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Geophysical Fluid Dynamics

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Abstract. The workshop "Geophysical Fluid Dynamics" addressed recent advances in analytical, stochastic, modeling and computational studies of geophysical fluid models. Of central interest were the reduced geophysical models, that are derived by means of asymptotic and scaling techniques, and their investigations by methods from the above disciplines. In particular, contributions concerning the viscous and inviscid geostrophic models, the primitive equations of oceanic and atmospheric dynamics, tropical atmospheric models and their coupling to nonlinear dynamics of phase changes moisture, thermodynamical effects, stratifying effects, as well as boundary layers were presented and discussed.

Mathematics Subject Classification (2010): 76-XX, 86A10, 35-XX, 60Hxx.

Introduction by the Organisers

This workshop was aiming to bringing together experts from diverse scientific disciplines with common interest in geophysical fluid dynamics and to encompass their scientific exchange concerning their investigation of various classes of geophysical fluid models. These models have been of great scientific interest due to the complex structure of their underlying coupled nonlinear dynamics. In particular, numerous scientific tools from mathematical analysis, stochastic dynamics, modeling and computational sciences have been developed to examine their quantitative and qualitative behaviours. As in the case of the Navier-Stokes equations, some of these detailed geophysical models still lack, however, basic understanding concerning global existence and uniqueness of smooth solutions. In oceanic and atmospheric dynamics, as well as in the theory of boundary layers, one often tends to derive and investigate reduced simplified models, whose derivations are based on formal asymptotic procedures. These simplified models bring up difficult analytical and physical questions concerning, e.g., the well-posedness, validity and stability of these models for the relevant spatial and temporal scales. Notably, identifying these relevant spatial and temporal scales of validity is already a major mathematical challenge. Thus, a major goal of this workshop was to bridge between recent theoretical advances, in those branches of mathematics relevant to geophysical flows, and the physical understanding of the observed underlying phenomena in those flows; and in particular to judge and validate the reliability of these simplified models for the relevant physical spatial and temporal scales of their derivation.

The mathematical investigation of these models involves many modern mathematical tools ranging from nonlinear partial differential equations and their stochastic counter parts, harmonic analysis, dispersive estimates and transport theory to evolution equations. As a first step one aims to prove the global well-posedness of the underlying equations. This represents also an important step in the development of numerical and computational schemes for simulation of these models.

Of particular interest in this context was also the understanding and analysis of boundary layers, such as Prandtl's boundary layer model. Furthermore, since certain solutions of the inviscid primitive equations exhibit blow up in finite time, several attempts were made to examine whether the fast rotation term, due to Coriolis force, has a stabilizing effect on these solutions that allows to prolong its life-span and whether certain smoothing techniques, which work well for the three-dimensional Euler equations, may be transferred to this setting. Moreover, modern tropical atmospheric moisture models, taking into account also phase transitions of the clouds' vapour/water, require the balance laws for energy and entropy. Rigorous verifications of the fact that these models are consistent with the second law of thermodynamics are important. Finally, recently developed downscaling data assimilation algorithms for weather and climate predications were considered.

The meeting ignited lively and productive interaction and exchange of ideas and was thus a very inspiring experience. Each lecture was allocated 40 minutes followed by a very lively and interactive discussion of 20 minutes. Moreover, the vibrant presence of young participants was very visible during the meeting. In particular, they were encouraged to present their work in a special evening session, which was fully attended by all participants.

In summary, the meeting brought together an excellent balanced mixture of scientist from the various scientific communities. In particular, several leaders from different disciplines met for the first time in person. Notably, the age, gender and geographic diversity of the participants was more than adequate.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons

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Workshop: Geophysical Fluid Dynamics

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Abstracts

The initial value problem for the Euler equations of incompressible fluids viewed as a concave maximization problem of optimal transport type

Yann Brenier

Let us fix a time interval $[0,T]$ and denote by D the periodic box $D=\mathbb{R}^d/\mathbb{Z}^d$. The Euler model of an incompressible fluid of unit mass density, moving in D during the time interval $[0,T]$, without external forces, assumes the existence of a square-integrable, divergence-free vector field V, over $[0,T]\times D$, such that (written in coordinates, with the usual implicit summation over repeated lower and upper indices) $\partial_t V^i + \partial_j(V^iV^j)$ is a gradient field. In weak form, this means

$$(1) \qquad \int_{[0,T]\times D} \partial_i\varphi(t,x)V^i(t,x)dxdt = 0,$$

for all smooth function φ over $[0,T]\times D$, which encodes that V is divergence-free and

$$(2) \qquad \int_{[0,T]\times D} \left(\partial_j A_i V^iV^j + \partial_t A_i V^i\right)(t,x)dxdt + \int_D P_0^i(x)A_i(0,x)dx = 0,$$

for all smooth divergence-free vector fields A on $[0,T]\times D$, vanishing at $t=T$, which includes (weakly) the initial condition that V is P_0 at time $t=0$, P_0 being a given L^2 divergence-free vector field on D.

Our goal is to solve, by a concave maximization method, the initial value problem for the Euler model with as initial condition a fixed divergence-free vector field P_0, square integrable over D and of zero spatial mean. The idea is very simple: we try to find a divergence-free vector field V, weak solution to the Euler equation with initial condition P_0, of minimal kinetic energy. This leads to the saddle-point problem

$$(3) \quad \mathcal{I}[P_0] = \inf_V \sup_{A,\varphi} \int_{[0,T]\times D} \frac{1}{2}|V|^2 + \partial_j A_i V^iV^j + (\partial_t A_i + \partial_i\varphi)V^i + \int_D P_0^i A_i(0,\cdot)$$

over all L^2 vector fields V on $[0,T]\times D$, all smooth divergence-free vector fields A vanishing at $t=T$, and all smooth real functions φ. We may interpret (A,φ) as Lagrange multipliers for the constraint that V is a weak solution to the Euler equations with initial condition P_0, in the sense of (1,2). Investigating problem (3) looks silly since the Euler equation to be solved is already included as a constraint! Furthermore, for smooth solutions of the Euler equation on the periodic box, the kinetic energy is constant in time and, therefore, depends only on the data P_0, so that...there seems to be nothing to minimize! However, for a fixed initial condition, weak solutions are not unique and the conservation of energy is generally not true as well known since the celebrated results of Scheffer, Shnirelman, De Lellis and Székelyhidi [4, 5, 3]. Therefore, since, in addition, weak solutions always exist, following Wiedemann [7], the minimization problem is definitely not meaningless.

In this talk, we mostly investigate the dual problem obtained by exchanging the infimum and the supremum in (3), leading to a concave maximization problem which can be shown to be solvable. The resulting maximization problem roughly reads

$$\sup_{(E,B)} \quad -\int_{[0,T]\times D} E\cdot(\mathbb{I}_d+2B)^{-1}\cdot E + 2P_0\cdot E$$

where $\mathbb{I}_d$ denotes the $d\times d$ identity matrix, E and B are respectively valued in $\mathbb{R}^d$ and in the space of $d\times d$ symmetric matrices, and subject to

$$\partial_t B = LE, \quad B(t=T,\cdot)=0,$$

L being a suitable first-order constant coefficient (pseudo-)differential operator on D, namely (in coordinates)

$$L^k_{ij}E_k = \frac{1}{2}(\partial_j E_i + \partial_i E_j) + \partial_i\partial_j(-\triangle)^{-1}\partial^k E_k, \text{ where } \partial^k = \delta^{kj}\partial_j \text{ and } \triangle = \delta^{ij}\partial_i\partial_j.$$

Surprisingly enough, this problem looks very similar to the Monge optimal mass transport problem with quadratic cost in its so-called "Benamou-Brenier" formulation [2, 1, 6], which would read

$$\inf_{\rho,Q}\int_{[0,T]\times D} Q\cdot\rho^{-1}\cdot Q,$$

where ρ and Q are respectively valued in $\mathbb{R}_+$ and $\mathbb{R}^d$ and subject to $\partial_t\rho+\partial_i Q^i=0$, while ρ is prescribed at $t=0$ and $t=T$. Presumably, the maximization problem can be treated by the same numerical method as the one used in [2]. Next, we check that any local smooth solution of the Euler equations can be recovered, from the maximization problem, for short enough times T. Finally, we make a connection between the maximization problem and the theory of sub-solutions to the Euler equations which has recently attracted a lot of interest after the celebrated work of De Lellis and Székelyhidi [3] in the framework of Convex Integration theory.

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Global solutions to the isentropic compressible Navier-Stokes equations with a class of large initial data

Daoyuan Fang
(joint work with Ting Zhang, Ruozhao Zi)

In this talk, we consider the global well-posedness problem of the isentropic compressible Navier-Stokes equations in the whole space $\mathbb{R}^N$ with $N \geq 2$. In order to better reflect the dispersive property of this system in the low frequency part, we introduce a new solution space that characterizes the behaviors of the solutions in different frequencies, and prove that the isentropic compressible Navier-Stokes equations admit global solutions when the initial data are close to a stable equilibrium in the sense of suitable hybrid Besov norm. As a consequence, the initial velocity with arbitrary $\dot{B}_{2,1}^{\frac{N}{2}-1}$ norm of potential part $\mathbb{P}^{\perp}u_0$ and large highly oscillating are allowed in our results. The proof relies heavily on the dispersive estimates for the system of acoustics, and a careful study of the nonlinear terms. This is the joint work with Ting Zhang and Ruozhao Zi.

Continuous data assimilation algorithms for geophysical models

Aseel Farhat
(joint work with Michael Jolly, Evelyn Lunasin, Edriss S. Titi)

Analyzing the validity and success of a data assimilation algorithm when some state variable observations are not available is an important problem meteorology and engineering. In this talk, we will present an improved continuous data assimilation (downscaling) algorithm for few fluid dynamics models that *does not require observations of all evolving state variables* of the system. Rather than inserting the observational measurements directly into the equations, *a feedback control term* is introduced that forces the model towards its reference solution. For the 2D incompressible Bénard convection problem, for example, our algorithm uses *only velocity measurements* (temperature measurements are not necessary). In the case of the 3D Planetary Geostrophic model, our algorithm requires *observations of temperature only*. The choice of the determining state variables for different dissipative fluid models depends on the structure of the equations in each model.

The talk is based on joint works with Michael Jolly (Indiana University), Evelyn Lunasin (The United States Naval Academy), and Edriss S. Titi (Weizmann Institute of Science and Texas A&M University).

