Arbeitsgemeinschaft: Topological Cyclic Homology

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1 April – 7 April 2018

ABSTRACT. Introduced by Bökstedt-Hsiang-Madsen in the nineties, topological cyclic homology is a manifestation of the dual visions of Connes-Tsygan and Waldhausen to extend de Rham cohomology to a noncommutative setting and to replace algebra by higher algebra. The cohomology theory that ensues receives a denominator-free Chern character from algebraic K-theory, used by Hesselholt-Madsen to evaluate the p-adic K-theory of p-adic fields. More recently, Bhatt-Morrow-Scholze have defined a "motivic" filtration of topological cyclic homology and its variants, the filtration quotients of which give rise to their denominator-free p-adic Hodge theory $\mathbb{A}\Omega$.

Introduction by the Organisers

Cyclic homology was introduced by Connes and Tsygan in the early eighties to serve as an extension of de Rham cohomology to a noncommutative setting. The negative version of cyclic homology receives a trace map from algebraic K-theory, which extends the classical Chern character and roughly records traces of powers of matrices. This trace map is a powerful rational invariant of algebraic K-theory. Indeed, a theorem of Goodwillie from 1986 shows that, rationally, the discrepancy for K-theory to be invariant under nilpotent extentions of rings agrees with that for negative cyclic homology; and a theorem of Cortiñas from 2006 shows similarly that, rationally, the discrepancy for K-theory to preserve cartesian squares of rings agrees with that for negative cyclic homology.

In the early seventies, Boardman and Vogt planted the seeds for the higher algebra that was only fully developed much later by Joyal and Lurie, and, later in the decade, Waldhausen extended Quillen's definition of algebraic K-theory from

the rings of algebra to the (connective $\mathbb{E}_1$-)rings of higher algebra. Waldhausen advocated that the initial ring $\mathbb{S}$ of higher algebra be viewed as an object of arithmetic and that the cyclic homology of Connes and Tsygan be developed with the ring $\mathbb{S}$ as its base. In his philosophy, such a theory should be meaningful integrally as opposed to rationally.

In 1985, Bökstedt realized Waldhausen's vision as far as Hochschild homology is concerned, and he named this new theory topological Hochschild homology. (A similar construction had been considered by Breen ten years earlier.) He also made the fundamental calculation that, as a graded ring,

$$\mathrm{THH}_*(\mathbb{F}_p) = \mathrm{HH}_*(\mathbb{F}_p/\mathbb{S}) = \mathbb{F}_p[x]$$

is a polynomial algebra on a generator x in degree two. By comparison,

$$\mathrm{HH}_*(\mathbb{F}_p/\mathbb{Z}) = \mathbb{F}_p\langle x\rangle$$

is the corresponding divided power algebra, and hence, Bökstedt's theorem supports Waldhausen's vision that passing from the base $\mathbb{Z}$ to the base $\mathbb{S}$ eliminates denominators. In fact, the base-change map $\mathrm{HH}_*(\mathbb{F}_p/\mathbb{S}) \to \mathrm{HH}_*(\mathbb{F}_p/\mathbb{Z})$ can be identified with the edge homomorphism of a spectral sequence

$$E_{i,j}^2 = \mathrm{HH}_i(\mathbb{F}_p/\pi_*(\mathbb{S}))_j \Rightarrow \mathrm{HH}_{i+j}(\mathbb{F}_p/\mathbb{S}),$$

so apparently the higher stable homotopy groups of spheres, which Serre had proved to be finite, are exactly the right size to eliminate the denominators in the divided power algebra.

The appropriate definition of cyclic homology relative to $\mathbb{S}$ was given in 1993 by Bökstedt-Hsiang-Madsen. It involves a new ingredient that is not present in the Connes-Tsygan cyclic theory, namely, a Frobenius. The nature of this Frobenius is now much better understood thanks to the work of Nikolaus-Scholze [18], and we will use this work as our basic reference. As in the Connes-Tsygan theory, the circle group $\mathbb{T}$ acts on topological Hochschild homology, and by analogy, we may define negative topological cyclic homology and periodic topological cyclic homology to be the homotopy fixed points and the Tate construction of this action, respectively:

$$\mathrm{TC}^-(A) = \mathrm{THH}(A)^{h\mathbb{T}} \qquad \text{and} \qquad \mathrm{TP}(A) = \mathrm{THH}(A)^{t\mathbb{T}}.$$

There is always a canonical map from homotopy fixed points to the Tate construction, but, after p-completion, the Frobenius gives rise to another such map and the Bökstedt-Hsiang-Madsen topological cyclic homology is the homotopy equalizer of these two maps:

$$\mathrm{TC}(A) \longrightarrow \mathrm{TC}^-(A) \underset{\mathrm{can}}{\overset{\varphi_p}{\rightrightarrows}} \mathrm{TP}(A).$$

Topological cyclic homology receives a trace map from algebraic K-theory, which is called the cyclotomic trace map, and Dundas-McCarthy-Goodwillie showed that the discrepancy for K-theory to be invariant under nilpotent extensions agrees integrally with that for topological cyclic homology. Similarly, by work of Geisser-Hesselholt and Dundas-Kittang, the discrepancy for K-theory to preserve cartesian squares of rings agrees integrally with that for topological cyclic homology.

Calculations of algebraic K-groups, or rather the homotopy groups of the p-adic completion of the K-theory spectrum, by means of the cyclotomic trace begin with the calculation that said trace map

$$K(\mathbb{F}_p) \to \mathrm{TC}(\mathbb{F}_p)$$

induces an isomorphism of p-adic homotopy groups in non-negative degrees. The Dundas-McCarthy-Goodwillie theorem together with continuity results of Suslin and Hesselholt-Madsen then show that the same is true for

$$K(\mathbb{Z}_p) \to \mathrm{TC}(\mathbb{Z}_p)$$

and, more generally, for finite algebras over the ring of Witt vectors in a perfect field of characteristic p. This was used by Hesselholt-Madsen in 2003 to verify the Lichtenbaum-Quillen conjecture for p-adic fields, by evaluating the relevant topological cyclic homology, and one of the goals of the Arbeitsgemeinschaft was to understand this calculation.

The theories TC^- and TP are of significant independent interest, since they are closely related to interesting p-adic cohomology theories, both new and old. The precise relationship was established only recently by work of Bhatt-Morrow-Scholze that defines "motivic filtrations" on THH and related theories, the graded pieces of which are p-adic cohomology theories such as crystalline cohomology and the $\mathbb{A}\Omega$-theory of [2]. For example, if X is a scheme smooth over a perfect field of characteristic p, then the jth graded pieces of TC, TC^-, and TP form a homotopy equalizer

$$\mathbb{Z}_p(j) \longrightarrow \mathrm{Fil}^j \, R\Gamma_{\mathrm{crys}}(X/W(k)) \overset{``\frac{\varphi_p}{p^j}"}{\underset{\mathrm{can}}{\rightrightarrows}} R\Gamma_{\mathrm{crys}}(X/W(k)).$$

A second goal of the Arbeitsgemeinschaft was to understand these filtrations.

Since the questions that we consider are mainly in the p-complete setting and for $\mathbb{E}_\infty$-algebras (in fact, usual commutative rings!), we largely restrict our attention to this case, and in particular work with p-typical cyclotomic spectra.

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Abstracts

Introduction to the Tate construction and cyclotomic spectra
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1. Overview

The aim of this lecture is to define the stable ∞-category of cyclotomic spectra following [3]. Let $\mathbb{T} = S^1$ be the unit circle, and for each prime p let $C_p \subset \mathbb{T}$ be the cyclic subgroup of order p. Informally, a cyclotomic spectrum is a spectrum X with $\mathbb{T}$-action and $\mathbb{T} \cong \mathbb{T}/C_p$-equivariant maps

$$\varphi_p \colon X \to X^{tC_p}.$$

Here $(\cdot)^{tC_p}$ denotes the Tate construction; it naturally carries an action of $\mathbb{T}/C_p \xrightarrow{\sim} \mathbb{T}$ (the identification under the map $z \mapsto z^p$). Most of this talk is simply formalizing this definition. We need to address three things:

(1) what is meant by a spectrum with $\mathbb{T}$-action;
(2) what the Tate construction is;
(3) how cyclotomic spectra form a stable ∞-category CycSp.

2. The Tate construction

2.1. **Equivariant objects.** We begin in a more general setting. All ∞-categorical language will be as in the books [1] and [2] of Lurie. In particular, an ∞-category $\mathcal{C}$ is a simplicial set having all inner horn liftings.

Definition 1. Let G be a group, and let $\mathcal{C}$ be an ∞-category. We define the ∞-category of G-equivariant objects in $\mathcal{C}$ by $\operatorname{Fun}(BG, \mathcal{C})$ (recall that for any ∞-category $\mathcal{C}$ and simplicial set K, $\operatorname{Fun}(K, \mathcal{C})$ is an ∞-category). We will often write this as $\mathcal{C}^{BG}$.

Concretely, a G-equivariant object of $\mathcal{C}$ is a functor (i.e. a map of simplicial sets) $BG \to \mathcal{C}$. The main example of interest to us is when $\mathcal{C} = \operatorname{Sp}$ is the ∞-category of spectra (in which case we follow [3] in referring to Sp^{BG} as "spectra with G-action" to avoid confusion with the usual language of equivariant homotopy theory), and when G is either a finite group or the circle group $\mathbb{T}$; in the latter case by $B\mathbb{T}$ we mean the classifying space of the topological group $\mathbb{T}$.

2.2. **Norm maps.** To motivate the Tate construction, recall the classical construction of Tate cohomology groups. Let G be a finite group, and let M be a

G-module. Then we obtain a norm map $\mathrm{Nm}_G \colon M_G \to M^G$ defined by $\mathrm{Nm}_G(m) = \sum_{g \in G} g \cdot m$. The Tate cohomology groups are then given by

$$\widehat{H}^i(G; M) = \begin{cases} H^i(G; M), i \geq 1, \\ \mathrm{Coker}(\mathrm{Nm}_G), i = 0, \\ \ker(\mathrm{Nm}_G), i = -1, \\ H_{-i-1}(G; M), i \leq -2. \end{cases}$$

A key consequence of this definition is that a short exact sequence of G-modules yields a long exact sequence (infinite in both directions) in Tate cohomology; moreover, the product in group cohomology extends to one on Tate cohomology (in all degrees).

We now explain an ∞-categorical analogue.

Definition 2. Let G be a group, and let $\mathcal{C}$ be an ∞-category. If $\mathcal{C}$ has colimits indexed by BG, then define the homotopy orbits functor by

$$(\cdot)_{hG} \colon \mathcal{C}^{BG} \to \mathcal{C}$$
$$F \mapsto \operatorname*{colim}_{BG} F.$$

Dually, if $\mathcal{C}$ has limits indexed by BG, then define the homotopy fixed points functor

$$(\cdot)^{hG} \colon \mathcal{C}^{BG} \to \mathcal{C}$$
$$F \mapsto \lim_{BG} F.$$

Examples 3. When can we construct a "norm" $\mathrm{Nm}_G \colon X_{hG} \to X^{hG}$ that is a natural transformation of functors $\mathcal{C}^{BG} \to \mathcal{C}$ (i.e. a morphism in $\mathrm{Fun}(\mathcal{C}^{BG}, \mathcal{C})$)?

The framing of this question is of course not satisfactory, since it articulates no desiderata for the transformation Nm_G. One elementary requirement would be that the general construction should specialize to the classical norm map when we take X to be the Eilenberg-Maclane spectrum of some G-module (for some finite group G). In what follows, we will give an elementary approach (following [3, §I.1] and [2, §6.1.6]) to constructing norm maps for finite G; for a different approach that works more generally, see [3, §1.4]. Later talks will make use of this latter approach, particularly since it gives a usable universal property.

In any case, the answer to this question depends heavily on $\mathcal{C}$. First recall that an ∞-category $\mathcal{C}$ is pre-additive (called semi-additive in [2]) if it is pointed, has finite products and coproducts, and these agree (see [3, Definition I.1.1] for a precise formulation).

Proposition 4. *Let $\mathcal{C}$ be a pre-additive ∞-category admitting limits and colimits indexed by classifying spaces of finite groups, and let $f \colon X \to Y$ be a map of Kan complexes that is a relative finite groupoid (see the proof for details). Then there is a natural transformation $\mathrm{Nm}_f \colon f_! \to f_*$ of functors $\mathcal{C}^X \to \mathcal{C}^Y$ (whose defining properties will be elaborated in the course of the construction).*

Proof. Before beginning the proof proper, we start with some preliminaries. Until otherwise noted, we let $\mathcal{C}$ be *any* ∞-category. Let $f\colon X \to Y$ be any map of Kan complexes. Then we obtain a pullback

$$f^*\colon \mathcal{C}^Y \to \mathcal{C}^X$$

(recall $\mathcal{C}^X = \mathrm{Fun}(X,\mathcal{C})$). If they exist, we let $f_!, f_*\colon \mathcal{C}^X \to \mathcal{C}^Y$ be the left and right adjoints of f^*; these exist provided $\mathcal{C}$ admits colimits and limits indexed by the simplicial sets $X \times_Y (Y_{/y})$.

Now we begin the proof of the proposition, but still until otherwise specified not putting any hypothese on $\mathcal{C}$ and f. The proof will be inductive, based on the following construction. For any map $f\colon X \to Y$ of Kan complexes, consider the diagonal $\delta\colon X \to X \times_Y^h X$ (homotopy pull-back). *Assume* $\delta_!$ and δ_* exist, and that there is a natural transformation $\mathrm{Nm}_\delta\colon \delta_! \to \delta_*$ that is an equivalence (in $\mathrm{Fun}(\mathcal{C}^X, \mathcal{C}^{X\times_Y^h X})$). Consider the homotopy pullback diagram

$$
\begin{array}{ccc}
X \times_Y^h X & \xrightarrow{\ p_0\ } & X \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\
X & \xrightarrow{\ f\ } & Y.
\end{array}
$$

We obtain (using the two adjunctions and the equivalence Nm_δ) a natural transformation

$$p_0^* \to \delta_*\delta^* p_0^* \simeq \delta_* \xrightarrow[\sim]{\ \mathrm{Nm}_\delta^{-1}\ } \delta_! \simeq \delta_!\delta^* p_1^* \to p_1^*,$$

and then again by adjunction a transformation $\mathrm{id}_{\mathcal{C}^X} \to p_{0,*}p_1^*$. If we moreover assume that $\mathcal{C}$ has enough limits and colimits that f_* and $f_!$ exist, then the pull-back diagram yields a natural transformation $f^*f_* \to p_{0,*}p_1^*$ that is also an equivalence (see [2, Lemma 6.1.6.3]; this uses the fact that the diagram is a homotopy pull-back). Combining these observations, we obtain a transformation $\mathrm{id}_{\mathcal{C}^X} \to f^*f_*$, and thus by adjunction a transformation $f_! \to f_*$. We define

$$(\star) \qquad\qquad\qquad \mathrm{Nm}_f\colon f_! \to f_*$$

to be this natural transformation between functors $\mathcal{C}^X \to \mathcal{C}^Y$.

Now we explain how the "inductive" argument of the previous paragraph lets us build step-by-step toward the proof of the proposition; the basic idea is that δ is more connected than f, so we can induct on the connectivity. In what follows, we will say that a map $f\colon X \to Y$ of Kan complexes is n-truncated if the homotopy fibers F_f are all n-truncated in the sense that $\pi_k(F_f) = 0$ for all $k \geq n+1$.

Step -1: Let f be (-1)-truncated, so all homotopy fibers are either empty or contractible. Then δ is an equivalence, so we obviously have an equivalence $\mathrm{Nm}_\delta\colon \delta_! \xrightarrow{\sim} \delta_*$. We claim that if $\mathcal{C}$ is pointed (there exist initial and final objects and the map(s) between them are equivalences), then $f_!$ and f_* exist, and Nm_f as in Equation $(\star)$ is an equivalence. We explain a couple of key points. We can reduce to the case $Y = \Delta^0$. Then X is either (a) contractible or (b) empty. If (a), then f is a homotopy equivalence, and $f_!$ and f_* are both homotopy inverse to f^*,

equivalent under Nm_f; if (b), then $\mathcal{C}^X \simeq \Delta^0$, and $f_!, f_* \colon \Delta^0 \to \mathcal{C}$ are identified with initial and final objects of, and Nm_f with an equivalence from an initial to a final object.

Step 0: Now assume f is only 0-truncated, and that $\mathcal{C}$ is pointed. By the Mayer-Vietoris-type sequence on homotopy groups $(\pi_n(X \times_Y^h X) \to \pi_n(X) \oplus \pi_n(X) \to \pi_n(Y) \to \pi_{n-1}(X \times_Y^h X) \to \cdots)$, the diagonal δ is (-1)-truncated, so by Step -1, $\mathrm{Nm}_\delta \colon \delta_! \to \delta_*$ exists and is an equivalence. Now further assume that $\mathcal{C}$ is pre-additive and f has finite homotopy fibers. Then $f_!$ and f_* exist (the requisite limits and colimits in $\mathcal{C}$ are finite products and coproducts), and by the construction of Equation $(\star)$, $\mathrm{Nm}_f \colon f_! \to f_*$ exists. Moreover, Nm_f is an equivalence, the key point being that for any objects X and Y of $\mathcal{C}$, the obvious map $X \sqcup Y \to X \times Y$ is an equivalence ($\mathcal{C}$ is pre-additive).

Step 1: Finally, assume f is only 1-truncated, and moreover is a *relative finite groupoid* in the sense that the homotopy fibers of f have finitely many connected components, each of which is equivalent to the classifying space of a finite group. Then (by Mayer-Vietoris) δ is 0-truncated with finite fibers, so if $\mathcal{C}$ is pre-additive, then Nm_δ exists and is an equivalence. If we assume moreover that $\mathcal{C}$ has limits and colimits indexed by classifying spaces of finite groups (or just for those arising as in the homotopy fibers of f), then $f_!$ and f_* exist, and the basic construction $(\star)$ yields a natural transformation $\mathrm{Nm}_f \colon f_! \to f_*$ (not necessarily an equivalence).

This completes the promised construction.   □

The construction of the last Proposition yields the following application:

Corollary 5. *Let G be a finite normal subgroup of a topological group H, and let $f \colon BH \to B(H/G)$ be the projection. Let $\mathcal{C}$ be a pre-additive ∞-category with limits and colimits indexed by BG. Then there is a natural transformation $\mathrm{Nm}_f \colon f_! \to f_*$ arising from the inductive construction of Proposition 4.*

Definition 6. In the setting of the Corollary, we will write $\mathrm{Nm}_G \colon (\cdot)_{hG} \to (\cdot)^{hG}$ for $\mathrm{Nm}_f \colon f_! \to f_*$. This is the promised norm map for the finite group G and the ∞-category $\mathcal{C}$.

2.3. **The Tate construction.** The Tate construction will take the "cofiber" of Nm_G. In order to make sense of this, we have to assume $\mathcal{C}$ has more structure.

Definition 7. Let $\mathcal{C}$ be a stable ∞-category with all colimits indexed by BG for some finite group G. The Tate construction is the functor

$$(\cdot)^{tG} \colon \mathcal{C}^{BG} \to \mathcal{C}$$

$$X \mapsto \mathrm{cofib}(\mathrm{Nm}_G \colon X_{hG} \to X^{hG}).$$

More generally, if G is a normal subgroup of H, then the same formula yields $(\cdot)^{tG} \colon \mathcal{C}^{BH} \to \mathcal{C}^{B(H/G)}$.

Remark 8. The Tate construction is exact, in the sense of stable ∞-categories: it sends fiber sequences to fiber sequences.

Remark 9. In this approach to constructing Nm_G, the comparison to the classical construction is somewhat tedious, since one has to trace through the inductive construction. For a G-module M, let HM be the Eilenberg-Maclane spectrum, regarded as an object of Sp^{BG}. The defining fiber sequence $(HM)_{hG} \xrightarrow{\mathrm{Nm}_G} (HM)^{hG} \to (HM)^{tG}$ yields a long-exact sequence in homotopy groups, where $\pi_n(HM_{hG}) \cong H_n(G; M)$ and $\pi_n(HM^{hG}) \cong H^{-n}(G; M)$. Once one identifies the map $\pi_0(\mathrm{Nm}_G) \colon \pi_0(HM_{hG}) \to \pi_0(HM^{hG})$ with the classical norm map, one recovers $\pi_n(HM^{tG}) \cong \widehat{H}^{-n}(G; M)$ for all integers n.

We include one more important fact about the multiplicativity of the Tate construction:

Theorem 10 (Theorem I.3.1 of [3])**.** *Let G be a finite group. The functor*

$$(\cdot)^{tG} \colon \mathrm{Sp}^{BG} \to \mathrm{Sp}$$

admits a lax symmetric monoidal structure making the natural transformation $(\cdot)^{hG} \to (\cdot)^{tG}$ lax symmetric monoidal. In fact, the space of pairs consisting of a lax symmetric monoidal structure on $(\cdot)^{tG}$ and a lax symmetric monoidal refinement of $(\cdot)^{hG} \to (\cdot)^{tG}$ is contractible.

3. Cyclotomic spectra

We can now construct the stable ∞-category of cyclotomic spectra, making precise the informal description in §1. In the notation of the previous section, let $\mathcal{C} = \mathrm{Sp}$ be the stable ∞-category of spectra, let $H = \mathbb{T}$ (the circle), and let G be any of the finite cyclic subgroups $C_p \subset \mathbb{T}$. For all p, we have the Tate construction

$$(\cdot)^{tC_p} \colon \mathrm{Sp}^{B\mathbb{T}} \to \mathrm{Sp}^{B(\mathbb{T}/C_p)} \cong \mathrm{Sp}^{B\mathbb{T}},$$

with the identification coming from the isomorphism of topological groups $\mathbb{T}/C_p \xrightarrow{\sim} \mathbb{T}$, $z \mapsto z^p$. To define CycSp, we need one more general categorical construction:

Definition 11. Let $F, G \colon \mathcal{C} \to \mathcal{D}$ be functors between ∞-categories. Define the lax equalizer of F and G to be the (1-categorical) pullback of simplicial sets

$$
\begin{array}{ccc}
\mathrm{LEq}(F, G) & \longrightarrow & \mathcal{D}^{\Delta^1} \\
\downarrow & & \downarrow {\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\
\mathcal{C} & \xrightarrow{(F, G)} & \mathcal{D} \times \mathcal{D}.
\end{array}
$$

We sometimes write $\mathrm{LEq}(\mathcal{C} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{D})$ for $\mathrm{LEq}(F, G)$. $\mathrm{LEq}(F, G)$ is itself an ∞-category ([3, Proposition II.1.5]).

For instance, an object of $\mathrm{LEq}(F, G)$ is a pair (c, f) consisting of an object c of $\mathcal{C}$ and a morphism $F(c) \to G(c)$ in $\mathcal{D}$. Mapping spaces in $\mathrm{LEq}(F, G)$ are computed as follows. Let X, Y be objects of $\mathrm{LEq}(F, G)$, given by pairs (c_X, f_X) and (c_Y, f_Y)

as above. Then the space $\mathrm{Map}_{\mathrm{LEq}(F,G)}(X,Y)$ is the equalizer (see [3, Proposition II.1.5(ii)])

$$(\star\star) \quad \mathrm{Map}_{\mathrm{LEq}(F,G)}(X,Y) \simeq \mathrm{Eq}\left(\mathrm{Map}_{\mathcal{C}}(c_X, c_Y) \underset{f_Y^* F}{\overset{f_X^* G}{\rightrightarrows}} \mathrm{Map}_{\mathcal{D}}(F(c_X), G(c_Y)) \right).$$

Definition 12. Consider the two functors $\mathrm{Sp}^{B\mathbb{T}} \to \prod_p \mathrm{Sp}^{B\mathbb{T}}$ given by $F = (\mathrm{id})_p$ and $G = ((\cdot)^{tC_p})_p$. The ∞-category of cyclotomic spectra CycSp is defined to be $\mathrm{LEq}(F,G)$, i.e.

$$\mathrm{CycSp} = \mathrm{LEq}\left(\mathrm{Sp}^{B\mathbb{T}} \underset{((\cdot)^{tC_p})_p}{\overset{(\mathrm{id})_p}{\rightrightarrows}} \prod_p \mathrm{Sp}^{B\mathbb{T}} \right).$$

Working with just a single prime p, we make an analogous definition of p-cyclotomic spectra:

$$\mathrm{CycSp}_p = \mathrm{LEq}\left(\mathrm{Sp}^{BC_{p^\infty}} \underset{(\cdot)^{tC_p}}{\overset{\mathrm{id}}{\rightrightarrows}} \mathrm{Sp}^{BC_{p^\infty}} \right)$$

(again using the identification $C_{p^\infty}/C_p \xrightarrow{\sim} C_{p^\infty}$).

