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## Mathematical Questions and Challenges in Quantum Electrodynamics and its Applications

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ABSTRACT. Quantum field theory (QFT) may be considered one of the most fundamental frameworks of theoretical physics. Quantum Electrodynamics (QED) is the part of QFT that describes the interaction between matter and light. Although it is one of the experimentally best tested theories, it yet faces many open mathematical questions and challenges. The mathematical rigorous framework of QED and the implications deriving from it is the topic of Workshop 1737 held at MFO from September 11 through 15, 2017, bringing together mathematicians and theoretical physicists to discuss topics such as high- and low-energy QED, external field QED, quantum optics, many-boson and many-fermion systems, transport properties in condensed matter.

*Mathematics Subject Classification (2010):* 81V10, 81V70.

### Introduction by the Organisers

MFO Workshop 1737 *Mathematical Questions and Challenges in Quantum Electrodynamics and its Applications*, organized by Volker Bach (TU Braunschweig), Miguel Ballesteros (UNAM Mexico City), Dirk-André Deckert (LMU Munich), and Israel Michael Sigal (U of Toronto) was attended by 51 mathematicians and theoretical physicists. They represented a broad spectrum of scientific expertise, including mainstream and off-mainstream directions of research, and also a rather well-balanced blend of junior and senior scientists. Notably, eight female scientist participated and six of them presented their research results in a talk.

The presentations of mathematical results directly related to quantum electrodynamics (QED) constituted the core of the workshop program. Several presentations dealt with central objects and questions of QED, such as propagators and their construction – especially in the presence of external fields, the infrared problem and the ultraviolet problem and their solution in terms of renormalization schemes - with formulations ranging from concrete models and suitable resummations to new variational approaches such as the principle of the fermion projector, and finally to quantum field theoretic scattering matrices and their convergence as formal power series. Moreover, effective theories of Hartree-Fock type, the polaron model or the Euler-Heisenberg action deriving from QED were formulated and analyzed. The work horse of quantum optics, the spin-boson model, was also subject of several presentations, linking QED to concrete questions of decoherence and thermalization of systems. Several other presentations dealt with the derivation of effective descriptions of many-particle systems and especially cold quantum gases and other topics of statistical mechanics like Lieb-Robinson bounds yielding new correlation estimates rounded off the programme.

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## Abstracts

### Derivation of the (magnetic) Euler-Heisenberg Lagrangian in QED

MATHIEU LEWIN

In quantum field theory, the vacuum is a fluctuating medium which behaves as a nonlinear polarizable material. A convenient way of describing this effect is to use an effective action. In Quantum Electrodynamics (QED), this method corresponds to integrating out the electronic degrees of freedom in the full QED functional integral. The effective action is a function of a classical electromagnetic field treated as an external one. In the case of a constant electromagnetic field, the effective action has a rather simple explicit expression, which has been first derived by Euler and Heisenberg in [3]. This functional has been used to make spectacular predictions, like the birefringence of the vacuum which has only been confirmed recently, in the neighborhood of some neutron stars [4].

For time-independent fields in the Coulomb gauge, the effective Lagrangian action takes the form [1]

$$\mathcal{L}(V, A) := -\mathcal{F}_{\text{vac}}(eV, eA) + e \int_{\mathbb{R}^3} (j_{\text{ext}}(x) \cdot A(x) - \rho_{\text{ext}}(x)V(x)) dx + \frac{1}{8\pi} \int_{\mathbb{R}^3} (|E(x)|^2 - |B(x)|^2) dx$$

where  $E = -\nabla V$  and  $B = \text{curl}A$  are the electric and magnetic fields, and  $\mathcal{F}_{\text{vac}}(eV, eA)$  is the ground state energy of the Dirac vacuum in the fixed electromagnetic potential  $(V, A)$ . This term gives rise to nonlinear and nonlocal corrections to the classical linear Maxwell equations.

In the *Local Density Approximation*, the complicated energy  $\mathcal{F}_{\text{vac}}(eV, eA)$  is replaced by a local functional, that is,

$$(1) \quad \mathcal{F}_{\text{vac}}(eV, eA) \simeq \int_{\mathbb{R}^3} f_{\text{EH}}(eE(x), eB(x)) dx$$

where  $f_{\text{EH}}(eE, eB)$  is by definition the energy per unit volume of the vacuum in a *constant* electromagnetic field  $(E, B)$ . This function has a complicated explicit expression which was derived in [3, 6, 5]. In particular, for  $E \neq 0$  it has an exponentially small imaginary part, which is interpreted as the electron-positron pair production rate. The situation is easier for  $E \equiv 0$ , which is the case studied in [2] and discussed in the talk. Then

$$(2) \quad f_{\text{EH}}(0, eB) = \frac{1}{8\pi^2} \int_0^\infty \frac{e^{-sm^2}}{s^3} \left( es|B| \coth(es|B|) - 1 - \frac{e^2 s^2 |B|^2}{3} \right) ds,$$

where  $m$  is the mass of the electron. The reduced Planck constant  $\hbar$  and the speed of light  $c$  are set equal to 1.

Our main result in [2] is the justification of (1) for  $E \equiv 0$  in the regime where  $B$  varies slowly in space. To be more precise, we assume that the magnetic field

takes the form  $B(\varepsilon x)$  with a fixed smooth function  $B$  and then look at the limit  $\varepsilon \rightarrow 0$ . This corresponds to the strong magnetic potential  $A_\varepsilon(x) = \varepsilon^{-1}A(\varepsilon x)$ .

The ground state vacuum energy has to be regularized, due to well known ultraviolet divergences. It was properly defined in [1], using the Pauli-Villars regularization, by

$$(3) \quad \mathcal{F}_{\text{vac}}^{\text{PV}}(eV, eA) := \frac{1}{2} \text{tr} \sum_{j=0}^2 c_j \left( |D_{m_j, 0}| - |D_{m_j, eV, eA}| \right),$$

where  $D_{m, eV, eA} := \sum_{j=1}^3 \alpha_j (-i\partial_{x_j} - eA_j(x)) + eV(x) + m\beta$  is the Dirac operator in the external fields. We have  $c_0 = 1$  and  $m_0 = m$ . The ultraviolet divergences are removed if the coefficients  $c_1$  and  $c_2$  satisfy the Pauli-Villars conditions  $\sum_{j=0}^2 c_j = \sum_{j=0}^2 c_j m_j^2 = 0$ , which are assumed to hold from now on. The masses  $m_1, m_2$  play the role of a cut-off.

**Theorem 1** (Derivation of the magnetic Euler-Heisenberg functional). *Let  $B \in C^0(\mathbb{R}^3, \mathbb{R}^3)$  be such that  $\text{div} B = 0$ ,  $B \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$  and  $\nabla B \in L^1(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$ . Let  $A$  be the associated magnetic potential in  $\dot{H}^1(\mathbb{R}^3)$  with  $\text{div} A = 0$ . Set finally  $A_\varepsilon(x) = \varepsilon^{-1}A(\varepsilon x)$ . Then, we have*

$$(4) \quad \varepsilon^3 \mathcal{F}_{\text{vac}}^{\text{PV}}(0, eA_\varepsilon) = \int_{\mathbb{R}^3} f_{\text{EH}}^{\text{PV}}(0, e|B(x)|) dx + O(\varepsilon)$$

where

$$(5) \quad f_{\text{EH}}^{\text{PV}}(b) := \frac{1}{8\pi^2} \int_0^\infty \left( \sum_{j=0}^2 c_j e^{-sm_j^2} \right) \left( sb \coth(sb) - 1 \right) \frac{ds}{s^3}$$

is the Pauli-Villars-regulated Euler-Heisenberg vacuum energy.

The original expression (3) can be recovered from (5) after removing the cut-off masses  $m_1, m_2$ , through charge renormalization, see the discussion in [2, Sec. 2.3]. After scaling, the result (4) corresponds to a semi-classical limit in a strong magnetic field.

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### Topological edge states for bosonic systems

HERMANN SCHULZ-BALDES

The aim of the talk was to show how by now well-known techniques for the description of the integer quantum Hall effect can be adapted to deal with the thermal quantum Hall effect in two-dimensional dirty superconductors and to prove the existence of topological edge modes in two-dimensional bosonic systems. This is achieved within the framework of fermionic and bosonic quadratic Hamiltonians described by the Bogoliubov-de Gennes (BdG) formalism. The structure of the presentation was:

- A short review of the integer quantum Hall effect with a particular focus on the bulk-edge correspondence, namely it is recalled that the Chern numbers of the Fermi projection of a given one-particle Hamiltonian is equal to the (quantized) current flowing in the edge channels of a half-space restriction of the Hamiltonian [6, 5].
- The Chern number of the Fermi projection of a general fermionic BdG Hamiltonian with pairing potential is introduced by the standard formula. In these superconductor systems, charge is not a conserved quantity and one is thus lead to study heat transport. While a rigorous derivation of the Kubo formula for the Hall heat conductance [9] is still an open question, it is possible to show that the Chern number dictates the low temperature behavior of the heat current flow in the edge channels of the half-space restrictions [3], similar as in the integer quantum Hall effect. This is called the thermal quantum Hall effect.
- The differences in the BdG formalism for bosonic quadratic Hamiltonians are stressed, notably the BdG Hamiltonian implementing the time evolution is now only a  $J$ -hermitian operator on the Krein space given by the particle-hole Hilbert space. This makes the spectral analysis more involved. Nevertheless, it is possible to define Chern numbers for all bands of the bosonic BdG Hamiltonian. This was first used in [8] for periodic systems, and is extended to disordered systems in the work [5]. While the Chern number is not directly linked to any observable quantity as in the integer quantum Hall effect (and other fermionic systems like spin quantum Hall systems [3]), it can be shown to be connected to the number of topological edge state in half-space versions of these models. The proof of this bosonic bulk-boundary correspondence is the main objective of [5] and confirms the numerical and heuristic results used in the literature on systems of magnons, photonic crystals, lattices of microwave resonators and cold atoms (see the literature in [5]).

- Another interesting aspect of quadratic bosonic systems is that the  $J$ -hermitian BdG Hamiltonian can have spectrum off the real axis. This typically appears after a Krein collision of (bands of) eigenvalues with opposite Krein signature (see *e.g.* [7]). This complex spectrum then leads to dynamical instabilities of the system. It is possible to construct two-dimensional models which are dynamically stable in the bulk (namely, real spectrum of the bulk BdG Hamiltonian), but for which the edge modes are dynamically unstable. This idea goes back to Barret [1] in one-dimensional models, and is also discussed in [5].

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## External field QED on Cauchy surfaces

FRANZ MERKL

(joint work with Dirk-André Deckert)

We consider the evolution of a second-quantized Dirac fermion field between Cauchy surfaces of Minkowski space-time in an external classical smooth electromagnetic four-vector potential. According to classical results of Shale, Stinespring, and Ruisenaars, the second-quantized Dirac time evolution between all equal-time hyperplanes can be implemented in the standard Fock space if and only if the spatial part of the four-vector potential vanishes. However, we show that an implementation is always possible between varying Fock spaces, associated to polarization classes that depend on the four-potential restricted to the tangent bundle of the Cauchy surface. The approach is Lorentz and gauge covariant. However, in general the admissible polarizations lead to inequivalent Fock space representations. Actually, finding polarization classes in a gauge covariant way was the leading geometric motivation behind the approach.



The implementation of second-quantized Dirac evolution between varying Fock spaces associated to Cauchy surfaces leaves a global phase undetermined. Identifying this phase and its remaining degrees of freedom with respect to appropriate constraints is the topic of ongoing research.

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*Introductory overview:*

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### The Sine Gordon model in perturbative AQFT

DOROTHEA BAHNS

(joint work with Kasia Rejzner and Klaus Fredenhagen)

Originally given as a formal power series in the coupling constant, the  $S$ -matrix of the Sine Gordon model is shown to be summable in the model’s UV finite regime on 2-dimensional Minkowski spacetime.

The estimates are achieved in the framework of perturbative algebraic quantum field theory, which – to put it in a nutshell – is a formalism to study models in perturbative quantum field theory without Fock space or the annihilation/creation operator formalism. Its main input are the (uniquely determined) advanced and retarded fundamental solutions of a (hyperbolic) PDE and the choice of a Hadamard function, i.e. a symmetric bisolution of the same PDE with certain properties, which is in general not unique, particularly not in curved spacetimes, but whose existence is guaranteed by abstract arguments for large classes of models. Using these functions (distributions), certain star products (in the sense of formal deformation quantization) are introduced on certain spaces of functionals, that capture the notion of e.g. normal ordered products and of the formal Dyson series for the  $S$ -matrix. This input also gives rise to a state (Hadamard state) on the algebra of observables (constructed from certain spaces of functionals) which in turn via a GNS construction leads to a representation thereof.

Renormalization in this framework is formulated in the Epstein Glaser setting, as an extension of distributions problem. In the present work however, i.e. the UV finite part of the sine Gordon model, no renormalization is necessary.

Beyond the published estimates [1], I discussed some work in progress in my talk. I addressed in particular the fact that a certain restriction on the testfunction that cuts off the interaction (infrared cutoff) is not necessary if one considers not a representation in a generic (massless) Hadamard state but a particular one,

which leads to a representation of the massless field in 2 dimensions as in [2]. I also gave hints on how to calculate the local field of von Neuman algebras from the Bogolubov formula in this framework. Future work should involve a better understanding of infrared problems and a comparison with the Euclidean framework.

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### **The Klein-Gordon equation on curved spacetimes and its propagators**

JAN DEREZIŃSKI

(joint work with Daniel Siemssen)

The Klein-Gordon equation (including an electromagnetic potential) has several natural Green's functions, often called propagators. The so-called Feynman propagator, used in quantum field theory, has a clear definition on static spacetimes. I discuss, partly on a heuristic level, its possible generalizations to the non-static case. I also describe a curious open problem about whether the Klein-Gordon operator is self-adjoint on the space of square-integrable functions on space-time. The answer is positive when the space-time is static, otherwise it seems to be a difficult question.

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## Adiabatic Pair Creation in Heavy Ion and Laser Fields

PETER PICKL

(joint work with Detlef Dürr and Johannes Nissen-Meyer)

The creation of an electron positron pair in an almost stationary very strong external electromagnetic field (a potential well) is often referred to as spontaneous pair creation ([1],[2],[3]).

This adiabatic phenomenon emerges straightforwardly from the Dirac sea interpretation of negative energy states: An adiabatically increasing field (a potential well of changing deepness) lifts a particle from the sea to the positive energy subspace (by the adiabatic theorem) where it hopefully scatters and when the potential is switched off one has one free electron and one unoccupied state – a hole – in the sea [4]-[6]. A better terminology – and the one we shall use here – is thus adiabatic pair creation (APC).

The planned generation of lasers and heavy ion colliders renews the hope to see electron-positron pair creation in strong classical fields.

In the talk the theory of adiabatic pair creation will be presented. In particular I will argue, that an external field approximation always goes hand in hand with a time-adiabatic situation.

Then adiabatic pair creation in pure electric fields shall be discussed. Using the adiabatic theorem I will show that APC is impossible if the external field lies beyond a certain critical threshold. In this case the bridge between the Dirac sea and the positive spectrum is missing. For an overcritical field, i.e. a field where such a bridge does exist, I will show that the probability of creating a pair goes to one in the adiabatic limit. This is done under the assumption that there is a bound state with energy zero for the critical potential, which is in fact the generic case.

After that it will be shown that pure laser based experiments are highly questionable. For pure laser field there is always a gap in the spectrum, thus assuming the validity of the adiabatic theorem for the situation of laser fields, one directly gets that there is no pair creation in the adiabatic limit for pure laser fields, in contrast to predictions in the physics literature.

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## Thermodynamical Stability and Dynamics of Lattice Fermions with Mean-Field Interactions

WALTER DE SIQUEIRA PEDRA

(joint work with Jean-Bernard Bru, Sébastien Breteaux, and Rafael Miada)

For lattice fermions we study the thermodynamic limit of the time evolution of observables when the corresponding finite-volume Hamiltonians contain mean-field terms (like, e.g., the BCS model). It is well-known that, in general, this limit does not exist in the sense of the norm of observables, but may exist in the strong operator topology associated to a well-chosen representation of the algebra of observables. We show that this is always the case for any cyclic representation associated to an invariant minimizer of the free energy density, if the Hamiltonians are invariant under translations. Our proof uses previous results on the structure of states minimizing the free energy density of mean-field models along with Lieb-Robinson bounds for the corresponding families of finite-volume time evolutions. This is a joint work with Jean-Bernard Bru, Sébastien Breteaux and Rafael Miada.

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## Irreversible dynamics emerging from quantum resonances

MARCO MERKLI

(joint work with Martin Könenberg)

We consider the dynamics of quantum systems which possess stationary states as well as slowly decaying, metastable states arising from the perturbation of bound states. Our main result is a decomposition of the time evolution propagator into: a sum of a stationary part, an exponentially (time) decaying part and a polynomially (time) decaying remainder. The decay rates and decay directions are determined by the resonance data of the Hamiltonian. Our approach is based on an elementary application of the Feshbach map to the resolvent representation of the propagator [1].

