Mathematisches Forschungsinstitut Oberwolfach

Report No. 9/2018

DOI: 10.4171/OWR/2018/9

Mini-Workshop: Chromatic Phenomena and Duality in Homotopy Theory and Representation Theory

Organised by
Tobias Barthel, Copenhagen
Henning Krause, Bielefeld
Vesna Stojanoska, Urbana

4 March – 10 March 2018

Abstract. This mini-workshop focused on chromatic phenomena and duality as unifying themes in algebra, geometry, and topology. The overarching goal was to establish a fruitful exchange of ideas between experts from various areas, fostering the study of the local and global structure of the fundamental categories appearing in algebraic geometry, homotopy theory, and representation theory. The workshop started with introductory talks to bring researches from different backgrounds to the same page, and later highlighted recent progress in these areas with an emphasis on the interdisciplinary nature of the results and structures found. Moreover, new directions were explored in focused group work throughout the week, as well as in an evening discussion identifying promising long-term goals in the subject. Topics included support theories and their applications to the classification of localizing ideals in triangulated categories, equivariant and homotopical enhancements of important structural results, descent and Galois theory, numerous notions of duality, Picard and Brauer groups, as well as computational techniques.

Mathematics Subject Classification (2010): 13D, 14C, 16D, 18E, 20C, 55M, 55N, 55P, 55U.

Introduction by the Organisers

A spur of flourishing interactions between algebra, geometry, and homotopy theory was initiated by the seminal work of Devinatz, Hopkins, and Smith classifying the thick subcategories of the homotopy category of finite spectra. It was an unprecedented structural result that in particular organized and vastly generalized previous computational advancements in chromatic homotopy theory, and inspired analogous classifications in other settings, which aid the understanding of localizations. Neeman after Hopkins, for example, related the thick and localizing subcategories of the derived category of a ring to certain subsets of the prime spectrum of the ring. These classical results and their various analogues have been streamlined into tensor-triangulated geometry as developed by Balmer in the last decade. Balmer's formalization of the theory has been crucial in the expansion of chromatic techniques to new areas, most notably modular representation theory.

The aim of this mini-workshop was to gather researchers from homotopy theory, algebraic and triangulated geometry, and modular representation theory, in order to showcase recent advancements, as for example the development of stratifications of triangulated categories, new classification results in modular representation theory and stable equivariant homotopy theory, as well as advances in our understanding of various duality phenomena. A second goal was to facilitate new collaborations among the researches; discussions about the interactions among these fields also led to the coining of a new term to encompass them, *prismatic algebra*.

The main activities during the first two days of the workshop were the introductory lectures by Dell'Ambrogio (on tensor-triangulated geometry), Pevtsova (on modular representation theory), Schlank (on chromatic homotopy theory), and Neeman (on Grothendieck duality's latest conceptualization). The aim was to bring the researchers from various backgrounds to a common ground, so that deeper exchanges can happen as the week progresses.

At the same time, four work groups were formed to explore open questions in depth and detail, and as the week progressed, more and more time was spent on group work. At the end of most days, we convened for a report on progress by each group. In addition to all this, we had five talks showcasing the latest developments in the area, and on Thursday evening we had an informal discussion about the past and future of this interactive field, sharing open problems and questions.

Introductory talks. Dell'Ambrogio's lectures introduced the machinery of tensor-triangulated (tt-)geometry, which starts with the Balmer spectrum, a space associated to a tt-category and is built from thick ideals. The geometry of this space is related to localizations of the category in question, as is made precise by a theory of supports. Pevtsova's and Schlank's lectures were, in a sense, case studies of the general theory from Dell'Ambrogio's talk, although historically they came first and were the motivation behind the development of the general theory. Pevtsova talked about the stable module category of a finite group scheme in positive characteristic, where the thick subcategories are related to the prime spectrum of the cohomology ring of the group scheme in question. Schlank discussed chromatic homotopy theory, whereby the thick subcategories in stable homotopy theory are related to points on the moduli stack of formal groups.

Neeman's talk addressed the question of Grothendieck duality; while parts of the foundational theory are classically formal, other parts have been accepted as messy. Neeman presented the classical formal sides of the theory, and supplemented this with his recent results streamlining the sticky points.

Research talks. Even though the case of the stable homotopy category was the original example of a determination of a Balmer spectrum, through the thick subcategory theorem of Devinatz, Hopkins, and Smith, classifying the thick tensor ideals of the equivariant stable homotopy category of a finite group is very recent work. Noel spoke about the latest advances in this problem, building on previous work by Balmer, Sanders, and Strickland.

The talks of Greenlees and Castellana both discussed recent progress in duality in homotopy-theoretic settings. Greenlees explored various duality patterns appearing in the local cohomology spectral sequences for topological modular forms and $A(2)$, an important subalgebra of the dual Steenrod algebra. Castellana presented a stratification result for homotopical (p-local compact) groups, generalizing a theorem of Benson, Iyengar, and Krause for finite groups. In a nutshell, the analogous structural results hold, but the proofs require additional tools from homotopy theory.

Grodal gave a cohomological classification of endotrivial modules for arbitrary finite groups, obtained by homotopical methods. This amounts to computing the Picard group of the stable module category, and the results tied together extensive previous work of Alperin, Carlson, Dade, Thevenaz, and others.

Balmer explored the notion of residue fields in tt-geometry, using for a compactly generated tensor triangulated category the embedding (modulo phantom maps) into a Grothendieck category via the restricted Yoneda functor.

Group work. Inspired by questions from Neeman's talk, Krause, Neeman, and Pevtsova formed a group to explore the question of strong generation in tt-categories using chromatic methods and the theory of supports. Finding an appropriate framework for Grothendieck's local duality in the affine case for Gorenstein rings was another topic, because one wants to formulate an analogue for finite dimensional algebras via the action of Hochschild cohomology.

Castellana, Greenlees, and Grodal did some computations of the singularity and cosingularity categories of the ring of cochains on a classifying space of a group, to get a feeling for some unexplored structural properties. This was mostly inspired by a recent preprint by Greenlees and Stevenson.

Balmer, Dell'Ambrogio, Ricka, Sanders, and Stojanoska discussed questions related to Gorenstein and Anderson duality. One was to formalize Gorenstein duality in an abstract tensor-triangulated setting, in a way that lends itself to studying descent for dualizing modules. Another was to use techniques from relative homological algebra to establish vast generalizations of Anderson duality.

Barthel, Beaudry, Heard, Noel, and Schlank introduced Brauer spectra into modular representation theory and worked on computations of Brauer groups of stable module categories using descent-theoretic methods among other techniques.

Finally, in the **informal evening discussion** on Thursday, the victories as well as challenges of this interactive field were discussed, including several open problems of more long-term and open-ended nature than in the group work part of the workshop.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Paul Balmer in the "Simons Visiting Professors" program at the MFO.

