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Classical and Quantum Mechanical Models of Many-Particle Systems

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Abstract. This workshop was dedicated to the presentation of recent results in the field of the mathematical study of kinetic theory and its natural extensions (statistical physics and fluid mechanics). The main models are the Vlasov(-Poisson) equation and the Boltzmann equation, which are obtained as limits of many-body equations (Newton's equations in the classical case and Schrödinger's equation in the quantum case) thanks to the mean-field and Boltzmann-Grad scalings. Numerical aspects and applications to mechanics, physics, engineering and biology were also discussed.

Mathematics Subject Classification (2010): Primary: 76P05 Rarefied gas flows, Boltzmann equation, 35Q40 PDEs in connection with quantum mechanics, 35Q92 PDEs in connection with biology and other natural sciences, 82C40 Kinetic theory of gases, 82C22 Interacting particle systems; Secondary: 35K65 Degenerate parabolic equations, 82C70 Transport processes, 45K05 Integro-partial differential equations, 82D10 Plasmas, 81S30 Phase space methods including Wigner distributions, 81S22 Open systems, reduced dynamics, master equations, decoherence.

Introduction by the Organisers

The workshop *Classical and Quantum Mechanical models of Many-Particle Systems*, organized by Anton Arnold (Vienna), Eric Carlen (New Brunswick) and Laurent Desvillettes (Paris) was well attended by 52 participants with broad geographic representation, and a significant number of women (11) and young researchers. We remark that most of the participation slots were actually filled in the first round of invitations. So there were very few replacements or cancellations (due to baby care duties, e.g.). One of the participating post-docs was

supported be the NSF-grant for "junior Oberwolfach fellows", three PhD-students (or young post-docs) were funded as "Oberwolfach Leibniz graduate students", and one participant (Franca Hoffmann, Caltech) benefited from a Simons Visiting Professorship. 26 participants gave a lecture, mostly of 45 minutes.

The organizers took an active step towards gender balance, which was reflected both in the invitation list and the scheduling of the talks (9 of which were given by women).

An important innovation in this edition of the meeting was the organization of a session (which was held after dinner) dedicated to the exposition of open problems, in which four senior scientists of the field (Yann Brenier, Pierre Degond, Tai-Ping Liu, Lorenzo Pareschi) presented their own ideas about the current scientific challenges met by the community in the field, and possible new directions of research. This session seems to have been greatly appreciated by the audience (almost everybody attended the session), and especially by the younger researchers.

A significant amount of the talks were dedicated to the perturbative theory of the Boltzmann equation (and closely related models). This subfield expanded greatly in the recent years. Very interesting (and important for the applications to realistic physics) new subtopics emerged. One can quote the talk by Chanwoo Kim, in which boundary conditions are precisely studied, and the results presented by Marc Briant, in which mixtures are considered, and the unexpected effect of the different masses of gas molecules is explained. The presentations by Lingbing He and Isabelle Tristani, dedicated to the perturbative theory of the non cutoff Boltzmann and Landau equations, are also a testimony of the vitality of this subfield.

The links between kinetic equations and macroscopic equations (Euler or Navier-Stokes equations, diffusive equations) is one of the oldest subjects in the field, and still attracts a lot of attention. The talk by Alexander Bobylev proposes an approach in which one goes beyond the first order in the Chapman-Enskog approximation without encoutering the problems coming out of the study of the Burnett equation. Christian Schmeiser talked about the limits leading to fractional diffusive equations (starting from kinetic equation). Closely connected to this theme were the presentations of Shih-Hsien Yu, who explained the latest developments of the works that he began some years ago on Green's functions, and the talks by Jean Dolbeault and Mustapha Mokhtar-Kharroubi, in which the results on the topic of hypocoercivity were described. The connection between the Boltzmann equation and fluid dynamics was further illustrated by Tai-Ping Liu in the open problem session, concerning e.g. regularization and boundary effects.

Several talks were concerned with semiclassical limits and mean field limits (i.e. the passage from N particle systems to kinetic models). The talks of Chiara Saffirio and François Golse dealt with the derivation of the (classical) Vlasov equation in the joint limit $N \to \infty$ and $\hbar \to 0$. While C. Saffirio's approach is based on Wigner functions, F. Golse introduced a novel (Wasserstein-like) distance between

(classical) probability densities and (quantum) density matrices. Alessia Nota presented the derivation of a linear Boltzmann and Landau equation from the Newtonian dynamics of a Lorentz Gas, i.e. a classical particle moving in the field of fixed, but randomly distributed scatterers. The talk of Pierre-Emmanuel Jabin considered the large system limit of stochastic particles (with interaction satisfying the Biot-Savart law, e.g.) towards a macroscopic limit, like the vortex dynamics for the 2d incompressible Navier-Stokes equations.

A group of talks was concerned with quantum particle systems, and in particular with bosonic systems. Miguel Escobedo and Ricardo Alonso discussed the formation and dynamics of Bose-Einstein condensates via global-in-time solutions to isotropic quantum Boltzmann equations. The talk of Anne Nouri dealt with existence, uniqueness and stability of solutions to an inhomogeneous quantum Boltzmann equation for (very cold) particles satisfying a fractional quantum statistics. Benjamin Schlein presented his recent results on the spectrum of Bose gases in the $N \to \infty$ limit (ground state and Bose-Einstein condensation).

Applications to biology (mainly to the study of collective movement of animals or microorganisms) of the methods of kinetic theory (especially entropy methods), fluid mechanics and statistical physics were discussed in the talks by Jose Carrillo, Franca Hoffmann, Esther Daus and Sara Merino-Aceituno. The presentation of Pierre Degond in the session on open problems was also devoted to a presentation of interesting issues in this topic.

In her talk, Claudia Negulescu presented a derivation of a reduced kinetic model for a strongly confined tokamak plasma in an adiabatic limit in order to facilitate numerical simulations. In the open problem session, Lorenzo Pareschi first gave an introduction to *uncertainty quantification*, with emphasis on examples from kinetic theory. Apart of modelling issues he discussed numerical difficulties and solution approaches (based on generalized Polynomial Chaos expansions, e.g.).

Connected methods for PDEs (homogeneisation, coagulation-fragmentation models, Lagrangian description) were finally presented in the talks by Julien Fischer, Yann Brenier and Maxime Breden. A presentation of new possible approaches of optimal transport methods for quantum particles was also provided by Yann Brenier in the framework of the session on open problems.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Franca Hoffmann in the "Simons Visiting Professors" program at the MFO.

Workshop: Classical and Quantum Mechanical Models of Many-Particle Systems

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Abstracts

Swarming models with local alignment effects: phase transitions & hydrodynamics

José A. Carrillo

We discussed a collective behavior model in which individuals try to imitate each others' velocity and have a preferred asymptotic speed. It is a variant of the well-known Cucker-Smale model, in which the alignment term is localized, refered below as LCSM. We showed that a phase change phenomenon takes place as diffusion decreases, bringing the system from a "disordered" to an "ordered" state. This effect is related to recently noticed phenomena for the diffusive Vicsek model. We analysed the expansion of the large friction limit around the limiting Vicsek model on the sphere leading to the so-called Self-Organized Hydrodynamics (SOH) introduced in [5]. This talk was based on the papers [1, 2, 3].

In more details, considering that f is the distribution in both space x and velocity v at time t, the model features a Cucker-Smale type term which aligns the velocity of individuals nearby in space, a term adding noise in the velocity, and a friction term which relaxes velocities back to norm one:

$$\partial_t f + v\nabla_x f = \nabla_v \cdot \left(\alpha(|v|^2 - 1)vf + (v - u_f)f + D\nabla_v f\right) . \tag{1}$$

where

$$u_f(t,x) = \frac{\int v f(t,x,v)\,dv}{\int f(t,x,v)\,dv}.$$

Here α and D are respectively the self-propulsion force and noise intensities. The first part of the talk discussed the transition by noise on the spatially homogeneous case, where the model reduces to

$$\partial_t f = \nabla_v \cdot \left(\alpha(|v|^2 - 1)vf + (v - u_f)f + D\nabla_v f\right) . \tag{2}$$

where

$$u_f(t) = \frac{\int v f(t,v)\,dv}{\int f(t,v)\,dv}, \tag{3}$$

and where $f = f(t,v)$ is the velocity distribution at time t. Stationary solutions $f(v) > 0$ to (2) for $u_f = \bar{u}$ are of the form

$$f_{\bar{u}}(v) = \frac{1}{Z}\exp\left(-\tfrac{1}{D}\left[\alpha\tfrac{|v|^4}{4} + (1-\alpha)\tfrac{|v|^2}{2} - \bar{u}\cdot v\right]\right), \tag{4}$$

with Z the normalization factor such that f has unit mass. Therefore, the set of stationary solutions of (2) can be parametrized by the set of mean velocities $\bar{u} \in \mathbb{R}^N$ such that

$$\mathcal{H}(\bar{u}, D) := \int_{\mathbb{R}^N} (v - \bar{u}) f_{\bar{u}}(v) dv = 0.$$

Notice that $\mathcal{H}(\bar{u}, D) = 0$ is equivalent to $\bar{u} = u_{f_{\bar{u}}}$ given by (3), and that $\bar{u} = 0$ is always a solution corresponding to radially symmetric stationary states.

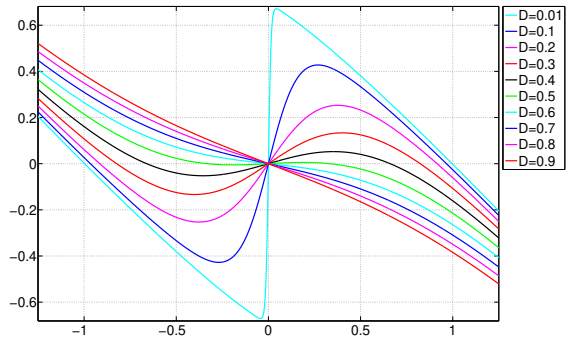

FIGURE 1. Plot of the one-dimensional $H(u, D)$ against u for $\alpha = 2$ and varying values of D.

By choosing the axis, we may assume without loss of generality that $\bar{u}$ points in the direction of the first axis or first vector e_1 of the canonical basis, and then let us denote the magnitude of $\bar{u}$ by $u \geq 0$. The full set of stationary solutions is obtained by composing $f_{\bar{u}}$ with any rotation in $\mathbb{R}^N$, and thus yields an $(N-1)$-dimensional family of stationary solutions for each $\bar{u} = ue_1$ satisfying $\mathcal{H}(\bar{u}, D) = 0$. Noticing that all components of $\mathcal{H}$ except for the first one vanish due to $f_{\bar{u}}(v)$, we can restrict our attention to the first component of $\mathcal{H}$. The real valued function whose roots have to be analyzed is the first component of $\mathcal{H}$, given by

$$H(u, D) = \int_{\mathbb{R}^N} (v_1 - u) f_{\bar{u}}(v) dv \,.$$

In figure 1, we plot $H(u, D)$ in one dimension as a function of u for varying values of D. It is clear from the figure that for small values of D, $H(u, D)$ has three roots, the zero root and two roots with identical speed; while for large values of D the only root is $u = 0$, and this can be deduced from the sign of $\frac{\partial H}{\partial u}(0, D)$. The main theorem about phase transitions showed is:

Theorem 1. (Phase Transition Driven by Noise [1]**)** *The nonlinear Fokker-Planck equation* (2)*, corresponding to the spatially homogeneous LCSM with noise, exhibits a phase transition in the following sense:*

(1) *For small enough D there is a function $u = u(D)$ with $\lim_{D\to 0} u(D) = 1$, such that $f_{\bar{u}}$ given by* (4) *with $\bar{u} = (u(D), 0, \ldots, 0)$ is a stationary solution of* (2)*.*
(2) *For large enough D the only stationary solution of* (2) *is the symmetric distribution given by* (4) *with $\bar{u} = 0$.*

We then elaborated in how to obtain a hydrodynamic description around the von Mises-Fisher equilibria obtained in the large friction limit $\alpha \to \infty$ in (1) when the noise parameter is below the critical noise on the sphere studied in [4]. The

precise asymptotic limit that we studied rewrites (1) as

$$\partial_t f^\varepsilon + \mathrm{div}_x(f^\varepsilon v) + \frac{1}{\varepsilon^2}\mathrm{div}_v\{f^\varepsilon(1-|v|^2)v\} = \frac{1}{\epsilon}Q(f)\,, \tag{5}$$

for all $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^{2N}$. The behavior of the family $(f^\varepsilon)_{\varepsilon>0}$, as the parameter ε becomes small, follows by analyzing the formal expansion $f^\varepsilon = f + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + ...$ Plugging the above Ansatz into (5), leads to the constraints

$$\mathrm{div}_v\{f(1-|v|^2)v\} = 0 \tag{6}$$

$$\mathrm{div}_v\{f^{(1)}(1-|v|^2)v\} = \mathrm{div}_v\{f(v-u[f]) + D\nabla_v f\} \tag{7}$$

and to the time evolution equations

$$\partial_t f + \mathrm{div}_x(fv) + \mathrm{div}_v\{f^{(2)}(1-|v|^2)v\} = \mathcal{L}_f(f^{(1)}) \tag{8}$$

with

$$\mathcal{L}_f(f^{(1)}) := \mathrm{div}_v\{f^{(1)}(v-u[f]) + D\nabla_v f^{(1)}\} - \mathrm{div}_v\left\{ f\,\frac{\int_{\mathbb{R}^N} f^{(1)}(v'-u[f])\,\mathrm{d}v'}{\int_{\mathbb{R}^N} f\,\mathrm{d}v'}\right\}$$

cutting the development at second order.

The main advantage for considering (5) is that the resolution of (5) for small ε will provide a solution supported near $\mathbb{R}^N \times \mathbb{S}^{N-1}$, which fits much better the modelling of the behavior of living organism systems, than a hard constraint on the sphere in velocity. But the price to pay is to deal with two Lagrange multipliers, appearing in (8), which have to be eliminated, thanks to the constraints (6) and (7). The first constraint was analyzed in detail in [2]. It exactly says that f is a measure supported in $\mathbb{R}^d \times (\{0\} \cup \mathbb{S}^{N-1})$. We denote by $\mathcal{M}_b^+(\mathbb{R}^N)$ the set of non negative bounded Radon measure on $\mathbb{R}^N$.

Proposition 1. *Assume that* $(1+|v|^2)F \in \mathcal{M}_b^+(\mathbb{R}^N)$. *Then* F *solves* $\mathrm{div}_v\{F(1-|v|^2)v\} = 0$ *in* $\mathcal{D}'(\mathbb{R}^N)$ *i.e.,*

$$\int_{\mathbb{R}^N} (1-|v|^2)v\cdot\nabla_v\varphi\,\mathrm{d}F(v) = 0, \quad \textit{for any } \varphi \in C_c^1(\mathbb{R}^N)$$

if and only if $\mathrm{supp}\,F \subset \{0\} \cup \mathbb{S}^{N-1}$.

We need to define a projection operator on the sphere. In order to do this we defined the average of a non negative bounded measure cf. [2].

Definition 1. *Let* $f \in \mathcal{M}_b^+(\mathbb{R}^N \times \mathbb{R}^N)$ *be a non negative bounded measure on* $\mathbb{R}^N \times \mathbb{R}^N$. *We denote by* $\langle f\rangle$ *the measure corresponding to the linear application*

$$\psi \to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \psi(x,v)\,\mathbf{1}_{v=0} f(x,v)\,\mathrm{d}v\mathrm{d}x + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \psi\left(x,\frac{v}{|v|}\right)\mathbf{1}_{v\neq 0} f(x,v)\,\mathrm{d}v\mathrm{d}x\,,$$

for all $\psi \in C_c^0(\mathbb{R}^N \times \mathbb{R}^N)$, *i.e.,*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\psi(x,v)\,\langle f\rangle\,(x,v)\,\mathrm{d}v\mathrm{d}x = \int_{v=0}\psi(x,v)f(x,v)\,\mathrm{d}v\mathrm{d}x + \int_{v\neq 0}\psi\left(x,\frac{v}{|v|}\right)f(x,v)\,\mathrm{d}v\mathrm{d}x,$$

for all $\psi \in C_c^0(\mathbb{R}^N \times \mathbb{R}^N)$.

A direct consequence of the definition is that any bounded, non negative measure, supported in $\mathbb{R}^N \times (\{0\} \cup \mathbb{S}^{N-1})$ is left unchanged by the average operator. Another property of the average operator is that it removes any measure of the form $\mathrm{div}_v\{f(1-|v|^2)v\}$, cf. Proposition 5.2 [2].

Proposition 2. *For any $f \in \mathcal{M}_b^+(\mathbb{R}^N \times \mathbb{R}^N)$ such that $\mathrm{div}_v\{f(1-|v|^2)v\} \in \mathcal{M}_b(\mathbb{R}^N \times \mathbb{R}^N)$, we have $\left\langle \mathrm{div}_v\{f(1-|v|^2)v\}\right\rangle = 0$.*

The above proposition plays a crucial role when eliminating the Lagrange multiplier $f^{(2)}$ in (8). Indeed, for doing that, it is enough to average both hand sides in (8). By the constraint (6), we know that f is supported in $\mathbb{R}^N \times (\{0\} \cup \mathbb{S}^{N-1})$, and thus is left invariant by the average. We check that $\langle \partial_t f\rangle = \partial_t \langle f\rangle = \partial_t f$, and thus, averaging (8) still leads to a evolution problem for f

$$\partial_t f + \langle \mathrm{div}_x(fv)\rangle = \left\langle \mathcal{L}_f(f^{(1)})\right\rangle. \tag{9}$$

Certainly, a much more difficult task is to eliminate the Lagrange multiplier $f^{(1)}$. We expect that this can be done thanks to the constraint in (7). The solvability of (7), with respect to $f^{(1)}$, depends on a compatibility condition, to be satisfied by the right hand side. Indeed, by Proposition 2, we should have

$$\langle \mathrm{div}_v\{f(v-u[f]) + D\nabla_v f\}\rangle = \left\langle \mathrm{div}_v\{f^{(1)}(1-|v|^2)v\}\right\rangle = 0$$

saying that f is a equilibrium for the average collision kernel $\langle Q(f)\rangle = 0$. The equilibria of the average collision kernel form a $N-1$-dimensional manifold, that is one dimension less than the equilibria manifold of the Fokker-Planck operator Q, see also [5]. For any $l \in \mathbb{R}_+, \Omega \in \mathbb{S}^{N-1}$, we introduce the von Mises-Fisher distribution

$$M_{l\Omega}(\omega)\,\mathrm{d}\omega = \frac{\exp\left(l\Omega\cdot\frac{\omega}{r}\right)}{\int_{\mathbb{S}^{N-1}}\exp\left(l\Omega\cdot\frac{\omega'}{r}\right)\,\mathrm{d}\omega'}\,\mathrm{d}\omega,\ \ \omega\in\mathbb{S}^{N-1}.$$

Proposition 3. *Let $F \in \mathcal{M}_b^+(\mathbb{R}^N)$ be a non negative bounded measure on $\mathbb{R}^N$, supported in $\mathbb{S}^{N-1}$. The following statements are equivalent:*
1. $\langle Q(F)\rangle = 0$, that is

$$\int_{v\neq 0}\left\{-(v-u[F])\cdot\nabla_v\left[\widetilde{\psi}\left(\frac{v}{|v|}\right)\right] + D\Delta_v\left[\widetilde{\psi}\left(\frac{v}{|v|}\right)\right]\right\}F\,\mathrm{d}v = 0,$$

for all $\widetilde{\psi}\in C^2(\mathbb{S}^{N-1})$.
2. There are $\rho\in\mathbb{R}_+, \Omega\in\mathbb{S}^{N-1}$ such that $F = \rho M_{l\Omega}\mathrm{d}\omega$ where $l\in\mathbb{R}_+$ satisfies

$$\frac{\int_0^\pi \cos\theta\, e^{l\cos\theta}\sin^{N-2}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\sin^{N-2}\theta\,\mathrm{d}\theta} = Dl. \tag{10}$$

The modulus of the mean velocity is not a coordinate on the equilibria manifold, but it is determined by the condition $|u| = Dl$ where l satisfies (10). Clearly $l = 0$ is a solution, which corresponds to the isotropic equilibrium

$$F = \rho M_{0\Omega}\,\mathrm{d}\omega = \rho\frac{\mathrm{d}\omega}{\bar{\omega}_d r^{N-1}}$$

where $\bar{\omega}_N$ represents the area of the unit sphere in $\mathbb{R}^N$.

Proposition 4. *Let $\lambda : \mathbb{R}_+ \to \mathbb{R}$ be the function given by*

$$\lambda(l) = \frac{\int_0^\pi \cos\theta e^{l\cos\theta}\sin^{N-2}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\sin^{N-2}\theta\,\mathrm{d}\theta}, \quad l \in \mathbb{R}_+, \quad N \geq 2.$$

The function λ is strictly increasing, the function $l \to \lambda(l)/l$ is strictly decreasing and verifies

$$\lambda(0) = 0, \quad \lambda'(0) = \lim_{l\searrow 0}\frac{\lambda(l)}{l} = \frac{1}{N}, \quad \lim_{l\to+\infty}\lambda(l) = 1.$$

If $D \geq \frac{1}{N}$, then the only solution of $\lambda(l) = Dl$ is $l = 0$. If $D \in]0, \frac{1}{N}[$, then there is a unique $l = l(D) > 0$ such that $\lambda(l) = Dl$.

In order to find the equations for the evolution of the density ρ and orientation Ω, we need to find $f^{(1)}$ from (7) in order to feed the terms needed in (9). However, we will see that this is not possible. We will need to introduce a notion of generalized collision invariants, quite related intuitively to the one introduced in [5, 4], in our functional setting of measures supported in $\mathbb{S}^{N-1}$ to avoid the computation of the full $f^{(1)}$. This is the main technical difficulty due to the measure functional setting since the precise definition of generalized collision invariant we need is more involved than in [5, 4]. Our main result established the macroscopic equations satisfied by the density ρ and orientation Ω, which parameterize the von Mises-Fisher equilibrium, obtained when passing to the limit for $\varepsilon \searrow 0$ in (5).

Theorem 2. *For any D such that $D \in]0, \frac{1}{N}[$, we denote by $l = l(D)$ the unique positive solution of $\lambda(l) = Dl$. Let $f^{\mathrm{in}} \in \mathcal{M}_b^+(\mathbb{R}^N \times \mathbb{R}^N)$ be a non negative bounded measure on $\mathbb{R}^N \times \mathbb{R}^N, N \geq 2$. For any $\varepsilon > 0$ we consider the problem*

$$\partial_t f^\varepsilon + \mathrm{div}_x(f^\varepsilon v) + \frac{1}{\varepsilon^2}\mathrm{div}_v(f^\varepsilon(1-|v|^2)v) = \frac{1}{\varepsilon}\mathrm{div}_v\{f^\varepsilon(v - u[f^\varepsilon]) + D\nabla_v f^\varepsilon\}$$

for all $(t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}^N$ with $f^\varepsilon(0) = f^{\mathrm{in}}$, $(x, v) \in \mathbb{R}^N \times \mathbb{R}^N$. Therefore the limit distribution $f = \lim_{\varepsilon\searrow 0} f^\varepsilon$, is a von Mises-Fisher equilibrium $f = \rho M_{l\Omega}(\omega)\,\mathrm{d}\omega$ on $\mathbb{S}^{N-1}$, where the density $\rho(t, x)$ and the orientation $\Omega(t, x)$ satisfy the macroscopic equations

$$\partial_t\rho + \mathrm{div}_x(\rho Dl\Omega) = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N \tag{11}$$

$$\partial_t\Omega + k_N\,(\Omega\cdot\nabla_x)\Omega + \frac{1}{l}(I_N - \Omega\otimes\Omega)\frac{\nabla_x\rho}{\rho} = 0 \tag{12}$$

with the initial conditions

$$\rho(0, x) = \int_{\mathbb{R}^N} f^{\mathrm{in}}(x)\,\mathrm{d}v, \quad \Omega(0, x) = \frac{\int_{\mathbb{R}^N} v f^{\mathrm{in}}(x)\,\mathrm{d}v}{\left|\int_{\mathbb{R}^N} v f^{\mathrm{in}}(x)\,\mathrm{d}v\right|}, \quad x \in \mathbb{R}^N$$

where

$$k_N = \frac{\int_0^\pi e^{l\cos\theta}\chi(\cos\theta)\cos\theta\sin^{N-1}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\chi(\cos\theta)\sin^{N-1}\theta\,\mathrm{d}\theta}$$

and χ *solves*

$$-D\frac{\mathrm{d}}{\mathrm{d}c}\left\{e^{lc}\chi'(c)(1-c^2)^{\frac{1}{2}}\right\} = e^{lc}, \quad c \in]-1,1[, \quad \chi(-1)=\chi(1)=0 \quad \textit{if } N=2$$

and

$$-D\frac{\mathrm{d}}{\mathrm{d}c}\left\{e^{lc}\chi'(c)(1-c^2)^{\frac{N-1}{2}}\right\} + (N-2)De^{lc}\chi(c)(1-c^2)^{\frac{N-5}{2}} = e^{lc}(1-c^2)^{\frac{N-2}{2}}$$
$$c \in]-1,1[, \ N \geq 3.$$

A nice practical implication of our main result is that this penalization procedure, by imposing asymptotically a cruise speed for particles, could lead to efficient and stable numerical schemes to compute the hydrodynamic equations (11)-(12). This is important due to the possible non-hyperbolicity of the system (11)-(12), see [4].

