Mathematisches Forschungsinstitut Oberwolfach

Report No. 26/2017

DOI: 10.4171/OWR/2017/26

Stochastic Analysis: Geometry of Random Processes

Organised by
Alice Guionnet, Lyon
Martin Hairer, Warwick
Grégory Miermont, Lyon

28 May – 3 June 2017

Abstract. A common feature shared by many natural objects arising in probability theory is that they tend to be very "rough", as opposed to the "smooth" objects usually studied in other branches of mathematics. It is however still desirable to understand their geometric properties, be it from a metric, a topological, or a measure-theoretic perspective. In recent years, our understanding of such "random geometries" has seen spectacular advances on a number of fronts.

Mathematics Subject Classification (2010): 60xx, 82xx.

Introduction by the Organisers

The workshop focused on the latest progresses in the study of random geometries, with a special emphasis on statistical physics models on regular lattices and random maps, and their connexions with the Gaussian free field and Schramm-Loewner evolutions.

The week opened with a spectacular talk by H. Duminil-Copin (based on his work with Raoufi and Tassion), presenting a new, groundbreaking use of decision trees in the context of statistical physics models in order to prove the sharpness of the phase transition, and vastly extending earlier results. Most notably, this new approach allows to circumvent the use of either the BK inequality or of planarity properties. In particular, it applies to the random cluster model $\mathrm{FK}(d, q)$ on $\mathbb{Z}^d$ for every parameter $q \geq 1$ and dimension $d \geq 2$.

A significant proportion of the talks focused on various and ever growing aspects of discrete random geometries, notably random planar maps, and their links with the planar Gaussian free field (GFF). N. Curien reported recent progress (notably with Bertoin, Budd, Kortchemski, Le Gall, Marzouk) on the properties of metrics, percolation and random walks on large random maps and their duals, paving the way to a (not yet formally identified) one-parameter family of random surfaces. J.-F. Le Gall presented how the positive excursions of the Brownian snake, defined with C. Abraham, can be used to construct the so-called Brownian disks and to analyse how these disks arise naturally in the Brownian map, or more precisely in the complement of its metric net introduced by Miller and Sheffield. J. Bouttier explained how the statistics of the nested loops in loop $O(n)$ models in maps can be analyzed from principles of analytic combinatorics and B. Duplantier discussed the continuum aspects of this analysis. Asaf Nachmias reported progress with Curien and Hutchcroft on the proof that the spectral dimension of causal random triangulations is 2 (a fact conjectured by Durhuus, Jonsson and Wheater). Louigi Addario-Berry reported on his joint work with Angel, Chapuy, Fusy, and Goldschmidt, dealing with the distribution of the Voronoi cells in the Brownian CRT, showing in this simpler context a version of a conjecture by Chapuy on the Brownian map. Jian Ding presented his results on the semi-discrete approach to random metrics (with Biskup, Dunlap, Goswami, Li, Zeitouni, Zhang), in which he studies natural metrics derived from the exponential of the discrete GFF in 2 dimensions, their scaling limits, and the fractal dimensions of the limiting spaces. In a related direction, O. Louidor discussed his recent progress with Biskup on the structure of the maxima of the discrete GFF.

On the "purely continuum side", Wendelin Werner described his joint work with Miller and Sheffield on percolation in conformal loop ensembles and its links with SLE/GFF couplings. This elaborate continuum percolation model on a random fractal is conjectured to describe the continuum limit of the loops in a discrete loop $O(n)$ models on random maps when conformally embedded in the plane. Titus Lupu (with Aru and Sepulveda) derived a continuum version of the isomorphism theorem for 2-dimensional GFF based on loop-soups, hence generalizing his approach based on Markov loops in graphs and making significant progress in the understanding of the level sets of GFF. Hao Wu showed the convergence of 2-dimensional Ising interfaces with mixed boundary conditions, by first constructing (with Beffara and Peltola) the limiting global multiple SLE, paving the way to the computation of connection probabilities. Continuing their series of works on the probabilistic, GFF-based constructions of the Liouville conformal field theory, Vincent Vargas, with Antti Kupiainen and Rémi Rhodes, showed how to derive the DOZZ formula from this approach, giving the 3-point structure constant in this theory. This is a quite striking result, which can be interpreted as a first result in the integrability properties of Gaussian multiplicative chaos measures (a.k.a. Liouville quantum gravity measures, or LQG). Christophe Garban initiated a study of a dynamical model admitting the LQG measures as stationary distributions, and showing similarities with the Sine-Gordon stochastic partial differential equations, with however notable differences due to the appearance of intermittency. Besides this tak, the topic of stochastic PDEs was at the core of Ismal Bailleul's talk, who presented his far-reaching generalization with Bernicot of paracontrolled calculus, in the context of manifolds and allowing higher order expansions.

In a new accomplishment in bridging the gap between the discrete and continuum approach to random planar geometries, S. Sheffield presented his very recent results with Gwynne and Miller on the convergence toward LQG surfaces of the Tutte embedding of discrete random maps models based on the discretization of his earlier "mating of trees" construction. This is at the same time the first convergence result of for some form of "conformal embedding" of discrete random planar maps towards LQG, and it provides in a sense the first one-parameter family of discrete models that approximate SLE curves for all values of κ. The proof is partly based on new techniques for random walks in random environment. Earlier in the week, E. Gwynne had presented his recent impressive series of papers with Miller showing Gromov-Hausdorff convergence of percolated random planar maps to SLE_6 on the Brownian map, a result that connects in a spectacular way the discrete and continuum world, since defining SLE-type curves on the Brownian map requires in a crucial way the Miller-Sheffield construction of the Brownian map based on the GFF.

Several talks focused on random interface models at the edge of integrable probability. In a groundbreaking progress in the study of first-passage percolation, Allan Sly (with Basu and Sidoravicius) was able to improve the upper bound on the variance of the interface fluctuations by a polynomial order, in models which are rotation-invariant, allowing to set apart the problem of the limit shape and to efficiently focus on the fluctuations by an elaborate multi-scale analysis. F. Toninelli reported his recent progress (with Chhita, Ferrari, Legras) on the study of hydrodynamic limits of a growth model in the anisotropic KPZ class. R. Kenyon showed limit shape results obtained with de Gier and Watson for the 5-vertex model with a weight per corners, a non-determinantal generalization of the dimer model which is still analyzable by Bethe Ansatz techniques. Nathanaël Berestycki described his very general result (with Laslier and Ray) for the height fluctuation of dimer models on Riemann surfaces, using original methods based on Miller and Sheffield's imaginary geometry. Vadim Gorin showed how techniques from integrable probability can be used to analyse the local limit of random sorting networks. Antti Knowles reported on his recent results with Benaych-Georges and Bordenave on the spectral radius of Erdoös-Rényi graphs and other sparse random matrices, showing a phase transition between a regime where the largest eigenvalues are described by the edge of the limiting spectral distribution (for degrees larger than $\log n$) and a regime with many outliers (with degrees much less than $\log n$). Still in the direction of random graphs, Christina Goldschmidt reported her ongoing work with Conchon-Kerjan on scaling limits of critical random graphs with i.i.d. fat-tailed degree distributions.

The topic of random optimization and random media was also well-represented. Nike Sun presented her series of groundbreaking papers with Ding, Sly and Zhang on random combinatorial optimisation problems, and in particular random constraint satisfaction problems, analyzing the fine structure of clustering and of replica symmetry breaking. Gérard Ben Arous reported on his recent work with Jagannath on slow mixing for dynamics on spin glass models, partly based on new large deviation principles related to the Franz-Parisi potential. Yan Fyodorov presented a number of motivating mathematical challenges for the understanding of a model of a directed polymer in random media. Jean-Christophe Mourrat showed his powerful quantitative approach to stochastic homogenization in $\mathbb{Z}^d$ (obtained with Armstrong and Kuusi) based on the fact that certain energy quantities behave are asymptotically local and additive functionals of the coefficient field, allowing one to obtain in particular a scaling limit result for the corrector. Omer Angel concluded this very intense and rich week with a talk on coexistence in two annihilating branching random walks, obtained with Ahlberg and Kolesnik.

Despite the slightly uncertain whether conditions during the week, the traditional Wednesday walk to Sankt Roman could take place as scheduled and was followed by the Barbecue evening at MFO.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

Workshop: Stochastic Analysis: Geometry of Random Processes

Table of Contents

Abstracts

Sharp phase transition for the random-cluster models via randomized algorithms

Hugo Duminil-Copin
(joint work with Aran Raoufi, Vincent Tassion)

In theoretical computer science, determining the computational complexity of tasks is a very difficult problem (think of P against NP). To start with a more tractable problem, computer scientists have studied *decision trees*, which are simpler models of computation. A decision tree associated to a Boolean function f takes $\omega \in \{0,1\}^n$ as an input, and reveals algorithmically the value of ω in different bits one by one. The algorithm stops as soon as the value of f is the same no matter the values of ω on the remaining coordinates. The question is then to determine how many bits of information must be revealed before the algorithm stops. The decision tree can also be taken at random to model random or quantum computers.

The theory of (random) decision trees played a key role in computer science (we refer the reader to the survey [BDW02]), but also found many applications in other fields of mathematics. In particular, random decision trees (sometimes called randomized algorithms) were used in [SS10] to study the noise sensitivity of Boolean functions, for instance in the context of percolation theory.

The OSSS inequality, originally introduced in [OSSS05] for product measure as a step toward a conjecture of Yao [Yao77], relates the variance of a Boolean function to the influence of the variables and the computational complexity of a random decision tree for this function. The first part of the talk consists in generalizing the OSSS inequality to the context of monotonic measures which are not product measures. A *monotonic* measure is a measure μ on $\{0,1\}^E$ such that for any $e \in E$, any $F \subset E$, and any $\xi, \zeta \in \{0,1\}^F$ satisfying $\xi \le \zeta$, $\mu[\omega_F = \xi] > 0$ and $\mu[\omega_F = \zeta] > 0$,

$$\mu[\omega_e = 1 \mid \omega_F = \xi] \le \mu[\omega_e = 1 \mid \omega_F = \zeta].$$

Theorem 1 ([DRT17a]). *Fix an increasing function $f : \{0,1\}^E \longrightarrow [0,1]$ on a finite set E. For any monotonic measure μ and any randomized algorithm T,*

$$\text{(1)} \qquad \mathrm{Var}_\mu(f) \;\le\; 2\sum_{e\in E} \delta_e(f,T)\,\mathrm{Cov}_\mu(f,\omega_e),$$

where $\delta_e(f,T)$ is the probability that e is revealed by the randomized algorithm before stopping.

We expect the previous result to have a number of applications. In this paper, we focus on applications in statistical physics. Here, we describe a specific application to the random-cluster model. Since random-cluster models were introduced by Fortuin and Kasteleyn in 1969, they have become the archetypal example of dependent percolation models and as such have played an important role in the study of phase transitions. The spin correlations of Potts models are rephrased as cluster connectivity properties of their random-cluster representations. This allows the use of geometric techniques, thus leading to several important applications. While the understanding of the model on planar graphs progressed greatly in the past few years [BD12, DST17, DGH+16, DRT17a], the case of higher dimensions remained poorly understood.

The model is defined as follows. Consider a finite subgraph $G = (V, E)$ of $\mathbb{Z}^d$ and introduce the boundary ∂G of G to be the set of vertices $x \in G$ for which there exists $y \notin G$ with $e = xy$ an edge of $\mathbb{E}$. A *percolation configuration* $\omega = (\omega_e)_{e \in E}$ is an element of $\{0,1\}^E$. A configuration ω can be seen as a subgraph of G with vertex-set V and edge-set given by $\{e \in E : \omega_e = 1\}$. Let $k(\omega)$ be the number of connected components in ω. Also set $|\omega|$ for the number of e with $\omega_e = 1$.

Fix $q, p > 0$. Let $\phi_{G,p,q}$ be the measure satisfying, for any $\omega \in \{0,1\}^E$,

$$\phi^f_{G,p,q}(\omega) = \frac{o^{|\omega|}(1-p)^{|E|-|\omega|}q^{k(\omega)}}{Z},$$

where Z is a normalizing constant introduced in such a way that $\phi_{G,p,q}$ is a probability measure. The measures $\phi_{G,p,q}$ and $\phi^{\mathrm{w}}_{G,p,q}$ are called the random-cluster measures on G with respectively free and wired boundary conditions. For $q \geq 1$, the measures $\phi_{G,p,q}$ can be extended to $\mathbb{Z}^d$ – the corresponding measure is denoted by $\phi_{\mathbb{Z}^d,p,q}$ – by taking the weak limit of measures defined in finite volume.

For notational convenience, we set $x \longleftrightarrow y$ if x and y are in the same connected component. We also write $x \longleftrightarrow Y$ if x is connected to a vertex in $Y \subset \mathbb{Z}^d$, and $x \longleftrightarrow \infty$ if the connected component of x is infinite. Finally, let Λ_n be the box of size n around 0 for the graph distance.

For $q \geq 1$, the model undergoes a phase transition: there exists $p_c = p_c(\mathbb{Z}^d) \in [0, \infty]$ satisfying

$$\theta(p) := \phi^{\mathrm{w}}_{\mathbb{Z}^d,p,q}[0 \longleftrightarrow \infty] = \begin{cases} = 0 & \text{if } p < p_c, \\ > 0 & \text{if } p > p_c. \end{cases}$$

The main theorem we will describe is the following.

Theorem 2 ([DRT17a])**.** *Fix $q \geq 1$ and consider the random-cluster model on $\mathbb{Z}^d$. Then,*

- *There exists $c > 0$ such that $\theta(p) \geq c(p - p_c)$ for any $p \geq p_c$ close enough to p_c.*
- *For any $p < p_c$ there exists $c_p > 0$ such that for every $n \geq 0$,*

$$\phi_{\mathbb{Z}^d,p,q}[0 \longleftrightarrow \partial\Lambda_n] \leq \exp[-c_p n].$$

For planar graphs, this result was proved for any $q \geq 1$ under some symmetry assumption in [DM16] (see also [MR16] for the case of planar slabs). On $\mathbb{Z}^d$, the result was restricted to large values of q [LMMS+91] and to the special cases of Bernoulli percolation ($q = 1$) [Men86, AB87, DT15] and the FK-Ising model ($q = 2$) [ABF87, DT15].

Theorem 2 extends to quasi-transitive weighted graphs and to finite range interactions (for the latter, simply interpret finite-range models as nearest-neighbor models on a bigger graph).

References

[AB87] M. Aizenman and D. J. Barsky. Sharpness of the phase transition in percolation models. *Communications in Mathematical Physics*, 108(3):489–526, 1987.

[ABF87] M. Aizenman, D. J. Barsky, and R. Fernández. The phase transition in a general class of Ising-type models is sharp. *J. Statist. Phys.*, 47(3-4):343–374, 1987.

[BD12] V. Beffara and H. Duminil-Copin. The self-dual point of the two-dimensional random-cluster model is critical for $q \geq 1$. *Probability Theory and Related Fields*, 153(3-4):511–542, 2012.

[BDW02] H. Buhrman and R. De Wolf. Complexity measures and decision tree complexity: a survey. *Theoretical Computer Science*, 288(1):21–43, 2002.

[DT15] H. Duminil-Copin and V. Tassion. A new proof of the sharpness of the phase transition for Bernoulli percolation and the Ising model. *Communications in Mathematical Physics*, 2015. arXiv:1502.03050.

