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Mini-Workshop: Positivity in Higher-dimensional Geometry: Higher-codimensional Cycles and Newton–Okounkov Bodies

Organised by
Mihai Fulger, Storrs
Alex Küronya, Frankfurt
Brian Lehmann, Chestnut Hill

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Abstract. There are several flavors of positivity in Algebraic Geometry. They range from conditions that determine vanishing of cohomology, to intersection theoretic properties, and to convex geometry. They offer excellent invariants that have been shown to govern the classification and the parameterization programs in Algebraic Geometry, and are finer than the classical topological ones. This mini-workshop aims to facilitate research collaboration in the area, strengthening the relationship between various positivity notions, beyond the now classical case of divisors/line bundles.

Mathematics Subject Classification (2010): 14Cxx, 14Exx, 14Fxx.

Introduction by the Organisers

The mini-workshop *Positivity in Higher-dimensional Geometry* brought together algebraic geometers with various interests circling around the idea of positivity. The participants invested the majority of their time in group work on open questions related to positivity of higher-codimensional numerical cycle classes on projective varieties, or convex geometry in the form of Newton–Okounkov bodies. There were also seven research talks given by Jian Xiao (Evanston), Victor Lozovanu (Hannover), Chung Ching Lau (Chicago), John Christian Ottem (Oslo), Catriona Maclean (Grenoble), Stefano Urbinati (Padova), and Nguyen-Bac Dang (Paris) in this order. The extended abstracts of their presentations appear in the sequel.

The topic of positivity is an active research subject in Algebraic Geometry:

- The position of the canonical class K_X of a projective variety X relative to the ample cone $\mathrm{Amp}(X) \subset \mathrm{N}^1(X)$ guides the Minimal Model Program.
- In Moduli Space theory, stability is behind boundedness for parameter spaces for varieties or for sheaves on them with prescribed numerical invariants.

Most of the techniques in contemporary research involve classical positivity properties of divisors/line bundles, or, by duality, curve classes. The primordial notion is ampleness. It can be characterized cohomologically, intersection theoretically (cf. [Kle66]), and geometrically. A geometric generalization is bigness. A divisor D is *big* if it is the sum $D = A+E$ of an ample divisor A and an effective divisor E. Its natural cohomological characterization involves an asymptotical construction:

$$D \text{ is big } \Leftrightarrow \mathrm{vol}\,(D) > 0,$$

where $\mathrm{vol}\,(D)$ measures the asymptotic growth of the dimension of the linear series $|mD|$ as m grows. More precisely

$$\mathrm{vol}\,(D) \stackrel{\mathrm{def}}{=} (\dim X)! \cdot \limsup_{m\to\infty} \frac{\dim H^0(X; \mathcal{O}_X(mD))}{m^{\dim X}}.$$

Recently, a convex geometric approach (cf. [LM09, KK12]) through the theory of Newton–Okounkov bodies has provided the ideal package for much of this information. To a complete flag of subvarieties

$$Y_\bullet : \quad X = Y_0 \supsetneq Y_1 \supsetneq \ldots \supsetneq Y_{\dim X} = \{x\} \in X$$

centered a closed point x that is smooth for all Y_i, one associates a rank $\dim X$ valuation-type function on the global sections on $\mathcal{O}_X(mD)$ for all m and a convex body $\Delta_{Y_\bullet}(D) \subset \mathbb{R}^{\dim X}$. It encodes many of the positivity properties of D:

- $\mathrm{vol}\,(D)$ is the normalized Euclidean volume of the Newton–Okounkov body $\Delta_{Y_\bullet}(D) \subset \mathbb{R}^{\dim X}$, cf. [LM09].
- The divisorial Zariski decomposition of D can be computed (cf. [Jow10, KL17, CHPW15]), from the knowledge of $\Delta_{Y_\bullet}(D) \subset \mathbb{R}^{\dim X}$ for all flags $Y_\bullet$ on X as above.
- The restricted volumes $\mathrm{vol}_{\,X|V}(D)$ can be determined (cf. [LM09]) from the knowledge of $\Delta_{Y_\bullet}(D) \subset \mathbb{R}^{\dim X}$ for all flags $Y_\bullet$ on X as above.
- The Seshadri constant $\varepsilon\,(D; x)$ can be found (cf. [KL17]) by working with infinitesimal flags centered at x.

In higher (co)dimension, historically the outlook has been negative and significant progress in the way of general results (as opposed to pathological examples) is of a more recent nature. There are two directions here: [Ott12, Lau16] study positivity from a cohomological perspective. A smooth subvariety $V \subset X$ of dimension d of a complex projective manifold is said to be ample if the relative $\mathcal{O}(1)$ on $\mathrm{Bl}_V X$ satisfies certain cohomology vanishing conditions (cf. [Ott12]). For example [Ott12] recovers Lefschetz-hyperplane-type results, while [Lau16] proves versions of Fujita vanishing in this context.

A numerical intersection theoretic perspective is adopted by Fulger–Lehmann in [FL17a, FL17b]. The mantra is that the geometry of cycles should be reflected in the geometry of convex cones inside the numerical groups $\mathrm{N}_d(X)$. Classical examples of such cones are the *pseudo-effective* cone $\overline{\mathrm{Eff}}_d(X) \subset \mathrm{N}_d(X)$, the closure of the convex cone generated by d-dimensional subvarieties of X. Another example is its dual $\mathrm{Nef}^d(X) \subset \mathrm{N}^d(X) \stackrel{\mathrm{def}}{=} \mathrm{N}_d(X)^\vee$ of classes $\beta \in \mathrm{N}^d(X)$ with $\beta(\alpha) \geq 0$ for all $\alpha \in \overline{\mathrm{Eff}}_d(X)$.

When $d = 1$, by [Kle66] the ample cone $\mathrm{Amp}(X)$ is the interior of the nef cone of divisors $\mathrm{Nef}^1(X) \subset \mathrm{N}^1(X)$. In particular $\mathrm{Nef}^1(X) \subseteq \overline{\mathrm{Eff}}_{\dim X-1}(X)$. This inclusion may fail for arbitrary d (cf. [DELV11]). Thus we may not expect good geometry from an arbitrary nef class. However [FL17a] prove that $\mathrm{Nef}^d(X)$ is full-dimensional in $\mathrm{N}^d(X)$ and contains complete intersection classes of ample divisors in its strict interior. As a corollary they show that $\overline{\mathrm{Eff}}_d(X) \subset \mathrm{N}_d(X)$ is a pointed cone.

The main goal of the workshop was to form diverse groups focused on open questions relating to cycles and/or convex geometry, facilitating future collaboration on the subject. The participants split into four groups, and their assignments are described below.

Problem 1.

(1) Is the convex cone generated by classes of ample subvarieties open in $\mathrm{N}_d(X)$?

This is an attempt to understand the relation between cohomological and numerical positivity in higher codimension, paralleling known results for divisor classes.

(2) Is there a convex geometric approach to higher cohomology functions similar to the Newton–Okounkov interpretation of $\mathrm{vol}\,(D)$? If D is a divisor on a projective variety, is there an object in convex geometry whose Euclidean volume is naturally equal to

$$\hat{h}^i(D) \stackrel{\mathrm{def}}{=} (\dim X)! \cdot \limsup_{m\to\infty} \frac{\dim H^i(X;\mathcal{O}_X(mD))}{m^{\dim X}}?$$

Can lim sup be replaced by lim? If this is true, then the following also holds

$$(D^{\dim X}) = \sum\nolimits_{i=0}^{\dim X} (-1)^i \hat{h}^i(D).$$

(3) Let X be a smooth toric variety and let $T \subset X$ be the open torus. The *movable* d-dimensional subvarieties of X are those that admit deformations through points of T. Is the closed cone $\overline{\mathrm{Mov}}_d(X) \subset \mathrm{N}_d(X)$ generated by numerical classes of such subvarieties rational polyhedral?

The cases of divisors and curves are known to be true, but their proofs rely on deep results that have no clear generalizations to arbitrary (co)dimension. The movable cone of divisors is rational polyhedreal because toric varieties are Mori Dream Spaces (cf. [HK10]). The movable cone of curves is dual to the effective cone of divisors (cf. [BDPP13] in the general case. A more elementary proof in the toric case appears in [Pay06]), which is easily seen to be rational polyhedral.