On the motion of compressible inviscid fluids driven by stochastic forcing

Eduard Feireisl
(joint work with Dominic Breit, Martina Hofmanová)

We consider the (barotropic) Euler system describing the motion of a compressible inviscid fluid driven by a stochastic forcing. Adapting the method of convex integration we show that the initial value problem is ill-posed in the class of weak (distributional) solutions. Specifically, we find a sequence $\tau_M \to \infty$ of positive stopping times for which the Euler system admits infinitely many solutions originating from the same initial data. The solutions are weak in the PDE sense but strong in the probabilistic sense, meaning, they are defined on an a priori given stochastic basis and adapted to the driving stochastic process.

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Could noise regularize 2D Euler equations?

Franco Flandoli

In recent years it has been discovered that suitable noise regularizes some classes of PDEs (beyond the well-known fact that regularizes ODEs: recall for intance [13] where it is proved well -posedness when the drift is just measurable bounded). By the sentence "regularization by noise" we mean that an equation with noise has better well-posedness properties compared to the same equation without noise; for instance, uniqueness may be restored by noise, or singuralities may be avoided. The more succesfull classes are linear transport type equations with irregular coefficients, both scalar [7] and vector valued [9]: with noise they have uniqueness and no-blow-up properties under weaker assumptions on the coefficients compared to the deterministic case. For nonlinear problems, restricting the attention to problems related to fluid-dynamics, one could mention for instance the better uniqueness properties of dyadic models with noise [1], or in a different direction some results for 3D Navier-Stokes equations [4], [10] (but in the present discussion we mainly concentrate on inviscid problems).

Concentrating on the particular case of the 2D Euler equations on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, we have investigated the properties of the equation, for the vorticity ω (and velocity u)

$$\partial_t \omega + u \cdot \nabla\omega + \nabla\omega \circ \xi = 0$$
$$\operatorname{div} u = 0, \qquad \nabla^{\perp} u = \omega.$$

The noise ξ is chosen in transport form, similarly to [7], [9] and other references. To avoid the counterexample to regularization by noise described in [7], the noise must be relatively complex from the view-point of the spatial structure: for instance, on

$$\xi(t,x) := \sum_{k\in\mathbb{Z}^2\setminus\{0\}} |k|^{-\alpha} e_k(x) \frac{dW_t^k}{dt}$$

where $e_k(x) = \frac{k^\perp}{|k|} e^{ik\cdot x}$ and W_t^k are independent Brownian motions. One can give a rigorous meaning to this equation and prove several theorems, for instance the existence in L^p [3] and the uniqueness in L^∞ [2] (for the vorticity ω); these are generalizations of known results of the deterministic case. Is it possible to prove more?

The first and only rigorous result deals with very particular solutions: the case of point vortices

$$\omega_t(dx) = \sum \omega_i \delta_{X_t^i}.$$

In the deterministic case there are examples of initial configurations which lead to collapse in finite time [11]; but for almost every initial configuration with respect to the Lebsgue measure, no collapse occurs and solutions are global. Under the previous noise (any α), it has been proved [8] that for every initial configuration collapse does not occur, with probability one; see also [5].

Concerning the case of function-valued solutions, a description of a potential approach by Girsanov transform, but still not applicable because of important difficulties, is given in [6]. In the talk given in Oberwolfach, the theory of Shnirelman [12] has been discussed, in the case of noise. The results under noise shown in the talk were not rigorous, just an heuristic indication. Recall that [12] constructs a counterexample to uniqueness of weak solutions of Euler equations. The heuristic result described in the talk is that Shnirelman's construction seems to be applicable also to the noise case as soon as $\alpha > 1$. On the contrary, for $\alpha = 1$, the noise seems to produce an obstruction to the proof of [12], a moderate indication that it could produce a regularization. The same noise, $\alpha = 1$, emerges from the Girsanov approach outlined in [6] and it is therefore proposed as an interesting case for future investigations.

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Multiscale asymptotics and analysis of moist atmospheric flows

Sabine Hittmeir
(joint work with R. Klein)

Model reductions in meteorology by scale analysis are inevitable and therefore have a long history in meteorology. The key technique for a systematic study of complex processes involving the interaction of phenomena on different length and time scales is multiple scales asymptotics. Of particular interest here are hot towers, which are large cumulonimbus clouds that live on small horizontal scales having a diameter of the order of one kilometer. It is common belief by now that these hot towers are to a great extent responsible for the vertical heat transport into the upper troposphere within the innertropical convergence zone. Due to their major contribution to the energy transport it is extremely important to develop a good understanding of their life cycles. Moreover these deep convective clouds constitute the building blocks of intermediate scale convective storms, which we study in a next step by incorporating the setting of organised convection into the multiscale approach. This requires not only the introduction of coordinates allowing for an individual tilt for each columnar cloud, but also new systematic averaging procedures, which enable us to quantify the modulation of the larger scale flow by the moisture processes in the small scale regions. This work is joint work with R. Klein.

While the just described multiscale asymptotics are purely formal, in collaboration with R. Klein, J. Li and E. Titi, we also proceed further in the rigorous analysis of the atmospheric flow models with moisture and phase transitions. We study the global existence and uniqueness of solutions for the moisture balance equations coupled to the thermodynamic equation building the basis for the above expansions, where in a first step we assume the flow field to be given.

Primitive Equations in L^p spaces and maximal regularity

Amru Hussein
(joint work with Yoshikazu Giga, Mathis Gries, Matthias Hieber, Takahito Kashiwabara)

The question of the well-posedness of the primitive equations L^p spaces is addressed from the perspective of maximal L^q-regularity. Maximal L^q-regularity for the linearized system one can be used to study many quasi-linear and semi-linear evolution equations. Here, focusing on the velocity equation, an explicit representation of the linearized Stokes-type operator in L^p as perturbation of the Laplacian is derived which proves maximal L^q-regularity. Estimating the non-linearity one can prove existence of strong solutions for arbitrarily large initial data in the Besov (trace space) $B^{2/p}_{pq}$ for $p, q \in (1, \infty)$ with $1/q+1/p \leq 1$ which can be extended globally using an H^2 *a priori* bound. That solutions are smoothened instantaneously to become real analytic follows directly methods used in maximal regularity theory. The flexibility of the method allows to include various boundary conditions. We aim to use this result also for our study of the limit case $p = \infty$ when constructing reference solutions.

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On the semigroup approach to the primitive equations

Takahito Kashiwabara
(joint work with Y. Giga, M. Gries, M. Hieber, A. Hussein)

1. Formulation of primitive equations as a parabolic problem

We are concerned with the primitive equations which describe large-scale motion of atmosphere or ocean:

$$(1) \quad \begin{cases} \partial_t v + u \cdot \nabla v - \Delta v + \nabla_H \pi = 0 & \text{in } \Omega \times (0, T), \\ \partial_z \pi = 0 & \text{in } \Omega \times (0, T), \\ \operatorname{div} u = 0 & \text{in } \Omega \times (0, T), \\ v(0) = a & \text{in } \Omega. \end{cases}$$

Here, $\Omega = (0, 1)^2 \times (-h, 0) =: G \times (-h, 0)$ is a box domain; $u = (v, w) = (v_1, v_2, w)$ denotes the three-dimensional velocity, and v and w are its horizontal and vertical components, respectively; π means the pressure. Differential operators are also separated into horizontal and vertical components, e.g., $\nabla = (\partial_x, \partial_y, \partial_z)$, $\nabla_H = (\partial_x, \partial_y)$.

We equip system (1) with the following boundary conditions:

$$\begin{aligned} \partial_z v = w = 0 \quad &\text{on} \quad \Gamma_u, \\ v = w = 0 \quad &\text{on} \quad \Gamma_b, \\ v, w, p \text{ are periodic} \quad &\text{on} \quad \Gamma_l, \end{aligned}$$

where $\Gamma_u = G \times \{z = 0\}, \Gamma_b = G \times \{z = -h\}, \Gamma_l = \partial G \times [-h, 0]$ denote the top, bottom, and lateral parts of the boundary, respectively.

Local-in-time strong solvability of (1) for $a \in H^1(\Omega)^2$ is established by [2] in the L^2-framework. Later, in [1] it is shown that the local strong solution can be extended globally in time. In this study, we extend those results to the L^p-setting by adopting an analytic semigroup approach.

In order to formulate (1) as a semilinear parabolic equation, we first eliminate w as

$$w = \int_z^0 \mathrm{div}_H v \, d\zeta,$$

which follows from $(1)_3$ and $w|_{\Gamma_u} = 0$. Since $w|_{\Gamma_b} = 0$, we obtain the constraint $\mathrm{div}_H \bar{v} = 0$, where $\bar{v} = \int_{-h}^0 v \, dz$ stands for the vertical average. Then, regarding π as a function defined in the 2D domain G, we rewrite (1) as

$$\partial_t v + v \cdot \nabla_H v + \int_z^0 \mathrm{div}_H \, d\zeta \, \partial_z v - \Delta v + \nabla_H \pi = 0, \quad \mathrm{div}_H \bar{v} = 0. \tag{2}$$

Next we incorporate the constraint $\mathrm{div}_H \bar{v} = 0$ into a functional analytical setting. For this we introduce the *hydrostatic Helmholtz projecter* $P : L^p(\Omega)^2 \to L^p(\Omega)^2$ by $Pf = f - \nabla_H q$, where q is the weak solution of $\Delta_H q = \mathrm{div}_H \bar{f}$. Then P becomes a bounded linear operator in $L^p(\Omega)^2$ for $p \in (1, \infty)$, and we obtain

$$X := \mathrm{Range}(P) = \{v \in L^p(\Omega)^2 \,|\, \mathrm{div}_H \bar{v} = 0 \text{ in } G, \ \bar{v} \cdot \nu_{\partial G} \text{ is anti-periodic on } \partial G\}.$$

We further define the *hydrostatic Stokes operator* by $Av = -P\Delta v$ with

$$D(A) = \{v \in W^{2,p}_{\mathrm{per}}(\Omega)^2 \,|\, \mathrm{div}_H \bar{v} = 0, \ \partial_z v|_{\Gamma_u} = v|_{\Gamma_b} = 0\},$$

where $W^{2,p}_{\mathrm{per}}(\Omega)$ denotes the $W^{2,p}(\Omega)$-functions with the periodic boundary conditions on Γ_l.

