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Differentialgeometrie im Großen

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Abstract. The topics discussed at the meeting were Kähler geometry, geometric evolution equations, manifolds of nonnegative curvature, metric geometry and geometric representations of groups. The choice of topics reflects current trends in the development of differential geometry.

Mathematics Subject Classification (2010): 53C.

Introduction by the Organisers

The workshop *Differentialgeometrie im Großen* was held July 2 - July 8, 2017. The participants were specialists in differential geometry and its neighboring fields, covering a broad spectrum of subareas which are in the focus of current developments.

The lectures during the five days of the meeting were roughly organized according to different themes.

The first day of the meeting began with three talks on the latest developments in the Ricci flow, followed by two afternoon talks on 3-manifolds and hyperbolic geometry.

The second day featured three morning talks on geometric flows and their applications in complex geometry and minimal surfaces. The afternoon saw talks on symmetric spaces and Higgs bundles.

Wednesday morning's talks discussed Einstein metrics in both general relativity and Kähler geometry. In the afternoon we had the traditional hike.

Thursday's talks were mainly devoted to aspects of Riemannian geometry and symmetric spaces. Finally, four talks were presented on the last day of the workshop on topics in Kähler and Riemannian geometry.

The meeting gave a good overview of the current developments, and showed significant progress in the field. The workshop was attended by researchers from around the world, ranging from graduate students to scientific leaders in their areas.

The atmosphere during the meeting was lively and open, and greatly benefited from the ideal environment at Oberwolfach.

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Workshop: Differentialgeometrie im Großen

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Abstracts

Ricci flow through singularities

Bruce Kleiner
(joint work with Richard Bamler)

After introducing singular Ricci flows, which are a kind of "Ricci flow through singularities" in dimension three, the following results were discussed:

1. (Lott-K.) For any compact Riemannian 3-manifold M, there exists a singular Ricci flow with initial condition M.

2. (Bamler-K.) For any compact Riemannian 3-manifold M, there is only one singular Ricci flow with intial condition M, up to equivalence.

3. (Bamler-K.) Let $\{\mathcal{M}^j\}$ be a sequence of Ricci flows with surgery (in the sense of Perelman) with surgery parameter δ_j. If the $\mathcal{M}^j s$ start from a fixed compact Riemannian manifold M, and $\delta_j \to 0$, then $\mathcal{M}^j$ converges to the singular Ricci flow with initial condition M as $j \to \infty$.

Ricci flow beyond non-negative curvature conditions

Esther Cabezas-Rivas
(joint work with Richard H. Bamler and Burkhard Wilking)

The search for invariant curvature conditions has been crucial in many applications of the Ricci flow. However, most of the known invariant curvature conditions are rather restrictive, because they impose strong positivity requirements, like that the scalar, Ricci or even sectional curvature is positive, which heavily constrains the topology of the underlying manifold.

In [1] we generalize most of the known Ricci flow invariant non-negative curvature conditions to less restrictive negative bounds that remain sufficiently controlled for a short time, that is, deteriorate under the flow by at most a controlled factor. These negative bounds hold for any metric after rescaling by a sufficiently large factor and hence we don't impose any topological restrictions.

The following theorem serves as an illustration for generalizations of a larger class of invariant curvature conditions, as presented in Theorem 4 below.

Theorem 1. *Given $n \in \mathbb{N}$ and a constant $v_0 > 0$, there exist positive constants $C = C(n, v_0) > 0$ and $\tau = \tau(n, v_0) > 0$ such that the following holds. Let (M^n, g) be a closed Riemannian manifold satisfying*

$$\mathrm{vol}_g \big(B_g(p,1)\big) \geq v_0 \quad \text{for all } p \in M \qquad \text{and} \quad \mathrm{Rm}_g \geq -\varepsilon \geq -1,$$

i.e. the lowest eigenvalue of Rm_g is bounded below by $-\varepsilon \in [-1, 0]$. Then the Ricci flow $g(t)$ with initial metric g exists until time τ, and we have the curvature bounds

$$\mathrm{Rm}_{g(t)} \geq -C\varepsilon \quad \text{and} \quad |\mathrm{Rm}_{g(t)}| \leq \frac{C}{t} \quad \text{for all } t \in (0, \tau].$$

Recall that we cannot expect that negative lower bounds for the curvature operator Rm are in general invariant under the Ricci flow (see [6] for a counterexample). But our almost preservation theorem implies a variety of smoothing and gap results of independent interest. For instance, we get a classification of closed manifolds with almost non-negative curvature operator in the non-collapsed case:

Corollary 2. *Given $n \in \mathbb{N}$ and positive constants D, v_0, there exists a constant $\varepsilon = \varepsilon(n, v_0, D) > 0$ such that the following holds. Any closed Riemannian manifold (M^n, g) with*

$$\operatorname{diam}_g(M) \leq D, \qquad \operatorname{vol}_g(M) \geq v_0 \quad \text{and} \qquad \operatorname{Rm}_g \geq -\varepsilon$$

also admits a metric of non-negative curvature operator.

Within the proof one can show that the metric whose existence is asserted in Corollary 2 is close to the original metric g in the Gromov-Hausdorff sense. This motivates the next smoothing result.

Corollary 3. *Let (X, d_X) be the Gromov-Hausdorff limit of a sequence $\{(M_i, g_i)\}_{i=1}^{\infty}$ of closed Riemannian manifolds satisfying*

$$\operatorname{vol}_{g_i}(M_i) \geq v_0, \qquad \operatorname{Rm}_{g_i} \geq \varepsilon_i, \qquad \operatorname{diam}_{g_i}(M_i) \leq D.$$

for some sequence $\{\varepsilon_i\} \subset (0,1]$ with $\varepsilon_i \to \varepsilon_\infty$, as $i \to \infty$. Then there exists $\tau = \tau(n, v_0) > 0$, a smooth manifold M_∞ and a smooth solution to the Ricci flow $(M_\infty, g_\infty(t))_{t\in(0,\tau)}$ which satisfies $\operatorname{Rm}_{g_\infty(t)} \geq \varepsilon_\infty$ and is coming out of the (possibly singular) space (X, d_X) in the sense that

$$\lim_{t\searrow 0} d_{GH}\big((X, d_X), (M_\infty, d_{g_\infty(t)})\big) = 0.$$

Moreover, for any choice of ε_∞, the space X is homeomorphic to the manifold M_∞ and the Riemannian distance $d_{g_\infty(t)}$ converges uniformly to a distance function d_0 on M_∞ as $t \searrow 0$ such that (M_∞, d_0) is isometric to (X, d_X).

By taking convergent sequences of manifolds as above one can generate a large variety of singular spaces that can be smoothed out by the Ricci flow with lower curvature bound.

Notice that the bound $\operatorname{Rm}_g \geq -\varepsilon$ can be rephrased by saying that the linear combination $\operatorname{Rm}_g + \varepsilon\, \mathrm{I}$, where I denotes the curvature operator of the unit round n-sphere, is non-negative definite. Hereafter we denote curvature conditions by $\mathcal{C}$ and we write $\operatorname{Rm}_g \in \mathcal{C}$ to indicate that Rm_g satisfies the corresponding curvature condition. Using this, we can extend Theorem 1:

Theorem 4. *Given $n \in \mathbb{N}$ and a constant $v_0 > 0$, there exist positive constants $C = C(n, v_0) > 0$ and $\tau = \tau(n, v_0) > 0$ such that the following holds. Let (M^n, g) be a complete Riemannian manifold with bounded curvature and consider one of the following curvature conditions $\mathcal{C}$:*

(1) *2-non-negative curvature operator*
(i.e. the sum of the lowest two eigenvalues is non-negative),

(2) *non-negative complex sectional curvature*
(i.e. weakly PIC_2*, meaning that taking the cartesian product with* $\mathbb{R}^2$ *produces a non-negative isotropic curvature operator),*
(3) *weakly* PIC_1 *(i.e. the cartesian product with* $\mathbb{R}$ *produces a non-negative isotropic curvature operator),*
(4) *non-negative bisectional curvature, in the case in which* (M, g) *is Kähler with respect to some complex structure* J.

Assume that

$$\mathrm{vol}_g\big(B_g(p,1)\big) \geq v_0 \quad \text{for all } p \in M \qquad \text{and} \quad \mathrm{Rm}_g + \varepsilon \mathrm{I} \in \mathcal{C},$$

for some $\varepsilon \in [0,1]$. *Then the Ricci flow* $g(t)$ *with initial metric* g *exists until time* τ, *is Kähler if* (M, g) *is Kähler, and we have the curvature bounds*

$$\mathrm{Rm}_{g(t)} + C\varepsilon\, \mathrm{I} \in \mathcal{C} \quad \text{and} \quad |\mathrm{Rm}_{g(t)}| \leq \frac{C}{t} \quad \text{for all } t \in (0, \tau].$$

In [1] we also obtain the corresponding versions of Corollary 2 and Corollary 3 for the curvature conditions (1)–(4) listed above.

The invariance of $\mathrm{Rm} \geq 0$ was first observed by Hamilton (see [7]) and further studied by Böhm and the third author (see [2]). Preservation of 2-non-negative curvature was originally proved by H. Chen [4]. The invariance of weakly PIC was first showed in dimension four by Hamilton [8]; the general case was obtained independently by S. Brendle and R. Schoen [3] and by H. T. Nguyen [9]. The invariant conditions weakly PIC_1 and PIC_2 were in turn introduced by Brendle and Schoen in [3] and play a key role in their proof of the differentiable sphere theorem. Finally, non-negative bisectional curvature is known to be preserved under the Kähler Ricci flow (cf. [10] and [11] for closed and complete manifolds with bounded curvature, respectively).

In dimension 3, Theorem 4 and Corollaries 2 and 3 were established by Simon in [12, 13] for the case of almost non-negative and 2-non-negative curvature operator, which in dimension 3 is equivalent to almost non-negative sectional and Ricci curvature, respectively.

We finish by highlighting that in [1] we additionally establish a local version of Theorem 4 in the case of non-negative curvature operator and non-negative complex sectional curvature. By applying this local result to a sequence of larger and larger balls, we obtain a short-time existence result on complete manifolds with possibly unbounded curvature, which generalizes the existence result in [5].

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Local Ricci flow and limits of three dimensional spaces with Ricci curvature bounded below

Miles Simon
(joint work with Peter Topping)

Given a three-dimensional Riemannian manifold containing a ball with an explicit lower bound on its Ricci curvature and a positive lower bound on its volume, we use a *local Ricci flow* to perturb the Riemannian metric on the interior to a nearby Riemannian metric with comparable lower bounds on Ricci curvature and volume, but additionally with uniform time dependant bounds on the full curvature tensor and all derivatives thereof. The new locally defined Riemannian manifold is then uniformly, in time, close to the initial ball with respect to distance, and furthermore we obtain bounds on the local Hölder/Lipschitz constants which describe the Hölder/Lipschitz equivalence of the metric spaces of the Riemannian manifolds in question.

One consequence is that we obtain a local bi-Hölder correspondence between the Gromov-Hausdorff limits of smooth non-collapsed manifolds with Ricci curvature bounded from below, and the manifolds themselves. This is more than a complete resolution of the three-dimensional case of the conjecture of Anderson-Cheeger-Colding-Tian, describing how Ricci limit spaces in three dimensions must be homeomorphic to manifolds, and we obtain this in the most general, locally non-collapsed case. This is joint work with Peter Topping, and the proofs build on results and ideas from recent papers of Raphael Hochard [1], and Simon-Topping [2]. The results presented in this talk can be found in the paper of Simon-Topping [3].

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On spherical CR structures on 3-manifolds

Pierre Will

(joint work with Antonin Guilloux and John Parker)

1. Context

The complex hyperbolic 2-space $\mathbf{H}^2_{\mathbb{C}}$ may be seen as the unit ball $\mathbf{B}^2_{\mathbb{C}} \subset \mathbb{C}^2$, equipped with its Bergman metric, which has pinched negative curvature. The group of holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$ is identified with PU(2,1), and the sphere at infinity $\partial_\infty \mathbf{H}^2_{\mathbb{C}}$ is just $\mathbb{S}^3$. A standard reference for complex hyperbolic geometry is Goldman's book [11] Let Γ be a discrete subgroup of the group of PU(2,1). The domain of discontinuity of Γ, denoted Ω_Γ, is the largest open subset of $\mathbb{S}^3$ on which Γ acts properly. It is the complement of the limit set of Γ, denoted Λ_Γ. The quotient $\Gamma\backslash\Omega_\Gamma$ is the *manifold at infinity* of Γ. Now, let M be a 3-manifold with fundamental group π and $\rho : \pi \longrightarrow$PU(2,1) be a representation of which image we denote by Γ. We say that ρ is a *spherical CR uniformization* of M if Γ is discrete, has isolated fixed points in $\mathbf{H}^2_{\mathbb{C}}$ and if the manifold at infinity of Γ is M. Note that certain isometries of $\mathbf{H}^2_{\mathbb{C}}$ fix pointwise complex lines, and in turn fix pointwise a circle in $\partial\mathbf{H}^2_{\mathbb{C}}$. This is the reason why we add the above condition on fixed points. We refer the reader to [4] for more details about these definitions.

The starting point of our work is a result due to R. Schwartz that describes an explicit spherical CR uniformisation of the Whitehead link complement (see [18,19] or the survey paper [17]). Schwartz's example is striking as it produces a complex hyperbolic manifold (the quotient of $\mathbf{H}^2_{\mathbb{C}}$ by the image of the representation) of which boundary at infinity is a real hyperbolic manifold. Only a handful of examples of such a situation is known to this day. Falbel and Deraux have described in [6] a spherical CR uniformisation of the figure eight knot complement, and Deraux proved in [5] that it is possible to deform it while keeping spherical CR uniformisations of the same manifold. Schwartz also proved and applied to his example a surgery theorem for these uniformisations, which implies that there is an infinite number of closed hyperbolic 3-manifolds that admit spherical CR uniformisations (Corollary 1.6 in [19]).

In fact, spherical CR uniformisations are examples of so-called $(\mathbb{S}^3, \mathrm{PU}(2,1))$-structures, in the sense of Ehresmann see [3]. It is therefore natural to study representations of π in PU(2,1), that arise as holonomies of these structures, and the corresponding character varieties. A direct approach to the character variety of

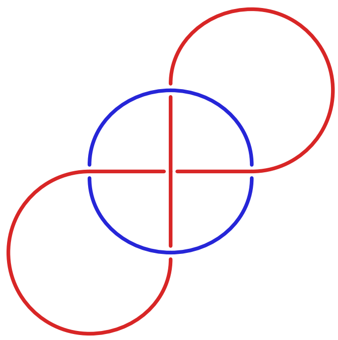

Figure 1. The Whitehead link. Its group has presentation $\pi = \langle u, v | [u, v][u, v^{-1}][u^{-1}, v^{-1}][u^{-1}, v] = 1 \rangle$.

representation in real forms of complex Lie groups have been described by Acosta in [1]. It is also natural to rather consider the SL(3,ℂ)-character variety. Indeed, spherical CR structures are examples of $\mathbb{C}P^2$-flag structures for which the target Lie group is PGL(3,ℂ) (we refer the reader to [2] for this aspect).