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Perturbative Cauchy theory for Landau and Boltzmann equations

Isabelle Tristani

(joint work with Kleber Carrapatoso, Frédéric Hérau, Daniela Tonon and Kung-Chien Wu)

1. Introduction

In this report, we present results about Landau equation and Boltzmann equation without angular cut-off. We here restrict to the case of the so-called hard potentials (even if for the Landau equation, we are indeed able to treat softer potentials). The results concerning the Landau equation are taken from a joint work with Kleber Carrapatoso and Kung-Chien Wu [1], the ones about the Boltzmann equation come from joint works with Frédéric Hérau and Daniela Tonon [4], [5].

In what follows, we consider particles described by their space inhomogeneous distribution density $f = f(t,x,v)$ with $t \in \mathbb{R}^+$ the time, $x \in \mathbb{T}^3$ the position and $v \in \mathbb{R}^3$ the velocity. We study the spatially inhomogeneous Boltzmann and Landau equations which take the form:

$$\partial_t f + v \cdot \nabla_x f = Q(f,f) \tag{1}$$

where Q stand for the Boltzmann or the Landau collision operator.

The main results are about the Cauchy theory of the nonlinear equations: we prove results of existence and uniqueness of solutions in a close-to-equilibrium regime in weighted Sobolev spaces with a polynomial weight. We also prove an exponential stability for such solutions, with a rate as close as we want to the optimal rate given by the semigroup decay of the corresponding linearized equation. The proof of those results are based on the study of the linearized problems and in particular, we investigate the regularization properties of the semigroups associated to the linearized operators.

2. Cauchy theory and exponential stability

Consider X a Sobolev space of type $H^3_x L^2_v(m)$ where $m(v) = \langle v\rangle^k$ with k large enough. We do not enter into details here but in the real definition of X, the polynomial weight is different according to the order of derivative in x. Also, we denote $\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$ the normalized Maxwellian equilibrium of our equations.

Let us mention that being a bit more precise in the previous statement, we can also get a result of uniqueness.

Our strategy is based on the study of the linearized equations, which is a standard strategy to develop a Cauchy theory in a close-to-equilibrium regime. However, we point out that our study of the nonlinear problems is then very tricky. Indeed, it is hard to see that the gain induced by the linear part of the equation is enough to directly control the loss due to the nonlinear part of the equation. It implies that the linear part is dominant and thus dictates the dynamics of the equation. In order to do that, we have to establish very accurate estimates on the collision operators. We also have to study very carefully the regularization properties of the semigroup associated to the linearized operator, the Boltzmann case is much more difficult and to perform this analysis, we see the linearized Boltzmann operator as a pseudo-differential one.

3. The linearized equations

3.1. **Semigroup decay.** The linearized operators around equilibrium are defined at first order through

$$\Lambda h := Q(\mu, h) + Q(h, \mu) - v \cdot \nabla_x h.$$

We study the linearized operator Λ in various weighted Sobolev spaces of type $H^n_x H^\ell_v(\langle v\rangle^k)$ up to $L^2_{x,v}(\langle v\rangle^k)$ for k large enough. In this type of space, we prove that there exist explicit constants $\lambda_* > 0$ and $C \geq 1$ such that

$$\forall\, t \geq 0, \quad \forall\, h \in \mathcal{E}, \quad \|\mathcal{S}_\Lambda(t)h - \Pi_0 h\|_{\mathcal{E}} \leq C\, e^{-\lambda_* t}\, \|h - \Pi_0 h\|_{\mathcal{E}},$$

where $\mathcal{S}_\Lambda(t)$ is the semigroup associated to Λ and Π_0 the projector onto the null space of Λ.

To prove this type of estimates, our strategy follows the one initiated by Mouhot in [7] for the homogeneous Boltzmann equation for hard potentials with cut-off.

This argument has then been developed and extended in an abstract setting by Gualdani, Mischler and Mouhot [2], and Mischler and Mouhot [6]. Thanks to that, we are able to get spectral/semigroup estimates on "large spaces" $\mathcal{E}$ using the already known spectral gap estimates for Λ on $H^\ell_{x,v}(\mu^{-1/2})$, for $\ell \geq 1$, from the paper of Mouhot and Neumann [8]. Roughly speaking, to do that, we have to find a splitting of Λ into two operators $\Lambda = \mathcal{A} + \mathcal{B}$ which satisfy some properties. The first part $\mathcal{A}$ has to be bounded, the second one $\mathcal{B}$ has to have some dissipativity properties, and also the operator $(\mathcal{A}\mathcal{S}_{\mathcal{B}}(t))$ is required to have some regularization properties.

3.2. **Regularization properties.** As mentioned above, in order to get our result of semigroup decay in large spaces, we need to understand the gain of regularity provided by the linearized operator. It is also important to get regularization estimates for the study of the nonlinear problem. Indeed, it is going to allow us to compensate the loss due to the nonlinear part of the equation and thus to close our estimates.

To summarize, the gain of regularity in velocity consists in an anisotropic gain of regularity for one derivative in the Landau case and a fraction of derivative for the Boltzmann case. Then, using the hypoellipticity features of the equations, we are able to obtain a similar gain of regularity in the space variable. To do that, we use Lyapunov functionals which were first introduced for the study of Fokker-Planck equation by Hérau in [3]. This method provides quantitative estimates in time of the short time regularization properties of the studied equation. In [1], we adapt the proof in order to get a similar result for the linearized Landau equation. And in [5], following the same strategy, we get a result of regularization first for the fractional Kolmogorov equation and then for the linearized Boltzmann one. We are thus able to substantially enlarge the range of application of this method of proof. Indeed, we develop it in more complicated contexts thanks to Fourier transform and/or pseudo-differential tools. Our proof for the fractional Kolmogorov equation is only based on Fourier transform (no need of pseudo-differential theory). On the contrary, the same question for the Boltzmann equation requires a very careful and sharp understanding of the Boltzmann operator itself. This precise analysis is done thanks to pseudo-differential theory and semi-classical ideas.

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The choice of representative volumes for random materials

Julian Fischer

The most widely employed method for determining the effective large-scale properties of materials with random heterogeneities on a small scale is the *representative volume element* (RVE) method: It basically proceeds by choosing a sample of the random material – the *representative volume element* – and computing its properties. To obtain an accurate approximation for the effective material properties, the sample should reflect the statistical properties of the material well. As a consequence, it is desirable to choose a large sample of the random material. However, an increased size of the sample comes with an increased computation cost; for this reason, there have been attempts in material science [6] and mechanics [5] towards capturing the statistical properties of the material in a better way in a sample of a fixed size.

Inspired by these attempts, in the context of diffusion in a random medium Le Bris, Legoll, and Minvielle [4] have devised a method capable of significantly increasing the accuracy of the RVE method for a given size of the sample. Their idea is to select among many samples of the random medium the one that captures certain statistical properties of the random medium best, i. e. the one that is "most representative" of the material. For example, for a composite of two constituent materials, in the simplest setting of their method they propose to select the material sample in which the volume fractions of the constituent materials match best with the volume fractions of the constituents in the overall material; see Figure 1 for an illustration of this approach.

In numerical examples with moderate ellipticity contrast, the method of Le Bris, Legoll, and Minvielle [4] has provided an increase in accuracy by a factor of about $3 - 10$ or equivalently a reduction in computational cost by a factor of about $10 - 50$. However, the analysis of the approach has essentially been limited to the one-dimensional setting, in which an explicit formula for the homogenization corrector is available, as well as a purely qualitative convergence result in the multidimensional setting in the limit of infinite size of the RVE.

In the recent work [3], we establish a rigorous mathematical justification of the approach of Le Bris, Legoll, and Minvielle. In particular, we quantify the gain in accuracy of the ansatz of Le Bris, Legoll, and Minvielle and prove that the method does never worse than a random selection of samples. Furthermore, we show that it fails to increase the accuracy for some (rather artificial) counterexamples of random media.

The diffusion in a random medium is described by the stationary or time-dependent diffusion equation

$$-\nabla\cdot(a\nabla u)=f \qquad \text{or} \qquad \partial_t u=\nabla\cdot(a\nabla u)$$

with a random diffusion coefficient field a. Assuming spatial homogeneity, ellipticity, and a finite range of dependence $\varepsilon\ll 1$ for the probability distribution of a (i. e. in particular fast decorrelation of the material properties on scales larger than ε), the theory of stochastic homogenization predicts that the diffusion in the random medium behaves on large scales like a diffusion with a constant effective diffusion coefficient a_{hom}: One may approximate the solution u by the solution u_{hom} of an effective diffusion equation

$$-\nabla\cdot(a_{\mathrm{hom}}\nabla u_{\mathrm{hom}})=f \qquad \text{or} \qquad \partial_t u_{\mathrm{hom}}=\nabla\cdot(a_{\mathrm{hom}}\nabla u_{\mathrm{hom}})$$

with a constant diffusion coefficient a_{hom}. In this context of diffusion in a random medium, the representative volume element method is employed to obtain an approximation a^{RVE} for the effective coefficient a_{hom}. Roughly speaking, the approximation a^{RVE} is computed by choosing a sample $a|_{[0,L\varepsilon]^d}$ of the random medium (where ε is the length of correlations in the medium and where $L\gg 1$), solving the equation for the homogenization corrector $-\nabla\cdot(a(e_i+\nabla\phi_i))=0$ on this sample, and averaging the fluxes

$$a^{\mathrm{RVE}}e_i:=\fint_{[0,L\varepsilon]^d} a(e_i+\nabla\phi_i)\,dx.$$

This approximation a^{RVE} is a random quantity, as it depends on the sample $a|_{[0,L\varepsilon]^d}$ of the random medium. In contrast, the macroscopic effective diffusion coefficient a_{hom} is deterministic. In fact, as shown by Gloria and Otto [1, 2] the leading-order contribution to the error $a^{\mathrm{RVE}}-a_{\mathrm{hom}}$ consists of random fluctuations: The order of the fluctuations of a^{RVE} is given by

$$\sqrt{\mathrm{Var}\,a^{\mathrm{RVE}}}\le CL^{-d/2},$$

while the systematic error is of higher order in the size L of the sample

$$\big|\mathbb{E}[a^{\mathrm{RVE}}]-a_{\mathrm{hom}}\big|\le CL^{-d}|\log L|^d.$$

As observed numerically by Le Bris, Legoll, and Minvielle [4], their method of selecting the material sample to obtain a "particularly representative sample" of the material increases the accuracy of approximations for a_{hom} by reducing the fluctuations. They also observed numerically that this strategy maintains the order of the systematic error. In the simplest case of their method, they select the sample $a|_{[0,L\varepsilon]^d}$ according to the following criterion on the spatial average of the coefficient field

$$\left|\fint_{[0,L\varepsilon]^d} a\,dx-\mathbb{E}\left[\fint_{[0,L\varepsilon]^d} a\,dx\right]\right|\le\delta L^{-d/2}\ll\sqrt{\mathrm{Var}\fint_{[0,L\varepsilon]^d} a\,dx} \tag{1}$$

for some $0<\delta\ll 1$. In a numerical example with moderate ellipticity contrast, this selection criterion achieves a numerical variance reduction by a factor of ≈ 10. By additionally considering a second statistical quantity derived from an expansion

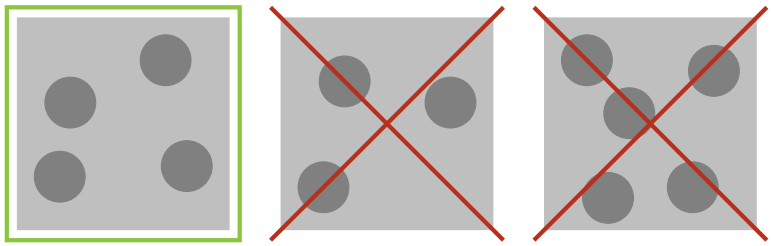

FIGURE 1. Among the three depicted material samples, the method of Le Bris, Legoll, and Minvielle would choose the first sample as the representative volume element and discard the other two, as the volume fraction of the inclusions in the first sample is closest to the overall material average. For a better illustration of the method, exaggeratedly small material samples are depicted.

of $a_{\rm hom}$ in the case of small ellipticity contrast $a \approx \mathrm{Id}$, in the same setting of moderate contrast they even achieve a variance reduction by a factor of ≈ 50.

In the recent work [3] we prove that the approach of Le Bris, Legoll, and Minvielle [4] indeed leads to a reduction of the fluctuations of the approximations $a^{\rm RVE}$: Denoting the approximation obtained by the selection criterion (1) by $a^{\rm SQS}$, we derive the estimate

$$\frac{\operatorname{Var} a^{\rm SQS}}{\operatorname{Var} a^{\rm RVE}} \le 1 - (1-\delta^2)|\rho|^2 + \frac{C}{\delta} L^{-d/2} |\log L|^p$$

where ρ denotes the Pearson correlation coefficient

$$\rho := \frac{\operatorname{Cov}\left[a^{\rm RVE}, \fint_{[0,L\varepsilon]^d} a\,dx\right]}{\sqrt{\operatorname{Var}[a^{\rm RVE}]}\sqrt{\operatorname{Var} \fint_{[0,L\varepsilon]^d} a\,dx}}.$$

Note that we also construct a counterexample for which this correlation is zero [3], in which case the method is shown to fail to give an advantage over random sampling. We also show in [3] that the method of Le Bris, Legoll, and Minvielle essentially preserves the order of the systematic error in the sense

$$\left|\mathbb{E}[a^{\rm SQS}] - a_{\rm hom}\right| \le \frac{C}{\delta} L^{-d} |\log L|^p$$

and that the tails (more precisely, the moderate deviations) of the probability distribution are reduced just as suggested by the variance reduction. Similar results are proven for the enhanced methods proposed by Le Bris, Legoll, and Minvielle.

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Comparison of strings and point particles: an Eulerian approach

Yann Brenier

In string theory (see [8], for instance), a (classical relativistic closed) string is a time-dependent loop $(t,s) \in \mathbb{R} \times \mathbb{T} \to (t, X(t,s)) \in \mathbb{R}^{1+d}$ (where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$) of extremal area in the ambient Minkowski space.
The area (also called "Nambu-Goto Action") over a time interval I reads

$$\mathcal{A}_I = -\int_I \int_{\mathbb{T}} \sqrt{\partial_s X^2 (1 - \partial_t X^2) + (\partial_t X \cdot \partial_s X)^2}\ dtds.$$

[In comparison, the action of a (free) relativistic point particle $t \to (t, X(t)) \in \mathbb{R}^{1+d}$ is its length, where the length over a time interval I reads

$$\mathcal{L}_I = -\int_I \sqrt{1 - |X'(t)|^2} dt.]$$

1. The Eulerian form of string motion

Let us first write, in Eulerian form, the free-motion of a string X that we assume to be smooth and not self-intersecting, with $\partial_s X$ never vanishing. We introduce the divergence-free vector-valued measure (or "1-current") $B = |B| b$, defined by

$$B(t,x) = \int_{\mathbb{T}} \delta(x - X(t,s)) \partial_s X(t,s) ds, \quad |B(t,x)| = \int_{\mathbb{T}} \delta(x - X(t,s)) |\partial_s X(t,s)| ds,$$

and assume there is a smooth "velocity field" $v = v(t,x) \in \mathbb{R}^d$ so that

$$\partial_t X(t,s) = v(t, X(t,s)).$$

Denoting by Π_b the projection along b, the area on a time interval I just reads

$$\mathcal{L}_I = -\int_{I \times \mathbb{R}^d} \sqrt{1 - \tilde{v}^2} |B|, \quad \tilde{v} = (\mathbb{I} - \Pi_b) v.$$

Then, introducing the "momentum" field

$$P(t,x) = |B(t,x)| p(t,x), \ \ p = \frac{\tilde{v}}{\sqrt{1 - \tilde{v}^2}}$$

we find, by standard Calculus of variations, the self-consistent set of equations

$$\partial_t B + \nabla \cdot \left(\frac{B \otimes P - P \otimes B}{R} \right) = 0, \quad \nabla \cdot B = 0, \quad R = \sqrt{B^2 + P^2}, \tag{1}$$

$$\partial_t P + \nabla \cdot \left(\frac{P \otimes P}{R} \right) = \nabla \cdot \left(\frac{B \otimes B}{R} \right). \tag{2}$$

This system admits an extra conservation law for the "energy" R which is a convex function of (B, P). This makes the system hyperbolic [5], but in a degenerate sense (since R is not strictly convex).

2. The Born-Infeld regularization of string motion

There is natural hyperbolic regularization of the string system, at least as $d = 3$, which comes from the Born-Infeld non-linear theory of Electromagnetism [1] , as shown in [2]. We get the system

$$\partial_t B + \nabla \times \left(\frac{B \times P + D}{R} \right) = 0, \quad \nabla \cdot B = 0, \tag{3}$$

$$\partial_t D + \nabla \times \left(\frac{D \times P - B}{R} \right) = 0, \quad \nabla \cdot D = 0, \tag{4}$$

$$\partial_t P + \nabla \cdot \left(\frac{P \otimes P}{R} \right) = \nabla \cdot \left(\frac{B \otimes B}{R} \right) + \nabla(\frac{1}{R}), \quad R = \sqrt{1 + B^2 + D^2 + P^2}, \tag{5}$$

which admits an extra conservation law for R which is now a strictly convex function of (B, D, P). The string equation is (consistently) obtained, when $d = 3$, in the regime where B and P are very large with respect to D and 1.

3. A heat equation for strings?

Following [4, 7], we can get a dissipative version of equation (3,4,5), just by first performing the quadratic change of time

$$t \to t^* = \frac{t^2}{2}, \quad B \to B^*, \quad P dt \to P^* dt^*, \quad D dt \to D^* dt^*,$$

which leads to a non-autonomous system, and next by dropping the higher order terms in $t^* << 1$. We obtain the new system (where all subscript * have been dropped):

$$\partial_t B + \nabla \times \left(\frac{B \times P + D}{R} \right) = 0, \quad \nabla \cdot B = 0, \tag{6}$$

$$P = \nabla \cdot \left(\frac{B \otimes B}{R} \right) + \nabla(\frac{1}{R}), \quad D = \nabla \times \left(\frac{B}{R} \right), \quad R = \sqrt{1 + B^2}. \tag{7}$$

For the string equation (which corresponds to the regime $|B| >> 1$), we find

$$\partial_t B + \nabla \times \left(\frac{B \times P}{R} \right) = 0, \quad \nabla \cdot B = 0, \tag{8}$$

$$P = \nabla \cdot \left(\frac{B \otimes B}{R} \right), \quad R = |B|, \tag{9}$$

which is nothing but the mean-curvature flow for "1-currents" (also called "curved-shortening" [3, 6]), and can be considered as a (geometric) heat equation for loops. Interestingly enough, the Born-Infeld regularization provides a kind of heat equation for strings that includes some diffusion effect in the whole space, in particular through the term $\nabla \times (R^{-1}\nabla \times (R^{-1}B))$, and not only along the loop itself (as for the mean-curvature flow). Then the energy dissipates according to

$$\frac{d}{dt}\int |B| = -\int \frac{D^2+P^2}{R}$$

(where D, P, R are defined by (7)) which provides, in our opinion, in the right-hand side, a kind of "Fisher information" for strings (that could be presumably used for a tentative Schrödinger -or Dirac- equation for strings, à la Madelung).

4. Comparison with point particules

We can reproduce our discussion in the case of point particles

$$t \in \mathbb{R} \to (t, X(t)) \in \mathbb{R}^{1+d}$$

with the length $-\int \sqrt{1-|X'(t)|^2}dt$ (substituting for the area of a string) and

$$\rho(t,x) = \int_{\mathbb{T}} \delta(x - X(t))dt.$$

Assuming the existence of a smooth velocity field v such that $X'(t) = v(t, X(t))$, and introducing $P = \rho v(1-v^2)^{-1/2}$, we find the Eulerian formulation of a free particle

$$\partial_t \rho + \nabla \cdot \left(\frac{P}{R}\right) = 0, \quad R = \sqrt{\rho^2 + P^2}, \quad \partial_t P + \nabla \cdot \left(\frac{P \otimes P}{R}\right) = 0.$$

A regularization à la Born-Infeld is again possible, leading, after a quadratic change of time, to the non-linear heat equation

$$\partial_t \rho = \nabla \left(\rho \nabla \log(\frac{\rho}{\sqrt{1+\rho^2}}) \right),$$

which also reads $\partial_t \rho = \triangle(\arctan\rho)$.

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Kinetic description for the Lorentz Gas with long range interactions

Alessia Nota
(joint work with Sergio Simonella, Juan J. L. Velázquez)

The problem of deriving macroscopic evolution equations from the microscopic laws of motion governed by Newton's laws of classical mechanics is one of the most challenging problems in mathematical physics. Here we consider a simpler model where a classical Newtonian particle moves in the three-dimensional space, in the field of forces produced by a Poisson distribution of fixed scatterers. This dynamical system is usually referred to as the Lorentz gas since it was proposed by H. A. Lorentz in 1905 (cf. [8]) to explain the motion of electrons in metals. In the classical setting, the scatterers are assumed to be short ranged and are modeled as elastic hard spheres.

In this talk we present the results obtained in [9] where we suppose that each scatterer generates a long range potential $Q_i\Phi(x-x_i)$ with different charges $Q_i \in \mathbb{R}$ and we assume that the potential Φ behaves typically as $|x|^{-s}$ for large $|x|$, with $s > 1/2$.

We first construct and study the random force field generated by a Poisson distribution of sources at finite density $x_1, x_2, \cdots$ in $\mathbb{R}^3$ that we denote as "generalized Holtsmark field" associated to Φ since their analysis was initiated by Holtsmark in [6] (cf. also [2]). To obtain translation invariant fields, for potentials decreasing for large $|x|$ as $|x|^{-s}$ with $s \leq 1$, some "electroneutrality" of the system is necessary. This is achieved either by means of the addition of a background with opposite charge density or using charges with positive and negative signs.

We then consider the dynamics of one tagged particle in such fields. Let $F(x, \varepsilon)$ be a generalized Holtsmark field associated to potentials of the form $\{\Phi(x, \varepsilon)\}_{\varepsilon>0}$ where ε is a small parameter tuning the mean free path ℓ_ε. Let $(x(t), v(t))$ be the position and velocity of the tagged particle. For each given scatterer configuration $\omega = \{(x_n, Q_{j_n})\}_{n\in\mathbb{N}}$, the evolution is given by

$$\frac{dx}{dt} = v \quad , \quad \frac{dv}{dt} = F(x, \varepsilon)\,\omega \tag{1}$$

with initial data

$$x(0) = x_0 \in \mathbb{R}^3, \quad v(0) = v_0 \in \mathbb{R}^3. \tag{2}$$

We denote by $T^t(x_0, v_0; \varepsilon; \omega)$ the Hamiltonian flow associated to (1)-(2). We will see that it is possible to obtain a rather detailed description of the dynamics of the tagged particle in the 'kinetic limit'.