Using A, one can formulate (2) as a semilinear parabolic equation

$$\partial_t v + Av = Fv := -P(v \cdot \nabla_H v + \int_z^0 \mathrm{div}_H \, d\zeta \, \partial_z v), \quad v(0) = a,$$

which is formally equivalent to the following integral equation by the Duhamel formula:

$$v(t) = e^{-tA} a + \int_0^t e^{-(t-s)A} Fv(s) \, ds, \quad \forall t \geq 0. \tag{3}$$

Below we construct a solution of (1) based on this representation (such solution is called a *mild solution* of PDEs).

2. Main result

The operator calculus e^{-tA} appearing in (3) is well-defined as an analytic semigroup, which follows from analysis of the resolvent problem for the linear primitive equations.

Theorem 1. *Let $\lambda \in \mathbb{C}$ such that $|\arg \lambda| < \pi - \epsilon$ with $\epsilon > 0$ and let $p \in (1, \infty)$. Then, for all $f \in X$ there exists a unique solution of $\lambda v + Av = f$ satisfying*

$$|\lambda| \|v\|_{L^p(\Omega)} + \|v\|_{D(A)} \leq C \|f\|_{L^p(\Omega)}.$$

Corollary 2. *For $1 < p < \infty$, A generates an analytic semigroup e^{-tA} in X.*

Now we state local-in-time well-posedness result, which is proved by the celebrated Fujita–Kato method. For this we need a complex function space $V_\theta := [X, D(A)]_\theta$ for $0 \leq \theta \leq 1$ to describe intermediate regularity between X and $D(A)$. The most important property of V_θ is $\|e^{-tA}a\|_{V_{\theta_1+\theta_2}} \leq Ct^{-\theta_1}\|a\|_{V_{\theta_2}}$.

Theorem 3. *Let $p \in (1, \infty)$ and $a \in V_{1/p}$. Then, for some T^* there exists a unique solution of (3) in $0 \leq t \leq T^*$ such that*

$$v \in C([0, T^*]; V_{1/p}) \cap C^1((0, T^*]; X) \cap C((0, T^*]; D(A)),$$

with $t^{1/2-1/(2p)}\|v(t)\|_{V_{1/2+1/(2p)}} \to 0$ as $t \to 0$. Furthermore,

$$T^* \geq (C\|a\|_{V_{1/p+\epsilon}})^{-1/\epsilon}, \quad \forall \epsilon \in (0, 1 - 1/p].$$

Remark 1. We may characterize $V_{1/p}$ as

$$V_{1/p} = \{v \in H^{2/p,p}_{\text{per}}(\Omega)^2 \,|\, \text{div}_H \bar{v} = 0,\ v|_{\Gamma_b} = 0\}.$$

If in particular $p = 2$, then this essentially agrees with the space for initial values utilized in [1, 2].

To obtain global-in-time well-posedness, we require an a priori bound for the solution of (1) in the H^2-norm. For the proof we basically follow the idea of [1], but the argument becomes more involved because of the Dirichlet boundary condition on Γ_b.

Theorem 4. *Let $v \in C^1([0, T]; L^2(\Omega)^2) \cap C([0, T]; H^2(\Omega)^2)$ be a solution of (1) with $a \in H^2(\Omega)^2$. Then there exists a continuous function $B = B(T, \|a\|_{H^2(\Omega)})$ such that*

$$\max_{0 \leq t \leq T} \|v(t)\|_{H^2(\Omega)} \leq B(T, \|a\|_{H^2(\Omega)}).$$

Combining Theorems 2 and 3, we are able to conclude our main theorem.

Theorem 5. *Let $p \in (1, \infty)$ and $a \in V_{1/p}$. Then, there exists a unique solution of (1) such that*

$$v \in C([0, \infty); V_{1/p}) \cap C^1((0, \infty); X) \cap C((0, \infty); D(A)).$$

Moreover we have exponential decay as $t \to \infty$: $\|v(t)\|_{D(A)} \leq Ce^{-ct}$.

For the detailed proofs of the theorems above, we refer to our papers [3, 4]. Finally, let us mention our ongoing work where we study the endpoint case corresponding to $p = \infty$. The difficulty in this case is that hydrostatic Helmholtz projector P is no longer bounded in L^∞-type spaces. However, we expect that the analytic-semigroup approach may also be applicable to this case. In particular we believe that a unique strong solution may be constructed for $a \in C(G; L^p(-h, 0))$. We would like to report more details of this result elsewhere.

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Clouds and the Climate System

Boualem Khouider

Convective clouds in the tropics are organized on multiple spatial and temporal scales, ranging from the convective cell of 1-10 km and a few hours to cloud clusters and super-clusters associated with mesoscale to planetary scale wave-like disturbances with life times of a few days to 1-2 months. Global climate models (GCM) used for climate and weather predictions solve the equations of fluid motion on grids ranging from 20 to 200 km on which the multiscale processes associated with tropical clouds and the corresponding interactions across scales are not resolved but represented by sub-grid models known as parametrization. This is a very challenging mathematical modeling problem and current GCMs poorly represent the rainfall patterns and climate variability associated with organized tropical convection. In this talk, I give an overview of this difficult problem and discuss the ideas behind the stochastic multicloud model which aims at designing a parameterization based on the theory of particle interacting systems to represent individual clouds of various types and their complex interactions with each other and with the climate system. Moreover, I show a few results of its successful implementation in a state of the art climate model to demonstrate the importance of using such a mathematical framework to represent clouds in climate and weather prediction models.

Analysis of a three-time-scale asymptotic problem in atmospheric fluid dynamics

Rupert Klein
(joint work with Didier Bresch, Martin Papke, Dennis Jentsch)

The analogue of the incompressible flow equations for atmospheric motions are fluid flow models of the "soundproof" type, usually called "anelastic" [1, 2] or pseudo-incompressible [3]. In the simplest setting of flow in a non-rotating system with realistic background stratification of the entropy (potential temperature), these flow models support advection and internal gravity waves as the main flow modes while sound waves are suppressed owing to a velocity divergence constraint. The full compressible flow equations do support sound propagation, of course, and the challenge in systematically deriving any of the soundproof models from the full compressible flow equations consists of a model reduction from three to the remaining two active modes.

In addition, for stratifications of potential temperature found in the troposphere (lowest $\sim$ 10 km of the atmosphere), sound, internal waves, and advection all feature asymptotically separated characteristic time scales, [4]. After a suitable nondimensionalization, sound waves have characteristic time $t_s = O(\varepsilon)$, where ε is the Mach number, the internal wave time scale is intermediate with $t_i = O(\varepsilon^\nu)$ with $0 < \nu < 1$, and the advective time scale is slowest with $t_a = O(1)$. Therefore, in removing the very fast sound modes by asymptotic arguments while leaving internal waves and advection intact one ends up with an equation system that still features the fast internal wave and slow advection time scales $t_i = O(\varepsilon^\nu)$ and $t_a = O(1)$, and can therefore not be understood as "the $\varepsilon = 0$ limit equations". As a consequence, a systematic justification of these soundproof limit models in such a realistic scenario cannot proceed along established lines by proving convergence of solutions of the full equations to those of some ε-free limit equation system in a suitable function space.

This presentation suggested the alternative route of showing that the full compressible and soundproof flow models have the same asymptotic behavior *over the slow advective time scale.* To this end, arguments from [4] were recalled that provide a formal asymptotic reasoning and reveal that the internal waves of the full compressible and soundproof models stay asymptotically close in spacial structure and phase over advective time scales if $0 \leq \nu < 2/3$.

In more recent work, presented last, we succeeded to demonstrate that the fast, non-constant coefficient linear subsystem of the full compressible equations that describes the propagation of sound and internal waves renders solutions controlled in arbitrary H^s Sobolev norms provided the initial conditions are suitably prepared. The key trick in overcoming the difficulty associated with non-constant coefficients relies on proving "near orthogonality" of eigenmodes of the fast linear system for large differences of the mode numbers.

It was shown furthermore, that acoustic and internal wave modes cannot resonate with each other for sufficiently small ε, and that the fast linear subsystem of the compressible equations feature a discrete spectrum. With these preliminaries, a rigorous proof of the validity of the inviscid pseudo-incompressible equations as a model for sound-free compressible motions in the atmosphere seems in reach.

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On Ocean Climate Models: Numerics, Physics and A Speculation

Peter Korn

The problem of modelling the circulation of the global ocean with its mathematical, physical and computational aspects is studied. After a brief review of the evolution of the field of ocean modelling, where the close connection to the evolution of high-performance computing is emphasized, we highlight the importance of efficient algorithms for the discretization of the partial differential equations of ocean dynamics. The quest for efficient discretizations on modern computing architectures has created a paradigm shift from *"structured"* grids with rectangular longitude-latitude cells towards *"unstructured grids"*. The reason is that structured grids create on the sphere an inhomogeneous tessellation which impedes the computational performance, while a homogenous tessellation of the sphere necessarily implies an unstructured grid, composed for example of triangular or hexagonal/pentagonal grid cells.