(4) Work of Lehmann [Leh16] and Lehmann–Xiao [LX16] has produced a good theory of positivity for curve classes in $N_1(X)$ by duality to the case of divisors. Can this be extended to a Newton–Okounkov-type convex body picture?

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Mini-Workshop: Positivity in Higher-dimensional Geometry: Higher-codimensional Cycles and Newton–Okounkov Bodies

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Abstracts

A notion of positivity of valuations in convex geometry

Nguyen-Bac Dang
(joint work with Jian Xiao)

My talk focuses on some positivity aspects of translation invariant valuations in convex geometry as described in [DX17]. Let $\mathcal{K}(E)$ be the set of convex bodies of an n-dimensional euclidian vector space E, endowed with the Gromov–Hausdorff metric. A valuation $\phi : \mathcal{K}(E) \to \mathbb{R}$ is a continuous, translation-invariant function which satisfies the following condition:

$$\phi(K \cup L) = \phi(K) + \phi(L) - \phi(K \cap L),$$

for all $K, L \in \mathcal{K}(E)$ such that $K \cup L \in \mathcal{K}(E)$ (see [AF14]). A valuation ϕ is homogeneous of degree k if $\phi(\lambda K) = \lambda^k \phi(K)$ for any $\lambda > 0$. The most common examples of homogeneous valuations of degree $n-k$ are given by the formula:

$$K \to V(L_1, \dots, L_k, K[n-k]),$$

with $L_1, \dots, L_k, K \in \mathcal{K}(E)$ and where $V(L_1, \dots, L_k, K[n-k])$ denotes the mixed volume of $L_1, \dots, L_k$ with K repeated $n-k$ times. We define the norm

$$||\phi|| = \sup_{K \subset B} |\phi(K)|,$$

where B is the unit ball in E. The set of valuations on E, denoted $\mathrm{Val}\,(E)$, endowed with the norm $|| \cdot ||$ is a Banach space.

Recall that if $\phi \in \mathrm{Val}\,(E)$ is a valuation and $g \in \mathrm{GL}\,(E)$, then the function $g \cdot \phi = \phi \circ g^{-1}$ is also a valuation. A valuation ϕ is called smooth if the map $g \in \mathrm{GL}\,(E) \to g \cdot \phi \in \mathrm{Val}\,(E)$ is a smooth map from $\mathrm{GL}\,(E)$ into the Banach space $\mathrm{Val}\,(E)$. For example, if $L_1, \dots, L_k \in \mathcal{K}(E)$ are k convex bodies with smooth and strictly convex boundary, then the valuation given by $K \to V(L_1, \dots, L_k, K[n-k])$ is smooth. It is a theorem due to Alesker ([Ale01]) that the set of smooth valuations is dense in $\mathrm{Val}\,(E)$. Denote by $\mathrm{Val}\,^\infty_{n-k}(E)$ the subspace of smooth homogeneous valuations of degree $n-k$, Bernig–Fu have proved that the vector space $\oplus_{k=0}^n \mathrm{Val}\,^\infty_k(E)$ ([BF06], [Ale11]) has a structure of graded algebra where the multiplication is given by the convolution of smooth valuations. In the case of mixed volumes, the convolution reduces to the following construction (see [BF06, see Corollary 1.3]). Take k, l such that $k + l \leqslant n$ and $\phi = V(L_1, \dots, L_k, -[n-k]) \in \mathrm{Val}\,^\infty_{n-k}(E)$, $\psi = V(L'_1, \dots, L'_{n-k}, -[n-l]) \in \mathrm{Val}\,^\infty_{n-l}(E)$ where $L_1, \dots, L_k, L'_1, \dots, L'_{n-l}$ are convex bodies with smooth and strictly convex boundary, then the convolution $\phi * \psi \in \mathcal{V}^\infty_{n-k-l}(E)$ is given by

$$\phi * \psi = V(L_1, \dots, L_k, L'_1, \dots, L'_l, -[n-k-l]).$$

A priori the convolution of arbitrary valuations in $\mathrm{Val}\,(E)$ is not well-defined (see [BF16, Theorem 2]). Our first aim is to exhibit a subspace of $\mathrm{Val}\,(E)$ containing the mixed volumes of all convex bodies and for which the convolution is still well-defined. To that end, we first introduce the cone $\mathcal{P}_{n-k}$ of all valuations

$$\phi_\mu(K) = \int_{\mathcal{K}(E)^k} V(L_1, \dots, L_k, K[n-k]) d\mu(L_1, \dots, L_k),$$

where $K \in \mathcal{K}(E)$ and where μ is a positive Radon measure on $\mathcal{K}(E)^k$ such that:

$$\int_{\mathcal{K}(E)^k} V(L_1, \dots, L_k, B[n-k]) d\mu(L_1, \dots, L_k) < +\infty. \tag{1}$$

Take a valuation ϕ in the vector space generated by $\mathcal{P}_{n-k}$, then the norm $||\phi||_{\mathcal{P}}$ is given by the formula:

$$||\phi||_{\mathcal{P}} = \inf \left\{ \epsilon > 0 \,\middle|\, \begin{array}{c} \forall K_1, \dots, K_{n-k} \in \mathcal{K}(E), \\ |\phi(K_1, \dots, K_{n-k})| \leqslant \epsilon V(B[k], K_1, \dots, K_{n-k}) \end{array} \right\},$$

where

$$\phi(K_1, \dots, K_{n-k}) := \left. \frac{\partial^{n-k}}{\partial t_1 \dots \partial t_{n-k}} \right|_{t_1 = \dots = t_{n-k} = 0} \frac{\phi(t_1 K_1 + \dots + t_{n-k} K_{n-k})}{(n-k)!}.$$

Let $\mathcal{V}_{n-k}$ be the completion of the vector space generated by $\mathcal{P}_{n-k}$ with respect to the norm $|| \cdot ||_{\mathcal{P}}$. Observe that for $k = n$, we have that $\mathcal{V}_0 = \mathbb{R}$ and for $k = 0$, $\mathcal{V}_n = \mathbb{R} \cdot \mathrm{vol}$ where vol is the volume in E.

Theorem 1. *There exists a unique continuous bilinear map* $* : \mathcal{V}_{n-k} \times \mathcal{V}_{n-l} \to \mathcal{V}_{n-k-l}$ *such that for any convex bodies* $L_1, \dots, L_k, L'_1, \dots, L'_l, K$ *one has that:*

$$\begin{aligned} &(K \to V(L_1, \dots, L_k, K[n-k])) * (K \to V(L'_1, \dots, L'_l, K[n-l])) \\ &= \left(K \to \frac{k! l!}{n!} \cdot V(L_1, \dots, L_k, L'_1, \dots, L'_l, K[n-k-l]) \right). \end{aligned}$$

It follows that the space $\oplus_{k=0}^n \mathcal{V}_k$ has a structure of graded algebra with unit and that the convolution maps $\mathcal{P}_{n-k} \times \mathcal{P}_{n-l}$ to $\mathcal{P}_{n-k-l}$. We now fix a linear map $g \in \mathrm{GL}(E)$ and explore the spectral properties of the natural action of g on the space of valuations, and more specifically on $\mathcal{V}_{n-k}$ where one can exploit the convolution of valuations.

Theorem 2. *Fix an integer* $0 \leqslant k \leqslant n$ *and denote by* $\phi = V(\Delta[k], -[n-k]) \in \mathcal{P}_{n-k}$ *and* $\psi = V(\Delta[n-k], -[k]) \in \mathcal{P}_k$ *where* Δ *is the standard* n*-dimensional simplex in* E*. For any linear map* $g \in \mathrm{GL}(E)$*, the dynamical degree*

$$\lambda_k(g) = \limsup_{p \to +\infty} ((g^p \cdot \phi) * \psi)^{1/p}$$

is well defined and is equal to

$$\lambda_k(g) = ||g_k|| = \frac{1}{|\det(g)|} \rho_1 \cdot \dots \cdot \rho_k,$$

where $||g_k||$ *denotes the norm of operator on* $(\mathcal{V}_{n-k}, ||\cdot||_{\mathcal{P}})$ *and where* $\rho_1 \geqslant \dots \geqslant \rho_k$ *are the absolute values of the eigenvalues of* g *put in decreasing order.*

Our proof gives an alternative approach to [Lin12, Theorem 6.2] and to [FW12, Corollary B]. Indeed, when $g \in \mathrm{SL}(E)$ has integer coefficients, then $\lambda_k(g)$ can be interpreted as the dynamical degree of the monomial map on $\mathbb{P}^n$ whose matrix is given by g. The following theorem is a first step in the understanding of the spectrum of the linear operators g_k.