Mini-Workshop: Chromatic Phenomena and Duality in Homotopy Theory and Representation Theory

Table of Contents

Abstracts

An introduction to tensor triangular geometry

Ivo Dell'Ambrogio

Tensor triangular geometry is the geometric study of tensor triangulated categories. As such, it can be seen as a way of extracting the intrinsic geometry of *prismatic algebra* – the stable intersection of topology, geometry, representation theory, etc. Of the possible ways to approach this subject, in this talk we focus on the one due to Paul Balmer, based on the notion of the *spectrum* of a tensor triangulated category. Our main goal is to set up some basic vocabulary and examples for the mini-workshop.

Tensor triangulated categories tend to arise in two flavors, 'big' ones, here denoted $\mathcal{T}$, and 'small' ones, denoted $\mathcal{K}$. Typically, the small $\mathcal{K}$ occur as the subcategory of compact-and-rigid objects, $\mathcal{K} = \mathcal{T}^c$, in a compactly generated $\mathcal{T}$.

Convention. To be more precise, in the following $\mathcal{K}$ will denote an essentially small tensor triangulated category, that is, a triangulated category equipped with a symmetric monoidal structure $\otimes\colon \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ with tensor unit object $\mathbf{1}$. For convenience here we assume $\mathcal{K}$ to be *rigid*, so that every object x admits a tensor dual $x^\vee$: $\mathcal{K}(x \otimes y, z) \cong \mathcal{K}(y, x^\vee \otimes z)$. By $\mathcal{T}$ we will denote a *rigidly-compactly generated* category, that is a compactly generated triangulated category which is tensor triangulated and such that its compact and rigid objects coincide; in all examples below, $\mathcal{K} = \mathcal{T}^c$ is the full tensor triangulated subcategory of these objects. Such $\mathcal{T}$ were called *unital algebraic stable homotopy categories* in the seminal work by Hovey-Palmieri-Strickland [10].

Examples abound in nature. Here are some fundamental ones:

Examples. (1) The ur-example is the stable homotopy category $\mathcal{T} = SH$, with $\mathcal{K} = SH^c$ the category of finite spectra.

(2) Generalising this (which is the case when $A = \mathbb{S}$ is the sphere spectrum), we have the derived category $\mathcal{T} = D(A)$ of any (sufficiently structured, that is E_4) commutative ring spectrum. This includes E_∞-rings, commutative dg-algebras, and of course ordinary (disctrete) commutative rings.

(3) Generalizing in the 'global' direction, we have the derived category $\mathcal{T} = D_{\mathrm{Qcoh}}(X)$ of any quasi-compact and quasi-separated scheme X, with $\mathcal{K} = D^{\mathrm{perf}}(X)$ the category of perfect complexes. Some classes of more general stacks also have similarly nice derived categories.

(4) In modular representation theory, we can take $\mathcal{T} = \mathrm{StMod}(kG)$ the stable module category of a finite group (or group scheme) G or the homotopy category of injective modules $\mathcal{T} = K(\mathrm{Inj}\, kG)$. The compact objects are respectively the stable category $\mathrm{stmod}(kG)$ and the bounded derived category $D^b(kG)$ of finite dimensional representations.

(5) We also have motivic, equivariant, $\mathbb{A}^1$-homotopic, etc. variants of the stable homotopy category.

Tensor triangular geometry starts with the following definition and theorem:

Definition (Balmer 2005)**.** The *spectrum* of $\mathcal{K}$, denoted $\mathrm{Spc}(\mathcal{K})$, is the set of tensor-ideal thick subcategories $\mathcal{P} \subset \mathcal{K}$ which are *prime*, that is proper ($\mathcal{P} \neq \mathcal{K}$) and satisfying $x \otimes y \in \mathcal{P} \Leftrightarrow x \in \mathcal{P}$ or $y \in \mathcal{P}$. It becomes a nice ('spectral') topological space when endowed with the *Zariski topology*, which has a basis of closed subsets given by the *supports* $\mathrm{supp}(x) := \{\mathcal{P} \mid x \notin \mathcal{P}\}$ of objects $x \in \mathcal{K}$.

Theorem (Balmer [1], Buan-Krause-Solberg [8])**.** *The lattice of thick tensor ideals of $\mathcal{K}$ is Hochster dual to the lattice of open subsets of the Balmer spectrum. Concretely, the maps $\mathcal{J} \mapsto \bigcup_{x\in\mathcal{J}} \mathrm{supp}(x)$ and $S \mapsto \{x \mid \mathrm{supp}(x) \subseteq S\}$ induce an inclusion-preserving bijection*

$$\big\{\textit{thick } \otimes\textit{-ideals } \mathcal{J} \subseteq \mathcal{K}\big\} \overset{\sim}{\leftrightarrow} \big\{\textit{dual-open subsets } S \subseteq \mathrm{Spc}(\mathcal{K})\big\}$$

where a subset $S \subseteq \mathrm{Spc}(\mathcal{K})$ is dual-open (or Thomason*) if it is a union of closed subsets, each of which has a quasi-compact open complement.*

By the above theorem, the Balmer spectrum always encodes a geometric classification of thick tensor-ideals.

The converse is also true: any (nice) space allowing such a classification must be homeomorphic to the spectrum. Thus we may convert known classification theorems into computations of the spectrum. For instance, it follows from the celebrated Thick Subcategory Theorem of Devinatz-Hopkins-Smith [9] that the space $\mathrm{Spc}(SH^c)$ has a beautiful description as the union of the chromatic towers for all prime numbers.

Subsequent work of Hopkins, Neeman [11] and Thomason [12] transposed the classification to commutative algebra and algebraic geometry. It follows that if A is a (discrete) commutative ring, the spectrum $\mathrm{Spc}(D(A)^c)$ is just the Zariski spectrum of A, and if X is a scheme $\mathrm{Spc}(D^{\mathrm{perf}}(X)) \cong X$ recovers the scheme.

One long-standing goal of axiomatic tensor triangular geometry was to invert the flow: to compute new spectra by tensor-triangular methods and to derive from this some new, non-obvious classification theorems. This has recently been achieved, in spectacular fashion, in the case of the (genuine) G-equivariant stable homotopy category $\mathcal{T} = SH(G)$:

Example. Balmer-Sanders [3] determined the spectrum $\mathrm{Spc}(SH(G)^c)$, as a set for all finite groups G, and as a space if the order is square-free. Then Barthel-Hausmann-Naumann-Nikolaus-Noel-Stapleton [4] computed the topology for all abelian G. This yields a very interesting new family of classification theorems.

Generally speaking, there are two strategies for approaching the computation of the spectrum in examples: one can try to approach this space from the left or from the right.

Approximation from the left. The assignment $x \mapsto \mathrm{supp}(x)$ from objects to closed subsets of $\mathrm{Spc}(\mathcal{K})$ is completely compatible with the tensor-triangular structure of $\mathcal{K}$ and is the universal such: given any other *support data* (T, σ)

consisting of a (spectral) space and such a compatible assignment σ, there is a unique continuous map

$$\lambda\colon T \longrightarrow \mathrm{Spc}(\mathcal{K})$$

such that $\sigma(x) = \lambda^{-1}\mathrm{supp}(x)$ for all objects. If one is particularly lucky, it may even be possible to extend σ to a support theory $\widetilde{\sigma}$ for *all* objects of $\mathcal{T}$, not just the compact ones. It is easy to construct some $\widetilde{\sigma}$ satisfying basic compatibilities, and if one works hard it may even satisfy the following special properties:

- *Tensor formula:* $\widetilde{\sigma}(x \otimes y) = \widetilde{\sigma}(x) \cap \widetilde{\sigma}(y)$ at least if one of x, y is compact.
- *Realization:* $U \subseteq T$ is quasi-compact open $\Leftrightarrow T \backslash U = \widetilde{\sigma}(x)$ for a compact x.
- *Detection of objects:* $\widetilde{\sigma}(x) = \emptyset$ implies that $x = 0$.