We explain the motivation for developing this theory by outlining recent results on the dynamics of the spin-boson model at large (arbitrary) coupling strength [2, 1, 4]. This model is being used to describe charge and excitation transport in quantum chemical processes found in living organisms, such as photosynthesis in plants and bacteria [3] (“Quantum Biology”). Experimentally found values of the parameters appearing in the model force us to consider a strong coupling between the spin and the thermal Bose field. It turns out that in this parameter regime, conventional mathematical techniques to analyze the spin-boson dynamics break down (spectral deformation methods). The work presented is mainly the content

of [1], where we establish a mathematical method to extract decay rates and decay directions in Hilbert space from Mourre theory, rather than from the technically much less demanding spectral deformation theory (which is not applicable in the present situation as mentioned above).

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### Renormalization of linearly coupled models

JACOB SCHACH MØLLER

(joint work with Andreas Wünsch)

Models of non-relativistic quantum matter linearly coupled to a scalar boson field, e.g. Fröhlich's polaron model and the Nelson model, may be ultraviolet renormalized, using several different renormalization schemes. If the dynamics of the quantum particle is not governed by the Laplacian, however, the choices narrow significantly. In this talk we revisit a resummation scheme on the level of resolvent expansions, due to Hepp and Eckmann, which permits the construction of an ultraviolet renormalized Hamiltonian via norm-resolvent convergence of the Hamiltonian with its self-energy subtracted.

### On the Nature of Energy Full Statistics: fluctuations' control and heavy-tailed distributions and ultraviolet regularization

ANNALISA PANATI

(joint work with Tristan Benoist and Renaud Raquépas)

We present a study of (free) energy statistical fluctuations in a perturbed system in the context of quantum statistical mechanics.

At the classical level, it is well understood that energy fluctuations are controlled by the strength of the perturbation. Consider a classical system whose dynamics is governed by an Hamiltonian  $H_\lambda = H_0 + \lambda V$ . Assume the system is initially at equilibrium with respect to the dynamics associated  $H_0$ . In other words, we think of  $H_0$  as the free energy and  $V$  as a perturbation.

Then, in classical physics, the total (free) energy variation  $\Delta Q_t := H_0(t) - H_0$

statistics<sup>1</sup> is equal to the  $\lambda V - \lambda V(t)$  statistics since  $H_\lambda(t) = H_\lambda$ . If the perturbation  $V$  is bounded, the total energy variation statistics  $\Delta Q_t$  is almost surely absolutely bounded and for  $t$  large enough,  $\frac{1}{t}\Delta Q_t$  almost surely concentrates in 0. Moreover,  $\mathbb{P}_{\Delta Q_t}$  is always light-tailed. If  $V$  is unbounded, bounds on  $|\Delta Q_t|$  are typically controlled by the strength parameter  $\lambda$ .

We want to consider the quantum regime. In the last two decades, it has become clear that the nature of work/energy in quantum physics is more subtle than its classical counterpart. The emerging picture is that in quantum mechanics work should not be understood as an observable. Instead, the work performed during a given time period is identified with the energy variation observed in a repeated measurement protocol, where a measurement is performed initially and at the end of a given period of length  $t > 0$ . This change of perspective opened a new area of research [4, 2].

The distribution of the measured energy variations,  $\mathbb{P}_t(\Delta E)$  is the energy *full (counting) statistics*, also called two-time measurement distribution. The name “full counting statistics” first appeared in the seminal work of Levitov and Lesovik on charge transport [7]. When considering energy, probability distributions after thermodynamic limit are typically continuous and the word “counting” can be misleading. To avoid confusion, we use the term “full statistics”.

A prominent reason to consider full statistics is the key result by Kuchan [6] and Tasak and Matsui [9] providing extension of the celebrated fluctuation relations to quantum systems. The significance of the fluctuation relations relies in their intimate connection to the second law of thermodynamics and they have been extensively studied (see [8] and [4] for an overview).

Here we consider free energy full statistics. Its behaviour turns out to be considerably different than in the classical setting. In particular, boundedness of the perturbation does not guarantee a light-tailed distribution. The key point of this contribution is to show tails of the energy full statistics are controlled by an appropriate notion of regularity rather than the strength of the perturbation. If the regularity condition is not satisfied the heat full statistics can be heavy-tailed. This phenomenon has no classical counterpart.

A regularity condition was already introduced in our partly co-authored work [1], where we considered quantum dynamical systems arising as the limit of a sequence of confined systems. Here we work directly with infinitely extended systems via algebraic theory.

Let  $\mathcal{O}$  be the  $C^*$ -algebra of observables and  $\mathbb{R} \ni t \rightarrow \tau^t \in \text{Aut}(\mathcal{O})$  denote the unperturbed dynamics over  $\mathcal{O}$ . Let  $V \in \mathcal{O}$ ,  $V = V^*$  denote the perturbation. We assume at time 0 the system is in a state  $\omega$  which is invariant for the free dynamics  $\tau^t$ . After the first measurement the system evolves through the perturbed dynamics  $\tau_V^t = \tau^t + i[-, V]$ .

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<sup>1</sup>Given a classical observable  $A$  and an initial state  $\rho$ , we call  $A$ -statistics the probability measure  $\mathbb{P}_A$  such that  $\int f(s)d\mathbb{P}_A(s) = \int f(A)d\rho$  for all  $f \in \mathcal{C}_b(\mathbb{R})$

The following theorems show that conditions on the regularity of the map

$$t \rightarrow \tau^t(V)$$

are sufficient to guarantee control over large fluctuations.

Let  $\mathbb{E}_t$  denote expectation with respect to  $\mathbb{P}_t$ .

**Theorem 1.** *Assume*

(1)  $t \rightarrow \tau^t(V)$  is  $n$  times norm-differentiable.

Then

(2) 
$$\sup_{t \in \mathbb{R}} \mathbb{E}_t(\Delta E_t^{2n+2}) < \infty.$$

In terms of large fluctuations, it is immediate to show that (2) yields

$$\mathbb{P}_t(\Delta E \geq tR) \leq C(tR)^{-2n+2}$$

for some  $C > 0$ .

**Theorem 2.** *Assume*

(3)  $t \rightarrow \tau^t(V)$  admits a bounded analytic extension to the strip  $\{z \in \mathbb{C} : |\Im z| < \frac{1}{2}\gamma\}$ ,

then

(4) 
$$\sup_{t \in \mathbb{R}} \mathbb{E}_t(e^{\gamma|\Delta E_t|}) < \infty.$$

Condition (4) for some  $\gamma > 0$  yields boundness of any moment  $\mathbb{E}(|\Delta Q_t|^m)$  is uniformly bounded in time. Moreover, in terms of large deviations, for any  $\epsilon > 0$ ,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P}(|\Delta Q| > t\epsilon) \leq -\gamma\epsilon$$

for any  $\gamma$  such that (3) holds.

Our core result is to show condition (1) is essentially optimal by providing examples. In other words, we provide examples where energy full statistics can be heavy tailed.

The first model we consider is a quantum dot interacting with a free Fermi gas. The model can be mapped also to the second quantization of a Wigner-Weisskopf atom. We can show for this model condition (1) is *equivalent* to (2).

Moreover, we can show the same mechanism occur in bosonic models where the perturbation  $V$  is unbounded. We consider an harmonic oscillator interacting with a Bose gas (which can also be mapped to bosonic version of the second Wigner-Weisskopf atom) and van Hove Hamiltonian.

We show that assumptions similar to (1) and (3) are equivalent (2) and (4). Note that conditions (1) and (3) are independent of  $\lambda$ . Once again this shows that the tails are controlled by the interaction's regularity rather than by its strength.

In the explicit model we analyze, conditions (1) and (3) translate immediately as a fast decay conditions on the ultraviolet regularization function  $f$ , appearing in the perturbation. One can think that in the quantum setting, it is energy transitions induced by the perturbation that are relevant for the fluctuations. These energy transition rates and the contribution of high energy frequencies are controlled by the ultraviolet regularization.

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### No-photon QED

CHRISTIAN HAINZL

Consider electrons in the field of external electric charges. If the electrons are described by the Dirac operator then one needs to cope with the negative energy spectrum of the Dirac operator. Since the detection of positrons it these energy states had to be taken seriously and the free vacuum was interpreted as the negative spectrum filled with electrons. These electrons redistribute, i.e., polarize the vacuum. In this way the corresponding potential contributes to an effective electric field und acts back on the vacuum and real particles. Such a situation was first treated by Dirac in his wonderful paper “Theory du Positron” [1] from 1934. He uses Hartree-Fock approximation where states are given by infinite rank projections,  $P$ , which include real electrons and virtual electron positron pairs. Its evolution is described by the Heisenberg equation of motion

$$i\dot{P} = [H, P], \quad H = D^0 - \alpha V$$

where  $D^0 = \vec{\alpha} \cdot \frac{1}{i} \nabla + \beta$  the free Dirac operator with mass equal to 1, and  $V$  satisfies the equation

$$-\Delta V(x) = \nu(x) + \rho_P(x)$$



where  $\nu$  is the distribution of the external charges and  $\rho_P$  is the density of  $P$  measured relative to the free vacuum  $P^0 = \chi_{(-\infty,0]}(D^0)$ . I.e.,  $\rho_P(x) = \text{Tr}_{\mathbb{C}^4}(P - P^0)(x, x)$ . Dirac calculated  $\rho_P$  to leading order and obtained that it is proportional to  $\alpha \ln \Lambda \nu(x)$ . He argued that realistic energies are not larger than  $137mc^2$  and therefore the polarization potential slightly diminishes the external potential. This was the first appearance of a charge renormalization procedure.

In joint works with M. Lewin and E. Séré [2, 3, 4] we implemented Dirac’s idea via Hamiltonian formalism under the assumption that without charges present the free vacuum is given by the projection on the negative energies of the free Dirac operator. More precisely, our starting Hamiltonian is given by

$$(1) \quad \mathbb{H} = \int dx : \Psi^*(x) D^0 \Psi(x) :_{P^0} + \alpha \int dx : \Psi^*(x) \phi(x) \Psi(x) :_{P^0} + \frac{\alpha}{2} \int dx \int dy \frac{:\Psi^*(x) \Psi(x) \Psi^*(y) \Psi(y):_{P^0}}{|\mathbf{x} - \mathbf{y}|},$$

with  $\phi(x) = \nu * \frac{1}{|x|}$ . For quasi free states  $\psi$  we obtained  $\langle \psi | \mathbb{H} | \psi \rangle = \mathcal{E}(P' - P^0)$ , where

$$(2) \quad \mathcal{E}(Q) = \text{Tr}(D^0 Q) - \alpha \int \rho_Q \phi + \frac{\alpha}{2} \iint \frac{\rho_Q(x) \rho_Q(y)}{|x - y|} dx dy - \frac{\alpha}{2} \iint \frac{|Q(x, y)|^2}{|x - y|} dx dy$$

with  $\rho_Q(x) = \text{Tr}_{\mathbb{C}^4} Q(x, x)$ , and  $Q = P' - P_0$ . The states  $Q$  are in general not trace-class. For that reason we restrict the momentum space to momenta  $|p| \leq \Lambda$ , i.e. we introduce an ultra-violet cut-off  $\Lambda$ . We further have to replace the formal trace  $\text{Tr}(D^0 Q)$  with the  $P^0$ -trace  $\text{tr}_{P^0}(D^0 Q)$ , where one only used the diagonal elements of the  $2 \times 2$  matrix obtained by the decomposition with respect to  $P^0$ . I.e.,  $\text{tr}_{P^0}(A) = \text{Tr}(1 - P^0)A(1 - P^0) + \text{Tr} P^0 A P^0$ . The state space contains all  $Q = P - P^0$  with  $0 \leq P \leq 1$  and  $Q$  Hilbert- Schmidt with finite  $P^0$ -trace.

We show in [3] that for any  $\Lambda$  there exists a minimizer  $\bar{Q}$  of  $\mathcal{E}$  such that  $\bar{P} = \bar{Q} + P^0$  is a projector satisfying the self-consistent equation

$$(3) \quad \bar{P} = \chi_{(-\infty,0)} \left( D^0 - \alpha \varphi + \alpha \rho_{\bar{Q}} * \frac{1}{|\cdot|} - \alpha \frac{\bar{Q}(x, y)}{|\mathbf{x} - \mathbf{y}|} \right).$$

Additionally, if  $\alpha$  and  $\nu$  satisfy certain constraints then this global minimizer  $\bar{Q}$  is unique and the associated polarized vacuum is neutral, i.e.  $\text{Tr}_{P^0}(\bar{Q}) = 0$ , meaning that no effective charge was created out of the vacuum.

The same can be done in charge-sectors [4], i.e., under the constraint  $\text{tr}_{P^0} Q = N$ , with an additional Lagrange-Parameter  $\mu$

$$(4) \quad \bar{P} = \chi_{(-\infty,\mu]} \left( D^0 - \alpha \varphi + \alpha \rho_{\bar{Q}} * \frac{1}{|\cdot|} - \alpha \frac{\bar{Q}(x, y)}{|\mathbf{x} - \mathbf{y}|} \right) = P + \gamma,$$

where  $\text{Tr} \gamma = N$ . One can interpret that  $\gamma$  represents the real electrons, and  $P$  the polarized vacuum. The total density  $\rho_{\bar{Q}}$  contains a logarithmic divergent term in  $\Lambda$ . This has to be *interpreted away* by renormalization.

To this aim we studied in [3] the reduced BDF-functional without Exchange term. We denote the total minimizer as  $Q_{\text{vac}}$

$$(5) \quad Q_{\text{vac}} = \chi_{(-\infty;0]}(F) - P_-^0, \quad F = D^0 + \alpha(\rho_{\text{vac}} - \nu) * \frac{1}{|x|}$$

where this equation can be reduced to an equation for the density. This reads

$$(6) \quad \rho_{\text{vac}}(x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\eta \operatorname{Tr} \left[ \frac{1}{D^0 + \alpha(\rho_{\text{vac}} - \nu) * \frac{1}{|x|} + i\eta} - \frac{1}{D^0 + i\eta} \right] (x, x).$$

potentials In terms of  $\rho = \rho_{\text{vac}} - \nu$  this leads to the equation

$$(7) \quad \alpha \hat{\rho} = -\frac{\alpha}{1 + \alpha B_\Lambda} \hat{\nu} + \frac{\alpha}{1 + \alpha B_\Lambda} C_\Lambda(k) \alpha \hat{\rho} + \frac{\alpha}{1 + \alpha B_\Lambda} \hat{F}_3[\alpha \rho].$$

where  $B_\Lambda \simeq \frac{2}{3\pi} \log(\Lambda)$  and the function  $C_\Lambda(k)$  has a well defined limit  $C(k)$ , which was calculated by Serber [8] and Uehling [9]. Defining the physical fine structure constant

$$\alpha_{\text{ph}} = \frac{\alpha}{1 + \alpha B_\Lambda}$$

we set  $\alpha_{\text{ph}} \rho_{\text{ph}} = \alpha \rho$ , which corresponds to charge and wavefunction renormalization and obtain

$$(8) \quad \alpha_{\text{ph}} \hat{\rho}_{\text{ph}} = -\alpha_{\text{ph}} \hat{\nu} + \alpha_{\text{ph}}^2 C_\Lambda(k) \hat{\rho}_{\text{ph}} + \alpha_{\text{ph}} \hat{F}_3[\alpha_{\text{ph}} \rho_{\text{ph}}].$$

Calculating the electric potential corresponding to the term  $\alpha_{\text{ph}}^2 C(k) \hat{\rho}_{\text{ph}}$  leads to the so called Uehling potential. We notice that the renormalized charge  $\alpha_{\text{ph}} \rightarrow 0$  when  $\Lambda \rightarrow \infty$ , independently of  $\alpha$ . Further it was proved in [3] that the unique polarized vacuum  $P_\Lambda$  of the *reduced* BDF model satisfies

$$\lim_{\Lambda \rightarrow \infty} \operatorname{Tr} (P_\Lambda - P_-^0)^2 = 0 \quad \text{and} \quad \lim_{\Lambda \rightarrow \infty} D(\rho_{P_\Lambda - P_-^0} - \nu, \rho_{P_\Lambda - P_-^0} - \nu) = 0.$$

In words, when  $\Lambda \rightarrow \infty$ , the vacuum polarization density *totally cancels the external density*  $\nu$ , for  $\rho_{P_\Lambda - P_-^0} \rightarrow \nu$ . But since  $P_\Lambda - P_-^0 \rightarrow 0$ , this means that in the limit  $\Lambda \rightarrow \infty$ ,  $P_\Lambda - P_-^0$  and its associated density become independent. Therefore, the minimization without cut-off makes no sense both from a mathematical and physical point of view. Indeed all this easily implies that when no cut-off is imposed and when  $\nu \neq 0$ , the infimum of the reduced BDF functional is *not attained*. This corresponds to the well known Landau-pole problem.