Theorem 3. *Consider* $g \in \mathrm{GL}(E)$.

(1) For any k, there exists a non-zero invariant valuation $\phi \in \mathcal{V}_{n-k}$ such that $g_k(\phi) = \lambda_k(g)\phi$.

(2) Suppose that $\lambda_k(g)^2 > \lambda_{k+1}(g)\lambda_{k-1}(g)$, then for any $\lambda_k(g)$ invariant valuation $\phi, \psi \in \overline{\mathcal{P}_{n-k}} \subset \mathcal{V}_{n-k}$, we have that $\phi * \psi = 0$.

(3) Suppose that $\lambda_1(g)^2 > \lambda_2(g)$, then there exists a unique (up to scaling) valuation $\phi \in \overline{\mathcal{P}_{n-k}} \subset \mathcal{V}_{n-1}$ which is $\lambda_1(g)$ invariant and the valuation ϕ lies in an extremal ray of $\overline{\mathcal{P}_{n-1}} \subset \mathcal{V}_{n-1}$.

Observe that in case (2) of Theorem 3, we also expect the eigenvalue $\lambda_k(g)$ to be simple and the operator g_k to exhibit a spectral gap.

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Fujita vanishing theorems for q-ample divisors and applications on subvarieties with nef normal bundle

Chung Ching Lau

All schemes in this talk are over a field of characteristic 0. It is natural to ask how the positivity of a subvariety affects the positivity of the underlying cycle. For example, Hartshorne's conjecture A states that if Y is a smooth subvariety with ample normal bundle of a smooth projective variety X, then $n[Y]$ moves in a large algebraic family for n large. This was disproved by Fulton and Lazarsfeld [FL82]. On the other hand, Fulton and Lazarsfeld [FL83] showed that if Y is a subvariety with nef normal bundle in a smooth projective variety X and if Z is an arbitrary subvariety of X with $\dim Y + \dim Z \geq \dim X$, then $\deg_H(Y \cdot Z) \geq 0$. Here H is an ample divisor.

Weakening the Serre vanishing condition, a line bundle $\mathscr{L}$ is defined to be *q-ample* if given any coherent sheaf $\mathscr{F}$, there is an m_0 such that

$$H^i(X, \mathscr{F} \otimes \mathscr{L}^{\otimes m}) = 0$$

for $i > q$ and $m > m_0$. We define $\mathscr{L}$ to be *q-almost ample* if for any given ample line bundle $\mathscr{A}$, there is M such that $\mathscr{L}^{\otimes m} \otimes \mathscr{A}$ is q-ample for $m \geq M$. After the work of [AG62, Som78, DPS96] on q-ample divisors, Totaro established the basic, yet not elementary properties of q-ample divisors [Tot13]. Another approach to partial ampleness is outlined in [dFKL07, Kür06]. Perhaps the most important property of q-ampleness divisor is the following

Theorem A ([DPS96, Tot13]). *Let X be a projective variety of dimension n. Then a line bundle $\mathscr{L}$ is $(n-1)$-ample if and only if $\mathscr{L}^\vee$ is not pseudo-effective.*

After the extensive work of Hartshorne [Har70], where he studied positivity properties of higher codimension subvarieties, Ottem discovered what is probably the right notion of an ample subscheme [Ott12]. He defined a subscheme of Y of codimension r of a projective scheme to be *ample* if the exceptional divisor in the blowup of X along Y is $(r-1)$-ample. It is a natural definition that generalizes many properties of ample divisors [Ott12, Corollary 5.6], which were predicted in Hartshorne's work, while at the same time includes the zero locus of a global section of an ample vector bundle [Ott12, Proposition 4.5].

We now move on to study a weaker positivity condition of a subscheme. Given an lci subvariety $Y \subset X$ with nef normal bundle, we would like to understand its positivity properties in terms of intersection theory.

Let $Y \subset X$ be an arbitrary subscheme of codimension r and let E be the exceptional divisor in $\mathrm{Bl}_Y X$. We say that Y is *nef* if E is $(r-1)$-almost ample. This definition is inspired by Ottem's definition of an ample subscheme [Ott12]. If Y is lci in X, then Y is nef if and only if Y has nef normal bundle.

We discuss two generalized versions of Fujita vanishing theorem from [Lau16] for q-ample divisors, improving [Kür13, Theorem C].

Theorem B. *Let X be a projective scheme of dimension n. Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be q_1- and q_2-ample line bundles on X respectively and let $\mathscr{F}$ be a coherent sheaf on X. Then given any lower bound M_2 on the exponent of L_2, there is a lower bound on exponent of $\mathscr{L}_1$, M_1, such that*

$$H^i(X, \mathscr{F} \otimes \mathscr{L}_1^{\otimes m_1} \otimes \mathscr{L}_2^{\otimes m_2} \otimes \mathscr{P}) = 0$$

for $i > q_1 + q_2$, $m_1 \geq M_1$, $m_2 \geq M_2$ and any nef line bundle $\mathscr{P}$ on X.

Our second version of Fujita vanishing theorem focuses on the vanishing of the top cohomology.

Theorem C. *With the same assumption as above, except that we assume $\mathscr{L}_2$ to be only q_2-almost ample and that $q_1 + q_2 < n = \dim X$. Then*

$$H^n(X, \mathscr{F} \otimes \mathscr{L}_1^{\otimes m_1} \otimes \mathscr{L}_2^{\otimes m_2} \otimes \mathscr{P}) = 0$$

for $m_1 \geq M_1$*,* $m_2 \geq M_2$ *and any nef line bundle* $\mathscr{P}$ *on* X*.*

Using this theorem, we show in [Lau16] that

Theorem D. *Let* $\iota : Y \hookrightarrow X$ *be a nef subvariety of codimension* r *of a projective variety* X*. Then the natural map* $\iota^* : \mathrm{N}^1(X)_{\mathbf{R}} \to \mathrm{N}^1(Y)_{\mathbf{R}}$ *induces* $\iota^* : \overline{\mathrm{Eff}}^1(X) \to \overline{\mathrm{Eff}}^1(Y)$ *and* $\iota^* : \mathrm{Big}(X) \to \mathrm{Big}(Y)$*.*

Sketch of proof. By Theorem A, it is equivalent to prove that if $\mathscr{L}$ is a line bundle on X such that $\mathscr{L}|_Y$ is $(n-r-1)$-ample, then $\mathscr{L}$ is $(n-1)$-ample. Let E be the exceptional divisor on the blowup of X along Y, $\mathrm{Bl}_Y X$. By considering the following short exact sequence on $\mathrm{Bl}_Y X$,

$$0 \to \pi^*(\mathscr{L}^{\otimes m} \otimes \mathscr{O}_X(-l)) \otimes \mathscr{O}_{\mathrm{Bl}_Y X}((k-1)E) \to \pi^*(\mathscr{L}^{\otimes m} \otimes \mathscr{O}_X(-l)) \otimes \mathscr{O}_{\mathrm{Bl}_Y X} X(kE) \to \pi^*(\mathscr{L}^{\otimes m} \otimes \mathscr{O}_X(-l)) \otimes \mathscr{O}_E(kE) \to 0,$$

it suffices to prove that there is an M such that $H^{n-1}(E, \pi^*(\mathscr{L}^{\otimes m} \otimes \mathscr{O}_X(-l)) \otimes \mathscr{O}_E(kE)) = 0$ for $m \geq M$ and $k \geq 1$. But by replacing $\mathscr{L}$ by a sufficiently large multiple, we may assume $\mathscr{L} \otimes \mathscr{O}_E(-E)$ is $(n-r-1)$-ample. Now we may rewrite the sheaf $\pi^*(\mathscr{L}^{\otimes m} \otimes \mathscr{O}_X(-l)) \otimes \mathscr{O}_E(kE) = \pi^* \mathscr{O}_X(-l) \otimes (\mathscr{L} \otimes \mathscr{O}_E(-E))^{\otimes m} \otimes \mathscr{O}_E((k+m)E)$ and apply Theorem A to obtain the desired vanishing of cohomology groups. □

When Y is a curve with nef normal sheaf, this is a result of Demailly–Peternell–Schneider [DPS96, Theorem 4.1]. We also show that nefness and ampleness are transitive properties without any assumptions on smoothness, thus generalizes Ottem's result [Ott12, Proposition 6.4].