Suppose that $f_0 \in \mathcal{M}_+\left(\mathbb{R}^3 \times \mathbb{R}^3\right)$ is the initial probability density for the tagged particle. To describe the evolution, we compute the probability density f_ε of finding the particle at a given point x with a given value of the velocity v at time $t \geq 0$. More precisely, our goal is to study the asymptotics of

$$f_\varepsilon(\ell_\varepsilon t, \ell_\varepsilon x, v) = \mathbb{E}[f_0(T^{-\ell_\varepsilon t}(\ell_\varepsilon x, v; \varepsilon; \cdot))] \tag{3}$$

as ε tends to zero, in several scaling limits where the mean free path ℓ_ε is much larger than the typical distance d between the scatterers.

To obtain a "kinetic limit" we require that

$$\lim_{\varepsilon\to 0}\frac{\ell_\varepsilon}{d}=\lim_{\varepsilon\to 0}\ell_\varepsilon=\infty, \tag{4}$$

which implies that the forces produced are small at distances of order one from the scatterers, and a second condition concerning the statistical independence of the particle deflections experienced over distances of order ℓ_ε. Then, on a macroscopic scale of space and time $f_\varepsilon\left(\ell_\varepsilon t,\ell_\varepsilon x,v\right)\to f(t,x,v)$, where f solves one of the following kinetic equations: 1) a linear Boltzmann equation, i.e.

$$\left(\partial_t f+v\partial_x f\right)(t,x,v)=\int_{S^2}B\left(v;\omega\right)\left[f\left(t,x,|v|\,\omega\right)-f\left(t,x,v\right)\right]d\omega\ , \tag{5}$$

where we denoted by ∂_x the three dimensional gradient and B is some nonnegative collision kernel (depending on Φ);

2) a linear Landau equation, i.e.

$$\left(\partial_t f+v\partial_x f\right)(x,v,t)=\kappa\Delta_{v_\perp}f\left(x,v,t\right)\ , \tag{6}$$

where $\Delta_{v_\perp}$ is the Laplace Beltrami operator on S^2 (sphere of radius $|v|=1$) and the diffusion coefficient $\kappa>0$ depends on Φ.

In the current literature a mathematical derivation of the kinetic equations (5) and (6) has been provided only in cases of compactly supported potentials, in the low density and weak coupling limits respectively (cf. [5, 7, 10]). Furthermore, the linear Landau equation (6) has been derived in [3] in the limit of grazing collisions and in an intermediate regime between the Boltzmann-Grad and the weak-coupling regime (see also [1]). In the cases of potentials with diverging support, it is possible to derive the Boltzmann equation (see [4]), but it is unclear even formally what kinetic behaviour has to be expected for generic $\left\{\Phi\left(x,\varepsilon\right)\right\}_{\varepsilon>0}$ and in particular for potentials of the form $\Phi\left(x,\varepsilon\right)=\frac{\varepsilon^s}{|x|^s}$.

To obtain the appropriate kinetic descriptions we estimate the diffusive timescale and identify conditions for the vanishing of correlations for different families of potentials (we refer to [9] where a detailed analysis for several families of long range potentials can be found).

Our strategy consists in splitting (smoothly) the potential $\Phi\left(x,\varepsilon\right)$ as

$$\Phi\left(x,\varepsilon\right)=\Phi_B\left(x,\varepsilon\right)+\Phi_L\left(x,\varepsilon\right)$$

where $\Phi_B\left(x,\varepsilon\right)$ is supported in a ball of radius $M\lambda_\varepsilon$ and $\Phi_L\left(x,\varepsilon\right)$ is supported at distances larger than $\frac{M\lambda_\varepsilon}{2}$ with M of order one but large and λ_ε the *Landau length*. At distances of order λ_ε, the particle is deflected an amount of order one by the interaction $\Phi_B\left(x,\varepsilon\right)$. The time between two consecutive collisions is of order of the 'Boltzmann-Grad timescale'

$$T_{BG}=\frac{1}{\left(\lambda_\varepsilon\right)^2}\ . \tag{7}$$

We compute the timescale T_L ('Landau timescale') in which the deflections produced by Φ_L become relevant. Due to (4), we have $T_{BG} \to \infty$ and $T_L \to \infty$ as $\varepsilon \to 0$. Moreover, we can characterize which of the following possibilities holds:

(i) if $T_{BG} \ll T_L$ as $\varepsilon \to 0$, i.e. the fastest process yielding particle deflections are binary collisions with single scatterers, the evolution of f will be given by (5);
(ii) if $T_{BG} \gg T_L$ as $\varepsilon \to 0$, i.e. the deflections due to the accumulation of a large number of small interactions yield a relevant change in the direction of the velocity before a binary collision takes place. Then f will evolve according to (6);
(iii) if T_{BG} and T_L are of the same order of magnitude, f will be driven by a Boltzmann-Landau equation.

In the case (ii) we can provide examples of potentials for which the deflections experienced by the particle do not decorrelate over times of order T_L. Hence, we cannot expect to have a single PDE describing the probability distribution in the particle phase space and the correlations between macroscopic deflections must be taken into account.

The form of equation derived for a given scaling limit depends on the decay, determined by the exponent s, as well as on the singularities of the potential.

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Global strong solutions of the Vlasov-Poisson-Boltzmann system in bounded domains

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(joint work with Yunbai Cao, Donghyun Lee)

Dynamics and collision processes of dilute charged particles can be modeled by the Vlasov-Poisson-Boltzmann system

$$\partial_t F + v \cdot \nabla_x F - \nabla_x \phi(t,x) \cdot \nabla_v F = Q(F,F). \tag{1}$$

The electrostatic potential is determined by the Poisson equation

$$-\Delta_x \phi(t,x) = \int_{\mathbb{R}^3} F(t,x,v)\mathrm{d}v - \rho_0 \ \text{ in } \Omega, \tag{2}$$

with the *zero* Neumann boundary condition $\frac{\partial \phi}{\partial n} = 0$ on $\partial\Omega$. Here in (2) a background density ρ_0 is a constant number.

The *characteristics (trajectory)* is determined by the Hamilton ODEs

$$\frac{d}{ds}\begin{bmatrix} X(s;t,x,v) \\ V(s;t,x,v)\end{bmatrix} = \begin{bmatrix} V(s;t,x,v) \\ -\nabla_x \phi(s, X(s;t,x,v))\end{bmatrix} \ \text{ for } -\infty < s,t < \infty, \tag{3}$$

with $(X(t;t,x,v), V(t;t,x,v)) = (x,v)$. For $(t,x,v) \in \mathbb{R}\times\Omega\times\mathbb{R}^3$, we define *the backward exit time* $t_{\mathbf{b}}(t,x,v)$ as

$$t_{\mathbf{b}}(t,x,v) := \sup\{s \geq 0 : X(\tau;t,x,v) \in \Omega \ \text{ for all } \tau \in (t-s,t)\}. \tag{4}$$

Furthermore, we define $x_{\mathbf{b}}(t,x,v) := X(t - t_{\mathbf{b}}(t,x,v); t,x,v)$ and $v_{\mathbf{b}}(t,x,v) := V(t - t_{\mathbf{b}}(t,x,v); t,x,v)$. For $\varepsilon > 0$ we define a *kinetic distance* $\alpha_\varepsilon(t,x,v)$

$$\chi\Big(\frac{t - t_{\mathbf{b}}(t,x,v) + \varepsilon}{\varepsilon}\Big)|n(x_{\mathbf{b}}(t,x,v)) \cdot v_{\mathbf{b}}(t,x,v)| + \Big[1 - \chi\Big(\frac{t - t_{\mathbf{b}}(t,x,v) + \varepsilon}{\varepsilon}\Big)\Big].$$

where $\chi : \mathbb{R} \to [0,1]$ is a monotone smooth function which equals 1 for $\tau \geq 1$ and zero when $\tau \leq 0$. Denote $w_\vartheta(v) = e^{\vartheta|v|^2}$.

Main Theorem. Assume a bounded open smooth domain $\Omega \subset \mathbb{R}^3$ is convex. Let $0 < \tilde{\vartheta} < \vartheta \ll 1$. Assume some 'natural' compatibility condition. There exist small constants $0 < \vartheta_0, \varepsilon \ll 1$ such that for all $0 < \varepsilon \leq \varepsilon_0$ if an initial datum $F_0 = \mu + \sqrt{\mu} f_0 \geq 0$ satisfies

$$\|w_\vartheta f_0\|_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)} < \varepsilon, \quad \|w_{\tilde{\vartheta}} \nabla_v f_0\|_{L^3(\Omega\times\mathbb{R}^3)} < \infty,$$

$$\|w_{\tilde{\vartheta}} \alpha_\varepsilon^\beta \nabla_{x,v} f_0\|_{L^p(\Omega\times\mathbb{R}^3)} < \varepsilon \ \text{ for } \ 3 < p < 6, \ \ 1 - 2/p < \beta < 2/3,$$

then there exists a unique global-in-time solution $F(t) = \mu + \sqrt{\mu} f(t)$ to the Vlasov-Poisson-Boltzmann system. Moreover there exists $\lambda_\infty > 0$ such that

$$\sup_{t\geq 0} e^{\lambda_\infty t}\|w_\vartheta f(t)\|_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)} + \sup_{t\geq 0} e^{\lambda_\infty t}\|\phi(t)\|_{C^2(\Omega)} < \infty,$$

$$\|w_{\tilde{\vartheta}} \alpha_\varepsilon^\beta \nabla_{x,v} f(t)\|_{L^p(\Omega\times\mathbb{R}^3)} < Ce^{Ct} \ \text{ for all } t \geq 0.$$

We note that the convexity of the domain is necessary for the singularity formation in the case of non-convex domains [5, 4].

For (3) is determinant $\left[\partial_t + v\cdot\nabla_x - \nabla_x\phi\cdot\nabla_v\right]\alpha_\varepsilon(t,x,v) = 0$. This crucial invariant property under the transport operator is the key for our approach. In [3], a different weight $\tilde{\alpha}$ was used and the extra $\langle v\rangle$-multiplier from $[\partial_t + v\cdot\nabla_x]\tilde{\alpha} \sim \langle v\rangle\tilde{\alpha}$ causes a super-exponential growth in $\tilde{\alpha}$-weighted $W^{1,p}$ estimate, which seems infeasible to prove a stability of μ. In the energy-type estimate of $\nabla_{x,v}f$ in $\alpha^\beta_{f,\varepsilon}$-weighted L^p-norm, the operator $v\cdot\nabla_x$ causes a boundary term to be controlled: $\int_0^t\int_{\partial\Omega}\int_{n\cdot v\le 0}|\alpha^\beta_{f,\varepsilon}\nabla_{x,v}f|^p|n\cdot v|\mathrm{d}v\mathrm{d}S_x\mathrm{d}s$. Due to the boundary singularity and the fact $\alpha_\varepsilon(t,x,v) = |n(x)\cdot v|$ on $x\in\partial\Omega$ and $n(x)\cdot v\le 0$, this integrand is integrable if

$$\beta > (p-2)/p \ \text{ so that } \ |n\cdot v|^{p\beta-p+1}\in L^1_{loc}.$$

In the energy-type estimate of $\nabla_{x,v}f$ we have two terms to be controlled:

$$\int_0^t\iint_{\Omega\times\mathbb{R}^3}\nabla_x^2\phi\nabla_v f\alpha_\varepsilon^{p\beta}|\nabla_{x,v}f|^{p-1}, \quad \int_0^t\iint_{\Omega\times\mathbb{R}^3}K\nabla_{x,v}f\alpha_\varepsilon^{p\beta}|\nabla_{x,v}f|^{p-1}. \tag{5}$$

The keystone to handle the first term of (5) is *a bound of* $\phi(t)$ in C^2_x. Unfortunately such estimate is a boarder line case of the well-known Schauder elliptic regularity theory when $\int_{\mathbb{R}^3} f\sqrt{\mu}\mathrm{d}v$ is merely continuous or bounded. If we have an α^β_ε-weighted L^p-estimate of $\nabla_{x,v}f$ for $p>3$ a priori then

$$\left\|\int_{\mathbb{R}^3}\nabla_x f\sqrt{\mu}\mathrm{d}v\right\|_{L^p_x(\Omega)} < C\sup_x\left\|\frac{\sqrt{\mu}}{\alpha^\beta_\varepsilon}\right\|_{L^{p^*}(\mathbb{R}^3)}\left\|\alpha^\beta_\varepsilon\nabla_x f\right\|_{L^p(\Omega\times\mathbb{R}^3)}, \tag{6}$$

which leads $C^{2,0+}$-bound of ϕ by the Morrey inequality as long as

$$\alpha_\varepsilon^{-\beta p^*}\in L^1_{loc} \ \text{ for some } \beta p^* > p - 2/p - 1. \tag{7}$$

We note that $\phi(t)$ has an exponential decay in weaker Hölder spaces $C^{1,1-}$ if f decays exponentially in L^∞. If the bound (6) grows at most *exponentially*,

$$\textbf{\textit{an exponential decay of}}\ \phi(t)\ \text{in}\ C^2_x \tag{8}$$

can be verified through an interpolation in Hölder spaces.

To prove (7), a major difficulty arises due to the *non-local* feature of α_ε, which is determined on the characteristics at $t - t_{\mathbf{b}}(t,x,v)$. To exam the integrability of $\alpha_\varepsilon^{-\beta p^*}$ we employ a change of variables

$$v\mapsto (x_{\mathbf{b}}(t,x,v), t_{\mathbf{b}}(t,x,v)). \tag{9}$$

By the direct computation the Jacobian is equivalent to

$$\frac{|t_{\mathbf{b}}|^3}{\alpha_\varepsilon}\det\left[\mathrm{Id}_{3\times 3} + \frac{1}{t_{\mathbf{b}}}\int_t^{t-t_{\mathbf{b}}}\int_t^s\nabla_v X(\tau;t,x,v)\nabla_x^2\phi_f(\tau;X(\tau;t,x,v))\mathrm{d}\tau\mathrm{d}s\right]. \tag{10}$$

Importantly we note that, for having a positive lower bound of (10) for all $t\ge 0$, it is indispensable to have the exponential decay of $\phi(t)$ in C^2 *plus*

$$|\nabla_v X(\tau;t,x,v)| < C|t-\tau| \ \text{ for all } \ \tau\le t. \tag{11}$$

In the presence of a time-dependent potential we have a *non-autonomous system* of $(\nabla_v X, \nabla_v V)$ from (3). We note that without (8) there could be an exponential growth factor in $|\nabla_v X(\tau;t,x,v)|$ so that the change of variables (9) is only valid

for a small time interval. To overcome this difficulty we discretize the time interval and derive an explicit formula of the discretized system, which leads to the linear growth bound and therefore verify (9).

If the power of weight in (7) is less than 1 then $\alpha_\varepsilon(t,x,v)$-factor in the Jacobian (10) cancels out the singularity and leads $\alpha_\varepsilon(t,x,v)^{1-\beta p^*}/t_{\mathbf{b}}^3$ instead. Then we carefully use a lower bound of $t_{\mathbf{b}} < C\frac{|x_{\mathbf{b}}-x|}{\max|V|}$ and a bound $\alpha < C\frac{|(x-x_{\mathbf{b}})\cdot n(x_{\mathbf{b}})|}{t_{\mathbf{b}}}$ to have, as long as $\beta p^* < 1$,

(12)
$$\int_{|v|<C1} \alpha_\varepsilon^{-\beta p^*}\mathrm{d}v < C\int_{\text{boundary}} \frac{|(x-x_{\mathbf{b}})\cdot n(x_{\mathbf{b}})|^{1-\beta p^*}}{|x-x_{\mathbf{b}}|^{3-\beta p^*}}\mathrm{d}x_{\mathbf{b}} + \text{good terms} < \infty.$$

This estimate is good to control the first term of (5) but has some restriction on p in controlling the second term of (5) due to $\frac{1}{|v-u|}$-singularity of K. They bound $K(\frac{1}{\alpha_\varepsilon^\beta}\alpha_\varepsilon^\beta\nabla_{x,v}f)$ by $\left\|\frac{1}{|v-u|}\frac{1}{\alpha_\varepsilon^\beta(u)}\right\|_{L_u^{p^*}} \times \|\alpha_\varepsilon^\beta\nabla_{x,v}f\|_{L_u^p}$. Then they view $\left\|\frac{1}{|v-u|}\frac{1}{\alpha_\varepsilon^\beta(u)}\right\|_{L_u^{p^*}}$ as $\left|\frac{1}{|v-\cdot|^{p^*}} * \frac{1}{\alpha_\varepsilon(\cdot)^{\beta p^*}}\right|^{1/p^*}$. Then by the Hardy-Littlewood-Sobolev inequality

$$\left\|\frac{1}{|v-\cdot|^{p^*}} * \frac{1}{\alpha_\varepsilon(\cdot)^{\beta p^*}}\right\|_{L^{p/p^*}} < C\left\|\frac{1}{\alpha_\varepsilon^\beta}\right\|_{L^{3/2}_{loc}} + \text{good terms}. \tag{13}$$

This causes a restriction $\beta < \frac{2}{3}$ from (12) and then $p < 6$.

Finally we use L^2-L^∞ bootstrap argument of [1, 2, 6] to derive an exponential decay of f in L^∞. The key of this process is to derive a positive lower bound of

$$\det\left(-(t-s)\mathrm{Id}_{3\times 3} - \int_t^s\int_t^\tau \frac{\partial X(\tau';t,x,v)}{\partial v}\nabla_x^2\phi(\tau', X(\tau';t,x,v))\mathrm{d}\tau'\mathrm{d}\tau\right),$$

except for a small set of s. Again as (10) we need (8) and (11). Finally they can prove an exponential growth bound of $\|\alpha_\varepsilon^\beta\nabla_{x,v}f\|_{L^p(\Omega\times\mathbb{R}^3)}$ by the Gronwall inequality and an exponential decay of f in L^∞ and therefore (8), which deduce (9) by an interpolation in Hölder spaces.

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From the Hartree dynamics to the Vlasov equation with singular interaction potentials

Chiara Saffirio

Consider the quantum dynamics of N fermions interacting through a real-valued two-body potential $V : \mathbb{R}^3 \to \mathbb{R}$ in a mean-field regime. It is decribed by the N-body Schrödinger equation

$$\begin{cases} i\,\varepsilon\,\partial_t\psi_{N,t} = \left[\sum_{j=1}^{N} -\varepsilon^2\,\Delta_{x_j} + \frac{1}{2N}\sum_{i,j=1}^{N} V(x_i - x_j)\right]\psi_{N,t}\,, \\ \psi_{N,0} = \psi_N\,, \end{cases} \tag{1}$$

where the wave function $\psi_{N,t} \in L^2_a(\mathbb{R}^{3N})$, the space of square integrable antisymmetric functions. In Eq.n (1) we assume $\varepsilon = N^{-1/3}$ to ensure the kinetic and potential energies of typical N-particle configurations to be of the same order of magnitude, as it is the case in a mean-field situation. We underline that the scale parameter ε in Eq.n (1) plays the exact same role of the Plank constant $\hbar$, showing that the mean-field approximation in the fermionic setting is naturally linked to a semiclassical limit.

We are concerned with the asymptotic limits of the dynamics driven by (1) as $N \to \infty$ and $\varepsilon \to 0$ at positive temperature. Relevant low energy states at positive temperature are mixed quasi-free states, that is the class of initial data we will restrict our attention to.

As N is sufficiently large, but fixed, the one-particle reduced density matrix associated to a solution of (1)

$$\gamma_N^{(1)} = \mathrm{tr}_{2\dots N}|\psi_{N,t}\rangle\langle\psi_{N,t}|$$

is expected to be well approximated by the one-particle fermionic operator $\omega_{N,t}$, $\mathrm{tr}\;\omega_{N,t} = N$, $0 \le \omega_{N,t} \le 1$, solution to the Hartree equation

$$i\,\varepsilon\,\partial_t\omega_{N,t} = \left[-\varepsilon^2\,\Delta + V * \rho_t, \omega_{N,t}\right]\,, \tag{2}$$

where ρ_t is a density function defined through the diagonal kernel of the operator $\omega_{N,t}$ by the relation $\rho_t(x) = N^{-1}\omega_{N,t}(x;x)$. For regular potentials, such an approximation has been shown to hold true under semiclassical assumptions on the sequence of initial data ω_N in [1], whereas nothing is known for singular potentials in the context of mixed states.

Since Eq.n (2) still depends on N, it is natural to investigate the limit $N \to \infty$, $\varepsilon \to 0$. A vast literature is available on this subject, leading as $\varepsilon \to 0$ to the Vlasov equation

$$\partial_t W_t(x,v) + v \cdot \nabla_x W_t(x,v) - \nabla_x(V * \varrho_t)\cdot\nabla_v W_t = 0\,, \tag{3}$$

where W_t is a density on the phase-space $\mathbb{R}^3 \times \mathbb{R}^3$ and ϱ_t is the spatial density defined as $\varrho_t(x) = \int W_t(x,v)\,dv$.

In particular, in [3] the semiclassical limit to the Vlasov equation has been investigated for general potentials, including the Coulomb interaction. The convergence has to be intended in an appropriate weak topology. As a limitation of the result,

no quantitative information on the rate of convergence is provided. This is an intrinsic restriction of compactness methods used in the proof.
Our result deals with singular potentials and establishes convergence to the Vlasov dynamics in strong topology (namely the trace norm topology), providing quantitative estimates. The price to pay is a restriction on the class of singular potentials we are allowed to consider. More precisely, we deal with inverse power law potentials with exponent $\alpha \in (0, 1/2)$. In particular, we cannot handle the Coulomb case.

Notations. Before stating our main Theorem, let us introduce some notations. We define the Wigner transform of the one-particle reduced density $\omega_{N,t}$

$$W_{N,t}(x,v) = \left(\frac{\varepsilon}{2\pi}\right)^3 \int \omega_{N,t}\left(x+\varepsilon\frac{y}{2}; x-\varepsilon\frac{y}{2}\right) e^{-iv\cdot y}\, dy$$

as a function on the phase-space $\mathbb{R}^3 \times \mathbb{R}^3$. The inverse Wigner transform is known as the Weyl quantization of the function $W_{N,t}$ and it is defined as

$$\omega_{N,t}(x;y) = N \int W_{N,t}\left(\frac{x+y}{2}, v\right) e^{iv\cdot\frac{x-y}{\varepsilon}}\, dv\,.$$

To compare the Hartree and the Vlasov dynamics, we perform a Weyl transformation of the Vlasov equation (3). Therefore the solutions of (2) and (3) are defined on the same space and thus comparable. As a result we obtain

$$i\,\varepsilon\,\partial_t \widetilde{\omega}_{N,t} = \left[-\varepsilon^2\Delta, \widetilde{\omega}_{N,t}\right] + A_t\,, \tag{4}$$

where $\widetilde{\omega}_{N,t}$ is the Weyl quantization of the solution to the Vlasov equation and A_t is an operator defined through its kernel

$$A_t(x;y) = \nabla\left(V * \varrho_t\right)\left(\frac{x+y}{2}\right)\cdot(x-y)\,\widetilde{\omega}_{N,t}(x;y).$$

Main result. Consider the inverse power law interaction potential $V(x) = 1/|x|^\alpha$, with $\alpha \in (0, 1/2)$. Let ω_N be a sequence of reduced density matrices on $L^2(\mathbb{R}^3)$, with $\mathrm{tr}\ \omega_N = N$, $0 \le \omega_N \le 1$ and with Wigner transform W_N satisfying the following assumptions:

i) $W_N \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$;
ii) let $m_0 > 6$. For $m < m_0$, $\int |v|^m |W_N(x,v)|\, dxdv$ is bounded uniformly in N;
iii) for all $R > 0$ and $T > 0$, $k = 1, \dots, 5$,

$$\begin{aligned}
&\sup\{|W_N(y+vt,w)| \ :\ |y-x| \le R,\ |v-w| \le R\} \\
&\qquad\qquad \in\ L^\infty((0,T)\times\mathbb{R}^3_x; L^1(\mathbb{R}^3_v))\,, \\
&\sup\{|\nabla W_N|(y+vt,w) \ :\ |y-x| \le R,\ |v-w| \le R\} \\
&\qquad\qquad \in\ L^\infty((0,T)\times\mathbb{R}^3_x; L^1\cap L^2(\mathbb{R}^3_v))\,, \\
&\sup\{(1+x^4)|\nabla^k W_N|(y+vt,w) \ :\ |y-x| \le R,\ |v-w| \le R\} \\
&\qquad\qquad \in\ L^\infty((0,T)\times\mathbb{R}^3_x; L^1\cap L^2(\mathbb{R}^3_v))\,.
\end{aligned}$$

We denote by $\omega_{N,t}$ a solution to the Hartree equation (2) with initial data ω_N and by $\widetilde{W}_{N,t}$ the solution to the Vlasov equation (3) with initial data $\widetilde{W}_{N,0} = W_N$. Moreover, we denote by $\widetilde{\omega}_{N,t}$ the Weyl quantization of $\widetilde{W}_{N,t}$.
Then, there exists a constant $C > 0$ such that

$$\text{tr } |\omega_{N,t} - \widetilde{\omega}_{N,t}| \leq C\, N\, \varepsilon + \text{l.o.t.} \tag{5}$$

Recall that we set tr $\omega_{N,t} = N$. In this sense, estimate (5) shows that $\omega_{N,t}$ and $\widetilde{\omega}_{N,t}$ are close in the large N limit, being their difference smaller than their trace norms by a factor $\varepsilon = N^{-1/3}$. Further, observe that on the one hand assumptions i) – iii) ensure existence, uniqueness and regularity of $\widetilde{W}_{N,t}$ by an adaptation of the well-posedness result provided in [4]. On the other hand, they are responsible for restricting the class of initial data to mixed states, preventing us to include the zero temperature case in our analysis.