In particular, an object of CycSp is a pair $(X, (\varphi_p)_p)$ of a spectrum with $\mathbb{T}$-action X and morphisms $\varphi_p \colon X \to X^{tC_p}$ for all primes p. There is no compatibility requirement as p varies.

General facts about lax equalizers translate to the following basic properties of CycSp (see [3, Proposition II.1.5] for details, and for additional, more technical, accessibility and presentability properties):

(1) A morphism in CycSp is an equivalence if and only if its image in $\mathrm{Sp}^{B\mathbb{T}}$ is an equivalence (which is the case if and only if its pullback along $\{1\} \to \mathbb{T}$ to Sp is an equivalence).
(2) CycSp is a stable ∞-category.
(3) The functor $\mathrm{CycSp} \to \mathrm{Sp}^{B\mathbb{T}}$ is exact and preserves small colimits.

Example 13. *The sphere spectrum $\mathbb{S} \in \mathrm{Sp}$ refines to an object of CycSp when given the trivial $\mathbb{T}$ action (i.e., when regarded as an object of $\mathrm{Sp}^{B\mathbb{T}}$ by pullback along $B\mathbb{T} \to *$). Namely, by adjunction there is a map (in Sp) $\mathbb{S} \to \mathbb{S}^{hC_p}$, and thus a map $\mathbb{S} \to \mathbb{S}^{tC_p}$. To add the necessary equivariant structure, we consider the composite (in Sp)*

$$\mathbb{S} \to \mathbb{S}^{h\mathbb{T}} \simeq (\mathbb{S}^{hC_p})^{h\mathbb{T}/C_p} \to (\mathbb{S}^{tC_p})^{h\mathbb{T}/C_p}.$$

Adjunction yields the $\mathbb{T} \cong \mathbb{T}/C_p$-equivariant structure on $\mathbb{S} \to \mathbb{S}^{tC_p}$.

Finally, since CycSp is a stable ∞-category we obtain mapping spectra functors

$$\mathrm{map}_{\mathrm{CycSp}}(\cdot, \cdot) \colon \mathrm{CycSp}^{\mathrm{op}} \times \mathrm{CycSp} \to \mathrm{Sp},$$

and we can then define the topological cyclic homology of a cyclotomic spectrum:

Definition 14. Let $(X, (\varphi_p)_p)$ be a cyclotomic spectrum. The integral topological cyclic homology $\mathrm{TC}(X)$ is the mapping spectrum $\mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}, X) \in \mathrm{Sp}$. Likewise, for an object (X, φ_p) of CycSp_p, the p-typical topological cyclic homology is $\mathrm{TC}(X, p) = \mathrm{map}_{\mathrm{CycSp}_p}(\mathbb{S}, X)$.

Combining the formula of Equation $(\star\star)$, Example 13, and the equivalence (see [2, Corollary 1.4.2.23]) $\mathrm{Fun}^{\mathrm{ex}}(\mathrm{CycSp}, \mathrm{Sp}) \xrightarrow{\Omega^\infty} \mathrm{Fun}^{\mathrm{lex}}(\mathrm{CycSp}, \mathrm{sSet})$, applied to $\mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}, \cdot)$, we deduce the following computation of $\mathrm{TC}(X)$:

Proposition 15 (Proposition II.1.9 of [3]). *Let $(X, (\varphi_p)_p)$ be a cyclotomic spectrum. Then writing* can *for the map*

$$\mathrm{can}\colon X^{h\mathbb{T}} \simeq (X^{hC_p})^{h(\mathbb{T}/C_p)} \simeq (X^{hC_p})^{h\mathbb{T}} \to (X^{tC_p})^{h\mathbb{T}},$$

there is a functorial fiber sequence computing $\mathrm{TC}(X)$:

$$\mathrm{TC}(X) \to X^{h\mathbb{T}} \xrightarrow{(\varphi_p^{h\mathbb{T}} - \mathrm{can})_p} \prod_p (X^{tC_p})^{h\mathbb{T}},$$

and analogously for p-cyclotomic spectra.

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Genuine cyclotomic spectra

Anna Marie Bohmann

In this talk, we discuss how the Nikolaus–Scholze definition of cyclotomic spectra [6], as outlined in the previous talk, relates to the previous definition of "genuine" cyclotomic spectra, as in Hesselholt–Madsen [3]. We focus on the comparison in the case of a chosen prime p. Thus, the main goal of this talk is to explain the following theorem:

Theorem 1 ([6, Theorem II.3.8, Theorem II.6.3]). *There is a functor $\mathrm{CycSp}_p^{\mathrm{gen}} \to \mathrm{CycSp}_p$ from the ∞-category of genuine p-cyclotomic spectra to the ∞-category of p-cyclotomic spectra in the sense of Nikolaus–Scholze that is an equivalence on the full sub-∞-categories of bounded below spectra.*

This functor, and the fact that it is an equivalence on bounded spectra, is a surprising and powerful result. The exegesis of this functor will be in three parts. First, we give a description of genuine cyclotomic spectra, which includes some indication of why even the existence of this functor is surprising, much less that it is an equivalence. Second, we explain how this functor arises. Lastly, we indicate why this functor restricts to an equivalence on bounded below spectra.

1. What are genuine cyclotomic spectra?

Each object of the ∞-category CycSp_p of p-cyclotomic spectra consists of a spectrum X with an action of C_{p^∞} and a map $X \to X^{tC_p}$ from X to the C_p-Tate construction on X. In contrast, the objects of the ∞-category $\mathrm{CycSp}_p^{\mathrm{gen}}$ consist of a *genuine C_{p^∞}-spectrum* X, in the sense of genuine equivariant stable homotopy theory, together with an *equivalence* $X \to \Phi^{C_p} X$ of X to its "geometric fixed points."

A genuine G-spectrum has an underlying spectrum with an action of G, but it is a much more structured object, as we will see below. We denote the ∞-category of spectra with G action by Sp^{BG} and the ∞-category of genuine G-spectra by $G\mathrm{Sp}$.[1]

There are several ways to describe genuine G-spectra [1, 4, 5, 7], but the salient features for us are the following. Recall that in Sp^{BG}, an equivalence $X \to Y$ is a weak equivalence that is also an equivariant map. This means that only well-defined notion of the "fixed points" of a spectrum $Y \in \mathrm{Sp}^{BG}$ is that of the homotopy fixed points, Y^{hG}. In $G\mathrm{Sp}$, it is harder for a map to be a weak equivalence and there are two additional well-defined notions of fixed points. These are the categorical fixed points, denoted X^G, and geometric fixed points, denoted $\Phi^G(X)$. The categorical fixed points are the most "fundamental," in that both geometric fixed points and homotopy fixed points can be defined in term of categorical fixed points. They are frequently just called the "fixed points." Categorical fixed points for genuine G-spectra fit into several nice adjunctions that parallel the adjunctions enjoyed by fixed points for G-spaces. However, categorical fixed points don't commute with smash product or taking suspension spectrum of a G-space. Geometric fixed points don't have these adjunction properties, but do commute with smash products and taking suspension spectra. This latter property is the origin of the name.

Remark 2. Note that if H is a subgroup of G and $X \in G\mathrm{Sp}$, there are also well-defined notions of categorical H-fixed points X^H, geometric fixed points $\Phi^H(X)$ and homotopy fixed points X^{hH}.

These more refined versions of fixed points allow one to give a characterization of a weak equivalence in $G\mathrm{Sp}$: a map $X \to Y$ in $G\mathrm{Sp}$ is a weak equivalence if the induced map on H-fixed points $X^H \to Y^H$ is a weak equivalence of spectra for all subgroups H of G. Since fixed points under the trivial group is simply the underlying spectrum, we see that a weak equivalence in $G\mathrm{Sp}$ is in particular a weak equivalence of underlying spectra-with-G-action. There is thus a forgetful functor

$$U \colon G\mathrm{Sp} \to \mathrm{Sp}^{BG}.$$

---

[1]Classically, "genuine G-spectra" are defined when G is a compact Lie group, which C_{p^∞} is decidedly not. The ∞-category of "genuine C_{p^∞} spectra" is defined as the limit of the ∞-categories of genuine C_{p^n}-spectra; see [6, p. 43].

In general, this functor forgets a lot of information—this is evident in the relaxation of the "weak equivalence" condition—which makes the equivalence part of Theorem 1 surprising.

At heart, the forgetful functor U is the functor of Theorem 1, but it is not apparent that forgetting defines a functor $\mathrm{CycSp}_p^{\mathrm{gen}} \to \mathrm{CycSp}_p$: if $X \to \Phi^{C_p}X$ is an object of $\mathrm{CycSp}_p^{\mathrm{gen}}$, forgetting yields a spectrum with C_{p^∞}-action and a map to the underlying spectrum with C_{p^∞}-action of $\Phi^{C_p}X$, rather than a map to the Tate spectrum X^{tC_p}.

2. The functor $\mathrm{CycSp}_p^{\mathrm{gen}} \to \mathrm{CycSp}_p$

We now show that there is a functor $\mathrm{CycSp}_p^{\mathrm{gen}} \to \mathrm{CycSp}_p$ arising from the forgetful functor $C_{p^\infty}\mathrm{Sp} \to \mathrm{Sp}^{BC_{p^\infty}}$. The basic ingredient is a comparison map between the underlying spectrum with C_{p^∞}-action of $\Phi^{C_p}X$ and the Tate spectrum X^{tC_p}. In order to define this comparison, we need a better understanding of homotopy fixed points, geometric fixed points and the Tate construction.

Definition 3. Let $E[\not\supseteq C_p]$ be a free contractible C_{p^∞}-space.[2] That is, $E[\not\supseteq C_p]$ is characterized by having fixed point spaces with homotopy types

$$(E[\not\supseteq C_p])^H \simeq \begin{cases} \emptyset & H = C_{p^n}, \ n \geq 1 \\ * & H \text{ trivial.} \end{cases}$$

Definition 4. Let X be a genuine C_{p^∞}-spectrum. For any $n \geq 1$, the C_{p^n}-homotopy fixed points of X can be defined as

$$X^{hC_{p^n}} = (F(E[\not\supseteq C_p]_+, X))^{C_{p^n}}.$$

The functor $F(E[\not\supseteq C_p]_+, -) \colon C_{p^\infty}\mathrm{Sp} \to C_{p^\infty}\mathrm{Sp}$ has an important interpretation. As a consequence of Definition 4, we see that spectra $F(E[\not\supseteq C_p]_+, X)$ in the image of this functor have the property that for any C_{p^n}, their categorical C_{p^n}-fixed points and their homotopy C_{p^n}-fixed points agree.

Definition 5. A genuine C_{p^∞}-spectrum $X \in C_{p^\infty}\mathrm{Sp}$ is called *Borel* if $X^{C_{p^n}} \simeq X^{hC_{p^n}}$ for all n.

Thus the image $F(E[\not\supseteq C_p]_+, -)$ consists of Borel spectra, and in fact this property characterizes the image: the image of the functor $F(E[\not\supseteq C_p]_+, -)$ is the full sub-∞-category of Borel spectra. Since homotopy fixed points are determined by the underlying homotopy type of a spectrum, a map of genuine C_{p^∞}-spectra that is an equivalence on underlying spectra with C_{p^∞}-action induces an equivalence after applying $F(E[\not\supseteq C_p]_+, -)$. This is the essence of the following theorem.

Theorem 6 ([6, Theorem II.2.7]). *The forgetful functor $U \colon C_{p^\infty}\mathrm{Sp} \to \mathrm{Sp}^{BC_{p^\infty}}$ admits a fully faithful right adjoint $B \colon \mathrm{Sp}^{BC_{p^\infty}} \to C_{p^\infty}\mathrm{Sp}$ whose essential image consists of the Borel spectra. We call B the* Borel construction *or* Borelification.

---

[2]We think of $E[\not\supseteq C_p]$ as a C_{p^∞}-space where the action by the family of all subgroups not containing C_p. This is the origin of the notation.

One way to conceptualize this theorem is that the Borel construction takes the only "fixed point data" that a spectrum with C_{p^∞}-action is entitled to have, its homotopy fixed points, and makes that the actual fixed point data. Since the homotopy type of a genuine C_{p^∞}-spectrum is determined by the homotopy type of all its fixed point spectra, this suffices to determine BY up to homotopy.

We next describe the C_p-geometric fixed points of a genuine C_{p^∞}-spectrum.

Definition 7. Let $\widetilde{E[\not\supseteq C_p]}$ be space defined by the cofiber sequence

$$\text{(1)} \qquad\qquad E[\not\supseteq C_p]_+ \to S^0 \to \widetilde{E[\not\supseteq C_p]}.$$

where the map $E[\not\supseteq C_p]_+ \to S^0$ is given by mapping the basepoint to the basepoint and $E[\not\subseteq C_p]$ to the non-base point of S^0.

Definition 8. If $X \in C_{p^\infty}\mathrm{Sp}$ is a genuine C_{p^∞}-spectrum, the C_p-geometric fixed points of X are defined to be the C_p-categorical fixed points $(\widetilde{E[\not\supseteq C_p]} \wedge X)^{C_p}$.

This definition of geometric fixed points relies on the fact that the only subgroup of C_{p^∞} not containing C_p is the trivial subgroup; $\widetilde{E[\not\supseteq C_p]}$ cannot be used to define $\Phi^{C_{p^n}}$ for $n > 1$.

For any $X \in C_{p^\infty}\mathrm{Sp}$, smashing the cofiber sequence (1) with X yields a cofiber sequence

$$E[\not\supseteq C_p]_+ \wedge X \to X \to \widetilde{E[\not\supseteq C_p]} \wedge X.$$

and then taking C_p-fixed points yields a cofiber sequence

$$(E[\not\supseteq C_p]_+ \wedge X)^{C_p} \to X^{C_p} \to (\widetilde{E[\not\supseteq C_p]} \wedge X)^{C_p}.$$

Using the Adams isomorphism, the left-hand term can be identified as the homotopy orbits X_{hC_p}, so this sequence takes the form

$$X_{hC_p} \to X^{C_p} \to \Phi^{C_p}X.$$

Smashing the cofiber sequence (1) with the map from X to the Borelification of X produces a map of cofiber sequences

(2)

$$
\begin{array}{ccccc}
E[\not\supseteq C_p]_+ \wedge X & \longrightarrow & X & \longrightarrow & \widetilde{E[\not\supseteq C_p]} \wedge X \\
\downarrow & & \downarrow & & \downarrow \\
E[\not\supseteq C_p]_+ \wedge F(E[\not\supseteq C_p]_+, X) & \longrightarrow & F(E[\not\supseteq C_p]_+, X) & \longrightarrow & \widetilde{E[\not\supseteq C_p]} \wedge F(E[\not\supseteq C_p]_+ X)
\end{array}
$$

and one can show that the left vertical map is a genuine weak equivalence. This is the heart of "Generalized Tate cohomology" in the sense of [2]. Notice that in the lower cofiber sequence all of the spectra are Borel, which yields the following proposition.

Proposition 9. *For a genuine C_{p^∞}-spectrum X, the underlying cofiber sequence of spectra with C_{p^∞}-spectra obtained by smashing the sequence (1) with the Borelification $F(E[\not\supseteq C_p]_+, X)$ and taking C_p-fixed points is the norm cofibration sequence*

$$X_{hC_p} \to X^{hC_p} \to X^{tC_p}$$

of [6, Definition I.1.3] *which defines the C_p-Tate construction on the underlying spectrum with C_{p^∞}-action of X.*

Thus the applying C_p-fixed points to the diagram (2) induces a natural map of cofiber sequences

$$
\begin{array}{ccc}
X_{hC_p} & \longrightarrow X^{C_p} & \longrightarrow \Phi^{C_p}X \\
\simeq \downarrow & \downarrow & \downarrow \\
X_{hC_p} & \longrightarrow X^{hC_p} & \longrightarrow X^{tC_p}
\end{array}
$$

Proposition 10. *The right vertical map above yields a natural transformation of functors*

$$
\begin{array}{ccc}
C_{p^\infty}\mathrm{Sp} & \xrightarrow{\Phi^{C_p}} & C_{p^\infty}\mathrm{Sp} \\
\downarrow & \Swarrow & \downarrow \\
\mathrm{Sp}^{BC_{p^\infty}} & \xrightarrow{(-)^{tC_p}} & \mathrm{Sp}^{BC_{p^\infty}}
\end{array}
$$

where the vertical maps are the forgetful functor.

This natural transformation is the comparison we need to define the forgetful functor $\mathrm{CycSp}_p^{\mathrm{gen}} \to \mathrm{CycSp}_p$. On objects, this functor takes a genuine C_{p^∞}-spectrum X with its equivalence $X \xrightarrow{\sim} \Phi^{C_p}X$ to the underlying spectrum with C_{p^∞}-action of X equipped with the map $X \to \Phi^{C_p}X \to X^{tC_p}$.

3. The equivalence

Proving that the functor $\mathrm{CycSp}_p^{\mathrm{gen}} \to \mathrm{CycSp}_p$ is an equivalence on the sub-∞-categories of bounded below spectra requires both formal work with adjoint functors and a nonformal calculational ingredient. In order to better understand this functor, we factor it through an ∞-category $\mathrm{CoAlg}_{\Phi^{C_p}}$ whose objects are spectra $X \in C_{p^\infty}\mathrm{Sp}$ equipped with a map $X \to \Phi^{C_p}X$ of genuine C_{p^∞}-spectra that is *not* required to be an equivalence. The adjoint functor theorem for presentable infinity categories provides right adjoints as follows:

Theorem 11 ([6, Lemma II.6.2, Theorem II.5.6]). *The forgetful functors* $\mathrm{CycSp}_p^{\mathrm{gen}} \xrightarrow{\iota} \mathrm{CoAlg}_{\Phi^{C_p}}$ *and* $\mathrm{CoAlg}_{\Phi^{C_p}} \xrightarrow{U} \mathrm{CycSp}_p$ *have right adjoints*

$$
\mathrm{CycSp}_p^{\mathrm{gen}} \underset{R_\iota}{\overset{\iota}{\rightleftarrows}} \mathrm{CoAlg}_{\Phi^{C_p}} \underset{B}{\overset{U}{\rightleftarrows}} \mathrm{CycSp}_p.
$$

To show that $\mathrm{CycSp}_p^{\mathrm{gen}} \to \mathrm{CycSp}_p$ is an equivalence on the full sub-∞-categories of bounded below spectra, it therefore suffices to show that composite $U\iota$ reflects equivalences and the counit $U\iota RB$ is an equivalence on bounded below spectra. In fact, since genuine weak equivalences can be detected via equivalences on all geometric fixed points, for spectra known to be genuine cyclotomic, underlying equivalences must be genuine equivalences. Hence $U\iota$ reflects equivalences.

We discuss the two right adjoints in turn, starting with the adjoint B to the forgetful functor $U \colon \mathrm{CoAlg}_{\Phi^{C_p}} \to \mathrm{CycSp}_p$. Although it is not immediately obvious, on bounded below spectra this adjoint is given by applying the Borelification functor of Theorem 6. This is our excuse for the abuse of notation in calling both adjoints B.

Suppose $Y \to Y^{tC_p}$ is a cyclotomic spectrum in CycSp_p. Applying B to this map yields a map $BY \to B(Y^{tC_p})$ of genuine C_{p^∞}-spectra. The Tate construction depends only on underlying Borel-type data, so $B(Y^{tC_p}) \simeq (BY)^{tC_p}$. As we saw in Proposition 10, there is a map $\Phi^{C_p}BY \to BY^{tC_p}$, so in order to produce an object in $\mathrm{CoAlg}_{\Phi^{C_p}}$, it suffices to prove that this map is a genuine equivalence on bounded below spectra.

For any C_{p^∞}-spectrum X, the map of cofiber sequences (2) yields a pullback diagram

$$
\begin{array}{ccc}
X & \longrightarrow & \widetilde{E[\not\supseteq C_p]} \wedge X \\
\downarrow & & \downarrow \\
F(E[\not\supseteq C_p]_+, X) & \longrightarrow & \widetilde{E[\not\supseteq C_p]} \wedge F(E[\not\supseteq C_p]_+, X)
\end{array}
$$

which after applying C_{p^n}-fixed points becomes

$$
(3) \qquad\qquad
\begin{array}{ccc}
X^{C_p^n} & \longrightarrow & (\Phi^{C_p}X)^{C_{p^{n-1}}} \\
\downarrow & & \downarrow \\
X^{hC_{p^n}} & \longrightarrow & X^{tC_{p^n}}
\end{array}
$$

Note that the bottom row of this square depends on only on the underlying spectrum C_{p^∞}-action of X because it arose from $F(E[\not\supseteq C_p]_+, X)$.

Now if $X = BY$ is a Borel spectrum, for each $n \geq 1$, the left map is an equivalence, and so we have an equivalence of spectra $(\Phi^{C_p}BY)^{C_{p^{n-1}}} \xrightarrow{\sim} BY^{tC_{p^n}}$. To prove that the map $\Phi^{C_p}BY \to BY^{tC_p}$ is an equivalence, we must show that it induces an equivalence on $C_{p^{n-1}}$-categorical fixed points for all $n \geq 1$, and since for Borel spectra, categorical and homotopy fixed points agree, it suffices to prove the following lemma.

Lemma 12. *If* $Y \in \mathrm{Sp}^{BC_{p^\infty}}$ *is bounded below, then* $Y^{tC_{p^{n+1}}} \simeq (Y^{tC_p})^{hC_{p^n}}$.

This equivalence follows from a key calculational lemma.

Lemma 13 (Tate Orbit Lemma, [6])**.** *If* $Y \in \mathrm{Sp}^{BC_{p^2}}$ *is bounded below, then the norm map* $Y_{hC_{p^2}} \to (Y_{hC_p})^{hC_{p^2}/C_p}$ *is an equivalence and thus*

$$(Y_{hC_p})^{tC_{p^2}/C_p} \simeq *$$

The Tate Orbit Lemma follows from an induction argument starting with Eilenberg–MacLane spectra.