Theorem E. *Let* X *be a projective scheme of dimension* n*. If* Y *is an ample (resp. nef) subscheme of* X *and* Z *is an ample (resp. nef) subscheme of* Y*, then* Z *is ample (resp. nef) in* X*.*

We then study the cycle classes of nef subvarieties. We use this new notion of nef subvarieties to introduce the notion of the weakly movable cone, $\overline{\mathrm{WMov}}_d(X)$. We define it as the closure of the convex cone that is generated by pushforward of cycle classes of nef subvarieties of dimension d via proper surjective morphisms. We show that the weakly movable cone shares similar properties to that of the movable cone of d-cycles, $\overline{\mathrm{Mov}}_d(X)$.

Theorem F. *Let* X *be a projective variety of dimension* n*. For* $1 \leq d \leq n-1$*,*

(1) $\overline{\mathrm{Mov}}_d(X) \subseteq \mathrm{WMov}_d(X)$ *and* $\overline{\mathrm{Mov}}_1(X) = \overline{\mathrm{WMov}}_1(X)$*.*
(2) $\overline{\mathrm{Eff}}^1(X) \cdot \overline{\mathrm{WMov}}_d(X) \subseteq \overline{\mathrm{Eff}}_{d-1}(X)$*.*
(3) *Let* H *be a big Cartier divisor,* $\alpha \in \overline{\mathrm{WMov}}_d(X)$*. If* $H \cdot \alpha = 0$*, then* $\alpha = 0$*.*
(4) $\mathrm{Nef}^1(X) \cdot \overline{\mathrm{WMov}}_d(X) \subseteq \overline{\mathrm{WMov}}_{d-1}(X)$*.*

Analogous statements of 2, 3 and 4 hold for the movable cone [FL17, Lemma 3.10]. One can ask whether in general the two cones $\overline{\mathrm{Mov}}_d(X)$ and $\overline{\mathrm{WMov}}_d(X)$ are the same. This is true if and only if the cycle class of any nef subvariety lies in the movable cone. This question is closely related to the Hartshorne's conjecture A.

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Continuity and convexity in algebraic geometry

Victor Lozovanu

(joint work with Alex Küronya)

I will denote by X a smooth complex projective variety of dimension n, by $V \subseteq X$ a subvariety of dimension d, and by D a big $\mathbb{Q}$(or $\mathbb{R}$)-divisor on X. How do we determine the geometry of the class of D relative to the subvariety V?

1.1. **Intersection numbers.** The classical approach is through intersections

$$\mathrm{N}^1(X)_{\mathbb{R}} \ \ni \ D \ \rightsquigarrow \ (D^d \cdot V) \ \in \ \mathbb{R} \ .$$

This is a continuous, homogeneous and log-concave function of degree d. Furthermore, for ample D, we have asymptotic descriptions:

$$(D^d \cdot V) \ \approx \ \frac{\dim_{\mathbb{C}}(H^0(V, \mathcal{O}_V(mD)))}{m^d/d!} \ \approx \ \frac{\sharp\Big(D_1 \cap \ldots \cap D_d \cap V\Big)}{m^d/d!} \ \text{ for al } m \gg 0 \ ,$$

where each $D_i \in |mD|$ is general. The first equality is asymptotic Riemann–Roch. This approach is not as successful for non-ample divisors D, where $(D^d \cdot V)$ may be negative. For a big divisor D, the *stable base locus*

$$\mathbf{B}(D) \stackrel{\text{def}}{=} \mathrm{Bs}(|mD|)$$

for $m \gg 0$ may not be a numerical invariant. To correct this, Nakamaye introduced two other invariants that are invariants of the numerical class of D. First is the *augmented base locus*

$$\mathbf{B}_+(D) \stackrel{\text{def}}{=} \mathbf{B}(D - \epsilon A),$$

where A is any ample divisor and $0 < \epsilon \ll 1$. It is also called the non-ample locus, since for D big, D is ample if and only if $\mathbf{B}_+(D) = \emptyset$. Second is the *restricted base locus* (or non-nef locus)

$$\mathbf{B}_-(D) \stackrel{\text{def}}{=} \bigcap_{m \geq 1} \mathbf{B}(D + \frac{1}{m} A) \ .$$

They sit in a chain $\mathbf{B}_-(D) \subseteq \mathbf{B}(D) \subseteq \mathbf{B}_+(D)$.

1.2. **Volumes of divisors.** In [ELMNP09], the authors provide a more refined approach. They introduce the *restricted volume* of D with respect to V:

$$\mathrm{vol}_{X|V}(D) \stackrel{\text{def}}{=} \limsup_{m \to \infty} \frac{\dim\Big(\mathrm{Im}\big(H^0(X, \mathcal{O}_X(mD)) \to H^0(V, \mathcal{O}_V(mD))\big)\Big)}{m^d/d!} \ .$$

This is also a numerical invariant, homogeneous and log concave function of degree d. When D is ample, $\mathrm{vol}_{X|V}(D) = (D^d \cdot V)$. Furthermore, in the big case we have geometric interpretations of the restricted volume similar to the ample case. If $V \nsubseteq \mathbf{B}_+(D)$, then

$$\begin{aligned} \mathrm{vol}_{X|V}(D) &= \limsup_{m \to \infty} \frac{\dim_{\mathbb{C}}\Big(H^0\big(V, \mathcal{O}_V(mD) \otimes \mathscr{J}(X; ||mD||)|_V\big)\Big)}{m^d/d!} \\ &= \limsup_{m \to \infty} \frac{\sharp\Big(D_1 \cap \ldots \cap D_d \cap V \setminus \mathbf{B}(D)\Big)}{m^d/d!} \ . \end{aligned}$$

Here $\mathscr{J}(X, ||mD||)$ is the asymptotic multiplier ideal of mD, encoding the "bad" singularities of this class. The geometric description of the volume is quite powerful. The first equality allows us to compute the volume by knowing the shape and the "scheme structure" of $\mathbf{B}_+(mD) \cap V$ for all $m > 0$. The second does so by knowing information only generically, i.e. on $V \setminus \mathbf{B}_+(D)$.

Theorem 1 (Continuity from [ELMNP09])**.** *Suppose that either $V \nsubseteq \mathbf{B}_+(D)$ or V is an irreducible component of $\mathbf{B}_+(D)$, then for any sequence of big divisors $(D_n)_{n \in \mathbb{N}} \to D$, we have that* $\mathrm{vol}_{X|V}(D_n) \to \mathrm{vol}_{X|V}(D)$.

It is worth pointing out that there are examples where the restricted volume is not continuous exactly when both conditions are not satisfied. The proof of this theorem is very technical. Generalizing it is a central problem in Kähler geometry, or in characteristic p, or over number fields. One simple reason for its appeal is that it has many applications. For example, a corollary is a one line proof of a theorem of Nakamaye, which states that

$$\mathbf{B}_+(D) \;=\; \bigcup_{(D^d\cdot V)=0} V \;\text{ whenever } D \text{ is big and nef.}$$

This statement is a very nice example of the interesting bridge that seem to exists between how sections vanish and intersection numbers.