[DGH+16] H. Duminil-Copin, M. Gagnebin, M. Harel, I. Manolescu, and V. Tassion. Discontinuity of the phase transition for the planar random-cluster and Potts models with $q > 4$. *arXiv preprint arXiv:1611.09877*, 2016.

[DM16] H. Duminil-Copin and I. Manolescu. The phase transitions of the planar random-cluster and potts models with $q \geq 1$ are sharp. *Probability Theory and Related Fields*, 164(3-4):865–892, 2016.

[DRT17a] H. Duminil-Copin, A. Raoufi, and V. Tassion. Sharp phase transition for the random-cluster models via decision trees. *arXiv:1705.03104*, 2017.

[DRT17a] H. Duminil-Copin, A. Raoufi, and V. Tassion. Exponential decay of connection probabilities for subcritical Voronoi percolation in $\mathbb{R}^d$. *arXiv:1705.07978*, 2017.

[DST17] H. Duminil-Copin, V. Sidoravicius, and V. Tassion. Continuity of the phase transition for planar random-cluster and Potts models with $1 \leq q \leq 4$. *Communications in Mathematical Physics*, 349(1):47–107, 2017.

[LMMS+91] L. Laanait, A. Messager, S. Miracle-Solé, J. Ruiz, and S. Shlosman. Interfaces in the potts model i: Pirogov-sinai theory of the fortuin-kasteleyn representation. *Communications in Mathematical Physics*, 140(1):81–91, 1991.

[Men86] M. V. Menshikov. Coincidence of critical-points in the percolation problems. *Doklady akademii nauk sssr*, 288(6):1308–1311, 1986.

[MR16] I. Manolescu and A. Raoufi. The phase transitions of the random-cluster and potts models on slabs with $q \geq 1$ are sharp. 2016.

[OSSS05] R. O'Donnell, M. Saks, O. Schramm, and R. Servedio. Every decision tree has an influential variable. *FOCS*, 2005.

[SS10] O. Schramm and J. Steif. Quantitative noise sensitivity and exceptional times for percolation. *Annals of Mathematics*, 171:619–672, 2010.

[Yao77] A. C. Yao. Probabilistic computations: Toward a unified measure of complexity. In *Proceeding of the 18th Annual Symposium on Foundations of Computer Science*, pages 222–227. IEEE, 1977.

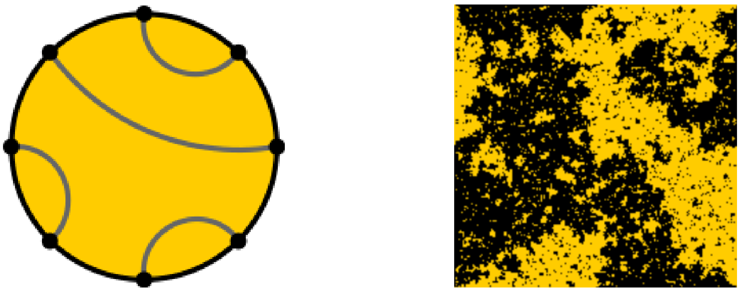


FIGURE 1. Simulation of the critical Ising model with alternating boundary conditions and the corresponding planar link pattern.

Hypergeometric SLE and Convergence of Critical Planar Ising Interfaces

HAO WU

Conformal invariance and critical phenomena in two-dimensional statistical physics have been active areas of research in the last few decades, both in the mathematics and physics communities. Conformal invariance can be studied in terms of correlations and interfaces in the critical models, and both approaches have been successful. This talk concerns conformally invariant probability measures on curves that should describe scaling limits of interfaces in critical lattice models (with suitable boundary conditions).

Consider the critical lattice model in simply connected domain with Dobrushin boundary conditions, the scaling limit of the interface should be a random curve known as the chordal SLE_κ (Schramm-Loewner evolution), uniquely characterized by a single parameter $\kappa \geq 0$ together with conformal invariance and domain Markov property [Sch00]. In this case, such scaling limit results have been rigorously established for many models: Percolation [Smi01, CN07], the Loop-Erased Random Walk and the Uniform Spanning Tree [LSW04], level lines of the discrete Gaussian Free Field [SS09, SS13], and the critical Ising and FK-Ising models [CS12, CDCH+14].

Consider the critical lattice model in rectangle with alternating boundary conditions, a pair of interacting interfaces appears. The scaling limit of such pair should be a pair of random curves with conformal invariance, domain Markov property and symmetry. We proved in [Wu17] that the pair of interacting simple random curves can be classified by these three properties, and the corresponding process is Hypergeometric SLEs: $\mathrm{hSLE}_\kappa(\nu)$ for $\kappa \in (0, 4]$ and $\nu < \kappa - 6$. Moreover, for the critical Ising model on the square lattice in rectangles with alternating boundary conditions, the scaling limit of the pair of interacting interfaces converges weakly to hSLE_3 processes.

In general, interfaces of critical lattice models with alternating boundary conditions with $2N$ marked points corresponds to a collection of N interacting random curves. In [PW17, BPW17], we construct global multiple interacting SLEs and show that they are characterized by the conditional laws: there exists a unique probability measure on N interacting random curves $(\eta_1, \ldots, \eta_N)$ such that the conditional law of η_j given $\{\eta_1, \ldots, \eta_{j-1}, \eta_{j+1}, \ldots, \eta_N\}$ is SLE_κ for all $j \in \{1, \ldots, N\}$. As a consequence, the collection of interfaces in the critical Ising model with alternating boundary conditions converges to the collection of global multiple SLE_3's.

It is also natural to ask questions about the global behavior of the interfaces, such as their crossing or connection probabilities. In fact, such a crossing probability, known as Cardy's formula, was a crucial ingredient in the celebrated proof of the conformal invariance of the scaling limit of critical percolation [Smi01, CN07]. In Figure 1, a simulation of the critical Ising model with alternating boundary conditions is depicted. The figure shows one possible connectivity of the interfaces separating the black and yellow regions, but when sampling from the Gibbs measure, other planar connectivities can also arise. One may then ask with which probability do the various connectivities occur. The answer is known for Loop-Erased Random Walks ($\kappa = 2$) and for the double-dimer model ($\kappa = 4$) [KW11], whereas e.g. the cases of the Ising model ($\kappa = 3$) and percolation ($\kappa = 6$) are still unknown to our knowledge.

In general, scaling limits of these connection probabilities should be encoded in certain quantities related to multiple SLEs, known as pure partition functions. In [PW17], we construct the pure partition functions of multiple SLEs for all $\kappa \in (0, 4]$ and show that they are smooth, positive, and (essentially) unique. In the spirit of [KW11], we find algebraic formulas for the pure partition functions in this case and show that they give explicitly the connection probabilities for the level lines of the Gaussian Free Field with alternating boundary conditions.

References

[BPW17] Vincent Beffara, Eveliina Peltola, and Hao Wu. Uniqueness of Global Multiple SLEs for $\kappa \leq 4$, 2017+.

[CDCH+14] Dmitry Chelkak, Hugo Duminil-Copin, Clément Hongler, Antti Kemppainen, and Stanislav Smirnov. Convergence of ising interfaces to schramm's sle curves. *Comptes Rendus Mathematique*, 352(2):157–161, 2014.

[CN07] Federico Camia and Charles M. Newman. Critical percolation exploration path and SLE_6: a proof of convergence. *Probab. Theory Related Fields*, 139(3-4):473–519, 2007.

[CS12] Dmitry Chelkak and Stanislav Smirnov. Universality in the 2D Ising model and conformal invariance of fermionic observables. *Invent. Math.*, 189(3):515–580, 2012.

[KW11] Richard W. Kenyon and David B. Wilson. Boundary partitions in trees and dimers. *Trans. Amer. Math. Soc.*, 363(3):1325–1364, 2011.

[LSW04] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.*, 32(1B):939–995, 2004.

[PW17] Eveliina Peltola and Hao Wu. Global and Local Multiple SLEs for $\kappa \leq 4$ and Connection Probabilities for Level Lines of GFF, 2017.

[Sch00] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000.

[Smi01] Stanislav Smirnov. Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. *C. R. Acad. Sci. Paris Sér. I Math.*, 333(3):239–244, 2001.

[SS09] Oded Schramm and Scott Sheffield. Contour lines of the two-dimensional discrete Gaussian free field. *Acta Math.*, 202(1):21–137, 2009.

[SS13] Oded Schramm and Scott Sheffield. A contour line of the continuum Gaussian free field. *Probab. Theory Related Fields*, 157(1-2):47–80, 2013.

[Wu17] Hao Wu. Hypergeometric SLE and Convergence of Critical Planar Ising Interface, 2017.

Paracontrolled calculus for singular PDEs

Ismael Bailleul

The study of a whole class of stochastic singular partial differential equation has been revolutionized recently by the works of M. Hairer on regularity structures [1, 2, 3]. Equations of this class share the common default of involving some product(s) of distributions that do not make analytical sense, so the first question is to make sense of the equation before even thinking of solving it. The theory of regularity structures offers a complete framework for the study of these problems that splits the task into two distinct parts, with different aims and methods. In an analytical part, one reformulates the equation in an abstract setting, under some sub-criticality assumption that ensures the existence of a unique solution to the abstract problem. One has a tool for associating a concrete distribution/function to the abstract solution. The second part of the work consists in showing that this concrete solution happens to be the unique limit of a sequence of solutions to some classically well-posed equations, similar to the initial equation, in which the noise has been regularized and some additional terms have been inserted in the equation, typically some terms involving constants diverging as the regularization parameter goes to 0. This second part of the framework requires the development of a heavy algebraic setting, and the use of some tools from the constructive approach to quantum field theory, to prove that some random distributions indexed by the regularization parameter converge when the latter tends to 0.

At the same time that M. Hairer built his theory, Gubinelli, Imkeller and Perkowski designed in [4] another method for studying a number of singular equations. It has the advantage to be much lighter in terms of formalism and to rest on previously available analytic tools – unlike Hairer's theory where everything is built from scratch. On the down side, it provides only a first order expansion theory for solutions of PDEs that limitates its potential scope of applications – unlike the theory of regularity structures that is a kind of arbitrarily high expansion theory. The paracontrolled approach to singular PDEs designed by Gubinelli, Imkeller and Perkowski has however been very successful, as the works [5] on the KPZ equation, [6, 7, 8] on the Φ^4_3 scalar equation from quantum field theory, or the work [9] on the Laplacian with white noise potential, testify, amongst others.

In a series of recent works, joint with F. Bernicot [10, 11, 12], we introduced a semigroup approach to paracontrolled calculus that extends its scope from equations set on a Euclidean space, or a torus, to equations set on manifolds, and turned the machinery into an arbitrary high order expansion framework, bridging partially the gap between the paracontrolled and the regularity structures approaches. These developments build on the introduction of a general Taylor-like expansion formula generalising Bony's paralinearisation formula, and on a number of continuity results for some correctors/commutators built from the paraproduct and resonant operators, together with the crucial introduction of a modified paraproduct that is intertwined with a natural paraproduct via the heat operator and its inverse. Part of this technology can be developed on general metric measured spaces satisfying some homogeneity conditions. Having some tools for studying singular PDEs on manifolds open the road to investigating in a mathematically plain framework Parisi and Wu's paradigm of stochastic quantization of gauge fields and to look at the problem of Euclidean quantization of Yang-Mills theory from a dynamical point of view.

References

[1] Hairer, M., A theory of regularity structures. *Invent. Math.*, **198**(2):269–504, 2014.

[2] Bruned, Y. and Hairer, M. and Zambotti, L., Algebraic renormalisation of regularity structures. arXiv:1610.08468, 2016.

[3] Chandra, A. and Hairer, M., An analytic BPHZ theorem for regularity structures. arXiv:1612.08138, 2016.

[4] Gubinelli, M. and Imkeller, P. and Perkowski, N., Paracontrolled distributions and singular PDEs. *Forum of Math., Pi*, **3**, 2015.

[5] Gubinelli, M. and Perkowski, N., KPZ reloaded. *Comm. Math. Phys.*, **349**(1):165–269, 2017.

[6] Catellier, R. and Chouk K., Paracontrolled Distributions and the 3-dimensional Stochastic Quantization Equation. *Ann. Probab.*, 2017.

[7] Mourrat, J.-C. and Weber, H., Global well-posedness of the dynamic Φ^4_2 model in the plane. *Ann. Probab.*, 2017.

[8] Mourrat, J.-C. and Weber, H., The dynamic Φ^4_3 model comes down from infinity. arXiv:1601.01234, 2016.

[9] Chouk, K. and Allez, R., The continuous Anderson hamiltonian in dimension two. arXiv:1511.02718, 2015.

[10] Bailleul, I. and Bernicot, F., Heat semigroup and singular PDEs. *J. Funct. Anal.*, **270**:3344–3452, 2016.

[11] Bailleul, I. and Bernicot, F., Space-time paraproducts for paracontrolled calculus, 3d-PAM and multiplicative Burgers equations. arXiv:1506.08773, 2015, to appear in *Ann. ENS.*

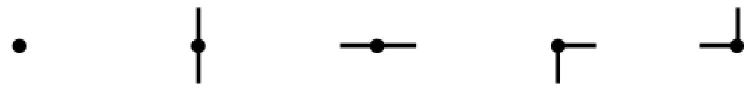
[12] Bailleul, I. and Bernicot, F., Higher order paracontrolled calculus. arXiv:1609.06966, 2016.

[13] Parisi, G. and Wu, Y.S., Perturbation theory without gauge fixing. *Sci. Sinica*, **24**(4):484–496, 1981.

Limit shapes beyond dimers

Richard Kenyon
(joint work with Jan de Gier, Sam Watson)

We discuss the "5-vertex model", a configuration model of edges on $\mathbb{Z}^2$ with local pictures of the five types:



This model is a generalization of the lozenge tiling model, or "monotone non-intersecting lattice path model", which is the uniform measure on the above set of configurations. Here we change the measure so that a configuration of lattice paths has probability proportional to r^c where r is a constant and c is the number of corners: vertices of the last two types above.

We compute the explicit free energy, surface tension function, and the Euler-Lagrange equation describing the limit shapes for the height function in the model. Since the model is not determinantal we use the "Bethe Ansatz" technique, which here has the convenient property that the Bethe roots lie on a family of curves called Cassini ovals, defined as the set of $z \in \mathbb{C}$ for which (for some constants α, β)

$$\alpha \log |z| + \beta \log |1 - z| = 1.$$

The DOZZ formula

Vincent Vargas
(joint work with Francois David, Antti Kupiainen, Rémi Rhodes)

Liouville Conformal Field Theory is a family of conformal field theories introduced by Polyakov in his 1981 seminal work " Quantum geometry of bosonic strings" [5]. Via the celebrated KPZ relation [3], LCFT is expected to describe the scaling limit of large planar maps properly embedded in the sphere.

Recently, using the theory of Gaussian multiplicative chaos, we gave in [1] a rigorous probabilistic formulation of LCFT based on the path integral formulation suggested by Polyakov. More recently, we also initiated in [4] the study of the local conformal structure of LCFT (BPZ equations, Ward identities, etc...) in order to relate LCFT in the path integral formulation to the so-called conformal bootstrap approach.

The conformal bootstrap approach is based on an explicit expression for the 3 point structure constants of the theory, the famous DOZZ formula discovered by Dorn-Otto [2] and Zamolodchikov-Zamolodchikov [6] in the middle of the 90's and a general recursive procedure to construct the higher order correlations. The bootstrap approach is expected in the physics literature to be related to the path integral formulation; however, the exact link between both approaches is still controversial.