Proof of Lemma 12. Consider $Y_{hC_{p^{n-2}}}$ as a spectrum with C_{p^2}-action; note that $(Y_{hC_{p^{n-2}}})_{hC_{p^2}} = Y_{hC_{p^n}}$. Thus the Tate orbit lemma implies that the norm map $Y_{hC_{p^n}} \to (Y_{hC_{p^{n-1}}})^{hC_p}$ is an equivalence. By induction, we deduce that the norm map $Y_{hC_{p^n}} \to (Y_{hC_p})^{hC_{p^{n-1}}}$ is an equivalence. We then have a diagram of norm sequences

$$
\begin{array}{ccccc}
Y_{hC_{p^n}} & \longrightarrow & Y^{hC_{p^n}} & \longrightarrow & Y^{tC_{p^n}} \\
{\scriptstyle \sim}\downarrow & & {\scriptstyle \sim}\downarrow & & \downarrow \\
(Y_{hC_p})^{hC_{p^{n-1}}} & \longrightarrow & (Y^{hC_p})^{hC_{p^{n-1}}} & \longrightarrow & (Y^{tC_p})^{hC_{p^{n-1}}}
\end{array}
$$

where the middle vertical map is an equivalence by definition and the left vertical equivalence is the one we've just constructed. $\square$

Hence for Y bounded below, the map $\Phi^{C_p}BY \to BY^{tC_p}$ induces an equivalence

$$(\Phi^{C_p}BY)^{C_{p^{n-1}}} \to (BY^{tC_p})^{C_{p^{n-1}}} \simeq (BY^{tC_p})^{hC_{p^{n-1}}}$$

on all fixed points and therefore is an equivalence of genuine C_{p^∞}-spectra. Thus, when $Y \to Y^{tC_p}$ is a bounded below cyclotomic spectrum,

$$BY \to B(Y^{tC_p}) \simeq \Phi^{C_p}(BY)$$

is an object of $\mathrm{CoAlg}_{\Phi^{C_p}}$. This shows that the right adjoint to the functor $U \colon \mathrm{CoAlg}_{\Phi^{C_p}} \to \mathrm{CycSp}_p$ is given by Borelification as desired.

The second right adjoint, $R \colon \mathrm{CoAlg}_{\Phi^{C_p}} \to \mathrm{CycSp}_p^{\mathrm{gen}}$, takes place purely in the world of genuine C_{p^∞}-spectra. It replaces a genuine C_{p^∞}-spectrum X equipped with a map $X \to \Phi^{C_p}X$ with a genuine C_{p^∞}-spectrum for which this map is an equivalence. The existence of this adjoint follows from the adjoint functor theorem and results about presentability of and existence of colimits in the ∞-categories in question. A detailed description is given in [6, Theorem II.5.6].

Instead of giving a detailed characterization of the functor $R \colon \mathrm{CoAlg}_{\Phi^{C_p}} \to \mathrm{CycSp}_p^{\mathrm{gen}}$, we unpack the structure of a genuine cyclotomic spectrum that is bounded below. We show that the data of such a spectrum can be reconstructed from underlying data, which makes the equivalence of ∞-categories of Theorem 1 plausible. In fact, this reconstruction is ultimately what proves that the counit of the adjunction in Theorem 11 is an equivalence on bounded below spectra; see [6, Sections II.5 and II.6].

Let X be a genuine cyclotomic spectrum that is bounded below. Since $X^{tC_{p^n}}$ depends only on the underlying spectrum of X, we can invoke Lemma 12 to see that the pullback square (3) takes the form

$$
\begin{array}{ccc}
X^{C_{p^n}} & \longrightarrow & (\Phi^{C_p}X)^{C_{p^{n-1}}} \\
\downarrow & & \downarrow \\
X^{hC_{p^n}} & \longrightarrow & (X^{tC_p})^{hC_{p^{n-1}}}
\end{array}
$$

and thus

$$
X^{C_{p^n}} \simeq X^{hC_{p^n}} \times_{(X^{tC_p})^{hC_{p^{n-1}}}} (\Phi^{C_p}X)^{C_{p^{n-1}}}
$$

Note that since $\Phi^{C_p}X \simeq X$, all of the geometric fixed points $\Phi^{C_{p^n}}X$ are also bounded below, and thus via induction we can describe $X^{C_{p^n}}$ using an iterated pullback diagram

$$
\begin{array}{ccccc}
X^{C_{p^n}} & \longrightarrow & & & \Phi^{C_{p^n}}X \\
\downarrow & & & & \downarrow \\
 & & \cdots \longrightarrow & & (\Phi^{C_{p^n}}X)^{tC_p} \\
 & & & & \downarrow \\
 & (\Phi^{C_p}X)^{hC_{p^{n-1}}} & \longrightarrow & ((\Phi^{C_p}X)^{tC_p})^{hC_{p^{n-2}}} & \\
\downarrow & \downarrow & & & \\
X^{hC_{p^n}} & \longrightarrow (X^{tC_p})^{hC_{p^{n-1}}} & & &
\end{array}
$$

In fact, this description holds for any bounded below C_{p^∞}-spectrum all of whose geometric fixed points are also bounded below. When we additionally assume X is genuine cyclotomic so that there are equivalences $X \xrightarrow{\sim} \Phi^{C_p}X \xrightarrow{\sim} \Phi^{C_{p^2}}X \xrightarrow{\sim} \cdots$, the fixed point spectrum $X^{C_{p^n}}$ becomes the iterated pullback

$$
(4) \qquad X^{C_{p^n}} \simeq X^{hC_{p^n}} \times_{(X^{tC_p})^{hC_{p^{n-1}}}} X^{hC_{p^{n-1}}} \times_{(X^{tC_p})^{hC_{p^{n-2}}}} \cdots \times_{X^{tC_p}} X.
$$

Here the maps from left to right are those coming from the norm cofibration sequence and the maps from right to left are those induced on homotopy fixed points by $X \xrightarrow{\sim} \Phi^{C_p}X \to X^{tC_p}$.

This pullback shows that the categorical fixed point spectra of X, which determine its genuine homotopy type, are in fact recoverable from the Borel, or underlying, data of X. This is essence of why the counit $U\iota BR$ of the adjunction in Theorem 11 is an equivalence on bounded below spectra: For bounded below cyclotomic spectra, the "extra" structure built in by the right adjoints was actually already present!

We can also use the iterated pullback of (4) to see the key structure of a cyclotomic spectrum that is necessary to form topological cyclic homology. Recall that classically, topological cyclic homology is built from maps R and F of the form

$R, F\colon X^{C_{p^n}} \to X^{C_{p^{n-1}}}$. The map F is given by inclusion of fixed points, and the map R is defined using the cyclotomic structure:

$$R\colon X^{C_{p^n}} \to (\Phi^{C_p} X)^{C_{p^{n-1}}} \xrightarrow{\sim} X^{C_{p^{n-1}}}.$$

Using the maps R, one defines TR as the inverse limit over the maps R: $\mathrm{TR} = \varprojlim_R X^{C_{p^n}}$. Then TC is defined as the equalizer

$$\mathrm{TC} \longrightarrow \mathrm{TR} \underset{F}{\overset{\mathrm{id}}{\rightrightarrows}} \mathrm{TR}\ .$$

Under the iterated pullback description (4) of $X^{C_{p^n}}$, the map $R\colon X^{C_{p^n}} \to X^{C_{p^{n-1}}}$ is given by forgetting the first factor of the iterated pullback. The map $F\colon X^{C_{p^n}} \to X^{C_{p^{n-1}}}$ is given by forgetting the last factor and composing with the maps $X^{hC_{p^n}} \to X^{hC_{p^{n-1}}}$ at each level. Thus, we see that all of the necessary data for defining TC of a bounded below genuine cyclotomic spectrum is captured by the underlying spectrum with C_{p^∞}-action.

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The Tate diagonal

Fabian Hebestreit

The goal of the talk was to construct the Tate diagonal and establish its basic properties, following [NiSc 17, Section 3] and [LNRo 12]. Among them is the simplest case of Segal's conjecture, the connection to which I spell out in the final section (it did not feature in the Arbeitsgemeinschaft again).

Throughout, the words category, functor, natural transformation and so on will refer to their $(\infty, 1)$-categorical incarnations. Following Nikolaus and Scholze I will denote the wedge and smash product of spectra by $\oplus$ and $\otimes$ instead of the usual $\vee$ and $\wedge$, respectively.

1. Construction of the Tate diagonal

First a bit of notation: Given a prime number p and a spectrum X, we shall denote

$$D_p(X) = (X^{\otimes p})_{hC_p},$$
$$F_p(X) = (X^{\otimes p})^{hC_p},$$
$$T_p(X) = (X^{\otimes p})^{tC_p},$$

the homotopy orbits, homotopy fixed point and Tate construction of the p-fold smash power of X, with $C_p \subseteq S_p$ the subgroup generated by the p-cycles (itself cyclic of order p) acting by permuting the factors. If we denote the category of spectra by Sp these assignments give rise to evident functors

$$D_p, F_p, T_p \colon \mathrm{Sp} \longrightarrow \mathrm{Sp}.$$

The functor D_p is a version of the *extended power construction* discussed further in Talk number 5, whereas T_p is sometimes called the *topological Singer construction*. By definition of the Tate construction the norm map N naturally connects these functors into an objectwise exact sequence

$$D_p \overset{N}{\Longrightarrow} F_p \Longrightarrow T_p.$$

We shall now set out to construct the *Tate diagonal* Δ_p, a particular natural transformation

$$\mathrm{id} \Longrightarrow T_p \colon \mathrm{Sp} \longrightarrow \mathrm{Sp},$$

essentially from formal properties of the functor T_p. To this end we have:

Proposition 1. *The functor T_p is exact.*

Proof. Let $X_0 \to X_1 \to X_2$ be a fibre sequence and consider the 'obvious' filtration F of $X_1^{\otimes p}$ (by C_p-spectra), given informally by

$$F_s = \text{'im'} \left(\bigoplus_{\substack{I \in \{0,1\}^p \\ |I|=s}} X_{I_1} \otimes \cdots \otimes X_{I_p} \longrightarrow X_1^{\otimes p} \right)$$

with the induced C_p-action. More formally, the terms $X_{I_1} \otimes \cdots \otimes X_{I_p}$ assemble into a functor

$$X \colon \{0,1\}^p \longrightarrow \mathrm{Sp},$$

where the source is regarded as a category via its product ordering and the underlying spectrum of F_i is just the colimit of the full subcategory on the objects with at most i entries being 1. To obtain the C_p-action we extend X to a functor on the category $\{0,1\}^p_{hC_p}$ using the symmetry of the smash product and define F_i as the left Kan extension of X along the projection

$$\{0,1\}^p_{hC_p} \longrightarrow *_{hC_p} = BC_p.$$

Then $F_0 = X_0^{\otimes p}$, $F_p = X_1^{\otimes p}$ and for $0 \le s < p$ there are canonical exact sequences

$$F_{s-1} \longrightarrow F_s \longrightarrow \bigoplus_{\substack{I \in \{0,2\}^p \\ |I|=2s}} X_{I_1} \otimes \cdots \otimes X_{I_p}$$

of C_p-spectra, the action of the last term given by permutation of the wedge *and* smash factors.

Claim: The induced maps

$$F_0^{tC_p} \longrightarrow \cdots \longrightarrow F_{p-1}^{tC_p}$$

are all equivalences.

Granting the claim the exact sequence

$$F_{p-1}^{tC_p} \longrightarrow F_p^{tC_p} \longrightarrow X_2^{tC_p},$$

the functor $-^{tC_p} \colon \mathrm{Sp}^{C_p} \to \mathrm{Sp}$ is exact after all, becomes

$$\mathrm{T}_p(X_0) \longrightarrow \mathrm{T}_p(X_1) \longrightarrow \mathrm{T}_p(X_2)$$

as desired.

To prove the claim observe that by virtue of the exactness of $-^{tC_p}$ the cofibre of $F_{s-1}^{tC_p} \to F_s^{tC_p}$ is

$$\left(\bigoplus_{\substack{I \in \{0,2\}^p \\ |I|=2s}} X_{I_1} \otimes \cdots \otimes X_{I_p} \right)^{tC_p} \simeq \bigoplus_{\substack{J \in \{0,2\}^p/C_p \\ |J|=2s}} \left(\bigoplus_{I \in J} X_{I_1} \otimes \cdots \otimes X_{I_p} \right)^{tC_p}$$

and the J-th term on the right is the image of $X_{I_1} \otimes \cdots \otimes X_{I_p}$ under the induction functor $\mathrm{Sp} \to \mathrm{Sp}^{C_p}$ for any $I \in J$: To see this, note only that J is automatically a free C_p-orbit, since the only fixed points of the C_p-action on $\{0,2\}^p$ are the constant sequences both of which are excluded by the assumption $0 < s < p$. But the Tate construction vanishes on induced spectra. $\square$

We now obtain Δ_p from the following standard fact:

Observation 2. *For a stable category $\mathcal{C}$ and a category $\mathcal{D}$ admitting finite limits, the functor*

$$\Omega^\infty \colon \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathrm{Sp}(\mathcal{D}_*)) \longrightarrow \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$$

is an equivalence.

Here $\mathrm{Fun}^{(l)\mathrm{ex}}$ denotes the full subcategories of functors Fun that are (left) exact and

$$\mathrm{Sp}(\mathcal{D}) = \lim \left(\cdots \xrightarrow{\Omega} D_* \xrightarrow{\Omega} \mathcal{D}_* \xrightarrow{\Omega} \mathcal{D}_* \right)$$

the category of spectral objects in $\mathcal{D}$, with $\Omega^\infty \colon \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$ the projection to the last entry in the limit diagram.

For S the category of groupoids ('spaces'), we then have $\mathrm{Sp} = \mathrm{Sp}(\mathrm{S})$ and by definition the above precisely says that Ω^∞ witnesses $\mathrm{Sp}(\mathcal{D})$ being the *stabilisation* of $\mathcal{D}$.

As we shall need a construction from the proof later, let me provide a sketch:

Proof. The inverse functor is given by sending $F \in \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$ to

$$X \longmapsto (\dots, F(\Sigma^2 X), F(\Sigma X), F(X)),$$

pointed by the composite $* = F(*) \overset{F(0)}{\to} F(X)$ and with structure maps $F(\Sigma^n X) \to \Omega F(\Sigma^{n+1} X)$ obtained as an instance of the following general construction (by staring at the diagram below) of maps $G(X) \to \Omega G(\Sigma X)$ for any pointed functor G from a pointed category with finite colimits to one with finite limits:
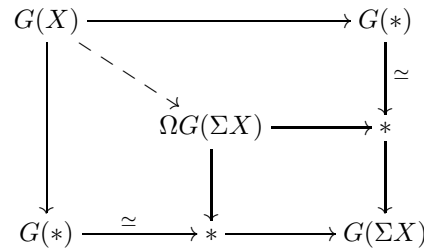
The outer sqaure is G applied to the defining pushout of ΣX and the inner square is the pullback defining its upper left corner. It is this general construction we shall need again below.

It is now readily checked that the functor we constructed above is indeed exact and an inverse to Ω^∞: For the former note that $\mathrm{Sp}(\mathcal{D})$ has an invertible loop functor, and is therefore stable, whence exactness for a functor into spectra objects is the same as left exactness, which can be tested at level 0, since Ω preserves limits. For the latter, note first that the composition starting and ending at $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$ is evidently the identity. For the other composition we observe that for the n-th component of an exact functor $F \colon C \to \mathrm{Sp}(\mathcal{D}_*)$ we have

$$F_n \simeq F_n \Omega^n \Sigma^n \simeq \Omega^n F_n \Sigma^n \simeq F_0 \Sigma^n,$$

as functors $\mathcal{C} \to \mathcal{D}_*$, which shows that F is recovered from its lowest piece by the recipe above, since the forgetful map

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D}_*) \longrightarrow \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$$

is obviously an equivalence (since every object in $\mathcal{C}$ is pointed and left exact functors preserve pointed objects). $\qquad\square$

Using the Yoneda lemma and the identification $\mathrm{Hom}(\mathbb{S}, -) \simeq \Omega^\infty \colon \mathrm{Sp} \to \mathrm{S}$ with $\mathbb{S}$ the sphere spectrum, we can now compute

$$\begin{aligned}
\mathrm{Nat}(\mathrm{id}, \mathrm{T}_p) &\simeq \mathrm{Nat}(\Omega^\infty, \Omega^\infty \mathrm{T}_p) \\
&\simeq \mathrm{Nat}(\mathrm{Hom}(\mathbb{S}, -), \Omega^\infty \mathrm{T}_p) \\
&\simeq \Omega^\infty \mathrm{T}_p(\mathbb{S}) \\
&\simeq \mathrm{Hom}(\mathbb{S}, \mathrm{T}_p(\mathbb{S})),
\end{aligned}$$

with the composite given by evaluating a natural transformation on the sphere. Observing that $\mathrm{T}_p(\mathbb{S}) \simeq \mathbb{S}^{\mathrm{t}C_p}$ (for the trivial C_p-action on $\mathbb{S}$) we set:

Definition 3. The Tate diagonal Δ_p is the essentially unique transformation which on the sphere is given by the canonical map

$$\mathbb{S} \longrightarrow \mathbb{S}^{\mathrm{h}C_p} \longrightarrow \mathbb{S}^{\mathrm{t}C_p}.$$

Here the first map is given by

$$\mathbb{S} \simeq \mathrm{F}(*, \mathbb{S}) \xrightarrow{r^*} \mathrm{F}(\mathrm{B}C_p, \mathbb{S}) \simeq \mathbb{S}^{\mathrm{h}C_p}$$

using the projection $r \colon \mathrm{B}C_p \to *$ (and F denoting function spectra). Let me warn the reader that for an arbitrary spectrum X, the map $\Delta_p \colon X \to \mathrm{T}_p(X)$ does *not* factor through $\mathrm{F}_p(X)$ (in particular, the functor F_p is not exact). This is for example the case for $\mathrm{H}\mathbb{Z}$, as is readily deduced from the computations in the proof of part ii) of the theorem below.

2. Properties of the Tate diagonal

Theorem 4. *We have:*

i) $\Delta_p \colon \mathrm{id}_{\mathrm{Sp}} \Longrightarrow \mathrm{T}_p$ *admits an essentially unique symmetric monoidal refinement.*

ii) Every transformation

$$\mathrm{id} \Longrightarrow \mathrm{T}_p^{\mathbb{Z}} \colon \mathcal{D}(\mathbb{Z}) \to \mathcal{D}(\mathbb{Z})$$

vanishes objectwise (where $\mathrm{T}_p^{\mathbb{Z}}(C) = (C^{\otimes_{\mathbb{Z}} p})^{\mathrm{t}C_p}$).

iii) For X bounded below, $\mathrm{T}_p(X)$ is p-complete and the induced map

$$\Delta_p \colon \widehat{X_p} \longrightarrow \mathrm{T}_p(X)$$

is an equivalence.

To make sense of the first statement, recall that T_p obtains a lax symmetric monoidal refinement from its decomposition into

$$\mathrm{Sp} \xrightarrow{-^{\otimes p}} \mathrm{Sp}^{C_p} \xrightarrow{-^{\mathrm{t}C_p}} \mathrm{Sp},$$

since the first functor becomes symmetric monoidal through the evident reshuffling of factors and the second functor was given a lax symmetric monoidal structure in the first talk.

The second statement says that the existence of the Tate diagonal is a feature of higher algebra, that is not present in classical derived categories. This dichotomy will be the ultimate source of extra structure, in particular the coveted Frobenius lifts, on topological, as opposed to non-topological, Hochschild homology and its derivates.

The third statement has non-trivial overlap with Segal's conjecture on the stable cohomotopy of classifying spaces, on which I will comment briefly in the third chapter. Either assertion becomes false upon dropping the assumption that X be bounded below, the counterexample KU is discussed after [NiSc 17, Theorem III.1.7].

Proof. To obtain i), I will discuss the following

Claim: The identity functor is initial among exact, lax symmetric monoidal functors $Sp \to Sp$.

To observe that the unique multiplicative transformation $\mathrm{id} \implies T_p$ obtained from this has the Tate diagonal as its underlying transformation simply observe that the defining composite

$$\mathbb{S} \longrightarrow \mathbb{S}^{hC_p} \longrightarrow \mathbb{S}^{tC_p}$$

is one of commutative ring spectra (and since $\mathbb{S}$ is initial among such it is unique with this property): For the first map this is clear, and for the second it is immediate from the symmetric monoidal structure of the transformation $-^{hC_p} \to -^{tC_p}$ obtained in the first talk.

To obtain the claim we follow [Ni 16]. It is (somewhat tautologically) true that for any symmetric monoidal category $\mathcal{C}$ with unit $\mathbb{I}$, the functor

$$\mathrm{Hom}(\mathbb{I}, -) \colon \mathcal{C} \longrightarrow S$$

is initial among all lax symmetric monoidal functors; the required transformation comes from Yoneda's lemma, as

$$\mathrm{Nat}(\mathrm{Hom}(\mathbb{I}, -), F) \simeq F(\mathbb{I}),$$

and as part of the data of a lax symmetric monoidal functor comes a map $* \to F(\mathbb{I})$. Now for a preadditive source category $\mathcal{C}$ for which the symmetric monoidal structure preserves products, the functor $\mathrm{Hom}(\mathbb{I}, -)$ naturally takes values in commutative cartesian monoid objects in S (i.e. $\mathbb{E}_\infty$-spaces), since it preserves products and every object of a preadditive category admits a unique cartesian monoid structure. Viewed as a functor

$$\mathrm{Hom}(\mathbb{I}, -) \colon \mathcal{C} \longrightarrow \mathrm{cMon}(S)$$

it then becomes initial among product preserving, lax symmetric monoidal functors. Similarly, for $\mathcal{C}$ additive, $\mathrm{Hom}(\mathbb{I}, -)$ lifts to commutative group objects in S (i.e. grouplike $\mathbb{E}_\infty$-spaces or equivalently connective spectra) and for $\mathcal{C}$ stably symmetric monoidal (i.e. stable with biexact symmetric monoidal structure) it lifts to

$$\mathrm{Hom}(\mathbb{I}, -) \colon \mathcal{C} \longrightarrow Sp$$

and becomes initial among all exact, lax symmetric monoidal functors.

A similar line of argument cannot be followed for the category $\mathcal{D}(\mathbb{Z})$, and indeed the identity is *not* initial among exact, lax symmetric monoidal endofunctors of $\mathcal{D}(\mathbb{Z})$; we shall obtain this as a corollary of the second part. The identity on $\mathcal{D}(\mathbb{Z})$ *is* initial among all exact, lax symmetric monoidal functors that are $\mathbb{Z}$-linear (that is the induced map on morphism spectra is $H\mathbb{Z}$-linear, e.g. a dg-functor), and part ii) therefore implies that $T_p^{\mathbb{Z}}$ is not $\mathbb{Z}$-linear.

To obtain ii) we shall consider the Eilenberg-Mac Lane functor $H\colon \mathcal{D}(\mathbb{Z}) \to \mathrm{Sp}$ and show:

Claim 1: Every transformation

$$\tau\colon H \Longrightarrow H \circ \mathrm{T}_p(\mathbb{Z})\colon \mathcal{D}(\mathbb{Z}) \to \mathrm{Sp}$$

is a $\mathbb{Z}/p$-multiple of

$$H \overset{\Delta_p H}{\Longrightarrow} \mathrm{T}_p \circ H \overset{\pi}{\Longrightarrow} H \circ \mathrm{T}_p^{\mathbb{Z}},$$

where π comes from the map

$$(H-)^{\otimes p} \longrightarrow H\left(-^{\otimes_{\mathbb{Z}} p}\right),$$

that is part of the lax symmetric monoidal structure H canonically carries, and the fact that the square

$$\begin{array}{ccc}
\mathcal{D}(\mathbb{Z})^{C_p} & \xrightarrow{\ -^{tC_p}\ } & \mathcal{D}(\mathbb{Z}) \\
\downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H} \\
\mathrm{Sp}^{C_p} & \xrightarrow{\ -^{tC_p}\ } & \mathrm{Sp}
\end{array}$$

commutes (essentially by construction of $-^{tC_p}$).

Claim 2: The transformation $a \cdot (\pi \circ \Delta_p H)$ lifts to a transformation

$$\mathrm{id} \Longrightarrow \mathrm{T}_p^{\mathbb{Z}}\colon \mathcal{D}(\mathbb{Z}) \to \mathcal{D}(\mathbb{Z})$$

only when $a = 0$.

