\overline{\mathbb{R}}, F{\restriction}[R, +\infty))$ defines the set $\mathbb{Z}$ (hence also all real projective sets).

Each of these can be studied by starting with $G := (zF'/F)'$ instead of F, for if $(\overline{\mathbb{R}}, G{\restriction}[R, +\infty))$ defines $\mathbb{Z}$, then so does $(\overline{\mathbb{R}}, F{\restriction}[R, +\infty))$, and if $(\overline{\mathbb{R}}, G{\restriction}[R, +\infty))$ is o-minimal, then so is $(\overline{\mathbb{R}}, F{\restriction}[R, +\infty))$ (by Pfaffian closure [6]). Some caution is in order: If $(\overline{\mathbb{R}}, F{\restriction}[R, +\infty))$ is o-minimal, then so is $(\overline{\mathbb{R}}, G{\restriction}[R, +\infty))$, but $(\overline{\mathbb{R}}, F{\restriction}[R, +\infty))$ can define $\mathbb{Z}$ while $(\overline{\mathbb{R}}, G{\restriction}[R, +\infty))$ does not.

Why work with G instead of F? For one thing, it is technically more convenient: $G = \sum a_n(z + a_n)^{-2}$, so we deal with a sum instead of a product. More interesting is that subtle asymptotics of F can be easier to detect in G. Indeed, a new possible outcome is witnessed:

(D). There exists $\alpha > 1$ such that, for every $R \in \mathbb{R}$, $(\overline{\mathbb{R}}, G{\restriction}[R, +\infty))$ defines $\alpha^{\mathbb{Z}} := \{\, \alpha^k : k \in \mathbb{Z} \,\}$, and every subset of $\mathbb{R}$ definable in $(\overline{\mathbb{R}}, G{\restriction}[R, +\infty))$ either has interior or is a finite union of discrete sets.

Some examples are in order.

It follows easily from the factorization of $\sin \pi z$ that $(\overline{\mathbb{R}}, F{\restriction}[R, +\infty))$ is interdefinable with $(\overline{\mathbb{R}}, e^x)$ if $\bar{a} = (n^2)$, and so (O) holds for F.

By further tricks based on the factorization of $\sin \pi z$, (PH) holds if $\bar{a} = (n^s)$ and s is an even integer > 2, even with G in place of F.

If $\alpha > 1$ and $\bar{a} = (\alpha^n)$, then (D) holds. This is far from obvious, but via some classical complex analysis, it is easy to see that $(\overline{\mathbb{R}}, G\restriction[R,+\infty))$ defines $\alpha^{\mathbb{Z}}$ and $G\restriction[R,+\infty)$ is definable in the structure $(\mathbb{R}_{\mathrm{an}}, \alpha^{\mathbb{Z}})$, which is known to satisfy the condition on the definable subsets of $\mathbb{R}$ (and much more; see [5]).

We currently do not know the status of any $\bar{a} = (n^s)$ for $s > 2$ that is not an even integer, but we suspect (O). The situation is subject to perturbation of the "base structure": By results from [3], there is an o-minimal structure $\mathfrak{R}$ on $\overline{\mathbb{R}}$ such that:

- if $1 < s \leq 2$ and $\bar{a} = (n^s)$, then $F\restriction[R,+\infty)$ is definable in $\mathfrak{R}$;
- if $1 < s < 2$ and $\bar{a} = (n^s)$, then $G\restriction[R,+\infty)$ is not definable in $(\mathbb{R}_{\mathrm{an}}, e^x)$;
- if $s > 2$ and $\bar{a} = (n^s)$, then (PH) holds with $\mathfrak{R}$ in place of $\overline{\mathbb{R}}$.

As establishing o-minimality can be rather difficult, a reasonable precursor is to check whether F generates a Hardy field. To put this another way, given $N \in \mathbb{N}$ and $p \in \mathbb{R}[X, Y_0, \ldots, Y_N]$, if the function $p(x, F, F', \ldots, F^{(N)})$ is not identically equal to 0, must its zero set be bounded above? By well-known Hardy field technology, it is enough to show that G generates a Hardy field. We are currently working to show that this holds for any $\bar{a} = (n^s)$ with $s > 2$ and not an even integer. Of course, we are also looking to detect when F does not generate a Hardy field. We have results in this direction if the growth of $\bar{a}$ is fast enough. To illustrate, if $\liminf_{n\to+\infty}(a_{n+1}/a_n) > 100$, then F does not generate a Hardy field and (PH) holds (we do not yet know in this generality what happens with G except that it does not generate a Hardy field).

An associated notion in this setting is whether F is differentially algebraic (DA), that is: Is there $N \in \mathbb{N}$ and $0 \neq p \in \mathbb{R}[X, Y_0, \ldots, Y_N]$ such that the function $p(x, F, F', \ldots, F^{(N)})$ is identically equal to 0? (Again, it suffices to work with G instead of F.) There is much on this topic in the literature, but something we could not find: If $s \in (1, +\infty) \setminus \mathbb{N}$ and $\bar{a} = (n^s)$, is G not DA? We think so, and are working on a proof. It is known [1] that G is not DA if s is an odd integer > 1, but the proof does not extend in any obvious way to the noninteger case. Generally speaking, faster growth of $\bar{a}$ is linked to a greater likelihood of failure of DA: It has long been known that F is not DA if its power series at the origin has sufficiently large gaps; it is easy to force this via fast enough growth of $\bar{a}$.

Aside from known results from classical complex analysis and the aforementioned work [3], there are two main techniques we employ: (1) For defining $\mathbb{Z}$, the main tool is "dimensional coincidence" [4] (but it would take us too far afield to explain this here). (2) For dealing with the (n^s), we have

$$\frac{F'(x)}{F(x)} = -\frac{1}{2x} + \frac{\csc(\pi/s)}{\pi/s} x^{\frac{1}{s}-1} + 2\sum_{n>0} \int_0^\infty \frac{\cos(2\pi n t)}{x + t^s}\, dt, \quad x > 0$$

(say, by Poisson summation). Hence, the point is to understand the sum, which can be approached by writing

$$2\sum_{n>0}\int_0^\infty \frac{\cos(2\pi nt)}{x+t^s}\,dt = \sum_{n>0}\int_0^\infty \frac{e^{i2\pi nt}}{x+t^s}\,dt + \sum_{n>0}\int_0^\infty \frac{e^{-i2\pi nt}}{x+t^s}\,dt$$

and proceeding by appropriate contour integrations.

So far, we have only considered the case $s > 1$. But of course, if $0 < s < 1$, then there is a canonical Weierstrass product with zero sequence (n^s). It is easy to see that for $s = 1$ the resulting $F{\restriction}(0,\infty)$ is interdefinable over $\overline{\mathbb{R}}$ with $\Gamma{\restriction}(0,\infty)$, which is known to be definable in the previously-mentioned structure $\mathfrak{R}$. Currently, it is open as to whether $\mathfrak{R}$ defines the canonical product (restricted to $(0,\infty)$) for the sequence (n^s) if $0 < s < 1$.

One can ask about $(\overline{\mathbb{R}}, F)$ (as opposed to the $(\overline{\mathbb{R}}, F{\restriction}[R,+\infty))$, but I am nearly sure it defines $\mathbb{Z}$ (no further assumptions on $\bar{a}$). More interesting, at least potentially, are the $(\overline{\mathbb{R}}, F{\restriction}(-\infty, R])$. Evidently, none of them are o-minimal, and it is easy to see that some of them define $\mathbb{Z}$ simply because their zero sets do (in particular, $\bar{a} = (n^s)$ for any $s > 1$). But perhaps some of them satisfy something like condition (D), say, if $\alpha > 1$ and $\bar{a} = \alpha^n$. To put this another way, what can be said about the expansion of $\overline{\mathbb{R}}$ by $\prod(1-\alpha^{-n}x)$, $x \geq 0$? Of course, $\alpha^{\mathbb{Z}}$ is definable, but what else is (or is not)?

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Counting Rational Points near Definable Sets

Philipp Habegger

The versatile theorem of Pila and Wilkie [3] counts rational points of bounded height that lie on a set that is definable in an o-minimal structure over $\mathbb{R}$. It extends earlier results of Jarník, Bombieri–Pila, and others. Let us fix the height of a rational number a/b with $a, b \in \mathbb{Z}$ coprime integers and $b \geq 1$ to be $\max\{|a|, b\}$. The height of $(q_1, \ldots, q_n)$, where each q_i is rational, is $\max_{1\leq i\leq n} H(q_i)$. Here is a special case of their theorem, for the sake of simplicity we avoid definable sets that contain real semi-algebraic curves.

Theorem 1 (Pila–Wilkie)**.** *Let $X \subset \mathbb{R}^n$ be definable in an o-minimal structure over $\mathbb{R}$ and suppose that X does not contain a real semi-algebraic curve. For all $\epsilon > 0$ there exists a constant $c = c(X, \epsilon) > 0$ such that*

$$\#\{q \in X \cap \mathbb{Q}^n : H(q) \leq T\} \leq cT^{\epsilon}$$

for all $T \geq 1$.

From the point of view of transcendence theory it is natural to ask if a qualitative statement can be made qualitative. For example, Lambert proved that π is irrational. In somewhat cumbersome terms, this can be reformulated as

$$\left|\pi - \frac{a}{b}\right| > 0$$

for all integers a and $b \neq 0$. Mahler made the irrationality of π qualitative by showing

$$\left|\pi - \frac{a}{b}\right| > \frac{1}{b^{42}},$$

at least for $b \geq 1$ large enough.

Following this line of thought motivates the following theorem [2]. Instead of counting points on a definable set, we count points that lie near one. We wish the proximity to be polynomial in terms of the height, just as in Mahler's result. For a point $x \in \mathbb{R}^n$ we let $|x|$ denote the maximum of the absolute value of the coordinates.

Theorem 2. *Let $X \subset \mathbb{R}^n$ be a closed set that is definable in a polynomially bounded o-minimal structure over $\mathbb{R}$ and suppose that X does not contain a real semi-algebraic curve. For all $\epsilon > 0$ there exist constants $c = c(X, \epsilon) > 0$ and $\lambda = \lambda(X, \epsilon) > 0$ such that*

$$\#\left\{q \in \mathbb{Q}^n : H(q) \leq T \text{ and there is } x \in X \text{ with } |q - x| \leq T^{-\lambda}\right\} \leq cT^{\epsilon}$$

for all $T \geq 1$.

We must assume that the ambient o-minimal structure is polynomially bounded to exclude counterexamples coming from the exponential function. For example,

$$\{(x, e^{-1/x}) : x \in (0, 1]\} \cup \{(0, 0)\} \tag{1}$$

is closed and definable in the structure over the reals generated by the graph of the exponential function restricted to the reals (which was identified as being o-minimal by Wilkie). The set (1) contains no real semi-algebraic curve. For an integer $b \geq 1$ the rational point $(1/n, 0)$ has distance at most e^{-n} to (1). This upper bound is less than any fixed power of T^{-1} when $T \leq 2H(1/n, 0) = n$ and n is large enough. Letting n range over integers in $[T/2, T]$ we get roughly $T/2$ rational points of height at most T that approximate (1) to any given fixed power of T^{-1}.

The proof of Theorem 2 follows the general induction scheme laid out by Pila and Wilkie [3]. Indeed, we can also allow X to contain a real semi-algebraic set. But to get the correct counting estimate, we shall avoid those x that are sufficiently close, in terms of T, to a connected real semi-algebraic set of positive dimension contained in X. One stumbling block towards applying the Pila–Wilkie strategy directly is that the constant c in our theorem is not uniform over a family of definable sets. This looks troubling, as the proof strategy requires a uniform statement to complete an induction step. The failure to being uniform can be traced back to the fact that the Łojasiewicz inequality is not uniform in a suitable sense.

Let us consider a simple example to see how uniformity fails in the Łojasiewicz inequality. We consider $X = [-2,2]^2$ and choose the function $f(y,x) = x^2+y^2-1$. The zero set of f is the unit circle Z. If $(y,x) \in X$ is such that $|f(y,x)|$ is small, then Łojasiewicz's Inequality, as in 4.14(2) [1], implies that (y,x) is close to Z. In fact, the distance $\mathrm{dist}((y,x),Z)$ of (y,x) to Z satisfies

$$\mathrm{dist}((y,x),Z) \le c|x^2+y^2-1|^{\delta}$$

for constants $c > 0$ and $\delta > 0$ that are independent of (y,x).

Things change when we consider $[-2,2]^2$ as a family parametrized by the first coordinate. We restrict $f(y,x)$ to the fiber $[-2,2]$ above y. The zero-set Z_y of f restricted to the fiber are the possible roots of x^2+y^2-1 as a polynomial in x.

By an application of the Łojasiewicz inequality mentioned above there are constants $c_y > 0$ and $\delta_y > 0$ such that

$$\min\{1, \mathrm{dist}(x, Z_y)\} \le c_y|x^2+y^2-1|^{\delta_y} \tag{2}$$

for all $(y,x) \in [-2,2]$; the distance is $+\infty$ if $Z_y = \emptyset$ and then the minimum is 1.

If $y > 1$ is fixed, $x \mapsto x^2+y^2-1$ does not vanish on $[-2,2]$, and so the left-hand side of (2) is 1. But for $x = 0$ the value $|y^2-1|$ is arbitrarily small as y approaches 1 from the right. Therefore, c_y and δ_y cannot *both* be bounded from below by a positive constant that is independent of y. At the core, the problem is that the modified distance $(y,x) \mapsto \min\{1, \mathrm{dist}(x,Z_y)\}$ is not continuous on $[-2,2]^2$. Indeed, Z_y is empty for $y > 1$ but non-empty if $y \in [-1,1]$. So on restricting to $x = 0$, the modified distance jumps from 1 to 0 as y converges to 1 from the right.

We remedy this problem by working with a replacement of Łojasiewicz's inequality that is tailored to our problem. Roughly speaking, when working in families, we need to have the freedom to jump to a nearby fiber. The number of possible fibers is bounded uniformly over the family.

If X is a subset of $\mathbb{R}^m \times \mathbb{R}^n$ and $y \in \mathbb{R}^m$, then we let X_y denote the projection of $X \cap \{y\} \times \mathbb{R}^n$ to $\mathbb{R}^n$. Here is a special case of our result (in fact we need to allow f to vary over polynomials of fixed degree as well).

Proposition 1. *Let f be a polynomial with real coefficients and in n variables. Let $X \subset \mathbb{R}^m \times \mathbb{R}^n$ be compact and definable in a polynomially bounded o-minimal structure. There exist $c = c(Z) \in (0,1]$ and a rational number $\delta = \delta(Z) \in (0,1]$ with the following property. If $y \in \mathbb{R}^m$ there are $y_1,\dots,y_N \in \mathbb{R}^m$ with $N \le c^{-1}$ such that for all $x \in Z_y$ with $|f(x)| \le c$ there is $i \in \{1,\dots,N\}$ and $x' \in Z_{y_i}$ with $f(x') = 0$ and $|(y_i,x')-(y,x)| \le |f(x)|^{\delta}$.*

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Curve-rational functions

Krzysztof Kurdyka
(joint work with János Kollár and Wojciech Kucharz)

In my talk I have explained some results from [1]. Let $X \subset \mathbb{R}^n$ be an algebraic set. We are interested in real-valued functions, defined on some subset of X, that are restrictions of regular functions or rational functions on X. First let me recall the precise definition.

Let $f\colon W \to \mathbb{R}$ a function defined on some subset $W \subset X$. We say that f is *regular at a point* $x \in W$ if and only if there exist two polynomials $p, q \in \mathbb{R}[x_1, \dots, x_n]$ such that $q(x) \neq 0$ and $f = p/q$ on $W \cap \{q \neq 0\}$. Moreover, f is called a *regular function* if it is regular at every point in W

Denoting by Y the Zariski closure of W in X, we see that f is regular at x if and only if $f|_{W\cap Y_x} = F_x|_{W \cap Y_x}$ for some regular function F_x defined on a Zariski open neighborhood $Y_x \subset Y$ of x.

We say that f is a *rational function* if there exist a Zariski open dense subset $Y^0 \subset Y$ and a regular function F on Y^0 with $f|_{W\cap Y^0} = F|_{W\cap Y^0}$. Clearly, each regular function on W is also a rational function.

While the definition makes sense for an arbitrary subset W, it is sensible only if W contains a sufficiently large portion of Y. The key examples of interest are open subsets and semialgebraic subsets, in particular the case $W = X$.

We are mainly interested in *continuous rational functions* on W, that is, continuous functions (for the Euclidean topology) which are also rational.

The function $f\colon \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(x,y) = \frac{x^3}{x^2+y^2} \quad \text{for } (x,y) \neq (0,0) \quad \text{and} \quad f(0,0) = 0,$$

is continuous rational but it is not regular at $(0,0)$.

The function $g(x,y) = 1/(1+x^2+y^2)$ is regular on $\mathbb{R}^2$.

Consider the curve $C = \{(x^3 - y^2 = 0\} \subset \mathbb{R}^2$ and the functions f, g defined on C by

$$f(x,y) = \frac{y}{x} \quad \text{for } (x,y) \neq (0,0) \quad \text{and} \quad f(0,0) = 0,$$
$$g(x,y) = \frac{x}{y} \quad \text{for } (x,y) \neq (0,0) \quad \text{and} \quad g(0,0) = 0.$$

Then f is continuous rational, whereas g is rational but it is not continuous at $(0,0)$.

Regular functions on $W = X$ are, of course in common use in algebraic geometry. On the other hand, continuous rational functions on $W = X$ have only recently become the object of serious research, see e.g., [2, 3, 4, 5] and [1] for an exhaustive bibliography on the subject.

Several examples discussed in [2, 4] show that continuous rational functions on $W = X$ behave in a rather unusual way. To eliminate some unexpected and undesirable phenomena a more restrictive notion of rational function was introduced in [2]. Following [2] we say that a function $f\colon W \to \mathbb{R}$ is *hereditarily rational* if for every real algebraic subset $Z \subset X$, the restriction $f|_{W\cap Z}$ is a rational function.

If X is nonsingular, then every continuous rational function on $W = X$ is hereditarily rational [2, Proposition 8], hence it is *regulous* in the sense of [3]. It is not the case for singular varieties. We now recall [2, Example 2]. The algebraic surface

$$S := (x^3 - (1 + z^2)y^3 = 0) \subset \mathbb{R}^3$$

is an analytic submanifold of $\mathbb{R}^3$ and the function $f\colon S \to \mathbb{R}$, defined by $f(x, y, z) = (1 + z^2)^{1/3}$, is analytic and semialgebraic. Furthermore, f is a continuous rational function on S since $f(x, y, z) = x/y$ on S without the z-axis. On the other hand, f restricted to the z-axis is not a rational function. Thus f is not hereditarily rational.

It turns out that hereditarily rational functions can be characterized by restrictions to irreducible real algebraic curves.

A function $f\colon W \to \mathbb{R}$ is said to be *rational on algebraic curves* if for every irreducible real algebraic curve $C \subset X$, the function $f|_{W\cap C}$ is rational. If, in addition, $f|_{W\cap C}$ is continuous, then f is said to be *continuous rational on algebraic curves* or *curve-rational* for short.

Our main result on curve-rational functions is the following.

Theorem 1. *Let X be a real algebraic set and let $W \subset X$ be a subset that is either open or semialgebraic. For a function $f\colon W \to \mathbb{R}$, the following conditions are equivalent:*

(1) *f is continuous and hereditarily rational.*
(2) *f is curve-rational.*

A function on $\mathbb{R}^n$ that is rational on algebraic curves need not be rational.

In [1, Section 4] we give a detailed description of relationships between hereditarily rational functions (not necessarily continuous) and functions rational on algebraic curves.

It is convenient to have the following local variants of the previous notions. A function $f\colon W \to \mathbb{R}$ is said to be *continuous rational on algebraic arcs* or *arc-rational* for short if for every point $x \in W$ and every irreducible real algebraic curve $C \subset X$, with $x \in C$, there exists an open neighborhood $U_x \subset W$ of x such that $f|_{U_x\cap C}$ is a continuous rational function.

Clearly, any curve-rational function is arc-rational. The converse does not hold for a rather obvious reason. For instance, consider the hyperbola $H := (xy - 1 =$

$0) \subset \mathbb{R}^2$. Any real-valued function on H that is constant on each connected component of H is arc-rational, but it must be constant to be rational.

Our main result on arc-rational functions concerns functions defined on connected open sets that avoid singularities. Let X be a real algebraic set. We say that an open subset $U \subset X$ is *smooth* if it is contained in $X \setminus S(X)$, where $S(X)$ stands for the singular locus of X.

Theorem 2. *Let X be a real algebraic set and let $U \subset X$ be a connected smooth open subset. For a function $f\colon U \to \mathbb{R}$, the following conditions are equivalent:*

(1) *f is continuous and hereditarily rational.*
(2) *f is arc-rational.*

The main properties of arc-rational functions on semialgebraic sets can be summarized as follows.

Theorem 3. *Let X be a real algebraic set and let $f : W \to \mathbb{R}$ be an arc-rational function defined on a semialgebraic subset $W \subset X$. Then f is continuous and there exists a sequence of semialgebraic sets*

$$W = W_0 \supset W_1 \supset \ldots \supset W_m = \emptyset$$

which are closed in W, such that f is a regular function on each connected component of $W_i \setminus W_{i+1}$, for $i = 0, \ldots, m-1$. In particular, f is a semialgebraic function.

We also establish a connection between arc-rational functions and, introduced earlier in [6], arc-analytic functions. A function $\varphi\colon V \to \mathbb{R}$, defined on a real analytic variety V, is said to be *arc-analytic* if $\varphi \circ \eta$ is analytic for every analytic arc $\eta\colon (-1, 1) \to V$.