In this talk, I will present a proof of the fact that the 3 point structure constants of our probabilistic construction of LCFT are indeed given by the DOZZ formula. This constitutes a first step in unifying both approaches to LCFT and settles a long-standing problem in theoretical physics. From a purely probabilistic point of view, this constitutes the first non trivial and rigorous integrability result on Gaussian multiplicative chaos measures.

References

[1] David F., Kupiainen A., Rhodes R., Vargas V.: Liouville Quantum Gravity on the Riemann sphere, *Communications in Mathematical Physics* **342** (3), 869?907 (2016).

[2] Dorn H., Otto H.-J.: Two and three point functions in Liouville theory, *Nuclear Physics B* **429** (2), 375-388 (1994).

[3] Knizhnik, V.G., Polyakov, A.M., Zamolodchikov, A.B.: Fractal structure of 2D-quantum gravity, *Modern Phys. Lett A*, **3**(8), 819-826 (1988).

[4] Kupiainen A., Rhodes R., Vargas V.: Local conformal structure of Liouville Quantum Gravity, arXiv:1512.01802.

[5] Polyakov A.M.: Quantum geometry of bosonic strings, *Phys. Lett.* **103B** 207 (1981).

[6] Zamolodchikov A.B., Zamolodchikov A.B.: Structure constants and conformal bootstrap in Liouville field theory, *Nuclear Physics B* **477** (2), 577-605 (1996).

Geometry of random planar maps and their duals

Nicolas Curien

We surveyed recent results and works in progress dealing with the geometry of large random planar maps and their dual maps. The setup is the following: given a non-zero weight sequence $\mathbf{q} = (q_1, q_2, ...) \in \mathbb{R}_+^{\mathbb{Z}_+}$ we define a measure $w_{\mathbf{q}}$ on the set $\mathcal{M}$ of all finite bipartite (i.e. all faces have even degree) rooted planar maps $\mathfrak{m}$ by the formula

$$w_{\mathbf{q}}(\mathfrak{m}) = \prod_{f \in \mathrm{Faces}(\mathfrak{m})} q_{\deg(f)/2}.$$

This measure has already been considered in [8]. We assume that $w_{\mathbf{q}}(\mathcal{M})$ is finite ($\mathbf{q}$ is said to be admissible) so that we can normalize $w_{\mathbf{q}}$ to make it a probability measure. We further assume that $\mathbf{q}$ is critical in the sense that $\sum w_{\mathbf{q}}(\mathfrak{m})|\mathfrak{m}|^2 = \infty$. Under some aperiodicity condition, we denote by $\mathfrak{M}_n$ a random planar map sampled according to $w_{\mathbf{q}}(\cdot \mid \text{size} = n)$ and study its geometric properties. By the invariance principle due to Le Gall [6], when $\mathbf{q}$ is finitely supported (and admissible, critical) we have the following convergence in the Gromov–Hausdorff sense

$$n^{-1/4} \cdot \mathfrak{M}_n \xrightarrow[n\to\infty]{(d)} c_{\mathbf{q}} \cdot \textbf{Brownian sphere},$$

where $\mathfrak{M}_n$ is viewed as a random compact metric space once endowed with the graph distance. The above convergence is believed to hold in a wider setting in particular when $\mathbf{q}$ is admissible, critical and satisfies the following asymptotic:

$$q_k \sim c \cdot \gamma^k \cdot k^{-a-1/2}, \qquad \text{for } a > 2.$$

In this regime, based on the work [5] it is tempting to conjecture that both $\mathfrak{M}_n$ and its dual map $\mathfrak{M}_n^\dagger$ converge after rescaling jointly towards the same Brownian sphere with different scaling constants (the work [5] establishes this fact for random triangulations). However, when the admissible and critical weight sequence $\mathbf{q}$ satisfies the last asymptotic with $a \in (1, 2)$ a totally different behavior happens: Le Gall and Miermont [7] have shown that the diameter of $\mathfrak{M}_n$ is typically of order $n^{\frac{1}{2a}}$ whereas the work [3] shows that the diameter of $\mathfrak{M}_n^\dagger$ is of order $n^{\frac{a-3/2}{a}}$ when $a \in (3/2, 2)$ and of logarithmic order when $a \in (1, 3/2)$. In the critical case $a = \frac{3}{2}$ the recent work [4] suggests that the diameter of $\mathfrak{M}_n^\dagger$ is of order $\log^2 n$. The scaling limits of these objets are not yet fully characterized when $a \in (3/2, 2)$ and recent work on growth-fragmentation processes [2, 1] may help building the limit random surfaces.

References

[1] J. Bertoin, T. Budd, N. Curien, and I. Kortchemski, *Martingales in self-similar growth-fragmentations and their applications to random planar maps*, arXiv:1605.00581.

[2] J. Bertoin, N. Curien, and I. Kortchemski, *Random planar maps & growth-fragmentations*, Ann. of Probab. (to appear), (2017).

[3] T. Budd and N. Curien, *Growth in infinite planar maps with high degrees*, Electronic Journal of Probability (2017).

[4] T. Budd, N. Curien, and C. Marzouk, *Infinite random planar maps related to cauchy processes*, arXiv:1704.05297.

[5] N. Curien and J.-F. Le Gall, *First-passage percolation and local perturbations on random planar maps*, arXiv:1511.04264, (2015).

[6] J.-F. Le Gall, *Uniqueness and universality of the Brownian map*, Ann. Probab., 41 (2013), pp. 2880–2960.

[7] J.-F. Le Gall and G. Miermont, *Scaling limits of random planar maps with large faces*, Ann. Probab., 39 (2011), pp. 1–69.

[8] J.-F. Marckert and G. Miermont, *Invariance principles for random bipartite planar maps*, Ann. Probab., 35 (2007), pp. 1642–1705.

Quantitative stochastic homogenization

Jean-Christophe Mourrat

Let $x \mapsto \mathbf{a}(x)$ be a random field on $\mathbb{R}^d$ taking values in the space of symmetric positive definite d-by-d matrices. The law of this random field is assumed to be invariant under translations. We also assume throughout that this random field has a finite range of dependence, and that there exists a constant $\Lambda \in (1, \infty)$ such that with probability one,

$$\forall x \in \mathbb{R}^d, \qquad I_d \le \mathbf{a}(x) \le \Lambda I_d, \tag{1}$$

We want to consider the situation where the correlation length of the coefficient field is much shorter than the typical length scale at which we want to describe the solution of an equation involving $\mathbf{a}$. In this case, it is reasonable to expect that the local fluctuations of the conductivity will be "averaged out" in the limit of large distances, resulting in an equivalent equation with constant, *homogenized* coefficients. One way to formulate this assumption of separation of scales is to introduce a small parameter $\varepsilon > 0$, and solve the family of problems

$$\begin{cases} -\nabla \cdot \left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right) \nabla u_\varepsilon\right) = 0 & \text{in } U, \\ u_\varepsilon = f & \text{on } \partial U, \end{cases} \tag{2}$$

where U is a given domain with Lipschitz boundary, and f is a fixed function with sufficient regularity. The statement of homogenization is that there exists a matrix $\overline{\mathbf{a}}$ which is constant in space, deterministic, and not depending on f or U, such that u_ε converges to $\overline{u}$ solution to

$$\begin{cases} -\nabla \cdot \overline{\mathbf{a}} \nabla \overline{u} = 0 & \text{in } U, \\ \overline{u} = f & \text{on } \partial U. \end{cases}$$

The matrix $\overline{\mathbf{a}}$ is usually called the homogenized or effective matrix. We may say that the solution operator associated with $-\nabla \cdot \mathbf{a}\left(\frac{\cdot}{\varepsilon}\right) \nabla$ converges to that of $-\nabla \cdot \overline{\mathbf{a}} \nabla$. The result is not specific to the choice of Dirichlet problems, and can be interpreted as a kind of law of large numbers. However, it is important to realize that the homogenized coefficients $\overline{\mathbf{a}}$ are *not* obtained as a simple average of the coefficient field $\mathbf{a}(x)$. They retain a non-trivial geometric information concerning the law of the coefficient field, informally related to how easy or difficult it is for the current to circumvent regions of low conductivity.

We are interested in making the convergence of u_ε to $\overline{u}$ quantitative. To begin with, we reduce the problem to a simpler setting. Since the solution $\overline{u}$ is $\overline{\mathbf{a}}$-harmonic, it is very smooth; over mesoscopic scales, it is essentially affine. We therefore focus on understanding the homogenization problem on simple domains, say on cubes, and with affine boundary condition. In other words, denoting by

$$\square_r := \left(-\frac{r}{2}, \frac{r}{2}\right)^d$$

and fixing $p \in \mathbb{R}^d$, we solve for the function $\phi_p^{(r)}$ such that

$$\begin{cases} -\nabla \cdot \mathbf{a}\left(p + \nabla \phi_p^{(r)}\right) = 0 & \text{in } U, \\ \phi_p^{(r)} = 0 & \text{on } \partial U. \end{cases}$$

One can show that $\nabla \phi_p^{(r)}$ converges in $L^2_{\text{loc}}(\mathbb{R}^d)$ to a well-defined limit as $r \to \infty$; we denote the limit by $\nabla \phi_p$. The function ϕ_p is well-defined up to an additive constant, and is called the "corrector" (in the direction p). It encodes the projection onto the vector p of the harmonic embedding of the coefficient field $x \mapsto \mathbf{a}(x)$ (which we can think of as a random Riemannian metric). We aim to assert that the large-scale oscillations of ϕ_p are small, or equivalently, that the gradient of ϕ_p displays large-scale cancellations. For convenience, we define the heat kernel

$$\Phi_{z,r}(x) := (4\pi r^2)^{-\frac{d}{2}} \exp\left(-\frac{|x-z|^2}{4r^2}\right),$$

and denote integrals against this heat kernel mask as

$$\int_{\Phi_{z,r}} f := \int_{\mathbb{R}^d} f(x)\, \Phi_{z,r}(x)\, dx. \tag{3}$$

In words, $\int_{\Phi_{z,r}} f$ is a spatial average of f centered at z and of length scale r. We simply write $\int_{\Phi_r} f$ for $\int_{\Phi_{0,r}} f$. We also introduce some convenient notation to measure the stochastic integrability of random variables. If X is a random variable and $s, \theta \in (0, \infty)$, then we define the statement

$$X \le \mathcal{O}_s(\theta)$$

to mean that

$$\mathbb{E}\left[\exp\left(\left(\theta^{-1} X_+\right)^s\right)\right] \le 2, \tag{4}$$

where $X_+ := X \vee 0$ is the positive part of X. This statement implies that the right tail of $\theta^{-1}X$ decay like $\exp(-c|x|^s)$ for some constant $c > 0$; the converse implication also holds with suitably modified constants. With Armstrong and Kuusi [2], we proved the following result.

Theorem 1 (Optimal estimates on correctors)**.** *For every $s < 2$, there exists $C < \infty$ such that, for every $r \ge 1$ and $p \in B_1$,*

$$\left| \int_{\Phi_r} \nabla \phi_p \right| \le \mathcal{O}_s\left(C r^{-\frac{d}{2}}\right), \tag{5}$$

$$\left| \int_{\Phi_r} \mathbf{a}(p + \nabla \phi_p) - \overline{\mathbf{a}} p \right| \le \mathcal{O}_s\left(C r^{-\frac{d}{2}}\right), \tag{6}$$

and

$$\left| \int_{\Phi_r} \frac{1}{2}(p + \nabla\phi_p) \cdot \mathbf{a}(p + \nabla \phi_p) - \frac{1}{2} p \cdot \overline{\mathbf{a}} p \right| \le \mathcal{O}_s\left(C r^{-\frac{d}{2}}\right), \tag{7}$$

The statements (5), (6) and (7) are all consistent with the idea that the fields under consideration (gradient, flux and energy respectively) rescale as random fields with a short range of correlation. The limitation $s < 2$ is easily understood in this light: by the central limit theorem, we can at best hope for Gaussian-type tail estimates, which correspond to the critical exponent $s = 2$. However, this intuition is truly consistent up to the critical scaling for the energy field only. The proof is indeed based on the fact that certain energy quantities are in fact *local, additive* functions of the coefficient field. The argument thus focuses on showing this first. It is then immediate to deduce that these energy quantities satisfy central limit theorems. Once the behavior of the energy field is understood, one can then identify the (non-trivial) critical correlations of the correctors by showing that

$$r^{\frac{d}{2}} (\nabla \phi_p)(r\,\cdot) \xrightarrow[r\to\infty]{\text{(law)}} \nabla \mathbf{\Psi}_p, \tag{8}$$

where Ψ_p is a variant of the Gaussian free field. The reference [1] offers a much more detailed presentation of these results and their consequences.

References

[1] S. Armstrong, T. Kuusi, and J.-C. Mourrat. *Quantitative stochastic homogenization and large-scale regularity.* Preliminary version available at http://perso.ens-lyon.fr/jean-christophe.mourrat/lecturenotes.pdf (2017).
[2] S. Armstrong, T. Kuusi, and J.-C. Mourrat. The additive structure of elliptic homogenization. *Invent. Math.*, 208(3):999–1154, 2017.

Universality of height fluctuations for the dimer model on Riemann surfaces

Nathanaël Berestycki
(joint work with Benoit Laslier, Gourab Ray)

In a series of recent results with Benoit Laslier and Gourab Ray, we established the convergence as the mesh size tends to zero of the centered height function associated to dimer models to a universal scaling limit (independent of the underlying microscopic details of the graph) in various situations. This includes in particular the following two cases: (a) dimer model on simply connected domains with boundary conditions lying in some arbitrary plane of fixed slope, and (b) Riemann surfaces. More precisely:

(a) On the hexagonal lattice with mesh size δ, for any simply connected domain D with locally connected boundary (which is essentially an optimal regularity assumption), for any fixed plane $P \subset \mathbb{R}^3$, there exists a sequence D_δ which converge to D and such that the height function h_δ restricted to the boundary ∂D^δ lies within $O(1)$ of the plane P, and $h_\delta - \mathbb{E}(h_\delta)$ converges in distribution (with respect to the Sobolev $H^{-1-\varepsilon}(D)$ for any $\varepsilon > 0$) to $(1/\chi)h_{\text{GFF}}$, where $\chi = 1/\sqrt{2}$ and h_{GFF} is a Gaussian free field with Dirichlet boundary conditions.

(b) Let S be a Riemann surface with finitely many holes and such that around every point on the boundary there is a neighbourhood which is conformally equivalent to the upper half plane. The height function in this case consists of a one-form called the scalar field, which encodes the local fluctuations as in the simply connected case, and an "instanton" component, which describes the height gap across nontrivial loops. We obtain convergence of both components towards universal, conformally invariant limits in a fairly general setup, for the moment when the surface is of low complexity (zero Euler characteristic, i.e., an annulus or a torus). This answers questions raised by Dubédat [2] and Dubédat and Ghessari [3], who obtained a particular case of this result on the torus (under an additional assumption of isoradiality on the graph) and described the limiting field as a compactified Gaussian free field.

Our setup is that of Temperleyan graphs G on S. That is, G is the superposition of a graph Γ drawn on S, its dual $\Gamma^\dagger$, and intermediary vertices on each edge of Γ. On surfaces of low complexity (zero Euler characteristic), we obtain an extension of Temperley's bijection which shows that dimer configurations on G are in bijection with oriented cycle-rooted spanning forests (or CRSF for short) of Γ. A key step of our proof is then to show that on any nice Riemann surface a uniform CRSF converges in the sense of Schramm, which generalises work of Kassel and Kenyon [5]. Contrary to previous works on the dimer model, our approach does not rely on exact computations, instead the arguments are based on the *imaginary geometry* couplings between Gaussian free field and SLE curves known to occur on simply connected domains [4, 6].