Together these claims show that the underlying transformation in spectra of any $\mathrm{id} \Rightarrow \mathrm{T}_p^{\mathbb{Z}}$ vanishes, whence by general non-sense about derived categories (and the fact that $\mathbb{Z}$ has projective dimension 1) the same holds objectwise for the original transformation (though it is not clear that it is trivial itself): Every $C \in \mathcal{D}(\mathbb{Z})$ is formal (i.e. quasi-isomorphic to the sum of its homology groups placed in the corresponding degree) and for abelian groups M and N

$$\pi_0 \mathrm{Hom}_{\mathcal{D}(\mathbb{Z})}(M[m], N[n]) = \mathrm{Ext}_{\mathbb{Z}}^{n-m}(M, N)$$

vanishes unless $m = n$ or $n + 1$, both of which can be detected on homology (directly in case of the former and by the Bockstein sequence in case of the latter) and therefore on the underlying transformation in spectra.

To obtain claim numero uno note first that $\mathrm{H} \circ \mathrm{T}_p^{\mathbb{Z}} \colon \mathcal{D}(\mathbb{Z}) \to \mathrm{Sp}$ is exact ($\mathrm{T}_p^{\mathbb{Z}}$ is by the same argument as for T_p), so that

$$
\begin{aligned}
\mathrm{Nat}(\mathrm{H}, \mathrm{HT}_p^{\mathbb{Z}}) &\simeq \mathrm{Nat}(\Omega^\infty \mathrm{H}, \Omega^\infty \mathrm{HT}_p^{\mathbb{Z}}) \\
&\simeq \mathrm{Nat}(\mathrm{Hom}(\mathbb{Z}, -), \Omega^\infty \mathrm{HT}_p^{\mathbb{Z}}) \\
&\simeq \Omega^\infty \mathrm{HT}_p^{\mathbb{Z}}(\mathbb{Z}) \\
&\simeq \Omega^\infty \mathrm{H}\mathbb{Z}^{tC_p}.
\end{aligned}
$$

This gives

$$
\pi_0 \mathrm{Nat}(\mathrm{H}, \mathrm{HT}_p^{\mathbb{Z}}) = \pi_0(\Omega^\infty \mathrm{H}\mathbb{Z}^{tC_p}) = \mathrm{H}_0(\mathbb{Z}^{tC_p}) = \widehat{\mathrm{H}}^0(C_p, \mathbb{Z}) = \mathbb{Z}/p.
$$

Now it is readily checked that $\pi \circ \Delta_p \mathrm{H}$ corresponds to 1 under this identification, for example since it is multiplicative. Let me warn the reader about a subtle point here: Composing the equivalence above with the identification

$$
\Omega^\infty \mathrm{H}\mathbb{Z}^{tC_p} \simeq \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}^{tC_p})
$$

results in a map

$$
\mathrm{Nat}(\mathrm{H}, \mathrm{HT}_p^{\mathbb{Z}}) \simeq \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}^{tC_p}) \xrightarrow{\ \mathrm{H}\ } \mathrm{Hom}(\mathrm{H}\mathbb{Z}, \mathrm{H}\mathbb{Z}^{tC_p}).
$$

In contrast to the case of endofunctors on spectra, this is *not* given by evaluating a transformation at $\mathbb{Z} \in \mathcal{D}(\mathbb{Z})$. Tracing through the diagrams shows that the two images of a transformation τ lie in the same component of $\mathrm{Hom}(\mathrm{H}\mathbb{Z}, \mathrm{H}\mathbb{Z}^{tC_p})$ if and only if

$$
\tau_{\mathbb{Z}} \colon \mathrm{H}\mathbb{Z} \longrightarrow \mathrm{H}\mathbb{Z}^{tC_p}
$$

happens to be $\mathrm{H}\mathbb{Z}$-linear and this need not be the case as the second claim shows. Incidentally, using iii) for the analogous calculation with T_p gives

$$
\pi_0 \mathrm{Nat}(\mathrm{id}, \mathrm{T}_p) = \pi_0(\Omega^\infty \mathbb{S}^{tC_p}) = \pi_0(\Omega^\infty \widehat{\mathbb{S}}_p) = \widehat{\mathbb{Z}}_p
$$

and composition with H and π induces the projection $\widehat{\mathbb{Z}}_p \to \mathbb{Z}/p$.

To obtain the second claim consider for a lift $\tau \colon \mathrm{id} \Rightarrow \mathrm{T}_p^{\mathbb{Z}}$ (and any preimage of a in the integers) the commutative diagram

$$
\begin{array}{ccccc}
\mathrm{H}\mathbb{Z} & \xrightarrow{\ a \cdot \Delta_p\ } & \mathrm{T}_p \mathrm{H}\mathbb{Z} & \xrightarrow{\ \mu^{tC_p}\ } & \mathrm{H}\mathbb{Z}^{tC_p} \\
\Big\| & & \Big\downarrow{\scriptstyle \pi} & & \Big\| \\
\mathrm{H}\mathbb{Z} & \xrightarrow{\ \mathrm{H}\tau\ } & \mathrm{H}T_p^{\mathbb{Z}}\mathbb{Z} & \xrightarrow[\ \sim\ ]{\mathrm{H}\mu^{tC_p}} & \mathrm{H}\mathbb{Z}^{tC_p}.
\end{array}
$$

It shows that $\mu^{tC_p} \circ a\Delta_p$ lies in the image of

$$
\mathrm{H} \colon \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}^{tC_p}) \longrightarrow \mathrm{Hom}(\mathrm{H}\mathbb{Z}, \mathrm{H}\mathbb{Z}^{tC_p}).
$$

But the components of the latter are easily determined:

$$\pi_0\mathrm{Hom}(\mathrm{H}\mathbb{Z}, \mathrm{H}\mathbb{Z}^{\mathrm{t}C_p}) = \pi_0\mathrm{Hom}(\mathrm{H}\mathbb{Z}, \tau_{\geq 0}\mathrm{H}\mathbb{Z}^{\mathrm{t}C_p})$$

$$= \prod_{i\in\mathbb{N}} \pi_0\mathrm{Hom}(\mathrm{H}\mathbb{Z}, \Sigma^{2i}\mathrm{H}\mathbb{Z}/p)$$

$$= (\mathcal{A}_p/\beta)\widehat{{}_I},$$

where $I \subseteq \mathcal{A}_p$ is the augmentation ideal of the Steenrod algebra and $\beta \in \mathcal{A}_p$ is the Bockstein homomorphism. Furthermore, Nikolaus and Scholze show that the value of $\mu^{\mathrm{t}C_p} \circ \Delta_p$ at $\mathrm{H}\mathbb{Z}$ corresponds to $\sum_{i\in\mathbb{N}} P^i$, only p-fold multiples of which lie in the image of

$$\mathrm{H}\colon \pi_0\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}^{\mathrm{t}C_p}) \longrightarrow \pi_0\mathrm{Hom}(\mathrm{H}\mathbb{Z}, \mathrm{H}\mathbb{Z}^{\mathrm{t}C_p})$$

since this map identifies with the unit map $\mathbb{Z}/p \to (\mathcal{A}_p/\beta)\widehat{{}_I}$.

For part iii) let us first verify that $X^{\mathrm{t}C_p}$ is indeed p-complete, when X is bounded below (which will verify that $\mathrm{T}_p X$ is p-complete, since smash powers of bounded below spectra remain bounded below). Now for any $\mathbb{Z}[C_p]$-module A

$$\pi_*(\mathrm{H}A^{\mathrm{t}C_p}) = \widehat{\mathrm{H}}^*(C_p; A)$$

is p-torsion, so $\mathrm{H}A^{\mathrm{t}C_p}$ is p-complete. But then inductively so is $(\tau_{\leq n}X)^{\mathrm{t}C_p}$ by exactness of $-^{\mathrm{t}C_p}$. We obtain the result from

$$X^{\mathrm{t}C_p} \simeq \lim(\tau_{\leq n}X)^{\mathrm{t}C_p},$$

since limits of p-complete spectra are themselves p-complete; to see the equivalence observe that it trivially holds for homotopy fixed points instead of the Tate construction (by commuting limits) and also for homotopy orbits (since their formation preserves connectivity).

For the proof of the second part we need to recall the fibre sequences

$$\mathrm{D}_p(X) \longrightarrow \mathrm{F}_p(X) \longrightarrow \mathrm{T}_p(X)$$

and the stabilisation maps $G(X) \to \Omega G(\Sigma X)$ from the observation in Section 1. Applying the latter to all entries of the former, shifting one to right and using the exactness of T_p yields a grid of fibre sequences

Claim: Taking the limit along the vertical maps gives an equivalence

$$\partial \colon \mathrm{T}_p(X) \longrightarrow \lim \Omega^{n-1} \mathrm{D}_p(\Sigma^n X).$$

The $\mathbb{Z}/p$-homology of $\mathrm{D}_p X$ is well-known, so granting the claim we can use the Adams spectral sequence to extract information about $\mathrm{T}_p X$. As doing so, however, requires taking a limit, I will simplify matters by assuming X to be of finite type (in addition to bounded below). Nikolaus and Scholze show how to remove this restriction a posteriori, by a pro-argument. It seems plausible to me that the finiteness assumption we make can be circumvented in the argument below by incorporated their argument directly, but I have not pursued this.

For X of finite type, we can now consider the induced maps on Adams spectral sequences (the vertical arrows indicating convergence)

$$
\begin{array}{ccc}
\mathrm{Ext}(\mathbb{Z}/p, \mathrm{H}_*(X, \mathbb{Z}/p)) & \xrightarrow{(\partial_n \circ \Delta_p)_*} & \mathrm{Ext}(\mathbb{Z}/p, \mathrm{H}_*(\Omega^{n-1}\mathrm{D}_p(\Sigma^n X), \mathbb{Z}/p)) \\
\Big\Downarrow & & \Big\Downarrow \\
\pi_*(\widehat{X_p}) & \xrightarrow{(\partial_n \circ \Delta_p)_*} & \pi_*(\Omega^{n-1}\mathrm{D}_p(\Sigma^n X)\widehat{_p}),
\end{array}
$$

where the Ext-groups are formed for the coaction of the dual Steenrod algebra $(\mathcal{A}_p)_*$. Taking the limit over n on the right hand side produces a comparison map

$$
\begin{array}{ccc}
\mathrm{Ext}(\mathbb{Z}/p, \mathrm{H}_*(X, \mathbb{Z}/p)) & \xrightarrow{\lim (\partial_n \circ \Delta_p)_*} & \mathrm{Ext}(\mathbb{Z}/p, \lim \mathrm{H}_*(\Omega^{n-1}\mathrm{D}_p(\Sigma^n X), \mathbb{Z}/p)) \\
\Big\Downarrow & & \Big\Downarrow \\
\pi_*(\widehat{X_p}) & \xrightarrow{(\Delta_p)_*} & \pi_*(\mathrm{T}_p X),
\end{array}
$$

where we have used the finiteness assumption to a) commute the limit with forming Ext (the homology of $\Omega^{n-1}\mathrm{D}_p(\Sigma^n X)$ is again of finite type, see below) and b) with forming homotopy groups, and c) to ensure that the limit of the spectral sequences remains a spectral sequence with the same convergence properties (these claims are verified in great detail in [LNRo 12]). Now as mentioned the homology of extended powers is entirely computable, indeed it only depends on the homology of X: There are functors

$$\mathrm{DL_n} \colon (\mathcal{A}_p)_*\text{-CoMod} \longrightarrow (\mathcal{A}_p)_*\text{-CoMod}$$

together with natural isomorphisms

$$\iota \colon \mathrm{DL_n}\mathrm{H}_*(-; \mathbb{Z}/p) \Longrightarrow \mathrm{H}_*(D_n-; \mathbb{Z}/p).$$

There are also natural maps $\mathrm{DL}_n(-)[n] \Rightarrow \mathrm{DL}_n(-[n])$, which model the homotopical stabilisation maps and allow us to form

$$\mathrm{R}(M) = \lim \mathrm{DL}_p(M[n])[1-n],$$

the *Singer construction*, so that

$$\lim \mathrm{H}_*(\Omega^{n-1}\mathrm{D}_p(\Sigma^n X), \mathbb{Z}/p) = \mathrm{R}(\mathrm{H}_*(X; \mathbb{Z}/p)).$$

Finally, there is a natural map $\epsilon\colon \mathrm{id} \Rightarrow \mathrm{R}$ such that

$$
\begin{array}{ccc}
& & H_*(X;\mathbb{Z}/p) \\
& \swarrow^{\epsilon} & \downarrow \lim\ (\partial_n \circ \Delta_p)_* \\
R_+(H_*(X;\mathbb{Z}/p)) & \xrightarrow{\ \iota\ } & \lim \mathrm{H}_*(\Omega^{n-1}\mathrm{D}_p(\Sigma^n X), \mathbb{Z}/p))
\end{array}
$$

commutes. The proof now ends unceremoniously with:

Theorem 5 (Adams-Gunawardena-Miller, Lin). *For every bounded below $(\mathcal{A}_p)_*$-comodule M the map ϵ induces an isomorphism*

$$
\mathrm{Ext}(\mathbb{Z}/p, M) \longrightarrow \mathrm{Ext}(\mathbb{Z}/p, \mathrm{R}(M)).
$$

It remains to verify the claim, but for this we simply compute the homotopy fibre:

$$
\lim \Omega^n \mathrm{F}_p(\Sigma^n X) \simeq \lim \Omega^n\big((\Sigma^n X)^{\otimes p}\big)^{\mathrm{h}C_p}
$$

$$
\simeq \big(\lim \Omega^n (\Sigma^n X)^{\otimes p}\big)^{\mathrm{h}C_p}
$$

and here all transition maps (and thus their limit) vanish before taking homotopy fixed points: Rewrite the last term as

$$
\lim \Omega^n (\Sigma^n X)^{\otimes p} \simeq \lim \mathbb{S}^{p(n-1)} \otimes X^{\otimes p}
$$

to see that the transition maps are induced by the diagonal $\mathbb{S}^1 \to \mathbb{S}^p$, which is null. $\qquad\square$

3. Relation to Segal's conjecture

Finally, let me explain how part iii) of the theorem above is related to Segal's conjecture. Neither the methods of this paragraph, nor the conjecture itself, were used in the remainder of the Arbeitsgemeinschaft.

Consider for a group G the map

$$
b\colon \mathrm{R}(G, \mathbb{C}) \longrightarrow K^0(\mathrm{B}G),
$$

determined by assigning to a complex G-representation V the class of the vector bundle $\mathrm{E}G \times_G V \to \mathrm{B}G$; here R denotes representation rings and K^0 the zeroeth topological K-group of Atiyah and Hirzebruch. Through the natural identification

$$
K^0 \cong \pi_0 \mathrm{F}(-, \mathrm{KU})
$$

the group $K^0(X)$ carries a filtration by the images of the various $\pi_0 \mathrm{F}(X, \tau_{\geq n}\mathrm{KU})$ and letting $I \subseteq \mathrm{R}(G, \mathbb{C})$ denote the augmentation ideal Atiyah showed:

Theorem 6. *For G finite, $K^0(\mathrm{B}G)$ is complete with respect to the filtration above, the map b is continuous for the I-adic topology on $\mathrm{R}(G, \mathbb{C})$ and the induced map*

$$
\mathrm{R}(G, \mathbb{C})\widehat{_I} \longrightarrow K^0(\mathrm{B}G)
$$

is an isomorphism.

The theorem actually holds for G a compact Lie-group and in that form is known as the *Atiyah-Segal completion theorem*. The theorem of Barratt-Priddy-Quillen in the form of an identification

$$\mathbb{S} \simeq K(\text{Fin}),$$

of the (algebraic) K-theory of the category of finite sets, lead Segal to conjecture that a similar completion theorem might hold for stable cohomotopy upon replacing vector bundles by finite sets. More formally, he conjectured that the tautological map $s \colon A(G) \to K(\text{Fin})^0(\text{B}G)$ should, via the identification above, give rise to an isomorphism

$$A(G)^{\widehat{}}_I \longrightarrow \mathbb{S}^0(\text{B}G)$$

for every finite group G; here $A(G)$ is the Burnside ring of G. His conjecture was eventually proven in full by Carlsson via a reduction to the case of elementary p-groups, the simplest case of which is implied by part iii) of the theorem in Section 2 (which in the case at hand was proven by Lin for $p = 2$ and Gunawardena for p odd).

To see the relation between the conjecture for $G = C_p$ and the theorem, we need a bit of genuinely equivariant homotopy theory. Since C_p has only one proper subgroup, there is for every genuine C_p-spectrum the bicartesian isotropy separation square

$$
\begin{array}{ccc}
X^{\mathrm{g}C_p} & \longrightarrow & X^{\phi C_p} \\
\downarrow & & \downarrow \\
X^{\mathrm{h}C_p} & \longrightarrow & X^{\mathrm{t}C_p},
\end{array}
$$

where I have denoted the genuine and geometric fixed points by the superscripts $\mathrm{g}C_p$ and ϕC_p, respectively. Denoting the category of genuine C_p-spectra by $\mathrm{Sp}^{\mathrm{g}C_p}$ the composite functor

$$\mathrm{Sp}^{\mathrm{g}G} \longrightarrow \mathrm{Sp}^G \xrightarrow{\Omega^\infty} \mathrm{S}^G$$

canonically factors through the category $\mathrm{S}^{\mathrm{g}G}$ of genuine G-spaces. The arising functor $\Omega^\infty \colon \mathrm{Sp}^{\mathrm{g}G} \longrightarrow \mathrm{S}^{\mathrm{g}G}$ admits a left adjoint (named Σ^∞ of course) and to translate Segal's conjecture we shall need two facts about it:

i) It commutes with taking geometric fixed points: There is a canonical natural equivalence

$$(\Sigma^\infty -)^{\phi G} \simeq \Sigma^\infty(-^G).$$

ii) There is another canonical natural equivalence, tom Dieck's splitting,

$$(\Sigma^\infty -)^{\mathrm{c}G} \simeq \bigoplus_{H \leq G} \Sigma^\infty\big(\mathrm{E}W_G(H) \times_{W_G(H)} -^H\big),$$

where the wedge runs over conjugacy classes of subgroups and $W_G(H)$ denotes the Weyl group of H in G.

Now generally, the splitting implies that for the sphere spectrum with trivial G-action (which is the suspension spectrum of $* \in S^{gG}$) we have

$$\pi_0(\mathbb{S}^{gG}) \cong A(G).$$

Considering the isotropy separation square for the sphere with trivial C_p-action we obtain

$$
\begin{array}{ccc}
\mathbb{S} \oplus \Sigma^\infty \mathrm{B}C_p & \longrightarrow & \mathbb{S} \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \Delta_p} \\
\mathrm{F}(\mathrm{B}C_p, \mathbb{S}) & \longrightarrow & \mathrm{T}_p(\mathbb{S});
\end{array}
$$

and as indicated the left hand vertical map identifies with Segal's map (on π_0) and the right hand one with the Tate diagonal, whence the third part of our theorem implies that

$$\widehat{s_p} \colon \widehat{\mathbb{S}_p} \oplus (\Sigma^\infty \mathrm{B}C_p)\widehat{_p} \to \mathrm{F}(\mathrm{B}C_p, \mathbb{S})\widehat{_p}$$

is an equivalence. Now it is readily checked directly that s maps the summand in $\mathbb{S}^{cC_p}$ corresponding to the trivial subgroup by an equivalence to the summand $\mathrm{F}(*, \mathbb{S})$ of $\mathrm{F}(\mathrm{B}C_p, \mathbb{S})$ and the other summand of the target

$$\widetilde{\mathrm{F}}(\mathrm{B}C_p, \mathbb{S}) = \mathrm{fib}\big(\mathrm{F}(*, \mathbb{S}) \xrightarrow{r^*} \mathrm{F}(\mathrm{B}C_p, \mathbb{S})\big)$$

is p-complete: The reduced group cohomology of any finite p-group is p-torsion, thus by induction so is

$$\widetilde{\mathrm{F}}(\mathrm{B}C_p, \tau_{\leq n}\mathbb{S}))$$

and the statement follows by passing to the limit. From this discussion we obtain an equivalence

$$(\Sigma^\infty \mathrm{B}C_p)\widehat{_p} \to \widetilde{\mathrm{F}}(\mathrm{B}C_p, \mathbb{S})$$

and all that remains to verify is that the homotopy groups of the source can also be described by completion at the augmentation ideal of $A(C_p)$ instead of p. This is a general fact about Mackey functors (see [MaMC 82, Lemma 5]), but for π_0 can be checked by hand: We have

$$A(C_p) = \mathbb{Z}[T]/(T^2 - pT) \quad \text{and} \quad I = (T - p)$$

with T represented by any free and transitive C_p-set, and it is easily computed that $I^{m+1} = p^m I$, whence the I- and p-adic topologies agree on I.

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Topological Hochschild homology of $\mathbb{E}_\infty$-rings

Jonas McCandless

The main focus of this talk is to define topological Hochschild homology of $\mathbb{E}_\infty$-ring spectra and show that it admits the structure of a cyclotomic $\mathbb{E}_\infty$-ring spectrum. Let us first discuss Hochschild homology for a few moments. Let k be a commutative ring, and let R be an associative k-algebra. Let us for simplicity assume that R is flat over k. Define Hochschild homology $\mathrm{HH}(R/k)$ as the geometric realization of the following simplicial k-module

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} R \otimes_k R \otimes_k R \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} R \otimes_k R \substack{\longrightarrow\\\longrightarrow} R$$

where the two maps $R \otimes_k R \to R$ are given by $x \otimes y \mapsto xy$ and $x \otimes y \mapsto yx$ respectively, and the three maps $R \otimes_k R \otimes_k R \to R \otimes_k R$ are given by $x \otimes y \otimes z \mapsto xy \otimes z$, $x \otimes y \otimes z \mapsto x \otimes yz$, and $x \otimes y \otimes z \mapsto zx \otimes y$ respectively. There is a $\mathbb{Z}/(n+1)$-action on the term in simplicial degree n which essentially means that the geometric realization $\mathrm{HH}(R/k)$ aquires a $\mathbb{T}$-action. More precisely, the simplicial k-module above is the underlying simplicial k-module of a cyclic k-module in the sense of Connes. We can also think of $\mathrm{HH}(R/k)$ as an object of the derived category $D(k)$ of k-modules via the Dold-Kan correspondence. This works very well if R is a flat k-module, but if we want to compute something like $\mathrm{HH}(\mathbb{F}_p/\mathbb{Z})$, then we need to derive the tensor product. In fact, there is a more natural definition of $\mathrm{HH}(R/k)$. Recall that if R is any ring, then the derived ∞-category $\mathcal{D}(R)$ of R-modules inherits a symmetric monoidal structure from the usual symmetric monoidal structure on the category of chain complexes over R. Let us write $\otimes_R$ for this symmetric monoidal structure on $\mathcal{D}(R)$. We define Hochschild homology by

$$\mathrm{HH}(R/k) := R \otimes_{R \otimes_k R^{\mathrm{op}}} R$$

now as an object in the derived ∞-category of k-modules. Moreover, one shows as before that $\mathrm{HH}(R/k)$ naturally admits the structure of a $\mathbb{T}$-equivariant object of $\mathcal{D}(k)$. Let us now return to Hochschild homology of $\mathbb{F}_p$ as a $\mathbb{Z}$-algebra. We have the following lemma:

Lemma 1. *As a graded algebra the Hochschild homology of $\mathbb{F}_p$ over $\mathbb{Z}$ is*

$$\pi_* \mathrm{HH}(\mathbb{F}_p/\mathbb{Z}) = \Gamma_{\mathbb{F}_p}\{x\},$$

a divided polynomial algebra on a generator x in degree 2.