2. Results

In the rest of this exposition, we focus on the case where the 3-manifold M is the Whitehead link complement. See Figure 1 for a presentation of the fundamental group of its complement.

2.1. Finding representations of the Whitehead link group in SL(3,ℂ).

There exists a general method for finding representations of fundamental groups of triangulated hyperbolic 3-manifolds, which is inspired from the one described by Thurston in [20]. We refer the reader to [7], [2] or [10]. This method is highly non-trivial in terms of computation, and our ambition is more modest here. Here is a way of finding a nice class of representations in the case of the Whitehead link complement. Our first remark is the following

Lemma 1. *The group $\mathbb{Z}_3 * \mathbb{Z}_3$ is a quotient of π.*

Proof. This is due to the fact that the Whitehead link has a (non-hyperbolic) surgery σ_0 which is the connected sum of two Lens spaces $L(3)$. We refer to [12] for more details. □

Remark 1. (1) Seen as a hyperbolic manifold, the Whitehead link complement has two cusps $\mathcal{C}_1$ and $\mathcal{C}_2$. The corresponding boundary tori have longitude and meridian denoted by m_i and ℓ_i ($i = 1, 2$). Topologically, the surgery σ_0 is obtained by making trivial the two loops $m_i^3 \ell_i^{-1}$. In other words, σ_0 is the surgery with slopes $(-3, -3)$. In particular, this gives elements in the kernel of the representations in X_0.

(2) At the group-theoretic level the surgery σ_0 corresponds to the morphism defined by

$$\begin{array}{rcl} \pi & \longrightarrow & \mathbb{Z}_3 * \mathbb{Z}_3 \\ (u,v) & \longmapsto & (st^{-1}, st^{-1}s), \end{array} \tag{1}$$

where s and t are the generators of $\mathbb{Z}_3 * \mathbb{Z}_3$. In view of Lemma 1, any group generated by two order three elements in SL(3,$\mathbb{C}$) is the image of π by a representation.

The first result of this talk is the following

Theorem 1 (Guilloux-Will)**.** *The SL(3,$\mathbb{C}$)-character variety of the Whitehead link group has an algebraic component X_0 of (complex) dimension 4, which is formed by representations that factor through the surgery σ_0.*

Remark 2. (1) A key ingredient in the proof of Theorem 1 is the description of the SL(3,$\mathbb{C}$)-character variety of the rank 2 free group given by Lawton in [15].

(2) The only examples of SL(3,$\mathbb{C}$)-character varieties for link groups that have been described so far are those of torus knots (see [14]), and the one of the figure eight knot, that have been worked out independantly by Heusener-Muñoz-Porti in [13] and by Falbel-Guilloux-Koseleff-Roullier-Thistlethwaite in [8].

(3) The least expected dimension for (non-degenerate) components of the charater variety of the Whitehead link complement is 4. We refer the interested reader to [9] for more details, and a lower bound on the dimension of (non-degenerate) components of SL(n,$\mathbb{C}$) character varieties of 3-manifolds.

(4) The *geometric* component of the SL(3,$\mathbb{C}$)-character variety of π is the one that contains the (character of the) representation

$$\rho_{\text{geom}} : \pi \overset{\rho_{\text{hyp}}}{\longrightarrow} \mathrm{SL}(2,\mathbb{C}) \overset{\text{irr}}{\longrightarrow} \mathrm{SL}(3,\mathbb{C}),$$

where ρ_{hyp} is the holonomy representation of the hyperbolic structure on M and irr is the irreducible representation. The representation ρ_{geom} is faithful, and therefore the component X_0 in Theorem 1 isn't the geometric one since all representations factorising through the surgery of Lemma 1 are non-faithful.

2.2. **Analysing special representations in SU(2,1).** We know focus on the case of representations in the component X_0 of which image is contained in SU(2,1). In that case, we obtain actions of π on $\mathbf{H}^2_{\mathbb{C}}$ (and $\partial_\infty \mathbf{H}^2_{\mathbb{C}}$). Now, in the usual hyperbolic structure on M, the peripheral groups contain only parabolic elements. We will restrict ourselves to representations that have the same feature, and map element in periheral subgroups to parabolic isometries. The way the representations in X_0 were obtained has an important consequence : the images in SU(2,1) of the (rank 2) peripheral subgroups in π are cyclic (this follows directly from the first part of Remark 1). As observed in [12] and [16], our restriction about peripheral subgroups amounts to assume that the two words $b_1 = st$ and $b_2 = st^{-1}$ are mapped to parabolic isometries ($\rho(b_1)$ and $\rho(b_2)$ generate respectively the images of the cusps groups of $\mathcal{C}_1$ and $\mathcal{C}_2$. This class of groups is referred to as *diamond groups* by Schwartz in Chapter 15 of [19]. To state our result, let us state the following facts.

Remark 3. Parabolic elements in SU(2,1) come in two types.

(1) Type 1 parabolic elements preserve a complex totally geodesic embedded copy of the Poincaré disc (a PU(2,1)-image of the first axis of coordinates of the ball $\mathbf{B}^2_{\mathbb{C}}$), and rotate around it thourgh an angle θ. When $\theta = 0$, they are 2-step unipotent.
(2) Type 2 parabolic elements preserve a real copy of the Poincaré disc (a PU(2,1) image of $\mathbb{R}^2 \cap \mathbf{H}^2_{\mathbb{C}}$), and are 3-step unipotent.

The rotation angles of b_1 and b_2 give coordinates on the set of diamond groups :

Proposition 1. *Representations of $\mathbb{Z}_3 * \mathbb{Z}_3$ in SU(2,1) such that b_1 and b_2 are parabolic are classified up to PU(2,1)-conjugation by the two rotation angles θ_1 and θ_2 of $\rho(b_1)$ and $\rho(b_2)$.*

Note that, in Proposition 1, the case where one of the rotation angles θ_i is zero actually correspond to a type 2 parabolic element (thus 3-step unipotent). In fact, it is not difficult to see that in this context, unipotent type 1 parabolics cannot occur (see [16].

Theorem 2. *The representation $\rho_0 : \mathbb{Z}_3 * \mathbb{Z}_3 \longrightarrow$ SU(2,1) for which $\theta_1 = \theta_2 = 0$ is discrete, faithful, and is a uniformisation of the Whitehead link complement.*

Remark 4.
(1) In Theorem 2, faithfulness is meant when the source group is $\mathbb{Z}_3 * \mathbb{Z}_3$. Of course the representation isn't faithful as a representation of π.
(2) The uniformisation described by Schwartz in [18, 19] is not the same as the one above. In fact, in his case the boundary parabolics aren't both unipotent (actually, only one is).
(3) The proof of Theorem 2 is made by constructing a fundamental domain for the action of the group on $\mathbf{H}^2_{\mathbb{C}}$. Intersecting this fundamental domain with $\partial_\infty \mathbf{H}^2_{\mathbb{C}}$ we prove that the manifold at infinty is the Whitehead link complement by describing an octahedron with face identifications analogous to the one described in Thurston's notes [20]. It turns out that the representation ρ_0 has a great deal of symmetry. This makes the geometric analysis of its action much simpler than what Schwartz did in [18, 19].
(4) The character variety of $\mathbb{Z}_3 * \mathbb{Z}_3$ in SU(2,1) has been analysed by Acosta in [1]. It contains many points that are geometrically significant, among which the uniformisation of the figure eight knot complement obtained by Deraux and Falbel in [6], and its deformations [5].

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Finiteness and Rigidity for compact 3-manifolds with bounded entropy

Filippo Cerocchi

(joint work with Andrea Sambusetti)

We consider the set $\mathscr{M}^{\partial}_{ngt}(E,D)$ (resp. $\mathscr{M}_{ngt}(E,D)$) of compact — with possibly empty boundary and no spherical boundary components — (resp. closed), orientable, non-geometric, Riemannian 3-manifolds with torsionless fundamental group whose volume entropy and diameter are bounded from above by two positive constants E and D respectively. We recall that the entropy of a Riemannian manifold X is defined as

$$\mathrm{Ent}(X) = \limsup_{R\to+\infty} \frac{1}{R} \log \Big(\mathrm{Vol}(\widetilde{B}(\tilde{x}, R)) \Big)$$

where $\widetilde{B}(\tilde{x}, R)$ is the ball of radius R centered at the point $\tilde{x} \in \widetilde{X}$ in the Riemannian universal cover of X. We prove the following:

Systolic estimate. For any $X \in \mathscr{M}^{\partial}_{ngt}(E, D)$ we have:

$$\mathrm{sys}\,\pi_1(X) \geq \frac{1}{E} \cdot \log\left(1 + \frac{4}{e^{26\,E\,D} - 1}\right) =: s_0(E; D) \tag{1}$$

The previous estimate stems from a general result concerning the existence of rank 2 free sub(semi)groups with a prescribed generator in torsion free groups acting non-elementarily and acylindrically on a simplicial tree (see [CS1] section §2). The existence of such actions for compact, non-geometric 3-manifold groups has been established by Wilton and Zalesskii in [Wi-Za] (see also [Cer2]). Inequality (1) generalizes an analogous lower bound for the homotopy 1-systole of manifolds whose fundamental group splits as a free product ([Cer1]).

As a byproduct of Gromov's Isosystolic Inequality ([Gro1]) and of inequality (1) we obtain the following:

Volume estimate. Let $X \in \mathscr{M}_{ngt}(E, D)$ and assume that X is not homeomorphic to the connected sum of k copies of $S^2 \times S^1$. Then there exists a constant c (independent of X) such that

$$\mathrm{Vol}(X) \geq c \cdot (s_0(E, D))^3 \tag{2}$$

Let us make few comments about these results:

- (i) the lack of S^2 in the boundary of the manifolds in $\mathscr{M}^{\partial}_{ngt}(E, D)$ is not necessary for the systolic estimate, but is necessary for all the subsequent rigidity and finiteness statements;
- (ii) the torsionless assumption is necessary in the systolic estimate (see [Cer1]);
- (iii) for any geometric manifold whose interior does not admit a complete metric locally isometric to $\mathbb{H}^3$ there exists a sequence of metrics collapsing the systole with uniformly bounded entropy and diameter;
- (iv) to the knowledge of the author the existence of a uniform estimate of type (1) for the entire class of 3-manifolds of hyperbolic type is still an open problem, and the estimates that we possess depends on the injectivity radius of the hyperbolic metric (see [BCG2]);
- (v) the volume estimate does not hold for the connected sum of k-copies of $S^2 \times S^1$. Actually, it is possible to show that on such a manifold there exists a sequence of metrics with uniformly bounded entropy and diameter, systole bounded away from zero and arbitrarily small volume.
- (vi) an analogous systolic estimate holds in general for any compact n-manifold whose fundamental group is torsionless and admits a k-acylindrical splitting (see [CS1]).

For a family $\mathscr{M}$ of compact Riemannian manifolds which satisfy a uniform lower bound on the homotopy 1-systole there exists a critical distance δ_0, depending on this lower bound, such that if $X, Y \in \mathscr{M}$ and $d_{GH}(X,Y) < \delta_0$ then $\pi_1(X) \cong \pi_1(Y)$ (see [Tus], [So-We]). We thus obtain a local π_1-rigidity result:

π_1-rigidity. Let $\delta_0(E,D) = \frac{s_0(E,D)}{40}$. If $X, Y \in \mathscr{M}^{\partial}_{ngt}(E,D)$ and $d_{GH}(X,Y) < \delta_0$, then $\pi_1(X) \cong \pi_1(Y)$.

Since we are dealing with compact 3-manifolds the local rigidity of the fundamental group provides stronger informations:

Local rigidity statements. Let $X, Y \in \mathscr{M}^{\partial}_{ngt}(E,D)$ and let $d_{GH}(X,Y) < \delta_0$:

(i) if X and Y are both irreducible then they are homotopy equivalent;
(ii) if X is irreducible with incompressible boundary then X and Y are homotopy equivalent;
(iii) if X is irreducible and closed then X and Y are diffeomorphic.

Remarks.

(1) There exists a sequence $\{(X_k, Y_k)\}_{k\in\mathbb{N}}$ of pairs of non homotopy equivalent, reducible, orientable, closed Riemannian 3-manifolds whose entropy and diameter are uniformly bounded in k such that $d_{GH}(X_k, Y_k) \to 0$.
(2) The incompressibility of the boundary is necessary in (ii).

Even though in the reducible case as well as in presence of boundary components a rigidity statement for the diffeomorphism type does not hold, we can still prove the local finiteness of the diffeomorphism type. In [Swa] Swarup proved that there exists only a finite number of pairwise non-homeomorphic irreducible, compact 3-manifold with a given fundamental group. As a consequence we have:

Local finiteness. Let $X \in \mathscr{M}^{\partial}_{ngt}(E,D)$, the number of diffeomorphism types in $B_{GH}(X,\delta_0) \cap \mathscr{M}^{\partial}_{ngt}(E,D)$ is finite.

Combining this local finiteness result with Gromov's Precompactness Theorem we provide a simple proof of the following:

Global finiteness with a lower Ricci curvature bound. Let $\mathscr{R}_{ngt}(k,D)$ be the set of closed, orientable, non-geometric Riemannian 3-manifolds with torsionless fundamental group, satisfying $\mathrm{Ricci}_X \geq -2\,k^2$ and $\mathrm{diam}(X) \leq D$. Then $\mathscr{R}_{ngt}(k,D)$ contains a finite number of diffeomorphism types.

This result is to compare with Zhu's finiteness theorem [Zhu]. We pay the price of dropping the assumption of a lower bound on the volume by restricting ourselves to the class of non-geometric 3-manifolds with torsionless fundamental group. A more refined finiteness result will appear in [CS2]. We replace the lower Ricci curvature bound by a much weaker upper bound on the entropy and we drop the torsionless assumption.

Global finiteness. Let $\mathscr{M}^{\partial}_{ng}(E,D)$ be the set of compact — with possibly empty boundary and with no spherical boundary component —, orientable, non-geometric, Riemannian 3-manifolds, whose entropy and diameter are uniformly bounded by E and D respectively. The number of diffeomorphism types in $\mathscr{M}^{\partial}_{ng}(E,D)$ is finite.

It is worth noticing that the proof of the latter result does not rely on the systolic estimates or on the consequent local rigidity results. Finally we announce the following precompactness result which will also appear in [CS2]:

Precompactness Theorem. Let $\mathscr{M}^{-}_{ng}(E,D)$ be the set of closed, orientable, nonpositively curved Riemannian 3-manifold with entropy and diameter bounded respectively by E and D. The set $\mathscr{M}^{-}_{ng}(E,D)$ is precompact in the Gromov-Hausdorff topology.