The proof relies on a generalized version of Fefferman-de la Llave representation formula, which allows to write the interaction potential in the following way

$$\frac{1}{|x-y|^\alpha} = \frac{4}{\pi} \int_0^\infty \frac{dr}{r^{4+\alpha}} \int e^{-|x-z|^2/r^2}\, e^{-|y-z|^2/r^2}\, dz\,. \tag{6}$$

By performing the time derivative of tr $|\omega_{N,t} - \widetilde{\omega}_{N,t}|$, using Eq.n (4) and formula (6), a Grönwall type estimate leads to (5).

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On the Nordheim equation "after condensation"

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(joint work with E. Cortés)

The Nordheim equation (cf.[5]) for an isotropic dilute gas of bosons may be written:

$$\begin{aligned} &\partial_t f(t,x_1) = I_3(f)(t,x_1) \\ &I_3(f)(t,x_1) = \iint_{D(x_1)} W(x_1,x_3,x_4) q(f)(x_1,x_3,x_4) dx_3 dx_4 \\ &q(f) = [(1+f_1)(1+f_2)\, f_3 f_4 - (1+f_3)(1+f_4) f_1 f_2] \end{aligned} \tag{1}$$

where $f_k = f(t, x_k)$, $k = 1, 2, 3, 4$ and

$$W(x_1, x_3, x_4) = \frac{\min\left\{\sqrt{x_1}, \sqrt{x_2}, \sqrt{x_3}, \sqrt{x_4}\right\}}{\sqrt{x_1}} \quad , \quad x_2 = x_3 + x_4 - x_1$$
$$\text{for all } x_1 \geq 0: \quad D(x_1) = \{x_3 \geq 0,\ x_4 \geq 0;\ x_3 + x_4 \geq x_1\}.$$

If we define the new variable:

$$G(t, x) = \sqrt{x} f(t, x).$$

the weak formulation of (1) in terms of G reads as follows:

$$(2) \quad \frac{d}{dt} \int\limits_{[0,\infty)} G(t, x)\, \varphi(t, x)\, dx = \mathcal{Q}_3(G, \varphi),\ \forall \varphi \in C_b^2([0, \infty))$$
$$\mathcal{Q}_3(G, \varphi) = \iiint\limits_{[0,\infty)^3} \frac{G_1 G_2 G_3}{\sqrt{x_1 x_2 x_3}} \Phi\left[\varphi_4 + \varphi_3 - \varphi_1 - \varphi_2\right] dx_1 dx_2 dx_3$$
$$+C \iiint\limits_{[0,\infty)^3} \frac{G_1 G_2}{\sqrt{x_1 x_2}} \Phi\left[\varphi_4 + \varphi_3 - \varphi_1 - \varphi_2\right] dx_1 dx_2 dx_3$$
$$\Phi = \min\{x_1, x_2, x_3, x_4\},\ x_4 = x_1 + x_2 - x_3.$$

The global existence of solutions G to such weak formulation was first proved in [7] for initial data f_0 such that $\sqrt{x} f_0$ is a non negative bounded measure on $[0, \infty)$ with finite moment of order one. The solutions where also proved to conserve the moment of order zero and one.

On the other hand, it has been proved in [3] that, given any $M > 0$, $E > 0$ there exists initial data $f_0 \in L^\infty\left(\mathbb{R}^+; (1 + x)^\gamma\right)$ with $\gamma > 3$, satisfying

$$(3) \qquad \int_{\mathbb{R}^+} f_0(x) \sqrt{x} dx = N, \quad \int_{\mathbb{R}^+} f_0(x)\, x^{3/2} dx = E$$

and such that there exists a global weak solution f and positive times $0 < T_* < T_0$ such that:

$$(4) \qquad \sup_{0<t\leq T_*} \|f(t, \cdot)\|_{L^\infty(\mathbb{R}^+)} < \infty$$

$$(5) \qquad \sup_{T_*<t\leq T_0} \int_{\{0\}} \sqrt{x} f(t, x)\, dx > 0.$$

In terms of the measure G, the condition (5) shows that a Diracl delta has formed in finite time. Once the Dirac mass at the origin $G(t, \{0\})\delta_0$ has appeared we are interested in its behavior and the way it interacts with the solution $G(t, x)$ for $x > 0$. That question has been considered by several authors both in the literature of physics (cf. [2], [4], [9]) and of mathematics (cf. [6], [1]).

Let us then write:

$$G(t) = n(t)\delta_0 + g(t), \quad n(t) = G(t, \{0\}).$$

and consider only a simplified problem, keeping only the term of the weak formulation (2) involving the condensate. We then obtain the following system:

$$\frac{d}{dt}\int_{[0,\infty)}\varphi(x)g(t,x)dx = n(t)(\mathscr{Q}_3(g,\varphi) - T(g)(t)\varphi(0)), \quad \forall\varphi\in C_b^1([0,\infty)) \tag{6}$$

$$n'(t) = -n(t)\big(M_{1/2}(g)(t) - T(g)(t)\big). \tag{7}$$

where the limit:

$$T(g)(t) = \lim_{\varepsilon\to 0}\iint\limits_{(0,\infty)^2}\frac{\Lambda_{\varphi_\varepsilon}(x,y)}{\sqrt{xy}}g(t,x)g(t,y)dxdy,\ \ \varphi_\varepsilon(x) = \varphi\left(\tfrac{x}{\varepsilon}\right)$$

is proved to exist for φ any non negative convex function, $\varphi\in C_b^{1,1}[0,\infty)$ such that $\varphi = 1$ and $\lim_{x\to\infty}\sqrt{x}\varphi_\varepsilon(x) = 0$.

We denote $\mathscr{M}_r^+([0,\infty))$ the space of nonnegative measures on $[0,\infty)$ such that

$$\int_{[0,\infty)}(1+x^r)G(x)dx < \infty.$$

We first show the following existence result.

Theorem *For all initial data $G_0\in\mathscr{M}_1^+([0,\infty))$, there exists a solution*

$$G\in C((0,\infty);\mathscr{M}_r^+([0,\infty)),\ \forall r>0$$

of (6), (7) such that $G(0) = G_0$. This solution satisfies also:

$$N = \int\limits_{[0,\infty)}G(t,x)dx,\ \ E = \int\limits_{[0,\infty)}xG(t,x)dx.$$

$$\forall\varepsilon>0,\forall t>0,\exists C(\varepsilon,t)>0;\forall r>0:$$

$$(i)\ \int_0^t\int_{(0,r]}G(s,x)x^{-\frac{1}{2}+\varepsilon}dxds\le C(\varepsilon,t)r^\varepsilon$$

$$(ii)\ \lim_{\varepsilon\to 0}C(\varepsilon,t) = \infty.$$

The second result is about the term $T(g)$. It is known that if $g(x) = x^{-1/2}$, then $T(g) = \pi/6$ (cf. [8]), and the same holds if $\lim_{x\to 0}\sqrt{x}g(x) = C>0$ (cf. [10]). It was observed also that if $g(t)\in L^1(0,\infty)$ and $x=0$ is a Lebesgue point of $g(t)$ then $T(g(t)) = 0$ (cf. [6]).

Theorem 2. *If G is the weak solution obtained in Theorem 1, then*

$$T(g)(t)>0,\ \ \forall t>0.$$

The las result is the following property of the function $n(t)$.

Theorem 3. *Suppose that the initial data f_0 is such that N and E defined in (3) satisfy:*

$$\frac{E}{N^{5/3}} > 6^{2/3}.$$

Then,

$$\int_0^\infty n(t)dt < \frac{3N^{\frac{3}{2}}M_2(0)}{E(E^{3/2}-6N^{5/2})}.$$

The quantity $E/N^{5/3}$ is, up to some constants, the quotient of the temperature of the data G_0 and the critical temperature:

$$\frac{T}{T_c} = \frac{(2\pi)^{\frac{1}{3}}}{3}\frac{\zeta(3/2)^{5/3}}{\zeta(5/2)}\frac{E}{N^{5/3}}$$

$$\frac{(2\pi)^{\frac{1}{3}}}{3}\frac{\zeta(3/2)^{5/3}}{\zeta(5/3)} \approx 2,27202, \text{ and } \frac{(2\pi)^{\frac{1}{3}}}{3}\frac{\zeta(3/2)^{5/3}}{\zeta(5/3)}6^{2/3} \approx 7,5020.$$

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The Cauchy problem for a quantum Boltzmann system for bosons at very low temperature

Ricardo Alonso

(joint work with Irene Gamba, Minh-Binh Tran)

A system that describes the dynamics of a Bose-Einstein condensate and a cloud of quasi-particles consists of a quantum Boltzmann equation describing the cloud and the Gross-Pitaevskii equation describing the condensate wave function

$$\begin{aligned} \partial_t f(t,p) &= Q(f,n_c)(t,p)\,, \\ \frac{\mathrm{d}n_c}{\mathrm{d}t}(t) &= -\int \mathrm{d}p\, Q(f,n_c)(t,p)\,. \end{aligned} \tag{1}$$

Here $f(t,p)$ and $n_c(t)$ are the quasi-particle density and the condensate mass respectively. The equations are posed for all time $t > 0$ and velocity $p \in \mathbb{R}^3$. For

technical reasons we consider only radially symmetric data $f_0 := f_0(|p|)$ for the quasi-particles. The initial mass of the Bose-Einstein condensate is necessarily positive $n_c(0) > 0$ for the validity of the model.
We solve the Cauchy problem for the system (1) in the so-called low temperature regime. In such regime, the Bogoliubov excitation energy is well approximated by

$$E(p) := \sqrt{\frac{|p|^4}{4m^2} + \frac{gn_c}{m}|p|^2} \approx \sqrt{\frac{gn_c}{m}}\,|p| =: c\,|p|\,,$$

and the transition probability kernel is approximated, up to first order, by

$$|\mathcal{M}|^2 = \frac{9c}{64\pi^2 m n_c^2}|p||p_1||p_2|\,.$$

In the aforementioned formulas m stands for the mass of the particles and g is an interaction coupling constant. We develop a local existence, with an explicit minimal interval for existence, and uniqueness result for general data with finite energy, that is $\int \mathrm{d}p|p|f_0 < \infty$, by means of abstract ODE's theory in Banach spaces proposed in [3] in the context of kinetic equations. In fact, we generalize such approach to operators with memory which are needed in the present context. In addition, we give an explicit threshold in terms of f_0 on the minimal value of the initial mass of the condensate $n_c(0)$ for global existence and stability of the condensate and prove qualitative properties of the quasi-particle density distribution's tails. In conclusion, our proof shows that all the conditions imposed in (f, n_c) for the validity of the model in its formal derivation are met.

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Kinetic modelling of strongly magnetized tokamak plasmas with mass disparate particles. The electron Boltzmann relation.

Claudia Negulescu

The subject matter of the present work is the formal obtention of the electron Boltzmann relation, from the underlying kinetic description of strongly confined tokamak plasmas with mass disparate particles. At the same time, the ions reach also an asymptotic limit, namely the gyrokinetic or hydrodynamic models.

Fusion plasmas are weakly collisional, due to high temperatures and low densities, such that the kinetic framework is the appropriate approach for their detailed description. In particular this amounts to solve a coupled system of two Vlasov or Boltzmann equations for the ion/electron distribution functions $f_{i,e}$ together with the Poisson equation for the electrostatic potential ϕ (or Maxwell's equations in the electromagnetic case). The difficulty with a fully kinetic treatment is however the high-dimensionality of the phase-space ($6D$). Furthermore the presence of multiple spatio-temporal scales makes the problem even more complicated, rendering it inaccessible for numerical simulations. In particular, these various scales impose the use of very small time and space steps in numerical simulations in order to follow all the microscopic motions. In our particular case, apart the strong magnetic field, it is the small mass ratio $m_e/m_i \approx 10^{-4}$ of the particles which induces disparate scales and hence difficulties; for a typical tokamak plasma with similar electron and ion temperatures the electron dynamics is faster than the ion dynamics, the ratio of the thermal velocities being given by $v_{th,e}/v_{th,i} = \sqrt{\frac{m_i}{m_e}} \approx 10^2$. This fact poses rather restrictive time-step constraints related to the fast electron motion, when a standard discretization of the bi-kinetic system is used, meaning that the numerical stability requires a CFL-condition of the type $v_{th,e}\Delta t \leq \Delta x$.

The fully-kinetic system contains however too many irrelevant spatio-temporal scales for the study of many interesting plasma processes. To redress this situation, a more macroscopic approach has to be adopted, eliminating the unnecessary fast dynamics and keeping the complete low-frequency physics. Such a macroscopic or reduced model is obtained via an asymptotic analysis, letting some specific parameters tend towards zero. In our particular case, we are only interested in following the plasma evolution on the large ion time-scales. At these time-scales the electrons attain a certain macroscopic thermal equilibrium, namely the electron Boltzmann-regime. In some words, this Boltzmann relation is obtained by assuming zero electron inertia ($m_e \to 0$) and zero viscosity in the "parallel" electron equation of motion ("parallel" with respect to the strong magnetic field), leading to the relation

$$\nabla_{||} p_e = -q\, n_e\, \mathbf{E}_{||}\,, \qquad \mathbf{E} = -\nabla\phi\,. \tag{1}$$

This relation indicates that the pressure-gradient and the electrostatic forces acting on the electrons (parallel to the magnetic field) are in balance. Moreover, rapid parallel thermal conduction assures that $\nabla_{||} T_e \sim 0$, such that with the thermodynamic equation of state $p_e = n_e\, k_B\, T_e$, one obtains

$$n_e(t,\mathbf{x}) = c(t,\mathbf{x}_\perp)\, \exp\left(\frac{q\,\phi(t,\mathbf{x})}{k_B\, T_e(t,\mathbf{x}_\perp)}\right)\,, \quad \mathbf{x} = \mathbf{x}_\perp + \mathbf{x}_{||} \in \mathbf{R}^3\,,\ \ t \in \mathbf{R}^+\,. \tag{2}$$

Equation (2) is the so-called Boltzmann relation or adiabatic response, relating the electron density to the electric potential. Here $c(t,\mathbf{x}_\perp)$ and $T_e(t,\mathbf{x}_\perp)$ are functions to be determined from the remaining transport equations as well as initial and boundary conditions; they do not depend on the parallel coordinate $\mathbf{x}_{||}$. Once c

and T_e are known, the relation (2) can be inserted into the Poisson equation for the electrostatic potential, which can then be coupled to a model for the ion dynamics (kinetic or fluid). Such a procedure is extensively applied in plasma simulations, with the purpose to study the ITG (ion-temperature-gradient) and TEM (trapped electron mode) micro-instabilities and associated turbulences, and this because the adiabatic electron response leads to large reductions in computational costs. Its practical use is due to the time-scale separation between ion and electron dynamics.

The aim of the here presented work was to understand how to obtain from a kinetic electron description the Boltzmann relation (2) in the so-called adiabatic limit. This asymptotic passage from the kinetic to the adiabatic regime can be very interesting for simulations in regions where the electron adiabatic response is violated, for example near the edge of the tokamak. There, one has to go back to the more precise kinetic electron description, where the need to find the way how to couple these two regimes via a suitable limit procedure.

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On the global dynamics of the inhomogeneous Boltzmann equaiton without angular cutoff: Hard potentials and Maxwellian molecules

Ling-Bing He

(joint work with Jin-Cheng Jiang)

We consider the global dynamics of the original Boltzmann equation without angular cutoff on the torus for the hard potentials and Maxwellian molecules. The new idea to solve the problem is the energy-entropy method which characterizes the propagation of the regularity, H-theorem and the interplay between the energy and the entropy. Our main results are as follows:

(1). We present a unified framework to prove the well-posedness for the original Boltzmann equation for both angular cutoff and without cutoff in weighted Sobolev spaces with polynomial weights. As a consequence, we obtain the propagation of

the regularity and an explicit formula for the asymptotics of the equation from angular cutoff to non-cutoff.

(2). We describe the global dynamics of the equation under the almost optimal assumption on the solution which ensures that the Boltzmann collision operator behaves like a fractional Laplace operator for the velocity variable. In particular, we obtain a new mechanism for the convergence of the solution to its equilibrium with quantitative estimates.

(3). We prove that any global and smooth solution to the equation is stable, i.e., any perturbed solution will remain close to the reference solution if initially they are close to each other. Here we remove the assumption that perturbed solution and the reference solution should have the same associated equilibrium.

Our approach incorporates almost all the fundamental properties of the equation: the entropy production inequality, the immediately appearance of pointwise lower bound of the solution, the smoothing property of the positive part of the collision operator, averaging lemma for the transport equation, the Povnzer inequality for L^1 moment and the sharp bounds for the collision operator.

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Quantitative estimates of propagation of chaos for stochastic systems with $W^{-1,\infty}$ kernels

Pierre-Emmanuel Jabin

(joint work with Zhenfu Wang)

We derive quantitative estimates proving the propagation of chaos for large stochastic systems of interacting particles. We obtain explicit bounds on the relative entropy between the joint law of the particles and the tensorized law at the limit. We have to develop for this new laws of large numbers at the exponential scale. But our result only requires very weak regularity on the interaction kernel in the negative Sobolev space $\dot{W}^{-1,\infty}$, thus including the Biot-Savart law and the point vortices dynamics for the 2d incompressible Navier-Stokes.

To be more specific we consider large systems of N indistinguishable point-particles given by the coupled stochastic differential equations (SDEs)

$$dX_i = F(X_i)dt + \frac{1}{N}\sum_{j\neq i} K(X_i - X_j)\,dt + \sqrt{2\sigma_N}\,dW_t^i, \quad i = 1, \cdots, N \tag{1}$$

where for simplicity $X_i \in \Pi^d$, the d-dimensional torus, the W^i are N independent standard Wiener Processes (Brownian motions) in $\mathbb{R}^d$ and the stochastic term in (1) should be understood in the Itô sense.

Our goal is hence to derive explicit, quantitative estimates comparing System (1) to the *mean field limit* $\bar{\rho}$ solving

$$\partial_t \bar{\rho} + \mathrm{div}_x(\bar{\rho}\,[F + K \star_x \bar{\rho}]) = \sigma\,\Delta\bar{\rho}. \tag{2}$$

Such estimates in particular imply the propagation of chaos in the limit $N \to \infty$. But precisely because they are quantitative, they also characterize the reduction of complexity of System (1) for large and finite N.

A guiding motivation of interaction kernel K in our work is given by the Biot-Savart law in dimension 2, namely

$$K(x) = \alpha\,\frac{x^\perp}{|x|^2} + K_0(x), \tag{3}$$

where $x^\perp$ denotes the rotation of vector x by $\pi/2$ and where K_0 is a smooth correction to periodize K on the torus represented by $[-1/2,\ 1/2]^d$.

If $F = 0$, the limiting equation (2) becomes

$$\partial_t \omega + K \star_x \omega \cdot \nabla_x \omega = \sigma\,\Delta\omega, \tag{4}$$

where we now write on $\omega(t,x)$, using the classical notation for the vorticity of a fluid. Eq. (4) is invariant by the addition of a constant $\omega \to \omega + C$. We may hence assume that $\int_{\Pi^d} \omega = 0$ and Eq. (4) is then equivalent to the 2d incompressible Navier-Stokes system.

Following the basic approach introduced in [1], our main idea is to use relative entropy methods to compare the *coupled law* $\rho_N(t, x_1, \ldots, x_N)$ of the whole system (1) to the tensorized law

$$\bar{\rho}_N(t, x_1, \ldots, x_N) = \bar{\rho}^{\otimes^N} = \Pi_{i=1}^N \bar{\rho}(t, x_i),$$

consisting of N independent copies of a process following the law $\bar{\rho}$, solution to the limiting equation (2).

As our estimates carry over ρ_N, we do not consider directly the system of SDEs (1) but instead work at the level of the Liouville equation

$$\partial_t \rho_N + \sum_{i=1}^N \mathrm{div}_{x_i}\left(\rho_N\,\left(F(x_i) + \frac{1}{N}\sum_{j=1}^N K(x_i - x_j)\right)\right) = \sum_{i=1}^N \sigma_N\,\Delta_{x_i}\rho_N, \tag{5}$$

where and hereafter we use the convention that $K(0) = 0$. The law ρ_N encompasses all the statistical information about the system. Given that it is set in $\Pi^{d\,N}$ with $N >> 1$, the observable statistical information is typically contained in the marginals

$$\rho_{N,k}(t, x_1, \ldots, x_k) = \int_{\Pi^{d\,(N-k)}} \rho_N(t, x_1, \ldots, x_N)\,dx_{k+1} \ldots\, dx_N. \tag{6}$$

Our method revolves around the control of the rescaled relative entropy

$$\mathcal{H}_N(\rho_N \mid \bar{\rho}_N)(t) = \frac{1}{N}\int_{\Pi^{d\,N}} \rho_N(t, X^N)\,\log\frac{\rho_N(t, X^N)}{\bar{\rho}_N(t, X^N)}\,dX^N, \tag{7}$$

while our main result is the explicit estimate

Theorem 1. *Assume that* $div F \in L^\infty(\Pi^d)$, *that* $K \in \dot{W}^{-1,\infty}(\Pi^d)$ *with* $div K \in \dot{W}^{-1,\infty}$. *Assume that* $\sigma_N \geq \underline{\sigma} > 0$. *Assume moreover that* ρ_N *is an entropy solution to Eq.* (5). *Assume finally that* $\bar{\rho} \in L^\infty([0,\ T],\ W^{2,p}(\Pi^d))$ *for any* $p < \infty$ *solves Eq.* (2) *with* $\inf \bar{\rho} > 0$ *and* $\int_{\Pi^d} \bar{\rho} = 1$. *Then*

$$\mathcal{H}_N(\rho_N \mid \bar{\rho}_N)(t) \leq e^{\bar{M}\,(\|K\|+\|K\|^2)\,t} \left(\mathcal{H}_N(\rho_N^0 \mid \bar{\rho}_N^0) + \frac{1}{N} + \bar{M}(1 + t\,(1 + \|K\|^2))\,|\sigma - \sigma_N| \right),$$

where we denote $\|K\| = \|K\|_{\dot{W}^{-1,\infty}} + \|div K\|_{\dot{W}^{-1,\infty}}$ *and* $\bar{M}$ *is a constant which only depends on*

$$\bar{M}\left(d,\ \underline{\sigma},\ \inf \bar{\rho},\ \|\bar{\rho}\|_{W^{1,\infty}},\ \sup_{p \geq 1} \frac{\|\nabla^2 \bar{\rho}\|_{L^p}}{p},\ \frac{1}{N} \int_{\Pi^{d\,N}} \rho_N^0\ \log \rho_N^0,\ \|div F\|_{L^\infty} \right).$$

It is then possible to derive from Theorem 1 the strong propagation of chaos as per

Corollary 1. *Under the assumptions of Theorem 1, if* $\mathcal{H}_N(\rho_N^0 \mid \bar{\rho}_N^0) \to 0$ *as* $N \to \infty$, *then over any fixed time interval* $[0,\ T]$

$$\mathcal{H}_N(\rho_N \mid \bar{\rho}_N) \longrightarrow 0, \quad as\ N \to \infty.$$

As a consequence considering any finite marginal at order k, *one has the strong propagation of chaos*

$$\|\rho_{N,k} - \bar{\rho}^{\otimes^k}\|_{L^\infty([0,\ T],\ L^1(\Pi^{d\,k}))} \longrightarrow 0.$$

Finally in the particular case where $\sup_N N\,\mathcal{H}_N(\rho_N^0 \mid \bar{\rho}_N^0) = H < \infty$, *and where* $\sup_N N\,|\sigma_N - \sigma| = S < \infty$, *then one has that, for some constant* C *defined in Theorem 1,*

$$\|\rho_{N,k} - \bar{\rho}^{\otimes^k}\|_{L^\infty([0,\ T],\ L^1(\Pi^{d\,k}))} \leq \frac{C}{\sqrt{N}}. \tag{8}$$

This applies in particular to the Biot-Savart law and the vortex dynamics for 2d Navier-Stokes since we can write

$$K = \operatorname{div} V, \quad V = \begin{bmatrix} -\phi \arctan \frac{x_1}{x_2} + \psi_1 & 0 \\ 0 & \phi \arctan \frac{x_2}{x_1} + \psi_2 \end{bmatrix},$$

where one can choose ϕ smooth with compact support in the representative of the torus $(-1/2,\ 1/2)^2$, and $(\psi_1,\ \psi_2)$ a corresponding smooth correction to periodize V. Therefore K satisfies the assumptions of Theorem 1.