References

[1] N. Berestycki, B. Laslier & G. Ray. (2016). Universality of fluctuations in the dimer model. arXiv:1603.09740.

[2] J. Dubédat (2015). Dimers families of Cauchy–Riemann operators I. *J. Amer. Math. Soc.*, 28, 1063–1167.

[3] J. Dubédat & R. Gheissari (2015). Asymptotics of height change on toroidal Temperleyan dimer models. *J. Stat. Phys.* 159.1: 75–100.

[4] J. Dubédat (2009). SLE and the free field: partition functions and couplings. *J. Amer. Math. Soc.*. 22(4):995–1054.

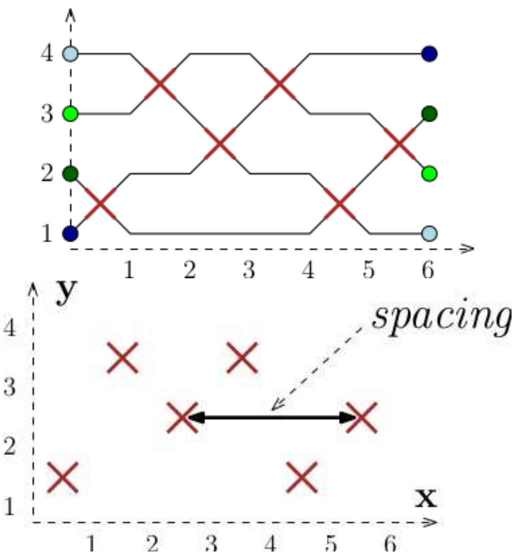
[5] A. Kassel & R. Kenyon (2017). Random curves on surfaces induced from the Laplacian determinant. *Ann. Probab.*,45, No.2, 932–964.

[6] J. Miller & S. Sheffield (2013). Imaginary geometry IV: interior rays, whole-plane reversibility, and space-filling trees. arXiv preprint arXiv:1302.4738.

Local limits of random sorting networks

Vadim Gorin

A sorting network of rank n is a shortest path between $12..n$ and $n..21$ in the Caley graph of the symmetric group spanned by swaps of adjacent letters. A sorting network can be interpreted as a wiring diagram, see the following figure.



Following [AHRV] we study a *uniformly random* sorting network of rank n. Random sorting networks have intriguing asymptotic behavior as $n \to \infty$, and we refer to [AHRV] for several exciting conjectures which are still open.

In the talk we discussed the bulk local limit of the swap process of random sorting networks. The limit object is defined through a deterministic procedure, a local version of the Edelman-Greene algorithm, applied to a two dimensional determinantal point process with explicit kernel. The latter describes the asymptotic joint law near 0 of the eigenvalues of the corners in the antisymmetric Gaussian Unitary Ensemble.

In particular, the limiting distribution of the first swap time at a given position is identified with the limiting distribution for the closest to 0 eigenvalue in the antisymmetric GUE. As a corollary, the asymptotic law of the spacings between swaps (see the right panel of the figure) is given by the Gaudin–Mehta distribution also known as *exact Wigner surmise* — the universal law for bulk spacings between eigenvalues of random real symmetric matrices.

The proofs rely on the determinantal structure and a double contour integral representation for the kernel of random Poissonized Young tableaux.

Detailed exposition of the results and proofs is in [GR], another approach to the local limits of sorting networks can be found in [ADHV].

References

[AHRV] O. Angel, A. Holroyd, D. Romik and B. Virág, Random sorting networks, *Advances in Mathematics* **215**(2) (2007), pp. 839–864.

[GR] V. Gorin, M. Rahman, Random sorting networks: local statistics via random matrix laws, arXiv:1702.07895

[ADHV] O. Angel, D. Dauvergne, A. E. Holroyd, and B. Virág, The Local Limit of Random Sorting Networks, arXiv:1702.08368.

Phase transitions in random constraint satisfaction problems

Nike Sun

(joint work with Jian Ding, Allan Sly, and Yumeng Zhang)

A *constraint satisfaction problem* (CSP) is specified by a set of variables subject to constraints. A classical example is the problem of giving a proper q-coloring to a given graph. Other well-studied examples include the graph independent set problem and the boolean k-satisfiability problem. We consider these problems in a randomized setting, and study the phase transitions that arise in the sparse (bounded constraint density) regime. We determine the exact satisfiability threshold in random regular k-NAE-SAT [3], random k-regular independent set [1], and random k-SAT for $k \geq k_0$ [2]. In the random regular k-NAE-SAT model we further compute the limiting free energy [6].

One main challenge presented by these models is the occurrence of "replica symmetry breaking", or lack of correlation decay — that is to say, variables can exhibit non-negligible dependencies across large distances in the constraint graph. In the sparse regime, this occurs even though the constraint graphs are locally tree-like (few short cycles), meaning that dependencies propagate across long cycles. It is believed that the lack of correlation decay is a consequence of a *condensation* phenomenon, where the solution space is dominated by large *clusters* (connected components in the hamming cube) of fluctuating weights. It is further proposed that in several interesting cases, when smaller solution clusters are preferentially weighted to a certain degree, the system *will* have correlation decay. This phenomenon is known as *one-step replica symmetry breaking* (1-RSB), in contrast with the full replica symmetry breaking (∞-RSB) which often arises in mean-field spin glasses. We refer to [4, 5] and refs. therein for details and for the precise formulations of these conjectures.

Our proof strategy is guided by the physics view of clustering and one-step replica symmetry breaking. To analyze satisfiability thresholds, we work with graphical models which give a succinct combinatorial representation to solution clusters. To analyze the free energy we work with a more elaborate graphical model which gives more direct access to the size-biased measure. The main technical work is to analyze first and second moments for these graphical models, which amount to solving certain non-convex optimization problems in a large number of variables. This is handled by *a priori* estimates (which give rough bounds), in combination with arguments relating the original graph optimization to a tree optimization, which is convex and allows for much more refined analysis.

References

[1] J. Ding, A. Sly, and N. Sun. Maximum independent sets on random regular graphs. arXiv:1310.4787.

[2] J. Ding, A. Sly, and N. Sun. Proof of the satisfiability conjecture for large k. In *Proc. 47th STOC*, pages 59–68, 2015. arXiv:1411.0650.

[3] J. Ding, A. Sly, and N. Sun. Satisfiability threshold for random regular NAE-SAT. In *Proc. 46th STOC*, pages 814-822, 2014. *Comm. Math. Phys.* 341(2):435–489, 2016.

[4] F. Krzakała, A. Montanari, F. Ricci-Tersenghi, G. Semerjian, and L. Zdeborová. Gibbs states and the set of solutions of random constraint satisfaction problems. *Proc. Natl. Acad. Sci.*, 104(25):10318–10323, 2007.

[5] M. Mezard and A. Montanari. *Information, physics, and computation.* Oxford University Press, 2009.

[6] A. Sly, N. Sun, and Y. Zhang. The number of solutions for random regular NAE-SAT. In *Proc. 57th FOCS*, pages 724–731. IEEE, 2016. arXiv:1604.08546.

Random planar metrics of Gaussian free fields

Jian Ding

The talk is a survey on some recent progresses on a number of metric aspects for planar Gaussian free fields, as explained briefly in what follows.

- Liouville first passage percolation at high temperatures, which aims at understanding the metric associated with Liouville quantum gravity through appropriate scaling limits of discrete approximations. This is a FPP metric where roughly speaking each vertex is given a weight that is exponential of the GFF value. With Subhajit Goswami, in [3] we gave an upper bound on the metric exponent, which seems to contradict with reasonable interpretations for Watabiki's prediction; with Alexander Dunlap, in [2] we showed there exists a subsequential scaling limit for such metric; with Fuxi Zhang, in [6] we showed that the geodesic of this metric is of dimension strictly larger than 1 and that such metric relies on subtle properties of the underlying fields. We also discuss some non-universality aspects for the Liouville FPP as in [5] joint with Fuxi Zhang as well as its implications to the heat kernel of Liouville Brownian motion as in [7] joint with Fuxi Zhang and Ofer Zeitouni.

- Chemical distance for level-set percolation. With Li Li, in [4] we showed that there exists a non-vanishing probability so that there exists a crossing of dimension 1 joining two boundaries of macroscopic annulus inside the box where the GFF values on this crossing are all positive.
- Effective resistances. With Marek Biskup and Subhajit Goswami, in [1] we showed that the effective resistances between u and v on the random network where the edge resistance is given by the exponential of the sum of GFF values at two endpoints, grows as as $|u-v|^{o(1)}$. This has applications for random walks on such random network.

References

[1] M. Biskup, J. Ding and A. Dunlap. Return probability and recurrence for the random walk driven by two-dimensional Gaussian free field. Preprint, available at https://arxiv.org/abs/1611.03901.
[2] J. Ding and A. Dunlap. Liouville first passage percolation: subsequential scaling limits at high temperatures. Preprint, available at http://arxiv.org/abs/1605.04011.
[3] J. Ding and S. Goswami. Upper bounds on Liouville first passage percolation and Watabiki's prediction. 2016. arXiv:1610.09998.
[4] J. Ding and L. Li. Chemical distances for level-set percolation of two-dimensional discrete Gaussian free fields. Preprint, available at http://arxiv.org/abs/1605.04449.
[5] J. Ding and F. Zhang. Non-universality for first passage percolation on the exponential of log-correlated Gaussian fields. Preprint, available at https://arxiv.org/abs/1506.03293.
[6] J. Ding and F. Zhang. Liouville first passage percolation: geodesic dimension is strictly larger than 1 at high temperatures. Preprint, available at https://arxiv.org/abs/1610.02766.
[7] J. Ding, F. Zhang and O. Zeitouni. On the Liouville heat kernel for k-coarse MBRW and nonuniversality. Preprint, available at https://arxiv.org/abs/1701.01201.

Convergence of percolation on random planar maps to SLE_6 on $\sqrt{8/3}$-Liouville quantum gravity

Ewain Gwynne
(joint work with Jason Miller)

It is expected that percolation interfaces on a large class of deterministic and random planar graphs converge in the scaling limit to Schramm-Loewner evolution (SLE) with parameter 6. So far, the only percolation model on a *deterministic* lattice for which this convergence has been proven is site percolation on the triangular lattice, using an argument due to Smirnov [11] which relies on an exact combinatorial identity and does not extend to other lattices.

In [5], we prove a version of this convergence for face percolation on random quadrangulations. In particular, suppose Q is a free Boltzmann quadrangulation with simple boundary and fixed perimeter $2l$, $l \in \mathbb{N}$, decorated by a critical ($p = 3/4$) face percolation configuration. That is, Q is sampled from the set of all quadrangulations with simple boundary of perimeter l with probability proportional to $12^{-\#\text{faces of } Q}$.

Theorem 1. *The quadrangulation Q, equipped with its graph metric, the counting measure on edges, and the chordal face percolation path between two marked boundary edges, converges in the scaling limit to chordal SLE_6 on an independent $\sqrt{8/3}$-Liouville quantum gravity disk, equivalently a Brownian disk, in the Gromov-Hausdorff-Prokhorov-uniform topology.*

The *Gromov-Hausdorff Prokhorov uniform (GHPU)* topology considered in Theorem 1 is a natural generalization of the Gromov-Hausdorff topology for curve-decorated metric measure spaces.

The proof of Theorem 1 is robust and, up to certain technical steps, extends to any percolation model on a random planar map which can be explored via peeling. For example, once it has been shown that the free Boltzmann *triangulation* converges in the scaling limit to the Brownian map, our proof immediately extends to give GHPU convergence of site percolation on such triangulations toward SLE_6 on $\sqrt{8/3}$-LQG.

Let us now discuss the limiting object in Theorem 1. For $\gamma \in (0, 2)$, a γ-Liouville quantum gravity (LQG) surface is, heuristically speaking, the random Riemann surface with metric tensor $e^{\gamma h}\, dx \otimes dy$, where h is some variant of the Gaussian free field and $dx \otimes dy$ is the Euclidean metric tensor. This does not make rigorous sense since h is a distribution, not a function, but one can rigorously define a γ-LQG surface in various ways. It was shown by Duplantier and Sheffield [3] that a γ-LQG surface can be defined as a random measure space with a conformal structure; and by Miller and Sheffield [8, 9, 10] that a $\sqrt{8/3}$-LQG surface also admits a metric space structure.

Certain special $\sqrt{8/3}$-LQG surfaces are equivalent (as metric measure spaces) to so-called *Brownian surfaces* (e.g., the Brownian map [6, 7]) which arise as the scaling limit of uniform random planar maps [9].

One such Brownian surface is the Brownian disk, which is the scaling limit of random planar maps with boundary [1]. The Brownian disk is equivalent to a $\sqrt{8/3}$-LQG surface called the *quantum disk*. Since the quantum disk has a canonical embedding into the unit disk $\mathbb{D}$, so does the Brownian disk. In particular, one can define an SLE_6 on the Brownian disk by embedding it into $\mathbb{D}$ then sampling an independent SLE_6 in $\mathbb{D}$.

The above definition is not useful for identifying scaling limits of discrete models since the manner in which the Brownian disk determines its embedding into $\mathbb{D}$ is not explicit. So, in [4] we prove a characterization theorem which says that SLE_6 on the Brownian disk is uniquely characterized by (a) its *topological* structure and (b) the fact that the complementary connected components of the curve are Brownian disks. This theorem is deduced from a more general characterization theorem for SLE_κ on γ-LQG for $\kappa \in (4,8)$, which is also proven in [4], using the equivalence of Brownian and $\sqrt{8/3}$-LQG surfaces.

The proof of Theorem 1 proceeds by checking tightness with respect to the GHPU topology, then verifying the hypotheses of our characterization theorem for a subsequential limit using the peeling procedure for the free Boltzmann quadrangulation. The hypothesis on the topology of the curve is the hardest part to check—for this we use a topological theorem from [2] and an "arm-counting" argument for face percolation.

References

[1] Jérémie Bettinelli and Grégory Miermont. Compact Brownian surfaces I: Brownian disks. *Probab. Theory Related Fields*, 167(3-4):555–614, 2017.

[2] Alexander Blokh, Lex Oversteegen, and E. D. Tymchatyn. On almost one-to-one maps. *Trans. Amer. Math. Soc.*, 358(11):5003–5014, 2006.

[3] Bertrand Duplantier and Scott Sheffield. Liouville quantum gravity and KPZ. *Invent. Math.*, 185(2):333–393, 2011.

[4] E. Gwynne and J. Miller. Characterizations of SLE_κ for $\kappa \in (4,8)$ on Liouville quantum gravity. *ArXiv e-prints*, January 2017.

[5] E. Gwynne and J. Miller. Convergence of percolation on uniform quadrangulations with boundary to SLE_6 on $\sqrt{8/3}$-Liouville quantum gravity. *ArXiv e-prints*, January 2017.

[6] Jean-François Le Gall. Uniqueness and universality of the Brownian map. *Ann. Probab.*, 41(4):2880–2960, 2013.

[7] Grégory Miermont. The Brownian map is the scaling limit of uniform random plane quadrangulations. *Acta Math.*, 210(2):319–401, 2013.

[8] J. Miller and S. Sheffield. Liouville quantum gravity and the Brownian map I: The QLE(8/3,0) metric. *ArXiv e-prints*, July 2015.