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Supersymmetric string vacua and geometric flows

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(joint work with Sebastien Picard and Xiangwen Zhang)

1. The Hull-Strominger system

It is a fundamental problem in string theory to identify string vacua which are still space-time supersymmetric. In 1986, C. Hull [7] and A. Strominger [14] independently proposed the following system of equations for supersymmetric vacua of the heterotic string. Let Y be a compact 3-dimensional complex manifold, equipped with a nowhere vanishing holomorphic $(3,0)$-form Ω. We look then for a vector bundle $E \to Y$, and Hermitian metrics ω on Y and $H_{\bar\alpha\beta}$ on E satisfying the equations

$$F^{2,0} = F^{0,2} = 0, \quad \omega^2 \wedge F^{1,1} = 0 \tag{1}$$

$$d^\dagger \omega = i(\partial - \bar\partial) \log \|\Omega\|_\omega \tag{2}$$

$$i\partial\bar\partial\omega - \frac{\alpha'}{4}(\mathrm{Tr}(Rm \wedge Rm) - \mathrm{Tr}(F \wedge F)) = 0 \tag{3}$$

Here $F^{p,q}$ denotes the (p,q)-components of the 2-form F. The expressions Rm and F denote the curvatures of the metrics ω and $H_{\bar\alpha\beta}$, viewed as 2-forms valued in $End(T^{1,0}(Y))$ and $End(E)$ respectively. The equation (1) implies that F should be the curvature of the Chern unitary connection E defined by $H_{\bar\alpha\beta}$. Since ω may not be Kähler, there is a one-parameter line of natural unitary connections on $T^{1,0}(Y)$ defined by ω, passing through the Chern unitary connection and the Bismut connection. We shall specify our choice as we go along. The expression $\|\Omega\|_\omega$ is the norm of Ω with respect to the metric ω, $\|\Omega\|_\omega^2 = i\Omega \wedge \bar\Omega(\omega^3/3!)^{-1}$.

The equation (1) is the familiar Hermitian-Yang-Mills equation for the Hermitian metric $H_{\bar\alpha\beta}$ on E. If the conformal class of ω is known, its solvability is, by the Donaldson-Uhlenbeck-Yau theorem extended to the Gauduchon case by Li and Yau and Lübke and Teleman, equivalent to the stability of E with respect to ω.

It was pointed out by Li and Yau [8] that the equation (2) is equivalent to

$$d(\|\Omega\|_\omega \omega^2) = 0. \tag{2'}$$

This is a conformal version of the notion of "balanced metric", defined by Michelsohn [9] as a metric on n-dimensional complex manifolds satisfying the condition $d(\omega^{n-1}) = 0$. The notion of balanced metric turns out to be natural in algebraic and complex geometry, as it is preserved under birational modifications [1]. The Hull-Strominger system can thus be interpreted as a notion of "canonical metric" for conformally balanced manifolds.

The equation (3) is the main equation accounting for both the novelty and the difficulty in solving the Hull-Strominger system. It originates from the famous Green-Schwarz anomaly cancellation mechanism required for the consistency of superstring theory. However, unlike more familiar notions of canonical metrics, which are defined by linear conditions in the curvature tensor, it involves its square of the curvature tensor.

Calabi-Yau Kähler manifolds can be viewed as a special solution of the Hull-Strominger system. Indeed, assume that Y is Kähler, with a nowhere vanishing holomorphic $(3,0)$-form Ω. Take $E = T^{1,0}(Y)$, ω a Kähler form on Y, and set $H_{\bar{\alpha}\beta} = 0$. Then the equation (3) is trivially satisfied. The equation (1) reduces to

$$Ric(\omega) = 0.$$

Since $Ric(\omega) = i\partial\bar{\partial}\log\|\Omega\|_\omega^2$, this implies that $\|\Omega\|_\omega$ is constant. The equation (2) follows then immediately from the fact that ω is Kähler. Thus the Hull-Strominger system reduces to finding a Kähler metric with vanishing Ricci curvature, which has been done by Yau in his solution of the Calabi conjecture. The fact that Calabi-Yau Kähler manifolds provide a solution of the Hull-Strominger system is just a rephrasing of the famous original work of Candelas, Horowitz, Strominger, and Witten [2].

Many special solutions to the Hull-Strominger system are now known. They were obtained by a wide variety of methods, including perturbations from Calabi-Yau solutions, formal duality constructions from physics, and geometric constructions (see e.g. [3] for recent results, and references therein). But the solution with the greatest influence on the present work is actually the first non-Kähler solution, found by Fu and Yau [5] by partial differential equations. By considering the toric fibrations $Y \to X$ on K3 surfaces X constructed by Goldstein and Prokushkin using an earlier construction of Calabi and Eckmann, Fu and Yau showed that the Hull-Strominger system can be reduced to a single equation for a scalar function u on X. This equation is a new equation of Monge-Ampère type, which they managed to solve, despite the fact that the right hand side involves both the unknown and its gradient [5, 6].

2. The Anomaly Flow

The goal of the present work is to develop a general method for finding general solutions of the Hull-Strominger system. An initial difficulty is to implement the conformally balanced condition (3), in the absence of a lemma such as the $\partial\bar{\partial}$-Lemma in the Kähler case. We circumvent this difficulty by introducing the following flow, which we call the Anomaly flow,

$$\partial_t(\|\Omega\|_\omega\omega^2) = i\partial\bar{\partial}\omega - \frac{\alpha'}{4}(\mathrm{Tr}(Rm\wedge Rm) - \mathrm{Tr}(F\wedge F)) \tag{4}$$

$$H^{-1}\partial_t H = -3\frac{\omega^2\wedge F}{\omega^3} \tag{5}$$

The equation (5) is the well-known Donaldson heat flow, whose stationary points satisfy the equation (1). The really new equation is the equation (4), the point of which is to preserve the conformally balanced condition, since the right hand side of (4) is a closed $(2,2)$-form. We have [10]

Theorem 1 Let $E \to Y$ be a holomorphic vector bundle over a 3-fold Y with nowhere vanishing holomorphic $(3,0)$-form Ω. Consider the flow (4),(5) with Chern unitary connection, and initial data $\omega(0)$ and $H(0)$, where $\omega(0)$ is conformally balanced. If $|\alpha' Rm(\omega(0))| < 1$, then the flow admits a solution in $t \in [0,T)$ for some $T > 0$.

The difficult questions for the flow are its long-time existence and convergence. For this, we need a more explicit expression for the flow in terms of a flow of metrics instead of $(2,2)$-forms $\|\Omega\|_\omega \omega^2$ [11]

Theorem 2 Consider the anomaly flow with a conformally balanced initial metric and the Chern unitary connection. Then the flow is given by

$$\partial_t g_{\bar{k}j} = \frac{1}{2\|\Omega\|_\omega}\Big\{ - \tilde{R}_{\bar{k}j} + g^{s\bar{r}} g^{p\bar{q}} T_{\bar{q}sj} \bar{T}_{p\bar{r}\bar{k}} - \alpha' g^{s\bar{r}} (R_{[\bar{k}s}{}^{\alpha}{}_{\beta} R_{\bar{r}j]}{}^{\beta}{}_{\alpha} - \Phi_{\bar{k}s\bar{r}j}) \Big\} \tag{6}$$

Here $\tilde{R}_{\bar{k}j} = g^{p\bar{q}} R_{\bar{q}p\bar{k}j}$ is the Chern-Ricci tensor, $i\partial\omega = \frac{1}{2} T_{\bar{k}jm} dz^m \wedge dz^j \wedge d\bar{z}^k$ is the torsion tensor, and we have set $\Phi = \mathrm{Tr}(F \wedge F)$. The bracket $[,]$ denote anti-symmetrization in each of the two sets of barred and unbarred indices.

The equation (6) suggests that, perhaps surprisingly, the Anomaly Flow can be interpreted as a next-order modification of the Ricci flow by quadratic terms in the curvature tensor.

3. Special Cases of the Anomaly Flow

The preceding comparison of the Anomaly flow with the Ricci flow shows that it will be a much more difficult flow to study. Nevertheless, the following partial results suggest that it is a well-behaved flow, which should provide a viable approach to the solution of the Hull-Strominger system.

(a) The case $\alpha' = 0$

This case can be viewed as an intermediate case between the Ricci flow and the full Anomaly flow. It is still of geometric interest, and its stationary points are pluriclosed metrics and hence Kähler, since they are also automatically conformally balanced. We have

Theorem 3 Assume that the flow exists for $t \in [0, \frac{1}{A}]$ and that

$$|Rm| + |DT| + |T|^2 \leq A, \quad z \in X.$$

Then for any $k \in \mathbf{N}$, there exists a constant C_k depending on a uniform lower bound for $\|\Omega\|_\omega$ so that

$$|D^k Rm| \leq C_k A t^{-\frac{k}{2}}, \qquad |D^{k+1} T| \leq C_k A t^{-\frac{k}{2}}.$$

This implies that the flow exists for all time $t \geq 0$, unless there is a finite time T and a sequence (z_j, t_j) with $t_j \to T$, and either $\|\Omega(z_j,t_j)\|_{\omega_j} \to 0$, or

$$(|Rm| + |DT| + |T|^2)(z_j, t_j) \to \infty.$$

(b) The case of Goldstein-Prokushkin fibrations

Let $(X, \hat{\omega})$ be a Calabi-Yau surface, with Ricci-flat metric $\hat{\omega}$, and holomorphic form Ω_X normalized so that $\|\Omega_X\|^2_{\hat{\omega}} = 1$. Given any two forms $\omega_1, \omega_2 \in 2\pi H^2(X, \mathbf{Z})$ with $\omega_1 \wedge \hat{\omega} = \omega_2 \wedge \hat{\omega} = 0$, Goldstein and Prokushkin (2004) construct a toric fibration $\pi : Y \to X$, equipped with a $(1,0)$-form θ on Y satisfying $\partial\theta = 0$, $\bar{\partial}\theta = \pi^*(\omega_1 + i\omega_2)$. Furthermore, the form $\Omega = \sqrt{3}\,\Omega_X \wedge \theta$ is a holomorphic nowhere vanishing $(3,0)$-form on Y, and for any scalar function u on X, the $(1,1)$-form $\omega_u = \pi^*(e^u \hat{\omega}) + i\theta \wedge \bar{\theta}$ is a conformally balanced metric on Y.

Fu and Yau [5,6] looked for a solution of the Hull-Strominger system on $Y, \pi^*(E)$ under the form $(\omega_u, \pi^*(H))$, where H is a Hermitian-Yang-Mills metric on a stable vector bundle $E \to (X, \hat{\omega})$. Then the only equation left to solve is the anomaly equation (3). In a key calculation, they showed that this equation descends to a Monge-Ampère type equation on X, which they showed admits a solution if and only if an integrability condition, depending only on the (X, ω_1, ω_2) data, is satisfied. A key test for the Anomaly flow is whether it can recapture the solution of Fu-Yau in this setting. In [12], we show that it does:

Theorem 4 Consider the Anomaly flow

$$\partial_t(\|\Omega\|_\chi \chi^2) = i\partial\bar{\partial}\chi - \frac{\alpha'}{4}\mathrm{Tr}(Rm(\chi) \wedge Rm(\chi) - F \wedge F)$$

on a Goldstein-Prokushkin fibration $\pi : Y \to X$, with initial data $\chi(0) = \pi^*(M\hat{\omega}) + i\theta\bar{\theta}$, where M is a positive constant. Assume the integrability condition on (X, ω_1, ω_2). Then there exists $M_0 > 0$, so that for all $M \geq M_0$, the flow exists for all time, and converges to a metric ω_∞ with $(\omega_\infty, \pi^*(H))$ satisfying the Hull-Strominger system.

The proof makes fundamental use of the maximum principle for non-linear heat equations, but with the diffusion operator given by

$$\Delta_F = F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}, \qquad F^{p\bar{q}} = g^{p\bar{q}} + \alpha'\|\Omega\|^3_\omega \tilde{\rho}^{p\bar{q}} - \frac{\alpha'}{2}(Rg^{p\bar{q}} - R^{p\bar{q}})$$

instead of just the Laplacian, as in the case of the Ricci flow. We choose the initial data so that $|\alpha' Rm(\chi)| << 1$ and the diffusion operator Δ_F is positive definite. The key and most difficult step is to prove that this condition is preserved along the flow. Remarkably, this can be done independently of the sign of α', so that we recover at one stroke both results of Fu and Yau for $\alpha' > 0$ [5] and $\alpha' < 0$ [6].

(c) The case of unimodular Lie groups

Finally, we discuss the case of the Anomaly flow on unimodular Lie groups, where all invariant metrics are automatically balanced, and the flow can be reduced as a system of ordinary differential equations. The stationary points of the flow have been found by Fei and Yau [4], and our main interest is in the behavior of the flow. In this case, we consider general unitary connections, defined by

$$\nabla_{\bar{k}} W^p = \partial_{\bar{k}} W^p + \kappa \bar{T}_{r\bar{k}}{}^p W^r$$

where κ is a real parameter. The values $\kappa = 0$ and $\kappa = 1$ correspond respectively to the Chern and the Bismut connection. We have [13]

Theorem 5 Assume that $\alpha'\tau > 0$, where $\tau = 2\kappa^2(2\kappa - 1)$.

(1) When $Y = \mathbf{C}^3$, any metric is a stationary point for the flow, and the flow is consequently stationary for any initial metric $\omega(0)$.

(2) When Y is nilpotent, there is no stationary point. Consequently the flow cannot converge for any initial metric. If the initial metric is diagonal, then the metric remains diagonal along the flow, the lowest eigenvalue is constant, while the other two eigenvalues tend to $+\infty$ at a constant rate.

(3) When Y is solvable, the stationary points of the flow are the metrics $g_{\bar{a}b}$

$$g_{\bar{1}2} = \overline{g_{\bar{2}1}} = 0, \qquad \frac{\alpha'\tau}{4} g^{3\bar{3}} = 1.$$

The Anomaly flow is asymptotically instable near any stationary point. However, the condition $g_{\bar{1}2} = \overline{g_{\bar{1}2}} = 0$ is preserved along the flow, and for any initial metric satisfying this condition, the flow converges to a stationary point.

(4) When $Y = SL(2, \mathbf{C})$, there is a unique stationary point, given by

$$g_{\bar{a}b} = \frac{\alpha'\tau}{2}\,\delta_{ab}.$$

The linearization of the flow at the fixed point admits both positive and negative eigenvalues. In particular, the flow is asymptotically instable.

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Steady Kähler-Ricci solitons on crepant resolutions of $\mathbb{C}^n/G$

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(joint work with Olivier Biquard)

The 'fixed points' of the Ricci flow are *steady Ricci solitons*. Such objects, natural generalizations of Ricci-flat metrics, are pairs (g, X) of a Riemannian metric and a vector field, satisfying the elliptic partial differential equation

$$\mathrm{Ric}(g) + \tfrac{1}{2}\mathcal{L}_X g = 0.$$

They arise in the study of the Ricci flow as models of singularities, and as backward limits of ancient solutions. They are also critical points, in a suitable sense [Has11], of the Perelman $\mathcal{F}$-functional, and thus can be considered canonical among all such pairs (g, X).