Theorem 4. *Let X be a real algebraic set and let $f : W \to \mathbb{R}$ be an arc-rational function defined on an open subset $W \subset X$. Then f is continuous and arc-analytic.*

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Hölder-Łojasiewicz inequalities for volumes of tame objects

Ta Lê Loi

Let $h : K \to \mathbb{R}^n$ be a continuous subanalytic map, where $K \subset \mathbb{R}^m$ is compact. Then the Hölder-Łojasiewicz inequality gives the following estimation

$$\|h(x) - h(y)\| \leq C\|x - y\|^\alpha, \ \forall x, y \in K,$$

where $C, \alpha > 0$.
Geometrically, when K is convex, then

$$\text{length}(h([x, y])) \leq C_1 \text{length}([x, y])^\alpha, \ \forall x, y \in K.$$

Besides, when $\text{int}(\{x \in K : \|h(x)\| = t\}) = \emptyset$, for all $t \geq 0$, we also have

$$\text{Volume}(\{x \in K : \|h(x)\| \leq t\}) \leq C_2 t^\beta = C_2 \text{length}([0, t])^\beta, \forall t \geq 0,$$

for some $C_2, \beta > 0$.

We are interested in the generalizations of the above estimations for volumes of images or pre-images of families of k-dimensional surfaces under certain mappings, via the volumes of the families involved.

• Clearly, in general, we can not get any useful estimation, e.g. we can meet phenomena like spirals, oscillations, fractals, or worse, Peano's curves.
• However, if the objects involved are tame then their properties can imply some useful inequalities. In this talk, we present some.
• The considering problem relates to some of others', among them are [5], [9], [2], [1], [12],[8], ...
• Since the parameterized integration of a family of functions definable in a structure is not in general belong to the same structure, one can not directly estimate the volumes of families of definable sets by integration.

To overcome this obstruction, we use:
- Tame properties of objects definable in o-minimal structures (such as certain uniform bounds for definable families, L- cell decompositions in families, Preparation and Parameterized rectilinearization of functions definable in polynomially bounded structures, certain stratification of definable sets, ...), see [3], [6], [10], [1], [7].
- Technics in Geometric Integration Theory (such as the area formula, the coarea formula, areas of projections, ...), see [4], [8].

In this talk we fix an o-minimal structure on $(\mathbb{R}, +, \cdot)$. "Definable" means definable in the structure. Let Φ denote the set of all odd, strictly increasing continuous definable bijection from $\mathbb{R}$ onto $\mathbb{R}$. We call $(S_t)_{t \in T}$ *definable family* of subsets of K if there is $S \subset T \times K$ be definable set and $S_t = \{x \in K : (t, x) \in S\}$, for $t \in T$. For each subset X of $\mathbb{R}^m$, let $\mathcal{H}^k(X)$ denote the *k-dimensional Hausdorff measure of X*.

The main results we give in this talk are the following.

Theorem 1 (Answers for the three questions of [11]). *Let $f : A \to \mathbb{R}^n$ be a continuous definable map on bounded set A.*
Then there exists a finite partition $f(A) = \cup_{i \in I} Z_i$, where Z_i's are definable subsets,

satisfying the following properties:
(i) For each $i \in I$, there exist complex K_i and definable homeomorphism $h_i : Z_i \times |K_i| \to f^{-1}(Z_i)$, such that $h_i(y, |K_i|) = f^{-1}(y)$, $\forall y \in Z_i$. In particular, the numbers of simplexes of the triangulations of the fibers of f are uniformly bounded.
(ii) For each $k \in \{0, \cdots, \dim A\}$, there exists $M_k > 0$ such that

$$\mathcal{H}^k(S_k(f^{-1}(y)) < M_k,\ \forall y \in f(A),$$

where $S_k(f^{-1}(y)) = \bigcup_{\Delta \in K_i, \dim \Delta \leq k} h_i(y, |\Delta|)$ (the k-skeleton).
(iii) Let

$$F^2(f) = \{(x, x') : x, x' \text{ are in a connected component of } f^{-1}(f(x))\}.$$

Then there exist $M > 0$ and definable map $\gamma : F^2(f) \times [0, 1] \to A$, such that $\gamma(x, x', 0) = x$, $\gamma(x, x', 1) = x'$, $\gamma(x, x', [0, 1]) \subset f^{-1}(f(x))$, and

$$\mathcal{H}^1(\gamma(x, x', [0, 1])) < M,\ \forall x, x' \in F^2(f).$$

Theorem 2 (Volumes of images)**.** *Suppose that the structure is polynomially bounded. Let $h : K \to \mathbb{R}^n$ be a continuous definable map on compact K. Then there exists $\alpha \in \Lambda$, $\alpha > 0$ satisfying the following:*
For any definable family $(S_t)_{t \in T}$ of subsets of K with $\dim S_t \leq k$, for all $t \in T$, there exists $C > 0$ such that

$$\mathcal{H}^k(h(S_t)) \leq C(\mathcal{H}^k(S_t))^\alpha, \forall t \in T.$$

Theorem 3 (Volumes of pre-images)**.** *Suppose that the structure is polynomially bounded. Let $h : K \to \mathbb{R}^n$ be a continuous definable map on compact $K \subset \mathbb{R}^m$. Let $(S_t)_{t \in T}$ be a definable family of subsets of $\mathbb{R}^n$ with $\dim S_t \leq k, \forall t \in T$. For each $d \in \{0, \dots, \dim K\}$, let $F_d(h) = \{y \in \mathbb{R}^n : \dim h^{-1}(y) \leq d\}$. Then for each definable closed subset B of $F_d(h)$, there exist $C, \alpha > 0$, such that*

$$\mathcal{H}^{d+k}(h^{-1}(S_t \cap B)) \leq C(\mathcal{H}^k(S_t))^\alpha,\ \forall t \in T.$$

Corollary (Volumes of sub-levels)*. Let $h : K \to \mathbb{R}^n$ be a continuous definable map, and $K \subset \mathbb{R}^m$ be a compact set. Suppose that int$(\{x \in K : \|h(x)\| = t\}) = \emptyset$, $\forall t \geq 0$. Then there exists $\varphi \in \Phi$, such that*

$$\mathcal{H}^m(\{x \in K : \|h(x)\| \leq t\} \leq \varphi(t),\ \forall t \geq 0.$$

In particular, if h is definable in a polynomially bounded structure, then there exist $C, \alpha > 0$ such that

$$\mathcal{H}^m(\{x \in K : \|h(x)\| \leq t\} \leq Ct^\alpha,\ \forall t \geq 0.$$

Conjecture. The analogous inequalities of that of Theorem 2 and Theorem 3 hold in arbitrary structures.

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Wilkie's conjecture for restricted elementary functions

Dmitry Novikov
(joint work with Gal Binyamini)

Let A be a set definable in some o-minimal structure. The Pila-Wilkie theorem (in its basic form) states that the number of rational points in the transcendental part of A grows sub-polynomially with the height of the points. The Wilkie conjecture stipulates that for sets definable in $R_{\exp}$, one can sharpen this asymptotic to polylogarithmic.

First, I describe a complex-analytic approach to the proof of the Pila-Wilkie theorem for subanalytic sets developed in [1]. Using Denef-van den Dries description of subanalytic sets, we reduce to the case of complex analytic set A. The key idea is to replace the traditional Gromov-Yomdin C^r-parameterization of A by covering of A by Weierstrass polydiscs.

Here is the definition: a product $\Delta = \Delta_z \times \Delta_w$ of two polydiscs is a Weierstrass polydiscs of A if $A \cap \{\partial\Delta_w \times \Delta_z\} = \emptyset$ and $\dim A = \dim \Delta_z$. In other words, A does not intersect the boundaries of the fibers of the projection $\pi : \Delta \to \Delta_z$. This implies that the restriction $\pi|_A : A \to \Delta_z$ is a finite map of degree $e(\Delta)$.

For any function f holomorphic in a Weierstrass polydisc of A, its restriction to A coincides with a restriction of a unique Weierstrass polynomials $g(z, w) = g_0(z)w^d + ... + g_d(z)$ of degree $d < e(\Delta)$, with explicit estimates on the norms of its coefficients $g_i(z)$. Using this representation instead of Taylor series, we get the same estimates for interpolation determinant as in the classical proof of

Bombieri-Pila, up to non-essential constant $e(\delta)$. However, the crucial advantage is the coarseness of the Weierstrass polydiscs: unlike Gromov-Yomdin C^r-parameterization, covering by Weierstrass polydiscs can be made uniform for A varying in any finite-dimensional analytic family $\{A_\epsilon\}$. This allows an easy induction by $\dim A$, contrary to the classical Pila-Wilkie proof.

This technique allows to prove a restricted version of Wilkie conjecture. Namely, let $A \subset \mathbb{R}^m$ be $\mathbb{R}^{RE}$-definable, where

$$\mathbb{R}^{RE} = (\mathbb{R}, <, +, \cdot, \exp|_{[0,\pi]}, \sin|_{[0,\pi]})$$

is a structure obtained by adding graphs of exp and sin over interval $[0, \pi]$ to semialgebraic functions.

Theorem 1 ([2]). *There exist integers $\kappa := \kappa(A)$ and $N = N(A, [\mathcal{F} : \mathbb{Q}])$ such that the number of rational points of height at most H lying in the transcendental part of A is at most $N \cdot (\log H)^\kappa$.*

The essential case is of a complex-analytic set A. The key point is that the set A is holomorphic-Pfaffian, and therefore its topological complexity can be effectively bounded in terms of its complexity β in the sense of fewnomials theory of Khovanskii, due to results of Gabrielov and Vorobiev. Using these bounds, we get an upper bound for ϵ-entropy of A, following Vitushkin. This allows to bound the number of Weierstrass polydiscs covering A by at most $poly(\beta)$.

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Density of algebraic points on Noetherian varieties

Gal Binyamini

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and denote by $\mathbf{x} := (x_1, \ldots, x_n)$ a system of coordinates on $\mathbb{R}^n$. A collection of analytic functions $\boldsymbol{\phi} := (\phi_1, \ldots, \phi_\ell) : \bar{\Omega} \to \mathbb{R}^\ell$ is called a (complex) *real Noetherian chain* if it satisfies an overdetermined system of algebraic partial differential equations,

$$\frac{\partial \phi_i}{\partial x_j} = P_{i,j}(\mathbf{x}, \boldsymbol{\phi}), \qquad \begin{array}{c} i = 1, \ldots, \ell \\ j = 1, \ldots, n \end{array}$$

where $P_{i,j}$ are polynomials. We call ℓ the *order* and $\alpha := \max_{i,j} \deg P_{i,j}$ the *degree* of the chain. If $P \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ is a polynomial of degree β then $P(\mathbf{x}, \boldsymbol{\phi}) : \Omega \to \mathbb{R}$ is called a *real Noetherian function* of degree β.

We call the set of common zeros of a collection of real Noetherian functions of degree at most β a *real Noetherian variety* of degree β. We call a set defined by a finite sequence of Noetherian equations or inequalities a *basic semi-Noetherian set*, and a finite union of such sets a *semi-Noetherian* set. We define the *complexity* β of a semi-Noetherian set (more precisely the formula defining it) to be the maximum of the degrees of the Noetherian functions appearing in the definition, plus the total number of relations. Finally, we define the *Noetherian size* of $\boldsymbol{\phi}$, denoted $\mathcal{S}(\boldsymbol{\phi})$, to be

$$\mathcal{S}(\boldsymbol{\phi}) := \max_{x\in\Omega} \max_{\substack{i=1,\dots,\ell \\ j=1,\dots,n}} \{|x_j|, |\phi_i(\mathbf{x})|, \|P_{i,j}\|_\infty\}$$

where $\|P\|_\infty$ denotes the maximum norm on the coefficients of P.

When we say that a quantity can be *explicitly estimated* in terms of the Noetherian parameters, we mean that it admits an explicit upper bound in terms of the parameters $n, \ell, \alpha, \mathcal{S}(\boldsymbol{\phi}), \beta$. Our main result, stated below, is an effective version of the Pila-Wilkie counting theorem [1] for semi-Noetherian sets with the constants explicitly estimated in terms of the Noetherian parameters. Our interest in the question of effectivity in this context is motivated by the applications of the Pila-Wilkie theorem in arithmetic geometry. Namely, we show that many functions of interest in arithmetic geometry fall within the Noetherian class, including elliptic and abelian functions, modular functions and universal covers of compact Riemann surfaces, Jacobi theta functions, periods of algebraic integrals, and the uniformizing map of the Siegel modular variety $\mathcal{A}_g$. We thus effectivize the (geometric side of) Pila-Zannier strategy for unlikely intersections in those instances that involve only compact domains.

Main statements. For a set $A \subset \mathbb{R}^n$ we define the *algebraic part* A^{alg} of A to be the union of all connected semialgebraic subsets of A of positive dimension. We define the *transcendental part* A^{trans} of A to be $A \setminus A^{\text{alg}}$. Recall that the *height* of a (reduced) rational number $\frac{a}{b} \in \mathbb{Q}$ is defined to be $\max(|a|, |b|)$. More generally, for $\alpha \in \mathbb{Q}^{\text{alg}}$ we denote by $H(\alpha)$ its absolute multiplicative height. For a vector $\boldsymbol{\alpha}$ of algebraic numbers we denote by $H(\boldsymbol{\alpha})$ the maximum among the heights of the coordinates. For a set $A \subset \Omega$ we denote the set of $\mathbb{Q}$-points of A by $A(\mathbb{Q}) := A \cap \mathbb{Q}^n$ and denote

$$A(\mathbb{Q}, H) := \{\mathbf{x} \in A(\mathbb{Q}) : H(\mathbf{x}) \le H\}.$$

The following is a basic form of our main theorem, which gives an effective version of the Pila-Wilkie theorem [1] for semi-Noetherian sets.

Theorem 1. *Let $X \subset \Omega$ be a semi-Noetherian set of and $\epsilon > 0$. There exists a constant N, explicitly estimated in terms of the Noetherian parameters, such that for any $H \in \mathbb{N}$ we have*

$$\#X^{\text{trans}}(\mathbb{Q}, H) \le N \cdot H^\epsilon.$$

This theorem is a direct corollary of the following more general statement. First, we consider *algebraic* points of a fixed degree $k \in \mathbb{N}$ instead of rational points. Toward this end we introduce the notation

$$A(k) := \{\mathbf{x} \in A : [\mathbb{Q}(x_1) : \mathbb{Q}], \dots, [\mathbb{Q}(x_n) : \mathbb{Q}] \le k\},$$
$$A(k, H) := \{\mathbf{x} \in A(k) : H(\mathbf{x}) \le H\}.$$

Second, we obtain a more accurate description of the part of X^{alg} where algebraic points of a given height may lie. Toward this end we introduce the following notation.

Definition 1. *Let A, W be two subsets of a topological space. We denote by*

$$A(W) := \{w \in W : W_w \subset A\}$$

the set of points of W such that A contains the germ of W around w, i.e. such that w has a neighborhood U_w with $U_w \cap W \subset A$.

In particular, when $W \subset \mathbb{R}^n$ is a connected positive dimensional semialgebraic set then we have $X(W) \subset X^{\mathrm{alg}}$. The following is the general form of our main theorem, which gives an effective version of the more general form of the Pila-Wilkie theorem established in [2].

Theorem 2. *Let $X \subset \Omega$ be a semi-Noetherian set and $\epsilon > 0$. There exists constants d, N, explicitly estimated in terms of the Noetherian paraemeters, with the following property. For every $H \in \mathbb{N}$ there exist at most NH^ϵ smooth connected semialgebraic sets $\{S_\alpha\}$ of complexity at most d such that*

$$X(k, H) \subset \bigcup_\alpha X(S_\alpha).$$

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O-minimality and Manin's conjecture

Marta Pieropan

(joint work with Christopher Frei)

A conjecture of Manin predicts an asymptotic formula for the number of rational points of bounded anticanonical height on Fano varieties over number fields. I report on a method to verify the conjecture on given varieties that was developed in joint work with C. Frei and successfully applied to a singular del Pezzo surface of degree 4. Further applications are work in progress.

1. Manin's conjecture

Let X be a Fano variety over a number field k. Then the anticanonical bundle on X defines a height function $H : X(k) \to \mathbb{R}_{\geq 0}$ that satisfies the Northcott property, i.e., $\#\{x \in X(k) : H(x) \leq B\} < \infty$ for all $B \in \mathbb{R}_{\geq 0}$. Hence, it is natural to investigate the asymptotic behavior of the function $N_{X,H}(B) := \#\{x \in X(k) : H(x) \leq B\}$.

Conjecture 1 (Manin [7]). *There exists an open subset $U \subseteq X$ such that*

$$N_{U,H}(B) \sim CB(\log B)^{r-1} \quad \text{as} \quad B \to \infty,$$

where C is a constant and r is the rank of the Picard group of X.

The conjecture has been verified for some families of varieties and many single cases using various techniques including analysis of the height zeta function and universal torsor method. See the introduction of [9] for an overview of results and techniques.

2. The method

Unlike other approaches, the universal torsor method for Manin's conjecture does not require geometric restrictions on the varieties it applies to. It consists of two main steps: a parameterization step that involves torsors and Cox rings, and a counting step via ad hoc lattice point counting techniques. The idea behind the method (that is, parameterization followed by lattice point counting) has been widely used to verify Manin's conjecture for many varieties over $\mathbb{Q}$, but only recently it has been developed for varieties over other number fields; see [3, 4, 5, 6, 8, 9, 10]. In in [9] the parameterization step has been investigated in full generality, while the counting step has been worked out for the following variety.

Theorem 1 ([9]). *Let k be a number field. Then Conjecture 1 holds for the surface defined by*

$$x_0x_3 - x_2x_4 = x_0x_1 + x_1x_3 + x_2^2 = 0$$

in $\mathbb{P}^4_k$.

2.1. **Parameterization via torsors.** The parameterization step, inspired by the pioneering work of Salberger [11], is based on the parameterization properties of torsors under tori [2, (2.7.2)] and uses Cox rings to compute the parameterization explicitly. After the parameterization step the verification of the conjecture for the variety under consideration is reduced to counting integral points of bounded height in some open subset of an affine scheme up to the action of a torus of rank r; that is, counting lattice points in a semialgebraic set up to the action of $(\mathcal{O}_k^\times)^r$, where $\mathcal{O}_k^\times$ is the group of units of the ring of integers of k.

2.2. **Fundamental domain and definable sets.** If $\mathcal{O}_k^\times$ is infinite, the choice of a suitable fundamental domain for the action, as in Schanuel [12], leads to counting lattice points in a definable set in the o-minimal structure $\mathbb{R}_{\exp}$ introduced by Wilkie [13]. Theorem 1 is then proven using a careful application of a result of Barroero and Widmer [1] for counting lattice points in definable sets in o-minimal structures.

3. Work in progress

The aim of this project is to study the application of o-minimality to the counting step. The main difficulty is that there is no canonical choice of the fundamental domain. Yet, the choice of a fundamental domain that is good enough to proceed with the counting strongly depends on the geometry of the variety and of the parameterization.

We are working to determine geometric conditions that ensure the existence of a fundamental domain that allows the application of o-minimality. Such conditions would be encoded in the combinatorial data that relate the generators of the Cox ring of the variety and the Picard group, and that determine the $(\mathcal{O}_k^\times)^r$-action. The purpose is to produce new examples of varieties for which Manin's conjecture can be verified via o-minimality techniques and, if possible, to find a systematic way of application.

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Recognizing (ultra)differentiable functions on closed sets

Armin Rainer

Recognizing (ultra)differentiable functions on open sets. Let $f : U \to \mathbb{R}$ be a function defined in an open set $U \subseteq \mathbb{R}^d$. Then f induces a map $f_* : U^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$, $f_*(c) = f \circ c$, whose invariance properties encode the regularity of f:

(i) f is smooth ($\mathcal{C}^\infty$) if and only if $f_*\mathcal{C}^\infty(\mathbb{R}, U) \subseteq \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$; due to [1].
(ii) f is $\mathcal{C}^{k,\alpha}$ if and only if $f_*\mathcal{C}^\infty(\mathbb{R}, U) \subseteq \mathcal{C}^{k,\alpha}(\mathbb{R}, \mathbb{R})$; see [5], [4], [8].
(iii) f is $\mathcal{C}^M$ if and only if $f_*\mathcal{C}^M(\mathbb{R}, U) \subseteq \mathcal{C}^M(\mathbb{R}, \mathbb{R})$, where M is a non-quasianalytic weight sequence; see [9].

By $\mathcal{C}^{k,\alpha}$ ($k \in \mathbb{N}$, $\alpha \in (0, 1]$) we denote the class of $\mathcal{C}^k$-functions whose partial derivatives of order k satisfy a local α-Hölder condition. Let us now define $\mathcal{C}^M$.

Ultradifferentiable functions of class $\mathcal{C}^M$. Let $M = (M_k)$ be a positive sequence. The *Denjoy–Carleman class* $\mathcal{C}^M(U, \mathbb{R}^m)$ is the set of all $f \in \mathcal{C}^\infty(U, \mathbb{R}^m)$ such that for all compact $K \subseteq U$,

$$\exists C, \rho > 0\, \forall n \in \mathbb{N}\, \forall x \in K : \|f^{(n)}(x)\|_{L_n(\mathbb{R}^d,\mathbb{R}^m)} \le C\rho^n n!\, M_n. \tag{1}$$

For the constant sequence $M_k = 1$, we recover the real analytic class $\mathcal{C}^\omega(U, \mathbb{R}^m)$.

We will impose some regularity properties on M: An increasing log-convex sequence $M = (M_k)$ with $M_0 = 1$ is called a *weight sequence*. A weight sequence M is called *non-quasianalytic* if

$$\sum_k \frac{M_k}{(k+1)M_{k+1}} < \infty; \tag{2}$$

otherwise it is said to be *quasianalytic*. We say that M has *moderate growth* if there is a constant $C > 0$ such that $M_{j+k} \le C^{j+k} M_j M_k$ for all j, k.

If M is a weight sequence, then $\mathcal{C}^M$ contains $\mathcal{C}^\omega$ and is stable under composition. By the Denjoy–Carleman theorem, M is non-quasianalytic if and only if there are $\mathcal{C}^M$-functions with compact support. Clearly, (iii) fails for quasianalytic weight sequences M. The moderate growth condition will be important below.