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On a Boltzmann equation for Haldane statistics

Anne Nouri

(joint work with Leif Arkeryd)

The study of quantum quasi-particles at low temperatures including their statistics, is a frontier area in modern physics. In a seminal paper Haldane [5] proposed a definition based on a generalization of the Pauli exclusion principle for fractional quantum statistics.

The talk is devoted to a study of quantum quasi-particles obeying Haldane statistics in a fully non-linear kinetic Boltzmann equation model with large initial data on a torus. Strong L^1 solutions are obtained for the Cauchy problem. The main results concern existence, uniqueness and stabililty. Depending on the space dimension and the collision kernel, the results obtained are local or global in time. We consider the Cauchy problem associated to the Boltzmann equation in a torus $[0,1]^k$, $k \in \{1,2,3\}$, for quantum particles obeying the Haldane statistics,

$$(1)\quad \partial_t f(t,x,v) + \bar{v}\cdot\nabla_x f(t,x,v) = Q(f)(t,x,v),\ (t,x,v) \in [0,+\infty[\times[0,1]^k \times \mathbb{R}^3,$$

$$(2)\quad f(0,x,v) = f_0(x,v),$$

where

$$\bar{v} = (v_1)(\text{ resp. } \bar{v} = (v_1,v_2),\ \text{resp. } \bar{v} = v) \text{ for } k = 1(\text{ resp. } k = 2,\ \text{resp. } k = 3).$$

The collision operator Q is given by

$$Q(f)(v) = \int_{\mathbb{R}^3\times S^2} B(\mid v - v_* \mid, n)\big(f'f'_*F_\alpha(f)F_\alpha(f_*) - ff_*F_\alpha(f')F_\alpha(f'_*)\big)dv_*dn,$$

with the filling factor F_α defined by

$$F_\alpha(f) = (1-\alpha f)^\alpha (1+(1-\alpha)f)^{1-\alpha}.$$

Strong solutions to the space-homogeneous case were obtained in [1]. Global in time strong solutions to the space-inhomogeneous case were obtained in [2] in a periodic slab (k= 1) for two-dimensional velocities. There the proof depends on the two-dimensional velocities setting. In the present paper we prove local in time well-posedness of the Cauchy problem for $k = 1$ and collision kernels similar to those used in [2], and for $k \in \{1,2,3\}$ global in time well-posedness under the supplementary assumption of very soft potential [6]. Extensions to the Fermi-Dirac equation (for fermions) and Boltzmann Nordheim equation (for bosons) are discussed.

1. The main results.

With $cos\theta = n \cdot \frac{v-v_*}{|v-v_*|}$, the kernel $B(|v-v_*|,\theta)$ is assumed measurable with

$$0 \le B \le B_0, \tag{3}$$

for some $B_0 > 0$. It is also assumed for some $\gamma, \gamma', c_B > 0$, that

$$B(|v-v_*|,\theta) = 0 \text{ for } |\cos\theta| < \gamma', \text{ for } 1 - |\cos\theta| < \gamma', \text{ and for } |v-v_*| < \gamma, \tag{4}$$

and that

$$\int B(u,\theta)d\theta \ge c_B > 0 \text{ for } u \ge \gamma. \tag{5}$$

The initial datum $f_0(x,v)$,

$$\text{periodic in } x, \text{is a measurable function with values in} [0, \frac{1}{\alpha}], \tag{6}$$

and such that for some positive constants c_0 and $\tilde{c}_0$,

$$(1+|v|^2) f_0(x,v) \in L^1([0,1]^k \times \mathbb{R}^3), \tag{7}$$

$$\int \sup_{x\in[0,1]^k} f_0(x,v)dv = c_0, \tag{8}$$

$$\int \sup_{x\in[0,1]^k} |v|^2 f_0(x,v)dv = \tilde{c}_0, \tag{9}$$

$$\text{for any subset } X \text{ of } \mathbb{R}^3 \text{ of positive measure, } \int_X \inf_{x\in[0,1]^k} f_0(x,v)dv > 0. \tag{10}$$

Denote by

$$f^\sharp(t,x,v) = f(t,x+t\bar{v},v),\ (t,x,v) \in [0,+\infty[\times[0,1]^k \times \mathbb{R}^3,\ \bar{v} = (v_1,\cdots,v_k) \in \mathbb{R}^k. \tag{11}$$

Strong solutions to (1) are considered in the following sense.

Definition 1. *f is a strong solution to (1) on the time interval I if*

$$f \in \mathcal{C}^1(I; L^1([0,1]^k \times \mathbb{R}^3)),$$

and

$$\frac{d}{dt} f^\sharp = \big(Q(f)\big)^\sharp, \quad \text{on } I \times [0,1]^k \times \mathbb{R}^3. \tag{12}$$

The main results are given in the following theorems.

Theorem 1.
Under the assumptions (3)-(8) and (10), there is a time $T > 0$, so that there exists a unique periodic in x, strong solution $f \in \mathcal{C}^1([0,T[; L^1([0,1] \times \mathbb{R}^3))$ of (1)-(2). It depends continuously in $\mathcal{C}([0,T[; L^1([0,1] \times \mathbb{R}^3))$ on the initial L^1-datum. It conserves mass, momentum and energy.

Theorem 2.
Under the assumptions (3)-(10) and the supplementary assumption of very soft collision kernels [6]*,*
(13)
$B(u,\theta) = B_1(u)B_2(\theta)$ *with* $|B_1(u)| \leq c|u|^{-3-\eta}$ *for some* $\eta > 0$*, and* B_2 *bounded,*

there exists a unique periodic in x*, strong solution*
$f \in \mathcal{C}^1([0,\infty[; L^1([0,1]^k \times \mathbb{R}^3))$ *of (1)-(2) for* $k \in \{1,2,3\}$*. For any* $T > 0$*, it continuously depends in* $\mathcal{C}([0,T]; L^1([0,1]^k \times \mathbb{R}^3))$ *on the initial* L^1*-datum.*
It conserves mass, momentum and energy.

The proofs of both theorems rely on the control of the mass density

$$\int \sup_{x \in [0,1]^k \times \mathbb{R}^3} f(t,x,v)dv,$$

and an introduction of an initial layer where $f(t,\cdot,\cdot) \leq \frac{1}{\alpha} - bt$ for some positive constant b.

2. Remark and Extensions.

Theorem 1 is restricted to the slab case, since its proof uses an estimate for the Bony functional only valid in one space dimension.
Theorems 1 and 2 also hold with the same proofs in the fermion case where $\alpha = 1$, in particular giving strong solutions to the Fermi-Dirac equation.
Theorems 1 and 2 also hold with a limit procedure when $\alpha \to 0$ in the boson case where $\alpha = 0$, in particular giving strong solutions to the Boltzmann Nordheim equation [4].

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Multispecies mixtures close to equilibrium

Marc Briant
(joint work with Esther S. Daus)

1. The Multispecies model

The multispecies Boltzmann equation rules the dynamics of a dilute gas on the torus $\mathbb{T}^3$ composed of N different species of chemically non-reacting mono-atomic particles, and undergoing solely binary collisions. It reads, stated on $\mathbb{R}^+ \times \mathbb{T}^3 \times \mathbb{R}^3$,

$$\forall\, 1 \leq i \leq N, \quad \partial_t F_i(t,x,v) + v \cdot \nabla_x F_i(t,x,v) = Q_i(\mathbf{F})(t,x,v).$$

Note that the distribution function of the system is given by the vector $\mathbf{F} = (F_1, \dots, F_N)$, with F_i describing the i^{th} species at time t, position x and velocity v.
The Boltzmann operator $\mathbf{Q}(\mathbf{F}) = (Q_1(\mathbf{F}), \dots, Q_N(\mathbf{F}))$ is given for all i by $Q_i(\mathbf{F}) = \sum_{j=1}^{N} Q_{ij}(F_i, F_j)$,where Q_{ij} describes interactions between particles of either the same $(i = j)$ or of different $(i \neq j)$ species and are local in time and space. The core difference with the monospecies case are the laws of elastic collisions that are no longer symmetric in v and v_*:

$$m_i v + m_j v_* = m_i v' + m_j v'_*,$$
$$\frac{1}{2} m_i |v|^2 + \frac{1}{2} m_j |v_*|^2 = \frac{1}{2} m_i |v'|^2 + \frac{1}{2} m_j |v'_*|^2 .$$

2. Loss of symmetry and perturbative regime

At first sight, it seems that the multispecies Boltzmann system is hardly more than a vectorial version of the classical monospecies Boltzmann equation ($N = 1$). However, if we have been able to recover the latest results known for $N = 1$ to general N and masses m_i, the extraspecies collisions generate a lot of breaking of standard symmetries. In particular it implies conservation of the total number density of each species, but preservation momentum and energy in the whole only:

$$\forall t \geq 0, \quad c_{\infty,i} = \int_{\mathbb{T}^3 \times \mathbb{R}^3} F_i(t,x,v)\, dxdv \quad (1 \leq i \leq N)$$
$$u_\infty = \frac{1}{\rho_\infty} \sum_{i=1}^{N} \int_{\mathbb{T}^3 \times \mathbb{R}^3} m_i v F_i(t,x,v)\, dxdv$$
$$\theta_\infty = \frac{1}{3\rho_\infty} \sum_{i=1}^{N} \int_{\mathbb{T}^3 \times \mathbb{R}^3} m_i |v - u_\infty|^2 F_i(t,x,v)\, dxdv,$$

where $\rho_\infty = \sum_{i=1}^{N} m_i c_{\infty,i}$. Note that this already shows intricate interactions between each species and the total mixture itself.

There exists a multi-species version of the classical H-theorem [3] which implies that the global equilibrium is the normalized Maxwellian

$$\boldsymbol{\mu} = (\mu_i)_{1\leq i\leq N} \quad \text{with} \quad \mu_i(v) = c_{\infty,i}\left(\frac{m_i}{2\pi}\right)^{3/2} e^{-m_i\frac{|v|^2}{2}}.$$

We can now construct a perturbative Cauchy theory and show exponential trend to equilibrium of solutions of the form $F_i(t,x,v) = \mu_i(v) + f_i(t,x,v)$ for all i :

$$\partial_t \mathbf{f} + v\cdot\nabla_x \mathbf{f} = \mathbf{L}(\mathbf{f}) + \mathbf{Q}(\mathbf{f}).$$

We built a Cauchy theory in $L^1_v L^\infty_x(1+|v|^k)$ with an explicit threshold $k > k_0$, thus recovering the latest results for monospecies case [4] when $N = 1$. However, new features have to be taken into account.

3. Methodology and differences with monospecies framework

Very little is known about any rigorous Cauchy theory for multi-species gases with different masses. We want to mention [1], where a linear compactness result was proved in $L^2_v(\boldsymbol{\mu}^{-1/2})$, and [2] where they proved that the operator $\mathbf{L}$ has a spectral gap in $L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)$ when $m_i = m_j$.

Our strategy can be decomposed into four main steps and we now describe each of them and their link to existing works.

Step 1: Spectral gap for the linear operator in $L^2_v\left(\boldsymbol{\mu}^{-\mathbf{1/2}}\right)$. Following [2] we exhibit and explicit spectral gap by showing that the cross-interactions between different species do not perturb too much the spectral gap that is known to exist for the diagonal operator L_{ii} (single-species operators).

Step 2: $L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$ theory for the full perturbed linear operator. The next step is to prove that $\mathbf{G} = \mathbf{L} - v\cdot\nabla_x$ generates a C^0-semigroup in $L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)$ with exponential decay. Unfortunately, a direct use of the spectral gap of $\mathbf{L}$ shows

$$\frac{d}{dt}\left\|e^{\mathbf{G}t}\mathbf{f}\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})} \leq -2\lambda_L\left\|e^{\mathbf{G}t}\mathbf{f} - \pi_{\mathbf{L}}\left(e^{\mathbf{G}t}\mathbf{f}\right)\right\|^2_{L^2_{x,v}(\boldsymbol{\mu}^{-1/2})},$$

where $\pi_{\mathbf{L}}$ is the orthogonal projection in $L^2_v\left(\boldsymbol{\mu}^{-1/2}\right)$ onto $\mathrm{Ker}(\mathbf{L})$. However, $\mathbf{G}$ has hypocoercive properties in the sense [8][6] :$\Delta\pi_L(f) \sim \partial^2\pi_L^\perp f +$ higher order terms, which can be combined with elliptic estimates to control the fluid part by the microscopic part in Sobolev spaces H^s. To avoid going to Sobolev regularity, the idea [5] is to integrate against test functions that contains a weak version of the elliptic regularity of $\pi_{\mathbf{L}}(\mathbf{f})$, recovered thanks to the transport part.

Step 3: $L^\infty_{x,v}\left(\langle v\rangle^\beta\boldsymbol{\mu}^{-1/2}\right)$ theory for the full nonlinear equation. Now we drag this L^2 theory up to a L^∞ setting. Indeed, following [7] we control the L^∞-norm by the L^2-norm along the characteristics. We have the following implicit Duhamel representation of its i^{th} component along the characteristics

$$S_{\mathbf{G}}(t)_i = e^{-\nu_i(v)t} + \int_0^t e^{-\nu_i(v)(t-s)}K_i\left[\mathbf{S}_{\mathbf{G}}(s)\right]\,ds.$$

As $\mathbf{K}$ is a kernel operator, its L^∞-norm can be transformed into a L^2 norm of $\mathbf{f}$ as long as we can invert the characteristic trajectories. There are two important

differences with the monospecies case. First, we compute new Carleman representation of the collision operator - the lack of symmetry between v' and v'_* compared to v generates new Carleman admissible sets (some become spheres) . Second, the decay of the exponential weight differs from one species to the other so one must understand that $\mathbf{K}$ mixes the exponential rate of decay among the cross-interaction between species.

Step 4: Extension to polynomial weights and $L^1_v L^\infty_x$ space. To get rid off the strong exponential weight we develop an analytic and nonlinear version of the recent [4]. The main strategy is to find a decomposition of the full linear operator $\mathbf{G} = \mathbf{G_1} + \mathbf{A}$, with $\mathbf{G_1}$ hypodissipative and $\mathbf{A}$ "regularizing". Basically, we decompose the perturbative equation into a system:

$$\begin{aligned} \partial_t \mathbf{f_1} &= \mathbf{G_1}(\mathbf{f_1}) + \mathbf{Q}(\mathbf{f_1}+\mathbf{f_2}, \mathbf{f_1}+\mathbf{f_2}) \\ \partial_t \mathbf{f_2} + v\cdot\nabla_x \mathbf{f_2} &= \mathbf{L}(\mathbf{f_2}) + \mathbf{A}(\mathbf{f_1}) \end{aligned}$$

The first equation is solved in $L^\infty_{x,v}(m)$ or $L^1_v L^\infty_x(m)$ with the initial data $\mathbf{f_0}$ thanks to the hypodissipativity of $\mathbf{G_1}$. The regularity of $\mathbf{A}(\mathbf{f_1})$ allows us to use Step 3 and thus solve the second equation with null initial data in $L^\infty_{x,v}\left(\langle v\rangle^\beta \boldsymbol{\mu}^{-1/2}\right)$. Here again recent estimates done for the monospecies case can be recovered with mixing methods for the multispecies case, including a generalized Povzner-type inequality (controlling the evolution of moments through a collision).

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Moments estimates for the discrete coagulation-fragmentation equations with diffusion

Maxime Breden
(joint work with Laurent Desvillettes, Klemens Fellner)

Coagulation-fragmentation equations describe the dynamics of clusters coalescing to build larger clusters and breaking apart into smaller pieces. Coagulation models were first introduced by Smoluchowski (see [6, 7]) and then complexified to take into account other effects like fragmentation. These models are used in numerous and diverse applications at very different scales, in physics (smoke, sprays), chemistry (polymers), or biology (hematology, animal collective behavior).

In this talk, I will consider discrete (in size) models, i.e. we assume that clusters can be of size $i \in \mathbb{N}^*$, and we denote by $c_i = c_i(t,x)$ the concentration of clusters of size i at time t and position x. We also assume that the clusters are confined in a smooth bounded domain Ω of $\mathbb{R}^N$. The concentrations c_i satisfy the following set of equations, for all $i \in \mathbb{N}^*$,

$$\begin{cases} \partial_t c_i - d_i \Delta_x c_i = Q_i(c) + F_i(c), & \text{on } [0,T] \times \Omega, \\ \nabla_x c_i \cdot \nu = 0 & \text{on } [0,T] \times \partial\Omega, \\ c_i(0,\cdot) = c_i^{in} & \text{on } \Omega. \end{cases}$$

The coagulation terms $Q_i(c)$ and the fragmentation terms $F_i(c)$ respectively write:

$$Q_i(c) = Q_i^+(c) - Q_i^-(c) = \frac{1}{2}\sum_{j=1}^{i-1} a_{i-j,j} c_{i-j} c_j - \sum_{j=1}^{\infty} a_{i,j} c_i c_j,$$

$$F_i(c) = F_i^+(c) - F_i^-(c) = \sum_{j=1}^{\infty} B_{i+j} \beta_{i+j,i} c_{i+j} - B_i c_i.$$

The nonnegative parameters B_i, $\beta_{i,j}$ and $a_{i,j}$ represent the total rate B_i of fragmentation of clusters of size i, the average number $\beta_{i,j}$ of clusters of size j produced due to fragmentation of a cluster of size i, and the coagulation rate $a_{i,j}$ of clusters of size i with clusters of size j. The fragmentation of one cluster into smaller pieces should conserve mass, clusters of size 1 should not fragment further and the coagulation rates should be symmetric, so for all $i, j \in \mathbb{N}^*$, we impose

$$i = \sum_{j=1}^{i-1} j\beta_{i,j}, \quad B_1 = 0, \quad a_{i,j} = a_{j,i} \quad \text{and} \quad a_{i,j}, B_i, \beta_{i,j} \geq 0.$$

Formally, it is easy to see that the total mass of the system should be conserved, i.e.

$$\int_\Omega \sum_{i=1}^{\infty} i c_i(t,x) dx = \int_\Omega \sum_{i=1}^{\infty} i c_i^{in}(x) dx, \quad \forall t \geq 0. \tag{1}$$

However, in some situations (1) does in fact not hold, as the total mass decreases strictly in finite time, a phenomenon called *gelation* (see for instance [5, 4]).

The usual way to get information about the occurrence or absence of gelation is to study moments of the solution. However, the presence of diffusion (and more precisely of diffusion rates d_i that depend upon the size i) makes it harder to obtain moment estimates. For instance, the first order moment (i.e. the mass) satisfies

$$\partial_t \left(\sum_{i=1}^{\infty} i c_i \right) - \Delta_x \left(\sum_{i=1}^{\infty} i d_i c_i \right) = 0,$$

which does not give a *true* equation on $\sum_{i=1}^{\infty} i c_i$ if the d_i are not all equal.

In this talk, I will explain how so-called *improved duality lemma* (see [3]) can be used in the context of coagulation-fragmentation equations to obtain new moment estimates. I will then present two mains consequences of these moment estimates, namely a new regularity result (obtention of strong solutions), and a proof that strong enough fragmentation can prevent gelation even for superlinear coagulation, which extends a know result in the homogeneous case to the inhomogeneous (i.e. diffusive) model.

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A Kinetic Wave Turbulence Model for Stratified Flows

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(joint work with Leslie Smith and Minh Binh Tran)

While resonant wave interactions, under the assumption of weak nonlinearity, can be characterized by Zakharov kinetic equations (cf. [3, 4, 9]) the spectral energy transfer on the resonant manifold, which is a set of wave vectors k, k_1, k_2 satisfying

$$k = k_1 + k_2, \quad \text{and the exact resonance relation} \quad \omega_k = \omega_{k_1} + \omega_{k_2}, \tag{1}$$

where the frequency ω is given by the dispersion relation between the wave frequency ω and the wavenumber k. However, it is known that exact resonances

do not capture some important physical effects, such as energy transfer to non-propagating wave modes with zero frequency, a near resonant condition needs to be assumed, given by the momentum transfer relation $k = k_1 + k_2$, and the θ- *broadening* relation $|\omega_k - \omega_{k_1} - \omega_{k_2}| < \theta(f, k)$, accounting for broadening of resonant surfaces, depending on the wave density $f = f(k, t)$. When such near resonances are included in the dynamics, numerical studies have demonstrated the formation of the anisotropic, non-propagating wave modes in dispersive wave systems relevant to geophysical flows (cf. [5, 6]). Such non-local near-resonance turbulence kinetic model for internal wave interactions is described by (cf. [3]),

$$\partial_t f(t,k) + 2\nu|k|^2 f(t,k) \ = Q[f](t,k), \qquad f(0,k) = f_0(k), \tag{2}$$

in which $f(t, k)$ is the nonnegative wave density at wavenumber $k \in \mathbb{R}^d$, $d \geq 2$. Following [8], the term $2\nu|k|^2 f$ corresponds to the viscous damping term, where ν is the viscosity coefficient. This equation is a three-wave form, in which the collision operator is of the form

$$Q[f](k) \ = \ \iint_{\mathbb{R}^{2d}} \Big[R_{k,k_1,k_2}[f] - R_{k_1,k,k_2}[f] - R_{k_2,k,k_1}[f]\Big] dk_1 dk_2 \tag{3}$$

with

$$R_{k,k_1,k_2}[f] := |V_{k,k_1,k_2}|^2 \delta(k - k_1 - k_2)\mathcal{L}_f(\omega_k - \omega_{k_1} - \omega_{k_2})(f_1 f_2 - f f_1 - f f_2), \tag{4}$$

This collisional form has been derived by Wyld in [7] following a the Feynman and Dyson techniques to obtain a one-loop diagramatic approach to acoustics by the analysis of forming the hierarchy of equations for the spectral cumulants (correlation functions of the wave amplitudes) of underlying non linear PDE models, such as Navier Stokes flow type models. The transition probability (or collision kernel) term V_{k,k_1,k_2} we used is modeled after [3]

$$V_{k,k_1,k_2} \ = \mathfrak{C}\,(|k||k_1||k_2|)^{\frac{1}{2}}\,, \tag{5}$$

where $\mathfrak{C}$ is a physical constant. Nest, denoting by, the Coriolis F and buoyancy (Brunt-Vaisala) frequencies M, reference vertical wave number m determined from data, the gravitational constant g, and reference density ρ_0, the dispersion law is

$$\omega_k = \sqrt{\lambda_1 + \lambda_2|k|^2}, \qquad \text{for } \lambda_1 = F^2 \text{ and } \lambda_2 = \left(\frac{g}{\rho_0 M m 0}\right)^2. \tag{6}$$

The broadening effects $\mathcal{L}_f$ are modeled by the Cauchy Lorentz distribution

$$\mathcal{L}_f(\zeta) = \frac{\Gamma_{k,k_1,k_2}}{\zeta^2 + \Gamma^2_{k,k_1,k_2}}, \qquad \text{with} \quad \lim_{\Gamma_{k,k_1,k_2}\to 0} \mathcal{L}_f(\zeta) = \pi\delta(\zeta), \tag{7}$$

and in the zero limit, the term (4) models exact resonances (cf. [8, 9]) as becomes $R_{k,k_1,k_2}[f] := |V_{k,k_1,k_2}|^2 \delta(k - k_1 - k_2)\delta(\omega_k - \omega_{k_1} - \omega_{k_2})(f_1 f_2 - f f_1 - f f_2)$.