In this case, the left approximation λ must be a homeomorphism. Moreover, with extra work, this may lead to *stratification* (Benson-Iyengar-Krause [6]), that is a geometric classification of all localizing tensor ideals in $\mathcal{T}$.

This strategy was originally employed by Benson-Carlson-Rickard [5] on the stable module category of a finite group, yielding the remarkable computation

$$\mathrm{Spc}(\mathrm{stmod}\, kG) \cong \mathrm{Proj}(H^*(G; k))$$

and the result was generalized to all finite group schemes by Benson-Iyengar-Krause-Pevtsova [7]. The first stratification theorem was obtained by Neeman [11] for the derived category of a commutative noetherian ring.

Approximation from the right. In complete generality, there is also a continuous map

$$\rho\colon \mathrm{Spc}(\mathcal{K}) \longrightarrow \mathrm{Spec}_{\mathrm{Zar}}(R_{\mathcal{K}})$$

where $R_{\mathcal{K}} = \mathrm{End}_{\mathcal{K}}(\mathbf{1})$ is the endomorphism ring of the unit, which is automatically commutative and thus has a Zariski spectrum. This map has graded versions and other variants, and they all localize well with respect to the action of $R_{\mathcal{K}}$ (or its graded version, etc.) on $\mathcal{K}$ induced by the tensor product. As a result one can reduce the computation of the spectrum to the – usually simpler – local case, with respect to localization over $\mathrm{Spec}_{\mathrm{Zar}}(R_{\mathcal{K}})$. This is part of the strategy for the above topological examples and several others.

We have barely scratched the surface here. In particular we have not mentioned what one can *do* with the spectrum (other than obtaining classification results). For this we refer to the early but extensive survey Balmer [2], and for more recent developments we refer to the mini-workshop's other talks.

References

[1] P. Balmer, *The spectrum of prime ideals in tensor triangulated categories*, J. Reine angew. Math. **588**, (2005) 149–168.

[2] P. Balmer, *Tensor triangular geometry*, Proceedings ICM, Hyderabad (2010), Vol. II, 85–112.

[3] P. Balmer, B. Sanders, *The spectrum of the equivariant stable homotopy category of a finite group*, Invent. Math. **208** (2017), no. 1, 283–326.

[4] T. Barthel, M. Hausmann, N. Naumann, T. Nikolaus, J. Noel, N. Stapleton, *The Balmer spectrum of the equivariant homotopy category of a finite abelian group*, preprint 2017, arXiv:1709.04828

[5] D. J. Benson, J. F. Carlson, J. Rickard, *Thick subcategories of the stable module category*, Fund. Math. **153** (1997), no. 1, 59–80.
[6] D. Benson, S. Iyengar, H. Krause, *Local cohomology and support for triangulated categories*, Ann. Sci. Éc. Norm. Supér. (4) **41** (2008), no. 4, 573–619.
[7] D. Benson, S. Iyengar, H. Krause, J. Pevtsova, *Stratification for module categories of finite group schemes*, J. Amer. Math. Soc. **31** (2018), no. 1, 265–302.
[8] A. B. Buan, H. Krause, Ø. Solberg, *Support varieties: an ideal approach*, Homology, Homotopy Appl. **9** (2007), no. 1, 45–74.
[9] M. J. Hopkins, J. H. Smith, *Nilpotence and stable homotopy theory II*, Ann. of Math. (2) **148** (1998) no. 1, 1–49.
[10] M. Hovey, J. H. Palmieri and N. P. Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc. **128** (1997).
[11] A. Neeman, *The chromatic tower for $D(R)$*, With an appendix by Marcel Bkstedt. Topology **31** (1992), no. 3, 519–532.
[12] R. W. Thomason, *The classification of triangulated subcategories*, Compositio Math. **105**, no. 1 (1997).

Tensor triangular geometry for finite group schemes

Julia Pevtsova

The purpose of this extended abstract is to give a snapshot of what was covered in a two lecture survey given at the beginning of the mini-workshop on "Chromatic phenomena and duality in homotopy theory and representation theory". My task was to focus on the (modular) representation part of the story. I narrowly interpreted the "chromatic phenomena in representation theory" as the description of the structure of thick and localizing subcategories in the stable module - and related - categories of a finite group scheme. Using the language introduced by Paul Balmer, one can reformulate this as a question about the spectra of the corresponding categories, or more generally, about their properties with respect to tensor triangular geometry.

We start with some terminology. A finite group scheme G defined over a field k is a representable functor:

$$G : \{\text{comm } k\text{-algebras}\} \to \{\text{groups}\}$$

such that the representing algebra $k[G]$ is finite dimensional as a vector space over k. For what follows, we assume that k has positive characteristic p. Dualizing the coordinate algebra, we get the *group algebra* kG which is a finite dimensional cocommutative Hopf algebra. This correspondence gives an equivalence of categories

$$\left\{ \begin{array}{c} \text{finite group} \\ \text{schemes} \end{array} \right\} \sim \left\{ \begin{array}{c} \text{finite dimensional co-} \\ \text{commutative Hopf algebras} \end{array} \right\}$$

Examples of these structures include finite groups, restricted Lie algebras and Frobenius kernels of algebraic groups. Representations of a finite group scheme G are equivalent to representations of its group algebra kG. Since the latter is Frobenius, one can construct the stable module categories $\operatorname{Stmod} G$ and $\operatorname{stmod} G$

of all and finite dimensional representations of G which are tensor triangulated categories. Hence, one can study tensor triangular geometry, or *tt-geometry*, in this context.

Following the fundamental principles of tt-geometry, we seek to construct a *support* map

$$\operatorname{supp} : \operatorname{Stmod} G \to X$$

for a topological space X which captures the basic structure of our category. This is where cohomology enters into the picture as we set

$$X = \mathsf{Proj}\, H^*(G, k).$$

By a theorem of Friedlander and Suslin, the (graded commutative) cohomology ring $H^*(G, k)$ is a finitely generated k-algebra; hence, X is a projective variety of finite type.

It is known, for example by the work of I. Dell'Ambrogio, that for the support map to classify the tensor ideals in $\operatorname{stmod} G$, which is the category of *compact* objects in $\operatorname{Stmod} G$, it suffices to show that it satisfies the following list of properties.