On closed fat sets with Hölder boundary. What about (i), (ii), and (iii) for functions defined in *non-open* subsets $X \subseteq \mathbb{R}^d$? For arbitrary $X \subseteq \mathbb{R}^d$ we define

$$\mathcal{A}^\infty(X) := \{f : X \to \mathbb{R} : f_*\{c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^d) : c(\mathbb{R}) \subseteq X\} \subseteq \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})\},$$
$$\mathcal{A}^M(X) := \{f : X \to \mathbb{R} : f_*\{c \in \mathcal{C}^M(\mathbb{R}, \mathbb{R}^d) : c(\mathbb{R}) \subseteq X\} \subseteq \mathcal{C}^M(\mathbb{R}, \mathbb{R})\},$$
$$\mathcal{A}^\infty_M(X) := \{f : X \to \mathbb{R} : f_*\{c \in \mathcal{C}^M(\mathbb{R}, \mathbb{R}^d) : c(\mathbb{R}) \subseteq X\} \subseteq \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})\}.$$

If $X \subseteq \mathbb{R}^d$ is a non-empty open set, then (i) and (iii) amount to

$$\mathcal{A}^\infty(X) = \mathcal{C}^\infty(X), \quad \mathcal{A}^M(X) = \mathcal{C}^M(X). \tag{3}$$

Clearly, some restrictions on X are necessary if one hopes for identities as in (3) on non-open sets X, not to mention definitions of $\mathcal{C}^\infty$ and $\mathcal{C}^M$. We will say that a non-empty closed subset $X \subseteq \mathbb{R}^d$ is *fat* if it has dense interior, i.e., $X = \overline{\operatorname{int}(X)}$. For such X we define (see also Remark 2 below)

$$(4)\quad \mathcal{C}^\infty(X) := \left\{ f : X \to \mathbb{R} \;\middle|\; \begin{array}{c} f|_{\operatorname{int}(X)} \in \mathcal{C}^\infty, \\ \forall n \in \mathbb{N} : (f|_{\operatorname{int}(X)})^{(n)} \text{ extends continuously to } X \end{array} \right\}.$$

For a weight sequence $M = (M_k)$, let

$$(5)\qquad \mathcal{C}^M(X) := \{ f \in \mathcal{C}^\infty(X) : (1) \text{ holds for all compact } K \subseteq X \}.$$

Question 1. *When do we have $\mathcal{A}^\infty(X) = \mathcal{C}^\infty(X)$ and $\mathcal{A}^M(X) = \mathcal{C}^M(X)$?*

Interestingly, the analogue for finite differentiability (ii) fails even on the closed half-space, which is a consequence of Glaeser's inequality. That the identities in Question 1 are not always true is shown by the following example.

Example 1. *Let $p : [0,\infty) \to [0,\infty)$ be a strictly increasing $\mathcal{C}^\infty$-function which is infinitely flat at* 0. *Consider the* ∞-flat cusp $X = \{(x,y) \in \mathbb{R}^2 : x \ge 0,\ 0 \le y \le p(x)\}$ *and the function $f : X \to \mathbb{R}$ defined by $f(x,y) = \sqrt{x^2+y}$. Then $f \notin \mathcal{C}^\infty(X)$, but $f \in \mathcal{A}^\infty(X)$. The latter follows from a division theorem of* [6].

On the positive side, [7] proved that $\mathcal{A}^\infty(X) = \mathcal{C}^\infty(X)$ holds for convex sets X with non-empty interior. We will extend this result to a larger family of sets.

Let $\mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R}$ with Euclidean coordinates $x = (x', x_d)$. Let $\alpha \in (0,1]$, and $r, h > 0$. Consider the *truncated open cusp*

$$\Gamma_\alpha(r,h) := \{ (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : |x'| < r,\ h(|x'|/r)^\alpha < x_d < h \}.$$

An open set $U \subseteq \mathbb{R}^d$ is said to have the *uniform cusp property of index* α (we write (UCP_α) for short), if for each $x \in \partial U$ there exist $\epsilon, r, h > 0$ and $A \in \mathrm{O}(d)$ such that for all $y \in \overline{U} \cap B(x,\epsilon)$ we have $y + A\Gamma_\alpha(r,h) \subseteq U$.

Remark 1. *A bounded open set $U \subseteq \mathbb{R}^d$ has (UCP_α) if and only if U has equi-α-Hölder boundary; cf.* [3]. *In particular, U has (UCP_1) if and only if it is a Lipschitz domain. If $\alpha < 1$ then the Hausdorff dimension of ∂U can be larger than $d-1$.*

Theorem 1. *Let $M = (M_k)$ be a non-quasianalytic weight sequence. Let $X \subseteq \mathbb{R}^d$ be a closed fat set. If* $\operatorname{int}(X)$ *has* (UCP_α) *for some* α, *then*

$$(6)\qquad \mathcal{A}^\infty(X) = \mathcal{A}^\infty_M(X) = \mathcal{C}^\infty(X).$$

If $\operatorname{int}(X)$ *has* (UCP_1), *then*

$$(7)\qquad \mathcal{A}^M(X) = \mathcal{C}^M(X).$$

On closed fat subanalytic sets. Using rectilinearization of subanalytic sets we obtain the following consequences of Theorem 1.

Theorem 2. *Let $M = (M_k)$ be a non-quasianalytic weight sequence. Let $X \subseteq \mathbb{R}^d$ be a closed fat subanalytic set. There is a locally finite collection of real analytic* *mappings* $\varphi_j : U_j \to \mathbb{R}^d$*, where each* φ_j *is the composite of a finite sequence of local blow-ups with smooth centers and* U_j *is open in* $\mathbb{R}^d$*, such that, for all* j*,*

$$\varphi_j^* \mathcal{A}^\infty(X) \subseteq \mathcal{C}^\infty(\varphi_j^{-1}(X)), \tag{8}$$

$$\varphi_j^* \mathcal{A}^M(X) \subseteq \mathcal{C}^M(\varphi_j^{-1}(X)). \tag{9}$$

If f is $\mathcal{C}^\infty$, φ real analytic, and the composite $f \circ \varphi$ is $\mathcal{C}^M$, then in general f need not be $\mathcal{C}^M$. Under suitable conditions one can however expect that f is $\mathcal{C}^{M^a}$ for some positive integer a independent of M (where $(M^a)_k := (M_k)^a$). Combining a result of [2] (which makes this precise) with Theorem 2 we deduce the following.

Let $M = (M_k)$ be a weight sequence. Let $X \subseteq \mathbb{R}^d$ be a closed fat set. We define

$$\mathcal{A}^{\widehat{M}}(X) := \bigcap_{a>0} \mathcal{A}^{M^a}(X) \quad \text{and} \quad \mathcal{C}^{\widehat{M}}(X) := \bigcap_{a>0} \mathcal{C}^{M^a}(X).$$

Theorem 3. *Let* $M = (M_k)$ *be a weight sequence of moderate growth such that* M^a *is non-quasianalytic for all* $a > 0$*. Let* $X \subseteq \mathbb{R}^d$ *be a closed fat subanalytic set. Then*

$$\mathcal{C}^\infty(X) \cap \mathcal{A}^{\widehat{M}}(X) = \mathcal{C}^{\widehat{M}}(X). \tag{10}$$

Example 2. *The sequence* $M_k = k!$ *satisfies the assumptions of Theorem 3. In that case* $\mathcal{C}^{\widehat{M}}$ *is the intersection of all Gevrey classes.*

Remark 2. *Often a function on a closed set* $X \subseteq \mathbb{R}^d$ *is declared to be* $\mathcal{C}^\infty$ *if it is the restriction of a* $\mathcal{C}^\infty$*-function on* $\mathbb{R}^d$*. For general closed fat sets, this differs from the notion of smoothness defined in* (4)*. But in the cases considered here (i.e.,* $\mathrm{int}(X)$ *has* (UCP_α) *for some* α*, or* X *is subanalytic) the two notions coincide.*

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Sobolev sheaves on subanalytic sites.

Adam Parusiński

We present some results of a recent Astérisque article [1], where a construction of Sobolov sheaves is given.

Sheaves on manifolds are suited to treat local problems, but many spaces one naturally encounters in analysis are not of local nature. For instance the spaces of functions or distributions with the tempered growth at the boundary do not form a sheaf. This is because for arbitrary open subsets U and V of a real analytic manifold M there are no constants C, N such that $dist(x, M \setminus (U \cup V))^N \leq C \max\{dist(x, M \setminus U), dist(x, M \setminus V)\}$. But this property holds, by Łojasiewicz Inequailty, provided U and V are relatively compact and subanalytic. Thus the subanalytic topology (in the sense of Grothendieck) on real analytic manifolds allows one to overcome this difficulty and to define the sheaves of functions or distributions with tempered growth. These sheaves were used by Kashiwara [2] to prove the Riemann-Hilbert correspondence between regular holonomic D-modules and the derive category of constructible complexes of sheaves.

However, the subanalytic topology is still too rough to treat more sophisticated spaces of analysis, such as the Gevrey functions of a given order at the boundary or the Sobolev spaces, for which the growth has to be controlled by the distance to the boundary with a fixed exponent. In order to handle these spaces P. Schapira and J.-P. Schneiders introduced in "Construction of sheaves on the subanalytic site" of [1], the linear subanalytic topology, a refinement of the subanalytic one. Recall that a Grothendieck topology is defined not by giving the open sets but by the admissible open coverings. Thus a finite family U_i of open relatively compact subanalytic subsets of M is a linear covering if there is a constant C such that

$$dist(x, M \setminus (\bigcup_i U_i)) \leq C \max_i dist(x, M \setminus U_i). \tag{1}$$

The complexes of sheaves in the subanalytic linear topology define the objects in the derived category of complexes of sheaves in the classical subanalytic topology. This functor has good properties thanks to the following result proven in the paper "Regular subanalytic cover" of [1].

Theorem 1. *Let U be an open relatively compact subanalytic subset of M. Then there exist a finite cover $U = \bigcup_i U_i$ by open subanalytic sets such that :*

(1) *every U_i is subanalytically homeomorphic to an open n-dimensional ball;*
(2) *the property* (1) *holds.*

The proof of this theorem is based on the regular projection theorem, see [5] and [6]. It is not clear whether this theorem holds in an arbitrary o-minimal structure, even if we assume it polynomially bounded, since its proof in [6] uses Puiseux Theorem with parameters in an essential way.

Let M be a real analytic manifold and let $s \in \mathbb{R}$. In "Sobolev spaces and Sobolev sheaves" of [1], G. Lebeau defines for U open and relatively compact in M, the Sobolev space $H^s(U)$. For U Lipschitz (i.e. with Lipschitz boundary) and

$s = k \in \mathbb{N}$, it coincides with the usual definition $H^k(U) = \{f \in L^2(U); \partial^\alpha f \in L^2(U),$ for all $|\alpha| \leq k\}$. Then the problem of construction of a Sobolev sheaf can be expressed as follows:

Does there exist a complex $\mathcal{H}^s$ of sheaves defined in the derived category of sheaves in the subanalytic Grothendieck topology on M that for U Lipschitz and relatively compact, $\mathcal{H}^s(U)$ coincide with the complex concentrated in degree 0 *and equal to $H^s(U)$?*

A trivial example is $s = 0$ since $U \to H^0(U) = L^2(U)$ is a sheaf (for a finite topology this means that for any two open U and V the Mayer-Vietoris sequence is exact

$$0 \to H^s(U \cup V) \to H^s(U) \oplus H^s(V) \to H^s(U \cap V) \to 0).$$

Similarly, in the subanalytic topology $U \to H^s(U)$ is a sheaf for $s \in]-1/2, 1/2[$. This follows from the following result proven in the paper "Regular subanalytic cover" of [1].

Theorem 2. *The algebra $\mathcal{S}(M)$ is generated by characteristic functions of open subanalytic Lipschitz balls.*

Here $\mathcal{S}(M)$ denotes the algebra of integer valued functions on M generated by charcteristic functions of relatively compact open subanalytic subsets of M. By *an open subanalytic Lipschitz ball* we mean a relatively compact open subanalytic subset of M such that its closure is subanalytically bi-Lipschitz homeomorphic to the unit ball of $\mathbb{R}^n$. The proof of Theorem 2 is based on the regular decompostion theorem, see [6], [3], [7], and therefore, by [4], [8], this theorem holds in an arbitrary o-minimal structure.

The paper "Sobolev spaces and Sobolev sheaves" of [1] contains also the construction of Sobolev sheaves for $s \leq 0$. The proof uses in an essential way Theorem 1 so it is not clear whether this construction generalizes to an arbitrary o-minimal structure. The existence of Soboloev sheaves for $s \geq 1/2$ is still an open problem.

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Unlikely intersections in products of families of elliptic curves

Fabrizio Barroero
(joint work with Laura Capuano)

Let n be an integer with $n \geq 2$ and let E_λ denote the elliptic curve with Legendre equation

$$Y^2 = X(X-1)(X-\lambda).$$

Consider a curve $\mathcal{C} \subseteq \mathbb{A}^{2n+1}$, defined over a number field k with coordinate functions $(x_1, y_1, \dots, x_n, y_n, \lambda)$, λ non-constant, such that for every $j = 1, \dots, n$, the points $P_j = (x_j, y_j)$ lie on the elliptic curve E_λ. As c varies in $\mathcal{C}(\mathbb{C})$, the specialized points $P_j(c) = (x_j(c), y_j(c))$ will be lying on the specialized elliptic curve $E_{\lambda(c)}$. We implicitly exclude the finitely many c with $\lambda(c) = 0$ or 1, since in that case $E_{\lambda(c)}$ is not an elliptic curve. Suppose moreover that there are no integers $a_1, \dots, a_n \in \mathbb{Z}$, not all zero, such that

$$a_1 P_1 + \dots + a_n P_n = O, \tag{1}$$

identically on $\mathcal{C}$. Nevertheless, for some $c \in \mathcal{C}(\mathbb{C})$ new relations might arise over $\mathbb{Z}$ or an eventually larger $\mathrm{End}(E_{\lambda(c)})$.

Theorem 1 (B., Capuano, 2016)**.** *Under the above assumptions, there are at most finitely many $c \in \mathcal{C}(\mathbb{C})$ such that the points $P_1(c), \dots, P_n(c)$ satisfy two independent $\mathbb{Z}$-relations on $E_{\lambda(c)}$.*

We remark that the case $n = 2$ of the above theorem is covered by the main proposition of [6] in the more general setting of a curve defined over $\mathbb{C}$.

Moreover, in [11] Rémond and Viada proved an analogue of Theorem 1 for a power of a constant elliptic curve with CM, where one must allow the coefficients $a_1, \dots, a_n$ in (1) to lie in the larger endomorphism ring. For the general case of powers of a constant elliptic curve, the result follows from works of Viada [13] and Galateau [3]. If $n = 2$ we get a very special case of Raynaud's Theorem [10], also known as the Manin-Mumford Conjecture.

One may ask if finiteness holds if we impose only one relation. This is not the case. Indeed, there are infinitely many λ such that a point with fixed algebraic abscissa is torsion (see Notes to Chapter 3 in [14]). On the other hand, the values of λ such that at least one relation holds are "sparse", as follows from a well-known theorem of Silverman [12] which implies that the absolute Weil height of such values is bounded. In particular, there are at most finitely many c yielding one relation in a given number field or of bounded degree over $\mathbb{Q}$.

Our proof follows the general strategy introduced by Pila and Zannier in [9] and used by Masser and Zannier in various articles, e.g. [6]. In particular, we consider the elliptic logarithms $z_1, \dots, z_n$ of $P_1, \dots, P_n$ and the equations

$$z_j = u_j f + v_j g,$$

for $j = 1, \dots, n$, where f and g are suitably chosen basis elements of the period lattice of E_λ. If we consider the coefficients u_j, v_j as functions of λ and restrict them to a compact set, we obtain a subanalytic surface S in $\mathbb{R}^{2n}$. The points of $\mathcal{C}$ that yield two independent relations on the elliptic curve will correspond to points of S lying on linear varieties defined by equations of some special form and with integer coefficients. In case $n = 2$, one faces the simpler problem of counting rational points with bounded denominator in S. For this, a previous result of Pila [7] suffices together with the fact that the surface is "sufficiently" transcendental. In the general case we adapted ideas of Pila building on a previous work of him [8] and obtained an upper bound of order T^ϵ for the number of points of S lying on subspaces of the special form mentioned above and rational coefficients of absolute value at most T, provided S does not contain a semialgebraic curve segment. Under the hypothesis that no identical relation holds on $\mathcal{C}$, using a result of Bertrand [1], we are able to show that there are no such semialgebraic curve segments.

Now, to conclude the proof, we use works of Masser [4], [5] and David [2] and exploit the boundedness of the height to show that the number of points of S considered above is of order at least T^δ for some $\delta > 0$. Comparing the two estimates leads to an upper bound for T and thus for the coefficients of the two relations, concluding the proof.

Now, as mentioned above, we might have relations arising over a CM ring. More recently we proved the following.

Theorem 2 (B., 2017)**.** *Under the assumptions of Theorem 1, there are at most finitely many $c \in \mathcal{C}(\mathbb{C})$ such that $E_{\lambda(c)}$ has complex multiplication and there exists $(a_1, \ldots, a_n) \in \operatorname{End}\left(E_{\lambda(c)}\right)^n \setminus \{0\}$ with*

$$a_1 P_1(c) + \cdots + a_n P_n(c) = O.$$

The proof of this statement follows the same Pila-Zannier strategy but Silverman's Theorem cannot be applied in this case. One then uses the theory of complex multiplication together with o-minimality and the above mentioned results of Masser and David to prove that the absolute value of the discriminant of $\operatorname{End}(E_{\lambda(c)})$ is uniformly bounded and thus obtaining the claim of Theorem 2.

Now we can formulate a result that contains the two theorems above.

Let $\mathcal{E} \to S$ be a non-isotrivial elliptic scheme over an irreducible, smooth, quasi-projective curve S, both defined over $\overline{\mathbb{Q}}$. Moreover, let $\mathcal{A} \to S$ be its n-fold fibered power. An irreducible subvariety of $\mathcal{A}$ is called special if it is an irreducible component of an algebraic subgroup of a CM fiber or an irreducible component of a flat subgroup scheme of $\mathcal{A}$. A subgroup scheme is called flat if every irreducible component dominates the base curve S.

Theorem 3. *Let $\mathcal{A}$ be as above and let $\mathcal{C}$ be an irreducible curve in $\mathcal{A}$ defined over $\overline{\mathbb{Q}}$ and not contained in a proper special subvariety of $\mathcal{A}$. Then $\mathcal{C}$ intersects at most finitely many special subvarieties of $\mathcal{A}$ of codimension at least 2.*

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Unlikely Intersections in families of abelian varieties and the polynomial Pell equation

Laura Capuano

(joint work with Fabrizio Barroero)

Fix a number field k and a smooth irreducible curve S defined over k. We consider a non-constant family $A \to S$ of abelian varieties defined over k of relative dimension g. This means that, for each $s \in S(\mathbb{C})$, the corresponding fiber A_s will be an abelian variety defined over $k(s)$ of dimension g. We may also think of this family as an abelian variety A defined over the function field $k(S)$. In [1] and [2] we studied the case of families of elliptic curves. In this talk, we will focus on the case $g \geq 2$. Moreover, we assume that the generic fiber of the family is simple, i.e., it does not contain any non-zero proper abelian subvariety.

Let $n \geq 1$ be an integer. We denote by A^n the n-fold fibered power $A \times_S \cdots \times_S A$ of A. As for A, we consider A^n as a family $A^n \to S$. Now, let $\mathcal{C}$ be a non-singular irreducible curve in A^n, also defined over k, and suppose that $\pi(\mathcal{C})$ dominates the base curve S, where π is the structural morphism $\pi : A^n \to S$. Generically, $\mathcal{C}$ will define a point in $A^n(k(\mathcal{C}))$ or, equivalently, n points $P_1, \ldots, P_n \in A(k(\mathcal{C}))$, while, for any $\boldsymbol{c} \in \mathcal{C}(\mathbb{C})$, we will have a specialized point of $A^n_{\pi(\boldsymbol{c})}(k(\boldsymbol{c}))$ or n points $P_1(\boldsymbol{c}), \ldots, P_n(\boldsymbol{c}) \in A_{\pi(\boldsymbol{c})}(k(\boldsymbol{c}))$.

Let R be the endomorphism ring of A. Note that this might be defined over a finite extension of $k(S)$, rather than over $k(S)$ itself. Every element of R specializes to an element of $\mathrm{End}(A_s)$ and this specialization map is injective, at least outside a finite number of points of S (see [6]). For our purposes, we can assume that there are no such points in S. By abuse of notation, we will denote again by R the specialization of $\mathrm{End}(A)$ in the various fibers. Note that, for some $s \in S(\mathbb{C})$, one may have $R \subsetneq \mathrm{End}(A_s)$.

The points $P_1, \dots, P_n$ defined by $\mathcal{C}$ might or might not satisfy one or more linear relations of the form $\rho_1 P_1 + \cdots + \rho_n P_n = O$, for some $\rho_1, \dots, \rho_n \in R$, where O is the origin of A. If they do, then clearly the same relations hold for all specializations $P_1(\boldsymbol{c}), \dots, P_n(\boldsymbol{c})$ in $A_{\pi(\boldsymbol{c})}(k(\boldsymbol{c}))$. On the other hand, for some specializations, some new relations might arise, with coefficients in R or in the possibly larger $\mathrm{End}\left(A_{\pi(\boldsymbol{c})}\right)$.

In this talk we consider the case in which no generic relation holds and prove that there are at most finitely many specializations such that the points satisfy a relation with coefficients in R. The main result is the following:

Theorem 1. *Let A^n and $\mathcal{C}$ be as above and suppose that the points $P_1, \dots, P_n$ are linearly R-independent. Then there are at most finitely many $\boldsymbol{c} \in \mathcal{C}(\mathbb{C})$ such that there exist $\rho_1, \dots, \rho_n \in R$, not all zero, with*

$$\rho_1 P_1(\boldsymbol{c}) + \cdots + \rho_n P_n(\boldsymbol{c}) = O,$$

on $A_{\pi(\boldsymbol{c})}$.

In case $n = 1$ one has a single point which is not generically torsion. Under this hypothesis, there are at most finitely many specializations such that the point is torsion. This was proved by Masser and Zannier in [7] for the case $g = 2$ and later for any $g > 1$. Analogous problems have been studied in the case of constant abelian varieties defined over the algebraic numbers. Ratazzi in [8] proved the theorem in case A is a simple abelian variety with complex multiplication. The proof of the Zilber-Pink Conjecture for a curve in any abelian veriety defined over a number field due to Habegger and Pila [5] removes Ratazzi's hypothesis on the endomorphism ring of A.