[9] J. Miller and S. Sheffield. Liouville quantum gravity and the Brownian map II: geodesics and continuity of the embedding. *ArXiv e-prints*, May 2016.

[10] J. Miller and S. Sheffield. Liouville quantum gravity and the Brownian map III: the conformal structure is determined. *ArXiv e-prints*, August 2016.

[11] Stanislav Smirnov. Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. *C. R. Acad. Sci. Paris Sér. I Math.*, 333(3):239–244, 2001.

Exponential number of equilibria and depinning threshold for a directed polymer in a random potential

Yan V. Fyodorov

(joint work with Pierre Le Doussal, Alberto Rosso, Christophe Texier)

Using the Kac-Rice approach, we show that the mean number $\langle \mathcal{N}_{\mathrm{tot}} \rangle$ of all possible equilibria of an elastic line (directed polymer), confined in a harmonic well and submitted to a quenched random Gaussian potential, grows exponentially with its length L: $\langle \mathcal{N}_{\mathrm{tot}} \rangle \sim \exp(r\,L)$. The growth rate r is found to be directly related to the generalized Lyapunov exponent of an associated Anderson localization problem

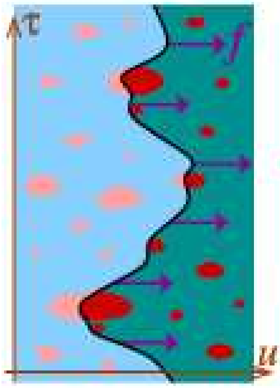


FIGURE 1. Elastic line in a disordered medium submitted to a uniform force field.

of a $1d$ Schrödinger equation in a random potential. For strong confinement, the rate r is small and given by a non-perturbative (instanton) contribution to the Lyapunov exponent. For weak confinement, the rate r is found proportional to the inverse Larkin length of the pinning theory. As an application, identifying the depinning with a landscape "topology trivialization" phenomenon, we obtain an upper bound for the depinning threshold f_c, in presence of an applied force.

The scaling limit of critical random graphs with i.i.d. power-law degrees

CHRISTINA GOLDSCHMIDT
(joint work with Guillaume Conchon-Kerjan)

This abstract is based on the paper in preparation [4]. Let $n \geq 1$ and fix a degree sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ such that $d_1, d_2, \ldots, d_n \geq 1$ and $\sum_{i=1}^n d_i$ is even. Assume that the set $\mathcal{G}(\mathbf{d})$ of graphs with vertex-set $[n] := \{1, 2, \ldots, n\}$ and such that vertex i has degree d_i is non-empty. Let G_n be an element of $\mathcal{G}(\mathbf{d})$ chosen uniformly at random. The standard method of generating such a graph is via the *configuration model* (see van der Hofstad [9] for a detailed description and precise references for the general results quoted below). Assign vertex i a number d_i of half-edges and then pair the half-edges uniformly at random. In general, this generates a multigraph M_n (i.e. a graph with self-loops and multiple edges), but if we condition it to be simple, it has the same law as G_n. An important feature of the configuration model is that we may generate the pairing edge-by-edge in an order that is convenient for analysis.

We study this model in the setting where the degrees are random variables $D_1, D_2, \ldots, D_n$ which are independent and identically distributed and such that

(1) $\mathbb{E}[D_1^2] = 2\mathbb{E}[D_1]$;
(2) $k^{\alpha+2}\mathbb{P}(D_1 = k) \to c$ as $k \to \infty$, for some constant $c > 0$ and $\alpha \in (1, 2)$.

If $\sum_{i=1}^n D_i$ is odd, simply throw away the last half-edge when we do the pairing. We observe that in this setting $\mathbb{P}(\mathcal{G}(\mathbf{D}) \neq \emptyset)$ converges to a strictly positive constant as $n \to \infty$, so that for sufficiently large n conditioning the multigraph to be simple makes sense. Condition (1) says that the graph is critical. Condition (2) entails that the degrees have finite variance but infinite third moment. The *finite* third moment setting has been extensively studied, in particular by Joseph [10] and Riordan [11] (for the component sizes) and Dhara, van der Hofstad, van Leeuwaarden and Sen [5] (for the metric structure). There the scaling limit is (up to constants) the same as that obtained for the critical Erdős–Rényi model in [1] (building on work of Aldous [2], who showed that the component sizes may be encoded as the lengths of excursions above the running minimum of a Brownian motion with parabolic drift).

Let us assume condition (2). Write $C_1^n, C_2^n, \ldots$ for the connected components of G_n listed in decreasing order of size, and $|C_1^n|, |C_2^n|, \ldots$ for those sizes. Let $(L_t, t \geq 0)$ be a spectrally positive α-stable Lévy process with Laplace transform

$$\mathbb{E}[\exp(-\lambda L_t)] = \exp(c_\alpha t \lambda^\alpha), \quad \lambda \geq 0, t \geq 0,$$

where $c_\alpha = \frac{c\Gamma(2-\alpha)}{\mu\alpha(\alpha-1)}$. Define a new process $(\tilde{L}_t, t \geq 0)$ via the following martingale change of measure: for any $t \geq 0$ and any bounded measurable test function f,

$$\mathbb{E}[f(\tilde{L}_s, 0 \leq s \leq t)] = \mathbb{E}\left[\exp\left(\frac{1}{\mu}\int_0^t (L_s - L_t)ds - \frac{c_\alpha t^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}}\right) f(L_s, 0 \leq s \leq t)\right]$$

Then the following result is an extension (from M_n to G_n) and reformulation of a theorem due to Joseph [10].

Theorem 1. As $n \to \infty$,

$$n^{-\alpha/(\alpha+1)}(|C_1^n|, |C_2^n|, \ldots) \xrightarrow{d} (\gamma_1, \gamma_2, \ldots)$$

in ℓ^2, where $\gamma_1, \gamma_2, \ldots$ are the ordered lengths of the excursions of $\tilde{L}$ above its running minimum.

Our main result describes the metric space scaling limits of the components themselves. Consider the components $C_1^n, C_2^n, \ldots$ of G_n as measured metric spaces, by endowing each with the graph distance and by assigning mass $n^{-\alpha/(\alpha+1)}$ to each vertex. The limit spaces have an explicit description. First, observe that excursions of the stable Lévy process L may be used to encode a forest of *stable trees*, via the so-called height process H (see Duquesne and Le Gall [7]). We may use the absolute continuity relation between $\tilde{L}$ and L to define a height process $\tilde{H}$ corresponding to $\tilde{L}$: for any $t \geq 0$ and any bounded measurable test function g,

$$\begin{aligned}&\mathbb{E}[g(\tilde{L}_s, \tilde{H}_s, 0 \leq s \leq t)] \\ &\quad = \mathbb{E}\left[\exp\left(\frac{1}{\mu}\int_0^t (L_s - L_t)ds - \frac{c_\alpha t^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}}\right) g(L_s, H_s, 0 \leq s \leq t)\right].\end{aligned}$$

Let us write

$$R_t = \tilde{L}_t - \inf_{0 \le s \le t} \tilde{L}_s, \quad t \ge 0.$$

Write $\varepsilon_1, \varepsilon_2, \dots$ for the excursions of R above 0, in decreasing order of length, and note that these are in one-to-one correspondence with the excursions $h_1, h_2, \dots$ of $\tilde{H}$ above 0 (again listed in decreasing order of length). In particular, both ε_i and h_i have length γ_i, for $i \ge 1$. Let $(\mathcal{T}_i, d_i, \mu_i)$ be the measured $\mathbb{R}$-tree encoded in the standard way by h_i, with canonical projection $p_i : [0, \gamma_i] \to \mathcal{T}_i$.

Conditionally on R, consider now a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}_+$ of intensity $\frac{1}{\mu} \mathbb{1}_{\{0 \le x \le R_t\}} dt dx$. (Equivalently, this is a rate $1/\mu$ Poisson point process in the region under the graph of the process R and above the horizontal axis.) Suppose that for $i \ge 1$, a number m_i of points fall under the excursion ε_i. If $m_i \ge 1$, let

$$(s_1^{(i)}, x_1^{(i)}), \dots, (s_{m_i}^{(i)}, x_{m_i}^{(i)})$$

be the points themselves. Then for $1 \le k \le m_i$, let

$$t_k^{(i)} = \inf \left\{ t \ge s_k^{(i)} : \varepsilon_i(t) = x_k^{(i)} \right\}.$$

Finally, for $i \ge 1$, if $m_i = 0$, let $\mathcal{C}_i = (\mathcal{T}_i, d_i, \mu_i)$; if $m_i \ge 1$, let $\mathcal{C}_i$ be the measured metric space obtained from $(\mathcal{T}_i, d_i, \mu_i)$ by identifying the pairs of points

$$\left(p_i(s_1^{(i)}), p_i(t_1^{(i)}) \right), \dots, \left(p_i(s_{m_i}^{(i)}), p_i(t_{m_i}^{(i)}) \right).$$

For a measured metric space (M, d, μ), write aM as a shorthand for (M, ad, μ). Let $\mathfrak{M}$ be the set of sequences of measure-preserving isometry classes of compact measured metric spaces, endowed with the product Gromov–Hausdorff–Prokhorov topology. Then we have the following scaling limit.

Theorem 2. As $n \to \infty$,

$$n^{-(\alpha-1)/(\alpha+1)} (C_1^n, C_2^n, \dots) \xrightarrow{d} (\mathcal{C}_1, \mathcal{C}_2, \dots)$$

in $\mathfrak{M}$.

Closely related results have recently been proved by Dhara, van der Hofstad, van Leeuwaarden and Sen [6] (for component sizes) and by Bhamidi, Dhara, van der Hofstad and Sen [3] (for metric structure).

We observe that the spanning trees $(\mathcal{T}_i, d_i, \mu_i), i \ge 1$ are measure-changed stable trees. Indeed, if $\tilde{\mathbb{e}}$ is an excursion of R conditioned to have length x and $\mathbb{e}$ is an excursion of $(L_t - \inf_{0 \le s \le t} L_s, t \ge 0)$ conditioned to have length x (and thus encoding a stable tree of size x), we have

$$\mathbb{E}[f(\tilde{\mathbb{e}}(t), 0 \le t \le x)] = \frac{\mathbb{E}\left[\exp\left(\frac{1}{\mu} \int_0^x \mathbb{e}(u) du \right) f(\mathbb{e}(t), 0 \le t \le x) \right]}{\mathbb{E}\left[\exp\left(\frac{1}{\mu} \int_0^x \mathbb{e}(u) du \right) \right]}.$$

This relation together with detailed knowledge of the stable trees from the literature entail that the distributional properties of the limit spaces $\mathcal{C}_1, \mathcal{C}_2, \dots$ are particularly tractable; these properties will be further explored in the paper [8] in preparation.

Acknowledgement. C.G.'s research is supported by EPSRC Fellowship EP/N004833/1.

References

[1] L. Addario-Berry, N. Broutin and C. Goldschmidt, *The continuum limit of critical random graphs*, Probability Theory and Related Fields **152** (2012), 367–406.

[2] D. Aldous, *Brownian excursions, critical random graphs and the multiplicative coalescent*, Annals of Probability **25** (1997), 812–854.

[3] S. Bhamidi, S. Dhara, R. van der Hofstad and S. Sen, *Universality for critical heavy-tailed network models: metric structure of maximal components*, `arXiv:1703.07145` (2017).

[4] G. Conchon-Kerjan and C. Goldschmidt, *Stable graphs: the metric space scaling limits of critical random graphs with i.i.d. power-law degrees*, in preparation.

[5] S. Dhara, R. van der Hofstad, J. van Leeuwaarden and S. Sen, *Critical window for the configuration model: finite third moment degrees*, `arXiv:1605.02868` (2016).

[6] S. Dhara, R. van der Hofstad, J. van Leeuwaarden and S. Sen, *Heavy-tailed configuration models at criticality*, `arXiv:1612.00650` (2016).

[7] T. Duquesne and J.-F. Le Gall, *Random trees, Lévy processes and spatial branching processes*, Astérisque **281** (2002), vi+147.

[8] C. Goldschmidt, B. Haas and D. Sénizergues, *Stable graphs: distributions and line-breaking constructions*, in preparation.

[9] R. van der Hofstad, *Random graphs and complex networks. Vol I.*, Cambridge University Press, Cambridge (2017), xvi+321.

[10] A. Joseph, *The component sizes of a critical random graph with given degree sequence*, Annals of Applied Probability, **24** (2014), 2560–2594.

[11] O. Riordan, *The phase transition in the configuration model*, Combinatorics, Probability and Computing **21** (2012), 265–299.

Dimer dynamics in the anisotropic KPZ class

Fabio Lucio Toninelli

Stochastic interface growth models aim at describing random growth phenomena observed in real experiments. From the mathematical point of view, a d-dimensional interface in $d+1$-dimensional space is described via a height function $h : G \mapsto \mathcal{V}$, with G being $\mathbb{Z}^d$ or another d-dimensional lattice and $\mathcal{V} = \mathbb{R}$ or $\mathbb{Z}$. The growth process itself is modeled by an irreversible Markov process with local update rules.

In reasonable models, one expects that:

- given any slope ρ, there exists a (unique?) stationary measure π_ρ with average slope ρ. More precisely, it is the law of *height gradients* that is stationary, while the height function itself grows on average linearly in time with some speed $v(\rho)$. The measure π_ρ is in general *not reversible* with respect to the dynamics.
- convergence to a hydrodynamic limit holds. If the height function at time zero approximates a smooth profile ϕ_0 (when the lattice mesh ϵ is sent to zero) then by rescaling space and time hyperbolically (i.e. setting by $x = \xi/\epsilon, t = \tau/\epsilon$), for $\epsilon \to 0$ the radom height $\epsilon h(x,t)$ converges to

a non-random $\phi(\xi,\tau)$ solving a first-order, generally non-linear, PDE of Hamilton-Jacobi type:

$$\partial_\tau \phi = v(\nabla_\xi \phi). \tag{1}$$

Such equations in general develop singularities (discontinuities of $\nabla_\xi \phi$) in finite time and for larger times one expects $\epsilon h(\xi/\epsilon, \tau/\epsilon)$ to converge to the *viscosity solution* of the PDE.

- the large space-time behavior of height fluctuations in the stationary process crucially depends on the convexity properties of the function $v(\cdot)$. Let us restrict to $d = 2$ here. It is conjectured [7] that if the Hessian of $v(\cdot)$ has signature $(+,-)$ (Anisotropic KPZ class) then asymptotically height fluctuations behave like those of the stochastic heat equation with additive noise and in particular the variance of height fluctuations grows like $\log \tau$. If instead the signature is $(+,+)$ or $(-,-)$ (isotropic KPZ class) then fluctuations should grow like some non-trivial power of time t.

It is difficult to prove such conjectures in specific models: given the irreversibility of the Markov process there is in general no natural guess for the stationary measures, and the hydrodynamic limit is mostly proven in situations [5] where it can be obtained via super-additivity methods, in which case the function $v(\cdot)$ is not explicitly known (but convexity of $v(\cdot)$ follows for free, so that these models should be in the isotropic KPZ class).