(1) $\operatorname{supp} k = X$, $\operatorname{supp}(0) = \emptyset$;
(2) "2 out of 3". If $M_1 \to M_2 \to M_3 \to$ is a triangle in $\operatorname{Stmod} G$, then $\operatorname{supp} M_2 \subset \operatorname{supp} M_1 \cup \operatorname{supp} M_3$;
(3) $\oplus$. $\operatorname{supp} M \oplus N = \operatorname{supp} M \cup \operatorname{supp} N$ for any $M, N \in \operatorname{Stmod} G$;
(4) Shift. $\operatorname{supp} M = \operatorname{supp} \Omega^{-1} M$ for $M \in \operatorname{Stmod} G$, and Ω^{-1} the Heller shift;
(5) Realization. For any closed subset $Y \subset X$, $\exists\ M \in \operatorname{stmod} G$ such that $\operatorname{supp} M = Y$;
(6) Detection. $\operatorname{supp} M = \emptyset \Leftrightarrow M \cong 0$ in $\mathsf{StMod}\, G$;
(7) Tensor product property. $\operatorname{supp} M \otimes N = \operatorname{supp} M \cap \operatorname{supp} N$ for any $M, N \in \operatorname{Stmod} G$.

In Balmer's terminology, this would say that this support theory is universal for $\operatorname{stmod} G$.

To achieve the explicit construction of the universal support theory for $\operatorname{Stmod} G$ one constructs not one, but two support theories. The first one is the theory of π-supports of Friedlander and Pevtsova which relies on the notion of a π-point. This construction is inspired by Carlson's rank variety for elementary abelian p-groups. The other approach, which takes its roots in the classical cohomological support variety, is the Benson-Iyengar-Krause theory of supports via local cohomology functors

$$\Gamma_{\mathfrak{p}} : \operatorname{Stmod} G \to \operatorname{Stmod} G$$

for $\mathfrak{p} \in \mathsf{Proj}\, H^*(G, k)$.

In a joint work with Benson, Iyengar and Krause we show that these two theories coincide for finite group schemes, thereby producing a universal support theory in that context. As an application, we classify localizing (and colocalizing) tensor ideals in $\operatorname{Stmod} G$ in the usual way: namely, we prove that there is one-to-one correspondence

$$\left\{ \begin{array}{c} \text{Localizing } \otimes\text{-ideals} \\ \text{subcategories of } \mathsf{StMod}\, G \end{array} \right\} \sim \left\{ \begin{array}{c} \text{subsets of} \\ \mathsf{Proj}\, H^*(G, k) \end{array} \right\}$$

given by support.

For this result we need to develop one other new technique, that of a "reduction to a closed point", relating the functors $\Gamma_{\mathfrak{p}}$ and $\Gamma_{\mathfrak{m}}$ for a point $\mathfrak{p}$ in $X = \mathsf{Proj}\, H^*(G, k)$ with a residue field $K = k(\mathfrak{p})$ and a closed point $\mathfrak{m}$ in X_K lying over $\mathfrak{p}$. This relationship, coming from commutative algebra, led to another application which was the last topic of my two lectures: namely, Gorenstein duality for $\operatorname{stmod} G$.

Categorical introduction to chromatic homotopy theory

Tomer Schlank

I gave two introductory talks on chromatic homotopy. The talks were given using the Balmer spectrum of the symmetric monoidal ∞-category of spectra as a starting point. We use this starting point to discuss types of finite complexes, and the notion of $K(n)$-local spectra. I then defined Morava E-theory as the Galois closure (in sense of Rognes) of the $K(n)$-local sphere. This way it is possible to present the main ingredients of chromatic homotopy theory from a purely categorical point of view (rather than using the Landweber exact functor theorem and the theory of formal groups). These ingredients include:

(1) The E_n-local category and E_n-localisation.
(2) The Morava stabiliser group and its action on Morava $E(n)$-theory.
(3) The chromatic fracture square.
(4) The chromatic convergence theorem.
(5) Telescopic localisation.
(6) The telescope conjecture.
(7) v_n-self maps and the $K(n)$-local sphere.
(8) Ambidexterity of the $K(n)$-local category.

Finally, using ambidexterity (more specifically the dualisability of the $K(n)$-localisation of the suspension spectra of classifying spaces of finite groups) I connected the theory back to the notion of a formal group law.

Grothendieck duality made simple—a brief survey

Amnon Neeman

There are two classical paths to the foundations of Grothendieck duality: one due to Grothendieck and Hartshorne [3] and (much later) Conrad [1], and a second due to Deligne [2] and Verdier [8] and (much later) Lipman [5]. The consensus has been that both are unsatisfactory. Until the recent past no one knew a clean way to set up the theory.

This changed dramatically about three years ago. However: even though the main articles have already appeared in print, only a few experts have been aware of the developments—the papers presenting the results have all focused not so much on the simple proofs of the old theorems, but rather on the technical advances made possible by the new insights. In the talk I took the opposite tack: the theorems presented were relatively small technical improvements on what may be found in Hartshorne [3], the emphasis was on the clean, modern approach to the proofs.

What is perhaps more remarkable is that, with one exception, the modern avenue to the foundations of Grothendieck duality was paved and ready for use already in the mid-1990s. The key new ingredient, which removed the last remaining obstacles, may well appear to be a small, minor step—especially when presented as part of the whole picture. In fact: in a talk I gave at Macquarie University in September 2017, presenting the results to a seminar of category theorists, Steve Lack reacted by asking why it took us so long to see our way through.

In the Oberwolfach talk I tried to explain this. The new insight might seem small in hindsight, but required quite a leap of imagination. It hinged on studying a certain map using the chromatic tools that formed the core subject of the workshop, and applying these tools to a morphism which—on the face of it—seems totally worthless.

To make this more concrete: the morphism—which proved to be key to the recent progress—has been around for 50 years now, and was dismissed as useless by some of the most eminent mathematicians of the era.

The reader is referred to [7] for a more extensive survey, and to [4, 6] for the published accounts [written for the experts] of recent progress.

References

[1] Brian Conrad, *Grothendieck duality and base change*, Lecture Notes in Mathematics, vol. 1750, Springer-Verlag, Berlin, 2000.

[2] Pierre Deligne, *Cohomology à support propre en construction du foncteur $f^!$*, Residues and Duality, Lecture Notes in Mathematics, vol. 20, Springer–Verlag, 1966, pp. 404–421.

[3] Robin Hartshorne, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966.

[4] Srikanth B. Iyengar, Joseph Lipman, and Amnon Neeman, *Relation between two twisted inverse image pseudofunctors in duality theory*, Compos. Math. **151** (2015), no. 4, 735–764.

[5] Joseph Lipman, *Notes on derived functors and Grothendieck duality*, Foundations of Grothendieck duality for diagrams of schemes, Lecture Notes in Mathematics, vol. 1960, Springer, Berlin, 2009, pp. 1–259.

[6] Amnon Neeman, *Traces and residues*, Indiana Univ. Math. J. **64** (2015), no. 1, 217–229.

[7] ———, *Grothendieck duality made simple*, (in progress).

[8] Jean-Louis Verdier, *Base change for twisted inverse images of coherent sheaves*, Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), Oxford Univ. Press, London, 1969, pp. 393–408.