As an application of Theorem 1, we discuss a function field variant of the classical Pell equation. As it is commonly known, this is an equation of the form $A^2 - DB^2 = 1$, to be solved in integers $A, B \neq 0$, where D is a positive integer. A famous theorem of Lagrange ensures that such an equation is solvable if and only if D is not a perfect square in $\mathbb{Z}$.

To obtain a polynomial analogue we replace $\mathbb{Z}$ with $K[X]$, for K a field to be specified later. For $D = D(X) \in K[X]$ of even degree $2d > 0$, we search for non-trivial solutions of $A^2 - DB^2 = 1$, where $A(X), B(X) \in K[X]$ and $B \neq 0$. For a survey on Pell equations in polynomials and related questions, see [9].

The problem in the polynomial case is more complicated, and depends heavily on the choice of the field K. When K is a finite field (of char $\neq 2$), the theory is completely analogous to the integer case, giving necessary and sufficient conditions for solvability. Here we consider K an algebraically closed field of characteristic 0, such as $\overline{\mathbb{Q}}, \overline{\mathbb{C}}$ or $\overline{\mathbb{C}(t)}$. We will call *Pellian* the polynomials $D(X)$ such that the associated Pell equation has a non-trivial solution.

A necessary condition for $D(X)$ to be Pellian is that $D(X)$ is not a square and has positive even degree $2d$. However, unlike the classical case, these conditions are also sufficient only if D has degree 2 while, for higher degrees, there are examples of non-square polynomials which are not Pellian. Indeed, the problem is equivalent to study whether a certain point has finite or infinite order in the Jacobian of a of a non-singular model of the hyperelliptic curve defined by $Y^2 = D(X)$.

As an application of their main result in [7] Masser and Zannier investigate the problem for the one-parameter family $D_t(X) = X^6 + X + t$. Clearly, if the family is identically Pellian, then D_{t_0} is Pellian for every specialization $t_0 \in \mathbb{C}$. It can be proved that this family is not identically Pellian, but the polynomial becomes Pellian for some specializations of the parameter t (for example for $t = 0$ we have $(2X^5 + 1)^2 - (X^6 + X)(2X^2)^2 = 1$ hence D_0 is Pellian). However, these are only "few exceptions", and they prove in fact that the set of values of the parameter for which the specialized polynomial is Pellian is actually finite. Of course, there is nothing special about the family $X^6 + X + t$; indeed, the result is true for any non-identically pellian squarefree $D \in \overline{\mathbb{Q}(t)}[X]$ of even degree at least 6 and such that the Jacobian of the curve $Y^2 = D(X)$ is simple. Analogous results for non-squarefree D appear in [3], [4] and in Harry Schmidt PhD thesis.

As an application of Theorem 1, here we consider an irreducible curve S defined over a number field k, and denote by K its function field $k(S)$. Let us also consider $D(X) \in K[X]$ a squarefree polynomial of degree $2d > 0$ and by $F(X) \in K[X]$ a non-constant polynomial of degree m. We want to study the solutions of the *"almost Pell equation"*, i.e.,

$$A^2 - DB^2 = F, \tag{1}$$

where $A, B \in K[X]$. We call trival a possible solution with $B = 0$. Note that this can happen if and only if F is a square. As before, if the equation (1) is identically solvable, then it will remain solvable for almost every point $\boldsymbol{s} \in S(\mathbb{C})$ (and the solutions will be nothing but the specializations of the general solutions). On the other hand, if it is not identically solvable, then we can still have points on the curve such that the specialized equation $A^2 - D_{\boldsymbol{s}_0}B^2 = F_{\boldsymbol{s}_0}$ has a solution in $\mathbb{C}[X]$, where we denote by $D_{\boldsymbol{s}_0}$ and $F_{\boldsymbol{s}_0}$ the polynomials in $k(\boldsymbol{s}_0)[X]$ obtained specializing the coefficients of D and F in $\boldsymbol{s}_0$. Analogously to the Pellian case, the existence of a non-trivial solution translates into the existence of certain linear relations on the Jacobian J_D of a non-singular model of the hyperelliptic curve defined by $Y^2 = D(X)$. From Theorem 1, one can deduce the following:

Theorem 2. *Let S, D and F be as above, with $d > 1$. Assume that J_D is identically simple and that $End(J_D) = \mathbb{Z}$. Then, if the equation $A^2 - DB^2 = F$ is not identically solvable, there are at most finitely many $\boldsymbol{s}_0 \in S(\mathbb{C})$ such that the specialized equation $A^2 - D_{\boldsymbol{s}_0}B^2 = F_{\boldsymbol{s}_0}$ has a solution.*

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Pfaffian functions and elliptic functions

Gareth Jones

(joint work with Harry Schmidt)

Khovanskii proved zero estimates for a class of real functions called *pfaffian functions*. These are functions satisfying differential equations of a certain form. For the precise definitions, see either the survey [1] by Gabrielov and Vorobojov, or Margaret Thomas's abstract in this collection. It is important to note that Khovanskii's theory concerns real functions. For it to apply to complex functions, we would need to know that the real and imaginary parts of the functions involved are pfaffian, as functions of two real variables.

Our results concern the Weierstrass elliptic functions $\wp, \zeta$ and σ associated to a lattice Ω in the complex plane. Macintyre [2] proved that the real and imaginary parts of $\wp$ are pfaffian, on suitable domains, and also that, on suitable domains, ζ can be defined using pfaffian functions. A similar result holds for σ. Building on Macintyre's work, we give explicit uniform definitions, in terms of pfaffian functions, for $\wp$ and ζ, on a uniform choice of fundamental domain for Ω. We prove that no such result holds for σ but we do give a uniform definition for

$$\phi(z) = \exp(-\frac{1}{2}z^2\eta_1/\omega_1 + \pi iz/\omega_1)\sigma_\Omega.$$

Here ω_1 is a certain period of Ω (see below) and η_1 is the associated quasi-period.

The precise statements of our results are rather cumbersome, so we do not give them here. Instead, we give a sample application. Let ω_1 and ω_2 be periods in Ω

such that $\tau = \frac{\omega_1}{\omega_2}$ lies in the upperhalf plane, with modulus at least 1 and real part bounded in modulus by $\frac{1}{2}$. Let $\mathcal{F}_\Omega = \{r_1\omega_1 + r_2\omega_2 : r_1, r_2 \in [0,1),\ \text{not both } 0\}$.

Theorem 1. [1] *Suppose that P is a polynomial in two variables, with complex coefficients, not identically zero and of total degree bounded by T. Then, on $\mathcal{F}_\Omega$, the function $g(z) = P(z, \wp(z))$ has at most*

$$7.4959 \times 10^{14} T^{11}$$

zeroes.

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Maximal compact subgroups of algebraic groups

Harry Schmidt
(joint work with Gareth Jones)

The maximal compact subgroup Γ of a complex commutative algebraic group G is a real analytic subgroup that contains all torsion points. If G is an algebraic torus it is a product of circles and in fact semi-algebraic. If G is an abelian variety $\Gamma = G$ and also semi-algebraic. However for G a non-split extension of an elliptic curve by $\mathbb{G}_a$, Γ is a "highly transcendental" variety of real dimension 2 and Corvaja, Masser and Zannier showed that the intersection of any curve with Γ is finite [1]. This can be viewed as a kind of sharpening of a Manin-Mumford statement. In their article they also asked for an effective refinement of their result and speculated on possible generalizations. In my talk I presented joint work with Gareth Jones in which we prove an explicit and uniform version of a generalization of their result to products of elliptic curves. The proof heavily relies on a theorem of Ax as well as Khovanskii's zero-estimates for Pfaffian functions.

In order to show the resemblence to a Manin-Mumford statement we display the theorem.

Theorem 1. *Let G be a universal vectorial extension of a product of g elliptic curves defined over the complex numbers. Let Γ be the maximal compact subgroup of G and V an algebraic subvariety of G of dimension at most g. Then the there exists a postive integer N such that*

$$V \cap \Gamma \subset \cup_{i=1}^{N} (H_i + a_i),$$

[1] This is work in progress: numbers can go up as well as down.

where H_i is an anti-affine subgroup of G of dimension strictly less then $2g$ and $a_i \in \Gamma$.

Here N depends polynomially on the degree $\deg(V)$ of V and in an explicit manner. For example, we computed that

$$N \leq 10^{250g^2} g^{51g^2} \max\{3, \deg(V)\}^{19g}.$$

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Effective Pila–Wilkie bounds for restricted Pfaffian surfaces

Margaret E. M. Thomas

(joint work with Gareth O. Jones)

The counting theorem of Pila and Wilkie [7] has become celebrated as one of the most important developments in o-minimality in recent years. It provides a subpolynomial bound on the number of rational points of bounded height lying on the 'transcendental parts' of sets definable in o-minimal expansions of the real field. It may be more precisely stated as follows. Suppose that $X \subseteq \mathbb{R}^n$ is a set definable in an o-minimal expansion of the real field. Set X^{alg}, its *algebraic part*, to be the union of all connected, infinite, semi-algebraic subsets of X, and set X^{trans}, its *transcendental part*, to be the complement of X^{alg} in X. Given a rational point $\overline{q} = (\frac{a_1}{b_1}, \dots, \frac{a_n}{b_n}) \in \mathbb{Q}^n$, where $\gcd(a_i, b_i) = 1$ for each $i = 1, \dots, n$, the height of $\overline{q}$ is $\max_{1 \leq i \leq n}\{|a_i|, |b_i|\}$. The Pila–Wilkie Theorem states that, for any positive real number ϵ and any $H \geq 1$, there are at most cH^{ϵ} rational points of height at most H lying on X^{trans}, where c is a positive real number depending on X and ϵ.

In fact, Pila and Wilkie proved several stronger statements, including the provision of a constant c which is uniform across the fibres of a definable family $Z \subseteq \mathbb{R}^m \times \mathbb{R}^n$. Analogous bounds were moreover established by Pila in [10] for algebraic points of bounded height and degree, where the constant c depends on X, ϵ and a bound k on the degree of the algebraic points. These results all share a common feature with the earlier work of Pila [8, 9] on subanalytic surfaces, in that the proof does not provide a method for computing the constant c in terms of ϵ, some definition of X and, if applicable, k. Indeed, at the level of generality of sets definable in o-minimal expansions of the real field, such an effective constant cannot be obtained; this is not even possible for the graphs of all one-variable, transcendental, restricted analytic functions. However, in certain cases, say when X can be defined using functions satisfying some reasonable algebraic differential equations, the question is valid, and indeed is interesting in view of the many applications of the Pila–Wilkie Theorem to diophantine geometry.

The main result presented, Theorem 1 below, is an effective version of Pila's result for subanalytic surfaces, under the assumption that the surface is the graph of a ‘Pfaffian’ function on the closed unit box in $\mathbb{R}^2$. An analytic function f on an open subset U of $\mathbb{R}^n$ is *Pfaffian* if it can be expressed in the form $f(\bar{x}) = P(\bar{x}, f_1(\bar{x}), \ldots, f_r(\bar{x}))$, where P is a polynomial over $\mathbb{R}$ in $n+r$ variables and $f_1, \ldots, f_r \colon U \to \mathbb{R}$ are analytic functions satisfying a triangular system of polynomial differential equations, i.e. there exist polynomials $P_{i,j} \in \mathbb{R}[Y_1, \ldots, Y_{n+i}]$, for each $i = 1, \ldots, r$ and $j = 1, \ldots, n$, such that $\frac{\partial f_i}{\partial x_j}(\bar{x}) = P_{i,j}(\bar{x}, f_1(\bar{x}), \ldots, f_i(\bar{x}))$, for all $\bar{x} \in U$. Moreover, we say that a Pfaffian function described in this way has *complexity* $B > 0$, if n, r, $\deg(P)$ and $\deg(P_{i,j})$, for all $i = 1, \ldots, r, j = 1, \ldots, n$, are at most B. Such functions were introduced by Khovanskii [5, 6], who showed that if $f \colon U \to \mathbb{R}$ is a Pfaffian function of complexity at most B, where U is an open box in $\mathbb{R}^n$, then the number of connected components of the zero set of f is bounded by a constant which is effectively computable from B.

Theorem 1. *Let B be a positive real number and let U be an open subset of $\mathbb{R}^2$ with $[0,1]^2 \subseteq U$. Suppose that $f : U \to \mathbb{R}$ is Pfaffian of complexity at most B. Let $\epsilon > 0$. There exists a positive constant c depending only on B and on ϵ, and effectively computable from them, with the following property. For all $H \geq 1$, the transcendental part of the graph of $f|_{[0,1]^2}$ contains at most cH^ϵ rational points of height at most H.*

There are two improvements in the constant obtained here over that which Pila’s theorem for subanalytic surfaces provides for such functions. Of course, one improvement is that the constant here is effective. The other is that it is uniform across the class of all Pfaffian functions of the same complexity. We contrast this with the recent work of Binyamini [1], which provides an effective constant in the more general setting of sets of all dimensions described by ‘Noetherian’ functions – these are functions defined in the same way as Pfaffian functions but without the triangularity assumption on the system of differential equations. While this setting encompasses that of Theorem 1, the constant obtained in [1] depends on input data other than the complexity defined here, such as the coefficients of the polynomials appearing in the system of differential equations. The uniformity that we obtain is a feature which promises to have some use in applications. It is also worth noting that the result applies whenever f is a real Pfaffian function, without further conditions being imposed.

Our proof follows the same structure as that of the proof of the Pila–Wilkie Theorem. In particular, we use a parameterization result, a partition of our surfaces into finitely many subsets described by functions with controlled derivatives. Here we cannot appeal directly to the o-minimal version of the parameterization of Yomdin and Gromov [12, 11, 2] that was proved by Pila and Wilkie. Indeed, this result is one of the main sources of ineffectivity in the Pila–Wilkie Theorem, for it involves the use of the compactness theorem (of first-order logic). Our main contribution is an effective version of this parameterization result in a certain setting. A complication in proving this is that, due to the inductive nature of the proofs, we must move outside the setting of Pfaffian functions themselves. Instead we have to work in the wider class of functions which are implicitly defined by (restricted) Pfaffian functions. We thus in fact obtain a version of Theorem 1 in this more general setting, when we combine our parameterization result with ideas from our previous paper [4]. For details, please see [3].

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André-Oort for Nonholomorphic Functions

Haden Spence

I begin by recalling some well-known facts about the modular j function. It is a holomorphic map $j : \mathcal{H} \to \mathbb{C}$, and is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, meaning that $j(\gamma\tau) = j(\tau)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. (Throughout, $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H}$ via Möbius transformations.)

The function j also has good behaviour with respect to $\mathrm{GL}_2^+(\mathbb{Q})$. For any $\tau \in \mathcal{H}$ which is quadratic over $\mathbb{Q}$, and therefore fixed by a nontrivial element of $\mathrm{GL}_2^+(\mathbb{Q})$, we have $j(\tau) \in \overline{\mathbb{Q}}$. Moreover, for each $N \in \mathbb{N}$, there is a polynomial $\Phi_N \in \mathbb{Z}[X, Y]$ such that

$$\Phi_N(j(g\tau), j(\tau)) = 0$$

identically, whenever $g \in \mathrm{GL}_2^+(\mathbb{Q})$ is a primitive integer matrix of determinant N.

These polynomials can be used to construct the *special subvarieties* of $\mathbb{C}^n$, which correspond to Shimura subvarieties of $\mathbb{C}^n$, viewed as a product of level 1 modular curves.

Definition 1. *A variety in $\mathbb{C}^n$ is a special subvariety if it is an irreducible component of some $V \subseteq \mathbb{C}^n$ defined by equations of the form*

- $\Phi_N(X_i, X_j) = 0$ *for some* N, $i \neq j$, *or*
- $X_i = j(\tau)$ *for some fixed quadratic* $\tau \in \mathcal{H}$.

Relatedly, we can define special subvarieties of $\mathcal{H}^n$.

Definition 2. *A subset of $\mathcal{H}^n$ is called a special subvariety if it is cut out by equations of the form*

- $\tau_i = g\tau_j$ *for some* $g \in \mathrm{GL}_2^+(\mathbb{Q})$, $i \neq j$, *or*
- $\tau_i = \tau$ *for some fixed quadratic* $\tau \in \mathcal{H}$.

With these definitions, we can state the classical Modular André-Oort theorem.

Theorem 1 (Pila, Modular André-Oort)**.** *Let $V \subseteq \mathbb{C}^n$ be an algebraic subvariety. Then V contains only finitely many maximal special subvarieties.*

This was proven by Pila in 2011, [2], using the now-standard Pila-Zannier o-minimal strategy. I will discuss an analogue of the above theorem, where j is supplemented by a certain nonholomorphic modular function.

Definition 3. *An* almost holomorphic modular form *(ahm form) of weight k is a function $f : \mathcal{H} \to \mathbb{C}$ of the form*

$$f(\tau) = \sum_{r=0}^{R} f_r(\tau) \left(\frac{1}{\operatorname{Im}\tau}\right)^r,$$

satisfying

$$f(\gamma\tau) = (c\tau + d)^k f(\tau),$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ *and each f_r is a holomorphic function with the property that*

$$f_r(x + iy) \text{ remains bounded as } y \to \infty \text{ (for each } x \text{ individually)}.$$

The prototypical ahm form (which is not just a modular form) is the "nonholomorphic Eisenstein series" E_2^*. It is defined as

$$E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi \operatorname{Im}\tau},$$

where E_2 is a holomorphic function from $\mathcal{H}$ to $\mathbb{C}$ derived from the weight 2 Eisenstein series. For more details on the theory of ahm forms, one can see the excellent survey by Zagier, [5].

There are no ahm forms of weight 0. For an André-Oort statement, one wants weight 0 objects, so we make the following definition: a function $f : \mathcal{H} \to \mathbb{C}$ is an ahm *function* if it is a quotient of two ahm forms of equal weight. Hence, like j, ahm functions are invariant under $\mathrm{SL}_2(\mathbb{Z})$. We will consider the ahm function

$$\chi^* = 1728 \cdot \frac{E_2^* E_4 E_6}{E_4^3 - E_6^2},$$

where E_4 and E_6 are the classical Eisenstein series. It is known that the *discriminant function* $\frac{1}{1728}(E_4^3 - E_6^2)$ does not vanish anywhere in $\mathcal{H}$, so χ^* has no singularities within $\mathcal{H}$.

Writing F^* for the field of ahm functions, it is known that

$$F^* = \mathbb{C}(j, \chi^*),$$

so χ^* is the only ahm function, besides j, that we need to worry about. It should be considered as the nonholomorphic analogue to j, and indeed it has many analogous properties. A result of Masser [1] shows that $\chi^*(\tau) \in \overline{\mathbb{Q}}$, for quadratic τ. One can also find analogues of the modular polynomials Φ_N: for each N there is a polynomial $\Psi_N \in \mathbb{Q}[X, Y, Z]$ such that

$$\Psi_N(\chi^*(g\tau), j(\tau), \chi^*(\tau)) = 0$$

for all primitive integer matrices $g \in \mathrm{GL}_2^+(\mathbb{Q})$ of determinant N.

Consider the variety $V_N' \subseteq \mathbb{C}^4$, defined by

$$\Phi_N(X_1, X_3) = 0, \qquad \Psi_N(X_2, X_3, X_4) = 0.$$

By the properties of Φ_N and Ψ_N, V_N' contains $(j, \chi^*)(G)$, where

$$(j, \chi^*) : \mathcal{H}^n \to \mathbb{C}^{2n}$$

is the map sending each coordinate τ to the pair $(j(\tau), \chi^*(\tau))$, and

$$G = \{(\tau, g\tau) : \tau \in \mathcal{H}\},$$

for some primitive integer matrix g, of determinant N. Now, V_N' has an irreducible component containing $(j, \chi^*)(G)$, which we call V_N; it is the Zariski closure of $(j, \chi^*)(G)$. This variety V_N is the main building block of a new type of special subvariety, called a $*$-*special subvariety* of $\mathbb{C}^{2n}$.

Definition 4. *A $*$-special subvariety of $\mathbb{C}^{2n}$ is the Zariski closure of $(j, \chi^*)(G)$, for G a special subvariety of $\mathcal{H}^n$. Equivalently, a $*$-special subvariety is an irreducible component of a variety in $\mathbb{C}^{2n}$ defined by equations of the form*

- $(X_{2i-1}, X_{2i}, X_{2j-1}, X_{2j}) \in V_N$, *for some N, $1 \le i, j \le n$, or*
- $(X_{2i-1}, X_{2i}) = (j, \chi^*)(\tau)$, *for some fixed quadratic $\tau \in \mathcal{H}$.*

Now I can state the main theorem: a natural analogue of Theorem 1.

Theorem 2 (S., Nonholomorphic Modular André-Oort)**.** *Let $V \subseteq \mathbb{C}^{2n}$ be an algebraic variety. Then V contains only finitely many maximal $*$-special subvarieties.*

I prove this in [4] using the usual Pila-Zannier strategy. There are three necessary components:

(1) Definability of (j, χ^*) in an o-minimal structure.
(2) Suitable control over the Galois orbits of $*$-special points $(j, \chi^*)(\tau)$.
(3) An Ax-Lindemann result to control the algebraic part of $(j, \chi^*)^{-1}(V)$.

The first is easy; it follows from the definability of j that the restriction of χ^* to a suitable fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$ is definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$.

The second component is also fairly easy; Masser proves in [1] that $\mathbb{Q}(\chi^*(\tau)) \subseteq \mathbb{Q}(j(\tau))$. Close inspection of the proof yields the following fact: if σ is a Galois automorphism acting on $j(\tau)$, for some quadratic τ, then

$$(j(\tau), \chi^*(\tau))^\sigma = (j(\tau'), \chi^*(\tau')),$$

for some other quadratic τ'. As a consequence, all the Galois control we need for $*$-special points follows from facts already known about j.

The third is by far the most difficult. Classical Ax-Lindemann results rely heavily on the holomorphicity of j, so proving an analogue for χ^* is much more challenging. However, by exploiting the nice shape of χ^*, one can perform some analytic tricks to get around this problem. Making use also of Pila's Ax-Lindemann result for the derivatives of j, [3], I prove the necessary Ax-Lindemann theorem as the bulk of the work in [4]; interested readers are encouraged to read that paper for details!