The seminal idea of Waldhausen was to replace the initial commutative ring $\mathbb{Z}$ with the sphere spectrum $\mathbb{S}$ and consequently work in the ∞-category Sp of spectra equipped with its smash product symmetric monoidal structure. This turns out to be a phenomenal idea and we are going to see some instances of this here. Later, we will see that topological Hochschild homology of the Eilenberg-MacLane spectrum $H\mathbb{F}_p$ is a polynomial ring $\mathbb{F}_p[x]$ which is due to Bökstedt. We are going to define topological Hochschild homology of $\mathbb{E}_\infty$-ring spectra and show that it admits the structure of a cyclotomic $\mathbb{E}_\infty$-ring spectrum. We will see that this crucially relies on the result that the Tate diagonal refines to a lax symmetric

monoidal transformation between lax symmetric monoidal functors as established in the previous talk. Let CAlg denote the ∞-category of $\mathbb{E}_\infty$-ring spectra and let $\mathcal{S}$ denote the ∞-category of spaces.

Proposition 2. *Let A be an $\mathbb{E}_\infty$-ring spectrum. The functor*

$$\mathrm{Map}_{\mathrm{CAlg}}(A, -)\colon \mathrm{CAlg} \to \mathcal{S}$$

corepresented by A admits a left adjoint.

Proof. This is an immediate consequence of the Adjoint Functor Theorem for presentable ∞-categories [2, Corollary 5.5.2.9]. The ∞-category $\mathcal{S}$ of spaces is presentable [2, Example 5.5.1.8], and the ∞-category CAlg of $\mathbb{E}_\infty$-rings is presentable [3, Corollary 3.2.3.5]. A functor corepresented by an object preserves small limits and is accesible [2, Proposition 5.5.2.7]. $\square$

We will now define topological Hochschild homology for $\mathbb{E}_\infty$-rings following [5, Chapter IV]. The more general definition of topological Hochschild homology for $\mathbb{E}_1$-rings will be given in a later lecture. It is a result due to McClure, Schwänzl, and Vogt [4] that our definition is equivalent to that one. See also [5, Proposition IV.2.2] for a proof. Let $A^{\otimes(-)}\colon \mathcal{S} \to \mathrm{CAlg}$ denote a left adjoint of $\mathrm{Map}_{\mathrm{CAlg}}(A, -)\colon \mathrm{CAlg} \to \mathcal{S}$.

Definition 3. Let A be an $\mathbb{E}_\infty$-ring spectrum. The topological Hochschild homology $\mathrm{THH}(A)$ of A is defined by $\mathrm{THH}(A) := A^{\otimes S^1}$.

We obtain a functor $\mathrm{THH}\colon \mathrm{CAlg} \to \mathrm{CAlg}$ determined by the construction $A \mapsto A^{\otimes S^1}$. If A is an $\mathbb{E}_\infty$-ring, then we have the functor $A^{\otimes(-)}\colon \mathcal{S} \to \mathrm{CAlg}$ and the map $* \to S^1$ of spaces induces a map $i\colon A \to \mathrm{THH}(A)$ of $\mathbb{E}_\infty$-rings. Another observation is that $\mathrm{THH}(A)$ admits the structure of a $\mathbb{T}$-equivariant object in the ∞-category of $\mathbb{E}_\infty$-rings. To see this we note that $\mathbb{T}$ acts continuously on S^1 by left multiplication so we obtain a map

$$S^1 \to \mathrm{Map}_{\mathcal{S}}(S^1, S^1) \to \mathrm{Map}_{\mathrm{CAlg}}(\mathrm{THH}(A), \mathrm{THH}(A))$$

of spaces by functoriality. Consequently, the functor $\mathrm{THH}\colon \mathrm{CAlg} \to \mathrm{CAlg}$ refines to a functor $\mathrm{THH}\colon \mathrm{CAlg} \to \mathrm{CAlg}^{B\mathbb{T}}$. To summarize, if A is an $\mathbb{E}_\infty$-ring, then $\mathrm{THH}(A)$ is a $\mathbb{T}$-equivariant $\mathbb{E}_\infty$-ring spectrum equipped with a non-equivariant map $i\colon A \to \mathrm{THH}(A)$ of $\mathbb{E}_\infty$-rings. Moreover, topological Hochschild homology of an $\mathbb{E}_\infty$-ring spectrum is initial with these properties. We summarize the discussion above as follows.

Proposition 4 (McClure-Schwänzl-Vogt [4])**.** *Let A be an $\mathbb{E}_\infty$-ring. If B is an $\mathbb{T}$-equivariant $\mathbb{E}_\infty$-ring equipped with a map $f\colon A \to B$ of $\mathbb{E}_\infty$-rings, then there exists a unique $\mathbb{T}$-equivariant map $\bar{f}\colon \mathrm{THH}(A) \to B$ of $\mathbb{E}_\infty$-ring spectra such that the following diagram*

$$
\begin{array}{ccc}
A & \xrightarrow{\;\;i\;\;} & \mathrm{THH}(A) \\
 & \searrow{\scriptstyle f} & \downarrow{\scriptstyle \bar{f}} \\
 & & B
\end{array}
$$

of $\mathbb{E}_\infty$*-rings commutes.*

Remark 5. Let us be very careful and write out the final statement of Proposition 4. Let $U\colon \mathrm{CAlg}^{B\mathbb{T}} \to \mathrm{CAlg}$ denote the forgetful functor induced by the inclusion $* \to B\mathbb{T}$. Given $f\colon A \to B$ as above. The existence of a unique $\mathbb{T}$-equivariant map $\bar{f}\colon \mathrm{THH}(A) \to B$ of $\mathbb{E}_\infty$-rings such that the diagram above commutes in CAlg means that there exists an edge $\bar{f}\colon \mathrm{THH}(A) \to B$ of $\mathrm{CAlg}^{B\mathbb{T}}$ which is unique up to contractible space of choice, and that there exists a 2-simplex in CAlg with edges i, $U(\bar{f})$, and f which we picture as follows

$$
\begin{array}{ccc}
 & U(\mathrm{THH}(A)) & \\
{}^{i}\nearrow & & \searrow^{U(\bar{f})} \\
A \xrightarrow{\quad f \quad} & & U(B).
\end{array}
$$

Next, we endow $\mathrm{THH}(A)$ with a cyclotomic structure following [5, Chapter IV]. Concretely, this means that we have to specify a $\mathbb{T}$-equivariant map

$$\varphi_p\colon \mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$$

of $\mathbb{E}_\infty$-rings for every prime number p, where $\mathrm{THH}(A)^{tC_p}$ is equipped with the residual $\mathbb{T}/C_p \simeq \mathbb{T}$-action. We need to recall some of material from the previous talks. Fix a prime number p. There is a functor $T_p\colon \mathrm{Sp} \to \mathrm{Sp}$ determined by the construction

$$X \mapsto (X \otimes \cdots \otimes X)^{tC_p}$$

where $X \otimes \cdots \otimes X$ denotes the p-fold tensor product equipped with the C_p-action given by cyclic permutation of the factors. In the previous talk we showed that T_p is an exact functor [5, Proposition III.1.1]. With this in mind let us briefly recall how the Tate diagonal was defined. Composition with the functor $\Omega^\infty\colon \mathrm{Sp} \to \mathcal{S}$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{Ex}}(\mathrm{Sp}, \mathrm{Sp}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{Lex}}(\mathrm{Sp}, \mathcal{S})$$

of ∞-categories. We also saw that evaluation on the sphere induces an equivalence

$$\mathrm{Map}_{\mathrm{Fun}^{\mathrm{Ex}}(\mathrm{Sp},\mathrm{Sp})}(\mathrm{id}_{\mathrm{Sp}}, T_p) \simeq \mathrm{Map}_{\mathrm{Sp}}(\mathbb{S}, T_p\mathbb{S}) \simeq \mathrm{Map}_{\mathrm{Sp}}(\mathbb{S}, \mathbb{S}^{tC_p})$$

of spaces. The Tate diagonal is a natural transformation $\Delta_p\colon \mathrm{id}_{\mathrm{Sp}} \to T_p$ which corresponds to the composite $\mathbb{S} \to \mathbb{S}^{hC_p} \to \mathbb{S}^{tC_p}$ under the equivalence above. For a more detailed discussion we refer the reader to [5, Section III.1 and III.3]. Recall that the functor $(-)^{tC_p}\colon \mathrm{Sp}^{BC_p} \to \mathrm{Sp}$ admits an essentially unique lax symmetric monoidal structure [5, Theorem I.3.1]. This means that the functor $T_p\colon \mathrm{Sp} \to \mathrm{Sp}$ inherits a lax symmetric monoidal structure. We proved the following result in a previous talk:

Proposition 6 ([5, Proposition III.3.1])**.** *There exists an essentially unique lax symmetric monoidal transformation*

$$\Delta_p\colon \mathrm{id}_{\mathrm{Sp}} \to T_p$$

such that the underlying natural transformation of functors is given by the Tate diagonal.

If A is an $\mathbb{E}_\infty$-ring equipped with a C_p-action, then A^{tC_p} admits the structure of an $\mathbb{E}_\infty$-ring since $(-)^{tC_p}$ admits a lax symmetric monoidal structure. It follows from Proposition 6 that the Tate diagonal

$$\Delta_p \colon A \to (A \otimes \cdots \otimes A)^{tC_p}$$

refines to a map of $\mathbb{E}_\infty$-rings. This will be crucial below.

We are now ready to endow topological Hochschild homology of an $\mathbb{E}_\infty$-ring with a cyclotomic structure. The p-fold tensor product $A \otimes \cdots \otimes A$ equipped with the C_p-action given by cyclic permutation of the factors is an induced C_p-object in the ∞-category of $\mathbb{E}_\infty$-rings with an action of C_p. More precisely, the forgetful functor $\mathrm{CAlg}^{BC_p} \to \mathrm{CAlg}$ admits a left adjoint given by the construction $A \mapsto A \otimes \cdots \otimes A$. By adjunction, there exists a C_p-equivariant map

$$\psi \colon A \otimes \cdots \otimes A \to \mathrm{THH}(A)$$

of $\mathbb{E}_\infty$-rings induced by the map $i \colon A \to \mathrm{THH}(A)$. Consequently, we obtain a map

$$A \xrightarrow{\Delta_p} (A \otimes \cdots \otimes A)^{tC_p} \xrightarrow{\psi^{tC_p}} \mathrm{THH}(A)^{tC_p}$$

of $\mathbb{E}_\infty$-rings. By the universal property of THH (Proposition 4), there exists a unique $\mathbb{T}$-equivariant map

$$\varphi_p \colon \mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$$

of $\mathbb{E}_\infty$-rings where the target is equipped with the residual $\mathbb{T}/C_p \simeq \mathbb{T}$-action such that the following diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\quad i \quad} & \mathrm{THH}(A) \\
\downarrow{\scriptstyle \Delta_p} & & \downarrow{\scriptstyle \varphi_p} \\
(A \otimes \cdots \otimes A)^{tC_p} & \xrightarrow{\psi^{tC_p}} & \mathrm{THH}(A)^{tC_p}
\end{array}
$$

of $\mathbb{E}_\infty$-rings commutes. The $\mathbb{T}$-equivariant map $\varphi_p \colon \mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$ of $\mathbb{E}_\infty$-rings is called the Frobenius map on $\mathrm{THH}(A)$. We have a Frobenius map $\varphi_p \colon \mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$ for every prime p and these endow the $\mathbb{T}$-equivariant $\mathbb{E}_\infty$-ring $\mathrm{THH}(A)$ with a cyclotomic structure.

Remark 7. The Frobenius maps $\varphi_p \colon \mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$ are a special feature of the topological theory. This is because the Tate diagonal is a special feature of spectra. More precisely, let $\mathcal{C}$ be a stable ∞-category which admits small limits and colimits and let G be a finite group. In this case we can define the Tate construction $(-)^{tG} \colon \mathcal{C}^{BG} \to \mathcal{C}$. In particular, we can define the Tate construction on the derived ∞-category $\mathcal{D}(\mathbb{Z})$ of $\mathbb{Z}$-modules. Recall that the derived ∞-category $\mathcal{D}(\mathbb{Z})$ admits a symmetric monoidal structure and we will denote the tensor product by $\otimes_\mathbb{Z}$. Let C be an object of $\mathcal{D}(\mathbb{Z})$. Just as above the construction $C \mapsto (C \otimes_\mathbb{Z} \cdots \otimes_\mathbb{Z} C)^{tC_p}$ which sends C to the Tate construction of C_p on the p-fold tensor product $C \otimes_\mathbb{Z} \cdots \otimes_\mathbb{Z} C$

equipped with the C_p action given by cyclic permutation of the factors refines to an endofunctor $T_p^{\mathbb{Z}} \colon \mathcal{D}(\mathbb{Z}) \to \mathcal{D}(\mathbb{Z})$ on the derived ∞-category of $\mathbb{Z}$-modules. However, the definition of the Tate diagonal relies on the universal property of the stable ∞-category of spectra. Nikolaus and Scholze prove the following theorem:

Theorem 8 ([5, Theorem III.1.10]). *Every natural transformation* $\mathrm{id}_{\mathcal{D}(\mathbb{Z})} \to T_p^{\mathbb{Z}}$ *of endofunctors on the derived ∞-category $\mathcal{D}(\mathbb{Z})$ of $\mathbb{Z}$-modules induces the zero map in homology. In particular, there is no lax symmetric monoidal transformation* $\mathrm{id}_{\mathcal{D}(\mathbb{Z})} \to T_p^{\mathbb{Z}}$.

The construction that we have just given of the Frobenius map on THH of an $\mathbb{E}_\infty$-ring relied crucially on the Tate diagonal admitting a lax symmetric monoidal structure. More precisely, if we consider the commutative diagram above defining the Frobenius maps $\varphi_p \colon \mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$, then the horizontal maps can be represented by maps of chain complexes. However, the vertical maps cannot be represented by maps of chain complexes by the theorem above.

The Frobenius maps $\varphi_p \colon \mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$ present on THH of an $\mathbb{E}_\infty$-ring are related to a refinement of the ordinary Frobenius homomorphism of commutative rings to a Frobenius-type map of $\mathbb{E}_\infty$-ring spectra called the Tate-valued Frobenius. Before we recall the definition of the Tate-valued Frobenius we make some preliminary remarks. If R is an $\mathbb{E}_\infty$-ring equipped with a C_p-action, then we have a C_p-equivariant multiplication map $m \colon R \otimes \cdots \otimes R \to R$ which is determined by $m \circ \eta \simeq \mathrm{id}_R$, where $\eta \colon R \to R \otimes \cdots \otimes R$ is the unit of the adjunction which exhibits the p-fold tensor product $R \otimes \cdots \otimes R$ as the induced C_p-object of CAlg^{BC_p}.

Definition 9 ([5, Definition IV.1.1]). Let p be a prime number. Let R be an $\mathbb{E}_\infty$-ring equipped with the trivial C_p-action. The Tate-valued Frobenius $\varphi_R \colon R \to R^{tC_p}$ of R is given by the composite

$$R \xrightarrow{\Delta_p} (R \otimes \cdots \otimes R)^{tC_p} \xrightarrow{m^{tC_p}} R^{tC_p}$$

map of $\mathbb{E}_\infty$-rings.

The $\varphi_R \colon R \to R^{tC_p}$ deserves to be called the Tate-valued Frobenius as the following example shows.

Example 10. *Let A be an ordinary commutative ring and let HA denote the Eilenberg-MacLane spectrum of A. The Eilenberg-MacLane spectrum HA admits the structure of an $\mathbb{E}_\infty$-ring. The Tate-valued Frobenius $\varphi_{HA} \colon HA \to HA^{tC_p}$ recovers the usual Frobenius homomorphism of A on π_0. We have that $\pi_0 HA \simeq A$ and $\pi_0 HA^{tC_p} \simeq \widehat{\mathrm{H}}^0(A, C_p) \simeq A/p$ and $\pi_0 \varphi_{HA} \colon A \to A/p$ is given by $a \mapsto a^p$.*

We now examine how the Frobenius map $\varphi_p \colon \mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$ on $\mathrm{THH}(A)$ is related to the Tate-valued Frobenius of A. By the universal property of $\mathrm{THH}(A)$, there exists a unique $\mathbb{T}$-equivariant map $\pi \colon \mathrm{THH}(A) \to A$ such

that the following diagram

$$A \xrightarrow{\ i\ } \mathrm{THH}(A)$$
$$\mathrm{id}_A \searrow \quad \downarrow \pi$$
$$A$$

of $\mathbb{E}_\infty$-rings commutes. We have the following:

Corollary 11 ([5, Corollary IV.2.4])**.** *Let A be an $\mathbb{E}_\infty$-ring. The composite map*

$$A \xrightarrow{i} \mathrm{THH}(A) \xrightarrow{\varphi_p} \mathrm{THH}(A)^{tC_p} \xrightarrow{\pi^{tC_p}} A^{tC_p}$$

of $\mathbb{E}_\infty$-rings is canonically equivalent to the Tate-valued Frobenius.

Proof. We have a commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\quad\quad i \quad\quad} & \mathrm{THH}(A) \\
\downarrow{\scriptstyle \Delta_p} & & \downarrow{\scriptstyle \varphi_p} \\
(A \otimes \cdots \otimes A)^{tC_p} & \xrightarrow{\psi^{tC_p}} \mathrm{THH}(A)^{tC_p} & \xrightarrow{\pi^{tC_p}} A^{tC_p}
\end{array}
$$

Consequently, it is enough to show that the composite map

$$A \otimes \cdots \otimes A \xrightarrow{\ \psi\ } \mathrm{THH}(A) \xrightarrow{\ \pi\ } A$$

is a C_p-equivariant map of $\mathbb{E}_\infty$-rings which is equivalent to the C_p-equivariant multiplication map $m\colon A \otimes \cdots \otimes A \to A$ of A, where A is equipped with the trivial C_p-action. Since $A \otimes \cdots \otimes A$ is an induced C_p-object of CAlg^{BC_p} it suffices to show that $\pi \circ i$ is equivalent to the identity id_A which is true by construction of π. This ends the proof. $\square$

I will end this talk with discussing the Tate-valued Frobenius on the 2-periodic complex K-theory spectrum KU. Recall that KU admits the structure of an $\mathbb{E}_\infty$-ring. We need to discuss some further properties of the Tate-valued Frobenius.

Lemma 12 ([5, Lemma IV.1.3])**.** *There is a unique lax symmetric monoidal factorization*

$$
\begin{array}{ccc}
 & & ((X \otimes \cdots \otimes X)^{tC_p})^{hW} \\
 & \nearrow & \downarrow \\
X & \xrightarrow[\Delta_p]{} & (X \otimes \cdots \otimes X)^{tC_p}
\end{array}
$$

of the Tate diagonal, where W is the Weyl group of C_p in Σ_p which is cyclic of order $p-1$. The action of W on $(X^{\otimes p})^{tC_p}$ is the residual action of the Σ_p-action on $X^{\otimes p}$.

Proof. First note that the functor $T_p^{hW}\colon \mathrm{Sp} \to \mathrm{Sp}$ given by

$$X \mapsto ((X \otimes \cdots \otimes X)^{tC_p})^{hW}$$

is exact since both T_p and $(-)^{hW}$ are exact. It also admits a lax symmetric monoidal structure since it is a composite of two functors which admit lax symmetric monoidal structures. The identity functor $\mathrm{id}_{\mathrm{Sp}}$ is initial among exact lax symmetric monoidal endofunctors on Sp by a result of Nikolaus ([6], Corollary 6.9). This shows the existence of the map $X \to ((X^{\otimes p})^{tC_p})^{hW}$. Commutativity of the diagram follows from the result that there is a unique, up to contractible choice, lax symmetric structure on the Tate diagonal. $\qquad\square$

An immediate consequence of this lemma is the following refinement of the Tate-valued Frobenius.

Corollary 13 ([5, Corollary IV.1.4]). *Let R be an $\mathbb{E}_\infty$-ring. The Tate-valued Frobenius is equivalent to the composite map*

$$R \to (R^{tC_p})^{hW} \to R^{tC_p}$$

of $\mathbb{E}_\infty$-rings.

We will also refer to the map $R \to (R^{tC_p})^{hW}$ as the Tate-valued Frobenius map. Why is this description of the Tate-valued Frobenius useful? One reason is that the homotopy fixed points become completely algebraic, that is

$$\pi_*(R^{tC_p})^{hW} \simeq (\pi_* R^{tC_p})^W.$$

To see this we look at the homotopy fixed point spectral sequence

$$E_{i,j}^2 = H^{-i}(W, \pi_j R^{tC_p}) \Rightarrow \pi_{i+j}(R^{tC_p})^{hW}.$$

Since $|\mathbb{F}_p^\times| = p - 1$ is invertible in $\pi_* R^{tC_p}$ we conclude that the E^2-page is concentrated on the line $i = 0$.

Let us now contemplate the Tate-valued Frobenius on the complex K-theory spectrum KU. Recall that $\pi_* \mathrm{KU} = \mathbb{Z}[\beta^\pm]$, where $|\beta| = 2$. We can compute the homotopy groups of the Tate construction of KU by a result due to Greenlees and May [1]. In this case we find that

$$\pi_* \mathrm{KU}^{tC_p} = \pi_* \mathrm{KU}((t))/((t+1)^p - 1) = \pi_* \mathrm{KU} \otimes_{\mathbb{Z}} \mathbb{Q}_p(\mu_p),$$

Recall that $\mathrm{Gal}(\mathbb{Q}_p(\mu_p)/\mathbb{Q}_p) \simeq W$ and the action of W on $\pi_* \mathrm{KU}^{tC_p}$ is precisely the Galois action. We conclude that

$$\pi_*(\mathrm{KU}^{tC_p})^{hW} \simeq (\pi_* \mathrm{KU}^{tC_p})^W \simeq \pi_* \mathrm{KU} \otimes_{\mathbb{Z}} \mathbb{Q}_p(\mu_p)^W \simeq \pi_* \mathrm{KU} \otimes_{\mathbb{Z}} \mathbb{Q}_p.$$

Very similarly one can show that if X is a compact object in the ∞-category of spaces, then

$$\pi_*((\mathrm{KU}^X)^{tC_p})^{hW} \simeq \pi_* \mathrm{KU}^X \otimes_{\mathbb{Z}} \mathbb{Q}_p$$

where KU^X denotes the spectrum of maps from $\Sigma_+^\infty X \to \mathrm{KU}$. We will end this talk by stating the following proposition which describes the Tate-valued Frobenius on KU.

Proposition 14 ([5, Proposition IV.1.12])**.** *Let X be a compact object in the ∞-category $\mathcal{S}$ of spaces. The Tate-valued Frobenius on KU^X is on π_0 given by*

$$\mathrm{KU}^0(X) \to \mathrm{KU}^0(X) \otimes_{\mathbb{Z}} \mathbb{Q}_p$$
$$V \mapsto \psi^p(V)$$

where $\psi^p \colon \mathrm{KU}^0(X) \to \mathrm{KU}^0(X)$ denotes the pth Adams operation, and $\psi^p(V)$ is considered as an element of $\mathrm{KU}^0(X) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ under the canonical inclusion $\mathrm{KU}^0(X) \to \mathrm{KU}^0(X) \otimes_{\mathbb{Z}} \mathbb{Q}_p$.

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Dyer-Lashof operations

Guozhen Wang

The Dyer-Lashof operations are tools to understand the homology of E_∞-algebras. In this talk, we will first give an account of the dual Steenrod algebra which is the homology ring of the Eilenberg-MacLane spectrum. Then we will give a general construction of the power operations and specialize to the case of ordinary homology which gives rise to the Dyer-Lashof operations. We will also give the basic properties of these operations. Finally we will give Steinberg's computations of the Dyer-Lashof operations on the dual Steenrod algebra, which will be used to determine the structure of the topological Hochschild homology of $\mathbb{F}_p$.