My current favourite duality pictures (the local cohomology theorems for tmf and $H^{*,*}(\mathcal{A}(2))$)

J.P.C. Greenlees

(joint work with R.R.Bruner, J.Rognes)

The aim of the talk was to consider duality properties related to tmf at the prime 2 and especially to make explicit and pictorial the duality this implies for coefficient rings.

The local cohomology theorem for tmf_*

Gorenstein duality for tmf gives a local cohomology spectral sequence for tmf_*. This takes the form

$$H^*_J(\mathrm{tmf}_*) \Rightarrow \Sigma^{-22}\pi_*(\mathbb{Z}^{\mathrm{tmf}}).$$

Here $\mathbb{Z}^{\mathrm{tmf}}$ denotes the Anderson dual of tmf and $J = (\beta_1, \beta_2)$ is an ideal of tmf_* with radical the ideal $\mathrm{tmf}_{>0}$ of positive degree elements, where β_1 (of degree 8) is essentially the Bott element and β_2 (of degree 192) is a periodicity element.

A picture was displayed showing explicitly what this means for the coefficient ring.

The local cohomology theorem for $H^{*,*}(\mathcal{A}(2))$**.**

The coefficient ring tmf_* can be calculated by an Adams spectral sequence, and the bigraded Ext group at the E_2-term (i.e., the cohomology $H^{*,*}(\mathcal{A}(2))$ of the algebra $\mathcal{A}(2) = \langle Sq^1, Sq^2, Sq^4\rangle$) enjoys a precisely analogous duality. The consequence of this duality is a Local Cohomology Spectral Sequence of bigraded algebras

$$H^*_{Jh}(H^{*,*}(\mathcal{A}(2)) \Rightarrow \Sigma^{-(23,0)}H^{*,*}(\mathcal{A}(2))^\vee,$$

where $Jh = (h_0, g, w_1, w_2)$ is a bihomogeneous ideal whose radical is the augmentation ideal of $H^{*,*}(\mathcal{A}(2))$, and where $(23, 0)$ refers to the Adams grading.

A picture was displayed showing the striking duality this implies for the bigraded algebra $H^{*,*}(\mathcal{A}(2))$.

References

[1] R.R.Bruner, J.P.C.Greenlees and J.Rognes "The Local Cohomology Theorems for tmf and $H^{*,*}(\mathcal{A}(2))$ In preparation, 31pp

[2] R.R.Bruner and J.Rognes "The coefficient ring tmf_*." In preparation

[3] J.P.C.Greenlees "Homotopy invariant commutative algebra over fields" CRM IRTATCA lectures (2015), (to appear) 61pp, arXiv:1601.024737

[4] J.P.C.Greenlees and V.Stojanoska "Anderson and Gorenstein duality" Preprint (2017), 21pp, arXiv: 1705.02664

Stratification and duality for homotopical groups

Natàlia Castellana
(joint work with Tobias Barthel, Drew Heard, Gabriel Valenzuela)

Let G be a finite group or a connected Lie group, and k a field of characteristic p. Benson–Iyengar–Krause (see [4]) developed the notion of stratification of a triangulated category by a Notherian commutative ring, using support theoretic techniques, which captures both the classification of thick and localizing subcategories. For G finite, this machinery was used to prove stratification results for $\mathrm{StMod}(kG)$, $K(\mathrm{Inj}kG)$ and $D(C^*(BG,k))$. For G a connected compact Lie group, Benson and Greenlees [3] proved that $D(C^*(BG,k))$ is stratified by the canonical action of $H^*(BG,k)$.

In this talk, new examples of stratification results will be presented coming from homotopical generalizations of classifying spaces of compact Lie groups at a prime p called p-local compact groups, introduced by Broto-Levi-Oliver [6]. They generalize previous results showing the statement only depends on the p-local information (in a group theoretic sense) of G.

For a commutative ring spectrum R we write Mod_R for the category of R-modules. For $M, N \in \mathrm{Mod}_R$ we write $M \otimes_R N$ for the monoidal product of R, and $\mathrm{Hom}_R(M,N)$ for the spectrum of R-module morphisms between M and N.

Given a space X we write X_+ for the suspension spectrum $\Sigma^\infty_+ X$ and $C^*(X,R)$ for $F(X_+,R)$, the spectrum of R-valued cochains on X_+. If R is a commutative ring spectrum, then so is $C^*(X,R)$. Often $R = Hk$ will be the Eilenberg–MacLane spectrum of a discrete commutative ring k; we simply write $C^*(X,k)$.

A subcategory $\mathcal{T} \subseteq \mathcal{C}$ of a stable ∞-category is called thick if it is closed under finite colimits, retracts, and desuspensions, and $\mathcal{T}$ is called localizing (respectively colocalizing) if it is closed under all filtered colimits (respectively all filtered limits).

Definition 1. A commutative ring spectrum R is called Noetherian if π_*R is Noetherian.

A morphism $f\colon R \to S$ of commutative ring spectra induces a triple of adjoints $(\mathrm{Ind}, \mathrm{Res}, \mathrm{Coind})$ between Mod_R and Mod_S, where $\mathrm{Ind}\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ is induction $- \otimes_R S$, $\mathrm{Res}\colon \mathrm{Mod}_S \to \mathrm{Mod}_R$ is restriction along f, and $\mathrm{Coind}\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ is coinduction, given by $\mathrm{Hom}_R(S,-)$. We denote by res the induced morphism between the homogeneous prime ideal spectra $\mathrm{Spec}^h(\pi_*(S)) \to \mathrm{Spec}^h(\pi_*(R))$.

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be conservative if it reflects equivalences.

Definition 2. A morphism of Noetherian commutative ring spectra $f\colon R \to S$ is said to satisfy Quillen lifting if for any two modules $M, N \in \mathrm{Mod}_R$ such that there is $\mathfrak{p} \in \mathrm{res}\,\mathrm{supp}_S(\mathrm{Ind}\,M) \cap \mathrm{res}\,\mathrm{cosupp}_S(\mathrm{Coind}\,N)$, there exists a homogeneous prime ideal $\mathfrak{q} \in \mathrm{res}^{-1}(\{\mathfrak{p}\})$ with $\mathfrak{q} \in \mathrm{supp}_S(\mathrm{Ind}\,M) \cap \mathrm{cosupp}_S(\mathrm{Coind}\,N)$.

The motivating example of a morphism of ring spectra satisfying Quillen lifting is the following one.

Example 3. Let G be a compact Lie group, k a field of characteristic p, and let $\mathcal{E}(G)$ be a set of representatives of conjugacy classes of elementary abelian p-subgroups of G. It is a consequence of the strong form of Quillen stratification for group cohomology that the following morphism satisfies Quillen lifting,

$$C^*(BG,k) \to \prod_{E\in\mathcal{E}(G)} C^*(BE,k).$$

We isolate sufficient conditions for descent of stratification and costratification along a morphism $f\colon R\to S$ in the next theorem.

Theorem 4. Suppose that $f\colon R\to S$ is a morphism of Noetherian ring spectra satisfying Quillen lifting and such that induction and coinduction along f are conservative. If Mod_S is canonically stratified, then so is Mod_R. If f additionally admits an R-module retract, then canonical costratification descends along f as well.