The three components combine in a standard way to prove Theorem 2.

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An o-minimal Szemerédi-Trotter theorem

Saugata Basu

(joint work with Orit E. Raz)

The Szemerédi-Trotter theorem [19] on incidences between lines and points in the plane is a foundational result in discrete geometry and extremal combinatorics. The statement of the original theorem (slightly restated and generalized from affine to projective) is as follows. Let $V = \{(x, x^*) \in \mathbb{P}^2_{\mathbb{R}} \times \mathbb{P}^{2*}_{\mathbb{R}} \mid x^*(x) = 0\} \subset \mathbb{P}^2_{\mathbb{R}} \times \mathbb{P}^{2*}_{\mathbb{R}}$ denote the incidence variety.

Theorem 1. [19] *There exists a constant $C > 0$, such that any pair of finite sets $\mathcal{P} \subset \mathbb{P}^2_{\mathbb{R}}, \mathcal{P}^* \subset \mathbb{P}^{2*}_{\mathbb{R}}$,*

$$\mathrm{card}(V \cap (\mathcal{P} \times \mathcal{P}^*)) \leq C \cdot (\mathrm{card}(\mathcal{P})^{2/3} \cdot \mathrm{card}(\mathcal{P}^*)^{2/3} + \mathrm{card}(\mathcal{P}) + \mathrm{card}(\mathcal{P}^*)).$$

Remark 1. *Notice that Theorem 1 can be interpreted as a theorem about "unlikely intersections" – since for generic choices of $\mathcal{P}, \mathcal{P}^*$, the set $V \cap (\mathcal{P} \times \mathcal{P}^*)$ is empty. Theorem 1 has been generalized later in many different ways (for example, to algebraic curves instead of lines* [9, 17]*, incidences between points and algebraic hypersurfaces in higher dimensions* [5]*, replacing $\mathbb{R}$ by $\mathbb{C}$* [20, 22] *etc.).*

From semi-algebraic to o-minimal. It was shown in [3], that many quantitative results in the theory of arrangements of semi-algebraic sets could be generalized to the setting where the elements of the arrangements are restricted to be the fibers of some fixed definable map in some fixed o-minimal structure over a real closed field R. More recently, o-minimal generalizations of results in combinatorial geometry have become a very active topic of research [7] (see also the survey article [16, §6]).

In another direction, Fox et al. [10, Theorem 1.1] obtained a very far reaching generalization of Theorem 1, by extending it to the case of semi-algebraic curves of fixed description complexity. It is thus a natural question if incidence results, such as the Szemerédi-Trotter theorem, and its various generalizations can also be extended to the more general setting of o-minimal geometry. The following theorem is such a generalization.

We fix an o-minimal structure over a real closed field R [21].

Theorem 2. [4] *Let V be a definable subset of $P \times P^*$, where P, P^* are definable sets of dimension at most two. Then one of the following holds.*

(1) *There exists a constant $C = C(V, P, P^*) > 0$, which depends on V, P, P^*, such that for every finite subsets $\mathcal{P} \subset P$, $\mathcal{P}^* \subset P^*$,*

$$\operatorname{card}(V \cap (\mathcal{P} \times \mathcal{P}^*)) \leq C \cdot (\operatorname{card}(\mathcal{P})^{2/3} \cdot \operatorname{card}(\mathcal{P}^*)^{2/3} + \operatorname{card}(\mathcal{P}) + \operatorname{card}(\mathcal{P}^*)).$$

(2) *There exist definable subsets $\alpha \subset P$ and $\alpha^* \subset P^*$, with $\dim(\alpha), \dim(\alpha^*) \geq 1$, such that $\alpha \times \alpha^* \subset V$.*

Remark 2. *Notice that if the second alternative in Theorem 2 holds, then the first alternative (i.e. the Szemerédi-Trotter-type bound) cannot hold, since by choosing $\mathcal{P} \subset \alpha, \mathcal{P}^* \subset \alpha^*$, we can ensure that $\operatorname{card}(V \cap (\mathcal{P} \times \mathcal{P}^*)) = \operatorname{card}(\mathcal{P}) \cdot \operatorname{card}(\mathcal{P}^*)$. Simultaneously with our paper Chernikov, Galvin and Starchenko* [6] *also announced a similar result. Their result is more general than ours (it applies to general distal structures), but the techniques behind their proof are quite different.*

A few remarks about technique. Modern proofs of non-trivial incidence results (such as the Szemerédi-Trotter theorem) usually rely on some deeper algebro-geometric and/or topological results. There are some difficulties in extending these results to higher dimensions and to more general situations (such as to o-minimal structures). We give a summary of these methods and the difficulties in extending them.

Partitioning. One very effective method is to partition the space (R^2 in the case of Szemerédi-Trotter) efficiently into semi-algebraic subsets, such that each subset contains few of the given finite set of points, and each line (or curve from certain restricted family of curves) has non-trivial intersection with few of these subsets. The standard method to achieve this is via the so called "polynomial partitioning theorem" due to Guth and Katz [12]. The polynomial partitioning method and its various generalizations currently do not extend to the o-minimal case, since it is impossible to satisfy the second requirement mentioned above (an arbitrary definable curve can have arbitrarily large number of isolated intersection points with

algebraic curves of fixed degree [11]). In the semi-algebraic case, the polynomial partitioning technique has proved to be very effective – for example, the proof of [10, Theorem 1.1] uses this method. An older method of partitioning which has been used with some success in the semi-algebraic case is sometimes referred to as "trapezoidal decompositions or cuttings" (see for example [14]). They can be thought to be a generalization of the well known (in semi-algebraic and o-minimal geometry) cylindrical decomposition adapted to a given family of semi-algebraic or definable sets – even though the partition need not have a cylindrical structure. The main useful property (which is also called "distality" in model theory [7]) is that the sets in the partition should each be determined by a fixed number of the given definable sets in a fixed definable way. The existence of such decompositions giving rise to full cylindrical decomposition was proved in [3] with quantitative bounds. But this bound does not give any useful incidence results. More efficient bounds on "semi-cylindrical decomposition" for definable sets was proved by Barone [2, Theorem 4.0.9, Chapter 4] – who nearly recovers the best known results in the semi-algebraic case. However, even in the semi-algebraic case the best known quantitative bounds on such decomposition fail to be optimal in higher dimensions and tightening this is major open problem. In lower dimensions (≤ 4) optimal results about cuttings are known, and their extension to the o-minimal case is the main technique in the proof of the o-minimal version of Szemerédi-Trotter by Chernikov, Galvin and Starchenko [6].
Definable crossing number inequality. The proof of Theorem 2 in [4] does not use a partition argument – but rather a different tool, namely the crossing number inequality for finite graphs due to Ajtai et al. [1] and Leighton [13]. The proof can be seen as an adaptation of the proof of the original Szemerédi-Trotter theorem due to Szekely [18], along with certain techniques developed in [15]. The extension of the crossing number inequality to the o-minimal setting uses o-minimal homology theory and Alexander duality [8]. At present it is not clear how to extend this method to higher dimensions, because of the use of the crossing number inequality, since the definition of the crossing number of a graph seems to be an intrinsically planar notion.

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Michael's Theorem for Lipschitz Cells in O-minimal Structures

Wieslaw Pawlucki

(joint work with Małgorzata Czapla)

This is my joint work with Małgorzata Czapla. Assume that R is any real closed field and an expansion of R to some o-minimal structure is given. We will be talking about definable sets and mappings referring to this o-minimal structure. It will be convenient to adopt the following definition of a closed cell. A subset S of R^m ($m \in \mathbb{Z}$, $m > 0$) will be called a *closed (respectively, closed M-Lipschitz) cell* in R^m, where $M \in R$, $M > 0$, if (i) S is a closed interval $[\alpha, \beta]$ ($\alpha, \beta \in R$, $\alpha \leq \beta$), or $S = [\alpha, +\infty)$, or $S = (-\infty, \alpha]$ ($\alpha \in R$), or $S = R$, when $m = 1$ and (ii) $S = [f_1, f_2] := \{(y', y_m) : y' \in S', f_1(y') \leq y_m \leq f_2(y')\}$, where $y' = (y_1, \ldots, y_{m-1})$, S' is a closed (respectively, closed M-Lipschitz) cell in R^{m-1}, $f_i : S' \longrightarrow R$ ($i = 1, 2$) are continuous (respectively, M-Lipschitz) definable functions such that $f_1(y') \leq f_2(y')$, for each $y' \in S'$, or $S = [f, +\infty) = \{(y', y_m) :$

$y' \in S', y_m \geq f(y')\}$, or $S = (-\infty, f] = \{(y', y_m) : y' \in S', y_m \leq f(y')\}$, or $S = S' \times R$, where S' is as before and $f : S' \longrightarrow R$ is continuous (respectively, M-Lipschitz), when $m > 1$. Let $F : A \rightrightarrows R^m$ be a multivalued mapping defined on a subset A of R^n; i.e. a mapping which assigns to each point $x \in A$ a nonempty subset $F(x)$ of R^m. F can be identified with its graph; i.e. a subset of $R^n \times R^m$. If this subset is definable we will call F *definable*. F is called *lower semicontinuous* if for each $a \in A$ and each $u \in F(a)$ and any neighborhood U of u, there exists a neighborhood V of a such that $U \cap F(x) \neq \emptyset$, for each $x \in V$. The main result is the following theorem.

Theorem 1. *Let $F : A \rightrightarrows R^m$ be a definable multivalued, lower semicontinuous mapping defined on a definable subset A of R^n such that every value $F(x)$ is a closed M-Lipschitz cell in R^m, where a constant $M > 0$ is independent of $x \in A$. Then F admits a continuous definable selection $\varphi : A \longrightarrow R^m$.*

The following generalization of Theorem 1 is immediate.

Corollary 1. *Let $F : A \rightrightarrows R^m$ be a definable multivalued, lower semicontinuous mapping defined on a definable subset A of R^n. If there is a continuous definable mapping $\Phi : A \longrightarrow Aut(R^m)$ with values in the space of linear automorphisms of R^m such that $\Phi(x)(F(x))$ is a closed M-Lipschitz cell in R^m, then F admits a continuous definable selection $\varphi : A \longrightarrow R^m$.*

Applying Theorem 1 to semilinear sets and taking into account that every closed semilinear cell is Lipschitz and for every semilinear family of semilinear cells they are M-Lipschitz with common M, we obtain the following application generalizing a result of Aschenbrenner and Thamrongthanyalak

Corollary 2. *Let $F : A \rightrightarrows R^m$ be a semilinear multivalued, lower semicontinuous mapping defined on a semilinear bounded subset A of R^n such that every value $F(x)$ is a closed semilinear cell in R^m. Then F admits a continuous semilinear selection $\varphi : A \longrightarrow R^m$.*

A proof of Theorem 1 is by induction on m. Consider first the case $m = 1$. Then $F(x) = \{t \in R : f(x) \leq t \leq g(x)\}$, for each $x \in A$, where $f : A \longrightarrow R \cup \{-\infty\}$ and $g : A \longrightarrow R \cup \{+\infty\}$ are definable functions. It is easy to check that F is lower semicontinuous if and only if g is lower semicontinuous and f is upper semicontinuous. Therefore, the problem reduces to the following.

Theorem 2. *Let $f : A \longrightarrow R \cup \{-\infty\}$ and $g : A \longrightarrow R \cup \{+\infty\}$ be two definable functions such that $f(x) \leq g(x)$, for each $x \in A$, and f is upper semicontinuous while g is lower semicontinuous. Then there exists a definable continuous function $\varphi : A \longrightarrow R$ such that $f \leq \varphi \leq g$.*

This is a definable version of the Katětov-Tong Insertion Theorem. We prove it by induction on $\dim A$ using a definable version of the Tietze Theorem. Assume now that $m > 1$ and our theorem is true for $m - 1$. To make the induction hypothesis work we prove the following.

Proposition 1. *Under the assumptions of Theorem 1, let*

$$\pi : R^m \ni y = (y_1, \ldots, y_m) \longmapsto y' = (y_1, \ldots, y_{m-1}) \in R^{m-1}$$

be the natural projection. Let $\pi \circ F : A \rightrightarrows R^{m-1}$ denote the composition defined by the formula $(\pi \circ F)(x) = \pi(F(x))$.

Then F treated as a multi-valued mapping $F : \pi \circ F \rightrightarrows R$ is lower semi-continuous.

To finish the proof of Theorem 1, observe that the mapping $\pi \circ F$ is lower semi-continuous as a composition of a lower semicontinuous mapping with a continuous one, so by the induction hypothesis there exists a continuous definable selection φ' for $\pi \circ F$. By above Proposition $F|\varphi' : \varphi' \rightrightarrows R$ is lower semi-continuous; hence, by Theorem 2, it admits a continuous definable selection $\sigma : \varphi' \longrightarrow R$, which gives a required selection $\varphi = (\varphi', \sigma \circ (id_A, \varphi'))$. There exists an example of a semialgebraic mapping $G : A \rightrightarrows R^2$, with $A \subset R^2$, which is not only lower semicontinuous, but even continuous with respect to the Hausdorff distance in the space of definable, closed, bounded and nonempty subsets, and which does not admit a continuous selection, although its values $G(x_1, x_2)$ are M-Lipschitz cells but not with a constant M independent of (x_1, x_2). The dimension two is here minimal both from the point of the domain and the target.

Trajectories of analytic vector fields and o-minimality. The interlaced case

Fernando Sanz
(joint work with Olivier Le Gal, Patrick Speissegger)

Let X be a real analytic vector field with a singular point at the origin of $\mathbb{R}^n$. The talk is framed in the general question of describing qualitatively the dynamics of X around 0 by studying geometric properties of the trajectories which converge to the singular point. We mean by a trajectory here the image of an integral curve $\gamma : [0, \infty) \to \mathbb{R}^n$ of X (again denoted by γ) such that $\gamma(t) \neq 0$ for any t and such that $\lim_{t\to\infty} \gamma(t) = 0$.

We are interested in the following questions:

Question 1.- What finiteness properties, with respect to the family of analytic sets, do individual trajectories of X have?

Question 2.- What type of behavior may have a trajectory with respect to its neighboring trajectories?

To tackle Question 1, we start by assuming that the trajectory γ is *non-oscillating*, i.e., that for any analytic set H at $0 \in \mathbb{R}^n$, either γ is contained in H or γ cuts H only finitely many times. Non-oscillation is an a priori finiteness property and non trivial in general. In the planar case $n = 2$, it is equivalent to the existence of a *tangent* at the limit point by a Rolle's argument. But it is strictly stronger than existence of tangent in higher dimension, in fact stronger than existence of *iterated tangents*, i.e., accumulation to a single limit point under iterated punctual *blow-ups*. A significant example for $n = 3$ is the following (see [2]):

$$X_1 = (-x + y)\frac{\partial}{\partial x} + (-y - x)\frac{\partial}{\partial y} - z^2\frac{\partial}{\partial z}. \tag{1}$$

For this example, any trajectory in the half space $\{z > 0\}$, except for the z-axis $\Gamma = \{x = y = 0\}$, converges to the origin spiraling around Γ and being *asymptotic* to Γ. Thus, any such trajectory is oscillating although it has iterated tangents (those of the analytic curve Γ).

The spiraling behavior of example (1) is general for oscillating trajectories with iterated tangents in dimension three ([2]). This result permits to check the non-oscillating property in some situations for $n = 3$, typically when the linear part of the vector field at the singularity is non-nilpotent (see for instance [10, 3]). In general, we can assert that a trajectory is non-oscillating if it is asymptotic to a formal curve $\widehat{\Gamma}$ at $0 \in \mathbb{R}^n$ which is divergent and *transcendental* with respect to the analytic functions, i.e., considering a parametrization $\widehat{\Gamma} = (\hat{h}_1(t), \dots, \hat{h}_n(t)) \in \mathbb{R}[[t]]^n$, if $F(x_1, \dots, x_n)$ is a germ of an analytic function with $F(0) = 0$ and $F(\hat{h}_1(t), \dots, \hat{h}_n(t)) = 0$ then $F \equiv 0$.

Apart from the existence of iterated tangents, non-oscillating trajectories have other interesting properties. For instance, they can be parameterized by an analytic coordinate and the components of such a parametrization generate a *Hardy field* of one-variable real functions (see [1]).

Looking for stronger tame properties of non-oscillating trajectories and, in accordance with the topics of the workshop, we can formulate strong versions of Questions 1 and 2 in the following way:

Question 1'.- Given a non-oscillating trajectory γ, is the expansion $\mathbb{R}_{an}(\gamma)$ of the real field by the restricted analytic functions and γ o-minimal?

Question 2'.- In which conditions the expansion of $\mathbb{R}_{an}$ by a family of non-oscillating trajectories of X is o-minimal?

The main result presented in this talk deals with Question 1'. Before making a precise statement, let us review the paper [3], which tackles Question 2 for $n = 3$. An *integral pencil* of X is the family of all (germs of) trajectories of X at $0 \in \mathbb{R}^3$ which share the same sequence of iterated tangents. The main result in [3] is the following (exclusive) dichotomy for the relative behavior of trajectories in a given integral pencil $\mathcal{P}$:

a) Either any pair of distinct trajectories $\gamma, \gamma' \in \mathcal{P}$ is an *interlaced* pair. This means that, after parameterizing by a coordinate $\gamma(z) = (u(z), z)$, $\gamma'(z) = (v(z), z)$, the vector $u(z) - v(z)$ spirals around the origin in $\mathbb{R}^2$ while z goes to 0. We speak of an *interlaced pencil*.

b) Or for any pair of distinct trajectories in $\gamma, \gamma' \in \mathcal{P}$ there exists a subanalytic submersion f from a neighborhood of $\gamma \cup \gamma'$ onto $\mathbb{R}^2$ such that $f(\gamma) \cap f(\gamma') = \emptyset$. We speak of a *separated pencil*.

Moreover, if $\mathcal{P}$ is an interlaced pencil, then there exists a (unique) formal curve $\widehat{\Gamma}$ at $0 \in \mathbb{R}^3$, called the *formal axis* of the pencil, such that any member of $\mathcal{P}$ is asymptotic to $\widehat{\Gamma}$. The formal axis is necessarily divergent and, in fact, transcendental. A concrete example of an interlaced pencil can be obtained by a "perturbation" of the example X_1 in (1):

$$X_2 = (-x + y + z)\frac{\partial}{\partial x} + (-y - x)\frac{\partial}{\partial y} - z^2\frac{\partial}{\partial z}. \tag{2}$$

Under the perturbation, the z-axis from example X_1 in (1) becomes an invariant formal divergent curve $\widehat{\Gamma}$ of X_2 in (2) and every trajectory of X_2 in the half space $z > 0$ is asymptotic to $\widehat{\Gamma}$ at the origin. This guarantees that they are non-oscillating trajectories of an integral pencil. On the other hand, if we parameterize two such trajectories as $\gamma(z) = (u(z), z)$, $\gamma'(z) = (v(z), z)$ then the curve $z \mapsto (u(z)-v(z), z)$ is a trajectory of example X_1, which shows that the pair γ, γ' is an interlaced pair.

Now we can state the main result of the talk.

Main Theorem ([6]).- *Let $\mathcal{P}$ be an interlaced pencil of non-oscillating trajectories of an analytic vector field at $0 \in \mathbb{R}^3$. Then for any $\gamma \in \mathcal{P}$, the expansion $\mathbb{R}_{an}(\gamma)$ is o-minimal, model-complete and polynomially bounded.*

This theorem answers positively Question 1' for trajectories of an interlaced pencil. It is still open for three dimensional trajectories in a separated pencil. However, the answer to Question 1' is negative in general for trajectories in dimension $n \geq 5$ thanks to the following example, constructed in the paper [8]: consider two distinct trajectories $\gamma_1(z) = (u(z), z)$, $\gamma_2(z) = (v(z), z)$ of example X_2 in (2) and put $\gamma(z) = (u(z), v(2z), z)$. Using the (SAT) property of the formal axis $\widehat{\Gamma}$ of X_2 (which we recall below), one can show that γ is asymptotic to a formal transcendental curve at $0 \in \mathbb{R}^5$, and thus γ is non-oscillating. On the other hand, by the definition of interlaced pair, it is clear that γ can not generate an o-minimal expansion.

The problem of o-minimality of a family of trajectories (Question 2') is more difficult and only very particular results are known, except for the planar case $n = 2$ for which every non-oscillating trajectory is a pfaffian set and hence all of them generate an o-minimal structure (see [7, 11]). In dimension three, notice that two distinct trajectories of an interlaced pair cannot generate an o-minimal expansion, so Question 2' can only be formulated for trajectories in a separated pencil. We may mention the following recent contributions to this problem for $n = 3$:

- In the paper [5], we analyze a system of two linear ODEs of the form $Y' = A(x)Y + B(x)$ where $Y = (y_1, y_2)$ and $A(x), B(x)$ are real-valued matrices for x in some interval $(0, \varepsilon)$ and definable in an o-minimal expansion $\mathcal{R}$ of $\mathbb{R}$. It corresponds to a three-dimensional vector field in coordinates (x, y_1, y_2) of a particular form, but not necessarily with analytic coefficients, since there is no restriction on the o-minimal expansion $\mathcal{R}$. We prove that we have also the same dichotomy interlaced/separated for the whole family of solutions of the system and that, in the separated case, such a family generates an o-minimal expansion of $\mathcal{R}$ (in fact a reduct of the pfaffian closure of $\mathcal{R}$).

- In a recent article [4], yet unpublished, we show that if $\mathcal{P}$ is a separated pencil with a transcendental formal axis and $\gamma, \gamma' \in \mathcal{P}$, parameterized by a common analytic coordinate, then the components of γ and γ' belong to a Hardy field.

The proof of the Main Theorem follows a similar scheme as the one for the main result in the paper [8]. Let us just sketch here the principal steps.

Step 1. The vector field in final form.- After a finite number of blow-ups and ramifications, we may assume (see [3]) that the vector field X is written as a system of analytic ODEs of the form

$$x^{k+1}\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \sum_{j=0}^{k} x^j \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + x^{k+1}\Theta(x,y), \tag{3}$$

where $k \geq 1$, $b_r \neq 0$ for some $0 \leq r \leq k$ (some of the matrices above has non-real eigenvalues) and $a_s > 0$ for some $0 \leq s \leq k-1$. The system (3) has a unique formal solution $\widehat{H}(x) = (\widehat{H_1}(x), \widehat{H_2}(x)) \in \mathbb{R}[[x]]^2$ and the trajectories of the interlaced pencil $\mathcal{P}$ correspond to the solutions $(h_1(x), h_2(x))$ with $x \in (0,\varepsilon)$ which are asymptotic to $\widehat{H}(x)$.