Steady Ricci solitons which are not Ricci-flat must be noncompact. The few known examples include several which are Kähler, with holomorphic vector field: Hamilton's *cigar soliton* [Ham88] on $\mathbb{C}$, H.-D. Cao's generalizations [Cao96] on $\mathbb{C}^n$ and $K_{\mathbb{CP}^{n-1}}$, and further generalizations by Dancer-M. Wang and B. Yang. Non-Kähler examples include the well-known constructions of Bryant and of Ivey. All these examples – and, we believe, all known examples – are highly symmetric, and the soliton metric is given either explicitly or by solving an ODE.

In the new work outlined in this talk, we use PDE methods to construct new steady Kähler-Ricci solitons (M, ω, X), in all complex dimensions $n \geq 2$ (real dimensions $2n \geq 4$), of infinitely many topological types in dimensions 2 and 3 at least. Like all steady Kähler-Ricci solitons, they have first Chern class

$$c_1(M) = [\mathrm{Ric}(\omega)] = [-\tfrac{1}{2}\mathcal{L}_X\omega] = 0.$$

Our construction proceeds by taking Joyce's well-known family [Joy00] (see also [TY90,TY91]) of Ricci-flat Kähler metrics on *crepant resolutions* of orbifolds $\mathbb{C}^n/G$ (which automatically have $c_1(M) = 0$), and modifying their metrics near infinity by *gluing* them to a G-quotient of Cao's soliton on $\mathbb{C}^n$. This produces a Kähler-Ricci soliton metric, whose drift vector field is the radial vector field $X = -2r\partial/\partial r$, in each sufficiently small Kähler class on M.

Joyce's construction applies to crepant resolutions M of orbifolds $\mathbb{C}^n/G$, where G is a finite subgroup of $SU(n)$ which acts freely on $\mathbb{C}^n \setminus \{0\}$. We, in the soliton setting, require the additional technical hypothesis that the crepant resolution M of $\mathbb{C}^n/G$ be *equivariant* with respect to the action of the radial vector field X; that is, that X extend smoothly to a vector field on M.

The class of such manifolds M is quite broad. It includes the unique minimal resolution of each 2-dimensional such orbifold (the Kleinian singularities), and at least one crepant resolution of each 3-dimensional such orbifold. It also includes the crepant resolution $M = K_{\mathbb{CP}^{n-1}}$ of $\mathbb{C}^n/\mathbb{Z}^n$, in each dimension $n \geq 2$; in this case our construction recovers Cao's family of solitons on the manifolds $K_{\mathbb{CP}^{n-1}}$.

Such gluing constructions have been performed before in other geometric settings, with pioneering constructions performed by Kapouleas (minimal surfaces), Taubes (anti-self-dual metrics), and Joyce (metrics of special holonomy). Close antecedents for our work include Nguyen's recent construction [Ngu09, Ngu13] of mean curvature flow self-translators, and Biquard-Minerbe's construction [BM11] of noncompact Calabi-Yau surfaces of several different asymptotic behaviours.

The analogous construction problem for expanding Kähler-Ricci solitons has also been recently studied [Sie13, CD16]; see Conlon's abstract in this report.

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Minimal two-spheres in three-spheres

ROBERT HASLHOFER
(joint work with Dan Ketover)

The min-max method goes back to Birkhoff, who in 1917 proved:

Theorem 1 (Birkhoff [1]). *Any closed Riemannian two-sphere contains at least one closed geodesic.*

Loosely speaking, Birkhoff considered sweepouts of the two-sphere by closed curves, and argued that the longest slice in a sweepout that is pulled tight is a closed geodesic. There are also higher non-trivial families of curves one can consider to produce more geodesics:

Theorem 2 (Lusternik-Schnirelmann [4, 9]). *Any closed Riemannian two-sphere contains at least three simple closed geodesics.*

In one higher dimension, one can consider sweepouts of three-spheres by two-spheres, and hope to produce an embedded minimal two-sphere. In 1983, Simon and Smith carried this out (adapting the more general min-max theory of Almgren and Pitts to the case of surfaces with fixed topology) and proved:

Theorem 3 (Simon-Smith [10])**.** *Let M be a three-manifold diffeomorphic to $\mathbb{S}^3$. Then M contains an embedded minimal two-sphere.*

In analogy with the case of simple closed geodesics on two-spheres, there are also higher parameter families of two-spheres on three-spheres that one can consider. One might hope that the families detecting the relevant cohomology classes $\alpha, \ldots, \alpha^4$ produce via min-max four distinct minimal two-spheres. The major difficulty is the phenomenon of *multiplicity*. Namely, it could happen that the min-max spheres associated with the second, third and fourth family, just give the sphere associated to the first family counted with higher integer multiplicities.

Using combined efforts from min-max theory and mean curvature flow we prove:

Theorem 4 (Haslhofer-Ketover [5])**.** *Let M be a three-manifold diffeomorphic to $\mathbb{S}^3$ and endowed with a bumpy metric. Then M contains at least 2 embedded minimal two-spheres. More precisely, exactly one of the following alternatives holds:*

(1) M contains at least 1 stable embedded minimal two-sphere, and at least 2 embedded minimal two-spheres of index one.
(2) M contains no stable embedded minimal two-sphere, at least 1 embedded minimal two-sphere Γ_1 of index one, and at least 1 embedded minimal two-sphere Γ_2 of index two. In this case, $|\Gamma_2| < 2|\Gamma_1|$.

We note that White [11] previously proved the existence of at least 2 minimal two-spheres in the special case that M has positive Ricci curvature.

A natural family of examples to illustrate Theorem 4 are ellipsoids. Namely, given $a > b > c > d > 0$, consider the ellipsoid

$$E(a,b,c,d) := \left\{ \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} + \frac{x_4^2}{d^2} = 1 \right\} \subset \mathbb{R}^4.$$

Observe that E contains at least 4 minimal 'planar' two-spheres, which are given by the intersection with the coordinates hyperplanes $\{x_i = 0\}$. However, by the area estimate $|\Gamma_2| < 2|\Gamma_1|$, if $a \gg b$ the second minimal two-sphere $\Gamma_2(a) \subset E$ produced by Theorem 4 is not planar. Moreover, as $a \to \infty$, the minimal two-spheres $\Gamma_2(a)$ converge as varifolds to a minimal two-sphere with multiplicity two.

Let us now sketch the main ideas of the proof of Theorem 4.

If M admits a stable embedded minimal two-sphere, then the manifold is a kind of dumbbell. Considering 1-parameter sweep-outs of both halves and using [7] we show that each half contains an unstable two-sphere of index one in its interior.

Let us now consider the case that M does not contain any stable embedded minimal two-spheres. Using Simon-Smith's existence theorem (Theorem 3) we obtain 1 embedded minimal two-sphere Γ_1 of index one. Sliding the Simon-Smith sphere a bit to both sides we can decompose $M = D_1 \cup Z \cup D_2$ where Z is the short cylindrical region obtained by sliding the Simon-Smith sphere around, and D_1 and D_2 are smooth embedded 3-discs with mean convex boundary. To proceed, we prove the following general theorem establishing the existence of smooth mean convex foliations in three-manifolds:

Theorem 5 (Haslhofer-Ketover [5]). *Let $D \subset M^3$ be a smooth three-disc with mean convex boundary. Then exactly one of the following alternatives holds true:*

(1) There exists an embedded stable minimal two-sphere $\Gamma \subset Int(D)$.
(2) There exists a smooth foliation $\{\Sigma_t\}_{t\in[0,1]}$ of D by mean convex embedded two-spheres.

Let us first explain how to finish the proof of Theorem 4 using Theorem 5.

Recalling that $M = D_1 \cup Z \cup D_2$ and using the foliations of D_1 and D_2 produced by Theorem 5 we can build an optimal foliation of M, by which we mean a smooth foliation $\{\Sigma_t\}_{t\in[-1,1]}$ of M by two-spheres so that the Simon-Smith sphere sits in the middle of the foliation as Σ_0 and all other slices have less area. From the one parameter family $\{\Sigma_t\}$ we can then form a two parameter family $\{\Sigma_{s,t}\}$ detecting α^2 and such that

$$\sup_{s,t} |\Sigma_{s,t}| < 2|\Gamma_1|. \tag{1}$$

Roughly speaking $\Sigma_{s,t}$ looks like Σ_s connected to Σ_t along a small neck, which we open up near $(s,t) \approx (0,0)$, using the catenoid estimate from [8].

The area bound (1) ensures that min-max for our two-parameter family doesn't simply produce Γ_1 with multiplicity two. We conclude that there exists an embedded minimal two-sphere Γ_2 with $|\Gamma_1| < |\Gamma_2| < 2|\Gamma_1|$ and index two.

To obtain some intuition for Theorem 5 (which is of independent interest), imagine that the disc D evolves by mean curvature flow. Recall that mean-convexity is preserved under mean curvature flow. In the simplest possible scenario, the mean curvature flow of D remains smooth and either becomes extinct in finite time in a round point, giving the foliation from (2), or converges for $t \to \infty$ to a minimal embedded two-sphere, giving (1). Of course, in general the situation is more complicated since the mean curvature flow typically develops local singularities. One way to continue the flow through singularities is given by the level set method, and in fact our proof shows that case (2) happens if and only if the level set flow becomes extinct in finite time. The main issue however is that the foliation produced by the level set flow is in general singular.

To produce a smooth foliation instead of a singular foliation we use mean curvature flow with surgery. Mean curvature flow with surgery in general ambient manifolds has been constructed first by Brendle-Huisken [2]. However, since we also need a canonical neighborhood theorem for our application we instead extend the approach from Haslhofer-Kleiner [6] to the setting of general ambient manifolds. We then combine the existence theorem, the canonical neighborhood theorem, and methods from the recent topological application of mean curvature flow with surgery by Buzano-Haslhofer-Hershkovits [3] to produce the desired smooth foliation.

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Equivariant Minimal Surfaces in Symmetric Spaces of Non-Compact Type

Nicolas Tholozan

(joint work with Brian Collier and Jérémy Toulisse)

Let Σ be a closed surface of genus at least 2 and ρ a homomorphism from $\pi_1(\Sigma)$ to a semi-simple Lie group G. Let X denote the symmetric space G/K, where K is a maximal compact subgroup. Recall that X carries a G-invariant Riemannian metric g_X of non-positive curvature.

Since X is contractible, there always exist smooth ρ-equivariant maps from the universal cover $\widetilde{\Sigma}$ of Σ to X, and all these maps are homotopic through equivariant maps. A natural question is whether one can find a ρ-equivariant map whose image is a minimal surface. For quasi-Fuchsian representations into $\mathrm{PSL}(2, \mathbb{C})$, this question can be seen as a "Plateau problem at infinity", namely, finding a ρ-invariant minimal disc in the hyperbolic 3-space whose boundary at infinity is given by the limit circle of the representation.

A classical strategy to address this problem, which dates back at least to the works of Schoen–Yau [8] and Sacks–Uhlenbeck [7], consists in minimizing a certain energy functional on the Teichmüller space of complex structures on Σ.

To avoid complications let us assume from now on that the image of ρ is Zariski dense in G. Then, for any complex structure J on Σ, one can find a unique

ρ-equivariant *harmonic map* $f_{J,\rho}$ from $(\widetilde{\Sigma}, J)$ to X. Let $\mathbf{E}_\rho(J)$ denote the total energy of $f_{J,\rho}$. The following are equivalent:

(*i*) The map $f_{J,\rho}$ is conformal, that is, the pullback metric $f_{J,\rho}^* g_X$ is in the conformal class defined by J,

(*ii*) The map $f_{J,\rho}$ is a *branched minimal immersion*, meaning that $f_{J,\rho}$ in an immersion whose local image is a minimal surface in the complement of a finite subset of Σ,

(*iii*) The complex structure J is a critical point of the energy functional $\mathbf{E}_\rho$ defined on the Teichmüller space of Σ.

When the representation ρ is a quasi-isometric embedding, the energy functional is proper and thus admits a global minimum. The above strategy thus guaranties the existence of an equivariant branched minimal immersion for a wide class of representations with interesting geometric properties, including quasi-Fuchsian representations into $\mathrm{PSL}(2,\mathbb{C})$ (or more generally convex cocompact representations into Lie groups of rank 1) as well as *Hitchin representations* into real split Lie groups (see [5]) and *maximal representations* into Hermitian Lie groups (see [1]).

A drawback of this strategy is that the minimal surface obtained may not be embedded, or not even an immersion. Moreover, the energy functional may have several critical points, giving rise to several branched minimal immersions. For instance, one can find quasi-Fuchsian representations into $\mathrm{PSL}(2,\mathbb{C})$ with arbitrarily many branched minimal immersions (see [4]). Among these, only the global energy minimlizer is known to be an embedding.

In contrast, for Hitchin representations into real split Lie groups of rank 2 (namely $\mathrm{SL}(3,\mathbb{R})$, $\mathrm{Sp}(4,\mathbb{R})$ and G_2), Labourie proved in [6] that the branched minimal immersion is unique. For $\mathrm{SL}(3,\mathbb{R})$, it is known to be an embedding. In [3], we extend this result to maximal representations in rank 2.

Theorem 1. *Let ρ be a maximal representation of $\pi_1(\Sigma)$ into a Hermitian Lie group G of real rank 2. Then there exists a unique ρ-equivariant branched minimal immersion of $\widetilde{\Sigma}$ into the symmetric space of G. Moreover, this map is an embedding.*

Quasi-Fuchsian, Hitchin and maximal representations are the main exemples of *Anosov representations*, defined in [5]. These representations all admit a “limit circle” in some flag variety of G. Heuristically, the qualitative difference between quasi-Fuchsian representations on one side and Hitchin and maximal representations on the other side is that the limit circle of a quasi-Fuchsian representation can be arbitrarily “twisted”, while the limit curve of a Hitchin or maximal representation has more regularity (it is a Lipschitz curve with a certain “cyclic order”). This suggests that both Labourie’s and our result should be true in higher rank.

Let us finally explain briefly the steps of the proof of Theorem 1:

- We start by reducing to the case where $G = \mathrm{SO}(2,n)$ (using the classification from [1] of the possible Zariski closures of maximal representations).

- Let us now choose a complex structure J on Σ such that $f_{J,\rho}$ is a branched minimal immersion. We transcribe this property, as well as the maximality of ρ, in terms of the *Higgs bundle* on (Σ, J) associated to ρ.
- The particular structure of this Higgs bundle allows us to construct a ρ-equivariant *minimal spacelike embedding* $\hat{f}_{J,\rho}$ of $\widetilde{\Sigma}$ into the homogeneous space $\mathrm{SO}(2,n)/\mathrm{SO}(2,n-1)$, which is pseudo-Riemannian of signature $(2, n-1)$. Moreover, the map $f_{J,\rho}$ is the *Gauss map* of $\hat{f}_{J,\rho}$. This shows in particular that $f_{J,\rho}$ is an embedding.
- Finally, we prove that there is at most one equivariant minimal spacelike embedding by adapting an argument of Bonsante–Schlenker [2] for the case $n = 2$ (for which this pseudo-Riemannian space is the 3-dimensional *anti-de Sitter space*). The key fact here is that the "spacelike" condition gives a strong control on the minimal embedding, which allows to "maximize the distance" between two such embeddings and get a contradiction by a maximum principle.