In [6], Broto, Levi, and Oliver introduced the powerful concept of p-local compact groups as a common generalization of the notions of p-compact group [7] as well as fusion systems $\mathcal{F}$ on a finite group [5]. A p-local compact group $\mathcal{G}=(S,\mathcal{F})$ consists of a saturated fusion system on a discrete p-toral group S. This definition provides a combinatorial model of the p-local structure of a compact Lie group $(S,\mathcal{F})$. In order to recover the p-completion of the classifying space, extra structure is needed. But, the latter is uniquely determined (see [8]) which makes it possible to construct a (p-completed) classifying space $B\mathcal{G}$ associated to $\mathcal{G}$, thus making saturated fusion systems amenable to homotopical techniques. Broto, Levi and Oliver provide examples given by compact Lie groups with no restriction on the group of components as well as p-completions of finite loop spaces.

Checking that the conditions of Theorem 4 are satisfied for the morphism $\phi_{\mathcal{G}}\colon C^*(B\mathcal{G},\mathbb{F}_p)\to C^*(BS,\mathbb{F}_p)$ crucially relies on the construction of a transfer morphism to prove that Ind and Coind are conservative functors.

Theorem 5. Any p-local compact group $\mathcal{G}=(S,\mathcal{F})$ admits a stable transfer $C^*(BS,\mathbb{F}_p)\to C^*(B\mathcal{G},\mathbb{F}_p)$ of $C^*(B\mathcal{G},\mathbb{F}_p)$-modules.

One consequence of this theorem is that the cohomology ring $H^*(B\mathcal{G},\mathbb{F}_p)$ is Noetherian for any p-local compact group, extending the classical result for finite groups and compact Lie groups, and for p-compact groups (see [7]).

We then use Rector's general formalism [9] to generalize the F-isomorphism to p-local compact groups. This allows us to deduce a strong form of Quillen stratification from the F-isomorphism theorem, following Quillen's original argument.

Theorem 6. For any p-local compact group $\mathcal{G}$, there is an F-isomorphism

$$H^*(B\mathcal{G},\mathbb{F}_p)\to \varprojlim_{\mathcal{F}^e} H^*(BE,\mathbb{F}_p),$$

where $\mathcal{F}^e$ is the full subcategory of $\mathcal{F}$ on the elementary abelian subgroups of S. Moreover, the variety of $\mathcal{G}$ admits a strong form of Quillen stratification:

$$\mathcal{V}_{\mathcal{G}} \cong \coprod_{E\in\mathcal{E}(\mathcal{G})} \mathcal{V}^{+}_{E,\mathcal{G}},$$

where $\mathcal{E}(\mathcal{G})$ denotes a set of representatives of $\mathcal{F}$-isomorphism classes of elementary abelian subgroups of S.

Our main result is then a combination of the previous three theorems.

Theorem 7. If $\mathcal{G}$ is a p-local compact group, then $\mathrm{Mod}_{C^*(B\mathcal{G},\mathbb{F}_p)}$ is canonically stratified and costratified. In particular, there are bijections

$$\left\{\begin{matrix}\text{Localizing subcat.}\\ \text{of } \mathrm{Mod}_{C^*(B\mathcal{G},\mathbb{F}_p)}\end{matrix}\right\} \overset{\sim}{\leftrightarrow} \left\{\begin{matrix}\text{Subsets of}\\ \mathrm{Spec}^h(H^*(B\mathcal{G},\mathbb{F}_p))\end{matrix}\right\} \overset{\sim}{\leftrightarrow} \left\{\begin{matrix}\text{Colocalizing subcat.}\\ \text{of } \mathrm{Mod}_{C^*(B\mathcal{G},\mathbb{F}_p)}\end{matrix}\right\}$$

as well as

$$\left\{\begin{matrix}\text{Thick subcategories}\\ \text{of } \mathrm{Mod}^{\mathrm{compact}}_{C^*(B\mathcal{G},\mathbb{F}_p)}\end{matrix}\right\} \overset{\sim}{\leftrightarrow} \left\{\begin{matrix}\text{Specialization closed subsets of}\\ \mathrm{Spec}^h(H^*(B\mathcal{G},\mathbb{F}_p))\end{matrix}\right\}.$$

Finally, Benson and Greenlees [2] show that $C^*(BG,\mathbb{F}_p)$ is an absolute Gorenstein ring spectrum for any finite group G. Using methods from [1], we extend this result to p-compact groups. As an immediate consequence this implies the existence of a local cohomology spectral sequence for p-compact groups.

Theorem 8. Let $\mathcal{G}$ be a p-compact group of dimension w, then $\mathcal{G}$ is absolute Gorenstein, i.e., for each $\mathfrak{p} \in \mathrm{Spec}^h(H^*(B\mathcal{G},\mathbb{F}_p))$ of dimension d, the local cohomology at $\mathfrak{p}$ is given by

$$H^*_{\mathfrak{p}}C^*(B\mathcal{G},\mathbb{F}_p) \cong I_{\mathfrak{p}}[w+d],$$

where $I_{\mathfrak{p}}$ denotes the injective hull $I_{\mathfrak{p}}$ of $(H^*(B\mathcal{G},\mathbb{F}_p))/\mathfrak{p}$.

References

[1] T. Barthel, D. Heard, G. Valenzuela,*Local duality for structured ring spectra*, Journal of Pure and Applied Algebra **222** 2018, 433–463.

[2] D. Benson, J. Greenlees, *Localization and duality in topology and modular representation theory*, J. Pure Appl. Algebra **212** (2008), 1716–1743.

[3] D. Benson, J. Greenlees, *Stratifying the derived category of cochains on BG for G a compact Lie group*, J. Pure Appl. Algebra **218** (2014), 642–650.

[4] D. Benson, S. Iyengar, H. Krause, *Stratifying triangulated categories*, J. Topol. **4** (2011), 641–666.

[5] C. Broto, R. Levi, B. Oliver,*The homotopy theory of fusion systems*, J. Amer. Math. Soc. **16** (2003), 779–856.

[6] C. Broto, R. Levi, B. Oliver, *Discrete models for the p-local homotopy theory of compact Lie groups and p-compact groups*, Geom. Topol. **11** (2007), 315–427.

[7] W.G. Dwyer, C. Wilkerson, *Homotopy fixed-point methods for Lie groups and finite loop spaces*, Annals of Math. Second Series **139** (1994), 395–442.

[8] R. Levi, A. Libman, *Existence and uniqueness of classifying spaces for fusion systems over discrete p-toral groups*, L. London. Math. Soc. **91** (2015), 47–70-

[9] D.L. Rector, *Noetherian cohomology rings and finite loop spaces with torsion*, J. Pure Appl. Algebra **32** (1984), 191–217

The classification of thick tensor ideals in genuine *A*-spectra

Justin Noel

(joint work with Tobias Barthel, Markus Hausmann, Niko Naumann, Thomas Nikolaus, Nat Stapleton)

In this joint project [BHNNNS], we classify the tt-ideals in genuine A-spectra [LMS86] when A is a finite abelian group. The original result of this type was the case when A is the trivial group, which is the celebrated thick subcategory theorem of Hopkins and Smith [HS98] (see also [Hop87, Rav92]). The primary prerequisites for the new classification result are:

(1) The general classification of tt-ideals in a rigid tensor triangulated category $\mathcal{C}$, in terms of the Thomason subsets of the Balmer spectrum $\mathrm{Spc}(\mathcal{C})$ of prime tensor ideals [Bal05, Bal10b].
(2) The identification of the underlying set of the Balmer spectrum $\mathrm{Spc}(SH_G)$ for genuine G-spectra from [BS17].