Step 2. The SAT condition.- A polynomial $P \in \mathbb{R}[x]$ is said to be *positive* if $P(x) > 0$ for all sufficiently small $x > 0$. It is said to be *k-short* if $\deg P < (k+1)\operatorname{ord} P$. In the paper [8] it is introduced the following

Definition.- A tuple $\widehat{H}(x) \in \mathbb{R}[[x]]^n$ of formal power series is called k-SAT (for *Strongly Analytically Transcendental*) if for any tuple $P = (P_1, P_2, ..., P_l)$ of distinct positive k-short polynomials, the tuple

$$\widehat{H} \circ P := (\widehat{H}(P_1(x)), ..., \widehat{H}(P_l(x))) \in \mathbb{R}[[x]]^{nl}$$

is analytically transcendental, i.e., if $F(z_0, z_1, ..., z_{nl})$ is convergent and $F(x, \widehat{H} \circ P) \equiv 0$ then $F \equiv 0$.

Step 3. The SAT condition vs quasi-analyticity and o-minimality.- One of the main results that we use is the following one.

Theorem [8].**-** Let $\widehat{H}(x) \in \mathbb{R}[[x]]^n$ be a formal solution of a system of analytic ODEs of the form $x^{k+1}Y' = f(x,Y)$ where f is analytic and assume that $\widehat{H}(x)$ has the SAT property. Let $h(x) = (h_1(x), ..., h_n(x))$ for $x \in (0,\varepsilon)$ be a solution of the same system which is asymptotic to $\widehat{H}(x)$. Then the expansion $\mathbb{R}_{an}(h)$ of the real field by the restricted analytic functions and the components of h is o-minimal, model-complete and polynomially bounded.

For the proof of this result one considers the smallest family $\{\mathcal{C}_m\}$ of algebras of germs of functions at the origin of $\mathbb{R}^m$, for any m, which is closed by composition, partial derivatives, solution of implicit equations and monomial division and such that the components h_j of the solution h belong to $\mathcal{C}_1$. The property SAT is used to prove that these algebras are *quasi-analytic*, i.e., their non-zero elements admit non-zero formal Taylor expansion. Then the result follows from the results in the paper [9].

Step 4. Multisummability and the SAT condition.- It remains to prove that the formal solution $\widehat{H}(x) \in \mathbb{R}[[x]]^2$ of the system (3) has the SAT property. This is a technical part which uses the theory summability and multisummability of formal series. Particularly, the series $\widehat{H}(x)$ is k-summable and, being divergent, has at least one non-trivial Stokes phenomenum. We prove that if $P_1, ..., P_l$ are distinct k-short positive polynomials and if F is a non-zero convergent series in $1 + 2l$ variables then the series $F(x, \widehat{H}(P_1(x)), ..., \widehat{H}(P_l(x)))$, which is multisummable, has also one non-trivial Stokes phenomenum, and hence it can not be the zero series. The presence of a non-real eigenvalue in one of the matrices in expression (3) is crucial here in the proof, as well as the fact that the P_j are distinct k-short polynomials.

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Bi-Lipschitz classification of surface germs

Andrei Gabrielov

(joint work with Lev Birbrair, Alexandre Fernandes)

Let X be a germ at the origin of $\mathbb{R}^n$ of a two-dimensional surface definable in a polynomially bounded o-minimal structure over $\mathbb{R}$. The outer metric on X, such that $\text{dist}(x, y) = ||y - x||$, is induced from $\mathbb{R}^n$. Two such surface germs, X and Y, are bi-Lipschitz equivalent if there exists a homeomorphism $h : (X, 0) \to (Y, 0)$

such that the outer metric on X is equivalent to the metric on X induced from Y. The goal is to find a discrete (no moduli in definable families) invariant of bi-Lipschitz equivalence class of definable surface germs.

We start with a simpler problem of bi-Lipschitz classification of germs at the origin of definable functions $f : \mathbb{R}^2 \to \mathbb{R}$ up to contact equivalence. Two such function germs are bi-Lipschitz contact equivalent if there exists a bi-Lipschitz homeomorphism $h : (\mathbb{R}^3_{x,y,z}, 0) \to (\mathbb{R}^3_{x,y,z}, 0)$ commuting with the projection $\mathbb{R}^3_{x,y,z} \to \mathbb{R}^2_{x,y}$ and mapping the graph of f to the graph of g. The invariant consists of a partition of the neighborhood of the origin of $\mathbb{R}^2_{x,y}$ into Hölder triangles T_j (defined up to bi-Lipschitz equivalence and uniquely determined by the cyclically ordered sequence of the exponents β_j of T_j), and for each j the sign $s_j \in \{+, -, 0\}$ of $f|_{T_j}$, the interval Q_j of the exponents q of f on the arcs $\gamma \subset T_j$, and the affine width function $\mu_j(q) = a_j q + c_j$ on Q_j that measures how much an arc $\gamma \subset T_j$ may be deformed so that the exponent q of $f|_\gamma$ does not change. Such a partition is called a "pizza" with "slices" T_j and "toppings" β_j, s_j, Q_j, μ_j. These data (with some additional constraints due to continuity of f) constitute a complete discrete invariant of the by-Lipschitz contact equivalence class of function-germs in $\mathbb{R}^2$ ([1], Theorem 3.1).

For a germ X of a definable two-dimensional surface, a similar (but more complicated combinatorially) canonical (up to bi-Lipschitz equivalence) partition into normally embedded Hölder triangles T_j can be defined, so that any two triangles of the partition are either "transversal" (not tangent to each other) or "coherent" (bi-Lipschitz equivalent to a slice of a pizza and a graph of a definable function over that slice). The discrete invariant of the bi-Lipschitz equivalence class of X consists in the combinatorial structure of the partition, the tangency orders q_{jk} between the boundary arcs of any two triangles T_j and T_k of the partition, and the "toppings" $\beta_{jk}, Q_{jk}, \mu_{jk}(q)$ associated with each pair of coherent triangles T_j and T_k ([2], work in progress).

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Monotone functions and maps

Nicolai Vorobjov

(joint work with Saugata Basu, Andrei Gabrielov)

Fix an o-minimal structure over $\mathbb{R}$. In what follows, all sets, functions, and maps are assumed to be definable in this structure. Consider a set $X \subset \mathbb{R}^n$ and a cylindrical decomposition of $\mathbb{R}^n$ compatible with X. Each d-dimensional cylindrical cell $C \subset X$ of this decomposition is a topological cell, i.e., a homeomorphic image of $(0, 1)^d$. On the other hand, C is not necessarily a *topologically regular* cell (C is topologically regular if the pair $(\overline{C}, C)$ is homeomorphic to the pair $\big([0,1]^d,\ (0,1)^d\big)$). In tame topology it is desirable to be able to construct cylindrical decompositions consisting of topologically regular cells, or, at least, partitions into regular cells each of which is cylindrical (possibly with respect to different orders of coordinates).

We introduce a class of cylindrical cells, which are topologically regular and are defined by a property that is relatively easy to realise and check.

We first need a preliminary definition.

Definition 1. *Let a bounded continuous map* $\mathbf{f} = (f_1, \dots, f_k)$ *defined on an open bounded non-empty set* $X \subset \mathbb{R}^n$ *have the graph* $\mathbf{F} \subset \mathbb{R}^{n+k}$. *We say that* $\mathbf{f}$ *is* quasi-affine *if for any coordinate subspace* T *of* $\mathbb{R}^{n+k}$, *the projection* $\rho_T : \mathbf{F} \to T$ *is injective if and only if the image* $\rho_T(\mathbf{F})$ *is* n*-dimensional.*

The concept of a monotone map, defined below, is a far-reaching common generalisation of the usual univariate monotone continuous function and of the convex set. The most natural definition builds on univariate monotone functions [2], but turns out to be quite involved. The following is an equivalent definition, geometrically simpler, in which the idea of monotonicity is only implicit.

Definition 2. *Let a bounded continuous quasi-affine map* $\mathbf{f} = (f_1, \dots, f_k)$ *defined on an open bounded non-empty set* $X \subset \mathbb{R}^n$ *have the graph* $\mathbf{F} \subset \mathbb{R}^{n+k}$. *We say that the map* $\mathbf{f}$ *is* monotone *if for each affine coordinate subspace* S *in* $\mathbb{R}^{n+k}$ *the intersection* $\mathbf{F} \cap S$ *is connected.* Monotone cell *is the graph of a monotone map.*

The most important property of monotone cells is expressed by the following theorem.

Theorem 1 ([1, 2])**.** *All monotone cells are topologically regular.*

We state the following conjecture which might be very useful in topological applications [2, 3].

Conjecture 2. *Consider a compact set* $K \subset \mathbb{R}^n$ *and a function* $f : K \to \mathbb{R}$. *There exists a partition* $\mathcal{P}$ *of* K *into monotone cylindrical cells, possibly with respect to different orders of coordinates, such that for each cell* C *in* $\mathcal{P}$ *the restriction* $f|_C$ *is a monotone function.*

The conjecture holds (in a stronger version, for cylindrical decompositions) in the case of $\dim K \leq 2$ and arbitrary n [3]. There is an understanding how to prove the conjecture in the case of arbitrary $\dim K$ and $n = 3$. Some applications of this theorem can be found in [2, 3].

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Variants of mild parameterizations, in definable families

Raf Cluckers

(joint work with Jonathan Pila, Alex Wilkie)

We present work from [3] with new parameterization theorems for sets definable in the structure $\mathbb{R}_{an}$ (i.e. for globally subanalytic sets) which are uniform for definable families of such sets. The existence of C^r-parameterizations for various kinds of subsets of $\mathbb{R}^n$ has been proved in [6] [7] [4] (for semi-algebraic sets) and [5] (for sets in o-minimal structures), and has been used to study both entropy and rational points of height bounded by H. Our new work relates to two big conjectures, one a variant of Yomdin's question raized just below Remark 3.8 in [2], and one a variant of a conjecture by Wilkie on polylogarithmic bounds for the number of rational points of height no larger than H on definable sets in (certain reducts of) $\mathbb{R}_{an}^{\exp}$.

Among the results from [3] , we focus on the polynomial (in r) bound (depending only on the given family of definable sets) for the number of parameterizing C^r functions.

We then give some diophantine applications motivated by the question as to whether the $H^{o(1)}$ bound in the Pila-Wilkie counting theorem can be improved, at least for certain reducts of $\mathbb{R}_{an}$. More specifically, uniform $(\log H)^{O(1)}$ bounds for the number of rational points of height at most H on $\mathbb{R}_{an}$-definable Pfaffian surfaces follow. The presented techniques and results also work more generally for $\mathbb{R}_{an}^{pow}$ instead of $\mathbb{R}_{an}$. The recent work of [1] also relates to the variant of Wilkie's conjecture for $\mathbb{R}_{an}$-definable Pfaffian sets, and introduces different techniques to approach this conjecture.

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Small sets in dense pairs

Pantelis E. Eleftheriou

We consider expansions $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P\rangle$ of an o-minimal structure $\mathcal{M}$ by a set $P \subseteq M$, such that the geometric behavior on the class of all definable sets is tame. An important category of such structures is when every definable open set is already definable in $\mathcal{M}$ ([1, 2, 4, 5, 7]). Three main examples of this category are:

(1) Dense pairs
(2) Expansions of $\mathcal{M}$ by a dense independent set
(3) Expansions of real closed field $\mathcal{M}$ by a dense divisible subgroup P of $\langle M^{>0}, \cdot\rangle$ with the Mann property.

In [7], all these examples were put under a common perspective, and a cone decomposition theorem was proved for their definable sets. That theorem aimed to provide an understanding of all definable sets in terms of sets definable in $\mathcal{M}$ and 'P-bound' sets. Corollary 3 below further reduces the study of P-bound sets to that of subsets of some P^l definable in $\widetilde{\mathcal{M}}$.

Notation. We fix an o-minimal expansion $\mathcal{M} = \langle M, <, +, 0, \dots\rangle$ of an ordered group with a distinguished positive element 1. We denote by $\mathcal{L}$ its language, and by dcl the usual definable closure operator in $\mathcal{M}$. An '$\mathcal{L}_A$-definable' set is a set definable in $\mathcal{M}$ with parameters from A. We also fix some $P \subseteq M$ and denote $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P\rangle$. An '($A$-)definable set' is a set definable in $\widetilde{\mathcal{M}}$ with parameters from A. We drop the above indices 'A', if we do not want to specify the parameters. Finally, we let D denote a subset of M.

Definition 1 ([4])**.** *A set $X \subseteq M^n$ is called* P-bound *over A if there is an $\mathcal{L}_A$-definable function $h : M^m \to M^n$ such that $X \subseteq h(P^m)$.*

In the aforementioned examples, P-boundness amounts to a precise topological notion of smallness ([7, Definition 2.1]), as well as to the classical notion of P-internality from geometric stability theory ([7, Corollary 3.12]). In [7], we asked:

Question 2. *Is every P-bound set in definable bijection with a subset of P^n, for some n?*

The main difficulty in answering the above question is that in $\widetilde{\mathcal{M}}$, most 'choice properties' generally fail. For example, it is known that a dense pair does not eliminate imaginaries and does not admit definable Skolem functions ([2, Section 5]. If P is a dense independent set, then $\widetilde{\mathcal{M}}$ eliminates imaginaries but does not admit definable Skolem functions ([3]). We observe here (Corollary 3 below) that all that is needed in order to answer the above question is that the induced structure on P by $\mathcal{M}$ eliminates imaginaries.

Definition 2. *Let $D, P \subseteq M$. The* D-induced structure on P by $\mathcal{M}$*, denoted by $P_{ind(D)}$, is a structure whose language is*

$$\mathcal{L}_{ind(D)} = \{R_{\phi(x)}(x) : \phi(x) \in \mathcal{L}_D\}$$

and, for every tuple $a \subseteq P$,

$$P_{ind(D)} \models R_\phi(a) \;\Leftrightarrow\; \mathcal{M} \models \phi(a).$$

We prove (Proposition 1) that in our examples, $P_{ind(D)}$ eliminates imaginaries, for any $D \subseteq M$ which is dcl-independent over P. We work in a general setting. Consider the following properties for $\widetilde{\mathcal{M}}$ and D:

(OP) (Open definable sets are $\mathcal{L}$-definable.) For every set A such that $A \setminus P$ is dcl-independent over P, and for every A-definable set $V \subset M^n$, its topological closure $\overline{V} \subseteq M^n$ is $\mathcal{L}_A$-definable.

$(\mathrm{dcl})_D$ Let $B, C \subseteq P$ and

$$A = \mathrm{dcl}(BD) \cap \mathrm{dcl}(CD) \cap P.$$

Then

$$\mathrm{dcl}(AD) = \mathrm{dcl}(BD) \cap \mathrm{dcl}(CD).$$

$(\mathrm{ind})_D$ Every A-definable set in $P_{ind(D)}$ is the trace of an $\mathcal{L}_{AD}$-definable set.

Properties (OP) and $(\mathrm{ind})_D$ already appear in the literature and are known for our three examples ([7]). Property $(\mathrm{dcl})_D$ is introduced here. We prove the following results ([6]).

Theorem 1. *Suppose* (OP), $(\mathrm{dcl})_D$ *and* $(\mathrm{ind})_D$ *hold for $\widetilde{\mathcal{M}}$ and D. Then $P_{ind(D)}$ eliminates imaginaries.*

Corollary 3. *Suppose* (OP), $(\mathrm{dcl})_D$ *and* $(\mathrm{ind})_D$ *hold for $\widetilde{\mathcal{M}}$ and D, and let $X \subseteq M^n$ be a D-definable set. If X is P-bound over D, then there is a D-definable injective map $\tau : X \to P^l$.*

Proof. Let $h : M^m \to M^n$ be an $\mathcal{L}_A$-definable map such that $X \subseteq h(P^m)$, and consider the following equivalence relation E on M^m:

$$xEy \Leftrightarrow h(x) = h(y).$$

Note that $E \cap (P^m \times P^m)$ is an equivalence relation on P^m, which is $\emptyset$-definable in $P_{ind(D)}$. Since $P_{ind(D)}$ eliminates imaginaries, there is a $\emptyset$-definable in $P_{ind(D)}$ map $f : P^m \to P^l$, for some l, such that for every $x, y \in P^m$,

$$f(x) = f(y) \Leftrightarrow xEy.$$

Define $\tau : X \to P^l$, given by $\tau(h(x)) = f(x)$. Then τ is well-defined, injective and D-definable (in $\widetilde{\mathcal{M}}$). □

We verify $(\mathrm{dcl})_D$ in our three main examples.

Proposition 1. *Let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ be a dense pair, or an expansion of $\mathcal{M}$ by a dense independent set or by a dense divisible multiplicative group with the Mann Property. Let $D \subseteq M$ be* dcl*-independent over P. Then* $(\mathrm{dcl})_D$ *holds. Hence $P_{ind(D)}$ eliminates imaginaries.*

The assumption that D is dcl-independent over P is necessary. Namely, without it, $P_{ind(D)}$ need not eliminate imaginaries. However, even without it, we still obtain the following corollary, which in particular applies to our examples.

Corollary 4. *Suppose* (OP), $(\mathrm{dcl})_D$ *and* $(\mathrm{ind})_D$ *hold for $\widetilde{\mathcal{M}}$ and every $D \subseteq M$ which is* dcl*-independent over P. Let $X \subseteq M^n$ be an A-definable set. If X is P-bound over A, then there is an $A \cup P$-definable injective map $\tau : X \to P^l$.*

Allowing parameters from P is standard practice when studying definability in this context; see for example also [7, Lemma 2.5, Corollary 3.24].

Finally, we show that Theorems A and B are optimal also in the following way. Let D be dcl-independent over P. Suppose (OP) and $(\mathrm{ind})_D$ hold for $\widetilde{\mathcal{M}}$ and D. Then:

$$P_{ind(D)} \text{ eliminates imaginaries } \Leftrightarrow (\mathrm{dcl})_D.$$

If we do not assume (OP), the above two properties need not hold. We do not know whether they hold, if we assume (OP) and $(\mathrm{ind})_D$. Finally, (OP) does not imply $(\mathrm{ind})_D$, but we do not know whether $(\mathrm{ind})_D$ is necessary for P_{ind} to eliminate imaginaries.

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Weakly admissible lattices, o-minimality, and lattice point counting

Martin Widmer

(joint work with Niclas Technau)

Let $n \geq 2$, let $S = \prod_{i=1}^{n}[y_i, y_i + s_i]$ be an aligned box in $\mathbb{R}^n$, i.e., the cartesian product of n intervals, and let $\Lambda \in SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$, i.e., a unimodular lattice in $\mathbb{R}^n$. We write $\overline{s}$ for the geometric mean of the side lengths s_i and s_{max} for the longest side length. Our first goal is to find sharp upper bounds for the lattice point discrepancy

$$\mathcal{E}_\Lambda(S) := \left|\#\Lambda \cap S - \mathrm{Vol}(S)\right| . \tag{1}$$

The problem is uninteresting for the standard lattice $\Lambda = \mathbb{Z}^n$ since there may be many lattice points on the faces of S, showing that the general trivial estimate $\mathcal{E}_\Lambda(S) \ll_\Lambda (1+s_{max})^{n-1}$ is sharp in this case. In contrast to this trivial case we call a lattice Λ *weakly admissible* if no face of S contains more than one lattice point for any aligned box S. For a quantitative version of this notion let us consider the function

$$\nu_\Lambda(\rho) := \inf\{|x_1 \cdots x_n|; \mathbf{x} \in \Lambda, 0 < |\mathbf{x}| < \rho\}, \tag{2}$$

defined for $\rho > \gamma_n^{1/2}$, where γ_n denotes the Hermite constant. Then Λ is weakly admissible if and only if $\nu_\Lambda(\rho) > 0$ for all ρ. If $\nu_\Lambda(\rho)$ is even bounded away from 0 then we say Λ is *admissible.*

Theorem 1 ([4], Theorem 1.1)**.** *Let Λ be a weakly admissible lattice then we have*

$$\mathcal{E}_\Lambda(S) \ll_n \inf_{\gamma_n^{1/2} < \rho \le s_{max}} \left(\frac{\overline{s}}{\nu_\Lambda(\rho)^{1/n}} + \frac{s_{max}}{\rho} \right)^{n-1} . \tag{3}$$

We expect our bound to be sharp although we have succeeded to prove this only for $n = 2, 3$ (where for $n = 3$ we are using a more flexible notion of weak admissibility; c.f. [4, Theorem 2.2]).

Skriganov [1] proved bounds for $\mathcal{E}_\Lambda(S)$ provided the dual lattice $\Lambda^\perp$ (w.r.t the standard inner product) is weakly admissible. If $\Lambda^\perp$ is even admissible then he obtains $\mathcal{E}_\Lambda(S) \ll_\Lambda (\log(\overline{s} + 2))^{n-1}$ which is expected to be sharp.

The results based on Skriganov's method (see [3, Theorem 1]) can be compared with Theorem 1, provided one can compare $\nu_{\Lambda^\perp}(\rho)$ with $\nu_\Lambda(\rho)$. If $\nu_\Lambda(\rho)$ is bounded away from 0 then $\nu_{\Lambda^\perp}(\rho)$ is bounded away from 0, i.e., if $\Lambda^\perp$ is admissible then also Λ is admissible. In this case Skriganov's result is much more precise than Theorem 1. On the other hand, if $\nu_\Lambda(\rho) = \nu_{\Lambda^\perp}(\rho)$ and Λ is weakly admissible but not admissible then Theorem 1 is more precise, provided S is sufficiently distorted (see [4, Introduction] for a more accurate statement).

But for which lattices Λ do we actually have this equality of the $\nu_\Lambda(\cdot)-$functions? The following proposition shows that this happens, e.g., for every symplectic lattice, in particular, whenever $n = 2$.