The dynamic equations of the ocean are the so-called *"primitive equations"* that govern the motion of an incompressible fluid under the hydrostatic and Boussinesq approximation on the sphere and with a free surface. The discretization of these equations on unstructured grids poses new challenges to numerical geophysical fluid dynamics and has motivated new *structure-preserving* discretization approaches. A particular discretization on a triangular grid with a staggered distribution of variables was described [1]. The staggering (Arakawa C-type, also known as Marker-and-Cell approach) requires reconstructions to calculate quantities such as temperature fluxes at locations where there originally are not defined. These reconstructions affect crucially the properties of the discrete model. The key result of the presented discretization approach was that a *discrete Hilbert space structure*, provided by the class of *admissible reconstructions*, introduced in [1], allows to preserve important conservation laws [4], and enables us to control an inevitable computational mode while it leaves the favourable dispersion properties of the C-grid nearly intact [3]. Our discretization us implemented in the ocean general circulation model ICON-O and formulates the discrete ocean equations in weak form and relies on *admissible reconstructions* to create a discrete Hilbert

space structure.
The oceanic PDEs are supplemented by subgrid scale closures that take the specific physical conditions of the ocean into account. We consider isoneutral diffusion and the eddy parametrization of Gent-McWilliams (see e.g. [5]). Isoneutral diffusion is motivated by the observational evidence that mixing in the ocean takes place within isoneutral surfaces and not across them. While this is satisfied by isopycnal model by construction it has to be parametrized by ocean models using a z-coordinate as vertical axis. The eddy parametrization by Gent-McWilliams intends to capture the fact that mesoscale eddies advect tracers along isoneutral surfaces and reduce potential energy by flattening the isoneutral surfaces. The physically consistent discretization of these operators poses subtle numerical problems. We presented a discretization of isoneutral diffusion and eddy parametrization based on a variational formalism that has evolved naturally from the discretization methodology for the ocean primitive equations described above and that agains benefits from the discrete Hilbert space structure [2].
Results from numerical experiments were shown that support our numerical analysis and that in addition demonstrate the computational performance of the model that makes is a suitable instrument to address ocean modelling problems at high-resolutions. We conclude with a speculative outlook that within the next decade a computational barrier will be reached that -within the existing hardware technology- will not allow to further increase the spatial resolution of ocean models and at the same time run them for the long integration times that are demanded by the world ocean circulation. Possible remedies of the computational barrier include time-parallel algorithms or may create an increasing trend towards stochastic modelling approaches.
Open problems in this field are the lack of understanding of the impact of different discrete Hilbert space scalar products on the model dynamics, the extension of structure-preserving discretizations to higher-order accuracy and stochastic enabled structure-preserving numerical schemes.

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Local existence and blowup results for the Prandtl equations

Igor Kukavica
(joint work with Vlad Vicol, Fei Wang)

In 1980, van Dommelen and Shen provided a numerical simulation that predicted generation of a singularity in the Prandtl boundary layer equations from a smooth initial datum, for a nonzero Euler background flow. We provide a proof of the blowup by showing that a quantity related to the boundary layer thickness becomes infinite in a finite time. We will also briefly survey available local and global existence results and connections with the vanishing viscosity limit. The blowup result is joint with Vlad Vicol and Fei Wang.

Global well-posedness of the anisotropic primitive equations

Jinkai Li
(joint work with Chongsheng Cao, Edriss S. Titi)

The primitive equations are derived from the Navier-Stokes equations by applying the Boussinesq and hydrostatic approximations. Due to the strong turbulence in the horizontal direction of the oceanic and atmospheric dynamics, the horizontal viscosity, which is understood as the eddy viscosity, is assumed to be positive. Noticing that the primitive equations with full dissipation exist a unique global strong solution, while the inviscid primitive equations may develop finite time singularities, it is natural to investigate the global well-posedness or finite time blow-up of the primitive equations with partial viscosity or partial diffusivity. In this talk we show that the primitive equations with only horizontal viscosity have a unique global strong solution, as long as one still has either horizontal or vertical diffusivity. These are joint works with Chongsheng Cao and Edriss S. Titi [1, 2, 3, 4, 5].

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On Prandtl expansion for the Navier-Stokes flows in the half plane

Yasunori Maekawa

In this talk we give a rigorous justification of the classical Prandtl boundary layer expansion around the monotone and concave boundary layer shear flows in a Gevrey class. As a key step, we establish the estimate for the linearized system which is considered to be optimal in view of the well-known Tollmien-Schlichting instability.

Optimal mixing by incompressible flows

Anna L. Mazzucato
(joint work with Giovanni Alberti, Gianluca Crippa)

We consider a passive scalar ρ advected by an incompressible flow, that is, a solution of the linear transport equation:

$$\partial_t \rho + u \cdot \nabla \rho = 0, \tag{1}$$

where u is a given, divergence-free vector field in $\mathbb{R}^d$ or $\mathbb{T}^d$. We study *Lagrangian solutions* of the initial-value problem for (1), that is, solutions that can be obtained by transporting the initial condition ρ_0 with the flow $X(x,t)$ of u, i.e.,

$$\rho(x,t) = \rho_0(X^{-1}(x,t)),$$

when the flow exists at least a.e..

Properties of the flow are reflected in how well it mixes the scalar, as measured by the decay in time of a lengthscale, the *mixing length*. In two space dimensions, a functional mixing length can be defined in terms of the negative homogeneous Sobolev norm $\|\rho(\cdot,t)\|_{\dot{H}^{-1}}$. A related geometric mixing length can be introduced in terms of rearrangements of sets. We discuss examples [1, 2] of velocity fields u with Sobolev regularity $W^{1,p}$, $1 \leq p \leq \infty$ that achieve the theoretical rate of decay of the mixing scale, established in [3, 5], for a specific datum ρ_0. An independent construction was given in [6].

Our examples are geometric in flavor and rely on the construction of divergence-free vector fields that realize the evolution of sets preserving the area of their connected components, plus appropriate scaling arguments. These examples in particular show that regular Lagrangian flows with velocity field in $W^{1,p}$ with arbitrary index $1 \leq p < \infty$ can compress a segment to a point or expand a point to a segment in finite time. The associated ODEs are not uniquely solvable pointwise and, as a matter of fact, the flow can be discontinuous in Sobolev spaces (for a related example see [4]).

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Analysis of a feedback-control data assimilation algorithm

Cecilia F. Mondaini
(joint work with Ciprian Foias, Edriss S. Titi)

The general idea of data assimilation is to obtain a good approximation of the state of a certain physical system by combining observational data with dynamical principles inherent to the underlying mathematical model. It is widely used in many fields of geosciences, mainly for oceanic and atmospheric forecasting.

The main difference among various data assimilation algorithms lies on the type of method used for combining the observations with the theoretical model. One of these methods consists in adding a term to the theoretical model which gradually relaxes the solution towards the observations. This type of approach is called *nudging* or *newtonian relaxation*, and has been considered by many researchers in the past few decades ([7, 8, 9, 10]). In [1], the authors consider a similar nudging approach, but in a much more general context, which is applicable to a large class of dissipative PDEs and observational measurements. Their motivation comes from ideas in control theory, and the term which is added to the original model is called the *feedback-control term*.

As a paradigm, the authors in [1] consider a forecast model given by the 2D incompressible Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0. \tag{1}$$

where $\mathbf{u} = (u_1, u_2)$ and p are the unknowns, and represent the velocity vector field and the pressure, respectively; while $\mathbf{f}$ and ν are given, and represent the mass density of volume forces applied to the fluid and the kinematic viscosity parameter, respectively. Thus, $\mathbf{u}$ is called the reference solution, and its exact value is unknown.

Assuming continuous in time and error-free measurements, the algorithm consists in finding a solution $\mathbf{v}$ on $[t_0, \infty)$ of the following problem

$$\frac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f} - \beta(I_h(\mathbf{v}) - I_h(\mathbf{u})), \tag{2}$$

$$\mathbf{v}(t_0) = \mathbf{v}_0, \tag{3}$$

where β is the relaxation (nudging) parameter; h is the spatial resolution of the observations; I_h is a spatial interpolation operator; and $\mathbf{v}_0$ is an arbitrary initial condition. The purpose of the second term on the right-hand side of (2), the feedback-control term, is to force the coarse spatial scales of the approximating solution **v** toward those of the reference solution **u** corresponding to the measurements, while the viscosity term stabilizes the fine spatial scales and any spill-over on these scales caused by the feedback term. Indeed, the authors in [1] show that, within this idealized scenario of continuous in time and error-free measurements and under suitable conditions on β and h, **v** converges exponentially to **u** as time goes to infinity, in an appropriate norm.

Aiming at adapting the algorithm introduced in [1] to a more realistic situation, in the work [3] with C. Foias and E. Titi, we constructed a new data assimilation algorithm employing measurements collected discretely in time, which may be contaminated by systematic errors. We prove that, under suitable conditions on the spatial resolution, the time step between successive measurements and the relaxation parameter, the approximating solution converges exponentially to the reference solution up to a term which depends on the size of the errors. Also, we studied the stationary statistical behavior of our algorithm, obtaining results that yield information on averages of physical quantities of the unknown reference solution by using computable averages of the same quantities associated to the approximating solution. Such statistical approach has a great practical interest, since most applications of data assimilation are for turbulent flows, where one is usually interested in averages of the associated physical quantities due to their more regular behavior in comparison to instantaneous values.

In the work [4], we analyze the data assimilation algorithm introduced in [1] from a numerical analysis viewpoint. More specifically, we obtain an analytical estimate of the error committed when numerically solving the approximate model given in this data assimilation algorithm by using a post-processing technique for the spectral Galerkin method, inspired by the theory of approximate inertial manifolds ([2]). Most importantly, our results show that, under suitable assumptions on the relaxation parameter and the spatial resolution of the observations, the error estimate in this case is uniform in time, as opposed to previous results obtained when applying the same post-processing technique to, e.g., the 2D Navier-Stokes equations directly, where the error estimate grows exponentially in time ([5, 6]). This important difference is justified due to the presence of the feedback control term, that stabilizes the large scales of the approximate solution. Although we considered the 2D Navier-Stokes equations as a paradigm, our results apply equally to other dissipative evolution equations.

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A drop of water

Piotr B. Mucha
(joint work with Raphael Danchin)

The talk presented a solution to an open problem stated by P.L. Lions in his monograph concerning inhomogenous Navier-Stokes system in 90'. The subject is the following system

$$\begin{array}{lll} \rho_t + v \cdot \nabla v = 0 & \text{in} & \Omega \times (0,T), \\ \rho v_t + \rho v \cdot \nabla v - \Delta v + \nabla p = 0 & \text{in} & \Omega \times (0,T), \\ \operatorname{div} v = 0 & \text{in} & \Omega \times (0,T) \end{array} \tag{1}$$

with initial data for v, ρ – the velocity and density of the fluid.

The presented result proved that solutions initiated by initial density being a characteristic function of a set, with H^1 initial velocity are uniquely determined. We use a method of the shift of integrability, which allows to control the change of coordinates into the Lagrangian setting. The talk based on results joint with Raphael Danchin (Paris).

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Stability analysis for Compressible Navier-Stokes equations: theory and numerics

Antonin Novotny

(joint work with E. Feireisl, T. Gallouet, R. Herbin, R. Hošek, B.J. Jin, D. Maltese, Y. Sun)

In the first part of the talk, we introduce the notion of relative energy functional which measures a "distance" of a weak solution to the compressible Navier-Stokes equations to any other sufficiently smooth state of the fluid. We reformulate the thermodynamic stability conditions for the compressible fluids in terms of the relative energy inequality which describes the evolution of the relative energy functional. This process relies very much on the structure of the equations. The relative energy functional encodes most of the stability properties of the weak solutions to the compressible Navier-Stokes equations.