To complete the classification of the tt-ideals for (compact) genuine G-spectra, it suffices to identify the standard basic opens of $\mathrm{Spc}(SH_G)$. Namely, the basic opens can be enumerated by the compact G-spectra. For such a G-spectrum X, the corresponding open is identified by determining, for each prime p and each conjugacy class $H \subseteq G$ of subgroup, the smallest $n \geq 0$, such that $K(n)_*(\Phi^H X) \neq 0$ (if no such finite n exists, we will let this value be ∞). This last number, which we will denote $t_p(\Phi^H X)$, is called the type of $\Phi^H X$.

Balmer and Sanders show that one can always reduce to the case when G is a p-group. Moreover, when G is an abelian p-group, which we will henceforth assume, they show that it suffices to just determine how the type of X can vary from the type of $\Phi^G X$. So we first establish an inequality which bounds how the type can vary. To show the inequality is sharp we need to find suitable examples.

In more detail, we show that

$$t_p(\Phi^G X) + \dim(H_1(BG; \mathbb{F}_p)) \geq t_p(X).$$

This ends up being a consequence of a generalization of Kuhn's blue shift theorem for Tate cohomology [Kuh04]. This generalization identifies the acyclics for the geometric fixed points of Borel equivariant Lubin-Tate theories. This refines the results of [MNN15], which showed that if the group was not an abelian p-group generated by n or fewer elements, then the geometric fixed points of a height n Borel equivariant Lubin-Tate theory E would be contractible. This new generalization builds on the moduli-theoretic description of $E^0(BG)$ from [HKR00].

To show that this inequality is sharp and complete the classification, we need to find an X for which the inequality is an equality. We observe that such complexes have already been constructed in the work of [Aro98, ADL16, AL17].

References

[ADL16] Gregory Arone, William G. Dwyer, and Kathryn Lesh. Bredon homology of partition complexes. *Doc. Math.*, 21:1227–1268, 2016.

[AL17] Gregory Arone and Kathryn Lesh. Fixed points of coisotropic subgroups of Γ_k on decomposition spaces. 2017. Available at http://front.math.ucdavis.edu/1701.06070.

[Aro98] Gregory Arone. Iterates of the suspension map and Mitchell's finite spectra with A_k-free cohomology. *Math. Res. Lett.*, 5(4):485–496, 1998.

[Bal05] Paul Balmer. The spectrum of prime ideals in tensor triangulated categories. *J. Reine Angew. Math.*, 588:149–168, 2005.

[Bal10b] Paul Balmer. Tensor triangular geometry. In *International Congress of Mathematicians, Hyderabad (2010), Vol. II*, pages 85–112. Hindustan Book Agency, 2010.

[BHNNNS] Tobias Barthel, Markus Hausmann, Niko Naumann, Thomas Nikolaus, Justin Noel, and Nathaniel Stapleton. The Balmer spectrum of the equivariant homotopy category of a finite abelian group. Preprint `arXiv:1709.04828v2`, 2017.

[BS17] Paul Balmer and Beren Sanders. The spectrum of the equivariant stable homotopy category of a finite group. *Invent. Math.*, 208(1):283–326, 2017.

[Hop87] Michael J. Hopkins. Global methods in homotopy theory. In *Homotopy Theory-Proc. Durham Symposium 1985.* Cambridge University Pres, 1987.

[HKR00] Michael J. Hopkins, Nicholas J. Kuhn, and Douglas C. Ravenel. Generalized group characters and complex oriented cohomology theories. *J. Amer. Math. Soc.*, 13(3):553–594, 2000.

[HS98] Michael J. Hopkins and Jeffrey H. Smith. Nilpotence and stable homotopy theory. II. *Ann. of Math. (2)*, 148(1):1–49, 1998.

[Kuh04] Nicholas J. Kuhn. Tate cohomology and periodic localization of polynomial functors. *Invent. Math.*, 157(2):345–370, 2004.

[LMS86] L. Gaunce Lewis, J. Peter May, and Mark Steinberger. *Equivariant Stable Homotopy Theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer-Verlag, 1986.

[MNN15] Akhil Mathew, Niko Naumann, and Justin Noel. Derived induction and restriction theory. 2015. Available at http://front.math.ucdavis.edu/1507.06867.

[May96] J. Peter May. *Equivariant homotopy and cohomology theory*, volume 91 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996. With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner.

[Rav92] Douglas C. Ravenel. *Nilpotence and periodicity in stable homotopy theory*, volume 128 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1992. Appendix C by Jeff Smith.

Endotrivial modules via homotopy theory

Jesper Grodal

For G a finite group and k a field of characteristic p, an endotrivial module is a kG-module M such that $M \otimes M^* \cong k \oplus (\text{proj})$ as kG-modules. Isomorphism classes of indecomposable endotrivial modules form a group $T_k(G)$, and identify with the Picard group of the stable module category $\mathrm{Pic}(\mathrm{stmod}(kG))$. They occur in many parts of representation theory as "almost 1-dimensional modules".

I started my talk by briefly explaining the classification of these modules for S a finite p-group, due to seminal work of Dade [Dad78a, Dad78b], Alperin [Alp01] and Carlson–Thevenaz [CT04, CT05]. I then went on to explain how to calculate this group for G an arbitrary finite group, based on my preprint [Gro16].

The image of the restriction map $T_k(G) \to T_k(S)$, for S a p-Sylow subgroup of G, is known, at least as an abstract group, by an elaboration of the above-mentioned calculation of $T_k(S)$. So the main question in describing $T_k(G)$ for an arbitrary finite group G lies in understanding the kernel $T_k(G,S) = \ker(T_k(G) \to T_k(S))$. This subgroup consists of finitely generated kG–modules M, whose restriction to S have the form $M|_S \cong k \oplus \text{(free)}$, i.e., "Sylow-trivial" modules. The following theorem describes this group:

Theorem 1. [Gro16, Thm. A] Fix a finite group G and k a field of characteristic p dividing the order of G, and let $\mathscr{O}_p^*(G)$ denote the orbit category on non-trivial p–subgroups. The group $T_k(G,S)$ is described via the following isomorphism of abelian groups

$$\Phi : T_k(G,S) \xrightarrow{\sim} H^1(\mathscr{O}_p^*(G); k^\times).$$

The inverse map, which to a 1–cocycle constructs an endotrivial module, is very explicit, in terms of the so-called twisted Steinberg complex, and can also be viewed as a "derived induction" map.