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Asymptotic geometric properties of Higgs bundle moduli spaces

Jan Swoboda

(joint work with Rafe Mazzeo, Hartmut Weiss and Frederik Witt)

In this talk, we present our resent results from [5] on the asymptotic geometry of the L^2 (or Hitchin) metric g_{L^2} on the moduli space $\mathcal{M}$ of irreducible solutions to the Hitchin self-duality equations

$$\mathcal{H}(A, \Phi) := (F_A + [\Phi \wedge \Phi^{*_h}], \bar{\partial}_A \Phi) = 0 \tag{1}$$

on a hermitian vector bundle (E, h) of degree 0 and rank 2 over a compact Riemann surface X of genus $\gamma \geq 2$, modulo unitary gauge transformations. The equations

(1) are a system of nonlinear PDEs for a unitary connection A on E and a so-called Higgs field $\Phi \in \Omega^{1,0}(X, \mathfrak{sl}(E))$. Here, F_A denotes the curvature of A and Φ^{*_h} is the hermitian conjugate of Φ. The study of these equations and the associated moduli space was initiated in the seminal article [3]. There, g_{L^2} has been introduced as the 'Weil-Petersson type' metric

$$g_{L^2}((\dot{A}_1, \dot{\Phi}_1), (\dot{A}_2, \dot{\Phi}_2)) = \int_X \langle \dot{A}_1, \dot{A}_2 \rangle + \langle \dot{\Phi}_1, \dot{\Phi}_2 \rangle \, dA$$

for a pair of tangent vectors $(\dot{A}_i, \dot{\Phi}_i)$ satisfying the Coulomb gauge condition.

The moduli space $\mathcal{M}$ is a noncompact manifold with singularities of dimension $12(\gamma - 1)$. It can also be identified as the moduli space of stable Higgs pairs $\{(\bar{\partial}_E, \Phi)\,\text{stable} \mid \bar{\partial}_E \Phi = 0\}$ modulo complex gauge transformations, as well as the twisted character variety of irreducible representations of $\pi_1(X)$ into $\mathrm{GL}(2, \mathbb{C})$ modulo conjugation. The fact that g_{L^2} is a hyperkähler metric reflects these various incarnations.

Many topological and geometric properties of $\mathcal{M}$ are now understood, and in the past few years a detailed picture has started to emerge about its asymptotic geometric structure at infinity. A key role is played by the space $\mathcal{M}_\infty$ of 'limiting configurations' which are solutions of the decoupled equations

$$F_A = 0, \quad [\Phi \wedge \Phi^{*_h}] = 0, \quad \bar{\partial}_A \Phi = 0$$

obtained as a limiting form of the Hitchin equations (1), again modulo unitary gauge transformations. These were initially defined and studied in [4], there for the subset of solutions for which the corresponding holomorphic quadratic differentials $q = \det \Phi$ have simple zeroes (the so-called free region, denoted $\mathcal{M}'$), and later in greater generality by Mochizuki [6]. The subset $\mathcal{M}'_\infty$ of limiting configurations over the space of holomorphic quadratic differentials with simple zeroes is the main building block for the construction of diverging families of solutions in the free region [4] and provide a natural compactification of $\mathcal{M}'$.

Entirely distinct from those developments, a remarkable conjectural picture of the asymptotic geometry of $\mathcal{M}$ has emerged from physics, and appears in the recent work by Gaiotto, Moore and Neitzke [2]. That work develops a formalism of spectral networks on Riemann surfaces, out of which they present a construction of a hyperkähler metric g_{GMN} which they conjecture to be precisely the L^2 metric. We refer to the survey paper by Neitzke [7] for an overview of this construction. Briefly, they assert that

$$g_{\mathrm{GMN}} \sim g_{\mathrm{sf}} + \mathcal{O}(e^{-\beta t})$$

where g_{sf} is a particular semiflat metric on $\mathcal{M}'$ and the remainder denotes terms which decay at some exponential rate as a certain radial variable t tends to infinity. Here the term 'semiflat' appeals to the fact that g_{sf} is induced by a special Kähler metric g_{sK} on the base $\mathcal{B}'$ (the set of holomorphic quadratic differentials with simple zeroes) of the Hitchin fibration $\det\colon \mathcal{M}' \to \mathcal{B}'$, with each fiber $\det^{-1}(q)$ being an intrinsically flat, half-dimensional complex torus. We give a brief outline of the construction of the latter metric, referring to the foundational article [1] for details. By definition, the term 'special' refers to a Kähler metric together with a real, torsionfree, flat symplectic connection ∇. In the present situation, it is associated with the spectral data of the points in $\mathcal{B}'$: Let S_q denote the spectral curve for $q \in \mathcal{B}'$, which is a smooth two-sheeted branched cover of X. Each S_q carries the Seiberg-Witten differential, a holomorphic $(1,0)$-form $\lambda_{\mathrm{SW}}(q)$ which is odd under the involution on S_q interchanging the two sheets. Fixing a local symplectic basis $\alpha_1(q), \dots, \alpha_m(q), \beta_1(q), \dots, \beta_m(q)$, $m = 3\gamma - 3$ of the odd first homology group $H_1(S_q; \mathbb{Z})^{\mathrm{odd}}$, a pair of local holomorphic coordinates on $\mathcal{B}'$, flat with respect to ∇, are provided by the period integrals

$$z_i(q) = \int_{\alpha_i(q)} \lambda_{\mathrm{SW}}(q), \quad w_i(q) = \int_{\beta_i(q)} \lambda_{\mathrm{SW}}(q), \quad i = 1, \dots, m.$$

In terms of these, the special Kähler form equals

$$\omega_{\mathrm{sK}} = -\frac{1}{4} \sum_i dz_i \wedge d\bar{w}_i + d\bar{z}_i \wedge dw_i.$$

We note that the associated special Kähler metric g_{sK} is an incomplete cone metric; a Kähler potential is given by $K(q) = \frac{1}{2} \int_X |q| \, dA$.

These two seemingly different constructions of hyperkähler metrics on $\mathcal{M}'$ lead naturally to the challenge of understanding the relationship of the Gaiotto-Moore-Neitzke metric g_{GMN} and the L^2 metric g_{L^2}. This goal has been settled in the recent preprint [5], where the following two results have been established.

Theorem 1. *Consider the space $\mathcal{M}'_\infty$ of limiting configurations over the space of holomorphic quadratic differentials with simple zeroes. It is possible to define a renormalized L^2 metric on this space, and this L^2 metric on $\mathcal{M}'_\infty$ is naturally identified with the Gaiotto-Moore-Neitzke semiflat metric g_{sf}.*

We then interpret the construction of large elements in $\mathcal{M}'$ from [4] as giving a coordinate system on this moduli space near the boundary of its compactification provided by the elements of $\mathcal{M}'_\infty$. This can be used to compute the coefficients of g_{L^2}. Leaving aside the analytical details involved there, it leads to the following conclusion.

Theorem 2. *The L^2 metric admits an asymptotic expansion*

$$g_{L^2} = g_{\mathrm{sf}} + t^2 \sum_{j=0}^{\infty} t^{-(2+j)/3} G_j + \mathcal{O}(e^{-\beta t})$$

as $t \to \infty$. Here each G_j is a symmetric two-tensor on $\mathcal{M}'$, independent of the radial variable t, and $G_0 \neq 0$.

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Collapsing in the Einstein flow

John Lott

The Einstein equation of general relativity is a weakly hyperbolic equation on a Lorentzian 4-manifold (M, g). Given a foliation of M by spatial hypersurfaces diffeomorphic to X, one can reduce Einstein's equation to a flow on triples (h, K, L) consisting of a Riemannian metric h on X, a symmetric 2-tensor field K on X and a positive function L on X. The metric g is recovered as $g = -L(t)^2 dt^2 + h(t)$. This "Einstein flow" goes back to the 1930's.

The question that we address is the future asymptotics of an expanding universe. We make the following simplifications : no matter, no cosmological constant and compact X. The Einstein equation becomes the statement that (M, g) is Ricci flat.

A standard assumption for the foliation is that for each t, the mean curvature function $H = \sum_{i,j=1}^{3} h^{ij} K_{ij}$ on X is constant. We make this assumption and also assume that as a function of time, $H : [T_0, \infty) \to [H_0, 0)$ is bijective and increasing, where $H_0 < 0$. The fact that H is negative means that the volume forms of $(X, h(t))$ are expanding in t.

The Hubble time is the choice of time function $t = -\frac{3}{H}$. Borrowing terminology from Ricci flow, we say that a solution to the Einstein flow is type-III if $|Riem(M, g)| = O(t^{-2})$ as $t \to \infty$, and type-II otherwise.

Under a noncollapsing assumption, Anderson showed that the future behavior of a type-III Einstein flow is modelled by a flat Lorentzian cone over a hyperbolic 3-manifold [1]. We show that in the collapsing case, there are arbitrarily large future time intervals that are modelled by a flat spacetime or a Kasner spacetime.

There are examples of type-II expanding CMC Einstein flows [2, 3]. In this case, one can do a type-II blowdown as in Ricci flow. If the second fundamental form K of the Einstein solution satisfies $|K|^2 \leq const.H^2$ then we show that the blowdown limit is a flat static Einstein solution. Here the limit is in the sense of weak $W^{2,p}$ convergence. The interpretation is that there are increasing oscillations of the rescaled curvature tensor, that average it out to zero.

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Embeddings of the Heisenberg group and the Sparsest Cut problem

Robert Young

(joint work with Assaf Naor)

In this report, we describe some sharp bounds on bilipschitz embeddings of the Heisenberg group into the Banach space L_1. This solves a geometric question that arose from work in theoretical computer science. This question arose from the study of the Goemans–Linial algorithm for approximating the Sparsest Cut problem [7, page 158] [10], which is currently the best known algorithm for approximating Sparsest Cut. The Sparsest Cut problem can be formulated as an optimization problem over the L_1–metrics, i.e., matrices of the form $(\|v_i - v_j\|_1)_{ij}$ where $v_i \in L_1$ for all i. This problem is NP–hard, but Goemans and Linial found a fast approximate solution by minimizing over the class of *negative-type metrics*.

The accuracy of this solution depends on how close the set of negative-type metrics is to the set of L_1–metrics or, equivalently, how well a negative-type metric space embeds in L_1. If X is a metric space and $p \in [1, \infty]$, we let $c_p(X)$ be the smallest distortion of a map $X \to L_p$, that is, the infimum over all $D \in [1, \infty]$ for which there exists an embedding $f\colon X \to L_p(\mathbb{R})$ such that $d(x, y) \leq \|f(x) - f(y)\|_p \leq Dd(x, y)$ for every $x, y \in X$. Goemans and Linial asked:

Question 1. *Let $\alpha(n) = \sup_{X\in\mathcal{N}_n} c_1(X)$, where $\mathcal{N}_n$ is the set of n–point negative-type metric spaces. What are the asymptotics of $\alpha(n)$?*

The function $\alpha(n)$, known as the *Goemans–Linial integrality gap*, measures how the accuracy of the Goemans–Linial algorithm depends on the size of the problem.

Goemans and Linial asked whether $\alpha(n)$ is bounded, i.e., whether every negative-type metric space embeds bilipschitzly in L_1. This hope was dashed in the remarkable work [8], which proved a lower bound $\alpha(n) \gtrsim \sqrt[6]{\log\log n}$ by using an example based on the Unique Games Conjecture. A very different approach was introduced in [9] and [3], which showed that the Heisenberg group is bilipschitz equivalent to a negative-type metric space that does not embed in L_1 by a bilipschitz map and thus gave another proof that $\lim_{n\to\infty} \alpha(n) = \infty$. This approach was improved and quantified in [4], which showed that $\alpha(n) \gtrsim (\log n)^\delta$ for a small, but effective, universal constant $\delta > 0$.

In recent research, Naor and found sharp bounds on the L_1–distortion of the Heisenberg group, which imply sharp bounds on the Goemans–Linial integrality gap. Here, $H_{\mathbb{Z}}^{2k+1}$ denotes the $(2k+1)$–dimensional Heisenberg group

$$H_{\mathbb{Z}}^{2k+1} = \left\{ \begin{pmatrix} 1 & x_1 & \dots & x_k & z \\ 0 & 1 & 0 & 0 & y_1 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 1 & y_k \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; x_i, y_i, z \in \mathbb{Z} \right\}. \tag{1}$$

Theorem 1. *For every $k \geq 2$ and every $n > 0$, let $B_n(H^{2k+1}) \subset H^{2k+1}$ be the ball of radius n. Then $c_1(B_n(H^{2k+1})) \gtrsim \sqrt{\log n}$. Consequently, $\alpha(n) \gtrsim \sqrt{\log n}$.*

The best known upper bound on $\alpha(n)$ is $\alpha(n) \lesssim (\log n)^{\frac{1}{2}+o(1)}$ [1]; this matches our lower bound up to an iterated logarithm.

The proof relies on a new isoperimetric inequality that compares the perimeter of a set with its *vertical perimeter.* Let $S = \{X_1, \dots, X_k, Y_1, \dots, Y_k, Z\}$ be the standard generating set of $H_{\mathbb{Z}}^{2k+1}$, so that Z generates its center. We say that a set $\Omega \subset H_{\mathbb{Z}}^{2k+1}$ is vertical if $\Omega = Z\Omega$. For every subset $\Omega \subset H_{\mathbb{Z}}^{2k+1}$, we define

$$|\partial_{\mathsf{v}}\Omega| = \left(\sum_{t=1}^{\infty} \frac{|\Omega \bigtriangleup Z^t\Omega|^2}{t^2} \right)^{\frac{1}{2}},$$

where $A \bigtriangleup B$ is the symmetric difference of A and B and $|A|$ is the number of points in A. This measures how close Ω is to a vertical set.

We can decompose $|\partial_{\mathsf{v}}\Omega|$ into contributions from many scales. For $r > 0$, let

$$|\partial_{\mathsf{v}}^r\Omega| = \left(\sum_{t=r}^{2r-1} \frac{|\Omega \bigtriangleup Z^t\Omega|^2}{t^2} \right)^{\frac{1}{2}}.$$

If $\partial\Omega = \{g \in \Omega \mid gS \not\subset \Omega\}$ is the boundary of Ω with respect to S, then

$$|\partial_{\mathsf{v}}^r\Omega| \lesssim \left(\sum_{t=r}^{2r-1} \frac{|\partial\Omega|^2 d(1, Z^t)^2}{t^2} \right)^{\frac{1}{2}} \approx |\partial\Omega| \left(\sum_{t=r}^{2r-1} \frac{t}{t^2} \right)^{\frac{1}{2}} \lesssim |\partial\Omega|.$$

The ratio $V(r) = |\partial_{\mathsf{v}}^r\Omega|/|\partial\Omega|$ thus measures the verticality of Ω at a single scale, and $|\partial_{\mathsf{v}}\Omega| = \sqrt{\sum_{i=0}^{\infty} V(r)^2}$.