In the second part of the talk, we adapt the above procedure to some FV/FE and FV/FD numerical schemes calculating compressible flows in order to investigate: 1) Uniform error estimates of the discrete numerical solution with respect to a strong solution. 2) The uniform stability of these numerical schemes in the low Mach number regime.

The first part of the talk is based on several joint papers with E. Feireisl with contribution of B.J. Jin and Y. Sun. The second part of the talk is based on several joint works with T. Gallouet, R. Herbin, D. Maltese and further contribution of E. Feireisl and R. Hošek.

On the quasi-geostrophic equations on compact surfaces

Jan Prüss

We present a new approach to the quasi-geostrophic equations via the theory of quasilinear parabolic evolution equations. We can offer a complete picture of the dynamics in the so-called subcritical case, including the critical spaces for this problem, with very simple proofs. Our approach is based on the theory of semilinear parabolic evolution equations.

On two phase problem for the Navier-Stokes equations in the whole space

Yoshihiro Shibata

Let Ω_+ be a bounded domain in $\mathbb{R}^N$ and Γ its boundary that is a smooth compact hypersurface. Let $\Omega_- = \mathbb{R}^N \setminus \overline{\Omega_+}$ and two different incompressible viscous fluids occupy $\Omega_\pm$, respectively. Let $\Omega_{\pm t}$ and Γ_t be the time evolution of $\Omega_\pm$ and Γ for $t > 0$. Let $\mathbf{n}_t$ be the unit outer normal to Γ_t. Problem is to find domains

$\Omega_{\pm t}$, velocities $\mathbf{v}_\pm = (v_{1\pm}, \dots, v_{\pm N})$ and pressures $\mathfrak{p}_\pm$ satisfying the Navier-Stokes equations:

$$(1)\quad \begin{cases} \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{v}) - \mathfrak{p}\mathbf{I}) = 0,\ \mathrm{div}\,\mathbf{v} = 0 & \text{in } \bigcup_{0<t<T} \Omega_t \times \{t\}, \\ [[\mu\mathbf{D}(\mathbf{v}) - \mathfrak{p}\mathbf{I}]]\mathbf{n}_t = \sigma H(\Gamma_t)\mathbf{n}_t,\ [[\mathbf{v}]] = 0,\ V_{\Gamma_t} = \mathbf{v}\cdot\mathbf{n}_t & \text{on } \bigcup_{0<t<T} \Gamma_t \times \{t\}, \\ \mathbf{v}|_{t=0} = \mathbf{v}_0,\ \Omega_{\pm t}|_{t=0} = \Omega_\pm. \end{cases}$$

Here, $\Omega_t = \Omega_{+t} \cup \Omega_{-t}$, $h = h_\pm$ for $x \in \Omega_{t\pm}$, $\mu = \mu_\pm$ for $x \in \Omega_{\pm t}$, $\mu_\pm$ being positive constants representing viscosity coefficients, σ positive constant (coefficient of surface tension), $H(\Gamma_t)$ the doubled mean curvature of Γ_t, V_{Γ_t} the evolution speed of Γ_t in the $\mathbf{n}_t$ direction, $\mathbf{I}$ the $N \times N$ identity matrix, $\mathbf{D}(\mathbf{v}) = \nabla\mathbf{v} + {}^\top\nabla\mathbf{v}$ the doubled deformation tensor whose (i, j) component is $\partial_i v_j + \partial_j v_i$, and $[[f]](x_0) = \lim_{\substack{x\to x_0\\ x\in\Omega_+}} f(x) - \lim_{\substack{x\to x_0\\ x\in\Omega_-}} f(x)$, which is the jump quantity of f at $x_0 \in \Gamma_t$.

This problem has been studied by the following authors:

- V. Denisova, V. A. Solonnikov: in the L_2 frame work and the Hölder space framework.
  (Port.Math.,**71**(1)(2014), 1-24; J. Math. Sci. (N.Y.)**185**(5)(2012), 668–686, etc.)
- J. Pruess, G. Simonett, et al: L_p maximal regularity and Local well-posedness. L_p-maximal regularity + Spectral analysis for the Laplace-Bertrami operator, and Global well-posedness in the container. (Interface and Free boundaries, **12**(2010), 311–345; Progress in NDE and their Appl, **80** (2011),507–540; Birkhäuser Monographs in Math. 2016, ISBN:978-3-319-27698-4, etc.)

But, global well-posedness in unbounded domains has not yet been treated. In this abstract, the two global well-posedness results are announced in the case that $\sigma = 0$ and $\sigma > 0$.

First of all, we mention that **Maximal L_p-L_q regularity** for the two phase problem for the Stokes equations does hold in a uniformly C^2 ($\sigma = 0$ case) or C^3 ($\sigma > 0$ case) domain under the assumption that weak Neumann problem is uniquely solvable (cf. Pruess and Simonnet, Monographs mentioned above, and also Shibata and Shimizu, JDE **251** (2) (2011), 373–419, Maryani and Saito, Diff. Int. Eqns. **30**(1-2) (2017), 1–52, etc).

Thus, **Local well-posedness holds**. Here, it is important that p and q can be chosen differently to prove Global well-posedness for free boundary problem in unbounded domain. In fact, in the unbounded domain case, we can get only polynomially decay for suitable L_q space norm of solutions, so that we have to choose p rather large to guarantee the L_p summability in time, which will be seen below.

Global wellposedness, $\sigma > 0$ case

Let $B_R = \{x \in \mathbb{R}^N \mid |x| < R\}$ and $S_R = \{x \in \mathbb{R}^N \mid |x| = r\}$. We assume that
Assumption 1 $|\Omega_+| = |B_R| = R^N \omega_N/n$, where $|\cdot|$ denotes the Lebesgue measure and ω_N is the area of the unit sphere.
Assumption 2 $\displaystyle\int_\Omega x\,dx = 0$.
Assumption 3 $\Gamma = \{x = (R + \rho_0(R\omega))\omega \mid \omega \in S_1\}$ with given small function ρ_0 defined on S_R.

Let

$$\Gamma_t = \{x = (R + \rho(R\omega, t))\omega + \xi(t) \mid \omega \in S_1\}$$

where ρ is a unknown function and $\xi(t)$ is the barycenter point of the domain Ω_t defined by

$$\xi(t) = \frac{1}{|\Omega_+|}\int_{\Omega_{+t}} x\,dx.$$

Assume that $\Omega_+ \subset B_R$ with a large constant $R > 0$. Let $L \geq 3R$. Given $\rho \in W_q^{3-1/q}(S_R)$, let $H(\xi, t) \in H_q^3(\dot{B}_L)$ be a function such that $H|_{S_R} = R^{-1}\rho$, $\|H\|_{H_q^2(\dot{B}_L)} \leq C\|\rho\|_{W_q^{2-1/q}(S_R)}$, and $\|H\|_{H_q^3(\dot{B}_L)} \leq C\|\rho\|_{W_q^{3-1/q}(S_R)}$, where $\dot{B}_L = B_L \setminus S_R$.

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi(x) = 1$ for $|x| \leq L-2$ and $\varphi(x) = 0$ for $|x| \geq L-1$. We use the **Hanzawa transform** defined by

$$x = e_h(y, t) = y + \varphi(y)H(y, t)y + \xi(t) \quad \text{for } y \in B_R.$$

Let

$$\begin{aligned} \mathbf{u}(y, t) &= \mathbf{v} \circ e_h, \quad \mathfrak{q}(y, t) = \mathfrak{p} \circ e_h - \frac{(N-1)\sigma}{R}, \\ \Omega_t &= \{x = y + \varphi(y)H(y, t)y + \xi(t) \mid y \in B_R\}, \\ \Gamma_t &= \{x = (R + \rho(R\omega, t))\omega \mid \omega \in S_1\}. \end{aligned}$$

And then, problem (1) is transformed to

$$(2) \qquad \begin{cases} \partial_t \mathbf{u} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q}\mathbf{I}) = F(\mathbf{u}, H) & \text{in } \Omega \times (0, T), \\ \mathrm{div}\,\mathbf{u} = F_d(\mathbf{u}, H) = \mathrm{div}\,\mathbf{F}_d(\mathbf{u}, H) & \text{in } \Omega \times (0, T), \\ [[\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q}]]\omega - \sigma(\mathcal{B}_R\rho)\mathbf{n} = G(\mathbf{u}, \rho) & \text{in } S_R \times (0, T), \\ [[\mathbf{u}]] = 0 & \text{in } S_R \times (0, T), \\ \partial_t\rho - \mathbf{n} \cdot P\mathbf{u} = D(\mathbf{u}, \rho) & \text{on } S_R \times (0, T), \\ (\mathbf{u}, \rho)|_{t=0} = (\mathbf{u}_0, \rho_0) & \text{on } \Omega \times S_R. \end{cases}$$

Here, $\Omega = B_R \cup B^R$ with $B^R = \{x \in \mathbb{R}^N \mid |x| > R\}$, $\mathbb{R}^N = \Omega \cup S_R$,

$$\mathcal{B}_R\rho = R^{-2}(\Delta_{S_1} + N - 1)\rho, \quad P\mathbf{u} = \mathbf{u} - \frac{1}{|B_R|}\int_{B_R} \mathbf{u}(y)\,dy.$$

Δ_{S_1} is the Laplace-Beltrami operator on S_1, and $F(\mathbf{u}, H)$, $F_d(\mathbf{u}, H)$, $\mathbf{F_d}(\mathbf{u}, H)$, $D(\mathbf{u}, \rho)$ are nonlinear functions. Then, we have the following theorem that is our global well-posedness theorem in the case that $\sigma > 0$.