In [6] we studied a growth process for $d = 2$, introduced previously in [1], for which we could identify the stationary measures π_ρ as being the ergodic Gibbs measures on dimer coverings of the infinite hexagonal lattice. In this case, the speed function $v(\cdot)$ can be computed explicitly [2, 3] and its Hessian turns out to have signature $(+,-)$, a fingerprint of the anisotropic KPZ class. In accord with this picture, in [6, 3] it is proven that height fluctuation variance in the stationary state grows at most like the logarithm of time. Lastly, in [4] we proved that a hydrodynamic limit holds: with hyperbolic rescaling, the height converges to the viscosity solution of Eq. (1), under the technical assumption that the initial profile ϕ_0 is convex.

The results on stationary measures and growth of fluctuations in the stationary state have been extended [6, 3] to a different two-dimensional growth model, whose stationary measures are the ergodic Gibbs states on dimer coverings of $\mathbb{Z}^2$.

References

[1] A. Borodin and P.L. Ferrari. Anisotropic Growth of Random Surfaces in $2+1$ Dimensions. *Comm. Math. Phys.*, 325:603–684, 2014.

[2] S. Chhita and P.L. Ferrari. A combinatorial identity for the speed of growth in an anisotropic KPZ model. *preprint: arXiv:1508.01665*, 2015.

[3] S. Chhita, P.L. Ferrari and F. L. Toninelli, *Speed and fluctuations for some driven dimer models*, arXiv:1705.07641

[4] M. Legras, F.L. Toninelli, *Hydrodynamic limit and viscosity solutions for a 2D growth process in the anisotropic KPZ class*, arXiv:1704.06581

[5] F. Rezakhanlou, *Continuum limit for some growth models*, Stoch. Proc. Appl. **101** (2002), 1-41.

[6] F.L. Toninelli. *A $(2+1)$-dimensional growth process with explicit stationary measures*, To appear in Ann. Probab., arXiv:1503.05339, 2015.

[7] D.E. Wolf. Kinetic roughening of vicinal surfaces. *Phys. Rev. Lett.*, 67:1783–1786, 1991.

Conformal Loop Ensembles on Liouville Quantum Gravity

Wendelin Werner

(joint work with Jason Miller, Scott Sheffield)

In this talk based on the joint work [4] with Jason Miller and Scott Sheffield, we try to explain how to build on some of our "CLE percolations" results of [3] dealing with the trunk+loops decompositions of target invariant SLE-type processes and explorations of conformal loop ensembles (CLE) in order to derive results relating conformal loop ensembles with Liouville quantum gravity.

More precisely, when one explores a CLE_κ (for κ in the whole range of possible values $(8/3, 8)$) drawn on an independent well-chosen quantum wedge, it turns out that the quantum surfaces delimited by the CLE loops that are cut out in their order of appearance do form a Poisson point process of quantum srfaces. This extends to the case of loop-forming SLE-type curves some results from [2] and has various implications:

- It allows us for instance to derive the exact constants involved in the CLE version from [3] of the discrete Edwards-Sokal coupling that relates CLE_κ and its variants for $\kappa < 4$ to the corresponding $\text{CLE}_{16/\kappa}$. We show that the value of $q(\kappa)$ such that when one agglomerates $\text{CLE}_{16/\kappa}$ loops that are selected with a probability $1/q(\kappa)$, one constructs a CLE_κ is given by the formula $q = 4\cos^2(\pi\kappa/4)$.

- As these results mirror in the continuum setting the asymptotic features of peeling-tye explorations of decorated discrete planar maps (as for instance in recent work by Chen, Curien and Maillard cite CCM), they provide a roadmap to establish a connection between these discrete models and these "CLE on LQG".

References

[1] L. Chen, N. Curien, and P. Maillard. *The perimeter cascade in critical Boltzmann quadrangulations decorated by an $O(n)$ loop model.* ArXiv e-prints, 2017.

[2] B. Duplantier, J.P. Miller, and S. Sheffield. *Liouville quantum gravity as a mating of trees.* ArXiv e-prints, 2014.

[3] J.P. Miller, S. Sheffield, W. Werner, *CLE percolations.* ArXiv e-prints, 2016.

[4] J.P. Miller, S. Sheffield, W. Werner, *Conformal loop ensembles on Liouville quantum gravity.* in preparation.

Annihilating Branching Brownian Motions and Random Walks

Omer Angel
(joint work with Daniel Ahlberg, Brett Kolesnik)

We consider a family of processes of branching and annihilating particles in $\mathbb{R}$. Particles move by independent Brownian motions or random walks, in discrete or continuous time. Additionally, each particle branches at rate 1, independently of all others. When a particle branches, it is replaced by a (possibly random) number of new particles. These particles have positions which may be equal to the parent particle's position, or may be shifted relative to the parent by some offset, either random or deterministic. We call any such process a **branching random walk**.

Some examples of branching random walks include:

(1) Branching Brownian motion: the movements are given by independent Brownian motions, and particles branch into a pair of particles at the same position.
(2) Branching simple random walks: Particles make simple continuous time random walk, and branch into a pair of particles.
(3) Particles do not move without branching; when a particle at x branches, three offspring are located at x, $x+1$ and $x-1$. This example is of particular interest since it arises in the study of a certain growth and competition model in $\mathbb{Z}^2$, studied by Daniel Ahlberg, Simon Griffiths, Svante Janson and Rob Morris [1].

We assume throughout that the expected number of offspring is greater than 1, i.e. the branching process itself is super-critical and has a positive probability of survival. We also assume that the random walks are in the domain of attraction of Brownian motion.

The **annihilating branching random walk** (ABRW) is the following generalization of the branching random walk. Particles are of one of several types. Particles move and branch independently, and inherit the type of the parent. However, if a pair of particles of distinct types are at the same location at any time, they annihilate each other immediately. See Figure 1 for a simulation.

We say that multiple types **coexist** if at all times there exist particles of each type.

Theorem 1. *In any ABRW with finite initial condition, coexistence has positive probability.*

A process is **order preserving**, if neither the random walk, nor the branching mechanism allow for a particle of one type (or any of its offspring) to change order with a particle of another type. If the particles perform Brownian motion or simple random walk on $\mathbb{Z}$ and the branching does not displace particles, then this would be the case. If a two-type process is order preserving and initially all red particles are to the left of all blue particles, then this is maintained at all times. The **interface** between the types is defined by $I(t) = (I_- + I_+)/2$, where

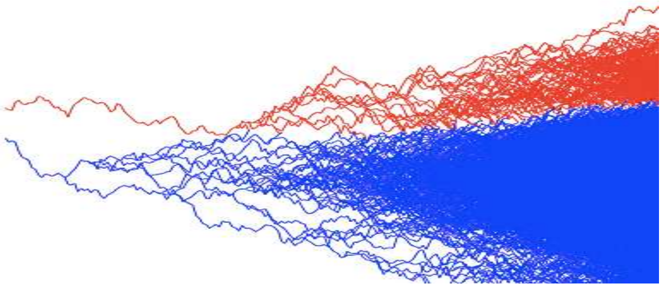


FIGURE 1. Branching and annihilating Brownian motions

$I_-(t)$ is the rightmost red particle and $I_+(t)$ is the leftmost blue particle. We have $I = \pm\infty$ if one of the types dies out, and $I(t)$ is not defined if both types die out.

Theorem 2. *If the ABRW is order preserving, then on the event of coexistence the limit* $S = \lim_{t\to\infty} \frac{1}{t} I(t)$ *exists, and its law has no atoms.*

The proofs extend also to give positive probability of coexistence of multiple types, as well as to higher dimension.

References

[1] Daniel Ahlberg, Simon Griffiths, Svante Janson, and Robert Morris. Competition in growth and urns. arXiv:1610.06479, 2016.

Nesting statistics in loop models, part I: $O(n)$ loop model on random planar maps

JÉRÉMIE BOUTTIER
(joint work with Gaëtan Borot, Bertrand Duplantier)

In this talk, I present the first part of the paper [1], devoted to the study of nestings in the $O(n)$ loop model on random planar maps. In the next talk, Bertrand will discuss nestings in Conformal Loop Ensembles on Liouville quantum gravity. In these two a priori independent settings, we will compute a same large deviations function, giving a strong confirmation that the second model describes indeed the scaling limit of the first. This is in some sense an extension of the Knizhnik-Polyakov-Zamolodchikov relations that are known to hold at the level of critical exponents.

For simplicity, we concentrate on the $O(n)$ loop model on triangulations with a boundary, see Figure 1. The unnormalized weight assigned to a configuration $\mathcal{C}$ is

$$w(\mathcal{C}) = n^{\#\text{loops}} g^{\#\text{empty triangles}} h^{\#\text{visited triangles}},$$

with n, g, h nonnegative real parameters ($w(\mathcal{C}) = n^3 g^{20} h^{36}$ in the above example).

In a configuration of perimeter ℓ, we pick a uniformly chosen vertex in the map and consider its *depth* P_ℓ, i.e., the number of loops separating it from the

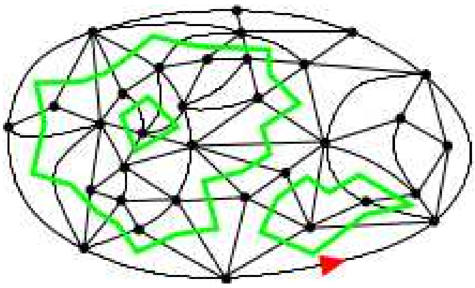


Figure 1. A configuration of the $O(n)$ loop model on a triangulation with a boundary of perimeter 8.

outer face. Our main result is that, at a "non generic critical point" of the $O(n)$ loop model (obtainable for $0 < n < 2$), P_ℓ grows logarithmically with ℓ and more precisely:

Theorem 1 (Central limit theorem)**.** *At a non generic critical point, we have*

$$\frac{P_\ell - \frac{p_{\mathrm{opt}}}{\pi}\ln \ell}{\sqrt{\ln \ell}} \xrightarrow[\ell\to\infty]{(d)} \mathcal{N}(0,\sigma^2)$$

where

$$p_{\mathrm{opt}} = \frac{n}{\sqrt{4-n^2}}, \qquad \sigma^2 = \frac{4n}{\pi(4-n^2)^{3/2}}.$$

Theorem 2 (Large deviations principle)**.** *At a non generic critical point, we have*

$$\mathbb{P}\left(P_\ell = \frac{\ln \ell}{\pi} p\right) \sim cst \cdot (\ln \ell)^{-1/2} \ell^{-\frac{1}{\pi}J(p)}, \qquad \ell \to \infty$$

where

$$J(p) = p \ln\left(\frac{2}{n}\frac{p}{\sqrt{1+p^2}}\right) + \mathrm{arccot}(p) - \arccos(n/2).$$

The function $p \mapsto J(p)$ is convex, nonnegative and vanishes at $p = p_{\mathrm{opt}}$ as it should. If, instead of letting ℓ tend to infinity, we condition on having a large number V of vertices, then both theorems remain valid upon replacing $\ln \ell$ by $c \ln V$ with c a simple constant (equal to 1 at a "dilute" critical point, and to $c = [1 - (1/\pi)\arccos(n/2)]^{-1}$ in the "dense" critical phase).

The proof consists in an exact computation of the probability generating function $\mathbb{E}(s^{P_\ell})$, and some saddle point analysis.

Nesting statistics in loop models, part II: CLE extreme nesting on Liouville quantum gravity

Bertrand Duplantier
(joint work with Gaëtan Borot, Jérémie Bouttier)

In a second part, an alternative description was given in terms of the nesting of the Conformal Loop Ensemble (CLE) on Liouville quantum gravity. The extreme nesting of CLE_κ, defined for $\kappa \in (8/3, 8)$ in the unit disk $\mathbb{D}$, has been studied by Miller, Watson and Wilson [2]. They found the multifractal spectrum of the set Φ_ν of points $z \in \mathbb{D}$ where the number of loops, $\mathcal{N}_\varepsilon(z)$, surrounding the ball $B_\varepsilon(z)$ scales as $[\nu + o(1)] \log(1/\varepsilon)$ as $\varepsilon \to 0$, where $\nu \in \mathbb{R}^+$. The almost sure Hausdorff dimension of this set is $\dim_{\mathcal{H}} \Phi_\nu = \min\{2, 2-\gamma_\kappa(\nu)\}$, where $\gamma_\kappa(\nu)$ is given in terms of some Legendre transform of a conjugate function, $\Lambda_\kappa(\lambda)$.

We transform this classical setting into a Liouville quantum gravity one, by defining a quantum ball centered at z, as the Euclidean ball $B_\varepsilon(z)$ as measured by $\delta := \int_{B_\varepsilon(z)} \mu_\gamma$, where the Liouville quantum measure,

$$\mu_\gamma := e^{\gamma h(z)} dz,$$

(with dz Lebesgue measure) is defined via a proper renormalization of the exponential of h, an instance of the Gaussian free field (a distribution on $\mathbb{D}$ with either Dirichlet or free boundary conditions) [3]. The parameter $\gamma \in (0, 2]$ is related to the SLE κ-parameter by $\gamma = \min\{\sqrt{\kappa}, 4/\sqrt{\kappa}\}$. CLE_κ is defined for $\kappa \in (8/3, 8)$, so we here have $\gamma \in (\sqrt{8/3}, 2]$.

We then look for the probability that a δ-quantum ball centered at $z \in \mathbb{D}$ is surrounded, as $\delta \to 0$, by a number $\mathcal{N}_z = (p + o(1)) \log(1/\delta)$ of CLE loops, with $p \in \mathbb{R}^+$. We find the Liouville quantum gravity large deviations form,

$$\mathbb{P}_{\mathcal{Q}} \left(\mathcal{N}_z \approx p \log(1/\delta) \right) \asymp \delta^{\Theta(p)},$$

where the symbol $\asymp$ stands for the equivalence of logarithms. The proof results from the convolution of the large deviations form of the classical nesting probability obtained in Ref. [2], $\mathbb{P}\left(\mathcal{N}_z \approx \nu \log(1/\varepsilon)\right) \asymp \varepsilon^{\gamma_\kappa(\nu)}$, with the independent asymptotic probability, $\mathbb{P}(\varepsilon|\delta)$, that a δ-quantum ball has a Euclidean radius ε, as obtained in Ref. [3]. A saddle point analysis, valid for $\delta \to 0$, yields the above large deviations form, with $\Theta(p)$ an explicit function of p depending on the parameter $\kappa \in (8/3, 8)$.

A well-known conjectural relation, between the critical $O(n)$-model and the CLE_κ parameters, states that $n = -2\cos(4\pi/\kappa) \in (0, 2]$, where $\kappa \in (8/3, 4]$ corresponds to the dilute phase and $\kappa \in (4, 8)$ to the dense phase. We then find that the LQG-large deviations function Θ exactly matches the large deviations function J obtained in Theorem 2 of Part I for the $O(n)$-model on a random planar map, with the rescaling identity, $\Theta(\frac{c}{2\pi} p) = \frac{c}{2\pi} J(p)$, where c stands for the same parameter as above, $c = \max\{1, \kappa/4\}$.

References

[1] G. Borot, J. Bouttier and B. Duplantier. *Nesting statistics in the $O(n)$ loop model on random planar maps.* arXiv:1605.02239 [math-ph].

[2] J. Miller, S. S. Watson, and D. B. Wilson. *Extreme nesting in the conformal loop ensemble.* Ann. Probab., 44(2):1013–1052, 2016. arXiv:1401.0217[math.PR].

[3] B. Duplantier and S. Sheffield. *Liouville quantum gravity and KPZ.* Invent. Math., 185:333–393, 2011. arXiv:0808.1560[math.PR].