Theorem 1 follows from the following isoperimetric inequality, which states that a set that fails to be vertical at many scales must have a large perimeter.

Theorem 2. *If $k \geq 2$ and $\Omega \subset H_{\mathbb{Z}}^{2k+1}$ is a finite subset, then $|\partial_{\mathsf{v}}\Omega| \lesssim |\partial\Omega|$.*

We prove Theorem 2 using a new technique based on uniform rectifiability. First, we show that Theorem 2 follows from the continuous inequality

$$\int_0^{\infty} \frac{\text{vol}(\Omega \bigtriangleup Z^t\Omega)^2}{t^2}\, dt \lesssim \text{area}(\partial\Omega)^2, \tag{2}$$

where Ω is a subset of the real Heisenberg group H^{2k+1}, i.e., the set of matrices of the form (1) with real coefficients, vol is Hausdorff $(2k+2)$–measure, and area is Hausdorff $(2k+1)$–measure.

Next, we define *intrinsic corona decompositions* in the Heisenberg group and use them to bound (2). These generalize David and Semmes's notion of corona decompositions for sets E in $\mathbb{R}^n$ [5]. A corona decomposition of E consists of a collection of Lipschitz graphs with small Lipschitz constants that approximate E at many scales. For example, a zigzag in the unit grid that connects $(0,0)$ to $(100,100)$ can be approximated by a diagonal line at large scales and by a union of horizontal and vertical lines at small scales. A more complicated curve (for instance, an iteration of the Koch snowflake) might need many approximating graphs over a large range of scales. A collection of approximating graphs is a corona decomposition if its total size is bounded by a *Carleson packing condition.* We develop analogous notions in the Heisenberg group, based on the intrinsic Lipschitz graphs defined by Franchi, Serapioni, and Serra Cassano [6].

Corona decompositions in $\mathbb{R}^n$ have been used to reduce the study of some types of singular integrals on subsets of $\mathbb{R}^n$ to the case of Lipschitz graphs. Likewise, when $\partial\Omega$ admits an intrinsic corona decomposition, we can use intrinsic corona decompositions to reduce (2) to an inequality for intrinsic Lipschitz graphs. When $k \geq 2$, this inequality follows from a bound on $c_2(H^3)$ proven in [2] – that is, we deduce an inequality about embeddings in L_1 from an inequality on embeddings of a lower-dimensional group in L_2!

Finally, we show that when $\Omega \subset H^{2k+1}$ satisfies $\operatorname{area}\partial\Omega < \infty$, then Ω can be decomposed into sets whose boundaries all admit intrinsic corona decompositions. This step can be divided into two parts. The first part is simpler; we decompose Ω into sets E_i such that E_i, ∂E_i, and $H^{2k+1} \setminus E_i$ are all Ahlfors regular. The second part is more involved; we use arguments based on the stability of monotone sets in H^{2k+1} to construct approximating graphs and prove bounds on their size.

This resolves the question of the embeddability of H^{2k+1} in L_1 when $k \geq 2$, but there are still open questions when $k = 1$. Our main inequality relies on reducing (2) to an inquality for functions on a subgroup of H^{2k+1}; when $k \geq 2$, we can reduce to H^3, but when $k = 1$, we cannot. In fact, work in progress suggests that H^3 has smaller L_1–distortion than H^5 — that $c_1(B_n(H^{2k+1})) \lesssim n^{\frac{1}{4}}$.

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Degeneration of Kähler-Einstein manifolds with negative scalar curvature

Jian Song

We conider any algebraic family of Kähler-Einstein manifolds of negative scalar curvature over a punctured disc. We show that the Kähler-Einstein manifolds converge, in pointed Gromov-Hausdorff topology to a unique finite disjoint union of complete metric spaces homeomorphic to a projective semi-log cannonical model with its locus of non log-terminal singularities removed.

Invisible boundary and the geodesic flow on metric spaces

Alexander Lytchak

(joint work with Vitali Kapovitch and Anton Petrunin)

In the talk I discuss the proof of the following theorem

Theorem 1. *Let X be the boundary of a convex body in $\mathbb{R}^{n+1}$. Then almost every direction in the tangent bundle of X is the starting direction of an infinite geodesic. The geodesic flow defined in this way preserves the Liouville measure.*

Even the existence of a single infinite geodesic on any convex surface is apparently a new result. The theorem arose in the attempt to prove the existence of a geodesic flow on any Alexandrov space. While we could not prove the result in full generalilty, we have reduced it to an analytic statement and have developed a new analytic tool which might be interesting in its own right beyond the realm of Alexandrov geometry.

Namely, for any Alexandrov space X (or more generally, any metric measure space) of Hausdorff dimension n, we consider the volume growth function $b_r(x) := vol(B(x,r))$, where $B(x,r)$ is the ball of radius r around x. The function $v_r(x) = 1 - \frac{b_r(x)}{\omega_n \cdot r^n}$ measures the volume deviation from the Euclidean volume. Considering v_r as the measure $V_r := v_r \cdot vol$ and letting r go to 0 we obtain information about infinitesimal average regularity of X. For a smooth Riemannian manifold without boundary V_r/r^2 converges to a multiple of the scalar curvature. More interestingly for our purposes, for a smooth Riemannian manifold with boundary V_r/r converges weakly to a multiple of the canonical measure on the boundary. This motivates the following definitions.

Definition 1. *We say that the metric measure space* (X, d, vol) *has locally finite mm-boundary if the signed Radon measures* V_r/r *are uniformly bounded, for* $0 < r < 1$*. We say that the mm-boundary vanishes if* V_r/r *weakly converges to* 0*.*

Our main result is motivated by the idea that on a smooth Riemannian manifold with boundary, the boundary is the obstacle for the existence of the geodesic flow.

Theorem 2. *For any Alexandrov space* X *the following holds true.*

(1) X *has locally finite mm-boundary.*
(2) The mm-boundary does not vanish if the topological boundary of X *is not empty.*
(3) If the mm-boundary vanishes then almost every tangent direction of X *is the starting direction of an infinite geodesic and the geodesic flow preserves the Liouville measure.*
(4) If X *is the boundary of the convex body or a general* 2*-dimensional Alexandrov space without boundary then the mm-boundary of* X *vanishes.*

Thus, if on an Alexandrov space with empty topological boundary the geodesic flow is not defined for all times, there must exist some topologically invisible boundary, namely our mm-boundary. This mm-boundary is a Radon measure on the Alexandrov space which is absolutely singular with respect to the volume and which vanishes on subsets with finite $(n-1)$-dimensional Hausdorff measure.

Asymptotic structure of self-shrinkers of mean curvature flow

Lu Wang

A surface, $\Sigma \subset \mathbb{R}^3$, is a *self-shrinker* if

$$\mathbf{H}_\Sigma + \frac{\mathbf{x}^\perp}{2} = \mathbf{0}. \tag{1}$$

Here $\mathbf{H}_\Sigma = -H_\Sigma \mathbf{n}_\Sigma = \Delta_\Sigma \mathbf{x}$, $\mathbf{n}_\Sigma$ is the unit normal of Σ, and $\mathbf{x}^\perp$ is the normal part of the position vector. They are a special class of solutions to mean curvature flow. Namely, let $\Sigma_t = \sqrt{-t}\,\Sigma$ for $t < 0$. Then $\{\Sigma_t\}_{t<0}$ moves by mean curvature vector, i.e.,

$$(\partial_t \mathbf{x})^\perp = \mathbf{H}_{\Sigma_t} \text{ for } \mathbf{x} \in \Sigma_t. \tag{2}$$

Mean curvature flow is the negative L^2-gradient flow of the area functional. Since there is no closed minimal surfaces in $\mathbb{R}^3$, the flow starting from any closed surface develops singularities in finite time. Combining Huisken's monotonicity formula [2] and Brakke's compactness theorem [1], Ilmanen [3] proved that all possible singularities at the first singular time of a compact mean curvature flow in $\mathbb{R}^3$ are modeled by self-shrinkers of finite genus.

From a variational view point, self-shrinkers are critical points of the *Gaussian surface area*

$$F[\Sigma] = (4\pi)^{-1} \int_\Sigma \mathrm{e}^{-\frac{|\mathbf{x}|^2}{4}} \, d\mathcal{H}^2 \tag{3}$$

where $\mathcal{H}^2$ is the 2-dimensional Hausdorff measure in $\mathbb{R}^3$. In other words, they are minimal with respect to the conformal change of the Euclidean metric on $\mathbb{R}^3$,

$$g_{ij} = \mathrm{e}^{-\frac{|\mathbf{x}|^2}{4}} \, \delta_{ij}. \tag{4}$$

However, this metric is incomplete, and the scalar curvature is negative and blows up approaching infinity. Thus the general theory of minimal submanifolds does *not* apply here.

Our goal is to investigate the moduli space of noncompact self-shrinkers in $\mathbb{R}^3$. Besides the planes and round cylinders, there exist a one-parameter family of noncompact shrinkers of high genus with one asymptotically conical end obtained from desingularizing the sphere and plane; cf. [5] and [6]. Previously we established some uniqueness theorems [8, 9] for noncompact self-shrinkers with given asymptotics at infinity. We now turn to studying the asymptotic behaviors of self-shrinkers. In [10] we confirmed a conjecture of Ilmanen [4, p. 39] for self-shrinkers of finite topology.

Theorem 1. *If Σ is a noncompact self-shrinker in $\mathbb{R}^3$ of finite topology and M is an end of Σ, then one of the following holds:*

- $\lim_{\tau\to 0^+} \tau M = C$ *in* $C^\infty_{loc}(\mathbb{R}^3 \setminus \{\mathbf{0}\})$ *for some regular cone* C.
- $\lim_{\tau\to 0^+} M - \tau^{-1}\mathbf{v} = \mathbb{R}_{\mathbf{v}} \times \sqrt{2}\,\mathbb{S}^1$ *in* $C^\infty_{loc}(\mathbb{R}^3)$ *for some nonzero vector* $\mathbf{v}$.

In particular, $\sup_{p\in\Sigma} |A_\Sigma(p)| < \infty$.

Here a regular cone means the link of the cone is a closed, smooth embedded curve in unit sphere, $\mathbb{R}_{\mathbf{v}}$ is the subspace of $\mathbb{R}^3$ spanned by $\mathbf{v}$, and A_Σ is the second fundamental form of Σ.

It is known that the tangent space at infinity of a self-shrinker is a cone $\hat{C}$; cf. [4, p. 8] and [7]. To prove Theorem 1, we need to address the regularity of $\hat{C}$ and that of the convergence as well. We appeal to the parabolic blow-up procedure to $\mathcal{M} = \{\sqrt{-t}\,M\}_{t<0}$ at time 0. Invoking the ϵ-regularity theorem of Brakke [1] we relate the regularity question to the multiplicity issue of tangent flows to $\mathcal{M}$. The technical heart is a so-called "sheeting" theorem for the convergence at time ∞ of the flow obtained from rescaling of $\mathcal{M}$. Its proof weaves analysis and geometry.

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Finsler geometry and quasigeodesics in higher rank symmetric spaces

Bernhard Leeb
(joint work with Misha Kapovich and Joan Porti)

The classical Morse Lemma for quasigeodesics asserts that, in negative curvature, quasigeodesics have rigid geometry in the sense that they are uniformly close to geodesics. To make this more precise, we recall that a map $q : I \to X$ from an interval into a metric space is an (L, A)-*quasigeodesic* if

$$\frac{1}{L} \cdot |t_1 - t_2| - A \leq d(q(t_1), q(t_2)) \leq L \cdot |t_1 - t_2| + A$$

for all $t_1, t_2 \in I$. If X is δ-hyperbolic in the sense of Gromov, then according to the Morse Lemma the image $q(I)$ of the quasigeodesic is contained in a tubular neighborhood of uniform radius $r = r(L, A, \delta)$ of a(ny) geodesic in X with the same endpoints $q(t_i)$.

In nonpositive curvature quasigeodesics are much more flexible and the Morse Lemma no longer holds, as the case of euclidean plane already shows. We restrict ourselves to the case of symmetric spaces $X = G/K$ of noncompact type (G is a noncompact connected semisimple Lie group and K a maximal compact subgroup) and rank ≥ 2. (The rank one case is covered by the classical Morse Lemma.) For simplicity, we assume that X is irreducible.

We obtain control on the geometry of quasigeodesics in X by imposing a *regularity* condition. We require that for sufficiently separated points on the quasigeodesic the segments connecting them are not almost singular of a certain type. There are different degrees of regularity which one can impose and they depend on the choice of a face $\tau \subseteq \sigma$ of the spherical Weyl chamber attached to X which we fix throughout our discussion. The face τ corresponds to a conjugacy class of parabolic subgroups $P < G$. The larger the face, respectively, the smaller the parabolic subgroups, the more restrictive will be the regularity condition.

In view of the natural identification $G\backslash(X \times X) \cong \Delta$ with the euclidean Weyl chamber Δ associated to X, we define the Δ-*distance* as the quotient map d_Δ :

$X \times X \to \Delta$ and regard it as a refined vector-valued metric on X. The Riemannian distance d_{Riem} on X is obtained by taking the euclidean norm, $d_{Riem} = \|d_\Delta\|$.

The euclidean Weyl chamber is the complete euclidean cone over the spherical Weyl chamber, $\partial_\infty \Delta \cong \sigma$. Let $\Theta \subset \sigma$ be a compact subset disjoint from the faces of σ which do not contain τ. We say that an oriented segment $xy \subset X$ is Θ-regular if its Δ-length $d_\Delta(x, y) \in \Delta$ points towards Θ. Moreover, we say that an (L, A)-quasigeodesic $I \to X$ is *τ-regular* if for some such Θ the segments connecting $q(t_1)$ and $q(t_2)$ are Θ-regular for all $t_1, t_2 \in I$ with $|t_1 - t_2|$ sufficiently large, cf. [KLP13, KLP17]. An equivalent notion had already been introduced by Benoist in deep his work [Be97], see in particular part (5) of Lemma 3.5.

Also τ-regular quasigeodesics do not satisfy the classical Morse Lemma literally, i.e. they are i.g. not uniformly close to geodesics with respect to the Riemannian metric on X. However, they turn out to be bounded perturbations of geodesics with respect to a suitable G-invariant "polygonal" *Finsler* metric d^τ_{Fins} depending on the face type τ. To describe it, we choose a vector $v \in \Delta$ pointing towards the interior of the face τ, then take the dual linear functional $\lambda = \langle v, \cdot \rangle$ on Δ and put

$$d^\tau_{Fins} = \lambda \circ d_\Delta,$$

see [KL15] for a detailed discussion. The metric d^τ_{Fins} is in general non-symmetric, but it is equivalent to d_{Riem} as a consequence of G-invariance. In particular, uniform d_{Riem}-quasigeodesics are uniform d^τ_{Fins}–quasigeodesics and vice versa.