1. The dual Steenrod algebra

Fix a prime p. Let $H\mathbb{F}_p$ be the Eilenberg-MacLane spectrum with coefficients in $\mathbb{F}_p$. Then it represents mod p cohomology: for any spectrum X, we have $H^*(X; \mathbb{F}_p) = [\Sigma^* X, H\mathbb{F}_p]$. We will abbreviate $H = H\mathbb{F}_p$ and $H^*(X) = H^*(X; \mathbb{F}_p)$.

By the Yoneda lemma, the set of natural transformations between $H^*(-)$ and itself is $H^*H = [\Sigma^* H, H]$. This forms an algebra under composition, which is called the Steenrod algebra.

Theorem 1. 　　(1) For $p = 2$, the Steenrod algebra H^*H is generated by the Steenrod squares Sq^i, $i = 1, 2, \ldots$, with $|Sq^i| = i$, under the Adem relations:

$$Sq^i Sq^j = \sum_{k=0}^{\lfloor i/2 \rfloor} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k$$

for $i < 2j$.

(2) *For odd prime p, the Steenrod algebra H^*H is generated by the Steenrod powers P^i, $i = 1, 2, \ldots$ and the Bockstein β, with $|P^i| = 2i(p-1)$ and $|\beta| = 1$, under the Adem relations:*

$$P^a P^b = \sum_i (-1)^{a+i} \binom{(p-1)(b-i)-1}{a-pi} P^{a+b-i} P^i$$

for $a < pb$ and

$$P^a \beta P^b = \sum_i (-1)^{a+i} \binom{(p-1)(b-i)}{a-pi} \beta P^{a+b-i} P^i$$
$$+ \sum_i (-1)^{a+i+1} \binom{(p-1)(b-i)-1}{a-pi-1} P^{a+b-i} \beta P^i$$

for $a \leq pb$.

We can also dualize to study operations on the homology. Now suppose X be a spectrum with finitely many cells under any dimension. Then we have $H_*(X)$ is canonically identified with the dual graded $\mathbb{F}_p$-vector space of $H^*(X)$.

For any operation $P \in H^i H$, its transpose acts on the homology of X:

$$P_* : H_*(X) \to H_{*-i}(X)$$

So we get a right action of the Steenrod algebra on the homology :

$$H_*(X) \otimes H^* H \to H_*(X)$$

We can further dualize and get a co-action of the dual of the Steenrod algebra (i.e. the homology H_*H of the Eilenberg-MacLane spectrum):

$$H_*(X) \to H_*H \otimes H_*(X)$$

To further understand the co-action formulations of homology operations, we first study the properties of the dual Steenrod algebra. Recall that H is an E_∞-ring spectrum with structure map

$$m : H \otimes H \to H$$

and

$$u : \mathbb{S} \to H$$

Here we use $\otimes$ to denote smash product over the sphere spectrum $\mathbb{S}$.

In particular, since $H \otimes H$ is a ring spectrum, $H_*H = \pi_*(H \otimes H)$ is a ring. It has the following operations:

(1) left unit map $H_* \to H_*H$ induced by

$$H \xrightarrow{\text{id} \otimes u} H \otimes H$$

and right unit map $H_* \to H_*H$ induced by

$$H \xrightarrow{u \otimes \text{id}} H \otimes H$$

They agree because H_* is the base field.

(2) co-multiplication $H_*H \to H_*H \otimes H_*H$ induced by

$$H \otimes H \xrightarrow{\text{id} \otimes u \otimes \text{id}} H \otimes H \otimes H$$

where $\pi_*(H \otimes H \otimes H)$ is identified with $H_*H \otimes H_*H$ via the isomorphism

$$H_*H \otimes H_*H \to \pi_*(H \otimes H \otimes H \otimes H) \xrightarrow{\text{id} \otimes m \otimes \text{id}} \pi_*(H \otimes H \otimes H)$$

(3) co-unit $H_*H \to H$ induced by the multiplication map m.

(4) conjugation (antipode) $\chi : H_*H \to H_*H$ induced by the twist map which exchanges the two factors:

$$H \otimes H \to H \otimes H$$

So H_*H is a graded commutative ring with a co-multiplication. One can check that these two are compatible and we get a Hopf algebra, i.e. $\text{Spec}(H_*H)$ becomes an affine (super) algebraic monoid. Moreover, the conjugation plays the role of an inverse, so it is actually an affine (super) algebraic group. For $p = 2$, the algebraic group represented by H_*H is the group of automorphisms of the additive formal group.

Remark 2. It turns out that the left unit and the right unit maps are equal for the Eilenberg-MacLane spectrum.

For a generalized homology theory, these two maps are different in general, and we get a Hopf algebroid instead of Hopf algebra.

The structure for the dual Steenrod algebra is studied by Milnor [3]:

Theorem 3. *The Hopf algebra H_*H is as follows:*

(1) For the prime 2,

$$H_*H = \mathbb{F}_p[\xi_1, \xi_2, \dots]$$

$$\psi(\xi_i) = \sum \xi_{i-k}^{2^k} \otimes \xi_k$$

(2) For p odd prime,

$$H_*H = \mathbb{F}_p[\xi_1, \xi_2, \dots] \otimes E[\tau_0, \tau_1, \dots]$$

$$\psi(\xi_i) = \sum \xi_{i-k}^{p^k} \otimes \xi_k$$

$$\psi(\tau_i) = \tau_i \otimes 1 + \sum \xi_{i-k}^{p^k} \otimes \tau_k$$

Remark 4. At the prime 2, the definition of the generators of H_*H is as follows:

Let $x : \Sigma^{\infty-1}\mathbb{R}P^\infty \to H$ be the non-trivial map. Let $e_i \in H_i(\mathbb{R}P^\infty)$ be the generators of homology. Then ξ_i is defined to be $x_*(e_{2^i})$.

One can check using the Steenrod actions on the cohomology of real projective space, that Sq^i is dual to ξ_1^i.

2. The Dyer-Lashof operations

First we define the notion of power operations in a general setting.

Let A and E be E_∞-ring spectra. We will define power operations on the A-homology of E.

Recall that A being E_∞ implies, among other things, that the multiplication map descends to the homotopy orbit:

$$\mu_n : D_n(A) = (A^{\otimes n})_{h\Sigma_n} \to A$$

and similarly for E.

Moreover, the spectrum $A \otimes E$ is also an E_∞-ring spectrum.

Let $x : S^m \to A \otimes E$ be an element of $A_m E$. Here $S^m = \Sigma^m \mathbb{S}$. We can construct the following map:

$$\tilde{x} : A \otimes D_n S^m \xrightarrow{\mathrm{id} \otimes D_n x} A \otimes D_n(A \otimes E) \xrightarrow{\mathrm{id} \otimes \mu_n} A \otimes A \otimes E \xrightarrow{m \otimes \mathrm{id}} A \otimes E$$

For any $e \in A_k(D_n S^m)$, we can define the power operation associated to e

$$\Theta^e : A_m E \to A_k E$$

by the formula:

$$\Theta^e(x) = \tilde{x}_*(e)$$

Now we specialize to the case when $A = H$ is the Eilenberg-MacLane spectrum.

First we consider the 2-primary case. By the homotopy orbit spectral sequence, $H_k(D_2 S^m)$ is 1-dimensional for $k \geq 2m$. (In fact, $D_2 S^m$ is homotopy equivalent to $\Sigma^m \mathbb{R}P_m^\infty$.) Let

$$e_r \in H_{r+2m}(D_2 S^m)$$

be the generator in degree $r + 2m$.

Definition 5. Define the Dyer-Lashof operations Q^r, $r \in \mathbb{Z}$, to be

$$Q^r(x) = \Theta^{e_{r-m}} x$$

for x in degree m.

From the definition, we have:

Theorem 6. *Let x be a homology class in degree m,*
   (1) $Q^r(x) = 0$ for $r < m$.
   (2) $Q^m(x) = x^2$.

As in the case of Steenrod operations, we also have the Cartan formula and the Adem relations:

Theorem 7. (*Cartan formula*)

$$Q^r(xy) = \sum_{i+j=r} (Q^i x)(Q^j y)$$

Theorem 8. (*Adem relations*) *If $r > 2s$ then*

$$Q^r Q^s = \sum_i \binom{i-s-1}{2i-r} Q^{r+s-i} Q^i$$

In addition to these, we also have the commutation relations between Steenrod operations and the Dyer-Lashof operations:

Theorem 9. (*Nishida relations*)

$$Sq_*^r Q^s = \sum_i \binom{2^n + s - r}{r - 2i} Q^{s-r+i} Sq_*^i$$

for any n large enough that the binomial coefficient is meaningful.

Remark 10. The usual definition of the Steenrod squares amounts to be the negative Dyer-Lashof operations on the Spanier-Whitehead dual of a suspension spectrum. One can check that in this case the Adems relations for the Dyer-Lashof operations becomes the Adem relations for the Steenrod squares.

For an odd prime p, we can define the Dyer-Lashof operations Q^i's similarly. Together with the Bockstein β, they generate the Dyer-Lashof algebra under the Adem relations.

Theorem 11. (*Adem relations*) *If $r > ps$,*

$$Q^r Q^s = \sum_i (-1)^{r+i} \binom{(p-1)(i-s) - 1}{pi - r} Q^{r+s-i} Q^i$$

If $r \geq ps$,

$$Q^r \beta Q^s = \sum_i (-1)^{r+i} \binom{(p-1)(i-s)}{pi - r} \beta Q^{r+s-i} Q^i$$
$$+ \sum_i (-1)^{r+i} \binom{(p-1)(i-s) - 1}{pi - r - 1} Q^{r+s-i} \beta Q^i$$

The odd primary Nishida relations is as follows:

Theorem 12. (*Nishida relations*)

$$P_*^r Q^s = \sum_i (-1)^{r+i} \binom{p^n + (p-1)(s-r)}{r - pi} Q^{s-r+i} P_*^i$$
$$P_*^r \beta Q^s = \sum_i \binom{p^n + (p-1)(s-r) - 1}{r - pi} \beta Q^{s-r+i} P_*^i$$
$$- \sum_i \binom{p^n + (p-1)(s-r) - 1}{r - pi - 1} Q^{s-r+i} P_*^i \beta$$

for any n large enough that the binomial coefficient is meaningful.

For the proof of these formulas, see [2] and [4]

3. Dyer-Lashof operations on the dual Steenrod algebra

We will give the formulas for the action of the Dyer-Lashof operations on the dual Steenrod algeba in this section.

First we consider the case for $p = 2$. To state the formula, let

$$\xi = t + \xi_1 t^2 + \xi_2 t^4 + \cdots + \xi_k t^{2^k} + \cdots$$

where t is an indeterminant. Set $\zeta_i = \chi(\xi_i)$ to be the conjugate of ξ_i.

Theorem 13. (*Steinberg*)
  (1) $t^{-1} + \xi_1 + \sum_{s>0} t^s Q^s \xi_1 = \xi^{-1}$.
  (2) $Q^{2^i-2} \xi_1 = \zeta_i$.
  (3) For $s > 0$, we have

$$Q^s \zeta_i = \begin{cases} Q^{s+2^i-2} \xi_1, & if\, s \equiv 0 \;or\; -1 \mod 2^i \\ 0, & otherwise \end{cases}$$

  (4) $Q^{2^i} \zeta_i = \zeta_{i+1}$.

Note that (2) and (4) are special cases of (1) and (3). Only (4) will be used in Bökstedt's calculations of $\mathrm{THH}(\mathbb{F}_p)$.

We will give the proof of (4) using the Nishida relations. The proof of the other formulas are similar, see [4].

To use the Nishida relations, first we note the following lemma, which follows from the fact that the Steenrod algebra is generated by the Sq^{2^i}'s:

Lemma 14. *The following are equivalent for $x, y \in H_* H$ with $|x| = |y| > 0$:*
  (1) $x = y$.
  (2) For any $P \in H^* H$, $< P, x >=< P, y >$.
  (3) For any admissible $I = (i_1, i_2, \ldots)$ with $|I| = |x|$, $Sq_*^I x = Sq_*^I y$. (Recall that such a sequence is admissible if $i_k \geq 2i_{k+1}$ for all k, and Sq^I's with I admissible forms a basis for the Steenrod algebra.)
  (4) For any $P \in H^* H$, $P_* x = P_* y$.
  (5) For any $k \geq 0$, $Sq_*^{2^k} x = Sq_*^{2^k} y$.

We will check the last condition for the two sides in (4):
By the Nishida relation, we have

$$Sq_*^1 Q^{2^i} \zeta_i = Q^{2^i-1} \zeta_i = \zeta_i^2$$

and for $k > 0$,

$$Sq_*^{2^k} Q^{2^i} \zeta_i = \binom{2^k + 2^i}{2^i} Q^{2^i - 2^k} \zeta_i = 0$$

On the other hand, by the co-action formula in $H_* H$ and the fact Sq^i is dual to ξ_1^i,

$$Sq_*^{2^k} \zeta_{i+1} = \begin{cases} \zeta_i^2, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

So we conclude:

$$Q^{2^i}\zeta_i = \zeta_{i+1}$$

Finally we state the formulas for the action of the Dyer-Lashof operations on the dual Steenrod algebra at an odd prime p. Let

$$\xi = t^2 + \xi_1 t^{2p} + \xi_2 t^{2p^2} + \cdots + \xi_k t^{2p^k} + \cdots$$

and

$$\tau = t + \tau_0 t^2 + \tau_1 t^{2p} + \cdots + \tau_k t^{2p^k} + \cdots$$

Theorem 15. (*Steinberg*)

(1) $t^{-1} + \tau_0 + \sum_{s>0} t^{2s(p-1)} Q^s \tau_0 = \xi^{-1}\tau$.

(2) $Q^{\frac{p^i-1}{p-1}}\tau_0 = (-1)^i \chi\tau_i$.

(3) *For $s > 0$, we have*

$$Q^s \chi\xi_i = \begin{cases} (-1)^i \beta Q^{s+\frac{p^i-1}{p-1}}\tau_0, & \text{if } s \equiv -1 \mod p^i \\ (-1)^{i+1}\beta Q^{s+\frac{p^i-1}{p-1}}\tau_0, & \text{if } s \equiv 0 \mod p^i \\ 0, & \text{otherwise} \end{cases}$$

and

$$Q^s \chi\tau_i = \begin{cases} (-1)^{i+1} Q^{s+\frac{p^i-1}{p-1}}\tau_0, & \text{if } s \equiv 0 \mod p^i \\ 0, & \text{otherwise} \end{cases}$$

(4) $Q^{p^i}\chi\xi_i = \chi\xi_{i+1}$. $Q^{p^i}\chi\tau_i = \chi\tau_{i+1}$.

where χ is the conjugation in $H_* H$.

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Bökstedt's computation of THH($\mathbb{F}_p$)
Eva Höning

In this talk we discuss the following theorem:

Theorem 1 (Bökstedt)**.** *We have*

$$\pi_* \operatorname{THH}(\mathbb{F}_p) = \mathbb{F}_p[u],$$

where u has degree $|u| = 2$.

We first explain why the homotopy of $\operatorname{THH}(\mathbb{F}_p)$ can be deduced from its $H\mathbb{F}_p$-homology: Recall that the $H\mathbb{F}_p$-homology of every spectrum X is a comodule over the dual Steenrod algebra $A_* = (H\mathbb{F}_p)_* H\mathbb{F}_p$. The coaction is given by

$$\pi_*(H\mathbb{F}_p \otimes X) \longrightarrow \pi_*(H\mathbb{F}_p \otimes S \otimes X) \xrightarrow{\hspace{2cm}} \pi_*(H\mathbb{F}_p \otimes H\mathbb{F}_p \otimes X)$$
$$\Big\downarrow \cong$$
$$\pi_*(H\mathbb{F}_p \otimes H\mathbb{F}_p \otimes_{H\mathbb{F}_p} H\mathbb{F}_p \otimes X)$$
$$\Big\uparrow \cong$$
$$\pi_*(H\mathbb{F}_p \otimes H\mathbb{F}_p) \otimes_{\mathbb{F}_p} \pi_*(H\mathbb{F}_p \otimes X).$$

We claim that $\pi_* \operatorname{THH}(\mathbb{F}_p)$ can be identified with the comodule primitives in $(H\mathbb{F}_p)_* \operatorname{THH}(\mathbb{F}_p)$, i.e. with the elements x in $(H\mathbb{F}_p)_* \operatorname{THH}(\mathbb{F}_p)$ that are mapped to $1 \otimes x$ under the coaction map. This follows from the following remark applied to $R = \operatorname{THH}(\mathbb{F}_p)$ and $B = H\mathbb{F}_p$.

Remark 2. Let B be an $\mathbb{E}_\infty$-ring spectrum and let R be an $\mathbb{E}_\infty$-B-algebra. Then we have a split equalizer

$$R \underset{s}{\overset{e}{\rightleftarrows}} B \otimes R \underset{t}{\overset{f}{\underset{g}{\rightrightarrows}}} B \otimes B \otimes R.$$

The map e is defined by "$r \mapsto 1 \otimes r$", the map s by "$b \otimes r \mapsto br$", the map f by "$b \otimes r \mapsto 1 \otimes b \otimes r$", the map g by "$b \otimes r \mapsto b \otimes 1 \otimes r$" and the map t by "$b \otimes c \otimes r \mapsto b \otimes cr$. If $\pi_*(B \otimes B)$ is flat over $\pi_*(B)$, we get a split equalizer

$$\pi_*(R) \overset{}{\underset{}{\rightleftarrows}} \pi_*(B \otimes R) \overset{f_*}{\underset{}{\rightrightarrows}} \pi_*(B \otimes B) \otimes_{\pi_*(B)} \pi_*(B \otimes R).$$

The map f_* is then given by $f_*(x) = 1 \otimes x$.

In order to prove Theorem 1 it thus suffices to show that we have

$$(H\mathbb{F}_p)_* \operatorname{THH}(\mathbb{F}_p) = A_* \otimes_{\mathbb{F}_p} \mathbb{F}_p[u],$$

where u is primitive. To compute the $H\mathbb{F}_p$-homology of $\operatorname{THH}(\mathbb{F}_p)$ we use the Bökstedt spectral sequence. For an $\mathbb{E}_\infty$-ring spectrum B it takes the form

$$E^2_{m,n} = \operatorname{HH}_{m,n}\big((H\mathbb{F}_p)_* B/\mathbb{F}_p\big) \Longrightarrow (H\mathbb{F}_p)_{m+n} \operatorname{THH}(B).$$

Here, $\operatorname{HH}_{m,n}(-/\mathbb{F}_p)$ denotes Hochschild homology over the ground ring $\mathbb{F}_p$, where m indicates the homological degree and n indicates the internal degree coming

from the grading of the input. The spectral sequence can be constructed by using that $\mathrm{THH}(B)$ is the geometric realization of a simplicial object and by using the skeleton filtration. It is an A_*-comodule spectral sequence. Angeltveit and Rognes have shown that $\mathrm{THH}(B)$ is an augmented commutative B-algebra, and in the stable homotopy category even a B-bialgebra [1, Theorem 3.9]. Since the structure maps are simplicial, the Bökstedt spectral sequence is an A_*-comodule $(H\mathbb{F}_p)_*B$-algebra spectral sequence, and if every page is flat over $(H\mathbb{F}_p)_*B$, it is an A_*-comodule $(H\mathbb{F}_p)_*B$-bialgebra spectral sequence [1, Proposition 4.2, Theorem 4.5]. This means that every page $E^r_{*,*}$ is a bigraded $(H\mathbb{F}_p)_*B$-algebra in the category of A_*-comodules. The differential d^r satisfies the Leibniz formula

$$d^r(xy) = d^r(x)y \pm x d^r(y),$$

and for the coaction ν^r on $E^r_{*,*}$ we have the formula

$$\nu^r \circ d^r = (\mathrm{id} \otimes d^r) \circ \nu^r.$$

Furthermore, the comodule algebra structure on the E^{r+1}-page is induced by the comodule algebra structure on the E^r-page, and the structure converges to the comodule algebra structure on $(H\mathbb{F}_p)_* \mathrm{THH}(B)$. If the flatness condition is satisfied, we in particular have a comultiplication ψ^r on each $E^r_{*,*}$ for which the co-Leibniz formula

$$\psi^r \circ d^r = (d^r \otimes \mathrm{id} \pm \mathrm{id} \otimes d^r) \circ \psi^r$$

holds.

Let $\sigma : \Sigma B \to \mathrm{THH}(B)$ be the composition

$$\Sigma B \longrightarrow \Sigma \mathrm{THH}(B) \longrightarrow \mathrm{THH}(B) \otimes \Sigma^\infty \mathbb{T}_+ \longrightarrow \mathrm{THH}(B).$$

Here, the first map is the suspension of the unit map and the last map is given by the $\mathbb{T}$-action on $\mathrm{THH}(B)$. To define the second map note that we have a cofiber sequence

$$\Sigma^\infty \infty_+ \longrightarrow \Sigma^\infty \mathbb{T}_+ \longrightarrow \Sigma^\infty \mathbb{T}.$$

There is a unique retraction map $\mathbb{T}_+ \to \infty_+$. Thus, the cofiber sequence is canonically split and we have a canonical section $\Sigma^\infty \mathbb{T} \to \Sigma^\infty \mathbb{T}_+$. Applying $H\mathbb{F}_p$-homology we get a map $\sigma_* \colon (H\mathbb{F}_p)_*B \to (H\mathbb{F}_p)_{*+1} \mathrm{THH}(B)$.

We also define a map $\sigma \colon (H\mathbb{F}_p)_*B \to \mathrm{HH}_{1,*}\big((H\mathbb{F}_p)_*B/\mathbb{F}_p\big)$ by sending a class x to the class that is represented in the standard Hochschild complex by $1 \otimes x$. Note that the classes σx are infinite cycles in the Bökstedt spectral sequence, because they have filtration degree one. It follows from the construction of the spectral sequence that $\sigma_*(x)$ is represented in the E^∞-page by the class σx.

To compute the differentials in the Bökstedt spectral sequence and the multiplicative extensions, we will need that σ_* commutes with the Dyer-Lashof operations.

Proposition 3 (Bökstedt/ Angeltveit-Rognes)**.** *For $x \in (H\mathbb{F}_p)_*B$ we have*

$$Q^k \sigma_*(x) = \sigma_* Q^k(x)$$

in $(H\mathbb{F}_p)_{*+2k(p-1)+1} \mathrm{THH}(B)$.

Proof. The following proof is due to Angeltveit and Rognes [1, Proposition 5.9]. Consider the diagram

$$\mathrm{THH}(B) \longrightarrow \mathrm{Map}(\Sigma^\infty \mathbb{T}_+, \mathrm{THH}(B)) \xleftarrow{\cong} \mathrm{THH}(B) \otimes \mathrm{Map}(\Sigma^\infty \mathbb{T}_+, S).$$

Here, $\mathrm{Map}(-,-)$ is the mapping spectrum and the first map is adjoint to the circle action. The spectra $\mathrm{Map}(\Sigma^\infty \mathbb{T}_+, \mathrm{THH}(B))$ and $\mathrm{Map}(\Sigma^\infty \mathbb{T}_+, S)$ are $\mathbb{E}_\infty$-ring spectra via the diagonal $\mathbb{T}_+ \to \mathbb{T}_+ \wedge \mathbb{T}_+$ and the multiplications of S and $\mathrm{THH}(B)$. The maps in the diagram are maps of $\mathbb{E}_\infty$-ring spectra. We denote the induced map in $H\mathbb{F}_p$-homology

$$(H\mathbb{F}_p)_* \mathrm{THH}(\mathbb{F}_p) \longrightarrow (H\mathbb{F}_p)_* \mathrm{THH}(\mathbb{F}_p) \otimes_{\mathbb{F}_p} (H\mathbb{F}_p)_* \mathrm{Map}(\Sigma^\infty \mathbb{T}_+, S)$$

by ρ_*. We have

$$(H\mathbb{F}_p)_* \mathrm{Map}(\Sigma^\infty \mathbb{T}_+, S) = (H\mathbb{F}_p)^{-*} \Sigma^\infty \mathbb{T}_+ = \mathbb{F}_p\{1\} \oplus \mathbb{F}_p\{l\},$$

where l is dual to the image of the identity under

$$\pi_1(\Sigma^\infty \mathbb{T}) \longrightarrow (H\mathbb{F}_p)_1 \Sigma^\infty \mathbb{T} \longrightarrow (H\mathbb{F}_p)_1 \Sigma^\infty \mathbb{T}_+.$$

If we denote by σ' the composition

$$\Sigma \mathrm{THH}(B) \longrightarrow \mathrm{THH}(B) \otimes \Sigma^\infty \mathbb{T}_+ \longrightarrow \mathrm{THH}(B),$$

we have

$$\rho_*(x) = x \otimes 1 + \sigma'_* x \otimes l.$$

By naturality of the Dyer-Lashof operations we get

$$Q^k(x \otimes 1 + \sigma'_* x \otimes l) = Q^k \rho_*(x) = \rho_*(Q^k(x)) = Q^k x \otimes 1 + \sigma'_* Q^k x \otimes l.$$

From the Cartan formula and the facts that $Q^i(l) = 0$ for $i \neq 0$ and $Q^0(l) = l$, it follows that the left side is equal to $Q^k x \otimes 1 + Q^k \sigma'_* x \otimes l$.   □

Remark 4. Using ρ_*, one can also show that σ_* is a derivation.