Theorem 1. Let $N \geq 3$. Let q_1 and q_2 be exponents such that $N < q_2 < \infty$ and $1/q_1 = 1/q_2 + 1/N$. Let b be a number such that $N/q_1 > b \geq N/(2q_2)$. Then, there exists an $\epsilon > 0$ such that if initial data $\mathbf{u}_0 \in B^{2(1-1/p)}_{q_2,p} \cap B^{2(1-1/p)}_{q_1/2,p} = D_{p,q_1,q_2}$ and $\rho_0 \in B^{3-1/p-1/q_2}_{q_2,p}(S_R)$ satisfy the smallness condition:

$$\|\mathbf{u}_0\|_{D_{p,q_1,q_2}} + \|\rho_0\|_{B^{3-1/p-1/q_2}_{q_2,p}(S_R)} \leq \epsilon$$

and the compatibility condition:

$$\operatorname{div} \mathbf{u}_0 = 0 \text{ in } \Omega, \quad [[\mu\mathbf{D}(\mathbf{u}_0)]]\omega - < [[\mu\mathbf{D}(\mathbf{u}_0)]]\omega, \omega >= 0 \text{ on } S_R,$$

then problem (2) admits unique solutions $\mathbf{u}$ and ρ with

$$\begin{aligned}
&\mathbf{u} \in L_p((0,\infty), H^2_{q_2}(\Omega) \cap H^2_{q_1/2}(\Omega)) H^1_p((0,\infty), L_{q_2}(\Omega) \cap L_{q_1/2}(\Omega)),\\
&\rho \in L_p((0,\infty), W^{3-1/p-1/q_2}_{q_2}(S_R)) \cap H^1_p((0,\infty), W^{2-1/p-1/q_2}_{q_2}(S_R))
\end{aligned}$$

possessing the estimate: $E(\mathbf{u},\rho)(0,\infty) \leq C\epsilon$. Here,

$$\begin{aligned}
&E(\mathbf{u},\rho)(0,T)\\
=&\| < t >^b (\mathbf{u}, H)\|_{L_\infty((0,T),H^1_\infty(\Omega)\times H^2_\infty(\dot{B}_L))} + \| < t >^{b-\frac{N}{2q_1}} \mathbf{u}\|_{L_p((0,T),H^1_{q_1}(\Omega))}\\
&+ \| < t >^{\frac{N}{2q_1}} \mathbf{u}\|_{L_\infty((0,T),L_{q_1}(\Omega))} + \| < t >^{b-\frac{N}{2q_2}} \partial_t(\mathbf{u}, H)\|_{L_p((0,T),L_{q_2}(\Omega)\times H^2_{q_2}(\dot{B}_L))}\\
&+ \| < t >^{b-\frac{N}{2q_2}} (\mathbf{u}, H)\|_{L_p((0,T),H^2_{q_2}(\Omega)\times H^3_{q_2}(\dot{B}_L))}.
\end{aligned}$$

Here, $< t >= (1+t^2)^{1/2}$.

Global wellposedness, $\sigma = 0$ case

In this case, we can not use the Hanzawa transform, because of the lack of regularity for the height function ρ. Thus, we use the partial Lagrange transform. Let $\varphi \in C^\infty_0(\mathbb{R}^N)$ such that $\varphi(x) = 1$ for $|x| \leq L-2$ and $\varphi(x) = 0$ for $|x| \geq L-1$ for $L \geq 3R$. Let $\mathbf{u}(\xi,s) = \mathbf{u}_\pm(\xi,s)$ for $\xi \in \Omega_\pm$ be the lagrange velocity fields, and the partial Lagrange transform is defined by

$$x = \xi + \varphi(\xi)\int_0^t \mathbf{u}(\xi,s)\,ds = X_{\mathbf{u}}(\xi,t) \quad \text{for } \xi \in \Omega_\pm.$$

There exists a small constant $\sigma > 0$ such that if

$$\int_0^T \|\nabla(\varphi(\cdot)\mathbf{u}(\cdot,s))\|_{L_\infty(\Omega)}\,ds \leq \sigma$$

then, the partial Lagrange transform is a diffeomorphism from $\Omega = \Omega_+ \cup \Omega_+ = \mathbb{R}^N \setminus \Gamma$ onto $\Omega_t = \{x = X_{\mathbf{u}}(\xi,t) \mid \xi \in \Omega\}$.

By the partial Lagrange transform, problem (1) is transformed to

$$
(3) \qquad \begin{cases} \partial_t \mathbf{u} - \mathrm{Div}\,(\mu \mathbf{D}(\mathbf{u}) - \mathfrak{q}\mathbf{I}) = F(\mathbf{u}), & \text{in } \Omega \times (0,T), \\ \mathrm{div}\,\mathbf{v} = f(\mathbf{u}) = \mathrm{div}\,\mathbf{f}(\mathbf{u}) & \text{in } \Omega \times (0,T), \\ [[\mu \mathbf{D}(\mathbf{u}) - \mathfrak{q}\mathbf{I}]]\mathbf{n} = \mathbf{g}(\mathbf{u}) & \text{on } \Gamma \times (0,T), \\ [[\mathbf{u}]] = 0 & \text{on } \Gamma \times (0,T), \\ \mathbf{u}|_{t=0} = \mathbf{v}_0, \quad \Omega_t|_{t=0} = \Omega, \end{cases}
$$

with suitable nonlinear functions $F(\mathbf{u})$, $f(\mathbf{u})$, $\mathbf{f}(\mathbf{u})$ and $\mathbf{g}(\mathbf{u})$. Then, we have the following theorem.

Theorem 2. Let $N \geq 3$ and let q_1 and q_2 be exponents such that $N < q_2 < \infty$ and $1/q_1 = 1/q_2 + 1/N$ and $q_1 > 2$. Let b, p and $p' = p/(p-1)$ be numbers satisfying the conditions:

$$
(4) \qquad \begin{aligned} &\frac{N}{q_1} > b > \frac{1}{p'}, \ \Big(\frac{N}{q_1} - b\Big)p > 1, \ \Big(b - \frac{N}{2q_2}\Big)p > 1, \ b > \frac{N}{2q_1}, \\ &\Big(\frac{N}{2q_2} + \frac{1}{2}\Big)p' < 1, \ bp' > 1, \ \Big(b - \frac{N}{2q_2}\Big)p' > 1, \ \frac{N}{q_2} + \frac{2}{p} < 1. \end{aligned}
$$

Then, there exists an $\epsilon > 0$ such that if initial data $\mathbf{v}_0$ satisfies the compatibility condition and the smallness condition: $\|\mathbf{v}_0\|_{B^{2(1-1/p)}_{q_2,p}(\Omega)} + \|\mathbf{v}_0\|_{B^{2(1-1/p)}_{q_1/2,p}(\Omega)} \leq \epsilon$, then problem (3) admits a unique solution $\mathbf{u} \in L_p((0,\infty), H^2_{q_2}(\Omega)^N) \cap H^1_p((0,\infty), L_{q_2}(\Omega)^N)$, possessing the estimate: $[\mathbf{u}]_\infty < C\epsilon$ with some constant $C > 0$ independent of ϵ. Here

$$
\begin{aligned} [\mathbf{u}]_T = \Big\{ &\int_0^T ((1+t)^b \|\mathbf{u}(\cdot,s)\|_{H^1_\infty(\Omega)})^p \, ds \\ &+ \int_0^T ((1+s)^{(b-\frac{N}{2q_1})} \|\mathbf{u}(\cdot,s)\|_{H^1_{q_1}(\Omega)})^p \, ds + (\sup_{0<s<T} (1+s)^{\frac{N}{2q_1}} \|\mathbf{u}(\cdot,s)\|_{L_{q_1}(\Omega)})^p \\ &+ \int_0^T ((1+s)^{(b-\frac{N}{2q_2})} (\|\mathbf{u}(\cdot,s)\|_{H^2_{q_2}(\Omega)} + \|\partial_t \mathbf{u}(\cdot,s)\|_{L_{q_2}(\Omega)}))^p \, ds \Big\}^{1/p}. \end{aligned}
$$

Precipitating Quasi-Geostrophic Equations and Potential Vorticity Inversion with Phase Changes

Leslie M. Smith

(joint work with Sam Stechmann)

A precipitating version of the quasi-geostrophic (QG) equations is derived systematically, starting from an idealized cloud-resolving model. The presence of phase changes of water from vapor to liquid and vice versa leads to important differences from the dry QG case. The precipitating QG (PQG) equations have two variables to describe the full system: a potential vorticity (PV) variable and a variable M including moisture effects. PV-and-M inversion allows the determination of all other variables, and it involves an elliptic partial differential equation (PDE) that is nonlinear due to phase changes between saturated and unsaturated regions. The phase interface location is unknown a priori from PV and M, and it is discovered as part of the inversion process. An energy conservation principle suggests that the model has a firm physical and mathematical foundation.

Our starting point is a model for precipitating dynamics with a Boussinesq dynamical core, linearized thermodynamics, and asymptotically fast, warm-rain cloud microphysics. The model was designed and analyzed by [1], and given by

$$\frac{D\mathbf{u}}{Dt} + f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla\left(\frac{p}{\rho_o}\right) + \hat{\mathbf{z}}\, g\left(\frac{\theta}{\theta_o} + R_{vd}q_v - q_r\right), \quad \nabla\cdot\mathbf{u} = 0 \tag{1a}$$

$$\frac{D\theta_e}{Dt} + w\frac{d\tilde{\theta}_e}{dz} = 0, \quad \frac{Dq_t}{Dt} + w\frac{d\tilde{q}_t}{dz} - V_T\frac{\partial q_r}{\partial z} = 0. \tag{1b}$$

The dynamical variables are the velocity $\mathbf{u} = (u, v, w)$, pressure p, equivalent potential temperature anomaly θ_e, and anomalous mixing ratio of total water q_t, all of which are functions of $\mathbf{x} = (x, y, z)$ and time t. The buoyancy $b = g\,(\theta/\theta_o + R_{vd}q_v - q_r)$ may be expressed as a function of θ_e, q_t and z as described below; V_T is the fall speed of rain, taken to be a constant value here for simplicity. Other notation and parameters are standard: $D/Dt = \partial/\partial t + \mathbf{u}\cdot\nabla$; f is the Coriolis parameter; background potential temperature $\theta_o \approx 300$ K; gravitational acceleration $g \approx 9.8$ m s^{-2}; $R_{vd} = (R_v/R_d) - 1 \approx 0.61$, where R_v, R_d are the gas constants for water vapor and dry air, respectively. All thermodynamic variables have been decomposed into background functions of altitude and anomalies. For example, the total equivalent potential temperature is $\theta_e^{tot}(\mathbf{x}, t) = \tilde{\theta}_e(z) + \theta_e(\mathbf{x}, t)$, where $\tilde{\theta}_e(z)$ is the background state. Though we will use constant $d\tilde{q}_t/dz$ and $d\tilde{\theta}_e/dz$ herein, extension to non-constant slopes is straightforward.