Moreover, the resonance broadening frequency is $\Gamma_{k,k_1,k_2} = \gamma_k + \gamma_{k_1} + \gamma_{k_2}$, where γ_k is computed in [3] using a one-loop diagram approximation, for a normalized

constant $\mathfrak{c}$ to unit and $\mathfrak{M}_0(t)$ the non-conserved density mass, to be modeled by

$$\gamma_k \approx \mathfrak{c}|k|^2 \int_{\mathbb{R}_+} |k|^2 |f(t,|k|)|d|k| \approx \int_{\mathbb{R}^3} f(t,k)dk := \mathfrak{M}_0(t)\,.$$

Thus, the broadening parameter is mass dependent, given by

$$\Gamma_{k,k_1,k_2} = (|k|^2 + |k_1|^2 + |k_2|^2) \int_{\mathbb{R}^3} f(t,k)dk \;=\; (|k|^2 + |k_1|^2 + |k_2|^2)\mathfrak{M}_0(t)\,. \tag{8}$$

Inspired by recent work of Alonso with two of the authors [1] on the Quantum Boltzmann equation for cold bosonic gases, whose equation can also be derived by diagrammatic techniques, we present the existence and uniqueness solution to a Cauchy problem for the model (2)-(7), whose complete proof can be found in [2]. The strategy consists in finding a suitable time invariant subspace S_T of the Banach space $L^1_N(\mathbb{R}^d)$ for which the weak turbulence equation has a unique strong solution, where this Banach space has norms defined by the n^{th} moment as the expectation of the n^{th}-power of the dispersion relation, that is for any given density g,

$$L^1_N(\mathbb{R}^d) := \{g \in L^1(\mathbb{R}^d),\quad s.t.\ \|g\|_{L^1_N} := \mathfrak{M}_N[g] = \int_{\mathbb{R}^d} \omega_k^N g(k)dk \;< \infty\} \tag{9}$$

The invariant subspace S_T is convex and positive and it is determined from a priori estimates of the initial value problem. Indeed, we show that for a given R_0, set

$$C_* := \frac{C_0(\lambda_1,\lambda_2)\left(1 + e^{(4\nu R_0^2 + 8R_0)T}\right)}{\|f_0(k)\chi_{R_0}\|_{L^1}}, \quad \text{and} \quad C^* := 4\nu R_0^2 + 8R_0,$$

and, for any $R^* > 0$, $R_* > 1$, and for $N, t > 0$, define the subset S_T

$$\mathcal{S}_T := \Big\{ f \in L^1_{N+3}\big(\mathbb{R}^d\big) \;:\; \mathbf{S1})\ f \geq 0;\ \mathbf{S2})\ \|f\|_{L^1_{N+3}} \leq c_0(t) := (2R_* + 1)e^{C_* T};$$
$$\mathbf{S3}) -\ \|f\|_{L^1} \geq c_1(t) := \tfrac{R^* e^{-C^* T}}{2}.\Big\}$$

Then the following results follows

Existence and Uniqueness Theorem . Let $N > 0$, and let $f_0(k) \in \mathcal{S}_0 \cap B_*(R_*)\backslash\overline{B_*(R^*)}$, where $B_*(R^*), B_*(R_*)$ with radius R^*, R_* on $L^1_{N+3}(\mathbb{R}^d)$. Then the weak turbulence flow model (2)-(7) has a unique strong solution $f(t,k)$ so that

$$0 \leq f(t,k) \in C\left([0,T); \mathcal{S}_T\right) \cap C^1\left((0,T); L^1_N(\mathbb{R}^d)\right). \tag{10}$$

Since T can be chosen arbitrarily large this is a unique global solution in time.

Comments on the proof: The proof uses tools of solving Ordinary Differential Inequalites (ODIs) in the Banach space $L^1_{N+3}\big(\mathbb{R}^d\big)$. Thus, estimates need to be derived for the positive and negative parts of collision or interacting operator $Q[f](t,k)$ in (7) as well as compatible initial data $f_0(k)$, both in the subspace S_T. The negative contribution from the viscous term $nu|k|^2 f(t,k)$ plays a crucial role to obtain estimates in S_T for $L^1_{N+3}(\mathbb{R}^d)$, as it necessary to show that there are positive constants $C_1 = C_1(\lambda_1,\lambda_2,N,R_0,C^*,C_*), C_2 = C_2(\nu)$ and C_3 such that it is the following ordinary differential inequality for $\mathfrak{M}_n[f](t)$

$$\frac{d}{dt}\mathfrak{M}_N[f] \;\leq\; C_1\mathfrak{M}_{N+1}[f] - C_2\mathfrak{M}_{N+2}[f],$$

that yields $\frac{d}{dt}\mathfrak{M}_N[f] \leq C_3\mathfrak{M}_N[f]$, and then a bound for $\mathfrak{M}_N[f] \leq Ce^{C'T}$. These estimates are sufficient to show that the flow model (2)-(7) is an Ordinary Differential Equation in $L^1_{N+3}(\mathbb{R}^d)$, that satisfies *(i) Holder Continuity, (ii) Sub-Tangent*, and *(iii) One-Sided Lipschitz* conditions in the invariant subset S_T. In fact conditions *(i)* and *(ii)* are not only sufficient for existence, but also characterize the set S_T, and *(iii)* yields uniqueness. This strategy to construct solutions was proposed in a set of unpublished notes by Bressan in 2006 for the classical Boltzmann equation for hard spheres in $3d$, and were used by [1] for solving the problem for the quantum Boltzmann equation for bosons. The argument and needed estimates do not make use of the concept of entropy, and exhibit that powerful upper and lower bounds in terms on statistical moments are sufficient for the existence theory.

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Hypocoercivity without confinement: mode-by-mode analysis and decay rates in the Euclidean space

Jean Dolbeault

(joint work with Emeric Bouin, Stéphane Mischler, Clément Mouhot, Christian Schmeiser)

Abstract.– L^2 *hypocoercivity results for scattering and Fokker-Planck type collision operators are obtained using decoupled Fourier modes. The rates are measured in a space with exponential weights and then extended to larger function spaces by a factorization method. Without confinement, sharp rates of decay are obtained.*

Let us consider the evolution equation

$$\frac{dF}{dt} + \mathsf{T}F = \mathsf{L}F \tag{1}$$

and assume that T and L are respectively anti-Hermitian and Hermitian operators on a complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ with norm $\|\cdot\|$. As in the *hypocoercivity* method of [4] for real valued operators, we consider the Lyapunov functional

$$\mathsf{H}[F] := \tfrac{1}{2}\,\|F\|^2 + \delta\,\mathrm{Re}\langle \mathsf{A}F, F\rangle$$

for some $\delta > 0$, with $\mathsf{A} := \big(1+(\mathsf{T}\Pi)^*\mathsf{T}\Pi\big)^{-1}(\mathsf{T}\Pi)^*$. Here * denotes the adjoint with respect to $\langle\cdot,\cdot\rangle$ and Π is the orthogonal projection onto the null space of L. We assume that positive constants λ_m, λ_M, and C_M exist, such that, for any $F \in \mathcal{H}$, the following properties hold:
(H1) *microscopic coercivity:* $-\langle \mathsf{L}F, F\rangle \geq \lambda_m\,\|(1-\Pi)F\|^2$,
(H2) *macroscopic coercivity:* $\|\mathsf{T}\Pi F\|^2 \geq \lambda_M\,\|\Pi F\|^2$,
(H3) *parabolic macroscopic dynamics:* $\Pi\mathsf{T}\Pi\,F = 0$,
(H4) *bounded auxiliary operators:* $\|\mathsf{A}\mathsf{T}(1-\Pi)F\| + \|\mathsf{A}\mathsf{L}F\| \leq C_M\,\|(1-\Pi)F\|$.
Then for any $t \geq 0$, if F solves (1) with initial datum F_0, we have

$$\mathsf{H}[F(t,\cdot)] \leq \mathsf{H}[F_0]\,e^{-\lambda_\star t}$$

where $\lambda_\star$ is characterized as the smallest $\lambda > 0$ for which there exists some $\delta > 0$ such that $(\delta\,C_M)^2 - 4\,\big(\lambda_m - \delta - \frac{2+\delta}{4}\,\lambda\big)\,\big(\frac{\delta\,\lambda_M}{1+\lambda_M} - \frac{2+\delta}{4}\,\lambda\big) = 0$ under the additional condition that $\lambda_m - \delta - \frac{1}{4}\,(2+\delta)\,\lambda > 0$.

This abstract hypocoercivity result applies to kinetic equations with various *collision operators* L whose null space is spanned by an *admissible local equilibrium* M, that is, a radially symmetric continuous function such that, additionally, M^{-1} has a growth faster than any polynomial as $|v| \to +\infty$, $\int_{\mathbb{R}^d} M\,\mathrm{d}v = 1$.

Here are two important examples:
▷ *Fokker-Planck operators with general equilibria:* $\mathsf{L}F = \nabla_v \cdot \big[M\,\nabla_v\,\big(M^{-1}\,F\big)\big]$ where M is such that $v \mapsto |\nabla_v\sqrt{M}|^2$ is integrable and a Poincaré inequality holds with respect to the measure $M\,\mathrm{d}v$.
▷ *Scattering collision operators:* $\mathsf{L}F = \int_{\mathbb{R}^d} \sigma(\cdot,v')\,\big(F(v')\,M(\cdot) - F(\cdot)\,M(v')\big)\,\mathrm{d}v'$. We assume that the *symmetry condition*

$$\int_{\mathbb{R}^d} \big(\sigma(v,v') - \sigma(v',v)\big)\,M(v')\,\mathrm{d}v' = 0$$

holds and that the *scattering rate* σ is such that $1 \leq \sigma(v,v') \leq \overline{\sigma}$ for some positive, finite $\overline{\sigma}$. The *microscopic coercivity* property follows from [3].

Next we consider a distribution function $f(t,x,v)$, where x denotes the position variable, $v \in \mathbb{R}^d$ is the velocity variable, and $t \geq 0$ is the time. We shall consider either $x \in \mathbb{T}^d \approx [0, 2\,\pi)^d$ or $x \in \mathbb{R}^d$. In order to perform a *mode-by-mode hypocoercivity* analysis, we introduce the Fourier representation with respect to x

$$f(t,x,v) = \int_{\mathbb{R}^d} \hat{f}(t,\xi,v)\,e^{-i\,x\cdot\xi}\,\mathrm{d}\mu(\xi)$$

where the measure $\mathrm{d}\mu$ is such that $\mathrm{d}\mu(\xi) = (2\pi)^{-d}\,d\xi$ and $d\xi$ is the Lesbesgue measure if $x \in \mathbb{R}^d$, and $\mathrm{d}\mu(\xi) = (2\pi)^{-d}\,\sum_{z\in\mathbb{Z}^d}\,\delta(\xi - z)$ is discrete for $x \in \mathbb{T}^d$.

Since the collision operator L does not depend on x, the kinetic equation

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f \tag{2}$$

is reduced to (1) applied to $F(t,v) = \hat{f}(t,\xi,v)$ for each mode ξ, where ξ is now considered as a parameter, and the transport operator $v\cdot\nabla_x$ is, in Fourier variables, the simple multiplication operator

$$\mathsf{T} F := i\,(v\cdot\xi)\,F\,.$$

With $\Theta = \int_{\mathbb{R}^d} |v\cdot\xi|^2\,M(v)\,\mathrm{d}v$, the operator A is now given by

$$\mathsf{A} F = \frac{-\,i\,\xi\cdot\int_{\mathbb{R}^d} v'\,F(v')\,\mathrm{d}v'}{1+\Theta\,|\xi|^2}\,M\,.$$

Under the above assumptions, for any $t\ge 0$, for any fixed ξ, with we have

$$\|F(t,\cdot)\|^2_{\mathrm{L}^2(\mathrm{d}\gamma)} \le 3\,e^{-\,\mu_\xi\,t}\,\|F_0\|^2_{\mathrm{L}^2(\mathrm{d}\gamma)}$$

where $\mathrm{d}\gamma = M^{-1}\,\mathrm{d}v$, $\mu_\xi := \frac{\Lambda\,|\xi|^2}{1+|\xi|^2}$, $\Lambda = \frac{\Theta}{3\,\max\{1,\Theta\}}\,\min\left\{1,\frac{\lambda_m\,\Theta}{\kappa^2+\Theta}\right\}$ with $\kappa = 2\,\overline{\sigma}\,\sqrt{\Theta}$ for scattering operators and $\kappa = 2\,\|\nabla_v\sqrt{M}\|_{\mathrm{L}^2(\mathrm{d}v)}/\sqrt{d}$ for Fokker-Planck operators. By the factorization result of [5], the same decay rate is obtained if we replace the measure $\mathrm{d}\gamma$ by

$$\mathrm{d}\gamma_k := \gamma_k(v)\,\mathrm{d}v \quad\text{where}\quad \gamma_k(v) = \pi^{d/2}\,\tfrac{\Gamma((k-d)/2)}{\Gamma(k/2)}\,\left(1+|v|^2\right)^{k/2}$$

for an arbitrary $k\in(d,+\infty)$. Using Parseval's identity, we obtain that the solution f of (2) on $\mathbb{T}^d\times\mathbb{R}^d$ with initial datum $f_0\in\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\gamma_k)$ such that $\iint_{\mathbb{T}^d\times\mathbb{R}^d} f_0\,\mathrm{d}x\,\mathrm{d}v = 1$ satisfies, for any $t\ge 0$,

$$\left\|f(t,\cdot,\cdot) - |\mathbb{T}^d|^{-1}M\right\|_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\gamma_k)} \le C_k\,\|f_0-f_\infty\|_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\gamma_k)}\,e^{-\,\Lambda\,t/4}$$

for some positive constant C_k.

On the whole Euclidean space $\mathbb{R}^d$, we consider the Lyapunov functional

$$f\mapsto \tfrac12\,\|f\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\gamma_k)} + \delta\,\langle\mathsf{A}f,f\rangle_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\gamma_k)}$$

where the operator $\mathsf{A} := \left(1+(\mathsf{T}\Pi)^*\mathsf{T}\Pi\right)^{-1}(\mathsf{T}\Pi)^*$ is now defined in the (x,v) variables using $\mathsf{T} := v\cdot\nabla_x$. We can use Plancherel's formula. However, it is clear that without an external potential of confinement, there is no Poincaré inequality to be expected. Replacing the *macroscopic coercivity* condition by *Nash's inequality*

$$\|u\|^2_{\mathrm{L}^2(\mathrm{d}x)} \le \mathcal{C}_{\mathrm{Nash}}\,\|u\|^{\frac{4}{d+2}}_{\mathrm{L}^1(\mathrm{d}x)}\,\|\nabla u\|^{\frac{2\,d}{d+2}}_{\mathrm{L}^2(\mathrm{d}x)}$$

allows us to prove that there exists a constant $C_k>0$ such that, for any $t\ge 0$,

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\gamma_k)} \le C_k\left(\|f_0\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\gamma_k)} + \|f_0\|^2_{\mathrm{L}^2(\mathrm{d}\gamma_k;\,\mathrm{L}^1(\mathrm{d}x))}\right)(1+t)^{-\frac{d}{2}}\,.$$

So far we did not assume any sign condition on f. Inspired by the properties of the solutions of the heat equation, a more detailed analysis shows that the zero average

solutions of (2) have an improved decay rate. Assume that $f_0 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}^d)$ with $\iint_{\mathbb{R}^d \times \mathbb{R}^d} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v = 0$ and let

$$\mathcal{C} := \|f_0\|^2_{\mathrm{L}^2(\mathrm{d}\gamma_{k+2};\,\mathrm{L}^1(\mathrm{d}x))} + \|f_0\|^2_{\mathrm{L}^2(\mathrm{d}\gamma_k;\,\mathrm{L}^1(|x|\,\mathrm{d}x))} + \|f_0\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\gamma_k)} < \infty\,.$$

Then there exists a constant $c_k > 0$ such that, for any $t \ge 0$,

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\gamma_k)} \le c_k\,\mathcal{C}\,(1+t)^{-(1+\frac{d}{2})}\,.$$

For details, see [1]. Further improved estimates will be available in [2].

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Global existence analysis of multi-species cross-diffusion systems in population dynamics

Esther S. Daus

(joint work with Xiuqing Chen, Ansgar Jüngel)

Introduction. Many applications in physics, chemistry and biology can be modeled by cross-diffusion systems, describing the evolution of the densities or the concentrations of a multi-component system. Formally, they are strongly coupled parabolic systems of the type

$$\partial_t u_i - \operatorname{div}\left(\sum_{j=1}^n A_{ij}(u)\nabla u_j\right) = f_i(u) \quad \text{in } \Omega,\ t>0, \quad i=1,\ldots,n, \tag{1}$$

with initial conditions and no-flux boundary conditions on a bounded domain $\Omega \subset \mathbb{R}^d$ ($d \ge 1$). The diffusion matrix $A(u)$ is in general non-diagonal and neither symmetric nor positive definite, and thus the existence analysis of these models is extremely challenging, see *e.g.* [3, 12, 13].

The SKT model. Our focus is on the existence analysis of a population dynamics model for competing species named after Shigesada, Kawasaki and Teramoto [19]. We are particularly interested in a generalized n-species version of

this SKT model, where the diffusion matrix has the form

$$(2) \quad A_{ij}(u) = \delta_{ij} p_i(u) + u_i \frac{\partial p_i}{\partial u_j}(u), \quad p_i(u) = a_{i0} + \sum_{k=1}^{n} a_{ik} u_k^s, \quad i,j = 1,\dots,n,$$

with a_{i0}, $a_{ij} \geq 0$ and $s > 0$. The functions p_i are the transition rates of the underlying random-walk model [13, 21], and a_{i0} are called diffusion, a_{ii} self-diffusion and a_{ij} cross-diffusion coefficients. The source terms f_i are of Lotka-Volterra competition type.

Historical background. The first global existence result for the classical SKT model (*i.e.* (2) with $s = 1$ and $n = 2$) was achieved by Kim [14] by studying the equations in one space dimension, neglecting self-diffusion, and assuming equal coefficients ($a_{ij} = 1$). Significant progress was made by Amann [1], who proved that a priori estimates in the $W^{1,p}$ norm with $p > d$ are sufficient for the solutions to general quasilinear parabolic systems to exist globally in time, moreover, he applied this result to the triangular case. The first result on global existence without restricting the diffusion coefficients (except assuming positivity) was achieved in [11] in one space dimension and in [6, 7] in several space dimensions. In [10], these results could be extended to the whole space. The existence of global classical solutions under suitable conditions on the coefficients was proved in, e.g., [15].

As a next step, nonlinear transition rates, but still for two species (*i.e.* (2) with $s > 0$ and $n = 2$), were analyzed. Desvillettes and co-workers managed to prove global existence in the sublinear case ($0 < s < 1$) [8] and later for the superlinear case ($s > 1$) under the weak cross-diffusion condition $((s-1)/(s+1))^2 a_{12}a_{21} \leq a_{11}a_{22}$ [9]. Similar results, but under a slightly stronger weak cross-diffusion hypothesis, were proved in the general paper on cross-diffusion systems [12].

Finally, first results for more than two species were considered. The existence of positive stationary solutions and the stability of the constant equilibrium was proved in [2, 18]. The existence of global weak solutions in one space dimension assuming a positive definite diffusion matrix was considered in [20], based on Amann's results. Using an entropy approach, the global existence of solutions was shown in [9] for three species under the condition $0 < s < 1/\sqrt{3}$.

Our results. In [5] we provide for the first time global-in-time weak solutions to generalized SKT reaction-cross-diffusion systems for an arbitrary number of competing population species (2) under very general conditions by refining the entropy method of [12]. Moreover, we explore a surprising relation between the monotonicity of the entropy and the detailed balance condition of an associated Markov chain. In particular, we show that global existence holds for system (2) under a detailed balance or weak cross-diffusion condition. The detailed balance condition is related to the symmetry of the mobility matrix, which mirrors Onsager's principle in thermodynamics. Under detailed balance (and without reaction), the entropy is nonincreasing in time, but counter-examples show that the entropy may increase initially if detailed balance does not hold.

Duality techniques. Certain duality techniques, tracing back for instance to M. Pierre [17], refined to several variants, for instance in [4] (suggested by F. Otto), were used in [16] to get rid of the space dependent lower bounds for s in [5].

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Hydrodynamics beyond Navier-Stokes

Alexander V. Bobylev

We consider the problem of regularization of classical (ill-posed) Burnett equations of hydrodynamics and show that this can be done by transformation to new hydrodynamic variables. It is also shown that the way of truncation of the Chapman-Enskog expansion is not unique (even at the Navier-Stokes level). We discuss the meaning of the above transformations. It is important to understand if there exists the optimal one. This question is studied in detail for the case of the linearized Boltzmann equation. It is shown that the corresponding Chapman-Enskog expansion has a special structure which allows to pass to diagonal equations of hydrodynamics (five independent equations). This is the meaning of the optimal transformation in the linear case. We also present in the talk the rigorous estimates for Navier-Stokes and Burnett level respectively. The diagonal Navier-Stokes equations are: three heat equations for two viscous and one thermal modes respectively, and also two Burgers (with viscosity) equations for sound modes. The accuracy of these equations has the order $O(Kn)$ uniformly in time, where Kn denotes the Knudsen number. The diagonal Burnett equations are: the same three heat equations for two viscous and one thermal modes, and two KdV-Burgers equations for sound modes. The solutions of these equations have the second order of approximation $O\left[(Kn)^2\right]$. Hence in that case we have a real improvement of results obtained at the Navier-Stokes level. More details can be found in recent paper[1].

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Equilibria of diffusing and self-attracting particles

Franca Hoffmann
(joint work with José A. Carrillo, Vincent Calvez)

We investigate a general class of aggregation-diffusion equations describing the evolution of a continuum density as a result of particle interaction of the form

$$\partial_t \rho = \frac{1}{N}\Delta \rho^m + 2\chi \nabla \cdot (\rho \, \nabla W * \rho) \tag{1}$$

for a density $\rho(t,x) \in L^1_+\left(\mathbb{R}^N\right) \cap L^m\left(\mathbb{R}^N\right)$ of unit mass defined on $\mathbb{R}_+ \times \mathbb{R}^N$. Here, the parameter $\chi \geq 0$ measures the interaciton strength and scales with the mass of ρ. Of recent interest is the attractive interaction potential $W(x) = \frac{|x|^k}{k}$ for $k \neq 0$ or $W(x) = \frac{1}{2\pi}\log|x|$ for $k = 0$, and thus non-linear diffusion enters in

competition with non-local attractive forces (see [4, 3, 6, 7, 5, 11, 10, 13, 16] and references therein). This model, although simple, exhibits a deep mathematical structure and has been intensively studied in various contexts arising in physics and biology. The three-dimensional configuration ($m = 1, k = -1$) is the so-called Smoluchowski-Poisson system arising in gravitational physics [12]. It describes macroscopically a density of particles subject to a self-sustained gravitational field. Without non-local particle interaction $\chi = 0$, model (1) reduces to the well-known porous medium equation ($m > 1$), heat equation ($m = 1$), or fast diffusion equation ($0 < m < 1$) [15, 14]. The two-dimensional logarithmic case ($m = 1, k = 0$) is the so-called Patlak-Keller-Segel (PKS) system in its simplest formulation (see [4] and references therein), modeling the collective motion of cells which are attracted by a self-emitted chemical substance due to chemotaxis. This mechanism has been widely observed in various biological fields (morphogenesis, bacterial self-organization, inflammatory processes among others), and existence and uniqueness of stationary states as well as convergence to equilibrium have been extensively studied (see [5] and references therein). However, much less is known for general $m > 0$ and $k \in (-N, \infty)$. For which choices of χ do stationary states exist? If they do, are they unique (up to dilation)? Do some or all solutions converge to them, and if yes, how fast and in which norm? Recent insights have been achieved by making a link to the functional

$$\begin{aligned}\mathcal{F}[\rho] &= \frac{1}{N(m-1)} \int_{\mathbb{R}^N} \rho^m(x)\,dx + \chi \iint_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y)\rho(x)\rho(y)\,dxdy \\ &:= \mathcal{H}[\rho] + \chi \mathcal{W}[\rho]\,,\end{aligned}$$

if $m \neq 1$ with $\mathcal{H}[\rho] = \frac{1}{N} \int_{\mathbb{R}^N} \rho \log \rho\,dx$ if $m = 1$. This is the free energy functional associated to the formal Wasserstein-2 gradient flow formulation of model (1) [2, 1], and $\mathcal{F}$ is non-increasing along the trajectories of the system. The competition of repulsive (diffusion) and attractive (interaction) forces, can also be observed on the level of the functional: taking dilations $\rho^\lambda(x) := \lambda^N \rho(\lambda x)$, one obtains

$$\mathcal{F}[\rho^\lambda] = \lambda^{N(m-1)} \mathcal{H}[\rho] + \lambda^{-k} \chi \mathcal{W}[\rho]\,.$$

In other words, the diffusion and aggregation forces are in balance if and only if $N(m-1) + k = 0$, which we coin as the *fair-competition regime*. Naturally, another set of questions arises: Are there minimisers to $\mathcal{F}$? If yes, are they regular enough to be stationary states of equation (1)? Conversely, are stationary states of equation (1) necessarily minimisers of $\mathcal{F}$?