The time-dependent equation corresponding to Dirac's original equation for  $P$  was investigated in [6].

In [5], see also [7], we posed the question of the correct free Dirac sea in the Hartree-Fock approximation including the interaction among the particles filling the Dirac sea. We studied this by starting from a formal Hamiltonian without any normal ordering, reducing to a box and letting the box tend to infinity. The result is a slightly changed projector satisfying the self-consistent equation

$$(9) \quad \begin{cases} \mathcal{P}_-^0 = \chi_{(-\infty;0]}(\mathcal{D}^0), \\ \mathcal{D}^0 = D^0 - \alpha \frac{(\mathcal{P}_-^0 - I/2)(x-y)}{|x-y|}. \end{cases}$$

In the presence of external fields we used this vacuum as starting point and calculated relative to the vacuum energy. This then led to a renormalized Bogolubov-Dirac-Fock-functional.

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**Supersymmetry and Ward identities: an alternative approach to renormalization.**

MARGHERITA DISERTORI

(joint work with Thomas Spencer and Martin Zirnbauer)

A major problem in QFT is to develop tools to implement some kind of multiscale analysis for theories where standard renormalization group techniques do not apply. In this context a possible tool may come from Ward identities. These are families of identities generated by the symmetries of the model and are often used in perturbative renormalization schemes to show, for example, that certain subclasses of Feynman diagrams cancel.

In this context, in collaboration with M. Zirnbauer and T. Spencer [2], we considered the so-called  $H^{2|2}$  supersymmetric nonlinear sigma model, introduced in [1] as a toy model for quantum diffusion. It is also a key ingredient in the construction and study of certain stochastic processes with memory (cf. [4] [5] [6]). For this model we constructed a multiscale procedure whose key ingredient is a infinite family of Ward identities generated by supersymmetry. We hope a similar strategy may extend to other models with and without supersymmetry.

1. A SUPERSYMMETRIC NONLINEAR SIGMA MODEL

Informally, one may view the construction below as a statistical mechanical 'Ising type' spin model, with spin taking values in a generalization of Minkowski  $2 + 1$

dimensional space. Let  $\Lambda \subset \mathbb{Z}^d$  be a cube. To each lattice point  $j \in \Lambda$  we attach a 'spin'  $S_j$

$$(1) \quad S_j = (t_j, x_j, y_j, \xi_j, \eta_j)$$

where  $\xi, \eta$  are odd elements and  $t, x, y$  are even elements of a real Grassmann algebra. The spin inner product is defined by the following generalization of Minkowski metric:

$$(2) \quad \langle S, S' \rangle = xx' + yy' - tt' + \xi\eta' - \eta\xi', \quad \langle S, S \rangle = x^2 + y^2 - t^2 + 2\xi\eta.$$

We introduce the finite volume 'Gibbs measure'

$$d\mu_\Lambda(S) := \frac{1}{Z_\Lambda} e^{-\mathcal{H}_\Lambda(S)} dS_\Lambda$$

where  $dS_\Lambda := \prod_{j \in \Lambda} dt_j dx_j dy_j d\xi_j d\eta_j$  and  $Z_\Lambda$  is the normalization constant. Finally the energy of each spin configuration is defined by

$$\mathcal{H}_\Lambda(S) := \beta \langle S, -\Delta_\Lambda S \rangle + \epsilon \sum_j (t_j - 1) = \beta \sum_{|i-j|=1} \langle S_i - S_j, S_i - S_j \rangle + \epsilon \sum_j (t_j - 1)$$

where  $\Delta_\Lambda$  is the discrete Laplacian, with some boundary conditions,  $\beta > 0$  is the inverse temperature and  $\epsilon > 0$  plays the role of a magnetic field or a mass term. Since  $\langle \cdot, \cdot \rangle$  is not positive definite the measure above is ill defined. To solve the problem we insert the nonlinear constraint

$$\langle S, S \rangle = -1 \quad \equiv \quad t^2 - x^2 - y^2 - 2\xi\eta = +1 \quad \equiv \quad t = \pm \sqrt{1 + x^2 + y^2 + 2\xi\eta}$$

Note that, neglecting for a moment the nilpotent term  $\xi\eta$ , this corresponds to require  $S$  to be a time-like vector. Below we select the manifold  $t > 0$ . One can check that with the constraint above it holds  $t_j - 1 \geq 0 \forall j$  and  $\langle S, -\Delta_\Lambda S \rangle \geq 0$  hence the measure is now well defined. The spin  $S$  is then parametrized by four independent variables, encoded in the vector  $u := (x, y, \xi, \eta)$ , with the following generalization of Euclidean metric  $(u, u') := xx' + yy' + \xi\eta' - \eta\xi'$ . The Gibbs measure with nonlinear constraint is then parametrized by

$$(3) \quad d\nu_\Lambda(u) := \frac{1}{Z_\Lambda} e^{-\mathcal{F}_\Lambda(u)} [du]_\Lambda$$

where  $[du]_\Lambda = \prod_j du_j [t_j(u)]^{-1}$  and  $t(u) = \sqrt{1 + (u, u)}$ . Finally

$$(4) \quad \mathcal{F}_\Lambda(u) := \beta(u, -\Delta_\Lambda u) - \beta(t(u), -\Delta t(u)) + \epsilon \sum_j (t_j(u) - 1)$$

Note that the spin  $S$  now lives on a hyperbolic plane parametrized by two even and two odd variables, hence the name  $H^{2|2}$  model.

**1.1. Heuristics and results.** At finite volume and  $\beta \gg 1$  the measure favours aligned spin configurations:  $S_i = S_j \forall i, j$ . In dimension  $d \geq 3$ , when the thermodynamic limit  $\Lambda \rightarrow \mathbb{Z}^d$  is taken together with  $\epsilon \rightarrow 0$  a transition has been established between a disordered phase at high temperature  $\beta \ll 1$  (cf. [3]) and an ordered phase at low temperature  $\beta \gg 1$  (cf. [2]).

2. SYMMETRIES

For  $H^{2|2}$  standard renormalization group tools are hard to apply. Instead the analysis is based on a sequence of Ward identities, generated by the symmetries. Indeed the measure  $d\nu(u)$  is invariant under global linear transformations  $u \rightarrow Au$  leaving the scalar product  $(\cdot, \cdot)$  invariant. An important role is played by transformations that mix even and odd elements of the Grassman algebra, the so called supersymmetric transformations. Their main feature can be informally stated as follows.

**Theorem 1** (Localization effect.). *Assume the function  $f(u_\Lambda)$  is smooth, decays fast enough at infinity and is invariant under supersymmetric transformations. Then*

$$(5) \quad \int f(u_\Lambda) \prod_j du_j = f(0).$$

Since  $f$  is arbitrary, we can construct a huge family of highly nontrivial identities. As a first application one may compute the normalization constant  $Z_\Lambda$ . By the localization theorem above, replacing  $f(u_\Lambda) = e^{-\mathcal{F}(u)} \prod_j [t_j(u)]^{-1}$  we get

$$Z_\Lambda = \int e^{-\mathcal{F}_\Lambda(u)} [du]_\Lambda = e^{-\mathcal{F}_\Lambda(0)} \prod_j [t_j(0)]^{-1} = 1$$

for any choice of the parameters  $\beta$  and  $\epsilon$ .

**2.1. Proof of Theorem 1.** For the general case see [2, App.C] and references therein. Here we will restrict to the following simple example. Assume there is only one lattice point, hence only one spin  $u = (x, y, \xi, \eta)$ . Let  $g \in C^\infty(\mathbb{R})$  with fast decay at infinity and consider  $f(u) := g((u, u)) = g(x^2 + y^2 + 2\xi\eta)$ . Then this function is invariant under supersymmetric transformations. We will prove (5) by two methods.

1) Proof by direct computation: Taylor expansion in  $\xi\eta$  yields

$f(u) = g(x^2 + y^2) + g'(x^2 + y^2)2\xi\eta$ . Then, passing to polar coordinates we get

$$I = \int f(u)du := \int dx dy \frac{1}{2\pi} \partial_\xi \partial_\eta f(u) = -2 \int_0^\infty g'(r^2) r dr = g(0) = f(0).$$

2) Proof by symmetry: Let  $I(\tau) := \int f(u) e^{-\tau(u, u)} du$ , with  $\tau \geq 0$ . Then  $I = I(0)$  and  $\lim_{\tau \rightarrow \infty} I(\tau) = f(0)$ . To conclude we need to prove  $I'(\tau) = 0$ . To see this consider the infinitesimal generator of the symmetry mixing even and odd elements given by  $Q := \bar{z} \partial_\xi - \eta \partial_z$ , where we abbreviated  $z = x + iy$  and  $\bar{z} = x - iy$ . One can check that  $Q^2 = 0$ ,  $Q(u, u) = 0$  and  $(u, u) = Qh$  where  $h := \frac{1}{2} z \xi$ . Then

$$I'(\tau) = - \int du f(Qh) e^{-\tau(Qh)} = - \int du Q [f h e^{-\tau Qh}] = 0.$$

## 3. MULTISCALE ANALYSIS

Let us see how symmetries can be used to infer bounds. Informally, by the localization theorem we get identities of the form

$$1 = \int [|x_j - x_k|^m (1 - mg_{jk}(u))] d\nu(u) \quad \forall m \geq 0.$$

For  $|j - k| = 1$  one can see that  $0 \leq g_{jk}(u) \leq \beta^{-1} \forall u$  configurations, hence

$$\int |x_j - x_k|^m d\nu(u) \leq \left(1 - \frac{m}{\beta}\right)^{-1} \leq 2 \quad \forall 0 < m \leq \frac{\beta}{2}.$$

For general  $j, k$  the argument above holds by restricting to the subset of 'good' configurations. To apply the localization theorem this set must be selected by a supersymmetric version of the corresponding characteristic function.

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### Lieb–Robinson Bounds for Multi–Commutators and Applications to Response Theory

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(joint work with Walter de Siqueria Pedra)

*Lieb–Robinson bounds* are upper-bounds on time-dependent commutators. They have first been derived in 1972 by Lieb and Robinson [1]. Nowadays, they are widely used in quantum information and condensed matter physics. Phenomenological consequences of Lieb–Robinson bounds have been experimentally observed in recent years, see [2].

As explained in [3] in the context of quantum spin systems, Lieb–Robinson bounds are expected to hold true for systems with short–range interactions. In [4] we define Banach spaces  $\mathcal{W}$  of short–range interactions and prove Lieb–Robinson bounds for the corresponding fermion systems. The spaces  $\mathcal{W}$  include density–density interactions resulting from the second quantization of two–body interactions defined via a real–valued and integrable interaction kernel  $v(r) : [0, \infty) \rightarrow$

$\mathbb{R}$ . The method of proof we use in [4] to get Lieb–Robinson bounds for non-autonomous  $C^*$ -dynamical systems related to lattice fermions is, up to simple adaptations, the one used in [3] for (autonomous) quantum spin systems.

Once the Lieb–Robinson bounds for commutators are established, we combine them with results of the theory of strongly continuous semigroups to derive properties of the infinite-volume dynamics. These allow us in [4] to extend Lieb–Robinson bounds to time-dependent *multi*-commutators. The new bounds on multi-commutators make possible rigorous studies of dynamical properties that are relevant for response theory of interacting fermion systems. In fact, by using the Lieb–Robinson bounds for multi-commutators, we extend in [6, 7] our results [5, 8, 9, 10] on free fermions to interacting particles with short-range interactions. This is an important application of such new bounds: The rigorous microscopic derivation of Ohm and Joule’s laws for *interacting* fermions, in the AC-regime.

The new bounds can also be applied to *non-autonomous systems*. Indeed, the existence of a fundamental solution for the non-autonomous initial value problem related to infinite systems of fermions with time-dependent interactions is usually a non-trivial problem because the corresponding generators are time-dependent unbounded operators. The Lieb–Robinson bounds on multi-commutators we derive in [4] yield the existence of fundamental solutions as well as other general results on non-autonomous initial value problems related to fermion systems on lattices with interactions which are non-vanishing in the whole space and time-dependent. This is done in a rather constructive way, by considering the large volume limit of finite-volume dynamics, without using standard sufficient conditions for existence of fundamental solutions of non-autonomous linear evolution equations. See [4].

Observe that the evolution equations for lattice fermions are not of parabolic type, in the precise sense formulated in [11], because the corresponding generators do not generate analytic semigroups. They seem to be rather related to Kato’s hyperbolic case [12, 13, 14]. Indeed, by structural reasons – more precisely, the fact that the generators are derivations on a  $C^*$ -algebra – the time-dependent generator defines a stable family of operators in the sense of Kato. Moreover, this family always possesses a common core. In some specific situations one can directly show that the completion of this core with respect to a conveniently chosen norm defines a so-called admissible Banach space  $\mathcal{Y}$  of the generator at any time, which satisfies further technical conditions leading to Kato’s hyperbolic conditions [12, 13, 14]. See also [15, Sect. 5.3.] and [16, Sect. VII.1]. Nevertheless, the existence of such a Banach space  $\mathcal{Y}$  is a priori unclear in the general case treated here.

For more details on these results, we refer to [4], which is written to be accessible to a wide audience, in particular to students in Mathematics with little Physics background. In particular, we introduce in [4] the algebraic setting for fermions, in particular the CAR  $C^*$ -algebra. Other standard objects (like fermions, bosons, Fock space, CAR, etc.) of quantum theory are also presented in [4], for pedagogical reasons.

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### Complete Bose-Einstein condensation in the Gross-Pitaevskii regime

SERENA CENATIEMPO

(joint work with Chiara Bocato, Christian Brennecke, and Benjamin Schlein)

Proving the existence of condensation for interacting bosons is a longstanding problem, tightly related to the occurrence of superfluidity in dilute Bose gases, as expected on the basis of the pioneering work of Bogoliubov [5]. So far, condensation has been proved only in a very special model [7] and the question of the validity of Bogoliubov theory is largely open [9, 17]. All attempts to improve Bogoliubov approximation encountered the difficulty of a singular perturbation theory, and a full non-perturbative construction for the bosonic model is far to be achieved, in spite of recent contributions to this program [1, 2]. More results are available if one considers scaling limits. In the mean field regime condensation has been proved with an  $N$  independent bound on the number of particles orthogonal to



the condensate (excitations) [16, 10, 8, 15]. This was also an important ingredient to show the validity of Bogoliubov theory for mean-field bosons [16, 10, 11, 8].

A more accurate model for the strong and rare interactions among bosons in cold atoms experiments is given by the Gross-Pitaevskii interaction  $V_N(x) = N^2V(Nx)$ . Correlations among the particles play a crucial role in the Gross-Pitaevskii regime, and this makes its study particularly challenging.

In the following we consider systems of  $N$  bosons in the three dimensional box  $\Lambda = [-1/2, 1/2]^{\times 3}$  with periodic boundary conditions, in the Gross-Pitaevskii limit. The Hamilton operator, acting on the space  $L_s^2(\Lambda^N)$  consisting of  $L^2(\Lambda^N)$  functions symmetric with respect to permutations of the particles, in second quantized form is

$$(1) \quad H_N = \sum_{p \in 2\pi\mathbb{Z}^3} p^2 a_p^* a_p + \frac{\kappa}{N} \sum_{p, q, r \in 2\pi\mathbb{Z}^3} \widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r},$$

where for every  $p \in 2\pi\mathbb{Z}^3$ ,  $a_p^*, a_p$  are the usual Fock space operators, creating and annihilating a particle with momentum  $p$ . A proof of condensation for the ground state of (1) (actually for the more general situation where particles are trapped by an external confining potential) was obtained in [12, 14], but with a bound on the rate of the convergence which is far from optimal. In this talk I report recent work in collaboration with C. Boccato, C. Brennecke and B. Schlein [3] where we give a proof of condensation valid for sufficiently small values of  $\kappa \geq 0$ , with a presumably optimal bound on the rate of the convergence. We also show that the ground state energy of (1) is  $4\pi a_0 N$  up to an error of order one, improving previous results obtained in [13, 14]. This is the content of the next theorem.

**Theorem 1.** *Let  $V \in L^3(\mathbb{R}^3)$  be non-negative, spherically symmetric and compactly supported and assume the coupling constant  $\kappa \geq 0$  to be small enough. Let  $\psi_N \in L_s^2(\Lambda^N)$  be a sequence with  $\|\psi_N\| = 1$  and such that*

$$(2) \quad \langle \psi_N, H_N \psi_N \rangle \leq 4\pi a_0 N + K$$

for some  $K > 0$ . Let  $\gamma_N^{(1)} = \text{tr}_{2, \dots, N} |\psi_N\rangle\langle\psi_N|$  be the one-particle reduced density associated with  $\psi_N$ . Then there exists a constant  $C > 0$ , depending on  $V$  and on  $\kappa$  but independent of  $K$  such that

$$(3) \quad N(1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle) \leq C(K + 1)$$

where  $\varphi_0(x) = 1$  for all  $x \in \Lambda$ . Furthermore, the ground state energy  $E_N$  of (1) is such that

$$|E_N - 4\pi a_0 N| \leq D$$

for a  $D > 0$  independent of  $N$  (depending only on  $V$  and  $\kappa$ ). Hence, the one-particle reduced density associated with the ground state of (1) satisfies (3), with  $K$  replaced by the constant  $D$ .