1.3. **Newton–Okounkov bodies (NObodies).** Based on an idea pioneered by Okounkov, [LM09, KK12] associate to a divisor a convex set. More precisely, suppose we are given an admissible flag $Y_\bullet$, where

$$Y_\bullet \;:\; X \;=\; Y_0 \;\supseteq\; Y_1 \;\supseteq\; \ldots \;\supseteq\; Y_n = \{x\}$$

is a full flag of (irreducible) subvarieties $Y_i \subseteq X$ with $\mathrm{codim}_X Y_i = i$ and the property that Y_i is smooth at the point x for all $0 \le i \le n$. Now for any $D' \ge 0$ effective $\mathbb{Q}$-divisor we can associate an integral vector

$$\nu_{Y_\bullet}(D') \;\stackrel{\text{def}}{=}\; (\nu_1, \nu_2, \ldots, \nu_n) \;\in\in \mathbb{Z}^n \;.$$

This vector is defined inductively. First, $\nu_1 \stackrel{\text{def}}{=} \mathrm{ord}_{Y_1}(D')$ and then the divisor D' naturally defines a non-trivial divisor $D'_1 = (D' - \nu_1 Y_1)|_{Y_1}$ on Y_1 and we proceed by induction from here. The *Newton–Okounkov body* of a big divisor D with respect to $Y_\bullet$ is defined to be

$$\mathbb{R}^n \;\supseteq\; \Delta_{Y_\bullet}(D) \;\stackrel{\text{def}}{=}\; \text{closed convex hull } \{\nu_{Y_\bullet}(D') | D' \equiv D \text{ effective } \mathbb{Q}-\text{divisor}\}.$$

Denote by $\pi : X' \to X$ the blow-up of a smooth point $x \in X$, with $E \simeq \mathbb{P}^{n-1}$ the exceptional divisor. Then one can consider an *infinitesimal flag* $\overline{Y}_\bullet$ on X' where $\overline{Y}_0 = X'$ and $\overline{Y}_i$ is a linear subspace of codimension $i-1$ in E. To it we associate what is now called *the infinitesimal NObody* $\Delta_{\overline{Y}_\bullet}(\pi^* D)$.

By [LM09], intersection numbers/restricted volumes appear naturally as euclidean volumes of slices of NObodies.

Theorem 2. *Let $V \subseteq X$ a subvariety as in Theorem 1 and denote by $\boldsymbol{0} \subseteq \mathbb{R}^d$ the origin. Then*

$$\mathrm{vol}_{X|V}(D) \;=\; d! \cdot \mathrm{vol}_{\mathbb{R}^d}\Big(\Delta_{Y_\bullet}(D) \cap \boldsymbol{0} \times \mathbb{R}^{n-d}\Big)$$

for any admissible flag $Y_\bullet$ with $Y_d = V$.

The importance of this theorem is two-fold. First, it gives a natural explanation of the continuity and convexity properties of intersection numbers/restricted volumes. Second, it gives practical meaning to these invariants. One asks: Can we study algebraic varieties by making use of the convex geometry of NObodies?

NObodies are numerical invariants. Thus, in [KL14, KL15a, KL17b] and separately [CHPW15] one gives very interesting interpretations of positivity properties of ampleness/nefness for a divisor in terms of NObodies. But before we proceed, define *the inverted standard simplex* of size $\xi > 0$ to be

$$\Delta_\xi^{-1} \stackrel{\text{def}}{=} \text{convex hull of } \{\mathbf{0}, \xi\mathbf{e}_1, \xi(\mathbf{e}_1+\mathbf{e}_2), \ldots, \xi(\mathbf{e}_1+\mathbf{e}_n)\} \subseteq \mathbb{R}^n ,$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is the standard basis vectors for $\mathbb{R}^n$. Also, Δ_ξ will stand for the standard simplex of length ξ.

Theorem 3. *Let D a big $\mathbb{R}$-divisor on X. Then the following are equivalent*

(1) D is ample.
(2) For every point $x \in X$ there exists an infinitesimal flag $Y_\bullet$ over x and $\xi > 0$ for which $\Delta_\xi^{-1} \subseteq \widetilde{\Delta}_{Y_\bullet}(D)$.
(3) For every point $x \in X$ there exists a flag $Y_\bullet$ at x and $\xi > 0$ for which $\Delta_\xi \subseteq \widetilde{\Delta}_{Y_\bullet}(D)$.

In the ample case, one can also recover the Seshadri constant $\varepsilon(D;x)$ from these convex sets.

Theorem 4. *(*[KL17b][Theorem C]*) Let D be an ample divisor and $x \in X$ a point. Then*

$$\varepsilon(D;x) \;=\; \max\{\xi | \Delta_\xi^{-1} \subseteq \Delta_{\overline{Y}_\bullet}(\pi^* D)\}$$

for any infinitesimal flag $\overline{Y}_\bullet$ at x.

In another direction, NObodies encode asymptotically how sections vanish along a flag. One is led to believe that they might play a role in finding singular divisor with certain singularities. It is worth to point out that many important results in algebraic geometry lie on this ability to find certain singular divisors. And the hope is that NObodies offer a visual understanding of the problem, facilitating the use of geometry of convex sets to understand positivity properties of intersections numbers. Some parts of this philosophy have first appeared in [KL15b]. For an exposition of this circle of ideas, the interested reader can consult [KL17a].

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Approximable algebras and a question of Chen's

Catriona Maclean

The Fujita approximation theorem [Fuj94] is an important result in algebraic geometry. It states that whilst the section ring associated to a big line bundle L on an algebraic variety X

$$R(L) \stackrel{\text{def}}{=} \oplus_m H^0(mL, X)$$

is typically not a finitely generated algebra, it can be approximated arbitrarily well by finitely generated algebras. More precisely, we have that

Theorem 1 (Fujita)**.** *Let X be an algebraic variety and let L be a big line bundle on X. For any $\epsilon > 0$ there exists a birational modification*

$$\pi : \hat{X} \to X$$

and a decomposition of $\mathbb{Q}$ divisors, $\pi^*(L) = A + E$ such that

- *A is ample and E is effective,*
- $\mathrm{vol}(A) \geq (1-\epsilon)\mathrm{vol}(L)$.

In [LM09], Lazarsfeld and Mustaţă used the Newton–Okounkov body associated to A to give a simple proof of Fujita approximation. The Newton–Okounkov body (NObody), constructed in [KK12] and [LM09], building on previous work of Okounkov [Oko03], is a convex body $\Delta_{Y_\bullet}(L, X)$ in $\mathbb{R}^d$ associated to the data of

- a d-dimensional variety X
- an admissible flag $Y_\bullet$ on X
- a big line bundle L on X.

This convex body encodes information on the asymptotic behaviour of the spaces of global sections $H^0(nL)$ for large values of L. Lazarsfeld and Mustaţă's simple proof of Fujita approximation is based on the equality of volumes of NObodies

$$\mathrm{vol}(L) = d!\mathrm{vol}(\Delta_{Y_\bullet}(L, X)) \tag{2}$$

where we recall that the volume of a big line bundle on a d-dimesional variety is defined by

$$\mathrm{vol}(L) = \lim_{n\to\infty} \frac{d!h^0(nL)}{n^d}.$$

One advantage of their approach to the Fujita theorem is that NObodies are not only defined for section algebras $R(L)$, but also for any graded sub-algebra of section algebras. Lazarsfeld and Mustaţă give combinatorical conditions (conditions 2.3-2.5 of [LM09]) under which equation 2 holds for a graded sub-algebra

$\mathbf{B} = \oplus_m B_m \subset R(L)$ and show that these conditions hold if the graded subalgebra $\mathbf{B}$ *contains an ample series.*[1]

Di Biagio and Pacenzia in [dBP16] subsequently used NObodies associated to restricted algebras to prove a Fujita approximation theorem for restricted linear series, ie. subalgebras of $\oplus_m H^0(mL|_V, V)$ obtained as the restriction of the complete algebra $\oplus_m H^0(mL, X)$, where $V \subset X$ is a subvariety.

In [Che10], Huayi Chen uses Lazarsfeld and Mustaţă's work on Fujita approximation to prove a Fujita-type approximation theorem in the arithmetic setting. In the course of this work he defines the notion of approximable graded algebras, which are exactly those algebras for which a Fujita-type approximation theorem hold.

Definition 2. An integral graded algebra $\mathbf{B} = \oplus_m B_m$ with $B_0 = k$ a field is approximable if and only if the following conditions are satisfied.