A Glauber dynamics for Liouville

Christophe Garban

In this talk, I introduced an SPDE which arises naturally in the context of Liouville quantum gravity. This SPDE is built to preserve the so-called Liouville measure which has been constructed recently on the two-dimensional sphere $\mathbb{S}^2$ and the torus $\mathbb{T}^2$ in the work by David-Kupiainen-Rhodes-Vargas. In its simplified form, the SPDE can be written as follows (on $\mathbb{R}\times\mathbb{T}^2$ or $\mathbb{R}\times\mathbb{S}^2$):

$$\partial_t X = \frac{1}{2}\Delta X + e^{\gamma X} + \sqrt{2\pi}\xi$$

where ξ is a space-time white noise. It belongs to the broad class of singular stochastic PDEs (which includes KPZ, dynamical Φ^4_d etc.) with some new features due to the presence of **intermittence**.

Our first main result identifies what is the a.s. Besov regularity of the non-linearity $e^{\gamma X}$ (or rather $: e^{\gamma X} :$ as it needs to be renormalized). It can be stated as follows:

Proposition 1. *Fix $\gamma < \hat{\gamma}_c = 2\sqrt{2}$. The positive measure $\Theta :=: e^{\gamma\Phi} :$ a.s. belongs to the space-time Besov space $\mathcal{C}^\alpha_s(\gamma)$ for any $\alpha < \bar{\alpha}(\gamma) := \frac{\gamma^2}{2} - 2\sqrt{2}\gamma$.*

The proof of this Proposition follows a classical scheme in multiplicative chaos which goes back to the works of Barral-Mandelbrot and Bacry-Muzy. There is one main difference : the lack of explicit $*$-scale invariant log-correlated field in parabolic space-time $\mathbb{R}\times\mathbb{R}^d$.

From this result, we single out two thresholds in γ:

(1) For all $\gamma < \gamma_{dPD} := 2\sqrt{2}-\sqrt{6}$, the non-linearity $: e^{\gamma X} :$ has a.s. a regularity larger than -1. This allows in particular to rely on the approach developed by Da Prato-Debussche to handle dynamical Φ^4_2.

(2) For all $\gamma < \gamma_2 := 2\sqrt{2}-2$, the non-linearity $: e^{\gamma X} :$ has a.s. a regularity larger than -2. In this sense, the value γ_2 corresponds to the critical parameter $\beta_c^2 = 8\pi$ in the so-called Sine-Gordon SPDE ($\partial u = \frac{1}{2}\Delta u + \sin(\beta u) + \xi$).

This brings us to our main statement.

Theorem 1.

- *For any $0 \leq \gamma < \gamma_{dPD} := 2\sqrt{2} - \sqrt{6}$, there exists T a.s. > 0 s.t. the solution X_ε to*
$$\begin{cases} \partial_t X_\varepsilon = \frac{1}{4\pi}\Delta X_\varepsilon - C_\varrho \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma X_\varepsilon} + \xi_\varepsilon \\ X_\varepsilon(0,\cdot) = \Phi_\varepsilon(0) + w \end{cases}$$
converges in probability to X in the space $\mathcal{C}_s^{-1}([0,T] \times \mathbb{T}^2)$.
- *For any $\gamma_{dPD} \leq \gamma < \gamma_2 = 2\sqrt{2} - 2$, using the positivity of the non-linearity, we obtain an intrinsic solution X to the SPDE (without showing the convergence in probability of $X_\varepsilon \to X$).*

Remark: Adding the α_i-log singularities at the punctures in the measure constructed in [DKRV14] does have an effect on the regularity of the non-linearity (at least in Besov spaces $\mathcal{C}_s^\alpha \equiv \mathcal{B}_{\infty,\infty}^\alpha$). The local existence result for the non-simplified Liouville SPDE still holds but with bounds in γ which depend on the strength of the coupling constants $\{\alpha_i\}$ at the punctures.

The spectral dimension of Causal triangulations is 2

Asaf Nachmias
(joint work with T. Hutchcroft, N. Curien)

Let T be a Galton-Watson tree with progeny distribution having mean 1 and finite variance, conditioned to survive forever. The random walk on this random rooted graph is well understood: the spectral dimension is 4/3 and the resistance from the root to the sphere of radius r is linear in r. Now from T create the map $M(T)$ by adding a horizontal line connecting each generation of the tree according to their location. $M(T)$ is known as the uniform infinite *Causal triangulation* in the physics literature.

We prove that M has spectral dimension 2 and that the resistance from the root to the sphere of radius r is $r^{o(1)}$. Due to relations to classical models of planar random maps we also study the map $M(T)$ where the progeny distribution of T has mean 1 and polynomial tails with exponent $\alpha \in (3/2, 2)$. We show that the root to sphere resistance in that case is at at most $r^{\frac{2-\alpha}{\alpha-1}}$. It is believed that the true answer is $r^{o(1)}$ for all $\alpha \in (1, 2)$.

Excursion theory for Brownian motion indexed by the Brownian tree and applications to Brownian disks

Jean-François Le Gall

We develop an excursion theory for Brownian motion indexed by the Brownian tree, which in many aspects resembles the classical Itô theory for linear Brownian motion.

The underlying Brownian tree that we consider is the tree coded by a Brownian excursion $(e_s)_{s\geq 0}$ under the so-called Itô excursion measure (note that the Itô excursion measure is a σ-finite measure). The tree $\mathcal{T}_e$ may be viewed as a scaled version of the CRT, for which $(e_s)_{s\geq 0}$ would be a Brownian excursion with duration

1. This tree is rooted at a particular vertex ρ. We then introduce Brownian motion indexed by $\mathcal{T}_e$: Roughly speaking we assign a label V_u to each vertex $u \in \mathcal{T}_e$, in such a way that the label V_ρ of the root is 0, labels vary like linear Brownian motion along line segments of the tree, and of course the increments of labels along two disjoint segments are independent.

Similarly as in the case of linear Brownian motion, we may then consider the connected components of the open set

$$\{u \in \mathcal{T}_e : V_u \neq 0\},$$

which we denote by $(\mathcal{C}_i)_{i\in I}$. These connected components are not intervals as in the classical case, but they are connected subsets of the tree $\mathcal{T}_e$, and thus subtrees of this tree. One then considers, for each component $\mathcal{C}_i$, the restriction $(V_u)_{u\in\mathcal{C}_i}$ of the labels to $\mathcal{C}_i$, and this restriction again yields a random process indexed by a continuous random tree, which we call the excursion E_i. Formally, E_i is viewed as a random element of the space $\mathcal{S}_0$ of all "snake trajectories" with initial point 0: an element of $\mathcal{S}_0$ is a continuous mapping $s \mapsto W_s = (W_s(t))_{0\leq t\leq\zeta_s}$ from a compact interval $[0,\sigma]$ into the space of all finite real paths starting from 0, which is such that $W_0 = W_\sigma$ is the trivial path consisting only of the point 0, and the snake property $W_s(t) = W_{s'}(t)$ for all $0 \leq t \leq \min\{\zeta_r : s \wedge s' \leq r \leq s \vee s'\}$ holds for every $s, s' \in [0,\sigma]$.

Our main results completely determine the "law" of the collection $(E_i)_{i\in I}$ (we speak about the law of this collection though we are working under an infinite measure). A first important ingredient of this description is an infinite excursion measure $\mathbb{M}_0$ on $\mathcal{S}_0$, which plays a similar role as the Itô excursion measure in the classical setting. We also need to introduce, for every $i \in I$, a quantity $\mathcal{Z}_i$ called the exit measure of E_i, that measures the size of the boundary of $\mathcal{C}_i$: $\mathcal{Z}_i$ may be defined by the approximation

$$\mathcal{Z}_i = \lim_{\varepsilon\to 0} \varepsilon^{-2}\mathrm{Vol}(\{u \in \mathcal{C}_i : |V_u| < \varepsilon\}) \tag{1}$$

where Vol is the usual volume measure on $\mathcal{T}_e$. Furtherore, we introduce a "local time exit process" $(\mathcal{X}_t)_{t\geq 0}$ such that, for every $t > 0$, $\mathcal{X}_t$ measures the quantity of vertices u of the tree $\mathcal{T}_e$ with label 0 and such that the total accumulated local time at 0 of the label process along the geodesic segment between ρ and u is equal to t.

Finally, one can define, for every $z > 0$, a conditional probability measure $\mathbb{M}_0(\cdot \mid \mathcal{Z} = z)$ which corresponds to the law of an excursion conditioned to have boundary size z (this is somehow the analog of the Itô measure conditioned to have a fixed duration in the classical setting).

Theorem 1. [1] *The distribution of $(\mathcal{X}_t)_{t>0}$ is the excursion measure of the continuous-state branching process with stable branching mechanism $\psi(\lambda) = \sqrt{8/3}\,\lambda^{3/2}$. Furthermore, the jumps of $(\mathcal{X}_t)_{t>0}$ are in one-to-one correspondence with the excursions $(E_i)_{i\in I}$, and for every $i \in I$ the boundary size of E_i coincides with the size z_i of the corresponding jump of $(\mathcal{X}_t)_{t>0}$. Finally,* *conditionally on the process* $(\mathcal{X}_t)_{t\geq 0}$*, the excursions* E_i*,* $i \in I$ *are independent, and, for every fixed* j*,* E_j *is distributed according to* $\mathbb{M}_0(\cdot \mid \mathcal{Z} = z_j)$*.*

There is a striking analogy with the classical setting, where excursions of linear Brownian excursion are in one-to-one correspondence with jumps of the inverse local time process, and the distribution of an excursion corresponding to a jump of size ℓ is the Itô measure conditioned to have duration equal to ℓ.

We apply the preceding results to the random metric space called the Brownian map, which may be obtained from the CRT by a gluing procedure involving Brownian motion indexed by the CRT. We first observe [3] that, for every $z > 0$, the measure $\mathbb{M}_0(\cdot \mid \mathcal{Z} = z)$ can be used to give a new approach to the Brownian disks introduced by Bettinelli and Miermont [2], via a construction that is much analogous to that of the Brownian map. By combining this new approach with our excursion theory and the construction of the Brownian map, we can show that Brownian disks are embedded in the Brownian map in various ways [3]. In particular, for every $h > 0$, the connected components of the complement of the ball of radius h centered at the distinguished point of the Brownian map are independent Brownian disks conditionally on their volumes and perimeters (these perimeters are defined by an approximation similar to (1)).

References

[1] C. Abraham, J.-F. Le Gall, Excursion theory for Brownian motion indexed by the Brownian tree. *J. Eur. Math. Soc. (JEMS)*, to appear.

[2] J. Bettinelli, G. Miermont, Compact Brownian surfaces I. Brownian disks. *Probab. Theory Related Fields* 167, 555-614 (2017)

[3] J.-F. Le Gall, Brownian disks and the Brownian snake. To appear.

Extreme and Large Values of the Discrete Gaussian Free Field

Oren Louidor

The present talk includes various new results concerning extreme and large values of the discrete Gaussian free field in dimension two. The main object of interest is the random vector $h_N = (h_N(x) : x \in D_N)$, where $D_N = ND \cap \mathbb{Z}^2$ and $D \subset \mathbb{R}^2$ is a "nice" domain. The law of h_N is taken to be Gaussian with 0 mean and covariance given by $G_{D_N} = (G_{D_N}(x,y) : x, y \in D_N)$, the discrete Green function on D_N with 0 boundary conditions on $\mathbb{Z}^2 \setminus D_N$.

A simple first moment calculation together with known asymptotics for G_{D_N} show that the mean number of $x \in D_N$ with $h_N(x) \geq m_N(\lambda)$, where $m_N(\lambda) := \lambda\sqrt{8/\pi}\log N$ and $\lambda \in (0,1)$ is $K_N(\lambda) := C_{\lambda,D} N^{2(1-\lambda^2)}/\sqrt{\log N}$ for some $C_{\lambda,D} > 0$ We show in [2] that in fact, there exists a non-degenerate random variable: $W_{\lambda,D}$, finite and positive almost surely, such that

$$\frac{\big|\{x : h_N(x) \geq m_N(\lambda)\}\big|}{K_N(\lambda)} \overset{N\to\infty}{\Longrightarrow} W_{\lambda,D}\,, \tag{1}$$

Moreover,

$$K_N(\lambda)^{-1} \sum_{x \in D_N} \delta_{\left(x/N, h_N(x) - m_N(\lambda)\right)} \overset{N\to\infty}{\Longrightarrow} Z_D^{\lambda}(\mathrm{d}x) \otimes \mathrm{e}^{-\lambda\sqrt{2\pi}t}\mathrm{d}t\,. \tag{2}$$

where Z_D^λ is the Liouville quantum gravity measure at parameter 2λ, with respect to the continuous Gaussian free field on D.

The above shows in particular that the global maximum of h_N is, on exponential order, $m_N(1) = \sqrt{8/\pi}\log N$. It turns out that the maximum and all other extreme values of the process actually center at distance $O(1)$ around $m_N := \sqrt{8/\pi}\log N - \sqrt{9/8\pi}\log\log N$. Indeed, extending the above to $\lambda = 1$, it was shown in [1] that

$$\sum_{x \in D_N} \delta_{\left(x/N, h_N(x) - m_N\right)} \overset{N\to\infty}{\Longrightarrow} \mathrm{CPPP}\big(Z_D(\mathrm{d}x) \otimes \mathrm{e}^{-\sqrt{2\pi}t}\mathrm{d}t, \nu\big)\,. \tag{3}$$

where Z_D is the Liouville quantum gravity measure at (the critical) parameter 2, ν is the law of a point process on $\mathbb{R} \times D$ and CPPP stands for clustered Poisson point process. The right hand side should be interpreted as a PPP on $\mathbb{R}\times D$ driven by the random intensity $Z_D(\mathrm{d}x) \otimes \mathrm{e}^{-\sqrt{2\pi}t}\mathrm{d}t$ describing the large N landscape of extreme values of h_N which are also local maxima. To these local maxima, a "cluster" chosen independently according to ν is attached in the limiting picture. The latter describes the large N asymptotics of the local configuration around each such extreme local maximum.

By studying the cluster distribution ν, one can obtain finer results concerning the extreme values of h. For instance, it was shown in [3] (for the closely related model of branching Browninan motion) that

$$\frac{\big|\big\{x \in D_N :\ h_N(x) \geq m_N - t\big\}\big|}{t\mathrm{e}^{-\sqrt{2\pi}t}} \overset{\mathrm{P}}{\longrightarrow} CZ_D(D) \tag{4}$$

when $N \to \infty$ and then $t \to \infty$, for some $C > 0$. Moreover, asymptotically (in N followed by t) with high probability, the total contribution to the numerator from (the clusters of) local maxima at level $m_N - s$ is uniform in $s \in [0, t]$ and that $1 - o(1)$ of this contribution comes from only $O(t^{-1})$ of the local maxima.

References

[1] M. Biskup and O. Louidor. Full extremal process, cluster law and freezing for two-dimensional discrete gaussian free field. *arXiv:1606.00510.*

[2] M. Biskup and O. Louidor. On intermediate level sets of two-dimensional discrete gaussian free field. *arXiv:1612.01424.*

[3] A. Cortines, L. Hartung, and O. Louidor. The structure of extreme level sets in branching Brownian motion. *arXiv:1703.06529.*

Geometric versions of isomorphism theorems for continuum GFF in 2D

Titus Lupu

The connection between the Gaussian free field (GFF) and Markov trajectories was known since the work of Kurt Symanzik [3], where, studying the Φ^4 model, he expresses the moments of a massive GFF in $\mathbb{R}^d$ as multiple integrals with respect to an infinite mass measure on Brownian loops. Further, the so called Dynkin's isomorphism and the generalized Ray-Knight theorems, give a more probabilistic version of this relation, where the square of the GFF is coupled to occupation times of Markov trajectories [4]. Le Jan gave a version of these isomorphisms where the whole square of the GFF is given by the occupation times of Markov loops in a Poisson point process, called Markov loop-soup, whose intensity is the one that appears in Symanzik [3].