The proof of the theorem is based on the combination of techniques used in [11] for the analysis of interacting bosons in the mean field regime with ideas developed in [4, 6] to study the time-evolution in the Gross-Pitaevskii regime. Following [11] we introduce a unitary map which factors out the Bose-Einstein

condensate described by  $\varphi_0$  and it let us focus on its orthogonal excitations. We observe that every normalized  $\psi_N \in L_s^2(\Lambda^N)$  can be represented uniquely as

$$(4) \quad \psi_N = \sum_{n=0}^N \alpha_N^{(n)} \otimes_s \varphi_0^{\otimes(N-n)}$$

for a sequence  $\alpha_N^{(n)} \in L_\perp^2(\Lambda)^{\otimes_s n}$ . Here  $L_\perp^2(\Lambda)^{\otimes_s n}$  denotes the symmetric tensor product of  $n$  copies of the orthogonal complement  $L_\perp^2(\Lambda)$  of  $\varphi_0$  in  $L^2(\Lambda)$ . We set

$$(5) \quad U_N : L_s^2(\Lambda^N) \rightarrow \mathcal{F}_+^{\leq N} \quad \text{through} \quad U_N \psi_N = \{\alpha_N^{(0)}, \alpha_N^{(1)}, \dots, \alpha_N^{(N)}\},$$

where  $\mathcal{F}_+^{\leq N} = \bigoplus_{n=0}^N L_\perp^2(\Lambda)^{\otimes n}$  denote the bosonic Fock space constructed over  $L_\perp^2(\Lambda)$ , truncated to sectors with at most  $N$  particles. The guiding idea for proving Theorem 1 is to show that states in  $L_s^2(\Lambda^N)$  with small energy correspond to states in  $\mathcal{F}_+^{\leq N}$  with a small number of excitations, this implying condensation. To this purpose we define the excitation Hamiltonian

$$\mathcal{L}_N = U_N H_N U_N^* : \mathcal{F}_+^{\leq N} \rightarrow \mathcal{F}_+^{\leq N}.$$

In the same spirit of Bogoliubov approximation, conjugating  $H_N$  with  $U_N$  extracts from the original interaction term in (1) contributions that are constant, quadratic and cubic in the creation and annihilation operators. In the mean field regime the expected value of the cubic and quartic terms in  $\mathcal{L}_N$  vanishes as  $N \rightarrow \infty$  on excitation vectors of the form  $U\psi_N \in \mathcal{F}_+^{\leq N}$  associated with low energy states  $\psi_N \in L_s^2(\Lambda^N)$ . On the contrary, in the Gross-Pitaevskii regime the cubic and quartic terms are crucial to recover the correct ground state energy of the system. This is due to the fact that states with a small energy in the Gross-Pitaevskii limit are characterized by a short scale correlation structure. Motivated by [4, 6] we approximate states with small energy in the Gross-Pitaevskii regime with vectors of the form  $\psi_N = U_N^* T \xi_N$  with  $\xi_N \in \mathcal{F}_+^{\leq N}$  and  $T$  an operator which implements correlations among particles. More precisely,

$$T = \exp \left\{ \frac{1}{2} \sum_{p \in \Lambda_+^*} \eta_p [b_p^* b_{-p}^* - b_p b_{-p}] \right\}$$

where  $b_p^* = a_p^* \sqrt{\frac{N - \mathcal{N}_+}{N}} = U a_p^* \frac{a_0}{\sqrt{N}} U^*$  can be interpreted as an operator exciting a particle from the condensate to its orthogonal complement, and  $b_p$  is its conjugate. The choice  $\psi_N = U_N^* T \xi_N$  for low energy states corresponds to the definition of a new (renormalized) excitation Hamiltonian

$$\mathcal{G}_N = T^* U_N H_N U_N^* T : \mathcal{F}_+^{\leq N} \rightarrow \mathcal{F}_+^{\leq N}.$$

While conjugation with  $T$  only create a finite number of excitations, it extracts additional energy of order  $N$  from the quartic term. Choosing  $\eta_p$  related to the solution of the scattering equation makes sure that the constant term in  $\mathcal{G}_N$  is exactly  $4\pi a_0 N$  and that all the other contributions can be bounded from below by  $\mathcal{N}_+$  up to errors of order one. As a consequence states of the form  $\psi_N =$

$U_N^* T \xi_N$  with small excitation energy correspond to excitation states  $\xi_N$  with small expectation for the  $\mathcal{N}_+$  operator and therefore they exhibit condensation.

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**From infrared problems to non-commutative recurrence**

ALESSANDRO PIZZO

(joint work with Wojciech Dybalski)

Let  $H_{P,\sigma}$  be the single-electron fiber Hamiltonian of the massless Nelson model of total momentum  $P$  and infrared cut-off  $\sigma > 0$ . We establish detailed regularity properties of the corresponding  $n$ -particle ground state wave functions  $f_{P,\sigma}^n$  as

functions of  $P$  and  $\sigma$ . In particular, we show that

$$(1) \quad |\partial_{P_j} f_{P,\sigma}^n(k_1, \dots, k_n)|, |\partial_{P_i} \partial_{P_j} f_{P,\sigma}^n(k_1, \dots, k_n)| \leq \frac{1}{\sqrt{n!}} \frac{(c\lambda_0)^n}{\sigma^{\delta_{\lambda_0}}} \prod_{i=1}^n \frac{\chi_{[\sigma,\kappa]}(k_i)}{|k_i|^{\frac{3}{2}}},$$

where  $c$  is a numerical constant,  $\lambda_0 \rightarrow \delta_{\lambda_0}$  is a positive function of the maximal admissible coupling constant which satisfies  $\lim_{\lambda_0 \rightarrow 0} \delta_{\lambda_0} = 0$  and  $\chi_{[\sigma,\kappa]}$  is the (approximate) characteristic function of the energy region between the infrared cut-off  $\sigma$  and the ultraviolet cut-off  $\kappa$ . While the analysis of the first derivative is relatively straightforward, the second derivative requires a new strategy. By showing a non-commutative recurrence relation we derive a novel formula for  $f_{P,\sigma}^n$  with improved infrared properties. In this representation  $\partial_{P_i} \partial_{P_j} f_{P,\sigma}^n$  is amenable to sharp estimates obtained by iterative analytic perturbation theory in a related paper. The bounds stated above are instrumental for scattering theory of two electrons in the Nelson model.

### Excitation Spectra of Bose Gases

BENJAMIN SCHLEIN

(joint work with Chiara Boccatto, Serena Cenatiempo, Christian Brennecke)

We consider systems of  $N$  bosons moving in the three dimensional box  $\Lambda = [0; 1]^{\times 3}$  and interacting through a two-body potential with scattering length of the order  $N^{-1}$  (Gross-Pitaevskii regime). The Hamilton operator acts on permutation symmetric functions in  $L_s^2(\Lambda^N)$ . In momentum space, it can be written in second quantized form as

$$(1) \quad H_N = \sum_{p \in \Lambda^*} p^2 a_p^* a_p + \frac{\kappa}{N} \sum_{p,q,r \in \Lambda^*} \widehat{V}(r/N) a_{p+r}^* a_q^* a_{q+r} a_p$$

where  $\Lambda^* = 2\pi\mathbb{Z}^3$ ,  $\kappa > 0$  is a coupling constant,  $\widehat{V}$  is the Fourier transform of a non-negative, spherically symmetric and compactly supported potential  $V \in L^3(\mathbb{R}^3)$ , and, for  $p \in \Lambda^*$ ,  $a_p^*$  and  $a_p$  are the usual creation and annihilation operators.

We are interested in the low-energy spectrum of (1). From [7] it is known that the ground state energy of (1) is such that

$$E_N = 4\pi a_0 N + o(N)$$

in the limit of large  $N$ . Here  $a_0$  denotes the scattering length of the interaction  $V$ , defined through the formula  $8\pi a_0 = \int V(x)f(x)dx$ , where  $f$  is the solution of the zero-energy scattering equation

$$(2) \quad \left[ -\Delta + \frac{1}{2}V \right] f = 0$$

with the boundary condition  $f(x) \rightarrow 1$  as  $|x| \rightarrow \infty$ . In [6], it was also shown that the ground state of (1) exhibits complete Bose-Einstein condensation in the zero-momentum mode  $\varphi_0$ , defined by  $\varphi_0(x) = 1$  for all  $x \in \Lambda$ .

To study the low-energy spectrum, it is useful to factor out the condensate and to focus on its excitations. Following [5], we introduce the unitary map  $U : L_s^2(\Lambda^N) \rightarrow \mathcal{F}_+^{\leq N} = \bigoplus_{j=0}^N L_{\perp\varphi_0}^2(\Lambda)^{\otimes_s j}$  requiring that  $U\psi = \{\alpha_0, \dots, \alpha_N\}$  if  $\psi \in L_s^2(\Lambda^N)$  can be written as

$$(3) \quad \psi = \alpha_0 \varphi_0^{\otimes N} + \alpha_1 \otimes_s \varphi_0^{\otimes(N-1)} + \dots + \alpha_N$$

with  $\alpha_j \in L_{\perp\varphi_0}^2(\Lambda)^{\otimes_s j}$  ( $L_{\perp\varphi_0}^2(\Lambda)$  is the orthogonal complement of  $\varphi_0$ ).

We can define the excitation Hamiltonian  $\mathcal{L}_N = UH_NU^* : \mathcal{F}_+^{\leq N} \rightarrow \mathcal{F}_+^{\leq N}$ . In the Gross-Pitaevskii regime, however, low-energy states develop a short scale correlation structure that is not taken into account by  $U$ . For this reason, it is convenient to conjugate  $\mathcal{L}_N$  with a generalized Bogoliubov transformation having the form

$$(4) \quad T = \exp \left( \frac{1}{2} \sum_{p \in \Lambda_+^*} \eta_p [b_p^* b_{-p}^* + \text{h.c.}] \right)$$

for appropriate coefficients  $\eta_p$ , related to the solution of (2). In (4),  $\Lambda_+^* = \Lambda^* \setminus \{0\}$  and, for  $p \in \Lambda_+^*$ , we introduced the modified creation and annihilation operators  $b_p^* = a_p^* \sqrt{(N - N_+)/N}$  and  $b_p = \sqrt{(N - N_+)/N} a_p$ . Compared with the standard creation and annihilation operators, they have the advantage of leaving the excitation space  $\mathcal{F}_+^{\leq N}$  invariant.

For the renormalized excitation Hamiltonian  $\mathcal{G}_N = T^* \mathcal{L}_N T$  we find

$$(5) \quad \mathcal{G}_N = C_N + \mathcal{Q}_N + \mathcal{C}_N + \mathcal{V}_N + \delta_N$$

where  $C_N$  is a constant,  $\mathcal{Q}_N$  is quadratic in creation and annihilation operators,

$$C_N = \frac{\kappa}{\sqrt{N}} \sum_{p, q \in \Lambda_+^* : q \neq -p} \widehat{V}(p/N) [b_{p+q}^* b_{-p}^* (b_q \cosh(\eta_q) + b_{-q}^* \sinh(\eta_q)) + \text{h.c.}]$$

$$\mathcal{V}_N = \frac{1}{2N} \sum_{p, q, r} \widehat{V}(r/N) a_{p+r}^* a_q^* a_{q+r} a_p$$

and  $\delta_N$  is an error term, negligible on low-energy states.

The expression (5) for the renormalized excitation Hamiltonian is not enough to compute its low-energy spectrum, since cubic and quartic contributions are still important in the limit of large  $N$ . To handle these terms, we conjugate  $\mathcal{G}_N$  with another unitary map of the form  $S = \exp(A)$ , where now  $A$  is cubic in (modified) creation and annihilation operators. With the appropriate choice of  $A$ , we obtain a new renormalized excitation Hamiltonian  $\mathcal{M}_N = S^* \mathcal{G}_N S$  which can be decomposed as

$$(6) \quad \mathcal{M}_N = \widetilde{C}_N + \widetilde{Q}_N + \mathcal{V}_N + \widetilde{\delta}_N$$

where  $\widetilde{C}_N$  and  $\widetilde{Q}_N$  are new constant and quadratic contributions and  $\widetilde{\delta}_N$  is, like  $\delta_N$  above, negligible on low-energy states. Using the decomposition (6) it is enough to diagonalize the quadratic part (with another Bogoliubov transformation) to read off the low-energy spectrum of (1) (the quartic term  $\mathcal{V}_N$  is positive and, therefore,

it is not very important for the analysis). The results are summarized in the next theorem.

**Theorem 1.** *Let  $V \in L^3(\mathbb{R}^3)$  be non-negative, spherically symmetric and compactly supported. Assume the coupling constant  $\kappa > 0$  to be small enough. Then, the ground state energy of the Hamilton operator (1) is given by*

$$(7) \quad E_N = 4\pi(N-1)a_N - \frac{1}{2} \sum_{p \in \Lambda_+^*} \left[ p^2 + 8\pi a_0 - \sqrt{|p|^4 + 16\pi a_0 p^2} - \frac{(8\pi a_0)^2}{2p^2} \right] + o(1)$$

in the limit  $N \rightarrow \infty$ . Here  $a_N$  is defined through the convergent Born series

$$(8) \quad \begin{aligned} 8\pi a_N = & \kappa \widehat{V}(0) + \sum_{k=1}^{\infty} \frac{(-1)^k \kappa^{k+1}}{(2N)^k} \\ & \times \sum_{p_1, \dots, p_k \in \Lambda_+^*} \frac{\widehat{V}(p_1/N)}{p_1^2} \left( \prod_{i=1}^{k-1} \frac{\widehat{V}((p_i - p_{i+1})/N)}{p_{i+1}^2} \right) \widehat{V}(p_k/N) \end{aligned}$$

Furthermore, the spectrum of  $H_N - E_N$  below a fixed threshold  $K > 0$  consists of eigenvalues given, in the limit  $N \rightarrow \infty$ , by finite sums of the form

$$(9) \quad \sum_{p \in \Lambda_+^*} n_p \sqrt{|p|^4 + 16\pi a_0 p^2} + o(1)$$

with  $n_p \in \mathbb{N}$  for all  $p \in \Lambda_+^*$ .

We conclude with some remarks.

- 1) Comparing (8) with the Born series for the scattering length  $a_0$ , we find

$$(10) \quad |4\pi a_0 N - 4\pi a_N N| \leq C$$

uniformly in  $N$ . The emergence of  $a_N$  instead of  $a_0$  in the ground state energy (7) can be thought of as a finite size effect, due to the restriction of the interacting particles to a box with volume one; by (10), this effect generates an order one shift to the energy.

- 2) Eq. (9) for the excitation energies was already predicted in [2] by Bogoliubov, who used the linearity of the dispersion  $\mathcal{E}(p) = (|p|^4 + 16\pi a_0 p^2)^{1/2}$  for small momenta to explain the emergence of superfluidity.
- 3) Theorem 1 extends previous results obtained in [9, 4, 5, 3, 8] for Bose gases in the mean field limit. Also, it extends recent results in [1], where similar expressions have been derived for intermediate regimes where the interaction decays in momentum on the scale  $N^\beta$ , for a  $0 < \beta < 1$  (the mean field regime corresponds to  $\beta = 0$ , (1) to  $\beta = 1$ ).
- 4) The idea of using a cubic phase has already been used in [10] to derive an upper bound to the ground state energy for an extended Bose gas in the thermodynamic limit.

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**Asymptotic completeness in dissipative scattering theory**

JÉRÉMY FAUPIN

(joint work with Jürg Fröhlich)

In the quantum-mechanical scattering theory for dissipative quantum systems, a typical example is a neutron interacting with a nucleus. When a neutron is targeted onto a complex nucleus, it may, after interacting with it, be elastically scattered off the nucleus or be absorbed by the nucleus, leading to the formation of a compound nucleus [1].