The family of (unparametrized) d^τ_{Fins}-geodesics in X depends only on τ and not on the choice of v. We refer to them as *τ-Finsler geodesics*. Riemannian geodesics are also τ-Finsler geodesics. However, there are more (in rank ≥ 2); due to the lack of strict convexity of balls, geodesic connections of pairs of points are non-unique. Nevertheless, the geometry of τ-Finsler geodesics is very restricted. Most importantly, they are contained in parallel sets (of type τ). In particular, if $\tau = \sigma$, they are contained in maximal flats.

The main result of [KLP14b], see Thm 1.3 there, can be paraphrased as follows:

Morse Lemma for regular quasigeodesics in symmetric spaces of higher rank: τ-Regular quasigeodesics in X are uniformly close to τ-Finsler geodesics.

The bound depends on the quasiisometry constants L, A and on Θ. In [KLP14b], this result had not been formulated in Finsler terms. The Finsler view point had only gradually emerged during our study of asymptotic and coarse properties of discrete isometry groups with "rank one behavior" acting on symmetric spaces and euclidean buildings, see also [KLP16, KL17, KLP17, KLP13, KLP14a, KL15]. The Higher Rank Morse Lemma is an important tool in our proof of the equivalence of various different characterizations for this family of subgroups. For instance, we showed that a finitely generated discrete subgroup $\Gamma < G$ is τ-Anosov, a dynamical condition introduced in [La06, GW12], if and only if it is τ-URU, that is, uniformly τ-regular and undistorted, thereby providing a simple characterization of Anosov subgroups in terms of coarse geometry.

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Sheared pleated surfaces and limiting configurations for Hitchin's equations

Michael Wolf

(joint work with Andreas Ott, Jan Swoboda and Richard Wentworth)

1. Introduction

In this extended abstract, we report on ongoing joint work with **Andreas Ott, Jan Swoboda**, and **Richard Wentworth**. Let $S = S_g$ denote a closed differentiable surface of genus g on which we will put various geometric structures.

We seek to interpret a stratum in the frontier of the character variety $\chi_g = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C})$ of (irreducible) genus g surface group representations into $\mathrm{PSL}(2,\mathbb{C})$. In particular, we refer to a recent work [4] of Mazzeo-Swoboda-Weiss-Witt. These authors fix a Riemann surface structure, say X, on S, and consider the moduli space $\mathcal{M}$ of stable PSL(2, $\mathbb{C}$-Higgs bundles (up to gauge equivalence) over X. They then consider those Higgs bundles for which the Higgs field, say Φ, has determinant $q = \det(\Phi) \in H^0(X, K_X^2)$, a holomorphic quadratic differential on X, to have but simple zeroes. Roughly, they continue from this restricted space to define a frontier for this portion of the moduli space by adjoining to the associated portion of χ_g a moduli space $\mathcal{M}_\infty$ consisting of (equivalence classes of) *limiting configurations*. These limiting configurations are pairs (Φ_∞, A_∞) of a singular Higgs field Φ_∞ and singular connection A_∞: together the pair satisfy a degenerate decoupled system of equations that is a limiting version of the Hitchin system. See [4] for complete details.

In this talk, we seek to address two questions:

(1) What is the dependence of this stratum of limiting conditions on the initial choice of Riemann surface X? For example, if (Φ_∞, A_∞) is a limit, under the correspondences above, of a sequence $\rho_n \in \chi_g$ of (of equivalence classes of) representations where we have chosen X as the background Riemann surface, and if $(\Phi'_\infty, A'_\infty)$ is an accumulation point of those classes of representations when we have chosen X' as a background Riemann surface, then how does $(\Phi'_\infty, A'_\infty)$ relate to (Φ_∞, A_∞)?

(2) The Hitchin theory (see [3]) proceeds via consideration of ρ_n-equivariant harmonic maps $u_n : \tilde{X} \to \mathsf{SL}(2,\mathbb{C})/\mathsf{SU}(2)$. The latter symmetric space is isometric to the hyperbolic three-space $\mathbb{H}^3$, so we seek an interpretation of the limiting configuration pair (Φ_∞, A_∞) in terms of hyperbolic-geometric objects.

We address these questions by relating limiting configurations (Φ_∞, A_∞) to classes of shearings of a pleated surfaces $\Sigma = (\tilde{f}, (S,\sigma), \rho, \lambda)$. Here the pleated surface Σ is defined by the following data: the surface S is equipped by a hyperbolic metric σ for which λ is a geodesic laminations, and the map $f : \tilde{S} \to \mathbb{H}^3$ is an isometry on complement $S \setminus \lambda$ of λ in S, as well as an isometry of λ onto its image (geodesic). For full details, see [1] and the papers referenced within.

Given a pleated surface $\Sigma = (\tilde{f}, S, \rho, \lambda)$ and a number s, we create a pleated surface $\Sigma_s = (\tilde{f}_s, S_s, \rho_s, \lambda)$ as follows. Set Ξ^λ_s to be the transverse cocycle associated to a left earthquake of S along λ. Then set $\Sigma_s = \Sigma_{s,\mu_{q,\mathrm{vert}}} = \Xi^\lambda_s \Sigma$, the result of shearing Σ along the lamination λ for a measure of $s\mu_{q,\mathrm{vert}}$, where here $\mu_{q,\mathrm{vert}}$ denotes the measure for the vertical foliation of q. Note that this operation results in a pleated surface Σ_s with the same bending cocycle as the original surface Σ. (Naturally, a similar construction of $\Sigma_{s,n} = \Xi^{\lambda_n}_s \Sigma_n$ can be made for laminations λ_n and measures $\mu_{q^1_n,\mathrm{vert}}$.)

2. The pleated surface for a limiting configuration

Let ρ_n denote a sequence of irreducible $SL(2,\mathbb{C})$ surface group representations which leave all compact sets in the character variety χ_g, converging to a limiting configuration (Φ_∞, A_∞) relative to a choice X of Riemann surface. Let $h_n : \tilde{X} \to \mathbb{H}^3$ denote the associated family of equivariant harmonic maps from the universal cover $\tilde{X}$ to hyperbolic 3-space $\mathbb{H}^3$, normalized by some fixed choice of frames.

Let $q_n = \det(\Phi_n)$ be the Hopf differential of the harmonic map h_n; here Φ_n refers to the Higgs field. Our assumption that the limiting configuration (Φ_∞, A_∞) has $\det \Phi_\infty$ a quadratic differential with simple zeroes implies that we may assume, for n sufficiently large, that the differential q_n also has only simple zeroes. We adopt the notation that $\tilde{X}$ denotes the universal cover of X, and $\tilde{q_n}$ (respectively $\tilde{q}$) denote the lifts to the universal cover $\tilde{X}$ of q_n (resp. q), and so on. Let $q^1_n = \frac{q_n}{\|q_n\|}$

denote the unit norm quadratic differential which is a multiple of the quadratic differential q_n. We might as well assume that $\|q\| = 1$ so that $q_n^1 \to q$.

Let $X^\times$ denote the complement in $\tilde{X}$ of the zeroes of $\tilde{q}$ and the horizontal leaves that emanate from those zeroes.

On $X^\times$, let π denote the natural map which takes horizontal leaves ℓ of q to their geodesic representatives $\pi(\ell)$ in the lamination λ, with an analogous definition for $\pi_n : X^\times \to \lambda_n$. We may extend this map to the horizontal leaves which emanate from the zeroes after some arbitrary choice of extending π (resp., π_n) by taking limits from the left. We continue to denote this map by π (resp. π_n).

Proposition 1. *There is a pleated surface $\Sigma = (\tilde{f}, S, \rho, \lambda)$ with the following properties. The measured lamination λ is projectively equivalent to the measured lamination naturally associated to the horizontal measured foliation of $q = \det(\Phi)$. Let $s = s(n) = 2E(h_n)^{\frac{1}{2}}$, and let $\Sigma_s = \Xi_s \Sigma$ be as in the previous paragraph. We then have the following estimates depending upon whether the Hopf differentials q_n are proportional or not.*

(i) Suppose that q_n^1 is independent of n. Then, for every ϵ, we may choose n sufficiently large so that the images $h_n(\tilde{X})$ are within distance ϵ of $\tilde{f}_s(\tilde{X})$; moreover, on the complement of any neighborhood of $q^{-1}(0)$, the map h_n nearly agrees with the projection $f_s \circ \pi$ from the punctured surface $\tilde{X}$ to the lamination λ, i.e. when $d_{|q_n^1|}(p, q_n^{-1}(0)) > \epsilon$, we have for n sufficiently large that $d_{\mathbb{H}^3}(h_n(p), \pi_n(p)) < \epsilon$.

(ii) In general, with no restriction on q_n^1 other than $q_n^1 \to q$, we conclude that for every (large) constant C and every ϵ, there is an n so that we have for n sufficiently large for points p so that $d_{|q_n^1|}(p, q_n^{-1}(0)) > \epsilon$, then $d_{\mathbb{H}^3}(h_n(p), \pi_n(p)) < 2s - C$.

Remarks 1.

1. In effect, the construction in this proposition results in a family ρ_s of representations defining the pleated surfaces Σ_s that track a subsequence of the representations induced by h_n.
2. One can understand the second statement in the proposition in the following way. A consequence of the first estimate is that if one takes a 'ray' of representations ρ_n whose Hopf differentials q_n are all multiples of a single unit quadratic differential q_n^1, then the harmonic map images $h_n(\tilde{X})$ are tracked very closely by shearings $\Sigma_s = \Xi_s \Sigma$ of a single pleated surface Σ. Thus, if one were to take a second family of representations ρ_n' whose Hopf differentials q_n' are all multiples of a single unit quadratic differential $q_n^{1'}$, then the harmonic map images $h_n'(\tilde{X})$ are tracked very closely by shearings $\Sigma_s' = \Xi_s \Sigma$ of a single pleated surface Σ'. But those shearings Σ_s and Σ_s' are bent along measured laminations which typically make some non-zero angle with other, so even for quadratic differentials q_n and q_n' whose zeroes are are close the distances between the images $h_n(p)$ and $h_n'(p)$ of a point p far from the zeroes will distance $d_{\mathbb{H}^3}(h_n(p), h_n'(p)) = 2s - O(1)$. This last estimate is because $h_n(p)$ will lie close to one geodesic and be moved by the shearing along that geodesic by a distance $s + O(e^{-cs})$ and $h_n'(p)$ will lie close to another distinct geodesic and be moved by the shearing along that geodesic by a distance $s + O(e^{-cs})$.

By elementary hyperbolic geometry, even if those geodesics intersect, the distance between the points $h_n(p)$ and $h_n^{'}(p)$ will be at least $2s - C_0$ for some absolute constant C_0.

The proof of Proposition 1 uses estimates on high energy harmonic maps to $\mathbb{H}^2$ and $\mathbb{H}^3$ from [8] and [5]: the constructions borrow heavily from the easier parts of Minsky's thesis [5].

3. Bending Cocycles

With this basic correspondence in hand, we rapidly sketch the remainder of the discussion.

The space of limiting configurations $\{(\Phi_\infty, A_\infty)\}$ fibers into Prym varieties which share a common singular Higgs field Φ_∞: in this construction a singular connection A_∞ differs from another singular connection A^0_∞ by a form $\alpha \in H^1(X^\times, L_{\Phi_\infty})$, where $L_{\Phi_\infty} = \{\gamma \in \mathfrak{su}(2) : [\gamma, \Phi_\infty] = 0\}$ is a line bundle over $X^\times$ (the Riemann surface X punctured at $\Phi_\infty^{-1}(0)$). There is an equivalence relation among the elements $\alpha \in H^1(X^\times, L_{\Phi_\infty})$ given by an integral relation among the periods of the forms. Again, see [4] for full details on the structure of the space of limiting configurations.

We show two results, which we summarize a bit informally, using only the terminology developed so far.

Proposition 2. *A form $\alpha \in H^1(X^\times, L_{\Phi_\infty})$ formally defines a bending cocycle $b_{[\alpha]}$ for a geodesic lamination $\lambda \subset S$ corresponding to the horizontal foliation for Φ_∞.*

Let $(\Phi_\infty, A^0_\infty)$ denote the limiting configuration corresponding to the Hitchin section of χ_g: these are also often referred to as the Fuchsian representations. Here the associated pleated surface from Proposition 1 has a vanishing bending cocycle.

Thus, associated to a form $\alpha \in H^1(X^\times, L_{\Phi_\infty})$, we now have two pleated surfaces (or more precisely, classes of shearings of pleated surfaces. The first, Σ^α is defined via Proposition 2 by bending the Fuchsian pleated surface along $\lambda = \lambda(\Phi_\infty)$ so that the resulting bending cocycle is α.

The second pleated surface Σ_α is obtained from $\alpha \in H^1(X^\times, L_{\Phi_\infty})$ by applying the construction of Proposition 1 to the limiting configuration $(\Phi_\infty, A^0_\infty + \alpha)$.

The main result in the talk is that

Theorem 1. *The pleated surfaces Σ_α and Σ^α agree up to shearing along λ.*

The proof involves giving a hyperbolic geometry interpretation of the bundle L_{Φ_∞} and the elements $\alpha \in H^1(X^\times, L_{\Phi_\infty})$ (cf. Donaldson [2]), and then combining these with some of the estimates on high energy harmonic maps as well as some elementary observations as to the geometry of highly sheared pleated surfaces.

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Harmonic quasiisometric maps

Yves Benoist
(joint work with Dominique Hulin)

We prove that a quasiisometric map $f : X \to Y$ between pinched Hadamard manifolds X and Y is always within bounded distance of a unique harmonic map $h : X \to Y$.

This result extends a previous recent result of M. Lemm and V. Markovic in [2] who were dealing with the case where both X and Y are equal to the same real hyperbolic space $\mathbb{H}^n$.

In this talk I followed carefully the argument as it is explained in [1].

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Hopf's Conjecture on the Euler characteristic holds if the manifold has an isometry group of rank 5

Burkhard Wilking
(joint work with Lee Kennard)

We consider isometric effective action of a 5-torus T^5 on a positively curved manifold. We then can analyze the rational topology of each fixed point component $F \subset M^{T^d}$. In fact F is either rational equivalent to rank 1 symmetric space to $S^2 \times \mathbb{H}P^\ell$ or $S^3 \times \mathbb{H}P^\ell$. As a consequence we can confirm that the Euler characteristic of the underlying manifold is positive in even dimensions.