We give a near-complete picture of the fair-competition regime in answering these questions. This talk focuses on the results in [6] and [7], which is joint work with José A. Carrillo and Vincent Calvez. We will summarise the results and some interesting open questions below.

Existence and Regularity in the Fair-Competition Regime

In the *Porous Medium Regime* $k \in (-N, 0)$ and $m = 1 - k/N \in (1, 2)$, we show that stationary states of (1) and minimisers of $\mathcal{F}$ only exist for a certain critical interaction strength $\chi = \chi_c$, an interesting mathematical phenomenon that also occurs for the classical PKS equation $k = 0$ and $m = 1$. In particular, there exist global minimisers of $\mathcal{F}$ and they are radially symmetric non-increasing, compactly supported and uniformly bounded. Further, we make use of well known functional inequalities (variants of the Hardy-Littlewood-Sobolev inequality) to show that stationary states and minimisers are equivalent in that case. For subcritical interaction strengths $\chi < \chi_c$, no stationary states exist as diffusion overcomes self-attraction. However, we can show existence of global minimisers for the rescaled functional Due to homogeneity, each global minimiser of $\mathcal{F}$ gives rise to a family of global minimisers for $\chi = \chi_c$ by dilation, but it is an open problem to show that there is a unique global minimiser modulo dilations. This uniqueness was proven in the Newtonian case in [16]. Further, in self-similar variables, we do not know if stationary states with second moment bounded are among global minimisers of the rescaled free energy $\mathcal{F}_{\mathbf{resc}}$ for the sub-critical regime $0 < \chi < \chi_c$, and whether these minimisers are unique. For $N = 1$, we fully answered these questions in [7], but it remains an open problem in higher dimensions.

For the *Fast Diffusion Regime* $k \in (0, N)$ and $m = 1 - k/N \in (0, 1)$, we show in [6] that no radially symmetric non-increasing stationary states and no radially symmetric non-increasing global minimisers of $\mathcal{F}$ exist for all values of the critical parameter χ, while we show the existence of continuous radially symmetric non-increasing self-similar profiles under the condition that diffusion is not too fast, that is for $k \in (0, 1]$. In this sense, we show that there is no criticality for $k > 0$. Numerical experiments in one dimension corroborate the absence of critical behaviour for $k > 0$.

A full proof of non-criticality involves the analysis of the minimisation problem in self-similar variables as for $k < 0$ showing that global minimisers exist in the right functional spaces for all values of χ and that they are indeed stationary states. We proved this result in one dimension in [7] by optimal transport techniques and one interesting direction of future investigation would be to generalise this result to higher dimensions.

In the fast diffusion regime, we show existence of self-similar profiles directly using a fixed point argument on a compact operator arising from the Euler-Lagrange equation, but we are lacking a suitable HLS-type inequality to show that self-similar profiles of (1) are global minimisers of $\mathcal{F}_{\mathbf{resc}}$. It seems natural that $\mathcal{F}_{\mathbf{resc}}$ should be bounded below and achieve its infimum for all values of χ as long as diffusion is not too fast. These are just some of the many interesting open questions for the fast diffusion fair-competition regime.

Convergence to Equilibrium in the One-Dimensional Fair-Competition Regime

In one dimension, the free energy functional $\mathcal{F}$ is the sum of the displacement-convex functional $\mathcal{H}$ and the displacement-concave functional $\mathcal{W}$, the overall convexity properties of $\mathcal{F}$ however are not known. In [7], we show uniqueness of minimisers of $\mathcal{F}$ in the porous medium regime $k \in (-1, 0)$ in the critical case $\chi = \chi_c$ and for minimisers of $\mathcal{F}_{\mathbf{resc}}$ in the sub-critical regime $\chi < \chi_c$. In the fast diffusion range $k \in (0, 1)$ when no critical parameter χ exists, we have uniqueness of minimisers of the rescaled free energy $\mathcal{F}_{\mathbf{resc}}$ – if they exist – for any $\chi > 0$. This means that $\mathcal{F}$ and $\mathcal{F}_{\mathbf{resc}}$ behave almost like convex functionals as the bad functional contribution is absorbed by the convex part. In particular, existence of a critical point implies uniqueness (up to dilations), and it seems there is an underlying convexity structure that is not yet well understood.
In [9], authors have proved the uniqueness (up to dilations) of the logarithmic HLS inequality which corresponds to a bound below for $\mathcal{F}$ in the case $k = 0$, $m = 1$. Their strategy is based on the so-called competing-symmetry argument initiated in [8]. We develop in [7] an alternative argument based on some accurate use of Jensen's inequality to get the same result for $\mathcal{F}$ with $k \in (-1, 0)$ in the critical case $\chi = \chi_c$. In self-similar variables when homogeneity has been broken, we derive an inequality similar to Jensen's to achieve a bound from below in the sub-critical regime $\chi < \chi_c$: for any stationary state $\bar{\rho}$ in self-similar variables, we have for any solution ρ,

$$\mathcal{F}_{\mathbf{resc}}[\rho] \geq \mathcal{F}_{\mathbf{resc}}[\bar{\rho}]\,, \tag{2}$$

with equality if and only if $\rho = \bar{\rho}$. It turns out that our method is also applicable to the fast-diffusion range $k \in (0, 1)$, and we are able to derive a type of reversed HLS inequality (2) for any $\chi > 0$.

As system (1) is the formal gradient flow of a convex + concave functional, one has to seek compensations. Such compensations do exist in our case, and in [7], we prove convergence in Wasserstein distance towards a unique (up to dilation) stationary state under suitable assumptions, in some cases with an explicit rate. It is of course extremely important to understand how the convex and the concave contributions are entangled.

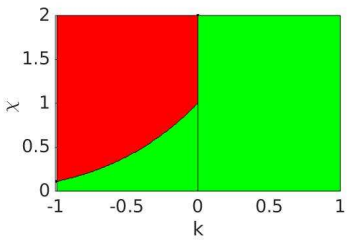

Figure 1. Regions of blow up (red) and convergence to self-similarity (green) in 1D.

Finally, we use an illuminating way to rewrite the energy functional $\mathcal{F}$ due to the particular form of the transport map to derive a numerical scheme for model (1). Our simulations confirm that there is no critical interaction strength for the fast diffusion regime $k > 0$, and in the porous medium regime $k < 0$ the scheme allows us to compute $\chi_c(k)$ numerically. Figure 1 gives an

overview of the asymptotic behaviour of solutions: The critical regime is characterised by the black line $\chi_c(k)$, $-1 < k \leq 0$, separating the red region (finite-time blow-up) from the green region (convergence to self-similarity).

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Invariant densities for conservative linear kinetic equations on the torus without spectral gaps

Mustapha Mokhtar-Kharroubi

This work is the continuation of a general theory, given in [3], on time asymptotics of conservative linear kinetic equations on the torus exhibiting a spectral gap. Our

aim here is to provide new functional analytic developments *in absence* of a spectral gap. Let us review first some results from [3]. We consider neutron transport-like equations

$$\frac{\partial f}{\partial t} + v.\frac{\partial f}{\partial x} + \sigma(x,v)f(t,x,v) = \int_V k(x,v,v')f(t,x,v')\mu(dv')$$

on $L^1(\mathcal{T}^n \times V)$ $(n \geqslant 1)$ where $\mathcal{T}^n := \mathbb{R}^n/(\mathbb{Z})^n$ is the n-dimensional torus under the *conservativity* assumption $\sigma(x,v) = \int_V k(x,v',v)\mu(dv')$ where $\mu(dv)$ is a velocity Radon measure on $\mathbb{R}^n$ with support V. The collision frequency $\sigma(.,.)$ is assumed (for simplicity) to be essentially bounded so that the collision operator

$$K : L^1(\mathcal{T}^n \times V) \ni \varphi(.,.) \to \int_V k(x,v,v')\varphi(x,v')\mu(dv') \in L^1(\mathcal{T}^n \times V)$$

is bounded on $L^1(\mathcal{T}^n \times V)$. If $K = 0$, the "collisionless" equation on $\mathcal{T}^n \times V$ is governed by a weighted shift C_0-semigroup $(U(t))_{t\geq 0}$

$$L^1(\mathcal{T}^n \times V) \ni \varphi \to e^{-\int_0^t \sigma(x-sv,v)ds}\varphi(x-tv,v) \in L^1(\mathcal{T}^n \times V) \quad (t \geqslant 0).$$

We denote by T the generator of $(U(t))_{t\geq 0}$. The type (or growth bound) of $(U(t))_{t\geq 0}$ is equal to

$$\omega(U) = -\lim_{t\to+\infty} \inf_{(x,v)\in\mathcal{T}^n\times V} t^{-1}\int_0^t \sigma(x+sv,v)ds,$$

see [3]. In particular $\omega(U) < 0$ *if and only if* there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\int_0^{C_1} \sigma(x+sv,v)ds \geqslant C_2 \quad \text{a.e. on } \mathcal{T}^n \times V,$$

see [3] (see also [1] for an earlier result in this direction). The full dynamics on $L^1(\mathcal{T}^n \times V)$ is governed by a C_0-semigroup $(W(t))_{t\geq 0}$ generated by $A := T + K$. Moreover, due to the conservativity condition, $(W(t))_{t\geq 0}$ is a *stochastic* (or markov) semigroup, i.e. mass-preserving on the positive cone, in particular its type is equal to *zero,* $\omega(W) = 0$. This perturbed semigroup $(W(t))_{t\geq 0}$ is given by a Dyson-Phillips series $W(t) = \sum_{j=0}^{\infty} U_j(t)$. We recall that any C_0-semigroup $(Z(t))_{t\geq 0}$ in a Banach space X admits an *essential* type $\omega_{ess}(Z)$ such that $r_{ess}(Z(t)) = e^{\omega_{ess}(Z)t}$ $(t > 0)$ where $r_{ess}(Z(t))$ is the *essential* spectral radius of $Z(t)$. We recall also that $(U(t))_{t\geq 0}$ and $(W(t))_{t\geq 0}$ have the *same essential type* if there exists $k \in \mathbb{N}$ such that $U_k(t)$ is a compact operator for all $t > 0$, (see [2] Chapter 2); in this case $\omega_{ess}(W) \leq \omega(U)$. In particular, if $\omega(U) < 0$ then $\omega_{ess}(W) < 0 = \omega(W)$ i.e. $(W(t))_{t\geq 0}$ exhibits a *spectral gap* and 0 is an isolated eigenvalue of $T + K$ with finite algebraic multiplicity. If $(W(t))_{t\geq 0}$ is irreducible then 0 is algebraically simple and is associated to a unique positive normalized eigenfunction u, i.e. a unique invariant density. In this case, there exist $\varepsilon > 0$ and $C > 0$ such that $\|W(t)\varphi - u\| \leq Ce^{-\varepsilon t}$ $(t \geq 0)$ for any density φ. The compactness of some $U_k(t)$ for all $t > 0$ holds for a general class of velocity measures $\mu(dv)$ (covering the usual Lebesgue measure on $\mathbb{R}^n$ or on spheres) and a general

class of collision operators (see [3] for the details). Our aim here is to deal with the critical case $\omega(U) = 0$, i.e.

$$\lim_{t\to+\infty} \inf_{(x,v)\in\mathcal{T}^n\times V} t^{-1}\int_0^t \sigma(x+sv,v)ds = 0.$$

In this case, the existence of an invariant density for $(W(t))_{t\geq 0}$ and the strong convergence of the latter to its ergodic projection are two key open problems. (We do *not* assume a detailled balance condition for which the existence of an invariant density is given for free.) Our general strategy consists in approximating $(W(t))_{t\geq 0}$ by a sequence of stochastic kinetic semigroups $\left\{\left(W^j(t)\right)_{t\geq 0}\right\}_j$ having a spectral gap and in analyzing the sequence of their invariant densities. First we characterize the condition $\omega(U) = 0$ by the existence of some $(\overline{x},\overline{v}) \in T^n \times V$ with $\overline{v} \neq 0$ such that $\sigma(\overline{x}+s\overline{v},\overline{v}) = 0\ \forall s \geq 0$ or by the existence of some $\overline{x} \in T^n$ such that $\sigma(\overline{x},0) = 0$ (if $0 \in V$), and define the set of degeneracy of σ as the set

$$\Xi := \{(x,v) \in T^n \times V;\ \sigma(x+sv,v) = 0\ \forall s \geq 0\}.$$

We prove that the projection of Ξ on V along T^n

$$\Pi = \{v;\ \exists x \in T^n, (x,v) \in \Xi\ \}$$

is never of full measure, i.e. $\Pi^c := V/\Pi$ has always a positive μ-measure. Our construction consists in approximating $k(x,v,v')$ by $k_j(x,v,v')$ ($j \in \mathbb{N}$) and $\sigma(x,v)$ by $\sigma_j(x,v) := \int_V k_j(x,v',v)\mu(dv')$ where k_j are *suitably chosen* (in connection with the set Π). In particular, since $\omega_j(U) < 0$ then $(W_j(t))_{t\geq 0}$ has a spectral gap and there exists $\varphi_j \in D_+(T_j)$ such that $T_j\varphi_j + K_j\varphi_j = 0$, $\left(\|\varphi_j\|_{L^1(T^n\times V)} = 1\right)$. One of our main results is: if $\mu(\Pi) = 0$ and if $k(y,v,v')$ *vanishes (in a suitable way) on* $T^n \times \Pi \times V$ then $(\varphi_j)_j$ converges in $L^1(\mathcal{T}^n \times V)$ to an invariant density of $(W(t))_{t\geq 0}$. The proof of this result is quite involved and relies on various mathematical preliminary results of independent interest. We show also the strong convergence of $(W(t))_{t\geq 0}$ as $t \to +\infty$ to its ergodic projection by using a functional analytic result relying on a "0-2" law for C_0-semigroups [4]. The assumption that $k(y,v,v')$ vanishes on $T^n \times \Pi \times V$ *cannot* be relaxed; indeed, we show, for space-homogeneous cross-sections, that without the assumption above, $(W(t))_{t\geq 0}$ has no invariant density and, for any density φ, *the total mass of* $W(t)\varphi$ *concentrates in the vicinity of the null-set* $T^n \times \{v;\sigma(v) = 0\}$ *as* $t \to +\infty$, *in particular* $\int_{T^n} W(t)\varphi dx$ *tends to a measure carried by the set* $\{v;\sigma(v) = 0\}$

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Excitation Spectra of Bose Gases

Benjamin Schlein

(joint work with Chiara Boccato, Serena Cenatiempo, Christian Brennecke)

We are interested in the low-energy spectrum of trapped Bose gases, consisting of N particles interacting through a repulsive short range potential with scattering length of the order N^{-1} (the Gross-Pitaevskii regime). To make our analysis simpler, we assume that the N bosons are confined in the box $\Lambda = [0;1]^{\times 3}$, with periodic boundary conditions. The Hamilton operator has the form

$$H_N = \sum_{j=1}^N -\Delta_{x_j} + \kappa \sum_{i<j}^N N^2 V(N(x_i - x_j)) \tag{1}$$

where $\kappa > 0$ is a coupling constant and V is non-negative, radially symmetric and compactly supported. The N dependence of the interaction ensures that its scattering length scales as N^{-1}.

Recall that the scattering length is defined through the zero-energy scattering equation

$$\left[-\Delta + \frac{\kappa}{2} V\right] f = 0 \tag{2}$$

with the boundary condition $f(x) \to 1$ as $|x| \to \infty$. Outside the support of V, f has the form

$$f(x) = 1 - \frac{a_0}{|x|}$$

for a constant $a_0 > 0$ which is known as the scattering length of κV. By scaling it is then easy to check that the scattering length of the rescaled potential $\kappa N^2 V(N.)$ is given by a_0/N.

The ground state properties of (1) have been studied in [8, 7]. In particular, it follows from [8] that the ground state energy E_N of (1) is such that

$$\lim_{N\to\infty} \frac{E_N}{N} = 4\pi a_0 \tag{3}$$

Moreover, in [7] it was shown that the ground state $\psi_N \in L^2_s(\Lambda^N)$ of (1) exhibits complete Bose-Einstein condensation; if γ_N denotes the reduced one-particle density matrix associated with ψ_N, then it was proven in [7] that

$$\gamma_N \to |\varphi_0\rangle\langle\varphi_0| \tag{4}$$

as $N \to \infty$. Here φ_0 denotes the zero-momentum mode, defined by $\varphi_0(x) = 1$ for all $x \in \Lambda$. Eq. (4) states that, in the state ψ_N, all particles, up to a fraction vanishing as $N \to \infty$, are in the same one-particle state φ_0.

Here, we would like to describe the low-energy spectrum of (1), including the ground state energy and the low-lying excitations. Since typical excitation energies are of order one, to reach this goal we need a much better resolution compared to (3), i.e. we need to compute energies up to errors that are small, as $N \to \infty$. The next theorem is the main result of [2].

Theorem 1. *Let $V \in L^3(\mathbb{R}^3)$ be pointwise non-negative, radially symmetric and compactly supported. Let $\kappa > 0$ be small enough. Then the ground state energy E_N of (1) is given by*

(5)
$$E_N = 4\pi a_N (N-1) - \frac{1}{2} \sum_{p \in \Lambda_+^*} \left[p^2 + 8\pi a_0 - \sqrt{|p|^4 + 16\pi a_0 p^2} - \frac{(8\pi a_0)^2}{2p^2} \right] + \mathcal{O}(N^{-1/4})$$

where $\Lambda_+^* = 2\pi\mathbb{Z}^3 \backslash \{0\}$ and

(6)
$$8\pi a_N = \kappa \widehat{V}(0) + \sum_{k=1}^{\infty} \frac{(-1)^k \kappa^{k+1}}{(2N)^k} \times \sum_{p_1,\ldots,p_k \in \Lambda_+^*} \frac{\widehat{V}(p_1/N)}{p_1^2} \left[\prod_{i=1}^{k-1} \frac{\widehat{V}((p_i - p_{i+1})/N)}{p_{i+1}^2} \right] \widehat{V}(p_k/N)$$

Moreover, the spectrum of $H_N - E_N$ below a threshold $\zeta > 0$ consists of eigenvalues given by finite sums of the form

$$\sum_{p \in \Lambda_+^*} n_p \sqrt{|p|^4 + 16\pi a_0 p^2} + \mathcal{O}(N^{-1/4}(1 + \zeta^3)) \tag{7}$$

with $n_p \in \mathbb{N}$ for all $p \in \Lambda_+^*$.

It is easy to check that the Born series on the r.h.s. of (6) converges absolutely, for $\kappa > 0$ small enough. In fact, the r.h.s. of (6) can be compared with the Born series for the scattering length a_0 defined through (2), which is given by

(8)
$$8\pi a_0 = \kappa \widehat{V}(0) + \sum_{k=1}^{\infty} \frac{(_1)^k \kappa^{k+1}}{2^k (2\pi)^{3k}} \int_{\mathbb{R}^{3k}} dp_1 \ldots dp_k \, \frac{\widehat{V}(p_1)}{p_1^2} \left[\prod_{i=1}^{k-1} \frac{\widehat{V}(p_i - p_{i+1})}{p_{i+1}^2} \right] \widehat{V}(p_k)$$

Estimating the difference between the r.h.s. of (6) and of (8), we show that $|a_N - a_0| \leq C N^{-1}$. Indeed, it seems (by numerical tests with some special choices of V) that $|a_N - a_0| \simeq N^{-1}$ is of this order of magnitude, not smaller. This means that, in (5), we cannot replace a_N by a_0. To the level of precision of (5), the ground state energy is sensitive to finite volume effects and it cannot be expressed just as a function of the infinite volume scattering length a_0. Notice, however, that these finite volume effects are expressed as an overall shift of the spectrum; they are not detected in the excitation energies (7).

The results of Theorem 1 (excluding the replacement of a_0 with a_N) have already been predicted by Bogoliubov in [3], who used the linear dispersion of the excitation resulting from (7) to explain the emergence of superfluidity.

first and second Born approximations have to be

Different choices of the N dependence of the interaction lead to mathematically simpler models, described by Hamilton operators of the form

$$H_N^\beta = \sum_{j=1}^N -\Delta_{x_j} + \frac{\kappa}{2N} \sum_{i<j}^N N^{3\beta} V(N^\beta (x_i - x_j)) \tag{9}$$

For $\beta = 1$, we recover the Gross-Pitaebvskii Hamilton operator (1). For $\beta = 0$, on the other hand, (9) describes a mean-field Hamiltonian: in contrast with the Gross-Pitaevskii regime, here collisions are frequent and weak. For $\beta = 0$, the low-energy spectrum of (9) has been determined in [6, 5, 6, 4, 9]. For $0 < \beta < 1$, it has been recently established in [1].

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Fractional diffusion as macroscopic limit of kinetic models

Christian Schmeiser

(joint work with Pedro Aceves Sanchez, Claude Bardos, Gaspard Jankowiak)

The derivation of diffusion equations from kinetic transport models has a long history. It is based on local equilibria in the form of directionally unbiased velocity distributions, where the diffusivity tensor is related to the tensor of second order moments of the equilibrium. It has been observed that for equilibrium distributions

with infinite second order moments the macroscopic behavior shows anomalous diffusion behavior.

In 2010, Antoine Mellet and coworkers have started a systematic study of the corresponding macroscopic limits [9, 10]. Recently the theory has been expanded in the PhD theses of Pedro Aceves-Sánchez (Vienna) and of Ludovic Cesbron (Cambridge). They have considered various kinetic models of the form

$$\varepsilon^\alpha \partial_t f + \varepsilon v \cdot \nabla_x f = Q_0(f) + \varepsilon^{\alpha-1} Q_1(f)\,,$$

where the null space of the leading order collision operator Q_0 is spanned by a fat-tailed equilibrium distribution $M(v) \approx c|v|^{-d-\alpha}$ as $|v| \to \infty$, with the space dimension d (i.e. $x, v \in \mathbb{R}^d$) and with $0 < \alpha < 2$. Whereas P. Aceves-Sánchez dealt with collision operators of the (scattering) form

$$Q_0(f) = \int_{\mathbb{R}^d} \sigma(v, v')(M(v)f(v') - M(v')f(v))dv'\,,$$

L. Cesbron considered fractional Fokker-Planck operators:

$$Q_0(f) = -(-\Delta_v)^{\alpha/2} f + \nabla_v \cdot (vf)$$

One of the goals was the derivation of fractional diffusion with drift, induced by the velocity bias modeled in Q_1. It has been shown for $\alpha \geq 1$ that $f(x, v, t) \to \rho(x, t)M(v)$ as $\varepsilon \to 0$, where the macroscopic density ρ satisfies an equation of the form

$$\partial_t \rho + (-\Delta_x)^{\alpha/2} \rho + \nabla_x \cdot (\rho u) = 0\,,$$

with a vector field u, when both Q_0 and Q_1 are scattering operators [3], when Q_0 is a scattering operator and $Q_1(f) = -E \cdot \nabla_v f$ is an acceleration term [2], and for a fractional Fokker-Planck operator Q_0 with an acceleration term Q_1 [1].