In [11], Feshbach, Porter and Weisskopf proposed a model describing the interaction of a neutron with a nucleus, allowing for the description of both elastic scattering and the formation of a compound nucleus. The force exerted by the nucleus on the neutron is modeled by a phenomenological potential of the form  $V - iW$ , where  $V, W$  are real-valued and  $W \geq 0$ . The nucleus is supposed to be localized in space, which corresponds to the assumption that  $V$  and  $W$  are compactly supported or decay rapidly at infinity. On  $L^2(\mathbb{R}^3)$ , the pseudo-Hamiltonian for the neutron is given by

$$H = -\Delta + V - iW.$$

Here, a linear operator  $H$  is called a pseudo-Hamiltonian if  $-iH$  generates a strongly continuous contractive semigroup  $\{e^{-itH}\}_{t \geq 0}$ . For any initial state  $u_0$ , with  $\|u_0\| = 1$ , the map  $t \mapsto \|e^{-itH}u_0\|$  is decreasing on  $[0, \infty)$ , and the quantity  $p_{\text{abs}} := 1 - \lim_{t \rightarrow \infty} \|e^{-itH}u_0\|^2$  gives the probability of absorption of the neutron by the nucleus, i.e., the probability of formation of a compound nucleus. The probability that the neutron, initially in the state  $u_0$ , eventually escapes from

the nucleus is given by  $p_{\text{scat}} := \lim_{t \rightarrow \infty} \|e^{-itH} u_0\|^2$ , and in the case where this probability is strictly positive, one expects that there exists an (unnormalized) scattering state  $u_+$  such that  $\|u_+\|^2 = p_{\text{scat}}$  and

$$\lim_{t \rightarrow \infty} \|e^{-itH} u_0 - e^{it\Delta} u_+\| = 0.$$

This model is referred to as the nuclear optical model, the term optical being used in reference to the phenomenon in optics of refraction and absorption of light waves by a medium. The model is empirical in that the form of the potentials  $V$  and  $W$  are determined by optimizing the fit to experimental data. Usually,  $V$  and  $W$  are decomposed into a sum of terms corresponding to the form of the expected interaction potentials in different regions of physical space, and sometimes a spin-orbit interaction term is included. We refer to e.g. [13] or [10] for a thorough description. A large range of observed scattering data can then be predicted by the model to a high precision.

Suppose that  $V$  and  $W$  belong to  $L_c^\infty(\mathbb{R}^3; \mathbb{R})$ , i.e., that  $V$  and  $W$  are bounded and compactly supported. This implies that  $H$  is a closed operator with domain  $\mathcal{D}(H) = H^2(\mathbb{R}^3)$ . Let  $\sigma(H)$  denote the spectrum of  $H$ . Since, for all  $u \in \mathcal{D}(H)$ ,  $\text{Im}(\langle u, Hu \rangle) \leq 0$ ,  $H$  is a dissipative operator and  $\sigma(H)$  is contained in the lower half-plane,  $\{z \in \mathbb{C}, \text{Im}(z) \leq 0\}$  (see, e.g., [7]). Because  $V$  and  $W$  are relatively compact perturbations of  $-\Delta$ , the essential spectrum of  $H$  is equal to  $[0, \infty)$ . Moreover,  $\sigma(H) \setminus [0, \infty)$  consists of a finite number of eigenvalues with finite algebraic multiplicities (see, e.g., [12, 18]). The eigenvalues  $\lambda$  of  $H$  are associated to generalized eigenvectors, i.e., vectors  $u \in \mathcal{D}(H^k)$  such that  $(H - \lambda)^k u = 0$  for some integer  $k$  less or equal to the multiplicity of  $\lambda$ .

The adjoint operator  $H^*$  is given by  $H^* = -\Delta + V + iW$ . Let  $\mathcal{H}_{\text{pp}}(H)$ , respectively  $\mathcal{H}_{\text{pp}}(H^*)$ , denote the vector space spanned by all the generalized eigenvectors of  $H$ , respectively of  $H^*$ , corresponding to isolated eigenvalues. Let

$$\mathcal{H}_d(H) := \{u \in \mathcal{H}, \lim_{t \rightarrow \infty} \|e^{-itH} u\| = 0\}.$$

Recall that, since  $V$  and  $W$  are bounded and compactly supported, resonances of  $H$  can be defined as poles of a meromorphic extension of the map

$$\{z \in \mathbb{C}, \text{Im}(z) > 0\} \ni z \mapsto (H - z^2)^{-1} : L_c^2(\mathbb{R}^3) \rightarrow L_{\text{loc}}^2(\mathbb{R}^3),$$

to the entire  $\mathbb{C}$ , where  $L_c^2(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3), u \text{ is compactly supported}\}$  and  $L_{\text{loc}}^2(\mathbb{R}^3) = \{u : \mathbb{R}^3 \rightarrow \mathbb{C}, u \in L^2(K) \text{ for all compact set } K \subset \mathbb{R}^3\}$  (see, e.g., [6]).

**Theorem 1** ([9]). *Let  $H = -\Delta + V(x) - iW(x)$  on  $L^2(\mathbb{R}^3)$  with  $W \geq 0$ ,  $W(x) > 0$  on some non-trivial open set and  $V, W \in L_c^\infty(\mathbb{R}^3; \mathbb{R})$ . Suppose that 0 is neither an eigenvalue nor a resonance of  $H_V = -\Delta + V(x)$ . Then*

$$\mathcal{H}_{\text{pp}}(H) = \mathcal{H}_d(H).$$

Moreover, the wave operator  $W_-(H, H_0) := \text{s-lim}_{t \rightarrow \infty} e^{-itH} e^{itH_0}$ , with  $H_0 = -\Delta$ , is asymptotically complete in the sense that

$$\text{Ran}(W_-(H, H_0)) = \mathcal{H}_{\text{pp}}(H^*)^\perp$$



if and only if  $H$  does not have real resonances. In this case, the restriction of  $H$  to  $\mathcal{H}_{\text{pp}}(H^*)^\perp$  is similar to  $H_0$  and there exist  $m_1 > 0$  and  $m_2 > 0$  such that, for all  $u \in \mathcal{H}_{\text{pp}}(H^*)^\perp$ ,

$$m_1 \|u\| \leq \|e^{-itH} u\| \leq m_2 \|u\|, \quad t \in \mathbb{R}.$$

Since  $W(x) > 0$  on some non-trivial open set, it follows from the unique continuation principle (see, e.g., [18]) that  $H$  does not have real eigenvalues. Moreover, it is likely that, generically,  $H$  does not have real resonances, implying that the wave operator  $W_-(H, H_0)$  is generically asymptotically complete. However, for any  $z_0 > 0$ , it is not difficult to construct smooth compactly supported potentials  $V$  and  $W$  such that  $-z_0$  is a resonance of  $H = -\Delta + V(x) - iW(x)$  (see [21]). The previous theorem underlines the importance of real resonances in the scattering theory of dissipative Schrödinger operators.

Our results generalize to abstract dissipative operators under suitable conditions (see [9]). The notion of resonances is replaced by a notion of spectral singularities related to that considered in [19] (see also [5]).

Mathematical scattering theory for dissipative operators on Hilbert spaces has been considered by many authors. Works related to ours included those by Martin [15], Davies [2, 3] and Neidhardt [17], for general abstract results, Mochizuki [16] and Simon [20], for Schrödinger operators, and by Kato [14], Wang and Zhu [22], and [8] for “weak coupling” results. The existence of the wave operators associated to  $H$  and  $H_0$  is established under various conditions. But proving their asymptotic completeness is a much more difficult problem which, to our knowledge, was solved only in some particular cases; (see [8, 14, 22] for weak coupling results). Our main results in [9] provide abstract conditions implying asymptotic completeness.

It should also be mentioned that scattering theory for dissipative operators on Hilbert spaces has important applications in the scattering theory of Lindblad master equations [4, 8, 9].

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## Relation between the Resonance and Scattering Theory in the massless Spin-Boson Model

FELIX HÄNLE

(joint work with Miguel Ballesteros and Dirk-André Deckert)

Inspired by results in scattering theory of quantum mechanics, such as the one of B. Simon [1] who derived an explicit formula for the scattering matrix and showed that the integral kernel of the transition matrix element has a meromorphic continuation to a certain subset of the complex plane with the only possible poles at the positions of the resonances, we established a similar connection in a model of quantum field theory, namely, the massless Spin-Boson model. Our work consists of two parts: In the first part we construct the resonance and the ground state of the model and in the second part we derive an explicit formula for the 2-body transition matrix elements. This provides a precise relation between the resonances and scattering theory and explains the well-known intensity profiles measured in scattering experiments in a rigorous manner.

1. DEFINITION OF THE MODEL

For sufficiently small coupling constants  $g > 0$ , we define the Spin-Boson Hamiltonian as

$$\begin{aligned}
 H &= \begin{pmatrix} e_1 & 0 \\ 0 & e_0 \end{pmatrix} \otimes \mathbf{1} + \mathbf{1} \otimes \int d^3k \omega(k) a(k)^* a(k) + g \sigma_1 \otimes (a(f) + a(f)^*) \\
 (1) \quad &= H_0 + gV
 \end{aligned}$$

where  $0 \leq e_0 < e_1 < \infty$  denote the atomic energy levels,  $\omega(k) = |k|$  is the dispersion relation of the massless scalar field,  $a, a^*$  are the annihilation and creation operators fulfilling the canonical commutation relations,  $\sigma_1$  is the first Pauli matrix and the form factor is of the form

$$(2) \quad f : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{R}, \quad k \mapsto e^{-\frac{k^2}{\Lambda^2}} |k|^{-\frac{1}{2} + \mu}.$$

The gaussian factor in the definition of  $f$  acts as an ultraviolet cut-off where  $\Lambda > 0$  is the ultraviolet cut-off parameter and the fixed number  $\mu > 0$  is a regularization of the infrared divergence of the Coulomb potential. Notice that  $H$  is densely defined on the Hilbert space

$$(3) \quad \mathcal{H} = \mathbb{C}^2 \otimes \mathcal{F}[\mathfrak{h}],$$

where  $\mathcal{F}[\mathfrak{h}]$  denotes the usual bosonic Fock space over  $\mathfrak{h} := L^2(\mathbb{R}^3, d^3k)$ .

Notice that the interaction term of the Hamiltonian  $V$  is infinitesimal small with respect to  $H_0$  on the domain  $\mathcal{D}(H_0)$ , and hence, it follows by the Kato-Rellich theorem that also  $H$  is self-adjoint on  $\mathcal{D}(H) = \mathcal{D}(H_0)$ .

2. GROUND STATE AND RESONANCE

In the following, we describe the construction of the resonance (and the ground state). It is proven in [2] that the only eigenvalue in the spectrum of  $H$  is given by  $\lambda_0 := \inf \sigma(H)$  and the rest of the spectrum is absolutely continuous. This implies that there is no stable excited state in the Spin-Boson model. Heuristically, the reason for this is that the atomic energy of the excited state  $e_1$  turns into a complex eigenvalue  $\lambda_1$  in a suitable Banach space with strictly negative imaginary part, the so-called resonance. The theory of resonances in models of quantum field theory are well-established thanks to many authors (see [3] among many others). We follow the approach described in [4]. We define a unitary transformation

$$(4) \quad u_\theta : \mathfrak{h} \rightarrow \mathfrak{h}, \quad \psi(k) \rightarrow e^{-\frac{\theta k^2}{2}} \psi(e^{-\theta} k)$$

for  $\theta \in \mathbb{R}$  and its second quantization  $U_\theta : \mathcal{F}[\mathfrak{h}] \rightarrow \mathcal{F}[\mathfrak{h}]$ . This yields a family of unitary equivalent Hamiltonians  $\{H^\theta\}_{\theta \in \mathbb{R}}$  which extends to an analytic family of type A  $\{H^\theta\}_{\theta \in \mathcal{S}}$  where  $\mathcal{S}$  is a suitable subset of the complex plane. For  $\theta \notin \mathbb{R}$  these Hamiltonians are not self-adjoint.

Usually, one would construct  $\lambda_0$  and  $\lambda_1$  via regular perturbation theory. However, this requires a spectral gap between  $e_0, e_1$  and the rest of the spectrum of  $H^\theta$  which is not true considering a massless scalar field. This infrared problem

can be controlled via the multiscale method described in [4]. We again define a family of Hamiltonians  $\{H^{(n),\theta}\}_{n \in \mathbb{N}}$  where  $n$  indicates an infrared cut-off by

$$(5) \quad H^{(n),\theta} = \begin{pmatrix} e_1 & 0 \\ 0 & e_0 \end{pmatrix} \otimes \mathbf{1} + \mathbf{1} \otimes \int_{|k| \geq \rho_n} d^3k \omega^\theta(k) a^*(k) a(k) + g\sigma_1 \otimes \left( a(f^{(n),\theta}) + a(f^{(n),\theta})^* \right)$$

where  $f^{(n),\theta} = \chi_{|\cdot| \geq \rho_n} f^\theta$ ,  $\omega^\theta(k) = e^{-\theta}|k|$  and  $\rho_n := \rho_0 \rho^n$  for suitable  $\rho_0 > 0$  and  $0 < \rho < 1$ . Note that  $H^{(n),\theta}$  is densely defined on the Hilbert space

$$(6) \quad \mathcal{H}^{(n)} := \mathcal{K} \otimes \mathcal{F}[\mathfrak{h}^{(n)}], \quad \mathfrak{h}^{(n)} := L^2(\mathbb{R}^3 \setminus \mathcal{B}_{\rho_n}), \quad \mathcal{B}_{\rho_n} := \{x \in \mathbb{R}^3 : |x| < \rho_n\}.$$

We show in an inductive scheme that for each  $n \in \mathbb{N}$ ,  $H^{(n),\theta}$  has two isolated eigenvalues  $\lambda_0^{(n)}$  and  $\lambda_1^{(n)}$  and we provide resolvent estimates in suitable regions around these eigenvalues. Finally, this allows us to prove the following theorem.

**Theorem 1** (Construction of the ground state and the resonance). *Let  $g > 0$ ,  $\nu > 0$  be sufficiently small and  $i \in \{0, 1\}$ . Then, the (complex) number  $\lambda_i := \lim_{n \rightarrow \infty} \lambda_i^{(n)}$  is an eigenvalue of  $H^\theta$  and its associated projection is given by  $P_i := \lim_{n \rightarrow \infty} P_i^{(n)}$  which is rank one. Further, the mappings  $\mathcal{S} \ni \theta \mapsto \lambda_i \equiv \lambda_i^\theta$  and  $\mathcal{S} \ni \theta \mapsto P_i \equiv P_i^\theta$  are analytic for a certain set  $\mathcal{S} \in \mathbb{C}^-$ .*

### 3. SCATTERING FORMULA

A rigorous framework for scattering theory has already been established in various quantum field theoretic models (see e.g. [5, 6, 7, 8]), and in particular, in the massless in the Spin-Boson model (see [5, 9]). In order to define scattering matrix elements, we define the asymptotic annihilation/ creation operators

$$(7) \quad a_\pm(h)^\# := s\text{-}\lim_{t \rightarrow \pm\infty} a_t(h)^\# \quad \text{where} \quad a_t(h)^\# := e^{itH} e^{-itH_0} a(h)^\# e^{itH_0} e^{-itH}$$

for  $h \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ . The existence of these strong limits are well-known thanks to the works mentioned above. For  $h, l \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ , we define the 2-body scattering matrix element  $S(h, l)$  and the 2-body transition matrix element  $T(h, l)$  as

$$(8) \quad S(h, l) := \langle a_+(h)^* \Psi_{\lambda_0}, a_-(l)^* \Psi_{\lambda_0} \rangle, \quad T(h, l) := S(h, l) - \langle h, l \rangle_2.$$

A long computation using the Laplace formula of the time evolution together with methods of complex analysis and the estimates obtained from the multiscale analysis lead to our main theorem:

**Theorem 2** (Scattering formula). *Let  $g > 0$  be sufficiently small and let  $0 < \nu < \nu' < \nu'$  for sufficiently small and fixed  $\nu$  and  $\nu'$ . Then,*

$$(9) \quad T(h, l) = \int d^3k d^3k' T(k, k') \delta(|k| - |k'|) \overline{h(k)} l(k')$$

for  $h, l \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$  and

$$(10) \quad T(k, k') = -2\pi i g^2 f(k) f(k') \left( \left\langle \sigma_1 \Psi_{\lambda_0}^{\bar{\theta}}, \frac{1}{H^{\bar{\theta}} - \lambda_0 - |k'|} \sigma_1 \Psi_{\lambda_0}^{\theta} \right\rangle - \left\langle \sigma_1 \Psi_{\lambda_0}^{\theta}, \frac{1}{H^{\bar{\theta}} - \lambda_0 - |k'|} \sigma_1 \Psi_{\lambda_0}^{\bar{\theta}} \right\rangle \right).$$

Again by using the results from multiscale analysis, we prove the the following result.

**Corollary 3.** *Let  $g > 0$  be sufficiently small and let  $0 < \nu < \nu < \nu'$  for sufficiently small and fixed  $\nu$  and  $\nu'$ . For sufficiently small  $\beta > 0$ , we define  $\mathcal{M}_\beta := (\Re\lambda_1 - \lambda_0 - \beta, \Re\lambda_1 - \lambda_0 + \beta)$ . Further, we set  $C := -2i \left\langle \sigma_1 \Psi_{\lambda_0}^{\bar{\theta}}, P_1^\theta \sigma_1 \Psi_{\lambda_0}^{\theta} \right\rangle$ . Then,*

$$(11) \quad T(k, k') = -2\pi i g^2 f(k) f(k') \left( \frac{C \Im\lambda_1}{|\lambda_1 - \lambda_0 - |k'||^2} + gR(k') \right) \quad \forall |k'| \in \mathcal{M}_\beta.$$

Moreover, there is a constant  $\tilde{C} > 0$  such that

$$(12) \quad |R(k')| \leq \frac{\tilde{C}}{|\lambda_1 - \lambda_0 - |k'||} \quad \forall |k'| \in \mathcal{M}_\beta.$$

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## Quantum Mean Field Asymptotics and Multiscale Analysis

SÉBASTIEN BRETEAUX

(joint work with Zied Ammari and Francis Nier)

In this work, we study how multiscale analysis and quantum mean field asymptotics can be brought together. In particular we study when a sequence of one-particle density matrices has a limit with two components: one classical and one quantum. The introduction of “separating quantization for a family” provides a simple criterion to check when those two types of limit are well separated.