I then went on to describe a number of explicit results about $T_k(G,S)$ that can be obtained with Theorem 1 as starting point, also taken from [Gro16]; they transform the calculation $T_k(G,S)$ to standard calculations in local group theory: I presented a positive solution to the so-called Carlson–Thevenaz conjecture, providing an explicit algorithm for computing $T_k(G,S)$ purely in terms of normalizers of p–subgroups and their intersections, that can easily be put on a computer. I also deduced other consequences such as that if the p–subgroup complex $\mathcal{S}_p(G)$ is simply connected, then $T_k(G,S)$ equals the one-dimensional characters of G, providing vanishing results for many classes of groups. The Carlson–Thevenaz conjecture comes out as a special case of more general "centralizer" and "normalizer" decompositions for $T_k(G,S)$, that express $T_k(G,S)$ in terms of p–local group theory packaged in different ways. E.g., the "centralizer decomposition" breaks $T_k(G,S)$ up in two parts, one only depending on the p–fusion in G, and one depending on the 1–dimensional characters on the centralizers of elementary abelian p–subgroups of rank one and two.

Finally, I described a number of explicit computations, both showing how existing results in the literature can be easily recovered by these methods, and computing $T_k(G,S)$ for a range of new groups, e.g, as a test case, the Monster sporadic simple group for all primes p.

References

[Alp01] J. L. Alperin. A construction of endo-permutation modules. *J. Group Theory*, 4(1):3–10, 2001.

[CT04] J. F. Carlson and J. Thévenaz. The classification of endo-trivial modules. *Invent. Math.*, 158(2):389–411, 2004.

[CT05] J. F. Carlson and J. Thévenaz. The classification of torsion endo-trivial modules. *Ann. of Math. (2)*, 162(2):823–883, 2005.

[Dad78a] E. C. Dade. Endo-permutation modules over p-groups. I. *Ann. of Math. (2)*, 107(3):459–494, 1978.

[Dad78b] E. C. Dade. Endo-permutation modules over p-groups. II. *Ann. of Math. (2)*, 108(2):317–346, 1978.

[Gro16] J. Grodal. Endotrivial modules for finite groups via homotopy theory. arXiv:1608.00499.

Homological residue fields

Paul Balmer

In Prismatic Algebra, one encounters a broad variety of tensor-triangulated categories T. The chromatic analysis of their compact-rigid objects T^c amounts to the determination of their triangular spectrum $Spc(T^c)$, whose points are triangular primes $P \subset T^c$.

We presented another approach to primes, by means of maximal Serre $\otimes$-ideal subcategories B of the module category $mod - T^c$ (the Freyd envelope of T^c). We proved with Krause and Stevenson [2] that every triangular prime P is the preimage under Yoneda

$$h : T^c \to mod - T^c$$

of one of those new homological primes B. It is an open question whether this B is unique. Remarkably, one can prove that B is unique (for a given P) in the standard examples from stable homoopy theory (including the equivariant versions), algebraic geometry (without noetherianity assumption), modular representation theory (including finite group schemes), etc. However, the proof is specific to each example and we do not know an abstract proof.

The associated *homological* spectrum $Spc^h(T)$ consisting of all homological primes (all maximal Serre $\otimes$-ideals) of $mod - T^c$, can be used to define supports for big objects, unconditionally (i.e. without supposing noetherianity of the triangular spectrum $Spc(T^c)$) as was done in the joint work with Favi [1]. To do this, one considers the big category $Mod - T^c$ of modules over T^c, of which the above $mod - T^c$ is the finitely presented part. There is a restricted Yoneda functor

$$h : T \to Mod - T^c$$

which is not faithful or full anymore (it kills phantom maps) but which remains a $\otimes$-functor. For every (big) object $X \in T$, one can then define its support $Supp^h(X) \subseteq Spc^h(T)$ as the set of homological primes B such that X does not vanish in the Gabriel quotient of $Mod - T^c$ by $\langle B \rangle$. This 'big support' lacks the fundamental properties needed of a good theory of support, notably the general tensor formula, but it is a good theory for ring objects (even for *weak* ring objects: objects equipped with a possibly non-associative, non-commutative multiplication admitting a one-sided unit).

It is work-in-progress to develop the properties of this big support and its relevance for the telescope property, i.e. for understanding the smashing subcategories of T.

References

[1] Paul Balmer and Giordano Favi. *Generalized tensor idempotents and the telescope conjecture*, Proc. Lond. Math. Soc. (3), 102(6):1161–1185, 2011.

[2] Paul Balmer, Henning Krause, and Greg Stevenson. *Tensor-triangular fields: Ruminations.* preprint, arXiv:1707.02167, 2017.

Reporter: Drew K. Heard

Participants

Prof. Dr. Paul Balmer
Department of Mathematics
University of California at Los Angeles
405, Hilgard Avenue
Los Angeles, CA 90095-1555
UNITED STATES

Dr. Tobias Barthel
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
2100 København Ø
DENMARK

Prof. Dr. Agnès Beaudry
Department of Mathematics
University of Colorado
Boulder, CO 80309-0395
UNITED STATES

Prof. Dr. Natàlia Castellana Vila
Departament de Matematiques
Universitat Autonoma de Barcelona
Campus UAB
08193 Bellaterra (Barcelona)
SPAIN

Prof. Dr. Ivo Dell'Ambrogio
Laboratoire Paul Painleve, UMR 8524
Université des Sciences et Techn. de Lille 1
Cité Scientifique, Bât. M2
59655 Villeneuve-d'Ascq Cedex
FRANCE

Prof. Dr. John Greenlees
Mathematics Institute
University of Warwick
Zeeman Building
Coventry CV4 7AL
UNITED KINGDOM

Prof. Dr. Jesper Grodal
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
2100 København Ø
DENMARK

Dr. Drew K. Heard
Department of Mathematics
Faculty of Natural Sciences
University of Haifa
Mount Carmel
31905 Haifa
ISRAEL

Prof. Dr. Henning Krause
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Prof. Dr. Amnon Neeman
Centre for Mathematics and its Applications
Mathematical Sciences Institute
Australian National University
Canberra ACT 0200
AUSTRALIA

Dr. Justin Noel
Fakultät für Mathematik
Universität Regensburg
Universitätsstrasse 31
93053 Regensburg
GERMANY

Prof. Dr. Julia Pevtsova
Department of Mathematics
University of Washington
Padelford Hall
Box 354350
Seattle, WA 98195-4350
UNITED STATES

Prof. Dr. Nicolas Ricka
UFR de Mathématique et d'Informatique
Université de Strasbourg
7, rue Rene Descartes
67084 Strasbourg Cedex
FRANCE

Dr. Beren Sanders
EPFL SV BMI UPHESS
MA B3 465 (Batiment MA)
Station 8
1015 Lausanne
SWITZERLAND

Dr. Tomer Schlank
Institute of Mathematics
The Hebrew University
Givat-Ram
91904 Jerusalem
ISRAEL

Prof. Dr. Vesna Stojanoska
Department of Mathematics
University of Illinois at
Urbana-Champaign
273 Altgeld Hall, MC-382
1409 West Green Street
Urbana IL 61801
UNITED STATES