(1) all the graded pieces B_m are finite dimensional over k.
(2) for all sufficiently large m the space B_m is non-empty
(3) for any ϵ there exists an p_0 such that for all $p \geq p_0$ we have that

$$\liminf_{n\to\infty} \frac{\dim(\mathrm{Im}(S^n B_p \to B_{np}))}{\dim(B_{np})} > (1-\epsilon).$$

In his paper [Che10], Chen asks whether any graded approximable algebra is in fact a subalgebra of the algebra of sections of a big line bundle. A counter-example was given to this is [Mac17], where a counter example is constructed in which the graded approximable algebra is equal to the section ring of an *infinite* divisor[2]. This begs the question : is any approximable algebra a subalgebra of the section ring of an infinite divisior?

Theorem 3. *Let* $\mathbf{B} = \oplus_m B_m$ *be a graded approximable algebra whose first graded piece* B_0 *is an algebraically closed field of characteristic zero. There is then a projective variety* $X(\mathbf{B})$ *and an infinite divisor* $D(\mathbf{B}) = \sum_{i=1}^{\infty} a_i D_i$ *such that* $a_i \to 0$ *and there is a natural inclusion of graded algebras*

$$\mathbf{B} \hookrightarrow \oplus_m H^0(X(\mathbf{B}), mD(\mathbf{B})).$$

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[1] I.e., if there exists an ample divisor $A \leq L$ such that $\oplus_m H^0(\lfloor mA \rfloor) \subset B$

[2] Infinite in this context meaning an infinite sum of Weil divisors with real coefficients $\sum_i a_i D_i$.

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Positivity of the diagonal

John Christian Ottem
(joint work with Brian Lehmann)

In algebraic geometry, projective varieties are studied and classified in terms of the positivity of their tangent bundle. The paper [LO17] proposes a parallel viewpoint to this, by studying a projective variety X in terms of the positivity of the diagonal Δ (as a higher codimension cycle) on the self-product $X \times X$. This is motivated by the fact that the normal bundle of Δ is the tangent bundle T_X of X, and one expects there to be interplay between the ampleness properties of T_X (as a vector bundle) and the cycle-type positivity of Δ. This perspective is quite vivid already for curves:

Example 1 (Curves)**.** Let C be a curve and let $\Delta \subset C \times C$ be the diagonal. We have the following table:

g	type	K_X	Δ
0	$\mathbb{P}^1$	$K_X < 0$	Δ is ample (it is a (1,1)-divisor on $\mathbb{P}^1 \times \mathbb{P}^1$.)
1	elliptic	$K_X = 0$	Δ nef but not big (any effective divisor is nef on an abelian surface, and $\Delta^2 = 0$)
≥ 2	general type	$K_X > 0$	Δ negative, can be contracted by the subtraction map $C \times C \to C - C \subset J(C)$.

In general, given a variety X, we have a canonical cycle class $[\Delta] \in N_n(X \times X)$, and there are several ways in which this can be 'positive' (nef, big, movable, ample,..). Our motivating question is:

How do the positivity properties of the cycle $[\Delta]$ *reflect the geometric properties of* X*?*

Here geometric properties will refer to algebro–geometric properties (e.g., rational curves) as well as topological invariants (e.g., $\pi_1(X)$, $H^k(X, \mathbb{Z})$,..). Intuitively, one can expect varieties with Δ positive to be more similar to projective spaces.

1.1. **Varieties with big diagonal.** Most of the notions of positivity for divisors have analogues for cycles of higher codimension. We define the effective cone $\mathrm{Eff}_k(X)$ as the cone spanned by effective k-cycles, and let $\mathrm{Nef}^k(X)$ be its dual cone, the nef cone. For most of the varieties in this note, numerical and cohomological equivalence coincide, so we will for simplicity think of these cones in $H_{2k}(X, \mathbb{R})$ and $H^{2k}(X, \mathbb{R})$ respectively. We say that a class is *big* if it lies in the interior of $\mathrm{Eff}_k(X)$, and *nef* it lies in $\mathrm{Nef}^k(X)$. So the question becomes: For which varieties have is $[\Delta]$ big or nef cycle on $X \times X$? Here is a sample theorem:

Theorem 2. *The only surfaces with nef and big diagonal are the projective planes, and the fake projective planes.*

It is interesting to compare this result to Mori's theorem, which states that the only smooth variety with ample tangent bundle is $\mathbb{P}^n$. In light of this, the above statement is somewhat surprising, given that these varieties really fall on the opposite sides of the spectrum in the classification; fake projective planes have *anti-ample* tangent bundles and are consequently of general type. So by switching to the perspective of numerical positivity of Δ, we also include varieties with the same cohomological behavior as projective space.

Example 3. Let us verify that Δ_X is nef and big for a (fake) projective space. By the Künneth formula we have $H^{2k}(X \times X) = \bigoplus_{p+q=k} \mathbb{R}(\pi_1^* h^p \cdot \pi_2^* h^q)$ where h is an ample divisor on X. Then the diagonal can be written as a sum

$$\Delta = \sum_{p+q=n} c_{pq}(\pi_1^* h^p \cdot \pi_2^* h^q)$$

with $c_{pq} > 0$. This class is obviously nef and big.

Some examples of fake projective planes are: (i) Odd dimensional quadrics; (ii) the 100 fake projective planes of [CS10]; (iii) the Del Pezzo quintic threefolds V_5; and (iv) the Fano threefolds V_{22}. Our results imply that these are in fact all the examples in dimension at most 3: that is, every variety of dimension ≤ 3 with nef and big diagonal is either $\mathbb{P}^2$, $\mathbb{P}^3$ or one of the varieties (i)–(iv).

So having big diagonal should be a quite restrictive condition. How would one prove that a given variety does not have big diagonal? This is equivalent to finding a nef class β having intersection product 0 with Δ; finding explicit classes like this can be non-trivial, especially given the examples of Debarre–Ein–Lazarsfeld–Voisin [DELV11]. One way of constructing β is via products of divisors. The following lemma is elementary, but turns out to be remarkably effective:

Lemma 4. *Let X be an n-dimensional smooth projective variety admitting a nef class $D \in N^1(X, \mathbb{R})$ such that $D^n = 0$. Then Δ is not big.*

The proof is straightforward: If D is the above divisor, a suitable product of the form $\beta = \pi_1^* D^k \cdot \pi_2^* D \cdot H^{n-k-1}$ (where $k > 0$ and H is ample), is a nef class which dots Δ to zero, and so Δ is not big. (Using a similar argument, one can show that the same statement holds also when $D \in H^{1,1}(X, \mathbb{R})$ is a (possibly non-algebraic) nef cohomology class.)

This lemma already puts strong restrictions on the possible varieties with big diagonal. For instance, varieties with big diagonal admit no maps to lower dimensional varieties. This implies in particular that the only smooth uniruled surface with big diagonal is $\mathbb{P}^2$. The main geometric implication of the diagonal being big is the following:

Theorem 5. *Let X be a smooth projective variety with big diagonal. Then $h^{k,0}(X) = 0$ for all $k > 0$.*

The proof is inspired by a proof of Voisin using the Hodge–Riemann relations to bound the effective cone. To give some details, we fix a Kahler form ω on $X \times X$ and assume that we have a non-zero closed $(k,0)$-form α on X. Now, the class

$$\beta = (-1)^{\frac{k(k-1)}{2}} i^k \, (\pi_1^*\alpha - \pi_2^*\alpha) \cup (\pi_1^*\bar{\alpha} - \pi_2^*\bar{\alpha}) \cup \omega^{n-k}$$

is represented by a non-zero (n,n)-form on $X \times X$, which by construction restricts to 0 on the diagonal. Moreover, since $(k,0)$-classes are automatically primitive, the Hodge–Riemann relations imply that the class of β is nef, which contradicts the bigness of Δ.