The total water $q_t^{tot} = q_v^{tot} + q_r^{tot}$ is the sum of contributions from water vapor q_v^{tot} and rain water q_r^{tot}, which can be recovered using

$$q_v^{tot} = \min(q_t^{tot}, q_{vs}^{tot}), \quad q_r^{tot} = \max(0, q_t^{tot} - q_{vs}^{tot}), \tag{2}$$

where $q_{vs}^{tot}(z)$ is the prescribed saturation mixing ratio. The buoyancy b has a functional form that changes depending on whether the phase is unsaturated or saturated:

$$b = b_u H_u + b_s H_s, \tag{3}$$

where H_u and H_s are Heaviside functions that indicate the unsaturated and saturated phases, respectively, and are therefore functions of q_t and $q_{vs}(z)$: $H_u = 1$ for $q_t < q_{vs}(z)$; $H_u = 0$ for $q_t \geq q_{vs}(z)$; $H_s = 1 - H_u$. In terms of q_t and θ_e, the variables b_u and b_s are given by

$$\frac{b_u}{g} = \left[\frac{\theta_e}{\theta_o} + \left(R_{vd} - \frac{L_v}{c_p\theta_o}\right)q_t\right], \quad \frac{b_s}{g} = \left[\frac{\theta_e}{\theta_o} + \left(R_{vd} - \frac{L_v}{c_p\theta_o} + 1\right)q_{vs} - q_t,\right] \tag{4}$$

and are both defined in unsaturated and saturated regions alike. The formulation (3)–(4) is convenient because it separates the continuous functional dependence within b_u and b_s from the discontinuous nature of the phase interface H_u and H_s.

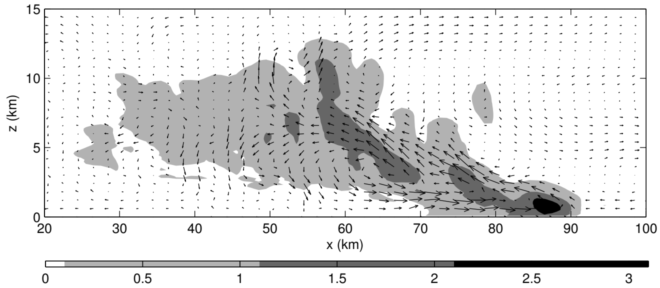

FIGURE 1. Simulation of a tropical squall line: vertical slice of time-averaged rain water (kg/kg) and velocity vectors in the frame of reference moving with the squall line.

In [1], it was demonstrated that (1)-(2) is able to capture regimes of organized convection on horizontal length scales of several hundred kilometers, such as a tropical squall line. Figure 1 shows that the simulated squall line has features of observed tropical squall lines, such as the tilted structure with ascending front-to-rear flow and the descending rear inflow [2].

QG scaling is appropriate for larger horizontal length scales $L \sim 1000$ km, and the purpose of the current work is to introduce a QG model with water included. A precipitating QG system will aid in the understanding of midlatitude storm dynamics, building upon dry QG, but including the important effects of latent heat release. Similar to the dry scaling, the non-dimensional Rossby number Ro and Froude numbers Fr_u, Fr_s are assumed to be small and comparable: $Ro = \epsilon$, $Fr_u = (L/L_u)Ro = O(\epsilon)$, $Fr_s = (L/L_s)Ro = O(\epsilon)$, $\epsilon \to 0$, where L_u (L_s) is the deformation radius for an unsaturated (saturated) regions. Small Ro corresponds to rapid rotation; small Fr_u, Fr_s imply, respectively, strongly increasing background temperature $\tilde{\theta}(z)$ and potential temperature $\tilde{\theta}_e(z)$ such that $d\tilde{\theta}(z)/dz \gg 1$, $d\tilde{\theta}_e(z)/dz \gg 1$. The background water profile $\tilde{q}_t(z)$ is assumed to decrease rapidly such that the ratio $(L_v/c_p)G_m \equiv -L_v d\tilde{q}_t(z)/dz\ (c_p d\tilde{\theta}_e(z)/dz)^{-1} = O(1)$, where the latent heat $L_v \approx 2.5 \times 10^6$ J kg^{-1} and specific heat $c_p \approx 10^3$ J kg^{-1} K^{-1}. Then all fields $f(\mathbf{x}, t)$ in the non-dimensional equations are expanded as $f = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \cdots$. The PQG model results from (1) keeping lowest-order and next-order balances.

Returning to the dimensional equations, the lowest-order balance is geostrophic and hydrostatic, given by

$$f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla_h \psi, \quad b_u^{(0)} H_u + b_s^{(0)} H_s = \frac{\partial \psi}{\partial z}, \tag{5}$$

where the stream function $\psi = p/\rho_o$, the vorticity $\zeta^{(0)} = \nabla_h^2 \psi$, and the lowest-order vertical velocity $w^{(0)} = 0$. The difference from dry QG is the phase interface in the hydrostatic balance relation, represented by the Heaviside functions H_u, H_s.

At next order, one finds

$$\frac{D_h\zeta^{(0)}}{Dt} = f\frac{\partial w^{(1)}}{\partial z}, \quad \frac{D_h\theta_e^{(0)}}{Dt} + \frac{d\tilde{\theta}_e}{dz}w^{(1)} = 0, \quad \frac{D_h q_t^{(0)}}{Dt} + \frac{d\tilde{q}_t}{dz}w^{(1)} = V_T\frac{\partial q_r^{(0)}}{\partial z}, \tag{6}$$

with the first-order vertical velocity $w^{(1)}$ appearing in all three equations. Denoting $B_e = d\tilde{\theta}_e/dz$ and eliminating $w^{(1)}$ gives the PQG system

$$\frac{D_h PV_e}{Dt} = -\frac{f}{B_e}\frac{\partial \mathbf{u}_h^{(0)}}{\partial z}\cdot\nabla_h\theta_e^{(0)}, \quad \frac{D_h M}{Dt} = V_r\frac{\partial q_r^{(0)}}{\partial z}, \tag{7}$$

where $\mathbf{u}_h$ is the horizontal velocity and we use the definitions

$$PV_e \equiv \zeta^{(0)} + \frac{f}{B_e}\frac{\partial\theta_e^{(0)}}{\partial z}, \quad M \equiv q_t^{(0)} + G_M\theta_e^{(0)}. \tag{8}$$

PV-and-M inversion to find the streamfunction ψ may be written as

$$PV_e = \nabla_h^2\psi +$$

$$\frac{\partial}{\partial z}\left[H_u\left(\frac{f^2}{N_u^2}\frac{\partial\psi}{\partial z} + \frac{L_v}{c_p}\frac{g}{\theta_0}\frac{f}{N_u^2}M\right)\right] + \frac{\partial}{\partial z}\left[H_s\left(\frac{f^2}{N_s^2}\frac{\partial\psi}{\partial z} + \frac{L_v}{c_p}\frac{g}{\theta_0}\frac{f}{N_s^2}q_{vs}(z)\right)\right], \tag{9}$$

where the buoyancy frequencies are $N_s^2 = (g/\theta_0)B_e$, $N_u^2 = N_s^2[1 + (L_v/c_p)G_M]$. Notice that the phase interface location is unknown a priori from PV and M, and is discovered as part of the inversion process. In a channel V with $w = 0$ on top and bottom and periodic boundary condition in (x, y), the PQG system (7)-(9) conserves energy $E = \int_V \varepsilon\, dV$ for ε given by

$$\varepsilon = |\nabla_h\psi|^2 + H_s\left[\frac{f^2}{N_s^2}\psi_z^2\right] + H_u\left[\frac{f^2}{N_u^2}\psi_z^2 + \left(\frac{L_v}{c_p}\frac{g}{\theta_0}\right)^2\frac{N_s^2}{N_u^2}D\left(M - \frac{N_u^2}{N_s^2}q_{vs}\right)^2\right] \tag{10}$$

where $D \equiv 1/(N_u^2 - N_s^2)$.

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Stochastic PDEs for Tropical Rainfall and the Madden-Julian Oscillation

Samuel N. Stechmann

Stochastic partial differential equations are presented for two phenomena of tropical atmospheric dynamics: (i) tropical rainfall and cloud clusters, and (ii) the Madden-Julian Oscillation (MJO) and tropical intraseasonal variability.

First, a linear stochastic model is presented for the dynamics of water vapor and tropical convection. Despite its linear formulation, the model reproduces a wide variety of observational statistics from disparate perspectives, including (i) a cloud cluster area distribution with an approximate power law; (ii) a power spectrum of spatiotemporal red noise, as in the "background spectrum" of tropical convection; and (iii) a suite of statistics that resemble the statistical physics concepts of critical phenomena and phase transitions. The physical processes of the model are precipitation, evaporation, and turbulent advection–diffusion of water vapor, and they are represented in idealized form as eddy diffusion, damping, and stochastic forcing. Consequently, the form of the model is a damped version of the two-dimensional stochastic heat equation. Exact analytical solutions are available for many statistics, and numerical realizations can be generated for minimal computational cost and for any desired time step. Given the simple form of the model, the results suggest that tropical convection may behave in a relatively simple, random way. Finally, relationships are also drawn with the Ising model, the Edwards–Wilkinson model, the Gaussian free field, and the Schramm–Loewner evolution and its possible connection with cloud cluster statistics. Potential applications of the model include several situations where realistic cloud fields must be generated for minimal cost, such as cloud parameterizations for climate models or radiative transfer models. This is work with Scott Hottovy and was published in the Journal of the Atmospheric Sciences in 2015.