We give examples of explicit computations of such limits, and how to check that the separating assumption is satisfied.

Let us give some details on our motivations. Over the past three decades, microlocal and semiclassical analysis provided interesting mathematical techniques for the study of quantum field theories and quantum many-body theory. In the present work we follow this fruitful stream of ideas and study the mathematical problem of defect of compactness for density matrices in the bosonic or fermionic Fock spaces. Previously, in a series of papers initiated in [1], the authors have introduced Wigner (or semiclassical) measures of density matrices in the bosonic Fock space and showed that it is a very useful tool to study the mean-field approximation of Bose gases. Moreover, it was noticed that a certain defect of compactness of density matrices is one of the difficulties that occurs in this context. So towards a better understanding of these concentration and defect of compactness phenomena we introduced here a multiscale analysis inspired by second microlocalization. We believe that this approach will be of interest to the study of the mean-field theory of Fermi and Bose gases. We indeed provide here some simple applications to the Bose and Fermi free gases and leave more involved applications to further investigations.

Let us briefly describe the main question we consider here. As mentioned before, in the analysis of general bosonic mean-field problems the following defect of compactness problem arises. In fact, if  $\varrho_\varepsilon$  are density matrices in the (fermionic or bosonic) Fock space and  $\gamma_\varepsilon^{(p)}$  are its  $p$ -particles reduced density matrices, one may have

$$(1) \quad \lim_{\varepsilon \rightarrow 0} \operatorname{Tr}[\gamma_\varepsilon^{(p)} \tilde{b}] = \operatorname{Tr}[\gamma_0^{(p)} \tilde{b}]$$

for any  $p$ -particle compact observable  $\tilde{b}$ , while it is not true for a general bounded  $\tilde{b}$ , e.g.

$$\lim_{\varepsilon \rightarrow 0} \operatorname{Tr}[\gamma_\varepsilon^{(p)}] > \operatorname{Tr}[\gamma_0^{(p)}].$$

This reflects the difference between the weak-\* convergence of trace-class operators and convergence with respect to the trace norm. In the fermionic case, it is even worse, because mean-field asymptotics cannot be described in terms of finitely many quantum states and the right-hand side of (1) is usually 0 while  $\lim_{\varepsilon \rightarrow 0} \operatorname{Tr}[\gamma_\varepsilon^{(p)}] > 0$ . From the analysis of finite dimensional partial differential

equations, it is known that such defect of compactness can be localized geometrically with accurate quantitative information by introducing scales and small parameters within semiclassical techniques (e.g. [6, 7]). We are thus led to introduce two small parameters  $\varepsilon > 0$  for the mean-field asymptotics and  $h > 0$  for the semiclassical quantization of finite dimensional  $p$ -particles phase space. The small parameter  $\varepsilon$  stands for  $1/n$ , where  $n \rightarrow \infty$  is the typical number of particles, while  $h$  is the rescaled Planck constant measuring the proximity of quantum mechanics to classical mechanics. Such scaling appears already in the mathematical physics literature with a specific relation between  $h$  and  $\varepsilon$  depending in the considered problem, see e.g. [5, 9, 8]. The combined analysis of this article is concerned with the general situation when  $\varepsilon = \varepsilon(h)$  with  $\lim_{h \rightarrow 0} \varepsilon(h) = 0$ . In order to keep track of the information at the quantum level especially in the bosonic case we also introduce finite dimensional multiscale observables in spirit of [2, 4, 3, 10].

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### Stability of quantum many-body systems with point interactions

ROBERT SEIRINGER

(joint work with Thomas Moser)

Models of particles with point interactions are ubiquitously used in physics, as an idealized description whenever the range of the interparticle interactions is much shorter than other relevant length scales. They were introduced in the early days of quantum mechanics as models of nuclear interactions, but have proved useful in many other branches of physics. While the two-particle problem is mathematically completely understood [1], for more than two particles the existence of a

self-adjoint Hamiltonian that is bounded from below and models pairwise point interactions is a challenging open problem. It is known that such a Hamiltonian can only exist for fermions with at most two components, due to the Thomas effect [5].

We consider here the simplest many-body system with point interactions, consisting of  $N$  (spinless) fermions of mass 1, interacting with another particle of mass  $m$ . The interaction is characterized by a parameter  $\alpha \in \mathbb{R}$ , where  $-1/\alpha$  is proportional to the scattering length of the pair interaction. Formally, the Hamiltonian of the system can be thought of as

$$(1) \quad H = -\frac{1}{2m}\Delta_{x_0} - \frac{1}{2}\sum_{i=1}^N \Delta_{x_i} + \gamma \sum_{i=1}^N \delta(x_0 - x_i)$$

where  $x_i \in \mathbb{R}^3$ , and  $\gamma$  represents an infinitesimal coupling constant. Models of this kind have been studied extensively in the literature (see, in particular, [3, 2]) and can be defined via a suitable regularization procedure. More precisely, the formal expression (1) can be given a meaning in terms of a suitable quadratic form  $\mathcal{F}_\alpha$ , defined as follows.

Let  $D(\mathcal{F}_\alpha)$  be the form domain defined by

$$(2) \quad \left\{ \psi = \phi + \mathcal{G}\xi \mid \phi \in H^1(\mathbb{R}^3) \otimes H_{\text{as}}^1(\mathbb{R}^{3N}), \xi \in H^{1/2}(\mathbb{R}^3) \otimes H_{\text{as}}^{1/2}(\mathbb{R}^{3(N-1)}) \right\}$$

where  $\mathcal{G}(k_0, k_1, \dots, k_N) = \left( \frac{1}{2m}k_0^2 + \frac{1}{2}\sum_{i=1}^N k_i^2 + \mu \right)^{-1}$  for some arbitrary  $\mu > 0$ ,  $\mathcal{G}\xi$  is short for the function with Fourier transform

$$(3) \quad \mathcal{G}(k_0, k_1, \dots, k_N) \sum_{i=1}^N (-1)^{i+1} \hat{\xi}(k_0 + k_i, k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_N)$$

and the subscript ‘‘as’’ stands for antisymmetric functions. Note that since  $\mathcal{G}\xi \notin H^1(\mathbb{R}^{3(N+1)})$  for  $\xi \neq 0$ , the decomposition of  $\psi$  as  $\psi = \phi + \mathcal{G}\xi$  is unique. Moreover, while  $\phi$  depends on  $\mu$ ,  $\xi$  is independent of the choice of  $\mu$ .

For  $\psi \in D(\mathcal{F}_\alpha)$ , we have

$$(4) \quad \begin{aligned} \mathcal{F}_\alpha(\psi) &= \left\langle \phi \left| -\frac{1}{2m}\Delta_{x_0} - \frac{1}{2}\sum_{i=1}^N \Delta_{x_i} + \mu \right| \phi \right\rangle - \mu \|\psi\|_{L^2(\mathbb{R}^{3(N+1)})}^2 \\ &+ N \left( \frac{2m}{m+1} \alpha \|\xi\|_{L^2(\mathbb{R}^{3N})}^2 + \mathcal{T}_{\text{diag}}(\xi) + \mathcal{T}_{\text{off}}(\xi) \right) \end{aligned}$$

where

$$(5) \quad \begin{aligned} \mathcal{T}_{\text{diag}}(\xi) &:= \int_{\mathbb{R}^{3N}} |\hat{\xi}(k_0, k_1, \vec{k})|^2 \mathcal{L}(k_0, k_1, \vec{k}) \, dk_0 \, dk_1 \, d\vec{k} \\ \mathcal{T}_{\text{off}}(\xi) &:= (N-1) \int_{\mathbb{R}^{3(N+1)}} \hat{\xi}^*(k_0 + s, t, \vec{k}) \hat{\xi}(k_0 + t, s, \vec{k}) \mathcal{G}(k_0, s, t, \vec{k}) \, dk_0 \, ds \, dt \, d\vec{k} \end{aligned}$$



and we used  $\vec{k} = (k_2, \dots, k_{N-1})$  for short. The function  $\mathcal{L}$  is given by

$$(6) \quad \mathcal{L}(k_0, k_1, \dots, k_{N-1}) := 2\pi^2 \left( \frac{2m}{m+1} \right)^{3/2} \left( \frac{k_0^2}{2(m+1)} + \frac{1}{2} \sum_{i=1}^{N-1} k_i^2 + \mu \right)^{1/2}$$

We are interested in the question of stability. We shall show that there exists a critical mass  $m^*$ , independent of  $N$ , such that stability holds for  $m > m^*$ . The value of  $m^*$  is determined by an optimization problem of a certain analytic function. A numerical evaluation of the expression yields  $m^* \approx 0.36$ .

To state our main result, we define, for any  $m > 0$ ,

$$(7) \quad \Lambda(m) = \sup_{s, K \in \mathbb{R}^3, Q > 0} \frac{s^2 + Q^2}{\pi^2(1+m)} \ell_m(s, K, Q)^{-1/2} \int_{\mathbb{R}^3} \frac{1}{t^2} \ell_m(t, K, Q)^{-1/2} \times \frac{|(s+AK) \cdot (t+AK)|}{\left[ (s+AK)^2 + (t+AK)^2 + \frac{m}{1+m} (Q^2 + AK^2) \right]^2 - \left[ \frac{2}{(1+m)} (s+AK) \cdot (t+AK) \right]^2} dt$$

where  $A := (2+m)^{-1}$  and

$$(8) \quad \ell_m(s, K, Q) := \left( \frac{m}{(m+1)^2} (s+K)^2 + \frac{m}{m+1} (s^2 + Q^2) \right)^{1/2}$$

Our main result is the following [4]:

**Theorem 1.** For any  $\xi \in H^{1/2}(\mathbb{R}^3) \otimes H_{\text{as}}^{1/2}(\mathbb{R}^{3(N-1)})$ ,  $\mu > 0$  and  $N \geq 2$ ,

$$(9) \quad \mathcal{T}_{\text{off}}(\xi) \geq -\Lambda(m) \mathcal{T}_{\text{diag}}(\xi)$$

In particular, if  $m$  is such that  $\Lambda(m) < 1$ , then  $\mathcal{F}_\alpha$  is closed and bounded from below by

$$(10) \quad \mathcal{F}_\alpha(\psi) \geq \begin{cases} 0 & \text{for } \alpha \geq 0 \\ -\left( \frac{\alpha}{2\pi^2(1-\Lambda(m))} \right)^2 \|\psi\|_{L^2(\mathbb{R}^{3(N+1)})}^2 & \text{for } \alpha < 0 \end{cases}$$

for all  $\psi \in D(\mathcal{F}_\alpha)$ .

We note that as a closed and bounded from below quadratic form,  $\mathcal{F}_\alpha$  gives rise to a unique self-adjoint operator for  $\Lambda(m) < 1$ . It is described in detail in [4].

The lower bound (10) is sharp as  $m \rightarrow \infty$ . For  $\alpha < 0$ ,  $-(\alpha/2\pi^2)^2$  equals the binding energy of the two-particle problem with point interactions. As  $m \rightarrow \infty$ , only one of the fermions can be bound, hence the ground state energy becomes independent of  $N$  in that limit.

It turns out that  $\Lambda(m) < 1$  if  $m \geq 0.36$ . It is in fact known that  $\mathcal{F}_\alpha$  is unbounded from below [2, Thm. 2.2] for any  $N \geq 2$  for  $m \leq 0.0735$ . In particular, the critical mass for stability satisfies  $0.0735 < m^* < 0.36$ .

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## Interacting models in perturbative AQFT

KASIA REJZNER

In my talk I gave an overview of perturbative algebraic quantum field theory (pAQFT) and explained how to construct interacting models in this framework. This approach is inspired by the axiomatic approach of Haag and Kastler [HK64], but allows for using perturbative methods to construct models that describe physically relevant systems, e.g. QED. Recently, these methods were also applied to integrable models (sine Gordon in 2D), where the perturbative series defining the formal  $S$ -matrix was shown to converge.

For more details about the pAQFT framework, see [FR15] and a recent book [Rej16].

## 1. MOTIVATION

The QED Lagrangian  $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$  is not quadratic in fields, so in order to quantize the corresponding field theory, practitioners of QFT use perturbative methods. Concrete computations are performed with the help of Feynman diagrams (to control the combinatorics) and the resulting quantities are formal power series in  $\hbar$  and the coupling constant. The pAQFT approach allows to put these formal calculations into a mathematically rigorous framework.

## 2. AQFT

The pAQFT framework is inspired by the Haag-Kastler axioms [HK64]. Here I present the “lite version” of these axioms, adapted to the context of pAQFT. Let  $\mathcal{O} \subset \mathbb{M}$  be a bounded, causally convex region of Minkowski spacetime. We assign to it  $\mathfrak{A}(\mathcal{O})$ , an involutive topological algebra over  $\mathcal{C}[[\hbar]]$  (or more restrictively, a  $C^*$ -algebra, as in the original framework). We require:

- (1) **Isotony**: if  $\mathcal{O}_1 \subset \mathcal{O}_2$ , then  $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$ .
- (2) **Causality**: if  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are spacelike separated, then  $[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = \{0\}$ .
- (3) **Time-slice axiom**: If  $\mathcal{N}$  contains a Cauchy surface for  $\mathcal{O}$ , then  $\mathfrak{A}(\mathcal{N}) \cong \mathfrak{A}(\mathcal{O})$ .

- (4) **Covariance:**  $\forall L \in \mathcal{P}$  elements of the connected component of the Poincaré group, there exist  $\alpha_L^\mathcal{O} : \mathfrak{A}(\mathcal{O}) \rightarrow \mathfrak{A}(L\mathcal{O})$ , satisfying natural compatibility conditions.

### 3. INTERACTING LOCAL NETS

A convenient method for constructing nets of algebras satisfying axioms from section 2 was presented in [FR15]. Let  $\mathcal{D}^n = \mathcal{C}_c^\infty(\mathbb{M}, \mathbb{R}^n)$  be the space of test functions. Assume we are given a family  $\{\mathcal{S}(f), f \in \mathcal{D}^n\}$  such that:

- **Causal factorisation:**  $\mathcal{S}(f + g + h) = \mathcal{S}(f + g)\mathcal{S}(g)^{-1}\mathcal{S}(g + h)$ , if the past of  $\text{supp } h$  has empty intersection with  $\text{supp } f$ .
- Translations act by  $\alpha_x(\mathcal{S}(f)) = \mathcal{S}(f_x)$ , where  $f_x(y) = f(y - x)$ .

Given a family  $\{\mathcal{S}(f)\}_{f \in \mathcal{D}^n}$  as above, we define the free local net  $\mathfrak{A}_0$ , by assigning to  $\mathcal{O}$  the  $C^*$ -algebra  $\mathfrak{A}_0(\mathcal{O})$  generated by the unitaries  $\mathcal{S}(f)$  with  $\text{supp } f \subset \mathcal{O}$ . **Isotony** and **Causality** hold by construction (causality follows from the causal factorisation property). In order to get the **Time-slice axiom** we need to take the quotient by an appropriate ideal.

The same family  $\{\mathcal{S}(f)\}_{f \in \mathcal{D}^n}$  can be used to define the *interacting local net*. To this end, we define *the relative S-matrix*:

$$(1) \quad f \mapsto \mathcal{S}_g(f) := \mathcal{S}(g)^{-1}\mathcal{S}(g + f)$$

for a given  $g \in \mathcal{D}^n$ . Relative  $S$ -matrices have following important properties:

- (1)  $\mathcal{S}_g$  satisfies *causal factorisation property*.
- (2) if  $\text{supp}(g - g')$  has an empty intersection with the past of  $\text{supp } f$ , then  $\mathcal{S}_g(f) = \mathcal{S}_{g'}(f)$ .
- (3) if  $\text{supp}(g - g')$  has an empty intersection with the future of  $\text{supp } f$ , then  $\mathcal{S}_g(f) = \text{Ad}_U(\mathcal{S}_{g'}(f))$ , where  $U = \mathcal{S}_{g'}(g - g')^{-1}$ .

The *interacting local net*  $\mathfrak{A}_g$  is defined by assigning to  $\mathcal{O}$  the  $C^*$ -algebra  $\mathfrak{A}_g(\mathcal{O})$  generated by the unitaries  $\mathcal{S}_g(f)$  with  $\text{supp } f \subset \mathcal{O}$ . Properties (2) and (3) guarantee that changing  $g$  outside  $\mathcal{O}$  leads to unitary equivalent nets and one can perform the *algebraic adiabatic limit* (see [FR15] for details).