We now consider the Bökstedt spectral sequence converging to $(H\mathbb{F}_p)_* \mathrm{THH}(\mathbb{F}_p)$. We restrict to the case, where p is an odd prime. For the rest of the talk we write $\otimes$ for $\otimes_{\mathbb{F}_p}$. Recall that the dual Steenrod algebra is given by

$$A_* = \mathbb{F}_p[\bar{\xi}_1, \bar{\xi}_2, \dots] \otimes \Lambda_{\mathbb{F}_p}\{\bar{\tau}_0, \bar{\tau}_1, \dots\}.$$

Here, $\bar{\xi}_i := \chi(\xi_i)$ and $\bar{\tau}_i := \chi(\tau_i)$, where ξ_i and τ_i are the classes defined in the previous talk. The generators have the degrees $|\bar{\xi}_i| = 2p^i - 2$ and $|\bar{\tau}_i| = 2p^i - 1$. One gets that the E^2-page of the Bökstedt spectral sequence is given by

$$E^2_{*,*} = A_* \otimes \Lambda_{\mathbb{F}_p}\{\sigma\bar{\xi}_1, \dots\} \otimes \Gamma_{\mathbb{F}_p}\{\sigma\bar{\tau}_0, \dots\},$$

where $\Gamma_{\mathbb{F}_p}\{-\}$ denotes the divided power algebra over $\mathbb{F}_p$. Note that there is one more $\sigma\bar{\tau}_i$ than there are $\sigma\bar{\xi}_i$'s. One can show that the comultiplication is given by

$$\begin{aligned}
\psi^2(\sigma\bar{\xi}_i) &= 1 \otimes_{A_*} \sigma\bar{\xi}_i + \sigma\bar{\xi}_i \otimes_{A_*} 1 \\
\psi^2(\sigma\bar{\tau}_i^{[k]}) &= \sum_{l+n=k} \sigma\bar{\tau}_i^{[l]} \otimes_{A_*} \sigma\bar{\tau}_i^{[n]}.
\end{aligned}$$

To compute the differentials we first need to prove the following fact about multiplicative extensions:

Lemma 5. *In $(H\mathbb{F}_p)_* \mathrm{THH}(\mathbb{F}_p)$ we have $(\sigma_* \bar{\tau}_i)^p = \sigma_* \bar{\tau}_{i+1}$ for all $i \geq 0$.*

Proof. Recall that we have $Q^{p^i}(\bar{\tau}_i) = \bar{\tau}_{i+1}$. By Proposition 3 we thus get

$$\sigma_* \bar{\tau}_{i+1} = \sigma_* \big(Q^{p^i}(\bar{\tau}_i) \big) = Q^{p^i}(\sigma_* \bar{\tau}_i).$$

Because of $2p^i = |\sigma_* \bar{\tau}_i|$ this is the same as $(\sigma_* \bar{\tau}_i)^p$. $\square$

Lemma 6. *In $(H\mathbb{F}_p)_* \mathrm{THH}(\mathbb{F}_p)$ we have $\sigma_*(\bar{\xi}_i) = 0$ for all $i \geq 1$.*

Proof. Let $i \geq 1$. Recall that the Bockstein homomorphism satisfies $\beta(\bar{\tau}_i) = \bar{\xi}_i$. By Lemma 5 and because β is a derivation we have

$$\begin{aligned}
\sigma_* \bar{\xi}_i &= \sigma_* \big(\beta(\bar{\tau}_i) \big) = \beta \big(\sigma_*(\bar{\tau}_i) \big) = \beta \big((\sigma_* \bar{\tau}_{i-1})^p \big) \\
&= p \cdot (\sigma_* \bar{\tau}_{i-1})^{p-1} \beta(\sigma_* \bar{\tau}_{i-1}) = 0.
\end{aligned}$$

$\square$

We now consider the spectral sequence again. Since the differentials are compatible with the bialgebra structure, a shortest non-zero differential in lowest total degree must map from an algebra indecomposable to a coalgebra primitive, i.e. an element x with

$$\psi(x) = 1 \otimes_{A_*} x + x \otimes_{A_*} 1$$

(see [1, Proposition 4.8]). The formulas for the comultiplication imply that the coalgebra primitives have filtration degree one. Using this one gets $d^i = 0$ for $i = 2, \ldots, p - 2$. Using Lemma 6 and the comodule bialgebra structure of the spectral sequence one gets

$$d^{p-1}(\sigma \bar{\tau}_i^{[p]}) = a_i \sigma \bar{\xi}_{i+1}$$

for a unit $a_i \in \mathbb{F}_p$. Note that this formula implies

$$d^{p-1}(\sigma \bar{\tau}_i^{[p+k]}) = a_i \sigma \bar{\xi}_{i+1} \sigma \bar{\tau}_i^{[k]}$$

for all $k \geq 0$: By induction on k one proves that

$$d^{p-1}(\sigma \bar{\tau}_i^{[p+k]}) - a_i \sigma \bar{\xi}_{i+1} \sigma \bar{\tau}_i^{[k]}$$

is zero. In the induction step this follows, because the class is a coalgebra primitive in filtration degree > 1. We get that

$$E_{*,*}^p = A_* \otimes \mathbb{F}_p[\sigma \bar{\tau}_0, \sigma \bar{\tau}_1, \ldots] / (\sigma \bar{\tau}_0^p, \sigma \bar{\tau}_1^p, \ldots).$$

Since this is generated as an $\mathbb{F}_p$-algebra by classes in filtration degree ≤ 1 the spectral sequences collapses at the E^p-page. With Lemma 5 we get

$$(H\mathbb{F}_p)_* \mathrm{THH}(\mathbb{F}_p) = A_* \otimes \mathbb{F}_p[\sigma \bar{\tau}_0]$$

as an A_*-algebra. Recall that the class $\bar{\tau}_0 \in A_*$ is mapped under the coaction map to

$$1 \otimes \bar{\tau}_0 + \bar{\tau}_0 \otimes 1 \in A_* \otimes A_*.$$

Since σ_* is a comodule map and a derivation, $\sigma_* \bar{\tau}_0$ is a comodule primitive. As explained at the beginning of the talk, this proves Theorem 1.

Finally, we want to outline the proof of the following theorem by Nikolaus and Scholze:

Theorem 7 (Nikolaus, Scholze)**.** *As $\mathbb{E}_\infty$-algebras in cyclotomic spectra we have*

$$\mathrm{THH}(\mathbb{F}_p) \simeq \tau_{\geq 0}(H\mathbb{Z}^{tC_p}).$$

We first explain what the cyclotomic structure on the right is. For this note that:

For every spectrum X there is a cyclotomic spectrum X^{triv} whose underlying spectrum is X and which has the trivial $\mathbb{T}$-action. The Frobenius maps are defined by

$$X \longrightarrow X^{hC_p} \longrightarrow X^{tC_p}.$$

Here, the first map is pullback along $BC_p \to *$. In the statement of Theorem 7 we equip $H\mathbb{Z}$ with this trivial cyclotomic structure.

For every connective cyclotomic spectrum there is a connective cyclotomic spectrum $\mathrm{sh}_p X$ whose underlying spectrum is $\tau_{\geq 0}(X^{tC_p})$ and which has the residual action. Since this spectrum is p-complete by [3, Lemma I.2.9], $\big(\tau_{\geq 0}(X^{tC_p})\big)^{tC_q}$ is zero for $q \neq p$ and we only have to define the Frobenius map for the prime p. For this note that the Frobenius map $X \to X^{tC_p}$ of X factors over the connective cover of X^{tC_p}, because X is connective. The Frobenius map of $\mathrm{sh}_p X$ is given by applying $\tau_{\geq 0}\big((-)^{tC_p}\big)$ to $X \to \tau_{\geq 0}(X^{tC_p})$. This defines the cyclotomic structure on $\tau_{\geq 0}(H\mathbb{Z}^{tC_p})$ in Theorem 7. Also note that the Frobenius map induces a map of cyclotomic spectra $X \to \mathrm{sh}_p(X)$.

We now roughly outline the proof of Theorem 7. Note that the proofs of Theorem 8 and Theorem 9 below rely on Bökstedt's theorem.

Theorem 8. *We have*

$$\pi_i \mathrm{TC}(\mathbb{F}_p) = \begin{cases} \mathbb{Z}_p, & i = -1, 0 \\ 0, & otherwise. \end{cases}$$

Proof. See [3, Corollary IV.4.10]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The theorem implies that we have a map

$$H\mathbb{Z} \longrightarrow H\mathbb{Z}_p = \tau_{\geq 0}\,\mathrm{TC}(\mathbb{F}_p) \longrightarrow \mathrm{TC}(\mathbb{F}_p).$$

We have an adjunction

$$(-)^{\mathrm{triv}} \colon \mathrm{Sp} \; \underset{\longleftarrow}{\overset{\longrightarrow}{\rule{2em}{0pt}}} \; \mathrm{CycSp} \colon \mathrm{TC}.$$

We thus get a map

$$H\mathbb{Z}^{\mathrm{triv}} \longrightarrow \mathrm{THH}(\mathbb{F}_p)$$

in cyclotomic spectra. Theorem 7 then follows from

Theorem 9. *We have*

$$\mathrm{sh}_p(H\mathbb{Z}^{\mathrm{triv}}) \xrightarrow{\simeq} \mathrm{sh}_p\big(\mathrm{THH}(\mathbb{F}_p)\big) \xleftarrow{\simeq} \mathrm{THH}(\mathbb{F}_p).$$

Proof. See [3, Corollary IV.4.13]. □

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Topological Hochschild homology of stable ∞-categories

THOMAS NIKOLAUS

We give a definition of topological cyclic homology for stable ∞-categories based on the notion of a 'trace theory' as invented by Kaledin. From a different perspective some of the constructions and results in this talk are also contained in the work of Blumberg–Mandell [2] and Ayala–Mazel-Gee–Rozenblyum [1]. We omit a lot of proofs and details in this note which will appear in [5].

1. UNSTABLE THH

We begin this note by recalling Connes cyclic category Λ. We first define a related category Λ_∞, the paracyclic category. It is defined as a full subcategory of the category of ordered sets with an order preserving $\mathbb{Z}$-action in which morphisms are non-decreasing equivariant maps. Then Λ_∞ consists of those objects which are isomorphic to $\frac{1}{n}\mathbb{Z}$ with the obvious ordering and the $\mathbb{Z}$-action by addition of integers. We denote the object $\frac{1}{n}\mathbb{Z}$ also by $[n]_\Lambda \in \Lambda_\infty$ and by definition every object in Λ_∞ is equivalent to one of those.

There is a canonical functor $\Delta \to \Lambda_\infty$ which sends a non-empty linearly ordered set S to the set $\mathbb{Z} \times S$ with lexicographic ordering and $\mathbb{Z}$ action by addition in the left factor. We define the cyclic category Λ by identifying certain morphisms in Λ_∞, namely the quotient by the relation $f \sim f + k$ for $k \in \mathbb{Z}$. The object of Λ corresponding to $\frac{1}{n}\mathbb{Z}$ is written as $[n]_\Lambda$. One should think of Λ as consisting of cyclic graphs:
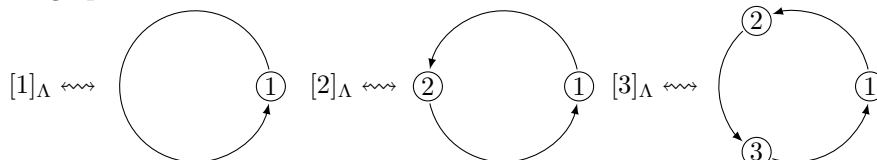
For every such cyclic graph with n vertices we have an associated 'free category' $\mathbb{T}_n \in \mathrm{Cat}$ and then a map $[n]_\Lambda \to [m]_\Lambda$ in Λ corresponds to a functor $\mathbb{T}_n \to \mathbb{T}_m$ of these categories that is a map of degree one of the circle after geometric realization.

Really the poset $\frac{1}{n}\mathbb{Z}$ with its $\mathbb{Z}$-action should be considered as the universal cover of such a graph/category. In particular this assignment gives us a functor

(1) $$\Lambda \to \mathrm{Cat} \qquad [n]_\Lambda \mapsto \mathbb{T}_n \ .$$

In a choice free way we can write this functor as sending a poset $P \in \Lambda$ with $\mathbb{Z}$-action to the quotient category $\mathbb{T}_P := P/\mathbb{Z}$ where the poset P is considered as a category and the quotient is taken in the category of categories.

For an ∞-category $\mathcal{D}$ we denote by $\mathcal{D}^\sim$ the maximal Kan complex inside of $\mathcal{D}$, i.e. the groupoid core.

Definition 1. For a small ∞-category $\mathcal{C}$ we define a space $\mathrm{uTHH}(\mathcal{C}) \in \mathcal{S}$ as the geometric realization of the cyclic space[1]

$$\Lambda^\mathrm{op} \to \mathcal{S} \qquad [n]_\Lambda \mapsto \mathrm{Fun}\,(\mathrm{N}\mathbb{T}_n, \mathcal{C})^\sim \ .$$

Remark 2. One can check that for a given ∞-category $\mathcal{C}$ there is an equivalence

$$\mathrm{Fun}\,(\mathrm{N}\mathbb{T}_n, \mathcal{C})^\sim \simeq \operatorname*{colim}_{c_1,\ldots,c_n \in \mathcal{C}^\sim} \prod_{i=1}^n \mathrm{Map}_\mathcal{C}(c_i, c_{i+1})$$

where $c_{n+1} = c_1$. This is the formula that we generalize for stable ∞-categories later.

Now we want to compare this definition of unstable topological Hochschild homology to the usual definition using the standard cyclic Bar construction.

Proposition 3. *If M is an associative monoid in the ∞-category $\mathcal{S}$ of spaces then the space $\mathrm{uTHH}(BM)$ for the associated ∞-category BM is equivalent to the geometric realization of the cyclic Bar construction of M:*

$$\cdots \mathrel{\substack{\textstyle\longrightarrow\\[-0.6ex]\textstyle\longrightarrow\\[-0.6ex]\textstyle\longrightarrow\\[-0.6ex]\textstyle\longrightarrow}} M \times M \times M \mathrel{\substack{\textstyle\longrightarrow\\[-0.6ex]\textstyle\longrightarrow\\[-0.6ex]\textstyle\longrightarrow}} M \times M \mathrel{\substack{\textstyle\longrightarrow\\[-0.6ex]\textstyle\longrightarrow}} M$$

An immediate consequence of our definition of unstable cyclic homology the way we defined it is the following well-known result of Goodwillie-Jones which is usually proven quite differently.

Corollary 4. *Let X be a space (i.e. a Kan complex) considered as an ∞-category. Then we have an equivalence*

$$\mathrm{uTHH}(X) \simeq LX \ .$$

where $LX = \mathrm{Map}(\mathbb{T}, X)$ is the free loop space of X.

Proof. For X an ∞-groupoid we have that

$$\mathrm{Fun}(\mathrm{N}\mathbb{T}_n, X)^\sim \simeq \mathrm{Fun}\,(|\mathrm{N}\mathbb{T}_n|, X) \simeq \mathrm{Map}\,(\mathbb{T}, X)$$

and the underlying simplicial object for varying n is constant. $\qquad\square$

---

[1]For a cyclic space $\Lambda^\mathrm{op} \to \mathcal{S}$ the geometric realization is defined as the colimit of the underlying simplicial space $\Delta^\mathrm{op} \to \Lambda^\mathrm{op} \to \mathcal{S}$.

Now the space $\mathrm{uTHH}(\mathcal{C})$ carries some extra structure: first it clearly has a $\mathbb{T}$-action induced by the fact that it is the geometric realization of a cyclic object[2]. Secondly the category $\mathbb{T}_{pn}$ comes with a canonical C_p-action by rotation (where C_p is the cyclic group with p elements) . Clearly we have (in Cat as well as in Cat_∞) that the homotopy orbits of this C_p-action on $\mathbb{T}_{pn}$ are given by $\mathbb{T}_n$. As a result we get an equivalence

$$\mathrm{Fun}(\mathrm{N}\mathbb{T}_n, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}(\mathrm{N}\mathbb{T}_{pn}, \mathcal{C})^{hC_p} \ .$$

One can show that the target of this map is also a cyclic object if we let n vary and this equivalence is an equivalence of cyclic objects. We omit the details here.

As soon as we know this it follows that we have a map of cyclic objects which after realization, and commuting the homotopy fixed points out of the colimit (recall that the colimit only depends on the underlying simplicial object), gives us a $\mathbb{T}$-equivariant map

$$\mathrm{uTHH}(\mathcal{C}) \to \mathrm{uTHH}(\mathcal{C})^{hC_p} \ .$$

This map is the key part of an 'unstable cyclotomic structure' on $\mathrm{uTHH}(\mathcal{C})$.

2. Stable ∞-categories and THH

In this section we want to define a variant of uTHH as discussed in the previous section, called $\mathrm{THH}(\mathcal{C})$ where we require $\mathcal{C}$ to be a stable ∞-category and such that $\mathrm{THH}(\mathcal{C})$ is a spectrum in contrast to the space $\mathrm{uTHH}(\mathcal{C})$. The idea is to define $\mathrm{THH}(\mathcal{C})$ as the geometric realization of a cyclic object which is informally given by

$$[n]_\Lambda \mapsto \underset{c_1,\ldots,c_n \in \mathcal{C}^\sim}{\mathrm{colim}} \bigotimes_{i=1}^n \mathrm{map}_{\mathcal{C}}(x_i, x_{i+1})$$

where $\mathrm{map}_{\mathcal{C}}(-,-)$ denotes the mapping spectrum in $\mathcal{C}$. This generalizes the description in Remark 2. The main point is to make this informal description into a homotopy coherent functor.

To this end we define an ∞-category Λ^{st} of 'labelled cyclic graphs'. Let us first give an informal description of this ∞-category and then the precise definition. Up to equivalence, an object in Λ^{st} is given by a cyclic graph labelled with stable ∞-categories $\mathcal{C}_1,\ldots,\mathcal{C}_n$ and colimit preserving functors $F_i : \mathrm{Ind}\mathcal{C}_{i+1} \to \mathrm{Ind}\mathcal{C}_i$ (a.k.a. profunctors or bimodules) for every i (as usual i is taken mod n). We will abbreviate such an object as $(F_1,\ldots,F_n)$ leaving the stable ∞-categories implicit or even as $\vec{F}$. Graphically such an object for $n = 5$ will be depicted as follows

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[2]See [4, Appendix T] for details
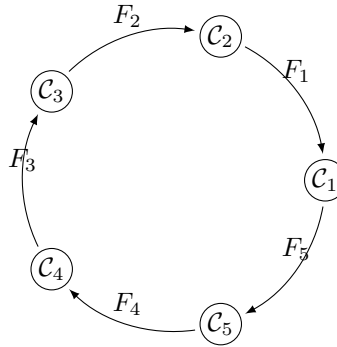
Here the functors are really profunctors but we draw them as ordinary arrows. The morphisms in Λ^{st} are generated by the following basic morphisms: composing adjacent functors, inserting identities, rotations and lax maps of diagrams of fixed shape. Lax means that for a given shape there are 2-cells allowed. For example if we have two endoprofunctors $F : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$ and $G : \mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{D})$ considered as objects in Λ^{st}
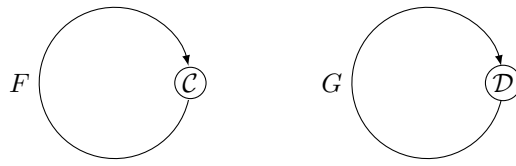
then such a morphism corresponds to a pair of a functor $\phi : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\eta : \phi_! \circ F \to G \circ \phi_!$ filling the diagram

$$
\begin{array}{ccc}
\mathrm{Ind}(\mathcal{C}) & \xrightarrow{\ F\ } & \mathrm{Ind}(\mathcal{C}) \\
\ \downarrow{\scriptstyle \phi_!} & \Swarrow_\eta & \downarrow{\scriptstyle \Phi_!} \\
\mathrm{Ind}(\mathcal{D}) & \xrightarrow{\ G\ } & \mathrm{Ind}(\mathcal{D})
\end{array} \quad .
$$

In particular for $\mathcal{C} = \mathcal{D}$ and $\phi = \mathrm{id}$ there are still morphisms of profunctors built into Λ^{st}. After this informal discussion we now make the definition of Λ^{st} precise:

Definition 5. Let $X \to S$ be a categorical fibration of simplcial sets. We say that that it is a flat stable fibration if it is a flat categorical fibration[3] , every fibre X_s for $s \in S$ is a stable ∞-category and for every edge $s \to s'$ in S the induced map $X_s^{\mathrm{op}} \times X_{s'} \to \mathcal{S}$ is excisive in every variable separately [4]. A functor $X \to X'$ of flat stable fibrations over S is a functor of fibrations over S which is fibrewise exact. We denote the ∞-category of flat stable fibrations over S (considered as a subcategory of the slice category) by $\mathrm{Stab}^\flat_{/S}$.

---

[3]This means that it is a categorical fibration and that for every 2-simplex $\Delta^2 \to S$ the induced map $X \times_S \Lambda^2_1 \to X \times_S \Delta^2$ is a categorical equivalence., see [3]

[4]Such a functor is equivalently given by a colimit preserving functor $\mathrm{Ind}X_{s'} \to \mathrm{Ind}X_s$

There is a functor

$$\chi : \Lambda^{\mathrm{op}} \to \mathrm{Cat}_\infty \qquad [n]_\Lambda \mapsto \mathrm{Stab}^\flat_{/\mathrm{N}\mathbb{T}_n}$$

and we let Λ^{st} be the coCartesian fibration over Λ^{op} classifying χ. We refer to objects of Λ^{st} as cyclic graphs of stable ∞-categories.

Note that the coCartesian morphisms of Λ^{st} over Λ^{op} are compositions of contraction, insertion and rotation morphisms. In the following we will study functors $T \colon \Lambda^{\mathrm{st}} \to \mathrm{Sp}$. In particular we will realize topological Hochschild homology as such a functor. Let us start by an example.