1.2. **Nefness of Δ.** Also nefness of Δ is imposes strong restrictions on the geometry of X. The primary examples here are the varieties with nef tangent bundle (which automatically have nef diagonals). Such varieties are expected to have very special properties (e.g., Campana–Peternell conjecture that they should be homogeneous space-fibrations over abelian varieties). As for bigness, to prove that Δ is not nef, one has to produce subvarieties of $X \times X$ that intersect Δ non-transversely. Some ways of producing such subvarieties include: (i) products of divisors; (ii) subvarieties linked to the diagonal via such products; (iii) graphs of (birational) automorphisms; and (iv) Δ itself. The latter condition implies in particular that a variety with nef diagonal must have non-negative Euler characteristic. One particularly useful criterion is the following:

Proposition 6. *Let X be a smooth variety. If Δ_X is nef, then every pseudoeffective class on X is nef.*

For instance, if S is a smooth surface with Δ then S is minimal (since (-1)-curves are not nef). By combining this result with Theorem 5, we obtain:

Corollary 7. *The only smooth projective surfaces with big and nef diagonal are the projective plane and fake projective planes.*

Indeed, if Δ is big and nef, then $h^{1,0}(X) = h^{2,0}(X) = 0$, and $\rho = h^{1,1}(X)$. If $\rho > 1$, X either has a non-nef pseudoeffective divisor (which contradicts Proposition 6), or there is a nef divisor with self-intersection 0 (contradicting Lemma 4). Hence $\rho = 1$, and X is a (fake) projective plane.

A similar, but more involved analysis shows that also a threefold with nef and big diagonal has to be a fake projective space. Here one needs the following additional ingredients: (i) $\chi(X) \geq 0$; (ii) $\chi(\emptyset_X) = \frac{1}{24}c_1c_2 = 1$; (iii) X is minimal; (iv) the Miyaoka inequality; and (v) The classification of Fano 3-folds. This indicates that a classification of varieties with nef and big diagonal in higher dimension will be more difficult; we pose the question whether a fourfold with nef and big diagonal must be a fake projective space.

1.3. **Examples.** The paper [LO17] studies how varieties with big or nef diagonal fit into the classification of varieties of low dimension. We conclude with a few examples illustrating these results:

Example 8.

(1) Quadrics have big diagonal if and only if the dimension is even.
(2) A Grassmannian has a big diagonal if and only if it is a projective space.
(3) Let X be a toric variety. Then Δ_X is nef if and only if X has nef tangent bundle if and only if X is a product of projective spaces. There are also toric threefolds other than $\mathbb{P}^3$ with big diagonal.

Example 9 (Surfaces)**.** Let us say a few words about how varieties with nef diagonal fit into the classification of surfaces. First of all, if S is a surface with nef diagonal, it must be minimal (otherwise there is a (-1)-curve, contradicting Lemma 6). We next consider the surfaces according to their Kodaira dimension κ. If $\kappa = -\infty$, S must either be $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or the ruled surface over an elliptic curve defined by a semistable rank 2 bundle. Each of these have nef tangent bundles and thus nef diagonal.

For $\kappa = 0$, abelian surfaces and hyperelliptic surfaces both have nef tangent bundles, and thus nef diagonal. Any Enriques surface admits an involution $i : S \to S$ exchanging the two sheets of a double cover $S \to T$. Intersecting Δ with the graph of the involution gives a negative number, so Δ is not nef. For K3 surfaces, we prove, using results of Bayer–Macrì [BM14], that the diagonal is not nef. In fact, we show that the diagonal is negative in a very strong sense: any effective cycle on $S \times S$ with class proporional to Δ, must itself be a multiple of Δ.

For $\kappa = 1$, we can consider the canonical map $\pi : S \to C$. By intersecting the diagonal with a cycle in from $\pi^{-1}(\Delta_C)$, one sees right away that the base C of the canonical map must have genus either 0 or 1. In the latter case, Δ has negative intersection with $\sigma \times \sigma$, where σ is a section of π. Furthermore, in [LO17] we give an example showing that it is in fact possible for Δ to be nef if π admits no sections.

Example 10 (Hypersurfaces)**.** Let X be a smooth hypersurface of degree ≥ 3 and dimension ≥ 2. An Euler characteristic computation shows that the diagonal of X is not nef. Bigness of the diagonal turns out to be more subtle. We show:

Theorem 11. *For a smooth Fano hypersurface of degree ≥ 3 and dimension ≤ 5, the diagonal is not big.*

The strategy here is to use the rational map $\mathbb{P}^{n+1} \times \mathbb{P}^{n+1} \dashrightarrow Gr(2, n+2)$, which is resolved by blowing up Δ. Using this map, we can pull back Schubert cycles from the Grassmannian, and intersect Δ with these to argue that the diagonal is not big.

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Newton–Okounkov bodies and Toric Degenerations of Mori dream spaces via Tropical compactifications

Stefano Urbinati
(joint work with Elisa Postinghel)

In this work we make a connection between the theory of Newton–Okounkov bodies (NObodies) and tropical geometry, sharing the aim of the recent preprints [KU16, KM16]. The results of the present paper yield a different point of view on how tropical geometry can be extremely helpful in birational geometry. Our principal aim is to study Mori dream spaces (MDS) via tropicalization. Via this connection we can describe a simple and computable way of reconstructing the movable cone of such varieties. The results obtained complete the picture introduced in [LS14, PSU15], where Minkowski bases for NObodies were given respectively for surfaces and for toric varieties, with respect to certain admissible flags. Let us first recall the main definitions. ([HK00, KSZ91])

1.1. **NObodies and toric degenerations.** Let X be a smooth complex projective variety of dimension n over an algebraically closed field k and let D be a Cartier divisor on X. Okounkov's construction associates to D a convex body

$$\Delta_{Y_\bullet}(D) \ \subseteq \ \mathbb{R}^n,$$

which we call the *NObody*. It depends on the choice of an *admissible flag* $Y_\bullet$

$$Y_0 = X \supset Y_0 \supset \ldots \supset Y_{n-1} \supset Y_n,$$

where Y_i is a subvariety of codimension i in X which is smooth at the point Y_n. Using the flag, one defines a rank n *valuation* $\nu = \nu_{Y_\bullet}$ which, in turn, defines a *graded semigroup* $\Gamma_{Y_\bullet} \subseteq \mathbb{N} \times \mathbb{N}^n$. The convex body $\Delta_{Y_\bullet}(D)$ is the intersection of $\{1\} \times \mathbb{R}^n$ with the closure of the convex hull of $\Gamma_{Y_\bullet}$ in $\mathbb{R} \times \mathbb{R}^n$.

One can also define the notion of *global NObody* of X which is the closed convex cone in $\mathbb{R}^n \times N^1(X)_\mathbb{R}$ whose fibers over any big divisor D on X coincides with the NObody $\Delta_{Y_\bullet}(D)$ of such divisor. We refer to [KK12, LM09] for details on this construction. NObodies are quite difficult to compute in general. They often are not polyhedral and, when polyhedral, they may be not rational. Even when the body is polyhedral, the semigroup $\Gamma_{Y_\bullet}$ need not be finitely generated.

For toric varieties NObodies turn out to be nice. In fact [LM09] prove that if X is a smooth *toric variety*, D is a T-invariant ample divisor on X and the Y_i's are T-invariant subvarieties, then $\Delta_{Y_\bullet}(D)$ is the lattice polytope associated with D in the usual toric construction. Moreover the global NObodies are rational polyhedral.

It is natural to investigate whether this construction behaves well for special classes of varieties. A consequence of the work contained in [BCHM10, HK00] is that divisors on MDS often have toric-like behavior, so it makes sense to pose the following question.

Question 1 ([LM09, Problem 7.1])**.** Let X be a smooth MDS. Does there exist an admissible flag with respect to which the global NObody of X is rational polyhedral?

In [And13], Anderson extended the connection between NObodies and toric varieties, by introducing a geometric criterion for $\Delta_{Y_\bullet}(A)$ to be a lattice polytope for A ample and, in this situation, by constructing an embedded *toric degeneration* of (X, A).

Theorem 2 ([And13, Theorem 5.8])**.** *Let A to be an ample divisor on X and assume the value semigroup associated to A with respect to the valuation induced by a complete flag is finitely generated. Then X admits a flat degeneration to a toric variety whose normalization is $X_{\Delta_{Y_\bullet}(A)}$.*

Question 3. For which varieties is it possible to find a flag such that the value semigroup of an ample divisor is finitely generated?

Notice that the latter is a very strong condition, and it is much stronger than the finite generation of the divisorial ring. In this work we give an affirmative answer to both Questions 1 and 3 for Mori dream spaces.