### 4. CONSTRUCTING EXAMPLES

Start with  $\mathcal{E} := \Gamma(E \rightarrow M)$ , the space of classical (off-shell) field configurations. The free classical theory is defined by the equation of motion  $P\varphi = 0$ , where  $\varphi \in \mathcal{E}$  and  $P$  is a normally hyperbolic operator. Let  $V(f)[\varphi] = \int V_x[\varphi]f(x)d^4x$ , where  $V_x[\varphi]$  is a local Lagrangian density, e.g.  $V_x[\varphi] = \varphi^4(x)$  and  $f$  is a test function. Define

$$(2) \quad \mathcal{S}(f) := e^{\frac{i}{\hbar}\lambda V(f)} = \sum_{n=0}^{\infty} \left(\frac{i\lambda}{\hbar}\right)^n \mathcal{T}_n(V(f)^{\otimes n}),$$

where  $\mathcal{T}_n$ s are called *time-ordered products*. These are multilinear maps on  $\mathcal{F}_{\text{loc}}$ , the space of local functionals on  $\mathcal{E}$ . To construct  $\mathcal{T}_n$ s, one uses Epstein-Glaser renormalization [EG73].

Take  $V(f_1), \dots, V(f_n)$  such that  $f_i, i = 1, \dots, n$  have pairwise disjoint supports. Define

$$\begin{aligned} \mathcal{T}_n(V(f_1) \otimes \dots \otimes V(f_n))(\varphi) \\ = e^{\hbar \sum_{1 \leq i < j \leq n} \langle \Delta_F, \frac{\delta^2}{\delta \varphi_i \delta \varphi_j} \rangle} V(f_1)[\varphi_1] \dots V(f_n)[\varphi_n] \Big|_{\varphi_1 = \dots = \varphi_n = \varphi}, \end{aligned}$$

where  $\Delta_F$  is the Feynman propagator. Renormalization is formulated as the problem of extending this definition of time-ordered products to  $f$ s with overlapping supports. The solution for scalar fields has been provided in [EG73] and later generalized in [Sch95, DF98, Hol08, FR12] to include QED and Yang-Mills theories. As a result of this construction, one obtains a family of unitaries satisfying the *causal factorisation property*, so the results of section 3 apply.

Recently, in [BR17], it was shown that the formal  $S$ -matrix (2) in the sine Gordon model in 2 dimensions converges in the low coupling regime. This opens up perspectives for non-perturbative results in pAQFT.

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### Persistence of Fermionic Exponential Decay and Spectral Gaps

MANFRED SALMHOFER

(joint work with Wojciech De Roeck)

The existence of a gap between the ground state energy and the energies of excited states of a quantum Hamiltonian has many important consequences, for instance the quantization of electric conductance [3, 5, 7]. Our work [4] addresses the existence of such a gap in interacting many-body systems.

We consider a class of quantum many-body systems of identical fermions on a space that is a finite set (e.g. a finite lattice)  $\Lambda$ , with a Hamiltonian  $H =$

$H_0 + H_I$ , where  $H_0$  is quasifree and self-adjoint, and  $H_I$  is given by a general multi-body interaction  $V$ . We prove that, if  $H_0$  has a fermionic covariance that decays exponentially in time, the truncated expectation values for  $H$  also decay exponentially in time, provided that  $V$  is small and falls off fast enough in space to be summable. We use this to show the persistence of the spectral gap under the interaction [4], uniformly in  $|\Lambda|$ . A different proof of a gap has recently been put forward in [6].

Our proof uses methods from constructive quantum field theory, namely a suitable Grassmann integral representation for expectation values, tree-determinant formulas that regroup perturbation expansions for fermionic systems, and determinant bounds. It is based on the results of [8, 9], where convergence of fermionic perturbation theory was proven in a general setting, and of [10], where  $\ell^1$ -clustering of truncated correlation functions of field operators was proven without multiscale analysis. The work reported here focuses on showing an exponential decay in Euclidian time and inferring from it the persistence of the gap. In the following, we describe the main statement and some essential ingredients of the proof of exponential decay informally.

Within the operator-algebraic setting of statistical mechanics [1], we consider the truncated correlation function  $\langle A(\tau); B \rangle_{\beta H} = \langle A(\tau)B \rangle_{\beta H} - \langle A \rangle_{\beta H} \langle B \rangle_{\beta H}$ , where  $A$  and  $B$  are observables (linear operators on the fermionic Fock space  $\mathcal{F}$ ),  $A(\tau) = e^{\tau H} A e^{-\tau H}$  is the Euclidian-time-evolved version of  $A$ , and  $\langle A \rangle_{\beta H} = \frac{1}{Z_{\beta H}} \text{Tr}_{\mathcal{F}} (e^{-\beta H} A)$  is the quantum statistical expectation value for inverse temperature  $\beta > 0$ , with the partition function  $Z_{\beta H}$  defined such that  $\langle 1 \rangle_{\beta H} = 1$ .

$H_0$  is the second quantization of a one-particle Hamiltonian  $H_0$  on  $\ell^2(\Lambda)$ , which is given by its matrix representation  $h_0(x, x')$ . By self-adjointness,  $h_0(x', x) = \overline{h_0(x, x')}$ . We do not need to assume translation invariance. The fermionic covariance is the time-ordered truncated two-point correlation for the system of free particles governed by  $H_0$ . It is given explicitly in terms of  $H_0$  as  $C(\tau, x; \tau', x') = \mathcal{C}(\tau - \tau', H_0)_{x, x'}$ , where

$$\mathcal{C}(t, H_0) = 1_{t \leq 0} (1 + e^{\beta H_0})^{-1} e^{-tH_0} - 1_{t > 0} (1 + e^{-\beta H_0})^{-1} e^{-tH_0}$$

We first prove that, if  $h_0(x, x')$  decays like a sufficiently high power of the distance between  $x$  and  $x'$  on  $\Lambda$ , and if  $H_0$  has a gap, there is  $\rho > 0$  such that

$$\alpha_\rho = \sup_{\tau, x} \sum_{x'} \int_{-\beta}^{\beta} d\tau' e^{\rho d(\tau, \tau')} \max\{|C(\tau, x; \tau', x')|, |C(\tau', x'; \tau, x)|\}$$

is bounded uniformly in  $|\Lambda|$ . Here  $d(\tau, \tau')$  is  $|\tau - \tau'|$  modulo  $\beta$ . During discussions at the workshop in Oberwolfach, Jean-Bernard Bru and Walter de Siqueira Pedra informed us that a similar decay estimate is also in their recent preprint [11].

An important further step in the convergence proof of fermionic perturbation theory is to bound determinants of  $n \times n$  matrices constructed from the fermionic covariances, for arbitrary  $n$ , by  $\delta^{2n}$ . The constant  $\delta$  is called the *determinant constant* of the fermionic covariance. It was introduced and shown to be finite for

a general class of translation-invariant  $H_0$  in [8]. In [4] we show that a simple generalization of the argument in [8] extends this bound to the covariance associated to an arbitrary self-adjoint  $H_0$ . Another proof of finiteness in this general setting, which uses noncommutative Hölder inequalities and yields optimal constants in the bounds, was given in [2].

Let  $V$  be an interaction, i.e. a sequence of functions  $v_{\bar{m},m}$  for  $\bar{m} \geq 0$  and  $m \geq 0$  that are antisymmetric with respect to permutations of their arguments. The contribution of  $v_{\bar{m},m}(\bar{x}_1, \dots, \bar{x}_{\bar{m}}; x_1, \dots, x_m)$  to  $H_I$  corresponds to a process where  $m$  particles at positions  $x_1, \dots, x_m$  are removed and  $\bar{m}$  particles at positions  $\bar{x}_1, \dots, \bar{x}_{\bar{m}}$  are created. A two-body potential corresponds to a special case where  $\bar{m} = m = 2$ . For  $h > 0$ , the norm of  $V$  is defined as

$$\|V\|_h = \sum_{\bar{m}, m=0}^{\infty} |v_{\bar{m},m}| h^{\bar{m}+m}$$

where  $|v_{\bar{m},m}|$  is the maximum of

$$\sup_{\bar{x}_1} \sum_{\bar{x}_2, \dots, \bar{x}_m} |v_{\bar{m},m}(\bar{x}_1, \dots, \bar{x}_{\bar{m}}; x_1, \dots, x_m)|$$

and

$$\sup_{x_m} \sum_{\bar{x}_1, \dots, \bar{x}_{m-1}} |v_{\bar{m},m}(\bar{x}_1, \dots, \bar{x}_{\bar{m}}; x_1, \dots, x_m)|$$

Our result about exponential decay is that if the fermionic covariance to  $H_0$  has determinant constant  $\delta$ , and if  $\alpha_\rho < \infty$  for some  $\rho > 0$ , then for all  $V$  with  $\alpha_\rho \|V\|_{1+\delta} < 1$ ,

$$|\langle A(\tau); B \rangle_{\beta(H_0+H_I)}| \leq K(A, B, \alpha_\rho, \|V\|_{1+\delta}) e^{-\rho d(\tau,0)}$$

where  $K$  is given explicitly in [4]. Because the conditions only involve  $\alpha_\rho$  and  $\delta$ , this bound holds for a very general class of fermionic models, which do not need to be translation-invariant, and the exponential decay in time does not require a hypothesis of exponential falloff of the interaction function in the spatial variables. It suffices to have spatial summability uniformly in  $|\Lambda|$ , and a spectral gap of  $H_0$ .

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**The causal action principle and fermionic loop diagrams**

FELIX FINSTER

The theory of *causal fermion systems* is an approach to describe fundamental physics. Giving quantum mechanics, general relativity and quantum field theory as limiting cases, it is a candidate for a unified physical theory (see [3] or the survey article [7]). From the mathematical perspective, causal fermion systems provide a general framework for describing and analyzing non-smooth geometries (see [5, 6] or the introductions in [4] and [3, Section 1.1.]). The dynamics is determined by a novel variational principle, called the *causal action principle* (see [8] and the references therein). As is worked out in the monograph [3], the causal action principle gives rise to the interactions of the standard model plus gravity on the level of second-quantized fermionic fields coupled to classical bosonic fields. The connection to second-quantized bosonic fields is made in [1, 2].

In the talk, I focus on a special case of the causal action principle where spinorial wave functions are varied in Minkowski space  $\mathcal{M}$ . Thus for given functions  $\psi_1, \dots, \psi_f \in C^0(\mathcal{M}, \mathbb{C}^4)$ , we introduce

$$\begin{aligned}
 P(x, y) &= - \sum_{a=1}^f \psi_a(x) \overline{\psi_a(y)} && \text{kernel of the fermionic projector} \\
 A_{xy} &= P(x, y) P(y, x) && \text{closed chain} \\
 \lambda_1^{xy}, \dots, \lambda_4^{xy} &\in \mathbb{C} && \text{eigenvalues of the closed chain} \\
 \mathcal{L}(x, y) &= \frac{1}{4} \sum_{i,j=1}^4 (|\lambda_i^{xy}| - |\lambda_j^{xy}|)^2 && \text{Lagrangian,}
 \end{aligned}$$

where  $\overline{\psi_a} := \psi_a^\dagger \gamma^0$  is the usual adjoint spinor. Then the *causal action principle* is to minimize the

$$\text{action} \quad \mathcal{S} = \int_{\Lambda} d^4x \int_{\Lambda} d^4y \mathcal{L}(x, y) + \kappa \int_{\Lambda} d^4x \int_{\Lambda} d^4y \sum_{i=1}^4 |\lambda_i^{xy}|^2$$

under variations of the functions  $\psi_1, \dots, \psi_f \in C^0(\mathcal{M}, \mathbb{C}^4)$ , respecting the

$$\text{trace constraint} \quad \int_{\Lambda} \text{Tr}_{\mathbb{C}^4}(P(x, x)) d^4x = \text{const.}$$

Here  $\Lambda$  is a compact subset of Minkowski space, and  $\kappa > 0$  is the Lagrange multiplier of the so-called boundedness constraint. The corresponding Euler-Lagrange equations are mentioned. The limiting cases  $\Lambda \rightarrow \mathcal{M}$  (removing the infrared regularization) as well as  $f \rightarrow \infty$ ,  $\kappa \searrow 0$  (removing the ultraviolet regularization) are discussed.

It is described how to compute the causal action for a regularized Dirac sea configuration in the vacuum and in the presence of an external potential. The connection to the light-cone expansion, the Hadamard expansion and the point-splitting renormalization method is made. It is explained why the usual quadratic and linear divergences of the vacuum polarization diagrams drop out of the causal action. I also outline how the remaining logarithmic pole is removed by the microlocal chiral transformation. An outlook on the coupled Dirac-Maxwell equations and second-quantized bosonic fields is given.

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## The Ground State Energy of Heavy Atoms

HEINZ SIEDENTOP

We reviewed results about the ground state of heavy atoms starting from the basic physical work of Thomas [5] and Fermi [1, 2] and the corresponding proof by Lieb and Simon [4].

We need some notation to formulate our result. We write  $D_{c,Z} := c\mathbf{p} \cdot \boldsymbol{\alpha} + c^2\beta - c^2 - Z/|\mathbf{x}|$  for the Coulomb-Dirac operator and set  $\mathfrak{H} := [\chi_{(-c^2, \infty)}(D_{c,Z})](L^2(\mathbb{R}^3 : \mathbb{C}^4))$ .

We now define the basic object, the relativistic Hamiltonian of an atom of nuclear charge  $Z$  and  $N$  electrons in the Furry picture: Assume  $Z/c < 1$  and



define  $F_{N,Z}$  to be the self-adjoint operator associated with the form

$$\mathcal{E} : \mathcal{S}(\mathbb{R}^{3N} : \mathbb{C}^{4^N}) \cap \bigwedge_{n=1}^N \mathfrak{H} \rightarrow \mathbb{R}$$

$$\psi \mapsto (\psi, (\sum_{n=1}^N D_{c,Z_n} + \sum_{0 \leq m < n \leq N} |x_m - x_n|^{-1})\psi).$$

We write  $E(Z)$  for the infimum of the spectrum of  $F_{Z,Z}$  and  $-e_{TF}$  for the Thomas-Fermi energy of hydrogen. Furthermore we define the Scott function: For  $\gamma \in (0, 1)$  write

$$\lambda_n^{S,H} : n\text{-th eigenvalue of } (p^2/2 - |x|^{-1}) \otimes 1_{\mathbb{C}^2}$$

$$\lambda_n^{D,H} : n\text{-th eigenvalue of } \alpha \cdot p + \beta - 1 - \gamma/|x|$$

$$s^D(\gamma) := \frac{1}{\gamma^2} \sum_{n=1}^{\infty} (\lambda_n^{S,H} - \lambda_n^{D,H})$$

for  $\gamma \in (0, 1)$ .

Our first result is on the energy:

**Theorem 1** (Handrek and Siedentop [3]). *Fix  $\gamma : Z/c \in (0, 1)$ . Then, as  $Z \rightarrow \infty$ ,*

$$(1) \quad E(Z) = -e_{TF}Z^{7/3} + (\frac{1}{2} - s^D(\gamma))Z^2 + o(Z^2).$$

For the second result we need the Coulomb scalar product

$$D(f, g) := \frac{1}{2} \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy \frac{\overline{f(x)}g(y)}{|x - y|}.$$

Furthermore we write  $\rho_{TF}$  for the Thomas-Fermi density of hydrogen. Then, our second result reads:

**Theorem 2** (Merz and Siedentop). *Assume  $\psi^Z$  a ground state of  $F_{Z,Z}$ ,  $\tilde{\rho}_{\psi^Z}$  the associated ground state density, and rescale*

$$\rho^Z(x) := Z^{-2} \tilde{\rho}_{\psi^Z}(Z^{-1/3}x)$$

Then

$$\lim_{Z \rightarrow \infty} D[\rho - \rho_{TF}] = 0.$$

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## Non-equilibrium almost-stationary states for interacting fermions on a lattice

STEFAN TEUFEL

Consider a family of Hamiltonians  $H_0^\Lambda$  for systems of interacting fermions on finite subsets  $\Lambda \subset \mathbb{Z}^d$  of the lattice  $\mathbb{Z}^d$  that has a spectral gap above the ground state uniformly in the system size  $|\Lambda|$ . We show in [T] that for a large class of perturbations  $V$  there exist *non-equilibrium almost-stationary states* (NEASS) for the perturbed Hamiltonian  $H = H_0 + V_\varepsilon$ , even if the perturbation closes the spectral gap. Almost-stationarity refers to the property that expectations of intensive quantities in these states are constant over long (super-polynomial in  $\frac{1}{\varepsilon}$ ) times up to small (super-polynomial in  $\varepsilon$ ) errors, uniformly in the size  $|\Lambda|$  of the system.