Proposition 1 ([3], Proposition 1)**.** *Let $\Lambda = A\mathbb{Z}^n$, and suppose there exist P, R both in $GL_n(\mathbb{Z})$ such that*

$$A^T P A = R,$$

and suppose P has exactly one non-zero entry in each column and in each row. Then, we have

$$\nu_{\Lambda^\perp}(\cdot) = \nu_\Lambda(\cdot).$$

If we only assume that Λ is weakly admissible then it is not possible to give a positive lower bound for $\nu_{\Lambda^\perp}(\cdot)$ solely in terms of $\nu_\Lambda(\cdot)$ and n since $\Lambda^\perp$ need not be weakly admissible (see [3, Example 4]). But even if we assume that Λ and $\Lambda^\perp$ are both weakly admissible it still is impossible to get a positive lower bound for $\nu_{\Lambda^\perp}(\cdot)$ in terms of $\nu_\Lambda(\cdot)$ and n as soon as the necessary condition $n \geq 3$ is fulfilled.

Theorem 2 ([3], Theorem 2)**.** *Let $n \geq 3$, and let $\psi : (0,\infty) \to (0,1)$ be non-increasing. Then, there exists a unimodular, weakly admissible lattice $\Lambda^\perp \subseteq \mathbb{R}^n$, and a sequence $\{\rho_l\} \subseteq (\gamma_n^{1/2}, \infty)$ tending to ∞, as $l \to \infty$, such that*

$$\nu_\Lambda(\rho) \gg \rho^{-n^2},$$

and

$$\nu_{\Lambda^\perp}(\rho_l) \leq \psi(\rho_l)$$

for all $l \in \mathbb{N} = \{1,2,3,\ldots\}$ and for all $\rho > \gamma_n^{1/2}$.

While Proposition 1 follows from a straightforward calculation Theorem 2 lies deeper and is based on a recent result of Beresnevich [2, Theorem 1] about the Hausdorff dimension of the set of badly approximable points on certain submanifolds of $\mathbb{R}^n$ which answers a longstanding question of Davenport.

None of all that has anything to do with o-minimality. But Theorem 1 holds for more general sets than aligned boxes.

We say a family $\mathcal{F}$ of bounded subsets of $\mathbb{R}^n$ has *Property (UTB)* if there exist $\kappa > 0$ and $M \in \mathbb{N}$ such that whenever $S \in \mathcal{F}$ and $A \in \mathrm{GL}_n(\mathbb{R})$ then the boundary $\partial(AS)$ is covered by the images of M maps $\phi_i : [0,1]^{n-1} \to \mathbb{R}^n$ $(1 \leq i \leq M)$, each satisfying a Lipschitz condition with constant $\kappa \cdot \mathrm{diam}(AS)$. Here $\mathrm{diam}(\cdot)$ denotes the diameter.

For any bounded set $S \subseteq \mathbb{R}^n$ we set $s_i := \mathrm{diam}(\pi_i S)$ where $\pi_i(\mathbf{x}) = x_i$ is the projection to the $i-th$ coordinate. With these definitions Theorem 1 remains valid for any set S that lies in a family $\mathcal{F}$ with Property (UTB), provided we replace $\ll_n$ in (3) by $\ll_{n,M,\kappa}$ (see [4, Theorem 2.2]).

This raises the problem of finding "large", "interesting" families $\mathcal{F}$ with Property (UTB). One can use integral geometry as in [5, Theorem 2.8] to show that any family $\mathcal{F}$ of bounded sets in $\mathbb{R}^2$ whose boundary is the path of a smooth, simple, closed curve that intersects no line in more than $\ll_{\mathcal{F}} 1$ points has Property (UTB). Another interesting family with Property (UTB) is the family of all bounded convex sets in $\mathbb{R}^n$ (see [5, Theorem 2.6]). Unsurprisingly, we can also use o-minimality to establish such families. For $Z \subset \mathbb{R}^{m+n}$ and $T \in \mathbb{R}^m$ we write $Z_T = \{\mathbf{x} \in \mathbb{R}^n; (T,\mathbf{x}) \in Z\}$ and call this the fiber of Z above T.

Proposition 2 ([4], Proposition 8.1)**.** *Suppose $Z \subset \mathbb{R}^{m+n}$ is definable in an o-minimal structure over $\mathbb{R}$, and assume further that all fibers Z_T are bounded sets. Then the family of all fibers Z_T has Property (UTB).*

The most important ingredients of the proof are the existence of definable Skolem functions, Pila and Wilkie's Reparameterization Lemma for definable families, and that for definable non-empty sets the dimension of the frontier is strictly smaller than the dimension of the set itself.

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Some applications of o-minimality in quantitative arithmetic geometry

Christopher Frei

(joint work with U. Derenthal, M. Pieropan, E. Sofos)

A conjectural program initiated by Manin [3] predicts the distribution of rational points of bounded height on certain classes of algebraic varieties over number fields. The case of del Pezzo surfaces, projective surfaces with ample anticanonical divisor and at most simple singularities, has attracted particular attention. For open subvarieties U of a del Pezzo surface X over a number field K and an anticanonical height function H on the rational points $X(K)$, we consider the counting function

$$N_{U,H}(B) := \#\{x \in U(K) \mid H(x) \leq B\}.$$

If $X(K) \neq \emptyset$, then Manin's conjecture predicts the existence of an open subvariety U, for which

$$N_{U,H}(B) \sim c_{X,H} B (\log B)^{\rho_X - 1}, \quad \text{as } B \to \infty,$$

with a positive constant $c_{X,H}$ and ρ_X the rank of the Picard group of X. A strategy successfully applied in proofs of special cases for many singular and some smooth del Pezzo surfaces over $\mathbb{Q}$ relies on two steps: first, the rational points $U(K)$ are parameterized by integral points of a higher-dimensional quasi-affine auxiliary variety. Then these integral points of bounded height are counted using geometry of numbers and analytic number theory. For the auxiliary variety in the first step, one frequently takes a universal torsor of X, given as an open subset of the spectrum of the cox ring $\mathrm{Cox}(X)$.

In joint work with U. Derenthal [2], we started first attempts at making this strategy available over a wider selection of base fields, focusing first on imaginary quadratic number fields. In the counting step, it is necessary to compare certain sums over lattice points in a region described by the height function H to certain integrals. This is complicated further by the presence of arithmetic functions that describe coprimality conditions arising from the fact that our auxiliary variety is quasi-affine. To deal with these problems using classical tools of analytic number theory, one requires strong monotonicity properties of parametric integrals of families of semialgebraic functions, which we established relying on the definability of these integrals in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$.

With M. Pieropan [4], we extended the strategy to arbitrary number fields K. Here, o-minimality is applied in a different way, in the form of a lattice point counting theorem in definable sets [1]. The sets under consideration are fundamental domains for certain actions of the unit group $\mathcal{O}_K^\times$, which are not semialgebraic but definable in $\mathbb{R}_{\exp}$. More details on this work are provided in M. Pieropan's abstract in this volume.

A further application of the lattice point counting theorem [1] arises in joint work with E. Sofos [5], where we prove lower bounds for certain generalized divisor sums over values of binary forms. These results imply the validity of the lower bound predicted by Manin's conjecture for all smooth del Pezzo surfaces over all number fields after a finite extension of the base field.

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The Zilber-Pink conjecture for pure Shimura varieties via o-minimality

Christopher Daw

(joint work with Jinbo Ren)

Using o-minimality, the Pila-Zannier strategy combines results from arithmetic and functional transcendence to prove finiteness theorems in arithmetic geometry. It originated in the paper [6] of Pila and Zannier in which the authors gave a new proof of the Manin-Mumford conjecture for abelian varieties.

The objective of [2] was to extend the Pila-Zannier strategy to the Zilber-Pink conjecture for (pure) Shimura varieties. The method generalises that of Habegger and Pila who in [3] obtained results for abelian varieties and products of modular curves.

Let S be a mixed Shimura variety. For any (irreducible) subvariety W of S, there exists a smallest special subvariety $\langle W\rangle$ of S containing W. In [7], Pink defined the defect of W to be

$$\delta(W) := \dim\langle W\rangle - \dim W.$$

In their article [3], Habegger and Pila made the following definition.

Definition 1. *Fix a subvariety V of S. A subvariety W of V is optimal in V if for any subvariety Y of V strictly containing W we have*

$$\delta(Y) > \delta(W).$$

This clearly generalises the notion of a maximal special subvariety. The André-Oort conjecture predicts that any subvariety of S contains only finitely many maximal special subvarieties. Hence, the following formulation of the Zilber-Pink conjecture is a natural generalisation of that statement.

Conjecture 1 (Zilber-Pink)**.** *Let V be a subvariety of S. Then V contains only finitely many subvarieties that are optimal in V.*

This formulation is equivalent to Zilber's conjecture regarding atypical intersections and it implies the conjecture of Pink in which V is intersected with special subvarieties of codimension exceeding the dimension of V.

1. Shimura varieties

By a Shimura variety, we refer to a variety of the form $\Gamma\backslash X$, where X is a hermitian symmetric domain and Γ is a congruence subgroup. For us, however, it is more useful to view X as the $G(\mathbb{R})^+$ conjugacy class of a morphism

$$\mathbb{C}^\times \to G(\mathbb{R})$$

of real Lie groups, where G is an algebraic group over $\mathbb{Q}$.

Special subvarieties arise as follows. Let x be any point on X and let M be the smallest subvariety of G defined over $\mathbb{Q}$ with the property that $x(\mathbb{C}^\times)$ is contained in $M(\mathbb{R})$. The $M(\mathbb{R})^+$ conjugacy class X_M of x is a hermitian symmetric subdomain of X and its image under

$$\pi : X \to \Gamma\backslash X$$

is an algebraic subvariety of $\Gamma\backslash X$. We refer to such a subvariety as a special subvariety of $\Gamma\backslash X$ and we refer to X_M as a pre-special subvariety of X.

Since $x(\mathbb{C}^\times)$ is contained in $M(\mathbb{R})$ and

$$M(\mathbb{R})^+ \to M^{\mathrm{ad}}(\mathbb{R})^+$$

is surjective, we have that X_M is equal to the $M^{\mathrm{ad}}(\mathbb{R})^+$ conjugacy class of x. In particular, for any direct product decomposition $M_1 \times M_2$ of M^{ad} defined over $\mathbb{Q}$, we have

$$X_M = X_1 \times X_2$$

and, for any point $x_1 \in X_1$, the image of $\{x_1\} \times X_2$ in $\Gamma\backslash X$ is again an algebraic subvariety of $\Gamma\backslash X$. We refer to such a subvariety as a weakly special subvariety of $\Gamma\backslash X$. In particular, any special subvariety of $\Gamma\backslash X$ is weakly special. We refer to $\{x_1\} \times X_2$ as a pre-weakly special subvariety of X.

2. The strategy

As in all instances of the Pila-Zannier strategy, the method can be broken into two parts. Both parts rely on o-minimality, though only the latter relies on the Pila-Wilkie counting theorems. The first part is geometric in nature and relies on a result from functional transcendence. We prove the following theorem that was obtained for products of modular curves by Habegger and Pila.

Theorem 1 (cf. [3], Proposition 6.6)**.** *Let V be a subvariety of S. There exists a finite set Σ of pre-special subvarieties of X such that if W is a subvariety of V that is optimal in V then W is an irreducible component of*

$$V \cap \pi(\{x_1\} \times X_2),$$

for some $x_1 \in X_1$, where $X_1 \times X_2 \in \Sigma$.

It is then straightforward to show that Conjecture 1 follows from...

Conjecture 2. *Let V be a subvariety of S. Then V contains only finitely many points that are optimal in V.*

At this point we enter the second phase of the strategy. Using the uniform (in families) version of Pila-Wilkie, we are able to show that Conjecture 2 holds under certain arithmetic hypotheses. The first of which is the so-called large Galois orbits conjecture (LGO), which says that the Galois orbit of an optimal point P should grow at least as quickly as a uniform positive power of the complexity of $\langle P \rangle$. Habegger and Pila obtained this conjecture in [4] for certain curves in products of modular curves but it is otherwise completely open.

The remaining hypotheses are concerned with the parametrisation of special subvarieties and optimal points. To use the Pila-Wilkie theorem, we need to control the heights of certain elements as well as the degrees of certain fields of definition. The height of a pre-special point in a fundamental set was bounded by Orr and the author in [1] but two hypotheses remain outstanding, though we are able to verify them both in a product of modular curves and hence give a new proof of Conjecture 1 under the LGO in that case.

3. Proof of Theorem 1

Recall that X can be realised as a bounded symmetric domain in $\mathbb{C}^N$ for some $N \in \mathbb{N}$. We define a subvariety of X to be any irreducible analytic component of the intersction of X with a subvariety of $\mathbb{C}^N$.

Fix a subvariety V of S. We say that a subset A of $\pi^{-1}(V)$ is an intersection component if it is an irreducible analytic component of the intersection of $\pi^{-1}(V)$ with a subvariety of X. If A is an intersection component, we let $\langle A \rangle_{\mathrm{Zar}}$ denote the smallest subvariety of X containing A i.e. the Zariski closure of A. We say that A is Zariski optimal if for any intersection component B strictly containing A, we have

$$\delta_{\mathrm{Zar}}(B) > \delta_{\mathrm{Zar}}(A).$$

The following conjecture is a problem in functional transcendence. Pila and Tsimerman gave a proof in [5] for the case when S is a product of modular curves. A proof of the full conjecture has recently been anounced by Mok, Pila, and Tsimerman.

Conjecture 3 (weak hyperbolic Ax-Schanuel)**.** *Let A be a Zariski optimal intersection component. Then $\langle A\rangle_{\mathrm{Zar}}$ is pre-weakly special.*

Using Conjecture 3, we can show that if a subvariety W of V is optimal in V then any irreducible analytic component of $\pi^{-1}(W)$ is a Zariski optimal intersection component. In particular, such a component is an irreducible component of the intersection of its Zariski closure with $\pi^{-1}(V)$. By Conjecture 3, the Zariski closure is pre-weakly special and, by virtue of the fact that the restriction of π to a fundamental set is definable in $\mathbb{R}_{\mathrm{an,exp}}$, we can choose Σ is bijection with a definable set. However, since the set of pre-special subvarieties is countablea and $\mathbb{R}_{\mathrm{an,exp}}$ is o-minimal, Σ must therefore be finite.

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Zeroes and $\mathbb{Q}$-points of analytic or oscillatory functions

Georges Comte

(joint work with Christopher Miller, Yosef Yomdin)

For $x = \dfrac{a}{b}, y = \dfrac{p}{q} \in \mathbb{Q}$ with $a \wedge b = p \wedge q = 1$, let us denote the height of (x, y) by

$$\mathrm{ht}(x, y) := \max\{|a|, |b|, |p|, |q|\},$$

For $f : [0, 1] \to \mathbb{R}$ (resp. $f : \bar{D}(0, 1) \to \mathbb{C}$) a C^∞ or an analytic function on a neighbourhood of $[0, 1]$ (resp. $\bar{D}(0, 1)$), let us denote by Γ_f the graph of f. Finally for $T \geq 1$ let us classically denote the number of $\mathbb{Q}$-points in Γ_f of height at most T by

$$\Gamma_f(\mathbb{Q}, T) := \{(x, f(x)) \in \Gamma_f \cap \mathbb{Q}^2; \mathrm{ht}(x, f(x)) \leq T\}.$$

We want to give, beyond the standard hypothesis of o-minimality made on Γ_f, conditions on f under which $\#\Gamma_f(\mathbb{Q}, T)$ is, in a certain sense, small. In this goal we introduce the following notation. For $d \in \mathbb{N}$, $\mathbf{P}_d \subset \mathbb{K}[X,Y]$ is the space of polynomials of degree $\leq d$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and

$$Z_d(f) := \sup_{P \in \mathbf{P}_d \setminus \{0\}} \#\{P(z, f(z)) = 0\} \in \mathbb{N} \cup \{\infty\}.$$

The set $Z_d(f)$ is the maximum number of intersection points between Γ_f and algebraic curves of degree $\leq d$. Call a bound for $Z_d(f)$ a *Bézout bound*. In case f is a polynomial function, $Z_d(f)$ is polynomially bounded in d (and $\deg f$) when the intersection $\Gamma_f \cap P^{-1}(0)$ is transverse and in general, when f is analytic, by the curve selection lemma one has the dichotomy

$$Z_d(f) < \infty \text{ or } \Gamma_f \text{ contains a semialgebraic curve (of dimension 1).}$$

We assume that for any $d \in \mathbb{N}$, $Z_d(f) < \infty$, that is f is a *transcendental function.*

The two following celebrated theorems 1 and 2, stating that $\#\Gamma_f(\mathbb{Q}, T)$ is subpolynomial, motivate this work.

Theorem 1 (see [2])**.** *When $f : [0,1] \to \mathbb{R}$ is analytic, and with the notation above,*

$$\forall \varepsilon > 0 \ \exists C_{f,\varepsilon} \geq 0 \ \ s.t. \ \ \forall T \geq 1, \ \ \#\Gamma_f(\mathbb{Q}, T) \leq C_{f,\varepsilon} T^{\varepsilon}.$$

In the more general case of definable sets in some o-minimal structure over the real field, we have:

Theorem 2 (see [9])**.** *For $X \subset \mathbb{R}^n$ an o-minimal set and obvious notation adapted to X,*

$$\forall \varepsilon > 0 \ \exists C_{X,\varepsilon} \geq 0 \ \ s.t. \ \ \forall T \geq 1, \ \ \#X^{trans}(\mathbb{Q}, T) \leq C_{X,\varepsilon} T^{\varepsilon},$$

where $X^{alg} := \{x \in X; \exists S$ semialgebraic of pure dimension 1, s.t. $x \in S \subset X\}$ and $X^{trans} := X \setminus X^{alg}$.

We give now conditions under which for a transcendental curve $\Gamma \subset \mathbb{R}^n$ (possibly oscillatory, with infinite length) the asymptotic of $\#\Gamma_f(\mathbb{Q}, T)$ is better than in Theorems 1 and 2 above (note that $n = 2$ for curves is enough). We first introduce specific parametrizations, called *slow* parametrizations.

Definition 1. *The C^∞-parametrization $\gamma = (f, g) : [a, +\infty[\to \mathbb{R}^2$ of a curve $\Gamma \subset \mathbb{R}^2$ is slow when*

(1) $\exists u \in \mathbb{R}, \forall x \geq a, \ |u - f(x)| \leq b(x) \searrow 0,$

(2) $\forall p \geq 0, \forall x \geq a, \ |\dfrac{f^{(p)}(x)}{p!}| \leq \varphi_p(x), \ |\dfrac{g^{(p)}(x)}{p!}| \leq \varphi_p(x),$

where $\exists$ constants A, B, C, D s.t. $\forall p \geq 1, \forall x \geq a \varphi_p(x) = D\Big(Ap^B \dfrac{\log^C x}{x}\Big)^p$.

Remark 1. *Functions satisfying* (2) *in Defintion 1 yield a subalgebra of the algebra $C^\infty([a, +\infty[)$, that is stable under derivation.*

Example 1. *functions of the form $g := h \circ \log^{\ell}$, where $\ell \geq 1$ and $\exists \alpha, \forall p \geq 0$, $|h^{(p)}(x)| \leq \alpha^p$, are slow.*

Definition 2. *$\varphi : [a, +\infty[\to \mathbb{R}$ is a height control function of γ when*

$$\forall T \geq 1, \quad \gamma^{-1}(\Gamma(\mathbb{Q}, T)) \subset [a, \varphi(T)].$$

Example 2. *For f slow, when $u \in \mathbb{Q}$ and f doesn't take the value u, one can take $\varphi(T) = b^{-1}(\frac{K}{T})$, and when $u \notin \mathbb{Q}$, and is not a U-number of degree 1 in Mahler's classification, one can take $\varphi(T) = b^{-1}(\frac{1}{T^K})$, for some K.*

Theorem 3 (see [3])**.** *Let γ be a slow parametrization of a transcendental curve Γ, with height control function φ, $T \geq 1$, $d \geq 1$ and*

$$\mathbf{B}_{d,A} := \sup_{P \in \mathbf{P}_d \setminus \{0\}} \# P^{-1}(0) \cap \gamma([a, A]),$$

then

$$\exists \delta, \nu \geq 0 \ \forall T \geq 1, \quad \#\Gamma(\mathbb{Q}, T) \leq \alpha \log^{\delta}(T) \times \log^{\nu}(\varphi(T)) \times \mathbf{B}_{\log T, \varphi(T)}.$$

Consequence. When $e^{\varphi(T)}$ and $\mathbf{B}_{d,A}$ are polynomially bounded in T, d, A then

$$\exists \alpha, \beta \geq 0 \ \ s.t. \ \ \forall T \geq 1, \quad \#\Gamma(\mathbb{Q}, T) \leq \alpha \log^{\beta}(T).$$

Example 3. *We use elementary functions composed with a power of* log*; it provides suitable $\mathbf{B}_{d,A}$ and φ (see [6], [3]) such as in the Consequence above.*

- *log-spirals: $\gamma(x) = (\frac{1}{x^F} \sin \circ \log^{\ell}, \frac{1}{x^G} \cos \circ \log^{q})$, $F, G > 0$, $\ell, q \in \mathbb{N}^*$.*
- *$\gamma(x) = (\log 2 + \dfrac{\arctan \log^2 x}{x^5(2 + \cos^3 \log x)}, \pi + \dfrac{\sin \log^2 x}{\sqrt{x}(1 + \log \log x)})$ etc.*
- *Graphs: $x \mapsto \sin \log^{\ell} x$ max. sol. of Euler equation $x^2 y'' + xy' + y = 0$ ($\sin \log^{\ell} x$ defines $\mathbb{Z}$ over $\mathbb{R} \iff \ell > 1$), and more generally the graph of any slow function built on elementary functions.*

Related open questions in model theory and number theory. Do we have a $\log^{\beta} T$-bound on $\#\Gamma(\mathbb{Q}, T)$ for

(1) sets definable in the expansion of $\mathbb{R}$ by any log-spiral and by restricted sin and exp (see [1])?
(2) $\emptyset$-definable sets of the expansion of $\mathbb{R}$ by any log-spiral?
(3) $\emptyset$-definable sets of $(\mathbb{R}, \sin \log)$? etc.