1.2. **Mori Dream Spaces and tropicalization.** MDS are special projective varieties for which the Minimal Model Program (MMP) is particularly simple, since they only admit a finte number of *small* $\mathbb{Q}$*-factorial modifications* (SQM's). The key property of these varieties is that they always admit a particularly nice embedding into toric varieties, see [HK00]. Given a MDS X and such an embedding $X \subseteq Z$ into a toric variety, let $T \subseteq Z$ be the maximal torus of the given toric variety. The main result of this paper can be summarized as follows:

Theorem 4. *Let $X \subset Z$ be a MDS with the embedding of* [HK00] *into a toric variety Z. Then the tropicalization of the variety restricted to T,* $\mathrm{Trop}(X|_T)$*, induces a model $h : \overline{X} \to X$, embedded in a toric variety $j : \overline{X} \subset \overline{Z}$, that dominates all the SQM's induced by the MMP. Moreover the fan of $\overline{Z}$ is supported on* $\mathrm{Trop}(X|_T)$*.*

The main ideas are inspired by the work of Tevelev [Tev07]. Note that in this way we recover a subscheme of the *Chow quotient* defined in [KSZ91]. Via this construction we are able to prove several consequences. First of all, the map h is given as an embedded map into toric varieties and this allows us to obtain a Minkowski basis for the NObodies on X. In particular we can reconstruct the movable cone of X. The main result is the following.

Theorem 5. *In the notation of Theorem 4, certain admissible flags on $\overline{Z}$ induce admissible flags on X, such that if $\{D_i^Z\}_{i\in I}$ is a set of generators of the nef cone of $\overline{Z}$, then $\{D_i := h_* j^* D_i^Z\}_{i\in I}$ is a Minkowski basis for X with respect to induced flag.*

Another important result of this paper is that the NObodies of the Minkowski basis elements are rational and polyhedral. Even more, the value semigroup is finitely generated. This implies the following result:

Theorem 6. *If X is a smooth MDS, the global NObody of X with respect to the flags obtained in Theorem 5 is a rational polyhedron.*

Moreover if A is an ample line bundle on X, (X, A) admits a flat embedded degeneration to a not necessarily normal toric variety whose normalisation is the toric variety defined the NObody of A.

The first statement of Theorem 6 answers affirmatively a question posed in [LM09]. The second statement is based on work of [And13].

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Polar transform and local positivity for curves

Jian Xiao

(joint work with Nicholas McCleerey)

Let X be a smooth projective variety of dimension n. Let L be a nef line bundle on X. One of the most important invariants measuring the local positivity of L at x is the local Seshadri constant, $s_x(L)$, introduced by [Dem92]. Another important local positivity invariant for divisors is the local Nakayama constant, $n_x(L)$, introduced by (see [Leh13, Definition 5.1], and also [Nak04] for related objects). It is clear that these invariants can extend to $(1,1)$ classes. By the general theory developed in [LX16], given a proper closed convex cone $\mathscr{C} \subset V$ of full dimension in a real vector space V, let $\mathrm{HConc}_1(\mathscr{C})$ be the space of real valued functions defined over $\mathscr{C}$ that are upper semicontinuous, homogeneous of degree one, strictly positive in the interior of $\mathscr{C}$, and 1-concave. Then one can study the polar transform $\mathcal{H} : \mathrm{HConc}_1(\mathscr{C}) \to \mathrm{HConc}_1(\mathscr{C}^*)$:

$$\mathcal{H}f : \mathscr{C}^* \to \mathbb{R},\ w^* \mapsto \inf_{v \in \mathscr{C}^\circ} \frac{w^* \cdot v}{f(v)},$$

where $f \in \mathrm{HConc}_1(\mathscr{C})$, and $\mathscr{C}^* \subset V^*$ is the dual of $\mathscr{C}$. This is a Legendre–Fenchel type transform with a "coupling" function given by the logarithm. It is clear that $s_x(\cdot) \in \mathrm{HConc}_1(\mathrm{Nef}^1(X))$ and $n_x(\cdot) \in \mathrm{HConc}_1(\overline{\mathrm{Eff}}^1(X))$.

We apply the polar transform to the following two geometric cases:

(1) $\mathscr{C} = \mathrm{Nef}^1(X),\ f = s_x,$

(2) $\mathscr{C} = \overline{\mathrm{Eff}}^1(X),\ f = n_x.$

Then we get two functions on the dual cones: $N_x(\cdot) \in \mathrm{HConc}_1(\overline{\mathrm{Eff}}_1(X))$ and $S_x(\cdot) \in \mathrm{HConc}_1(\overline{\mathrm{Mov}}_1(X))$. We show that these functions measure the local positivity for $(n-1, n-1)$ classes. In fact, we show that the polar of s_x behaves analogously to the function n_x, and similarly for the polar of n_x. Furthermore, this enables us to obtain a Seshadri type ampleness criterion for movable curves, that is, $\alpha \in \overline{\mathrm{Mov}}_1(X)^\circ$ if and only if there is a uniform constant $c > 0$ such that $S_x(\alpha) \geq c$ for every $x \in X$. This also enables us to give a characterization of the divisorial components of the non-Kähler locus of a big $(1,1)$ class. More precisely, assume that α is on the boundary of $\overline{\mathrm{Mov}}_1(X)$ and $S_p(\alpha) > 0$ at some point p, then $S_x(\alpha) = 0$ if and only if x lies on the the divisorial components of the non-Kähler locus of a big $(1,1)$ class determined by α. In particular, the vanishing locus of the function $x \mapsto S_x(\alpha)$ is a subvariety.

Independently, starting with a more geometric viewpoint which is similar to the original definition of $s_x(\cdot)$, M. Fulger [Ful17] has also studied Seshadri constants for movable curve classes. It turns out that this geometric definition for Seshadri constants is equivalent to the above $S_x(\cdot)$ given by polar transforms.

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Reporter: Aurel-Mihai Fulger

Participants

Prof. Dr. Ana-Maria Castravet
Department of Mathematics
Northeastern University
360 Huntington Avenue
Boston MA 02115
UNITED STATES

Prof. Dr. Nguyen-Bac Dang
Centre de Mathématiques Laurent
Schwartz
École Polytechnique
91128 Palaiseau Cedex
FRANCE

Dr. A. Mihai Fulger
Department of Mathematics
University of Connecticut
341 Mansfield Road
Storrs, CT 06269
UNITED STATES

Dr. Frank Gounelas
Institut für Mathematik
Humboldt-Universität Berlin
Unter den Linden 6
10117 Berlin
GERMANY

Prof. Dr. Alex Küronya
Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayer-Strasse 6-8
60325 Frankfurt am Main
GERMANY

Dr. Robert Laterveer
Institut de Recherche Mathématique
Avancee
Université de Strasbourg et CNRS
7, rue Rene Descartes
67084 Strasbourg Cedex
FRANCE

Dr. Chung-Ching Lau
Department of Mathematics
University of Utah
155 South 1400 East
Salt Lake City, UT 84112-0090
UNITED STATES

Prof. Dr. Brian Lehmann
Department of Mathematics
Boston College
Chestnut Hill, MA 02467-3806
UNITED STATES

Prof. Dr. Angelo Felice Lopez
Dipartimento di Matematica
Universita degli Studi Roma Tre
Largo S. L. Murialdo, 1
00146 Roma
ITALY

Dr. Victor Lozovanu
Institut für Mathematik
Universität Hannover
Postfach 6009
30060 Hannover
GERMANY

Prof. Dr. Diane Maclagan
Mathematics Institute
University of Warwick
Gibbet Hill Road
Coventry CV4 7AL
UNITED KINGDOM

Dr. Catriona Maclean
Laboratoire de Mathématiques
Université de Grenoble I
Institut Fourier
B.P. 74
38402 Saint-Martin-d'Hères Cedex
FRANCE

Matthias Nickel
Fachbereich Mathematik/Informatik
J.W.Goethe-Universität
60325 Frankfurt am Main
GERMANY

Dr. John Christian Ottem
Department of Mathematics
University of Oslo
P. O. Box 1053 - Blindern
0316 Oslo
NORWAY

Dr. Stefano Urbinati
Dipartimento di Matematica
Politecnico di Milano
Via Bonardi 9
20133 Milano
ITALY

Prof. Dr. Jian Xiao
Department of Mathematics
Northwestern University
Lunt Hall
2033 Sheridan Road
Evanston, IL 60208-2730
UNITED STATES