New examples of gradient expanding Kähler-Ricci solitons

Ronan J. Conlon
(joint work with Alix Deruelle)

A *Ricci soliton* is a triple (M, g, X), where M is a Riemannian manifold with a complete Riemannian metric g and a complete vector field X satisfying the equation

$$\mathrm{Ric}(g) - \frac{1}{2}\mathcal{L}_X g + \lambda g = 0 \tag{1}$$

for some $\lambda \in \{-1, 0, 1\}$. We call X the *soliton vector field*. A soliton is said to be *steady* if $\lambda = 0$, *expanding* if $\lambda = 1$, and *shrinking* if $\lambda = -1$. Moreover, if $X = \nabla^g f$ for some real-valued smooth function f on M, then we say that (M, g, X) is a *gradient* soliton. If g is Kähler with Kähler form ω, then we say that (M, g, X) is a *Kähler-Ricci soliton* if in addition to g and X satisfying (1), the vector field X is real holomorphic. In this case, one can rewrite the soliton equation as

$$\rho_\omega - \frac{1}{2}\mathcal{L}_X\omega + \lambda\omega = 0, \tag{2}$$

where ρ_ω is the Ricci form of ω. If g is a Kähler-Ricci soliton and if $X = \nabla^g f$ for some real-valued smooth function f on M, then we say that (M, g, X) is a *gradient Kähler-Ricci soliton.*

The study of Ricci solitons and their classification is important in the context of Riemannian geometry. For example, they provide a natural generalisation of Einstein manifolds. Also, to each soliton, one may associate a self-similar solution of the Ricci flow which are candidates for singularity models of the flow.

Given an *expanding* gradient Ricci soliton (M, g, X) with quadratic Ricci curvature decay and appropriate decay on the derivatives, one may associate to it a unique tangent cone (C_0, g_0) with a smooth link [2, 5, 10] which may be considered an initial condition of the Ricci flow $g(t)$, $t \geq 0$, associated to the soliton in the sense that $\lim_{t\to 0^+} g(t) = g_0$ as a Gromov-Hausdorff limit. We consider the converse to this statement, namely the following problem.

Problem. *For which metric cones C_0 is it possible to find an expanding (gradient) Ricci soliton with tangent cone C_0? Or more generally, given a metric cone (C_0, g_0), when is it possible to find a Ricci flow $g(t)$, $t \geq 0$, such that $\lim_{t\to 0^+} g(t) = g_0$ in the Gromov-Hausdorff sense?*

Deruelle [5, 6] has shown that one can always solve this problem when the link of the cone C_0 is a sphere with positive curvature operator bounded from below by the identity, and a result due to Lott and Wilson [9] shows that this question has a positive answer at the level of formal expansions. When the cone C_0 is Kähler, Siepmann [10] has shown that the above question always has an affirmative answer when C_0 is furthermore Ricci-flat and admits an equivariant resolution satisfying certain topological conditions. Our contribution is to remove the hypothesis of Ricci-flatness from Siepmann's result.

Theorem 1 (C.-Deruelle [3]). *Let C_0 be a Kähler cone with complex structure J_0, Kähler cone metric g_0, Ricci curvature $\mathrm{Ric}(g_0)$, and radial function r. Let $\pi : M \to C_0$ be a Kähler resolution of C_0 with complex structure J and exceptional set E such that*

(a) the complex torus action on C_0 generated by $J_0 r\partial_r$ extends to M so that $X = \pi^*(r\partial_r)$ lifts to M;
(b) $H^1(M) = 0$ or $H^{0,1}(M) = 0$ or $X|_A = 0$ for $A \subset E$ for which $H_1(A) \to H_1(E)$ is surjective.

Then for all $c > 0$, there exists a unique expanding gradient Kähler-Ricci soliton g_c on M with soliton vector field $X = \pi^*(r\partial_r)$, the lift of the vector field $r\partial_r$ on C_0, and with $\mathcal{L}_{JX} g_c = 0$, such that

$$|(\nabla^{g_0})^k(\pi_* g_c - cg_0 - \mathrm{Ric}(g_0))|_{g_0} \leq C(k) r^{-4-k} \quad \text{for all } k \in \mathbb{N}_0$$

if and only if

$$\int_V (i\Theta)^k \wedge \omega^{\dim_{\mathbb{C}} V - k} > 0 \tag{3}$$

for all positive-dimensional irreducible analytic subvarieties $V \subset E$ and for all $1 \leq k \leq \dim_{\mathbb{C}} V$ for some Kähler form ω on M and for some curvature form Θ of a hermitian metric on K_M.

We call a resolution of C_0 satisfying condition (a) here an *equivariant resolution*. Such a resolution of a complex cone always exists [8]. What is not clear a priori is if this resolution satisfies condition (3). From (2), one can see that (3) is in fact a necessary condition on M in Theorem 1 to admit an expanding Kähler-Ricci soliton. Furthermore, as remarked in [7], an asymptotically conical ("AC") Kähler manifold of complex dimension $n \geq 2$ can only have one end, hence in these dimensions having one end is also a necessary condition on M in Theorem 1 to admit AC Kähler-Ricci solitons.

As an application of Theorem 1, we obtain new examples of AC gradient expanding Kähler-Ricci solitons on the total space of certain holomorphic line bundles, thereby extending previous work of Dancer-Wang [4].

Corollary 2 (C.-Deruelle [3]). *Let L be a negative holomorphic line bundle over a compact Kähler manifold D, let $\pi : L \to L^\times$ denote the blowdown of the zero section of L, and let g_0 be a Kähler cone metric on $L^\times$ with Ricci curvature $\mathrm{Ric}(g_0)$ and with radial function r such that $\frac{1}{a} \cdot r\partial_r$ is the Euler vector field[1] on $L \setminus \{0\}$ for some $a > 0$.*

Then for all $c > 0$, there exists a unique expanding gradient Kähler-Ricci soliton g_c on the total space of L with soliton vector field $X = \pi^*(r\partial_r)$ a scaling of the Euler vector field on L by a, such that

$$|(\nabla^{g_0})^k(\pi_* g_c - cg_0 - \mathrm{Ric}(g_0))|_{g_0} \leq C(k) r^{-4-k} \quad \text{for all } k \in \mathbb{N}_0$$

if and only if $c_1(K_D \otimes L^*) > 0$.

[1] By the Euler vector field on a vector bundle E, we mean the infinitesimal generator of the homotheties of E.

Notice that Corollary 2 asserts that the total space of $L^{\otimes p}$ admits an expanding gradient Kähler-Ricci soliton asymptotic to a cone at infinity for any negative line bundle L over a compact Kähler manifold whenever p is sufficiently large. Corollary 2 follows from Theorem 1 after applying the adjunction formula and noting that the blowdown map $\pi : L \to L^{\times}$ is a Kähler equivariant resolution of $L^{\times}$ with respect to any positive scaling of the standard $\mathbb{C}^*$-action on these spaces and that X restricted to the zero section of L, that is, the exceptional set of the resolution π, vanishes, so that the final condition of hypothesis (b) of Theorem 1 is satisfied with $A = E$.

Our strategy of proof of Theorem 1 follows closely the work of Siepmann [10]. We first construct a background AC Kähler metric which serves as an "approximate" expanding Kähler-Ricci soliton. We then perturb this metric to a precise expanding gradient Kähler-Ricci soliton by solving a complex Monge-Ampère equation. Since we are missing a $\partial\bar{\partial}$-lemma (K_M has the wrong sign), we require hypothesis (b) of Theorem 1 to set up the complex Monge-Ampère equation which we then solve by implementing the continuity method as in the seminal work of Aubin [1] and Yau [11] on the existence of Kähler-Einstein metrics on compact Kähler manifolds, although we work with weighted function spaces in order to compensate for the non-compactness of our situation. We also work invariantly under the corresponding real torus action in order to obtain an a priori C^0-estimate on the radial derivative of solutions in the closedness part of the continuity method. As a consequence of this invariance, our expanding Kähler-Ricci solitons are also invariant under the corresponding real torus action from which it follows from hypothesis (b) of Theorem 1 that they are necessarily gradient Kähler-Ricci solitons.

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Sasaki-Einstein metrics and K-stability

Tristan C. Collins
(joint work with Gabor Székelyhidi)

Sasakian geometry is an odd-dimensional generalization of projective Kähler geometry, which has recently gained interest due to its role in both Kähler geometry, where Sasakian manifolds appears as links of tangent cones, and theoretical physics, where Sasakian manifolds play a role in the AdS/CFT correspondence. A Riemannian manifold (S, g) is Sasakian if there is a complex structure J on the metric cone $(S \times \mathbb{R}_{>0}, dr^2 + r^2 g)$ so that $r\partial_r$ is real holomorphic, and $(S \times \mathbb{R}_{>0}, dr^2 + r^2 g, J)$ is Kähler. Of particular interest are those Sasakian manifolds for which g is also Einstein. The objective of this talk is to address the question of when such metrics exist.

In general, Sasaki-Einstein metrics are obstructed. For example, if X is a Fano Kähler manifold (ie. $-K_X$ is ample), then the $U(1)$ bundle in K_X defined by a negatively curved metric is a Sasakian manifold, and this manifold is Einstein if and only if X is Kähler-Einstein with positive scalar curvature. The famous Yau-Tian-Donaldson conjecture, solved recently by Chen-Donaldson-Sun [1–3], predicts that the existence of a Kähler-Einstein metric on a Fano manifold is equivalent to K-stability, an algebro-geometric notion.

The aim of this talk is to introduce K-stability for Sasakian manifolds, and discuss the connection with Sasaki-Einstein metrics. We introduce K-stability [4], and discuss the connection with the Einstein-Hilbert functional, building on work of Martelli-Sparks-Yau [6]. The main theorem of this talk is that (G-equivariantly) K-stable Sasakian manifolds admit Sasaki-Einstein metrics [5]. By exploiting the G-equivariance to restrict the number of test configurations we explicitly check that the links of the Brieskorn-Pham singularities

$$Z_{p,q} = x^2 + y^2 + z^p + w^q$$

and K-stable provided $2p > q$ and $2q > p$. This produces infinitely many non-isometric Einstein metrics on the five sphere.

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The renormalized volume of quasifuchsian manifolds

Jean-Marc Schlenker

The renormalized volume of quasifuchsian manifolds can be considered as a special case of the renormalized volume of Poincaré-Einstein manifolds, see e.g. [6,8]. It is also strongly related, however, to the Liouville functional in complex analysis, see e.g. [13, 14]. It is also possible to define a corresponding Chern-Simons invariant, with the volume and Chern-Simons invariant being the real and imaginary part of a section of a complex line bundle, see [7].

Let M be a quasifuchsian manifold, homeomorphic to $S\times\mathbb{R}$, where S is a closed, oriented surface of genus $g\geq 2$. Its boundary at infinity $\partial_\infty M$ is the disjoint union of two copies of S, each endowed with a complex structure $(c_+,c_-)\in\mathcal{T}_S\times\mathcal{T}_{\bar S}$. Given two metrics h_+ and h_- on S compatible respectively with c_+ and c_-, it follows from work of Epstein [5] that there exists an equidistant foliation $(S_{\pm,r})_{r\geq r_0}$ of a neighborhood of infinity in M, unique up to the choice of r_0, such that if $I_{\pm,r}$ is the induced metric on $S_{\pm,r}$ then

$$\lim_{r\to\infty} 2e^{-2r}I_{\pm,r}=h_\pm\ .$$

We then define a quantity $W(h_+,h_-)$ as the constant term in the asymptotic expansion of the volume $V(r)$ of the region between $S_{+,r}$ and $S_{-,r}$:

$$V(r)=V_2e^{2r}+V_1r+W(h_+,h_-)+o(1)\ .$$

The renormalized volume $V_R(M)$ is then defined by maximizing $W(h_-,h_+)$ over the conformal classes at infinity, using the following statement.

Lemma 1. *Among metrics h_+,h_- of area $\pi|\chi(S)|$ in the conformal classes at infinity, $W(h_+,h_-)$ is maximal exactly when h_+ and h_- have constant curvature -2.*

Using the Bers simultaneous uniformization theorem, V_R can be considered as a function $V_R:\mathcal{T}_S\times\mathcal{T}_{\bar S}\to\mathbb{R}$. It follows from its relation to the Liouville functional (but can also be proved directly, see [10, 11]) that for $c_-\in\mathcal{T}_{\bar S}$ fixed, $V_R(\cdot,c_-):\mathcal{T}_S\to\mathbb{R}$ is a Kähler potential for the Weil-Petersson metric on $\mathcal{T}_S$.

The renormalized volume has a simple variational formula. In a first-order deformation of M, determined by a first-order variation $\dot c$ of the complex structure at infinity (considered as a Beltrami differential),

$$\dot V_R=\Re(\langle \dot c,q\rangle)\ . \qquad (1)$$

Here q is the Schwarzian derivative of the uniformization map on the universal covering of each connected component of the boundary at infinity of M, and $\langle,\rangle$ is the duality pairing between Beltrami differentials and holomorphic quadratic differentials. Specifically, if ∂_+M is the upper boundary at infinity of M and $\phi_+:\widetilde{\partial_+M}\to D$ is the uniformization map to the disk, then $q_+=-\mathcal{S}(\phi_+)$, where $\mathcal{S}$ denotes the Schwarzian derivative and $\mathcal{S}(\phi_+)$ is considered as a holomorphic quadratic differential, and similarly for ∂_-M.

More recently [12], building on the construction in [10], the renormalized volume was connected to the volume of the convex core, and bounded from above in terms of the Weil-Petersson distance between c_- and c_+.

Theorem 1. *Let V_C denote the volume of the convex core of M, and let m and l be the induced metric and measured bending lamination on the boundary of the convex core. Then, under the conditions above:*

(1) $V_R \leq V_C - L_m(l)/4 \leq V_R + C_g$, *where C_g is a constant depending only on the genus of S,*
(2) $V_R \leq 3\sqrt{\pi(g-1)}d_{WP}(c_-, c_+)$.

Note that $L_m(l)$ is bounded from above by a constant depending on the genus only, see [2]. Note also that those results were extended to convex co-compact hyperbolic manifolds, with significant differences for manifolds with compressible boundary, see [1].

A direct consequence is that $V_C \leq 3\sqrt{\pi(g-1)}d_{WP}(c_-, c_+) + C'_g$, where C'_g depends only on the genus of S. This adds some light to a result of Brock [3], who proved that the volume of the convex core is quasi-equivalent (with constants depending on the genus) to the Weil-Petersson distance between the conformal metrics at infinity.

Theorem 1 has applications to the geometry of closed hyperbolic manifolds and of the Weil-Petersson metric on moduli space.

Kojima and McShane [9] and Brock and Bromberg [4] show that given a pseudo-Anosov diffeomorphism $\phi : S \to S$, its entropy $\mathrm{ent}(\phi)$ is bounded from below by $1/3\pi|\chi(S)|$ times the hyperbolic volume of the mapping torus N_ϕ.

Brock and Bromberg [4] also give a number of explicit estimates on the systoles, and other geometric quantities of interest, on the moduli space $\mathcal{M}_S$ equiped with the Weil-Petersson metric. A particularly striking result is that the systole of $\mathcal{M}_S$ is bounded from below by $1/3\sqrt{\pi(g-1)}$ times the volume of the smallest closed hyperbolic 3-dimensional manifold.

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