The second goal was to derive the macroscopic limit on bounded position domains $\Omega \subset \mathbb{R}^d$. The limiting equation in the case of absorbing (i.e. homogeneous inflow) boundary conditions has the form [4]

$$\partial_t \rho(x, t) = c_{d,\alpha} P.V. \int_{S_\Omega(x)} \frac{\rho(y, t) - \rho(x, t)}{|x - y|^{d+\alpha}} dy - h_\alpha(x)\rho(x, t)$$

where $S_\Omega(x)$ is the largest star shaped subset of Ω with center in $x \in \Omega$ and $h_\alpha \to \infty$ as $x \to \partial\Omega$. It models a jump process with killing, where jumps are only allowed along straight line segments remaining in Ω. The more difficult case of specular reflection boundary conditions, in the case where Ω is a ball, has been treated in [6].

In the case of the classical diffusion limit ($\alpha = 2$), $Q_0(f) = O(\varepsilon)$ can be shown by an entropy estimate, which leads to proofs of compactness for the macroscopic density. This allows the treatment of nonlinearities (see e.g. [8, 11]). For $\alpha < 2$, only $Q_0(f) = O(\varepsilon^{\alpha/2})$ can be proven, which is not sufficient for the compactness results. Work in progress deals with equations of the form

$$\varepsilon^\alpha \partial_t f + \varepsilon v \cdot \nabla_x f = \nu(\rho_f)\left[\gamma(\rho_f, v) - f\right]\,,$$

where γ is a fat-tailed equilibrium profile satisfying $\int \gamma(\rho, v)dv = \rho$ and $\gamma(\rho, v) \approx F(\rho)|v|^{-d-\alpha}$ as $|v| \to \infty$. Formally, the limit ρ of the macroscopic density satisfies

$$\partial_t \rho + c_{\alpha,d} P.V. \int \frac{\nu(\rho(x))\nu(\rho(y))}{\hat{\nu}_\rho(x,y)^{1+\alpha}} \frac{F(\rho(x)) - F(\rho(y))}{|x-y|^{d+\alpha}} dy = 0$$

with $\hat{\nu}_\rho(x,y) = \int_0^1 \nu(\rho(sx + (1-s)y))ds$. This shows that there are various reasonable possibilities for the formulation of nonlinear fractional diffusion equations (see also [5, 7])

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Coupled Self-Organized Hydrodynamics and Stokes models for suspensions of active particles

Sara Merino-Aceituno

(joint work with Pierre Degond, Fabien Vergnet, Hui Yu)

We present an individual-based model for collective motion in a viscous fluid. The fluid is represented by the Stokes equation while the collective motion is represented by the Vicsek model (explained next). Collective motion gives a heuristic description of hydrodynamic interactions, i.e., interactions between the swimmers mediated by the fluid. The Stokes equation and the Vicsek model are coupled by taking into account the influence of the fluid on the swimmers and vice versa. Specially, the system of equations that we consider is the following, where all the quantities are dimensionless and where the stochastic differential equation (1b) must be understood in the Stratonovich sense; the unknowns are $(X_i(t), \omega_i(t))_{i\in\{1,\dots,N\}}$

(corresponding to the position $X_i \in \mathbb{R}^3$ of agent i and its orientation ω_i), $v(x,t)$, $p(x,t)$ (corresponding to the velocity of the fluid and its pressure, respectively):

$$\begin{cases} dX_i = u_i dt = v(X_i,t)dt + a\omega_i dt, & \text{(1a)} \\ d\omega_i = P_{\omega_i^\perp} \circ \left[\nu\overline{\omega_i}dt + \sqrt{2D}\, dB_t^i \ + \big(\lambda S(v) + A(v)\big)\omega_i dt \ \right], & \text{(1b)} \\ \bar{\omega}_i = \dfrac{J_i}{|J_i|} \text{ with } J_i = \displaystyle\sum_{k=1}^{N} K\left(\frac{|X_i - X_k|}{R}\right)\omega_k, & \text{(1c)} \\ -\Delta_x v + \nabla_x p = -\dfrac{b}{N}\displaystyle\sum_{i=1}^{N}\left(\omega_i \otimes \omega_i - \frac{1}{3}\mathrm{Id}\right)\nabla_x \delta_{X_i(t)}, & \text{(1d)} \\ \nabla_x \cdot v = 0. & \text{(1e)} \end{cases}$$

In this system a, ν, D, λ, R and b are constants. The symbol '$\otimes$' denotes the tensorial product and 'Id' the 3×3 identity matrix. The symbol $P_{\omega_i^\perp} = \mathrm{Id} - \omega_i \otimes \omega_i$ gives the orthonormal projection operator onto the sphere $\mathbb{S}^2$ at ω_i; the '$\circ$' symbol following it indicates that the Stochastic Differential Equation (1b) has to be understood in the Stratonovich sense. The terms $(B_t^i)_{t\geq 0}$, $i = 1, \ldots, N$ are independent Brownian motions in $\mathbb{R}^3$. The terms S, A are matrices that will be defined later. The operators Δ_x, ∇_x, $\nabla_x \cdot$ indicate the Laplacian, the gradient and the divergence in $\mathbb{R}^3$, respectively. The symbol δ_X is the delta distribution in $\mathbb{R}^3$ at $X \in \mathbb{R}^3$. Finally, $K = K(r) \geq 0$, $r \geq 0$, is a given sensing function.

The matrices A and S are the antisymmetric and symmetric parts of the linear flow $\nabla_x v$ (which is a matrix with components $(\nabla_x v)_{ij} = \partial_{x_i} v_j$, $i, j = 1, 2, 3$), respectively:

$$A(v) = \tfrac{1}{2}\left(\nabla_x v - (\nabla_x v)^T\right), \tag{2}$$

$$S(v) = \tfrac{1}{2}\left(\nabla_x v + (\nabla_x v)^T\right), \tag{3}$$

where the exponent 'T' indicates the transpose of the matrix.

The Vicsek model. Eqs. (1a)-(1c) correspond to the Vicsek model. In this model agent i moves at a constant speed a in the direction ω_i while trying to align its direction motion with the one of its neighbours (given by $\bar{\omega}_i$) up to some noise (given by the term dB_t^i). The coupling with the fluid is twofold: firstly, particles are dragged by the fluid, that is why the $v(X_i,t)$ term appears in the equation for X_i and, secondly, the fluid changes the orientation of the swimmers following Jeffery's equation:

$$\frac{d\omega_i}{dt} = P_{\omega_i^\perp}\left(\lambda S(v) + A(v)\right)\omega_i = \nabla_\omega\left[\lambda\frac{1}{2}\omega \cdot S\omega\right] + \frac{1}{2}(\nabla_x \times v) \times \omega, \tag{4}$$

which describes how a fluid changes the orientation of an spheroidal particle with aspect ration λ.

The Stokes equation. Eqs. (1d)-(1e) give the evolution for the velocity of the fluid v, where the right-hand side of Eq. (1d) is the coupling term with the swimmers. This term comes from what is called a force dipole. More details can be found in [1].

Results. In this talk we presented the motivation for presenting such a model; its coarse-grained analysis into continuum equations; stability analysis of the continuum equations; and extensions to consider inertial systems (Navier-Stokes equation), as well as, repulsion between swimmers. A detailed explanation of this along with a literature review can be found in Ref. [1].

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On the Mean-Field and Classical Limits of the N-Body Schrödinger Equation

François Golse
(joint work with Thierry Paul)

We are concerned with the following problem: to derive the Vlasov equation (with self-consistent potential) from the quantum dynamics of N identical particles in the large N, small $\hbar$ limit. Our approach is based on the definition of a pseudo-metric measuring the distance between classical and quantum densities that is analogous to the Monge-Kantorovich-Wasserstein distance with exponent 2 [9].

1. Comparing classical and quantum densities. A classical density P is a probability density on $\mathbf{R}^d \times \mathbf{R}^d$, whereas a quantum density is a bounded operator ρ on $\mathfrak{H} = L^2(\mathbf{R}^d)$ such that $\rho = \rho^* \geq 0$ and $\mathrm{trace}(\rho) = 1$. A *coupling* of P and ρ is a measurable function $Q \equiv Q(x,\xi)$ defined a.e. on $\mathbf{R}^d \times \mathbf{R}^d$ with values in the algebra $\mathcal{L}(\mathfrak{H})$, such that $Q(x,\xi) = Q(x,\xi)^* \geq 0$ for a.e. (x,ξ) and

$$\int_{\mathbf{R}^d \times \mathbf{R}^d} Q(x,\xi) dx d\xi = \rho\,, \qquad \mathrm{trace}(Q(x,\xi)) = P(x,\xi) \quad \text{for a.e. } (x,\xi)\,.$$

The *pseudo-metric* measuring the distance between P and ρ is

$$E_\hbar(P,\rho) := \sqrt{\inf_{Q \in \mathcal{C}(P,\rho)} \int_{\mathbf{R}^d \times \mathbf{R}^d} \mathrm{trace}(Q(x,\xi)^{1/2} C(x,\xi) Q(x,\xi)^{1/2}) dx d\xi} \in [0, +\infty]\,,$$

where $\mathcal{C}(P,\rho)$ designates the set of couplings of p and ρ, while the *transportation cost* $C(x,\xi)$ is the operator in the variable y parametrized by $(x,\xi) \in \mathbf{R}^d \times \mathbf{R}^d$ given by the formula

$$C(x,\xi) := |x-y|^2 + |\xi + i\hbar \nabla_y|^2\,.$$

We recall Schrödinger's coherent state at the point $(q,p) \in \mathbf{R}^d \times \mathbf{R}^d$:

$$|q,p\rangle(x) := (\pi\hbar)^{-d/4} e^{-|x-q|^2/2\hbar} e^{ip\cdot(x-q/2)/\hbar}\,.$$

For each Borel positive measure μ on $\mathbf{R}^d \times \mathbf{R}^d$, one defines the *Töplitz operator of symbol* μ as

$$\mathrm{Op}_\hbar^T[\mu] := \frac{1}{(2\pi\hbar)^d} \int_{\mathbf{R}^d \times \mathbf{R}^d} |q,p\rangle\langle q,p| \mu(dqdp)\,.$$

For each quantum density ρ on $\mathfrak{H}$ with integral kernel r, one defines the *Wigner transform of* ρ by the formula [8]

$$W_\hbar[\rho](x,\xi) := \tfrac{1}{(2\pi)^d} \int_{\mathbf{R}^d} r(x + \tfrac{1}{2}\hbar y, x - \tfrac{1}{2}\hbar y) e^{-iy\cdot\xi} dy\,.$$

(The right-hand side is to be understood as a Fourier-Plancherel transform.) It is easy to see that $W_\hbar[\rho]$ is real-valued, and that $W_\hbar[\rho]$ is in general not nonnegative. This positivity issue is solved by considering instead the *Husimi transform* of ρ, defined by the formula [8]

$$\widetilde{W}_\hbar[\rho] := e^{\hbar\Delta_{x,\xi}/4} W_\hbar[\rho] \geq 0\,.$$

Theorem A [4] Let P be a classical density such that

$$\int_{\mathbf{R}^d \times \mathbf{R}^d} (|x|^2 + |\xi|^2) P(x,\xi) dx d\xi < \infty\,. \tag{1}$$

(a) For each quantum density ρ on $\mathfrak{H}$

$$E_\hbar(P,\rho)^2 \geq \max(d\hbar, \mathrm{dist}_{MK,2}(P, \widetilde{W}_\hbar[\rho])^2 - d\hbar)\,.$$

(b) For each Borel probability measure μ on $\mathbf{R}^d \times \mathbf{R}^d$

$$E_\hbar(P, \mathrm{Op}_\hbar^T[(2\pi\hbar)^d \mu])^2 \leq \mathrm{dist}_{MK,2}(P,\mu)^2 + d\hbar\,.$$

The notation $\mathrm{dist}_{MK,2}$ designates the Wasserstein, or Monge-Kantorovich distance of exponent 2 defined in chapter 7 of [9].

2. Application to the mean-field limit. Let V be an even function such that $\nabla V \in W^{1,\infty}(\mathbf{R}^d)$. Consider the *N-body quantum Hamiltonian*

$$\mathcal{H}_N := \sum_{j=1}^N -\tfrac{1}{2}\hbar^2 \Delta_{x_j} + \frac{1}{N} \sum_{1 \leq j < k \leq N} V(x_j - x_k)\,.$$

A quantum density ρ_N on $\mathfrak{H}_N = \mathfrak{H}^{\otimes N} \simeq L^2(\mathbf{R}^{dN})$ is said to be *symmetric* if it commutes to all the unitary operators U_σ defined on $\mathfrak{H}_N$ by

$$U_\sigma \Psi(x_1, \ldots, x_N) := \Psi(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(N)})$$

as σ runs through the set of all permutations of $\{1, \ldots, N\}$. The *first marginal density* of a symmetric quantum density ρ_N on $\mathfrak{H}_N$ with integral kernel r_N is the quantum density on $\mathfrak{H}$, denoted by $\rho_{N:1}$, with integral kernel

$$\int_{\mathbf{R}^{d(N-1)}} r_N(x, z_2, \ldots, z_N, y, z_2, \ldots, z_N) dz_2 \ldots dz_N\,.$$

For each classical density f, denote by H_f the mean-field classical Hamiltonian

$$H_f(x,\xi) := \tfrac{1}{2}|\xi|^2 + \int_{\mathbf{R}^d \times \mathbf{R}^d} V(x-y) f(y,\eta) dy d\eta\,.$$

Theorem B [4] Let f^{in} be a classical density satisfying (1), and let $f \equiv f(t,x,\xi)$ be the solution of the Cauchy problem for the Vlasov equation

$$\partial_t f + \{H_f, f\} = 0\,, \qquad f\Big|_{t=0} = f^{in}\,.$$

On the other hand, let ρ_N^{in} be a symmetric quantum density on $\mathfrak{H}_N$, and let $\rho_N(t) := e^{-it\mathcal{H}_N/\hbar}\rho_N^{in}e^{it\mathcal{H}_N/\hbar}$, which is a symmetric quantum density on $\mathfrak{H}_N$ for all $t \in \mathbf{R}$. Then, setting $\Gamma := 2 + 4\max(1, \mathrm{Lip}(\nabla V)^2)$, one has, for each $t \geq 0$

$$E_\hbar(f(t), \rho_{N:1}(t))^2 \leq \frac{1}{N} E_\hbar((f^{in})^{\otimes N}, \rho_N^{in}(t))^2 e^{\Gamma t} + \frac{8\|\nabla V\|_{L^\infty}}{N}\frac{e^{\Gamma t}-1}{\Gamma}\,.$$

In particular, if $\rho_N^{in} := \mathrm{Op}_\hbar^T[(2\pi\hbar)^{dN}(f^{in})^{\otimes N}]$, one has

$$\mathrm{dist}_{MK,2}(f(t), \widetilde{W}_\hbar[\rho_{N:1}(t)])^2 \leq d\hbar(e^{\Gamma t}+1) + \frac{8\|\nabla V\|_{L^\infty}}{N}\frac{e^{\Gamma t}-1}{\Gamma}\,.$$

3. Extensions. Several remarks on the convergence rate obtained in Theorem B are in order.

(a) One can define a pseudo-metric between two quantum densities on $\mathfrak{H}$ by a similar procedure: see [3]. The pseudo-metric so obtained is used to establish a uniform as $\hbar \to 0$ convergence rate for the mean-field limit of the N-body quantum dynamics to the Hartree equation (the quantum analogue of the Vlasov equation). This result can be strengthened, and by an interpolation argument with the nonuniform in $\hbar$ convergence rate obtained from Cauchy-Kowalevski estimates on the BBGKY hierarchy, one arrives at a uniform in $\hbar \in (0,1)$ convergence rate for the mean-field limit of the N-body quantum dynamics to the Hartree equation. The missing details and a complete proof can be found in [7].

(b) One can extend Theorem A (a) to generalized Töplitz operators involving coherent states built on some non Gaussian $a \in \mathcal{S}(\mathbf{R}^d)$: see [5] for the elementary properties of the pseudo-distance introduced in [3] and its connection with this class of generalized Töplitz operators.

(c) Another approach of the mean-field limit of the quantum N-body problem involves a quantum analogue of the notion of N-particle empirical measure used to establish the mean-field limit of the N-body problem in classical mechanics [1, 2]. See [6] for a detailed presentation of this notion, and an application to the problem of finding a uniform in $\hbar \in (0,1)$ convergence rate for the mean-field limit.

(d) Finally, it could be interesting to study whether the pseudo-metric introduced here, or its analogue considered in [3] can be formulated in terms of some quantum variant of optimal transport — e.g. is there an analogue of the Benamou-Brenier variational formula (see chapter 8 of [9]) for this pseudo-distance?

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Singular-Regular decomposition of Green's function and its application to compressible Navier-Stokes equation

Shih-Hsien Yu
(joint work with Tai-Ping Liu)

In this talk we discuss an initial value problem for the compressible Navier-Stokes equation in the Lagrangian coordinate,

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x(v) = \mu(u_x/v)_x, \\ v(x,0) = v_0(x),\ u(x,0) = u_0(x). \end{cases} \tag{1}$$

There are extensive works on this subject, [2, 4, 3, 5, 6]. All those works require a high order regularity assumption on the initial data so that a priori estimate works to develop various nonlinear stability theories. The assumption on the regularities in the initial data (v_0, u_0) limits on the understanding of the nature of the mathematical model itself.

The goal the presentation is to show a decomposition of the Green's function $\boldsymbol{G}(x,t)$, for the PDE linearized around a constant state $(v_0, 0)$:

$$\begin{pmatrix} \partial_t & -\partial_x \\ -c^2\partial_x & \partial_t - \mu\partial_x^2 \end{pmatrix} \boldsymbol{G}(x,t) = 0 \text{ for } t > 0,\ x \in \mathbb{R},$$
$$\boldsymbol{G}(x,0) = \delta(x)\mathsf{I}_3,$$

into a singular-regular decomposition $\boldsymbol{G}(x,t) = \boldsymbol{G}^*(x,t) + \boldsymbol{G}^\sharp(x,t)$ through an asymptotic expansion of the semi-group

$$\hat{\boldsymbol{G}}(\eta,t) = \mathsf{M}_+(\eta)e^{\lambda_+(\eta)t} + \mathsf{M}_-(\eta)e^{\lambda_-(\eta)t}$$

at the Fourier variable η at ∞. (The decomposition was first introduced in [1] for a 3-D model.) It yields that there exist $C, K_0, K_1 > 0$ such that for $x \in \mathbb{R}$, $t > 0$ it is satisfied

$$\begin{cases} \left| \boldsymbol{G}^*(x,t) - \begin{pmatrix} \delta(x)e^{-t/\mu} & 0 \\ 0 & \chi(1-t)K_1 \frac{e^{-\frac{x^2}{K_0 t}}}{\sqrt{t}} \end{pmatrix} \right| \le O(1)e^{-(|x|+t)/C}, \\ \left| \boldsymbol{G}^\sharp(x,t) \right| \le O(1) \frac{e^{-\frac{(x+ct)^2}{C(t+1)}} + e^{-\frac{(x-ct)^2}{C(t+1)}}}{\sqrt{t+1}} + e^{-(|x|+t)/C}, \end{cases}$$

where $\boldsymbol{c}$ is the sound speed at rest, $\chi(x)$ is the Heaveside function. With this structure, one proceeds to remove the regularity assumption. One includes the solution of the nonlinear problem in the dissipation as part of the paramaters in the construction of the Green's function as a form of nonlinear Green's function. Finally, with the nonlinear Green's function one can rewrite the quasi-linear PDE as a system of integral equations of order zero with an integrable kernel in space-time domain; and kernel is constructed pointwise exponentially sharp in the space-time domain. The form of the integral equation requires no assumption on the regularity and the global pointwise structure of the solution is a simple consequence of the pointwise structure of the kernel functions, the precise informations in the initial data, and Picard's iteration to result in the theory:

Theorem. *Suppose that the initial data* $(v_0-1, u_0) = \varepsilon e^{-x}\chi(x)$ *and* $\boldsymbol{c} = 1$. *Then, there are* $\varepsilon_0 > 0$, C_0, *and* $K_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *the solution of* (1) *satisfies*

$$\begin{aligned}(|v-1|, |u|) \leq O(1)\varepsilon \frac{e^{-\frac{(x+t)^2}{t+1}} + e^{-\frac{(x-t)^2}{t+1}}}{\sqrt{1+t}} + O(1)\varepsilon e^{-(|x|+t)/C_0} \\ + O(1)\varepsilon^2 \left(\chi_{[-(t+1),(t+1)]} \left(\frac{1}{|x+t| + \sqrt{t+1}} + \frac{1}{|x-t| + \sqrt{t+1}} \right) \right),\end{aligned}$$

where $\chi_{[a,b]}$ *stands for characteristic function for the set* $[a, b]$.

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Open problems: Uncertainty quantification for kinetic equations

Lorenzo Pareschi

In spite of the vast amount of existing research, both theoretically and numerically, the study of kinetic equations has mostly remained deterministic and ignored uncertainty. In reality, many sources of uncertainties can arise in these equations:

- Incomplete knowledge of the interaction mechanism between particles.
- Imprecise measurements of the initial and boundary data.
- Other sources of uncertainty like forcing and geometry, etc.

Understanding the impact of these uncertainties is critical to the simulations of the kinetic systems to validate the models and to obtain more reliable predictions.

Uncertainty quantification in kinetic equations represents a computational challenge for many reasons. Simple tasks such as the estimation of statistical properties of the solution typically require multiple calls to a deterministic solver. A single solver call is already very expensive for such complex mathematical models. In addition, the schemes must deal with the intrinsic structural properties of the solution. Non-negativity of the distribution function, conservation of invariant quantities, entropy dissipation and steady states are essential in order to compute qualitatively correct solutions. We will survey some open problems and new research directions in this field.

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Open problems in collective dynamics

Pierre Degond

Collective dynamics has raised considerable interest in the last decade. It occurs in systems of self-propelled particles (such as animal groups) when patterns of motion emerge on scales much large than the inter-individual scales. Collective dynamics is often understood as a phase transition, as the collective states only appear for certain ranges of parameters, while the system stays in a disordered state for other values of these parameters. There has been considerable debate about the nature of such phase transitions among physicists. In an attempt to clarify the situation, Vicsek and co-authors have proposed a particle model [1] with a minimal set of parameters and have shown that phase transitions to order appear either at low noise or large densities. Recently, a kinetic model derived from the Vicsek model has been proposed and studied in [2]. It has given exact critical values of the parameters for these transitions to occur, as well as their nature (first or second order). It has rigorously proved nonlinear stability or instability of the equilibria and in the stable case, provided sharp convergence rates to these equilibria. However, this study is restricted to the spatially homogeneous case. The techniques relying on a variational structure of the problem can't be extended to the spatially inhomogeneous case, because the variational structure of the problem is lost in this case. Therefore, the mathematical study of equilibria and phase transitions for the spatially-nonhomogeneous kinetic Vicsek model is totally open and requires the forgeing of new mathematical tools that can replace the inapplicable variational techniques.

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Open problems: Different approaches to Quantum particles through optimal transportation methods

Yann Brenier

There have been several recent attempts to treat systems of quantum particles through the concept of optimal transportation. I would like to quote, in particular, the work of Carlen and Mas [2] on one hand and the work of Golse, Mouhot and Paul [3] on the other hand. It seems reasonable to think that these concepts, which look quite different, must be related in some way. In addition, there has been some years ago several concepts of optimal transport in the framework of free probability theory, in particular by Biane and Voiculescu [1], and in non commutative geometry (see [4] for a recent reference). So, in my opinion, it would be very fruitful to compare all these approaches.

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