During my PhD with Le Jan I found a way to relate not only the square, but also the sign of a GFF to a Markov loop-soup, in a way that the sign of the GFF is constant on each cluster. For this I considered metric graphs, obtained by replacing each edge in a discrete network by a continuous line of length equal the resistance, and the GFF on on these objects, obtained by interpolating the discrete GFF on vertices by independent Brownian bridges inside the edges. The process obtained this way is Markov, Gaussian and moreover it is continuous. Thus, since its square is strictly positive on each cluster of loops on the metric graph, it has to be of constant sign on each of it [2].

For the continuum GFF in dimension 2, one has a version of Le Jan's isomorphism with the centred occupation measure of a Brownian loop-soup and a renormalized (Wick) square of the GFF, where one subtracts the diverging expectation. However, in this setting, one has a stronger relations. In fact, the GFF lives entirely on clusters of Brownian loops in a loop-soup. This is explained in a collaboration with Juhan Aru end Avelio Seplveda at ETH [1]. There we introduce the first passage sets of a continuum GFF Φ in a (possibly multiply connected) domain D of $\mathbb{C}$. Informally, the first passage set of level $-a$ is

$$A_{-a} = \{z \in \bar{D} | \exists \gamma \text{ path from } z \text{ to } \partial D, \Phi \geq -a \text{ on } \gamma\}.$$

Above expression does not make immediate sense because Φ is a generalized function. However, A_{-a} can be constructed by iterating level lines of the GFF. A_{-a} has zero Lebesgue measure and is of Hausdorff dimension 2. A_{-a} is a local set of Φ, and it is non-thin: Φ on A_{-a} is a positive measure. We show that this measure, as a random variable, is measurable with respect its support A_{-a}, and conjecture that it is a Minkowski content in a right gauge. Finally, these first passage sets have same law as clusters of Brownian loops and boundary-to-boundary excursions connected to the boundary. For this, we use approximations by metric graphs.

References

[1] J. Aru, T. Lupu and A Seplveda. First passage sets of the 2D continuum Gaussian free field. Preprint arXiv:1706.07737, 2017.

[2] T. Lupu. From loop clusters and random interlacement to the free field. *Annals of Probability*, Volume 44, 2117-2146, 2016.

[3] K. Symanzik. *Euclidean quantum field theory.* Scuola intenazionale di Fisica "Enrico Fermi". XLV Corso. 152-223 Academic Press, 1969.

[4] A.-S. Sznitman. *Topics in occupation times and Gaussian free field.* Zurich lectures in advanced mathematics. European Mathemtical Society, 2012.

First passage percolation on rotationally invariant fields

Allan Sly

(joint work with Riddhipratim Basu, Vladas Sidoravicius)

Understanding the distance fluctuations is a major open problem in the study of first passage percolation. For two points at distance n, the Poincare inequality give a bound of $O(n)$. Work of Benjamnin, Kalai and Schramm [1], using methods from the analysis of Boolean functions, showed that in the Boolean case the variance is sub-linear, at most $O(n/\log n)$. While this remains the best bound, it is widely believed that the variance should scale $O(n^{2/3})$, known only for exactly solveable models of last passage percolation.

One key obstacle in the study of first passage percolation is a lack of understanding of its limit shape. While under quite general conditions the limit shape is expected to be smooth with uniformly bounded positive curvature, this is not known in any example [3]. In order investigate first passage percolation while avoiding this impediment we consider models with rotational symmetry. Examples of such models include continuous fields on the plane where the shortest path is given by minimizing integrals over continuous path between the endpoints or for graph distance in random geometric graphs. In such models the limit shape is of course the circle.

We take a multi-scale approach and show that the variance is at most $O(n^{1-\epsilon})$. This is done by a geometric approach to showing chaotic behaviour of the path, that is that after resampling a small fraction of the field the new path is almost completely disjoint from the original. Using the correspondence between chaos and super-concentration (variance bounds better than the Poincare inequality) this yields improved estimates. By carrying out this analysis on every scale we derive a polynomial improvement in the variance overall. To carry out this scheme we need to control the transversal fluctuations of the geodesic together with exponential concentration using an argument of Kesten [2] adapted to our multi-scale setting.

References

[1] Itai Benjamini, Gil Kalai, and Oded Schramm. Noise sensitivity of boolean functions and applications to percolation. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, (90):5–44, 1999.

[2] Harry Kesten. On the speed of convergence in first-passage percolation. *The Annals of Applied Probability*, pages 296–338, 1993.

[3] Charles M Newman. A surface view of first-passage percolation. In *Proceedings of the International Congress of Mathematicians*, volume 1, pages 1017–1023, 1995.

Voronoi cells in continuum random maps on any surface

Louigi Addario-Berry

(joint work with Omer Angel, Guillaume Chapuy, Éric Fusy, Christina Goldschmidt)

Let (T, d, μ) be the Brownian continuum random tree (CRT), with distance d and mass measure μ.

Theorem 1. *Let $X_1, \ldots, X_k \in T$ be k independent samples from μ in the tree T. Partition the tree into k Voronoi cells $C_1, \ldots, C_k$. Then the vector $(\mu(C_i))_{i \leq k}$ is a uniform partition of unity (i.e. uniformly distributed on the $k-1$ dimensional simplex).*

This surprising result extends to other settings. The unicellular continuum random map (CRM) is a generalization of the CRT to other surfaces. Unicellular CRMs arise as the scaling limits of uniform random unicellular maps embedded in surfaces. They are locally treelike, unlike the Brownian map. If the surface is the sphere, the unicellular CRM is the CRT itself. The unicellular CRMs are *universal*, in that many different measures on discrete maps will have a unicellular CRM as their limit.

Theorem 2. *Fix a surface S, and let (M, d, μ) be the unicellular CRM on S. Let $X_1, \ldots, X_k \in M$ be k independent samples from μ in M. Partition M into k Voronoi cells $C_1, \ldots, C_k$. Then the vector $(\mu(C_i))_{i \leq k}$ is a uniform partition of unity.*

Voronoi cells and their volumes depend only on metric structure and not on the embedding in the surface. Thus, if two random maps have the same distributions as measured metric spaces then the mass partitions induced by their Voronoi cells are identically distributed. Using this observation, Theorem 2 may be applied to deduce that Voronoi cells of randomly sampled points in a number of other continuum maps also yield uniform volume partitions.

Theorem 3. *The following continuum random maps also have uniformly distributed Voronoi cell volumes:*

(1) The continuum unicycle (map with a single cycle),
(2) The CRT with three uniform points identified (a theta-map),
(3) The CRT with any number of triplets of uniform points identified,

(4) The continuum barbell (map with two loops connected by an edge).

These cases are proved by considering unicellular CRMs on the projective plane, torus, torus of general genus g, and Klein bottle respectively. Note that case (iii) above does not involve a single structure for the map, but a random structure, depending on the relation between the locations of the different points glued. Thus case (iii) implies that certain mixtures of map structures have uniform Voronoi cell volumes.

Acknowledgements This work was started during the Eleventh Annual Workshop on Probability and Combinatorics, at McGill University's Bellairs Institute in Holetown, Barbados. It was motivated by third author's conjecture in [1], that the Brownian map on any fixed surface has uniform Voronoi cells.

References

[1] Guillaume Chapuy. On tessellations of random maps and the t_g-recurrence. *arXiv preprint arXiv:1603.07714*, 2016.

Reporter: Ewain Gwynne

Participants

Prof. Dr. Louigi Addario-Berry
Department of Mathematics and Statistics
McGill University
Burnside Hall
805 Sherbrooke Street West
Montreal QC H3A 2K6
CANADA

Prof. Dr. Omer Angel
Department of Mathematics
University of British Columbia
1984 Mathematics Road
Vancouver BC V6T 1Z2
CANADA

Dr. Juhan Aru
Departement Mathematik
ETH-Zentrum
8092 Zürich
SWITZERLAND

Prof. Dr. Ismael Bailleul
U. F. R. Mathématiques
I. R. M. A. R.
Université de Rennes I
Campus de Beaulieu
35042 Rennes Cedex
FRANCE

Dr. Vincent Beffara
Institut Fourier
Université Grenoble-Alpes
BP 74
100, rue des Mathématiques
38402 Saint-Martin-d'Hères Cedex
FRANCE

Prof. Dr. Gerard Ben Arous
Courant Institute of Mathematical Sciences
New York University
251, Mercer Street
New York NY 10012-1185
UNITED STATES

Prof. Dr. Nathanael Berestycki
DPMMS
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Prof. Dr. Jean Bertoin
Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190
8057 Zürich
SWITZERLAND

Prof. Dr. Jeremie Bouttier
Institut de Physique Théorique
CEA Saclay
91191 Gif-sur-Yvette Cedex
FRANCE

Prof. Dr. Nicolas Curien
Laboratoire de Mathématiques
Université Paris Sud (Paris XI)
Batiment 425
91405 Orsay Cedex
FRANCE

Prof. Dr. Jian Ding
Department of Statistics
University of Chicago
5747 S. Ellis Avenue
Chicago IL 60637
UNITED STATES

Dr. Julien Dubedat
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
UNITED STATES

Prof. Dr. Hugo Duminil-Copin
Institut des Hautes Etudes Scientifiques
(IHES), Le Bois-Marie
35, route de Chartres
91440 Bures-sur-Yvette
FRANCE

Prof. Dr. Bertrand Duplantier
Institut de Physique Théorique
CEA Saclay
91191 Gif-sur-Yvette Cedex
FRANCE

Prof. Dr. Yan V. Fyodorov
Department of Mathematics
King's College London
Strand
London WC2R 2LS
UNITED KINGDOM

Prof. Dr. Christophe Garban
Institut Camille Jordan
Université de Lyon I
43 bd. du 11 Novembre 1918
69622 Villeurbanne Cedex
FRANCE

Prof. Dr. Christina Goldschmidt
Department of Statistics
Lady Margaret Hall
University of Oxford
2 Fyfield Road
Oxford OX2 6QE
UNITED KINGDOM

Dr. Vadim Gorin
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139-4307
UNITED STATES

Prof. Dr. Alice Guionnet
Département de Mathématiques
U.M.P.A.
École Normale Superieure de Lyon
46, Allee d'Italie
69364 Lyon Cedex 07
FRANCE

Dr. Ori Gurel-Gurevich
Einstein Institute of Mathematics
The Hebrew University
Givat Ram
91904 Jerusalem
ISRAEL

Dr. Ewain Gwynne
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge, MA 02139-4307
UNITED STATES

Prof. Dr. Benedicte Haas
Département de Mathématiques
Institut Galilee
Université Paris XIII
99, Avenue J.-B. Clement
93430 Villetaneuse Cedex
FRANCE

Prof. Dr. Martin Hairer
Mathematics Institute
University of Warwick
Gibbet Hill Road
Coventry CV4 7AL
UNITED KINGDOM

Nina Holden
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge, MA 02139-4307
UNITED STATES

Dr. Adrien Kassel
École Normale Superieure de Lyon
UMPA UMR 5669 CNRS
46, Allee d'Italie
69364 Lyon Cedex 07
FRANCE

Prof. Dr. Richard Kenyon
Department of Mathematics
Brown University
Box 1917
Providence, RI 02912
UNITED STATES

Prof. Dr. Davar Khoshnevisan
Department of Mathematics
University of Utah
155 South 1400 East
Salt Lake City, UT 84112-0090
UNITED STATES

Dr. Antti Knowles
Section of Mathematics
University of Geneva
2-4 Rue du Lièvre
1211 Genève 4
SWITZERLAND

Prof. Dr. Igor Kortchemski
Centre de Mathématiques Appliquées
C.N.R.S.
École Polytechnique
Route de Saclay
91128 Palaiseau Cedex
FRANCE

Prof. Dr. Antti Kupiainen
Department of Mathematics and
Statistics
University of Helsinki
P.O. Box 68
00014 Helsinki
FINLAND

Prof. Dr. Jean-Francois Le Gall
Département de Mathématiques
Université de Paris-Sud
Bat. 425
91405 Orsay Cedex
FRANCE

Prof. Dr. Yves Le Jan
Laboratoire de Mathématiques
Université Paris Sud (Paris XI)
Batiment 425
91405 Orsay Cedex
FRANCE

Prof. Dr. Xue-Mei Li
Mathematics Institute
University of Warwick
Zeeman Building
Coventry CV4 7AL
UNITED KINGDOM

Prof. Dr. Oren Louidor
Faculty of Industrial Engineering
and Management, TECHNION
Israel Institute of Technology
Haifa 32000
ISRAEL

Dr. Titus Lupu
Departement Mathematik
ETH-Zentrum
Rämistrasse 101
8092 Zürich
SWITZERLAND

Dr. Pascal Maillard
Laboratoire de Mathématiques
Université Paris Sud (Paris XI)
Batiment 425
91405 Orsay Cedex
FRANCE

Prof. Dr. Grégory Miermont
UMPA
ENS de Lyon
46, Allée d'Italie
69364 Lyon Cedex 07
FRANCE

Dr. Jason P. Miller
Department of Pure Mathematics
and Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Dr. Jean-Christophe Mourrat
UMPA - ENS Lyon
46, Allée d'Italie
69364 Lyon Cedex 07
FRANCE

Prof. Dr. Asaf Nachmias
Department of Mathematics
University of British Columbia
Vancouver BC V6T 1Z2
CANADA

Dr. Ellen G. Powell
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Dr. Wei Qian
Department of Pure Mathematics
and Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Dr. Rémi Rhodes
LAMA
Université Paris-Est Marne La Vallée
Champs sur Marne
5, Boulevard Descartes
77454 Marne-la-Vallée Cedex 2
FRANCE

Dr. Avelio Sepulveda
Departement Mathematik
ETH-Zentrum
Rämistrasse 101
8092 Zürich
SWITZERLAND

Prof. Dr. Scott Sheffield
Department of Mathematics
MIT
Cambridge, MA 02139
UNITED STATES

Prof. Dr. Allan Sly
Department of Statistics
University of California, Berkeley
367 Evans Hall
Berkeley CA 94720-3860
UNITED STATES

Dr. Nike Sun
Department of Statistics
University of California, Berkeley
367 Evans Hall
Berkeley CA 94720-3860
UNITED STATES

Prof. Dr. Fabio Toninelli
Institut Camille Jordan
Université Claude Bernard Lyon I
43, Bd. du 11 Novembre 1918
69622 Villeurbanne Cedex
FRANCE

Prof. Dr. Hao Wu
Section of Mathematics
University of Geneva
2-4 Rue du Lièvre
1211 Genève 4
SWITZERLAND

Dr. Vincent Vargas
DMA UMR 8553 CNRS
École Normale Superieure
45, rue d'Ulm
75230 Paris Cedex 05
FRANCE

Prof. Dr. Lorenzo Zambotti
Laboratoire de Probabilities et Modeles
Aleatoires
Universite Paris VI et VII
2, place Jussieu
75005 Paris Cedex
FRANCE

Prof. Dr. Wendelin Werner
Departement Mathematik
ETH-Zentrum
Rämistrasse 101
8092 Zürich
SWITZERLAND