Remark 2. *When f is in some o-minimal structure, there exists $K_{f,d}$ s.t. $Z_d(f) \leq K_{f,d} < \infty$. On the other hand $Z_d(f)$ may be polynomially bounded in d while f is not o-minimal (see [5]). Even when f is analytic, the asymptotic of $Z_d(f)$ is difficult to predict: for any $\zeta \in]0, 1[$, there exists $f : D \to \mathbb{C}$ analytic such that for a sequence of degrees d going to ∞, $Z_d(f) \geq e^{d^{\zeta}}$ (see [10], [11], [7]). But again, on the other side, for f entire of finite order $:= \limsup_{r \to \infty} \dfrac{\log \log \max_{D_r} |f|}{\log r}$, for a certain sequence of degrees going to ∞, $Z_d(f) \leq Cd^2$ (best possible asymptotic).*

The conditions of the following Theorem 4 on the coefficients of the Taylor expansion of f at the origin guarantee a polynomial bound in d for $Z_d(f)$, $f : \bar{D}(0,1) \to \mathbb{C}$. It turns out that such a bound for $Z_d(f)$ implies that $\#\Gamma_f(\mathbb{Q}, T)$ is poly-log bounded in T (see [8]). Before Stating Theorem 4 we first define the sequence

$$b_d := \max_{P \in \mathbf{P}_d \setminus \{0\}} \#\{\mathrm{mult}_0 P(z, f(z))\}, \quad d \geq 1.$$

Recall that f is *hypertranscendental* when f satisfies no algebraic differential equation over $\mathbb{Z}$. For f hypertranscendental, let us now define the sequence

$$\eta_d := \max_{P \in \mathbb{Z}_d[X_0, \cdots, X_d] \setminus \{0\}} \{\mathrm{mult}_0 P(z, f(z), f'(z), \cdots, f^{(d)}(z))\}.$$

Note that $(b_d)_{d\geq 1}$ (resp. $(\eta_d)_{d\geq 1}$) measures the transcendency (resp. the hypertranscendency) of f, since the faster $(b_d)_{d\geq 1}$ (resp. $(\eta_d)_{d\geq 1}$) goes to ∞ the less f seems transcendental (resp. hypertranscendency). We have

Theorem 4 (see [4])**.** *For $f(z) = \sum_{k=0}^{\infty} a_k z^k$, in case $a_k \in \mathbb{Q}$ we denote by h_ℓ a bound for the denominators of $|a_0|, \cdots, |a_\ell|$. Then assuming one of the following conditions*

(1) $f \in \mathbb{Q}\{z\}$, $\exists R, S \in \mathbb{R}[X]$ *s.t.* $b_d \leq R(d)$, $h_l \leq e^{S(l)}$,
(2) $f \in \mathbb{Q}\{z\}$, $\exists R, S \in \mathbb{R}[X]$ *s.t.* $\eta_d \leq R(d)$, $h_l \leq e^{S(l)}$,
(3) $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \in \mathbb{R}\{z\}$, $n_k^2 < n_{k+1} \leq n_k^q$, *for* $q > 2$, $|a_k| \geq e^{-n_k^p}$, *for* $p > 0$,
(4) f *is a solution of a linear differential equation with coefficients in* $\mathbb{Q}[z]$ *with rational initial conditions,*
(5) f *is a random series,*

there exists $U \in \mathbb{R}[X]$ s.t. $\forall d \geq 1$, $Z_d(f) \leq U(d)$ and therefore (see [8]*) there exist $\alpha, \beta > 0$, s.t. $\forall T \geq 1$, $\#\Gamma_f(\mathbb{Q}, T) \leq \alpha \log^\beta T$.*

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Normalization and factorization of linear ordinary differential operators

Sergei Yakovenko

(joint work with Leanne Mezuman, Shira Tanny)

The local theory of linear ordinary differential equations exists in two closely related but different flavors.

One may consider singular point of a system of first order differential equations which have the form

$$t^{1+r}\dot{x} = A(t)x, \quad t \in (\mathbb{C}, 0),\ A(t) = A_0 + tA_1 + t^2A_2 + \cdots \in \mathbb{C}[[t]] \otimes \operatorname{Mat}(n, \mathbb{C}),$$

where the nonnegative integer $r \in \mathbb{Z}_+$ is the Poincaré rank; if $r = 0$, the singularity is called Fuchsian. On the space of such systems there is a natural group action, called the gauge equivalence: two systems defined by two matrix series $A(t), B(t)$ are equivalent, if there exists a matrix series $H \in \operatorname{GL}(n, \mathbb{C}[[t]])$ such that

$$t^{1+r}\dot{H} = HA - BH, \qquad H = H_0 + tH_1 + t^2H_2 + \cdots, \ \det H_0 \neq 0.$$

The simplest form, to which a system can be transformed by a (always formal in our settings) gauge equivalence, depends on the Poincaré rank and the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the leading matrix A_0. For several reasons it is more convenient to write the systems using the Euler derivation $\boldsymbol{\epsilon} = t\,\frac{\mathrm{d}}{\mathrm{d}t}$ rather than the usual derivation in t, denoted by the dot above.

Theorem 1 (H. Poincaré, H. Dulac)**.** *If no two eigenvalues of a Fuchsian system differ by a positive integer, $\lambda_i - \lambda_j \notin \{1, 2, 3, \ldots\}$, then the system is gauge equivalent to an Euler system $(\boldsymbol{\epsilon} - A_0)x = 0$.*

If d is the largest natural difference between the eigenvalues, then the Fuchsian system is equivalent to an integrable polynomial system $(\boldsymbol{\epsilon} - A_0 + \cdots + t^dA_d)x = 0$.

In the non-Fuchsian case with $r > 0$ we have the following result.

Theorem 2 (Diagonalization theorem, Hukuhara–Turritin–Levelt)**.** *If all eigenvalues of a non-Fuchsian system are pairwise different, $\lambda_i \neq \lambda_j$ for $i \neq j$, then the system can be transformed into a diagonal form $(t^r\boldsymbol{\epsilon} - B(t))x = 0$ with the diagonal matrix function $B(t) = \operatorname{diag}(\beta_1(t), \ldots, \beta_n(t))$, $\beta_i \in \mathbb{C}[[t]]$, $\beta_i(0) = \lambda_i$.*

The normal form in the resonant case is more complicated.

Another flavor of the local linear theory is that of linear ordinary differential operators of higher order. Such equations can be written using *differential operators* under the form $Ly = 0$, where $L = a_0(t)\boldsymbol{\epsilon}^n + a_1(t)\boldsymbol{\epsilon}^{n-1} + \cdots + a_{n-1}(t)\boldsymbol{\epsilon} + a_n(t)$ is an operator with the coefficients $a_i \in \mathbb{C}[[t]]$, $a_0 \not\equiv 0$, and $\boldsymbol{\epsilon} = t\frac{\mathrm{d}}{\mathrm{d}t}$ is the Euler derivation operator. Any such operator can be re-expanded in the form $L = \sum_{k=0}^{\infty} t^k p_k(\boldsymbol{\epsilon})$, $k \in \mathbb{Z}_+$, $p_k \in \mathbb{C}[\boldsymbol{\epsilon}]$, in other words, $\mathscr{W} = \mathbb{C}[[t]] \otimes_{\mathbb{C}} \mathbb{C}[\boldsymbol{\epsilon}]$ with the commutation identity $\boldsymbol{\epsilon}^j t^k = t^k(\boldsymbol{\epsilon} + k)^j$. An operator L is *Fuchsian*, if $\deg p_0 = n = \max_k \deg p_k$. Fuchsian operators form a subset $\mathscr{F} \subseteq \mathscr{W}$ closed by composition, albeit not a subalgebra.

There is no natural group acting on differential operators, but they form a (non-commutative) $\mathbb{C}$-algebra $\mathscr{W}$ with respect to composition.

Definition 1 (cf. [O, TY]). *Two operators $L, M \in \mathscr{W}$ are $\mathscr{F}$-equivalent, if there exist two operators $K, H \in \mathscr{F}$ such that $KL - MH = 0$ and $\gcd(H, L) = 1$.*

This means that the operator H acting by $u = Hy$ sends solutions y of the operator $Ly = 0$ to those of $Mu = 0$, while not vanishing on any one of them.

Theorem 3 (cf. [TY]). *If $L = p_0(\boldsymbol{\epsilon}) + tp_1(\boldsymbol{\epsilon}) + \cdots$ is a Fuchsian operator and no two roots of the polynomial $p_0 \in \mathbb{C}[\boldsymbol{\epsilon}]$ differ by a positive integer, $\lambda_i - \lambda_j \notin \{1, 2, 3, \dots\}$, then the operator is $\mathscr{F}$-equivalent to an Euler equation $M = p_0(\boldsymbol{\epsilon}) \in \mathbb{C}[\boldsymbol{\epsilon}] \subseteq \mathscr{F}$.*

If d is the largest natural difference between the roots, then the Fuchsian operator is $\mathscr{F}$-equivalent to a Liouville integrable operator $(\boldsymbol{\epsilon} - \beta_1(t)) \cdots (\boldsymbol{\epsilon} - \beta_n(t))$ with polynomial coefficients $\beta_i \in \mathbb{C}[t]$ of degrees $\leq d$, $\beta_i(0) = \lambda_i$.

In the non-Fuchsian case we look for an analog of the Diagonalization theorem, which would describe factorization of an operator $L \in \mathscr{W} \setminus \mathscr{F}$ into a composition of operators of smaller orders. The answer depends on the growth pattern of the degrees $\deg p_k$, $k = 0, 1, 2, \dots$ expressed in terms of the Newton diagram. If $L = \sum_{j,k} c_{jk} t^k \boldsymbol{\epsilon}^j$ is the double series (all powers of t appear to the right from powers of $\boldsymbol{\epsilon}$), then the support $\operatorname{supp} L = \{(j,k) \colon c_{jk} \neq 0\}$ is a subset in $\mathbb{Z}_+^2$, and the Newton polygon Δ_L is the convex hull of the origin $(0,0)$, the support $\operatorname{supp} L$ and its vertical translates by $(0,1)$. The Newton polygon is an epigraph of a piecewise-affine convex monotone function $\chi \colon [0, d] \to \mathbb{R}_+$ with corners only at the lattice points $\mathbb{Z}_+^2 \subseteq \mathbb{R}_+^2$. The set of values of its derivative (slopes) is called the Poincaré spectrum of L, $S(L) \subseteq \mathbb{Q}_+$.

The main property of the Newton polygon is the identity $\Delta_{LM} = \Delta_{ML} = \Delta_L + \Delta_M$ with respect to the Minkowski sum, which holds exactly in the same form as for the *commutative algebra of pseudopolynomials* $\mathscr{P} = \mathbb{C}[[t]] \otimes \mathbb{C}[\xi] = \mathbb{C}[[t]][\xi]$. The latter case is well known since Newton's invention of the "rotating ruler method" [N]. It turns out that one can derive directly the results for the non-commutative algebra $\mathscr{W}$ from those for $\mathscr{P}$.

Definition 2. *A* single-slope *operator $L = \sum a_{j,k}\, t^k \boldsymbol{\epsilon}^j$ is the operator for which the function χ is linear, $\chi(j) = rj$, $r = \frac{p}{q} \in \mathbb{Q}_+$, so that $S(L) = \{r\}$.*

A symbol *of a single-slope operator is the polynomial*

$$\sigma_L(t,\xi) = \sum_{k-rj=0} a_{jk}\, t^k \xi^j = \prod_{i=1}^{m} (\lambda_i - t^p \xi^q)$$

of degree $n = mq$, $n = \operatorname{ord} L$. The numbers $\lambda_i \in \mathbb{C}^*$ are characteristic roots.

A single-slope operator is totally resonant*, if all characteristic roots $\lambda_1, \ldots, \lambda_m \in \mathbb{C}^*$ of its symbol coincide, $\lambda_1 = \cdots = \lambda_m$, in particular, if $m = 1$.*

Theorem 4 (cf. [M, R, vdPS])**.**
1. *A non-Fuchsian operator L with the Poincaré spectrum $S(L) = \{r_1, \ldots, r_s\} \subseteq \mathbb{Q}_+$, $r_i \neq r_j$, admits factorization into single-slope terms $L = L_1 \cdots L_s$, $S(L_i) = r_i$.*
2. *A single-slope non-Fuchsian operator L with $S(L) = \{r\}$ and symbol σ admits factorization into totally resonant operators of the same slope.*
3. *In particular, if all characteristic roots $\lambda_1, \ldots, \lambda_m \in \mathbb{C}^*$ of the single-slope operator L are pairwise different, $\lambda_i \neq \lambda_j$, then this operator admits factorization into m operators of the first order $L_i = t^p \boldsymbol{\epsilon}^q - \beta_i(t)$, $\beta_i \in \mathbb{C}[[t]]$, $\beta_i(0) = \lambda_i$.*

The known proofs of this theorem are based on involved considerations in the non-commutative algebra $\mathscr{W}$. In contrast with that, we developed a direct approach that allows direct transfer of factorization results in the commutative algebra $\mathscr{P}$ or in the local algebra $\mathbb{C}[[t,s]]$ to the non-commutative case.

More specifically, we consider weighed quasihomogeneous polynomials in two variables with the weight $w(t^k\xi^j) = k - rj$, $r \in \mathbb{Q}_+$ and the *homological equations*

$$P_\alpha u_{\gamma-\alpha} + Q_\beta v_{\gamma-\beta} = S_\gamma, \quad \operatorname{supp} P_\alpha \subseteq \Delta,\ \operatorname{supp} Q_\beta \subseteq \Delta'',\ \operatorname{supp} S_\gamma \subseteq \Delta' + \Delta'',$$

which have to be solved with respect to the unknown quasihomogeneous polynomials $u_{\gamma-\alpha}, v_{\gamma-\beta}$ subject to the constraints $\operatorname{supp} u_{\gamma-\alpha} \subseteq \Delta''$, $\operatorname{supp} v_{\gamma-\beta} \in \Delta'$. Solvability of these equations for any weight γ and any right hand side S_γ depends on the Newton polygons Δ', Δ'' and the polynomials P_α, Q_β in a very nontrivial way, but can be derived from the factorization results in $\mathscr{P}$.

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Reporter: Christopher Daw

Participants

Dr. Fabrizio Barroero
Mathematisches Institut
Universität Basel
Spiegelgasse 1
4051 Basel
SWITZERLAND

Prof. Dr. Saugata Basu
Department of Mathematics
Purdue University
150 N. University Street
West Lafayette IN 47907-2067
UNITED STATES

Prof. Dr. Daniel Bertrand
Institut de Mathématiques
Université Pierre et Marie Curie
Case 247
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Dr. Gal Binyamini
Department of Mathematics
The Weizmann Institute of Science
P. O. Box 26
Rehovot 76100
ISRAEL

Dr. Laura Capuano
Mathematical Institute
University of Oxford
Andrew Wiles Building
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Prof. Dr. Antoine Chambert-Loir
UFR de Mathématiques
Université Paris Diderot
5, rue Thomas Mann
75205 Paris Cedex 13
FRANCE

Prof. Dr. Zoé Chatzidakis
Département de Mathématiques et
Applications
Ecole Normale Supérieure
45, Rue d'Ulm
75230 Paris Cedex 05
FRANCE

Dr. Derya Ciray
Fachbereich Mathematik und Statistik
Universität Konstanz
Postfach 216
78457 Konstanz
GERMANY

Prof. Dr. Raf Cluckers
Laboratoire Paul Painleve, UMR 8524
Université des Sciences et Technologie de
Lille 1
Cité Scientifique, Bat. M2
59655 Villeneuve-d'Ascq Cedex
FRANCE

Prof. Dr. Georges Comte
Laboratoire de Mathématiques
Université de Savoie - Mont Blanc
73376 Le Bourget-du-Lac Cedex
FRANCE

Dr. Christopher M. Daw
Department of Mathematics and
Statistics
University of Reading
Whiteknights
P.O. Box 220
Reading RG6 6AX
UNITED KINGDOM

Prof. Dr. Mario Jorge Edmundo
Departamento de Matemática
Universidade de Lisboa
Campo Grande, Edifício C6, Gabinete 6.1.
1749-016 Lisboa Codex
PORTUGAL

Prof. Dr. Pantelis Eleftheriou
Fachbereich für Mathematik und Statistik
Universität Konstanz
Box 216
78457 Konstanz
GERMANY

Dr. Christopher Frei
Institut für Analysis und Zahlentheorie
Technische Universität Graz
Kopernikusgasse 24/II
8010 Graz
AUSTRIA

Dr. James Freitag
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago
851 S. Morgan Street
Chicago IL 60607-7045
UNITED STATES

Prof. Dr. Andrei Gabrielov
Department of Mathematics
Purdue University
150 N. University Street
West Lafayette IN 47907-2067
UNITED STATES

Zeinab Galal
UFR de Mathématiques
Université Paris Diderot
Bâtiment Sophie Germain
8 Place Aurélie Nemours
75013 Paris Cedex 13
FRANCE

Prof. Dr. Philipp Habegger
Department für Mathematik und Informatik
Universität Basel
Spiegelgasse 1
4051 Basel
SWITZERLAND

Prof. Dr. Gareth Owen Jones
School of Mathematics
The University of Manchester
Manchester M13 9PL
UNITED KINGDOM

Prof. Dr. Tobias Kaiser
Fakultät für Mathematik und Informatik
Universität Passau
94030 Passau
GERMANY

Prof. Dr. Krzysztof Kurdyka
Laboratoire de Mathématiques
Université de Savoie
73376 Le Bourget-du-Lac Cedex
FRANCE

Dr. Olivier Le Gal
Département de Mathématiques
Université de Savoie
P.O. Box Boite Postale 1
73011 Chambery Cedex
FRANCE

Prof. Dr. Francois Loeser
Institut de Mathématiques de Jussieu
Université de Paris VI, Case 247
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Dr. Ta Le Loi
Department of Mathematics
University of Dalat
1 Phu Dong Thien Vuong
Dalat, Lamdong
VIETNAM

Prof. Dr. Angus John MacIntyre
7a West Castle Road
Edinburgh EH10 5AT
UNITED KINGDOM

Dr. Jana Marikova
Department of Mathematics
Western Illinois University
Macomb, IL 61455
UNITED STATES

Prof. Dr. Chris Miller
Department of Mathematics
The Ohio State University
100 Mathematics Building
231 West 18th Avenue
Columbus, OH 43210-1174
UNITED STATES

Dr. Dmitry Novikov
Department of Mathematics
The Weizmann Institute of Science
P.O. Box 26
Rehovot 76100
ISRAEL

Dr. Martin Orr
Imperial College London
Department of Mathematics
South Kensington Campus
681, Huxley Building
London SW7 2AZ
UNITED KINGDOM

Prof. Dr. Adam Parusinski
Laboratoire J.-A. Dieudonne
Université de Nice
Sophia Antipolis
Parc Valrose
06108 Nice Cedex 2
FRANCE

Prof. Dr. Wieslaw Pawlucki
Instytut Matematyki
Uniwersytet Jagiellonski
ul. Reymonta 4
30-059 Kraków
POLAND

Prof. Dr. Kobi Peterzil
Department of Mathematics and
Computer Sciences
University of Haifa
Mount Carmel
Haifa 31905
ISRAEL

Dr. Marta Pieropan
Fachbereich Mathematik und Informatik
Freie Universität Berlin
Arnimallee 3
14195 Berlin
GERMANY

Prof. Dr. Jonathan Pila
Mathematical Institute
University of Oxford
Andrew Wiles Building
Radcliffe Observatory Quarter
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Dr. Armin Rainer
Fakultät für Mathematik
Universität Wien
Oskar-Morgenstern-Platz 1
1090 Wien
AUSTRIA

Dr. Serge Randriambololona
Department of Mathematics
Galatasaray University
Ciragan Cad. No. 36
34357 Ortaköy, Istanbul
TURKEY

Prof. Dr. Jean-Philippe Rolin
UFR des Sciences et Techniques
Université de Bourgogne
B.P. 47970
9, Avenue Alain Savary
21078 Dijon Cedex
FRANCE

Dr. Zofia Rozen
Instytut Matematyki
Uniwersytet Jagiellonski
ul. Prof. Stanisława Łojasiewicza 6
30-348 Kraków
POLAND

Dr. Fernando Sanz
Departamento de Algebra, Análisis
Matemático, Geometría y Topología
Facultad de Ciencias
Universidad de Valladolid
47011 Valladolid
SPAIN

Prof. Dr. Thomas W. Scanlon
Department of Mathematics
University of California
Berkeley CA 94720-3840
UNITED STATES

Dr. Harry Schmidt
Mathematical Institute
Oxford University
24-29 St. Giles
Oxford OX1 3LB
UNITED KINGDOM

Dr. Tamara Servi
UFR de Mathématiques
Université Paris Diderot
Bâtiment Sophie Germain
BC 7012
75205 Paris Cedex 13
FRANCE

Prof. Dr. Patrick Speissegger
Department of Mathematics and
Statistics
McMaster University
1280 Main Street West
Hamilton ON L8S 4K1
CANADA

Haden Spence
Lincoln College
Oxford University
Turl Street
Oxford OX1 3DR
UNITED KINGDOM

Prof. Dr. Sergei S. Starchenko
Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556-4618
UNITED STATES

Dr. Margaret E. Thomas
Zukunftskolleg
Fachbereich Mathematik und Statistik
Universität Konstanz
Universitätsstrasse 10
Postfach 216
78457 Konstanz
GERMANY

Prof. Dr. Nicolai N. Vorobjov
Department of Computer Sciences
University of Bath
Bath BA2 7AY
UNITED KINGDOM

Dr. Martin Widmer
Department of Mathematics
Royal Holloway College
University of London
Egham TW20 0EX
UNITED KINGDOM

Prof. Dr. Alex Wilkie
Mathematical Institute
University of Oxford
Andrew Wiles Building
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Dr. Andrei Yafaev
Department of Mathematics
University College London
Gower Street
London WC1E 6BT
UNITED KINGDOM

Dr. Sergei Yakovenko
Department of Mathematics
The Weizmann Institute of Science
P. O. Box 26
Rehovot 76100
ISRAEL

Prof. Dr. Yosef N. Yomdin
Department of Mathematics
The Weizmann Institute of Science
P. O. Box 26
Rehovot 76100
ISRAEL