Second, a stochastic model is presented for the MJO, which is the dominant mode of variability in the tropical atmosphere on intraseasonal time scales and planetary spatial scales. Despite the primary importance of the MJO and the decades of research progress since its original discovery, a generally accepted theory for its essential mechanisms has remained elusive. In previous work, a minimal dynamical model has been proposed that recovers robustly the most fundamental MJO features of (i) a slow eastward speed of roughly 5 m/s, (ii) a peculiar dispersion relation with frequency approximately independent of wavenumber, and (iii) a horizontal quadrupole vortex structure. This model, the skeleton model, depicts the MJO as a neutrally stable atmospheric wave that involves a simple multiscale interaction between planetary dry dynamics, planetary lower-tropospheric moisture, and the planetary envelope of synoptic-scale activity. In the stochastic version of the model, it is shown that the skeleton model can further account for (iv) the intermittent generation of MJO events and (v) the organization of MJO events into wave trains with growth and demise, as seen in nature. The goal is achieved by developing a simple stochastic parameterization for the unresolved details of synoptic-scale activity, which is coupled to otherwise deterministic processes in the skeleton model. In particular, the intermittent initiation, propagation, and shut down of MJO wave trains in the skeleton model occur through these stochastic effects. This is work with Sulian Thual and Andrew J. Majda and was published in the Journal of the Atmospheric Sciences in 2014.

K41 solutions of the Euler equations

László Székelyhidi Jr.

Consider the energy cascade in homogeneous isotropic three-dimensional turbulence, postulated by Richardson, Kolmogorov, Onsager and others, interpreted as a statement about single (weak) solutions of the Navier-Stokes equations with viscosity ν. According to this picture

$$u = \bar{u} + \sum_{q=1}^{N_\nu} w_q^\nu + w_{diss}^\nu,$$

where w_q^ν is concentrated on frequency $\lambda_q \sim 2^q$ with kinetic energy $\delta_q \sim \lambda_q^{-2/3}$ in the inertial range $1 \leq q \leq N_\nu$. For any $Q \ll N_\nu$ the partial sum $u_Q = \bar{u} + \sum_{q=1}^{Q} w_q$ satisfies the Euler-Reynolds system

$$\partial_t u_Q + u_Q \cdot \nabla u_Q + \nabla p_Q = -\mathrm{div}\ R_Q,$$

where $R_Q \sim \langle w_{Q+1} \otimes w_{Q+1} \rangle \sim \delta_{Q+1}$. As $\nu \to 0$ (and $N_\nu \to \infty$), one expects (i) $w_q^\nu \to w_q$ in L^2, and (ii) $w_{diss}^\nu \to 0$ in L^2, where w_q is again concentrated on frequency λ_q with kinetic energy δ_q. Statement (i) is closely related to Onsager's conjecture, asserting the existence of weak solutions u of the Euler equations with the structure

$$u = \bar{u} + \sum_{q=1}^{\infty} w_q,$$

where $\|w_q\|_{L^\infty} \lesssim \delta_q^{1/2}$ and $\|\nabla w_q\|_{L^\infty} \lesssim \delta_q^{1/2}\lambda_q$ with $\delta_q \sim \lambda_q^{-2/3}$. In the talk we discuss the recently completed construction of dissipative weak solutions, following work by T. Buckmaster, S. Daneri, C. De Lellis, P. Isett, L. Székelyhidi and V. Vicol. These *K41 solutions* have, for any $\alpha < 1/3$, the structure above with

$$\lambda_q \sim a^{b^q}, \quad \delta_q \sim \lambda_q^{2\alpha},$$

with some large $a \gg 1$ and $1 < b < \frac{1-\alpha}{2\alpha}$.

Time periodic initial value problem for rotating stably stratified fluids

Ryo Takada
(joint work with Matthias Hieber, Alex Mahalov)

We consider the time periodic problem for the 3D Boussinesq equations, describing the motion of viscous incompressible fluids under the effects of both the rotation and the stable stratification:

$$\begin{cases} \partial_t v + (v \cdot \nabla) v = \Delta v - \Omega e_3 \times v - \nabla p + \theta e_3 + g & t > 0,\, x \in \mathbb{R}^3, \\ \partial_t \theta + (v \cdot \nabla)\theta = \Delta \theta - N^2 v_3 + h & t > 0,\, x \in \mathbb{R}^3, \\ \nabla \cdot v = 0 & t > 0,\, x \in \mathbb{R}^3, \\ v(0,x) = v_0(x), \quad \theta(0,x) = \theta_0(x) & x \in \mathbb{R}^3. \end{cases}$$

Here, $v = (v_1(t,x), v_2(t,x), v_3(t,x))^T$, $p = p(t,x)$ and $\theta = \theta(t,x)$ are the unknown functions, representing the velocity field, the scalar pressure and the thermal disturbance about a mean state in hydrostatic balance, respectively, while $g = (g_1(t,x), g_2(t,x), g_3(t,x))^T$ and $h = h(t,x)$ are given time periodic external forces. $N > 0$ is the Brunt-Väisälä (buoyancy) frequency for the constant stratification and $\Omega \in \mathbb{R} \setminus \{0\}$ is the angular frequency of the background rotation.

Making use of the gerenal approach to time periodic problem [1] and the dispersive nature due to both the rotation and the stable stratification [2], we give an explicit relation between the size of the time periodic external forces and the buoyancy frequency which ensures the unique existence of time periodic solutions to the above systems. In particular, it is shown that the size of the time periodic external forces can be taken large in proportion to the strength of the rotation and the stable stratification.

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Turbulent Mixing and Atmospheric Predictability

Joseph Tribbia

Atmospheric predictability is largely determined by the turbulent mixing of two scalar fields. At large scales the lagrangian conservation of potential vorticity is the determining dynamics that leads to quasi-geostrophic turbulence, a rapidly decaying kinetic energy spectrum and a roughly two day doubling time of errors. At the sub-mesoscale moist dynamics and the conservation of water species become the determining factors in setting the time scale for the rapid loss of predictability associated with the release of heat due to the phase changes of water.

The consequences of this dual dynamics for scale interactions and the ultimate limits of atmospheric predictability will be explored in this talk building on the parameterization of moisture in [1].

References

[1] J. Lambaerts, G. Lapeyre, V. Zeitlin, *Moist vs dry baroclinic instability in a simplified two-layer atmospheric model with condensation and latent heat release* , J. Atmos. Sci. **69** (2012), 1405–1426.

On the Incompressible Euler Equation with a Free Boundary

Amjad Tuffaha
(joint work with Igor Kukavica, Vlad Vicol, Fei Wang)

In this talk, we discuss results on the local-in-time existence and regularity of solutions to the shallow water wave model consisting of the 3D incompressible Euler equations on a free surface without surface tension, under minimal regularity assumptions on the initial data and the Rayleigh-Taylor sign condition. We give an overview of the result on local well-posedness in the rotational case when the initial datum u_0 satisfies $u_0 \in H^{2.5+\delta}$ and $\nabla\times u_0 \in H^{2+\delta}$, where $\delta > 0$ is arbitrarily small, under the Taylor condition on the pressure. This is based on joint works with Igor Kukavica from the University of Southern California, Vlad Vicol from Princeton university and Fei Wang from the University of Southern California.

Simple Models of Convection and Convective Parameterization

Geoffrey K. Vallis

We suggest a new way to parameterize convection, and perform some vey simple numerical test. The metod involves conditionally averaging the equation, thereby sampling the PDF of a quantity in a given grid box. One then obtains equations of motion that predict two (or more) values of a field at any given location, representing for example convecting fluid and stable fluid. In this way the equations of motion themselves can be used to parameterize convection that is subgrid-scale. Some numerical experiments that integrate the two-fluid model show promising results.

Think Globally, Act Locally: Numerical analysis with finite time scale separation in oscillatory PDEs

Beth A. Wingate

One of the fundamental issue standing in the way of designing new numerical methods capable of taking advantage of new computer architectures is embedded in the mathematical structure of the underlying PDEs. One such PDE that governs many physical applications, including weather and climate, plasma physics, etc, has the following mathematical form:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\epsilon} L(\mathbf{u}) + N(\mathbf{u}, \mathbf{u}) = D(\mathbf{u}), \quad \mathbf{u}(0) = \mathbf{u}_0, \tag{1}$$

where the linear operator L has pure imaginary eigenvalues, the nonlinear term $N(\mathbf{u}, \mathbf{u})$ is of polynomial type, the operator D encodes a form of dissipation, and ϵ is a small non-dimensional parameter. For notational simplicity, we let $\mathbf{u}(t)$ denote the spatial (vector-valued) function $\mathbf{u}(t, \cdot) = (u_1(t, \cdot), u_2(t, \cdot), \ldots)$. The operator $\epsilon^{-1}L$ results in time oscillations on an order $\mathcal{O}(\epsilon)$ time scale, and generally necessitates small time steps if standard explicit numerical integrators are used. Even implicit integrators need to use small time steps if accuracy is required.

Therefore, examining the low-frequency content of the PDEs and making best use of that in numerical algorithm development is essential if we are to make best use of the new computer architectures. The type of equation (1) is known as a fast singular limit, and as such we expect small scale oscillations will remain a part of the solution even when the nonlinearity, or 'phase scrambler' creates low frequency dynamics. We are called to see if we can use the long-time, low-frequency dynamics (think globally) to advance an accurate solution (act locally) for PDEs of the type (1).

In this talk I introduce a parareal-type method[3, 1] for equations of the form (1), where we have used the above strategy of using the long-time, low frequency dynamics to drive a locally accurate solution. I show that under certain regularity constraints this method has superlinear convergence [2] as $\epsilon \to 0$ and sketch the ideas behind a new proof for superlinear convergence, one that relies on the role of near-resonances inherent in the PDEs, for the case when ϵ is finite[4].

References

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[2] T. Haut and B. Wingate, *An asymptotic parallel-in-time method for highly oscillatory pdes*, SIAM Journal of Scientific Computing, 36 (2014), pp. A693–A736. arXiv:1303.6615.

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[4] A. Peddle and T. Haut and B. Wingate, *An asymptotic parallel-in-time method for highly oscillatory pdes with finite frequencies*, Submitted to SIAM Journal of Scientific Computing (2017).

The Mathematical theory of the primitive equations in the presence of humidity

Mohammed Ziane
(joint work with M. Coti Zelati, A. Huang, I. Kukavica, R. Temam)

The work presented is a collaboration with M. Coti Zelati, A. Huang, I. Kukavica, and R. Temam; A modification of the classical primitive equations of the atmosphere is considered in order to take into account important phase transition phenomena due to air saturation and condensation. We provide a mathematical formulation of the problem that appears to be new in this setting, by making use of differential inclusions and variational inequalities, and which allows to develop a rather complete theory for the solutions to what turns out to be a nonlinearly coupled system of non-smooth partial differential equations. Specifically we prove global existence of quasi-strong and strong solutions, along with uniqueness results and maximum principles of physical interest.

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