These NEASS are connected to the ground state of the unperturbed Hamiltonian by local unitary transformations. The class of allowed perturbations  $V_\varepsilon$  includes slowly varying potentials and small local Hamiltonians. Both types of perturbations need not be small in norm. Slowly varying potentials typically close the gap of  $H_0$ , but leave a local gap structure intact.

We also prove an adiabatic-type theorem for time-dependent NEASS associated with time-dependent perturbations. Based on this theorem, we show that when starting from the ground state of the unperturbed gapped system and then adiabatically turning on the perturbation (which might close the gap), then the final state of the corresponding Schrödinger evolution is given by a NEASS up to errors that are asymptotically smaller than any power of the adiabatic parameter. The NEASS that is finally reached is independent of the precise form of the switching function.

For non-interacting systems, the formalism of space-adiabatic perturbation theory allows to prove similar results, cf. [PST]. The results described here could thus be viewed as a generalisation of space-adiabatic perturbation theory to interacting systems, although the mathematical techniques are very different.

As mentioned before, we also prove an adiabatic-type theorem for time-dependent Hamiltonians  $H(t)$  and arbitrary initial states supported in super-adiabatic subspaces associated with gapped parts of the spectrum. It was shown in [BdRF] and elaborated on in [MT] that, as long as  $H(t)$  has a gapped part of the spectrum, the time-evolution of pure states supported initially in the corresponding spectral subspace follows adiabatically this time-dependent spectral subspace with error estimates that are uniform in the system size. Here we show much more. With a gapped part of the spectrum of the unperturbed Hamiltonian  $H_0(t)$  we associate a super-adiabatic subspace (a time-dependent generalisation of the NEASS) of the perturbed Hamiltonian  $H(t)$  (which no longer needs to be gapped) and show that

arbitrary states supported initially in the super-adiabatic subspace remain, under the adiabatic time-evolution, within the time-dependent super-adiabatic subspace with error estimates that are uniform in the system size. If the underlying spectral subspace of  $H_0(t)$  is more than one-dimensional and if the initial state is not just the projection onto the total super-adiabatic subspace, then the dynamics within the super-adiabatic subspace are typically non-trivial and generated by an effective Hamiltonian.

Our proofs heavily use and partly extend a technical machinery developed rather recently. This includes Lieb-Robinson bounds for interacting fermions [NSY, BD], the so called spectral-flow or quasi-adiabatic evolution [HW, BMNS], an adiabatic theorem for local observables in interacting lattice systems [BdRF], and its extension to intensive quantities and super-adiabatic tracking in [MT].

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### Quantum backflow in scattering situations (Young Researcher Session)

DANIELA CADAMURO

In classical (statistical) mechanics, point masses travel in the same direction as their momentum. For a particle moving without friction in one direction, its initial state is a probability distribution  $\sigma(x, p)$ , where  $x, p$  are position and momentum of the particle, with time evolution  $\sigma_t(x, p) = \sigma(x - pt, p)$ . The probability density  $\rho_t(x) = \int dp \sigma_t(x, p)$  to find the particle at a point  $x$  and the corresponding probability flux  $j_t(x) = \int dp p \sigma_t(x, p)$  fulfill the continuity equation, implying

$$(*) \quad \frac{d}{dt} \int_0^\infty dx \rho_t(x) = j_t(0).$$

We note that if  $\sigma_t(x, p) = 0$  for  $p < 0$  and all  $x$ , then  $j_t \geq 0$ , i.e., the right hand side of (\*) is non-negative. Therefore, the probability of finding the particle on the right of 0 is increasing in time, while it is decreasing on the left. In other words, if the particle has positive momentum (with probability 1) then the probability distribution moves to the right.

For a quantum mechanical particle the situation is different. Suppose that its state  $\varphi$  is chosen so that it still has momentum  $p > 0$  with probability 1. Can we still expect  $\langle J(x) \rangle_\varphi \geq 0$  (where  $\langle J(x) \rangle_\varphi$  denotes the expectation value of the probability flux operator  $J(x)$  in the state  $\varphi$ )? In some examples for  $\varphi$ , exactly this does *not* happen, which is called “*quantum backflow*”. The question is how large this effect is, and whether positivity is retained in some approximate sense.

As a first example, we consider a *free* particle (without external forces) moving in one dimension, whose state is a vector in the Hilbert space  $\mathcal{H} = L^2(\mathbb{R}, dp)$  with support in  $\mathbb{R}_+$  (positive momentum with probability 1). Its time evolution is given by  $\varphi_t(p) = \exp(-i\frac{\hbar^2}{2}t)p\varphi(p)$  ( $\hbar = m = 1$ ). The probability density and probability flux for the position of the particle are

$$\begin{aligned}\langle \rho(x) \rangle_\varphi &= \int dp dq \overline{\varphi(p)} \frac{1}{2\pi} e^{i(q-p)x} \varphi(q), \\ \langle J(x) \rangle_\varphi &= \int dp dq \overline{\varphi(q)} \frac{p+q}{4\pi} e^{i(q-p)x} \varphi(q)\end{aligned}$$

and the continuity equation still holds. We can find states with positive momentum such that the expectation value of the flux operator at a given point  $x$  is negative, see [1] for explicit examples. Hence, the probability of finding the particle is increasing in time to the left of  $x$ , and decreasing to the right (“backflow”). To compute lower bounds on the flux, some care is needed. As the expression above diverges for large momenta  $p, q$ , it makes sense only as a quadratic form. We therefore consider “smooth averages” over a small space interval:  $\langle J_f \rangle_\varphi = \int dx f(x) \langle J(x) \rangle_\varphi$  with  $f \in \mathcal{S}(\mathbb{R})$ ,  $f \geq 0$ .

We now ask about the spectrum of  $J_f$  restricted to wave functions of positive momentum  $\varphi = E_+ \varphi$  (where  $E_+$  is the projector onto positive momenta). The following result is due to Eveson, Fewster and Verch [1]:

**Theorem 1.** *For every  $f \in \mathcal{S}(\mathbb{R})$ ,  $f \geq 0$ , there exists  $c_f \geq 0$  such that  $\langle J_f \rangle_\varphi \geq -c_f \|\varphi\|^2$  whenever  $\varphi$  has compact support in  $\mathbb{R}_+$ .*

This “quantum inequality” shows that the backflow effect is limited. However, the result does not take into account any influence from external potentials. Our goal is to prove that quantum backflow is a *general* feature of quantum mechanical particles, even if they are not free. Can one expect such lower bounds in the presence of a potential? There may be a reflected wave, which behaves (for large times) like a free particle moving to the left – is this an obstacle to bounds?

We consider a scattering potential  $V$  and the Hamiltonian  $H = \frac{1}{2}P^2 + V(X)$ . The time evolution is now more intricate since it is not just a multiplication operator. However, for suitable potentials, the wavefunctions have a well-known

asymptotics at  $t \rightarrow \pm\infty$  (incoming/outgoing waves), which is an essential ingredient of *scattering theory*. This is of importance since a *prima facie* problem to the extension of the result [1] in the interacting situation is to understand what a particle “moving to the right” is. In fact, since  $[E_+, H] \neq 0$ , the space of “positive momentum wavefunctions” is not preserved under time evolution.

A replacement for this notion is provided in scattering theory by considering asymptotic (incoming) states with positive momentum. The Møller operator  $\Omega_{\text{in}}$  maps “interacting” states to “incoming” states. Using this, the question we ask is whether  $E_+ \Omega_{\text{in}} J_f \Omega_{\text{in}}^* E_+$  is bounded below. Our main result is [2]:

**Theorem 2.** *Suppose that  $\int dx (1 + |x|)|V(x)| < \infty$ . Then, for any  $f \in \mathcal{S}(\mathbb{R})$ ,  $f \geq 0$ , there exists a constant  $c_{V,f} > 0$  such that*

$$E_+ \Omega_{\text{in}} J_f \Omega_{\text{in}}^* E_+ \geq -c_{V,f} \cdot \mathbf{1}.$$

This applies to all usual short range potentials; examples are the square well potential (attractive or repulsive), the Pöschl-Teller type transparent potentials, and any measurable potential decaying like  $|x|^{-\alpha}$ ,  $\alpha > 2$ . A similar result holds for a delta potential (attractive or repulsive). This indicates that reflection does not present a problem to existence of lower bounds.

This result could potentially be generalized to other, physically interesting situations, including particles with spin or other degrees of freedom, several particles, 2- or 3-dimensional scattering. One could also investigate possible generalizations of the current  $J$ , for example consider the probability flux of only the “spin up” component in the case of a spin-1/2 particle, or identify a suitable  $J$  in a 3-dimensional situation. Another possible direction for generalization is to consider PDEs other than the Schrödinger equation, or more generally, recasting the theorem in the framework of abstract scattering theory. Finally, we remark that backflow may be one of the quantum mechanical effects directly verifiable in experiments, as an experiment to measure it has been proposed [3]. Moreover, it has relations to other “quantum inequalities” appearing in quantum field theory in connection with the energy density, which are relevant for the stability of spacetime; see, e.g., [4, 5].

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**Non-Locality of the Vacuum and Scattering of Atoms in Relativistic QED (Young Researcher Session)**

MAXIMILIAN DUELL

The particle spectrum of Quantum Electrodynamics (QED) combines massless particle excitations (photons) and massive particles (non-charged atoms or charged electrons). Due to infrared problems, long-range forces and non-localizability of charges, these cases are understood to be of increasing difficulty and require distinct methods.<sup>1</sup>

Our contribution addresses the scattering theory of massive Wigner particles, such as hydrogen atoms from the point of view of relativistic QED. In QED-like settings conventional Haag-Ruelle theory does not apply due to its dependence on isolated mass shells. So far all available results for massive particles are based on the spectral condition of Herbst [2]: suitable local operators  $A \in \mathfrak{A}(\mathcal{O})$  yield a one-particle vector  $\Psi_1 = E(H_m)A\Omega \neq 0$  and satisfy

$$(H) \quad \|E(H_m^\delta)A\Omega - \Psi_1\| \leq C\delta^\epsilon$$

for some  $C, \epsilon > 0$  and all sufficiently small  $\delta$ -neighbourhoods of the mass shell  $H_m$ . Here the local operator  $A$  acts on the vacuum vector  $\Omega$  and  $E = E_{(H, \mathbf{P})}$  are the energy-momentum spectral projections. Condition (H) is commonly justified heuristically by hinting at stability requirements or absence of long-range forces, but so far there are no rigorously established links between such physical ideas and (H).

In [1] we provide technical improvements leading to an alternative Haag-Ruelle construction for embedded mass shells based on a complementary condition with uniformly localized operator families  $A_\beta \in \mathfrak{A}(\mathcal{O})$ , where  $\beta > 0$  parametrizes the one-particle approximation error

$$(RS1) \quad \|A_\beta\Omega - \Psi_1\| \leq \beta,$$

$$(RS2) \quad \|A_\beta\| \leq \beta^{-\gamma}, \text{ (for some } \gamma > 0\text{)}.$$

The Reeh-Schlieder Theorem implies the existence of operator families  $(A_\beta)_{\beta>0}$  satisfying (RS1). As the localization region  $\mathcal{O}$  is fixed and does not need to be enlarged as  $\beta \rightarrow 0$ , this may be regarded as exploiting non-local correlations in the vacuum state  $\Omega$  for the construction of scattering states. In combination with the norm growth assumption (RS2), convergence and Fock structure of scattering states was established in [1]. In fact (H) implies a weakened version of (RS1), (RS2), but in general the two sets of conditions seem to be independent, and (RS2) does not appear to be a consequence of general assumptions. Further technical progress should allow to weaken conditions (RS1), (RS2), and we expect that our construction will also give new insights for the scattering-theoretic analysis of non-relativistic Quantum Electrodynamics. (Joint work with W. Dybalski).

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<sup>1</sup>A recent discussion of scattering of electrical charges and the more severe infrared problems expected for this case is given in [3].

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**Derivation of the Maxwell-Schrödinger equations from the Pauli-Fierz Hamiltonian (Young Researcher Session)**

NIKOLAI LEOPOLD

(joint work with Peter Pickl)

The corpuscular character of light is in a lot of situations subordinate and the second-quantized electromagnetic field is described by a classical field satisfying Maxwell’s equations. In this talk, we justify such an approximation in the mean field regime and derive the Maxwell-Schrödinger equations from the spinless Pauli-Fierz Hamiltonian. We consider a system, modeled by a wave function  $\Psi_{N,t} \in L^2(\mathbb{R}^{3N}) \otimes \mathcal{F}$ , of  $N$  identical charged bosons in interaction with a photon field. Here,  $\mathcal{F}$  denotes the bosonic Fock space over  $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ . The time evolution of  $\Psi_{N,t}$  is governed by the Schrödinger equation

$$(1) \quad i\partial_t \Psi_{N,t} = H_N \Psi_{N,t},$$

where

$$(2) \quad H_N = \sum_{j=1}^N \left( -i\nabla_j - \frac{\hat{A}_\kappa(x_j)}{\sqrt{N}} \right)^2 + \frac{1}{N} \sum_{1 \leq j < k \leq N} v(x_j - x_k) + H_f$$

is the Pauli-Fierz Hamiltonian.  $H_f$  denotes the free Hamiltonian of the photon field,  $\hat{A}_\kappa$  the quantized transverse vector potential and  $v$  causes a direct interaction between the charged particles. The mean field scaling  $1/N$  in front of the interaction potential and the scaling  $1/\sqrt{N}$  in front of the vector potential ensure that the kinetic and potential energy of  $H_N$  are of the same order. It is known that coherent states of photons behave like a classical field. However, it is unclear which initial configurations of charges will lead to the creation of a coherent state. We are interested in systems with many weakly correlated charges and consider initial conditions of the product form  $\varphi_0^{\otimes N} \otimes W(\sqrt{N}\alpha_0)\Omega$ , where  $\Omega$  denotes the vacuum in  $\mathcal{F}$  and  $W$  is the usual Weyl operator creating coherent states of photons. Due to the interaction correlations take place and the time evolved state will no longer have an exact product structure. In general, the photon state does not need to be coherent and behave like a classical field at later times. However, for large  $N$  we are able to show that the time evolved state can be approximated in trace norm distance of reduced density matrices by a state of product form  $\varphi_t^{\otimes N} \otimes W(\sqrt{N}\alpha_t)\Omega$ , where

$$(3) \quad |k|^{1/2} \alpha_t(k, \lambda) = \frac{1}{\sqrt{2}} \epsilon_\lambda(k) \cdot (|k| \mathcal{FT}[\mathbf{A}](k, t) - i \mathcal{FT}[\mathbf{E}](k, t))$$

and

$$(4) \quad (\varphi_t, \mathbf{A}(t), \mathbf{E}(t))$$

solve the Maxwell-Schrödinger system<sup>1</sup>

$$(5) \quad \begin{cases} i \partial_t \varphi_t(x) &= ((-i \nabla - (\kappa * \mathbf{A})(x, t))^2 + (v * |\varphi_t|^2)(x)) \varphi_t(x), \\ \nabla \cdot \mathbf{A}(x, t) &= 0, \\ \partial_t \mathbf{A}(x, t) &= -\mathbf{E}(x, t), \\ \partial_t \mathbf{E}(x, t) &= (-\Delta \mathbf{A})(x, t) - (1 - \nabla \operatorname{div} \Delta^{-1})(\kappa * \mathbf{j}_t)(x), \\ \mathbf{j}_t(x) &= 2 (\Im(\varphi_t^* \nabla \varphi_t)(x) - |\varphi_t|^2(x) (\kappa * \mathbf{A})(x, t)), \end{cases}$$

with initial datum

$$\begin{cases} \varphi_0, \\ \mathbf{A}(x, 0) = (2\pi)^{-3/2} \sum_{\lambda=1,2} \int d^3 k \frac{1}{\sqrt{2|k|}} \epsilon_\lambda(k) (e^{ikx} \alpha_0(k, \lambda) + e^{-ikx} \alpha_0^*(k, \lambda)), \\ \mathbf{E}(x, 0) = (2\pi)^{-3/2} \sum_{\lambda=1,2} \int d^3 k \sqrt{\frac{|k|}{2}} \epsilon_\lambda(k) i (e^{ikx} \alpha_0(k, \lambda) - e^{-ikx} \alpha_0^*(k, \lambda)). \end{cases}$$

This system of equations models the coupling of a non-relativistic particle to the classical electromagnetic field. The precise result is given in [Theorem I.1., [1]].

#### REFERENCES

- [1] N. Leopold and P. Pickl, *Derivation of the Maxwell-Schrödinger Equations from the Pauli-Fierz Hamiltonian*, arXiv:1609.01545, (2017)

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<sup>1</sup>Hereby,  $(\kappa * \mathbf{A})(x, t) = \int d^3 k e^{ikx} \tilde{\kappa}(k) \mathbf{A}(k, t)$  with  $\tilde{\kappa}(k) = (2\pi)^{-3/2} \mathbf{1}_{|k| \leq \Lambda}(k)$ .



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