Proposition 6. *There is a functor*

$$\mathrm{End} : \Lambda^{\mathrm{st}} \to \mathrm{Cat}_\infty$$

such that objectwise we have $\mathrm{End}(X \to \mathrm{N}\mathbb{T}_n) \simeq \mathrm{Fun}_{\mathrm{N}\mathbb{T}_n}(\mathrm{N}\mathbb{T}_n, X)$, *i.e. it is the* ∞-*category of sections of the flat stable fibration.*

For a cyclic graph of stable ∞-categories described by the list of profunctors $(F_1, \ldots, F_n)$ the ∞-category $\mathrm{End}(F_1, \ldots, F_n)$ has as objects sequences of objects $(c_i \in \mathcal{C}_i)$ together with morphisms $c_i \to F_i(c_{i+1})$. In particular for the cyclic graph given by the identity endoprofunctor on a stable ∞-category this is the category $\mathrm{End}(\mathcal{C}) \simeq \mathrm{Fun}(\mathrm{N}\mathbb{T}_1, \mathcal{C})$ of endomorphisms in $\mathcal{C}$.

Using the functor End we get a resulting functor

$$E: \quad \Lambda^{\mathrm{st}} \xrightarrow{\mathrm{End}} \mathrm{Cat}_\infty \xrightarrow{(-)^\sim} \mathcal{S} \xrightarrow{\Sigma^\infty_+} \mathrm{Sp} \ .$$

which is key for what follows. We have an equivalence

$$E(F_1, \ldots, F_n)^\sim \simeq \mathop{\mathrm{colim}}_{c_i \in \mathcal{C}_i^\sim} \bigotimes_{i=1}^{n} \Sigma^\infty_+ \mathrm{Map}_{\mathrm{Ind}(\mathcal{C}_i)}(c_i, F_i c_{i+1}) \ .$$

For every sequence of stable ∞-categories $(\mathcal{C}_1, \ldots, \mathcal{C}_n)$ one can construct a functor

$$\prod_{i=1}^{n} \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}\mathcal{C}_{i+1}, \mathrm{Ind}\mathcal{C}_i) \to \Lambda^{\mathrm{st}}$$

where $\mathrm{Fun}^{\mathrm{L}}$ denotes colimit preserving (equivalently left adjoint) functors. It sends the sequence $(F_1, \ldots, F_n)$ to the object denoted in the same way in Λ^{st}.

Definition 7. Let $T \colon \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ be a functor.

  (1) T is called *reduced* if for every sequence of stable ∞-categories $\mathcal{C}_1, \ldots, \mathcal{C}_n$ the restriction of T to a functor

$$\prod_{i=1}^{n} \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}\mathcal{C}_{i+1}, \mathrm{Ind}\mathcal{C}_i) \to \mathrm{Sp}$$

  is reduced in every variable separately, i.e. sends the zero functor to a zero object in Sp

(2) T is called *stable* if for every sequence of stable ∞-categories $\mathcal{C}_1, \ldots, \mathcal{C}_n$ the restriction of T to a functor

$$\prod_{i=1}^{n} \operatorname{Fun}^{\mathrm{L}}(\operatorname{Ind}\mathcal{C}_{i+1}, \operatorname{Ind}\mathcal{C}_i) \to \operatorname{Sp}$$

is exact in every variable separately, i.e. sends pushouts in $\operatorname{Fun}^{\mathrm{ex}}(\operatorname{Ind}C_{i+1}, \operatorname{Ind}C_i)$ to pullbacks in Sp.

Proposition 8. *The inclusions*

$$\operatorname{Fun}^{\mathrm{st}}(\Lambda^{\mathrm{st}}, \operatorname{Sp}) \subseteq \operatorname{Fun}^{\mathrm{red}}(\Lambda^{\mathrm{st}}, \operatorname{Sp}) \subseteq \operatorname{Fun}(\Lambda^{\mathrm{st}}, \operatorname{Sp})$$

admit left adjoints $T \mapsto T^{\mathrm{st}}$ and $T \mapsto T^{\mathrm{red}}$ such that

$$T^{\mathrm{st}}(F_1, \ldots, F_n) \simeq \varinjlim_k \Omega^{nk} T\big(\Sigma^k F_1, \ldots, \Sigma^k F_n\big) \ .$$

and $T^{\mathrm{red}}(F_1, \ldots, F_n)$ is given by the total cofibre of the n-cube

$$P\{1, \ldots, n\} \to \operatorname{Sp} \qquad S \subseteq \{1, \ldots, n\} \mapsto T(F_1', \ldots, F_n')$$

where F_i' is given by F_i for $i \in S$ and by the zero functor otherwise.

We will abusively denote the composite left adjoint

$$\operatorname{Fun}(\Lambda^{\mathrm{st}}, \operatorname{Sp}) \to \operatorname{Fun}^{\mathrm{st}}(\Lambda^{\mathrm{st}}, \operatorname{Sp})$$

given by first making the functor reduced and then stable also by $(-)^{\mathrm{st}}$. The results above show that the restriction of the transformation $T \to T^{\mathrm{st}}$ to the subcategory

$$\prod_{i=1}^{n} \operatorname{Fun}^{\mathrm{L}}(\operatorname{Ind}\mathcal{C}_{i+1}, \operatorname{Ind}\mathcal{C}_i) \subseteq \Lambda^{\mathrm{st}}$$

exhibit $T^{\mathrm{st}}\big|_{\prod_{i=1}^{n} \operatorname{Fun}^{\mathrm{L}}(\operatorname{Ind}\mathcal{C}_{i+1}, \operatorname{Ind}\mathcal{C}_i)}$ as the exact (i.e. 1-excisive and reduced) approximation of $T\big|_{\prod_{i=1}^{n} \operatorname{Fun}^{\mathrm{L}}(\operatorname{Ind}\mathcal{C}_{i+1}, \operatorname{Ind}\mathcal{C}_i)}$. This more or less directly implies the following result.

Proposition 9. *The stable approximation E^{st} to the functor $E : \Lambda^{\mathrm{st}} \to \operatorname{Sp}$ as constructed above is pointwise given by*

$$E^{\mathrm{st}}(F_1, \ldots, F_n) \simeq \operatorname*{colim}_{c_1, \ldots, c_n \in \mathcal{C}^{\sim}} \bigotimes_{i=1}^{n} \operatorname{map}_{\mathcal{C}}(x_i, F_i x_{i+1}) \ .$$

Now we can now define topological Hochschild homology through the usual cyclic Bar construction as

$$\operatorname{THH}(\mathcal{C}) = \big| \ldots E^{\mathrm{st}}(\operatorname{id}_\mathcal{C}, \operatorname{id}_\mathcal{C}, \operatorname{id}_\mathcal{C}) \mathrel{\substack{\textstyle\longrightarrow \\ \textstyle\longrightarrow \\ \textstyle\longrightarrow}} E^{\mathrm{st}}(\operatorname{id}_\mathcal{C}, \operatorname{id}_\mathcal{C}) \mathrel{\substack{\textstyle\longrightarrow \\ \textstyle\longrightarrow}} E^{\mathrm{st}}(\operatorname{id}_\mathcal{C}) \big| \ .$$

In fact we can define THH now more generally.

Definition 10. We more generally define $\operatorname{THH}(F_1, \ldots, F_n)$ as the realization of the simplicial object

$$\ldots \mathrel{\substack{\textstyle\longrightarrow \\ \textstyle\longrightarrow \\ \textstyle\longrightarrow}} E^{\mathrm{st}}(F_1, \operatorname{id}, F_2, \operatorname{id}, \ldots, F_n, \operatorname{id}) \mathrel{\substack{\textstyle\longrightarrow \\ \textstyle\longrightarrow}} E^{\mathrm{st}}(F_1, \ldots, F_n)$$

where the simplicial object is obtained from an appropriately constructed simplicial object in Λ^{st}. This gives us a functor
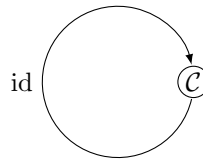
$$\mathrm{THH}\colon \Lambda^{\mathrm{st}} \to \mathrm{Sp}$$

which comes with a transformation $E^{\mathrm{st}} \to \mathrm{THH}$.

In particular we have a functor $\mathrm{Cat}^{\mathrm{st}}_\infty \to \mathrm{Stab}^\flat_{/\mathrm{N}\mathbb{T}_1} \to \Lambda^{\mathrm{st}}$ sending $\mathcal{C}$ to

$$(\mathrm{id}_{\mathcal{P}(\mathcal{C})}) = (\mathcal{C} \times \mathrm{N}\mathbb{T}_1 \to \mathrm{N}\mathbb{T}_1)$$

i.e. the cyclic graph

The composition with THH as defined above then gives THH for stable ∞-categories.

Definition 11. A functor $T : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ is called a *trace theory* if it sends coCartesian morphisms in Λ^{st} to equivalences in Sp. It is called a stable trace theory, if it additionally is stable (see Definition 7).

For a trace theory T we have equivalences $T(F \circ G) \simeq T(F, G) \simeq T(G, F) \simeq T(G \circ F)$. Thus it behaves like the usual cyclic invariance of the trace of an endomorphism. This is the reason for the naming. In fact, for a trace theory $T : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ the values $T(F)$ for a single functor already determine all values, since we have

$$T(F_1, \ldots, F_n) \simeq T(F_1 \circ \ldots \circ F_n) \ .$$

The main point of the notion of trace theory is that the combinatorics of Λ^{st} encode a homotopy coherent way of expressing the 'cyclic invariance'. Also note that every coCartesian morphism is a composition of rotation, contraction and insertion morphisms (since this is true in Λ). The rotations are equivalence in Λ^{st} and the insertions are one sided inverses to contractions. Therefore to check that something is a trace theory it suffices to check that the morphisms given by contraction of two adjacent profunctors induces an equivalence of spectra.

Theorem 12. *The functor* $\mathrm{THH} : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ *is a stable trace theory. Moreover the natural transformation* $E \to \mathrm{THH}$ *exhibits* $\mathrm{THH} : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ *as the universal stable trace theory under E.*

Proposition 13. *For every trace theory T and every functor $F : \mathrm{Ind}\mathcal{C} \to \mathrm{Ind}\mathcal{C}$ there is a C_p-action on $T(F \circ \ldots \circ F) \simeq T(F, \ldots, F)$ which extends to a $\mathbb{T}$-action for $F = \mathrm{id}$.*

Proof. The first claim is obvious since $T(F \circ \ldots \circ F)$ is by the trace property equivalent to $T(F, \ldots, F)$ which has a C_p-action since the object $(F, \ldots, F) \in \Lambda^{\mathrm{st}}$ carries a C_p-action. For the second we observe that $T(\mathrm{id}_{\mathcal{C}})$ is equivalent to

the colimit of the cyclic diagram $[n] \mapsto T(\mathrm{id}_{\mathcal{C}}, \ldots, \mathrm{id}_{\mathcal{C}})$ (with p-identities) since all structure maps are equivalences. Thus it gets an induced $\mathbb{T}$-action as the realization of a cyclic object. By subdividing this cyclic object we see that the action is compatible with the one on $T(\mathrm{id}, \ldots, \mathrm{id})$. $\square$

Proposition 14. *For every trace theory $T : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ the induced functor*

$$T : \mathrm{Cat}_{\infty}^{\mathrm{st}} \to \Lambda^{\mathrm{st}} \to \mathrm{Sp}$$

is Morita invariant, that is for an exact functor $F : \mathcal{C} \to \mathcal{D}$ of stable ∞-categories which is an equivalence after idempotent completion the induced map $T(\mathcal{C}) \to T(\mathcal{D})$ is an equivalence.

Proof. Being an equivalence after idempotent completion means that we have an inverse profunctor $G : \mathrm{Ind}\mathcal{D} \to \mathrm{Ind}\mathcal{C}$ such that $G \circ \mathrm{Ind}(F) \simeq \mathrm{id}_{\mathrm{Ind}\mathcal{C}}$ and $\mathrm{Ind}(F) \circ G \simeq \mathrm{id}_{\mathrm{Ind}\mathcal{D}}$. We then have

$$T(\mathcal{C}, \mathrm{id}) \simeq T(\mathcal{C}, G \circ \mathrm{Ind}(F)) \simeq T(\mathcal{D}, \mathrm{Ind}(F) \circ G) \simeq T(\mathcal{D}, \mathrm{id})$$

which shows the claim. $\square$

We recall that a Verdier sequence of stable ∞-categories is a sequence of stable ∞-categories $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ such that the composition is the zero functor (this is a property and not extra structure) and such that it is a fibre and cofibre sequence in $\mathrm{Cat}_{\infty}^{\mathrm{st}}$.

Proposition 15. *Let $T : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ be a stable trace theory. Then for every Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ of stable ∞-categories the induced sequence*

$$T(\mathcal{C}) \to T(\mathcal{D}) \to T(\mathcal{E})$$

is a fibre sequence of spectra.

Proof. The induced sequence

$$\mathrm{Ind}(\mathcal{C}) \xrightarrow{i} \mathrm{Ind}(\mathcal{D}) \xrightarrow{p} \mathrm{Ind}(\mathcal{E})$$

is a split Verdier sequence, that is there are right adjoints to R_p to p and R_i to i such that the unit $\mathrm{id} \to R_i \circ i$ as well as the counit $p \circ R_p \to \mathrm{id}$ are equivalences (these exist by the adjoint functor theorem). Then we get a cofibre sequence of functors

$$i \circ R_i \to \mathrm{id}_{\mathrm{Ind}(\mathcal{D})} \to R_p \circ p$$

using the properties of a Verdier sequence. Now using the trace property of T we get equivalences $T(\mathcal{C}) \simeq T(R_i \circ i) \simeq T(i \circ R_i)$ and $T(\mathcal{E}) \simeq T(p \circ R_p) \simeq T(R_p \circ p)$ and under these equivalences the sequence in question corresponds to the sequence $T(i \circ R_i) \to T(\mathrm{id}_{\mathrm{Ind}(\mathcal{D})}) \to T(R_p \circ p)$ which is a fibre sequence by stability of T. $\square$

Corollary 16. *For every stable ∞-category $\mathrm{THH}(\mathcal{C}) = \mathrm{THH}(\mathrm{id}_{\mathcal{C}})$ carries canonically a $\mathbb{T}$-action. For every functor $F : \mathrm{Ind}\mathcal{C} \to \mathrm{Ind}\mathcal{C}$ the spectrum $\mathrm{THH}(F^p) \simeq \mathrm{THH}(F, \ldots, F)$ carries a C_p-action. Moreover THH is Morita invariant and localizing .*

We now can form the spectrum $\mathrm{THH}(\vec{F}, \ldots, \vec{F})^{tC_p}$. This assignment again forms a functor $\Lambda^{\mathrm{st}} \to \mathrm{Sp}$ in a non-trivial way (similar to the discussion at the end of the first section). We again omit the details of the construction.

Example 17. *There is a natural transformation $E \to E^{hC_p}$ sending a sequence of morphisms $(\varphi_1, \ldots, \varphi_p)$ to the p-fold iterate. Formally there is an equivalence of the ∞-categories of sections*

$$\mathrm{End}(X \to \mathrm{N}\mathbb{T}_n) \simeq \mathrm{End}(w^* X \to \mathrm{N}\mathbb{T}_{pn})^{hC_p}$$

where $w : \mathbb{T}_{pn} \to \mathbb{T}_n$ is the projection. This induces the transformation in question (where we again omit the construction of the coherences in this text).

Proposition 18. *The functor $\vec{F} \mapsto \mathrm{THH}(\vec{F}, \ldots, \vec{F})^{tC_p}$ is a stable trace theory.*

Proof. Clearly it is a trace theory since for every contraction in F the induced morphism can (before taking Tate) be written as an p-fold composition of contractions and is thus an equivalence. For stability we use the usual fact about Tate construction of a multilinear functor being exact. $\qquad\square$

Now we get an induced transformation of functors $\Lambda^{\mathrm{st}} \to \mathrm{Sp}$ as in the diagram

$$
\begin{array}{ccc}
E & \longrightarrow & \mathrm{THH} \\
\downarrow & & \Big\downarrow{\scriptstyle \exists! \, | \, \varphi_p} \\
E^{hC_p} & \longrightarrow & \mathrm{THH}^{tC_p}
\end{array}
\quad .
$$

After evaluation on $\mathcal{C}$ this gives us a $\mathbb{T}$-equivariant map

$$\varphi_p : \mathrm{THH}(\mathcal{C}) \to \mathrm{THH}(\mathcal{C})^{tC_p} \ .$$

for every stable ∞-category $\mathcal{C}$. This shows that $\mathrm{THH}(\mathcal{C})$ is a cyclotomic spectrum. But the transformation is more general since it also gives some information for THH with coefficients: for an endofunctor $F : \mathrm{Ind}\mathcal{C} \to \mathrm{Ind}\mathcal{C}$ we get a map

$$\mathrm{THH}(F) \to \mathrm{THH}(F, \ldots, F)^{tC_p} \simeq \mathrm{THH}(F \circ \ldots \circ F)^{tC_p}$$

generalizing the cyclotomic Frobenius.

Corollary 19. *The functor* $\mathrm{THH} : \mathrm{Cat}^{\mathrm{st}}_\infty \to \mathrm{Sp}$ *extends to a functor*

$$\mathrm{THH} : \mathrm{Cat}^{\mathrm{st}}_\infty \to \mathrm{CycSp}$$

where CycSp *is the ∞-category of cyclotomic spectra. This functor is Morita invariant and localizing (i.e. sends Morita equivalences to equivalences and Verdier sequences to cofibre sequences). Therefore also the composite*

$$\mathrm{TC} : \mathrm{Cat}^{\mathrm{st}}_\infty \to \mathrm{CycSp} \to \mathrm{Sp}$$

is Morita invariant and localizing.

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TC of spherical group rings
Markus Land

The goal of this talk is to give a proof the following theorem due to Bökstedt-Hsiang-Madsen [1, Theorem IV.3.6].

Theorem 1. *Let M be an $\mathbb{E}_1$-group in spaces and p be any prime. Then, after p-completion, there is a pullback square of the form*

$$
\begin{array}{ccc}
\mathrm{TC}(\mathbb{S}[M]) & \longrightarrow & \Sigma\big(\Sigma_+^\infty L(BM)_{h\mathbb{T}}\big) \\
\downarrow & & \downarrow \\
\Sigma_+^\infty L(BM) & \xrightarrow{\mathrm{id}-\tilde{\varphi}_p} & \Sigma_+^\infty L(BM).
\end{array}
$$

Here, $L(BM)$ is the free loop space of the classifying space BM of M and the map $\tilde{\varphi}_p$ is the map induced by the map $x \mapsto x^p$ on S^1. Furthermore, $\mathbb{S}[M] = \Sigma_+^\infty M$ is the spherical group ring of M – an $\mathbb{E}_1$-ring spectrum.

To set the stage, we recall some definitions, see [1, Definition II.1.8 & Proposition II.1.9].

Definition 2. Let (X, φ_p) be a p-cyclotomic spectrum. Then its topological cyclic homology $\mathrm{TC}(X)$ is given by

$$
\mathrm{TC}(X) = \mathrm{map}_{\mathrm{CycSp}_p}(\mathbb{S}, X) \stackrel{\mathrm{Prop}}{\simeq} \mathrm{fib}\big(X^{h\mathbb{T}} \xrightarrow{\mathrm{can}-\varphi_p^{h\mathbb{T}}} (X^{tC_p})^{h\mathbb{T}}\big).
$$

The first step is to understand $\mathrm{THH}(\mathbb{S}[M])$. For this we recall from the last lecture that we have defined topological Hochschild homology for categories:

Definition 3. Let $\mathscr{C}$ be an ∞-category. Then, as constructed in the last lecture, its unstable topological Hochschild homology $\mathrm{uTHH}(\mathscr{C})$ is given by

$$
\mathrm{uTHH}(\mathscr{C}) = \operatorname*{colim}_{n \in \Delta} \mathrm{Fun}\big((n), \mathscr{C}\big)^{\sim} \in \mathcal{S}^{B\mathbb{T}}.
$$

As the association $n \mapsto (n)$ is a cocyclic category, it follows that $\mathrm{uTHH}(\mathscr{C})$ is in fact the geometric realization of a cyclic space, and as such is canonically endowed with a $\mathbb{T}$-action. Furthermore, for every prime p, in the last lecture, an unstable cyclotomic Frobenius, i.e. a $\mathbb{T}$-equivariant map

$$
\mathrm{uTHH}(\mathscr{C}) \xrightarrow{\psi_p} \mathrm{uTHH}(\mathscr{C})^{hC_p}
$$

was established, so that uTHH($\mathscr{C}$) is a cyclotomic object in $\mathcal{S}$, the ∞-category of spaces. If $\mathscr{C}$ is a stable ∞-category, or more generally a spectrally enriched ∞-category, then one similarly obtains THH($\mathscr{C}$) as a cyclotomic spectrum.

For an $\mathbb{E}_1$-ring spectrum R, its category of compact modules Mod_R^ω is a stable ∞-category, and one defines topological Hochschild homology of R via its category of compact modules:

$$\mathrm{THH}(R) = \mathrm{THH}(\mathrm{Mod}_R^\omega)$$

which is thus a cyclotomic spectrum. One then sets

$$\mathrm{TC}(R) = \mathrm{TC}(\mathrm{THH}(R)).$$

We will make use of the following lemma.

Lemma 4. (1) *Let M be an $\mathbb{E}_1$-space (not necessarily grouplike) and let $\mathbb{B}M$ be the ∞-category with one object and M as endomorphisms. Then there is an equivalence* $\mathrm{uTHH}(\mathbb{B}M) \simeq \mathrm{B}^{\mathrm{cyc}}(M)$.

   (2) *Let X be an ∞-groupoid. Then $\mathrm{uTHH}(X) \simeq L(X)$ as cyclotomic spaces, where the free loop space has the unstable cyclotomic Frobenius given by the map $L(X) \to L(X)^{hC_p}$ induced by the pth-power map on S^1.*

   (3) *Let $\mathscr{C}$ be an ∞-category. Then $\Sigma_+^\infty \mathrm{uTHH}(\mathscr{C}) \simeq \mathrm{THH}(\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathrm{Sp})^\omega)$ as cyclotomic spectra.*

Proof. Part (1) was discussed in the last lecture. For part (2) we observe the following: Since X is an ∞-groupoid, we have

$$\mathrm{Fun}((n), X)^\sim \simeq \mathrm{Fun}((n), X) \simeq \mathrm{Fun}((n)^{\mathrm{gp}}, X) \simeq \mathrm{Fun}(S^1, X) \simeq \mathrm{Map}(S^1, X)$$

where $(-)^{\mathrm{gp}}$ is the left adjoint to the inclusion of ∞-groupoids in ∞-categories. Concretely, it is the a functor that inverts all morphisms in an ∞-category. Thus $\mathrm{uTHH}(X)$ is the geometric realization of the constant simplicial space

$$\mathrm{Map}(S^1, X) = L(X).$$

It is *not* constant as a cyclic space, and in fact, it turns out that the induced $\mathbb{T}$-action on the geometric realization is the natural one through S^1. For (3) the idea is as follows. We define a spectrally enriched category $\Sigma_+^\infty \mathscr{C}$ given by applying the functor Σ_+^∞ to all mapping spaces. Then, essentially by definition we obtain

$$\Sigma_+^\infty \mathrm{uTHH}(\mathscr{C}) \simeq \mathrm{THH}(\Sigma_+^\infty \mathscr{C}).$$

To obtain the claim, it hence suffices to prove that $\Sigma_+^\infty \mathscr{C}$ and $\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathrm{Sp})^\omega$ are Morita equivalent, i.e. have equivalent spectral presheaves. For this we simply calculate

$$\mathrm{Fun}^{\mathrm{ex}}((\Sigma_+^\infty \mathscr{C})^{\mathrm{op}}, \mathrm{Sp}) \simeq \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathrm{Sp})$$

and

$$\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathrm{Sp})^\omega, \mathrm{Sp}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathrm{Sp}), \mathrm{Sp})) \simeq \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathrm{Sp}) \otimes \mathrm{Sp}$$
$$\simeq \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathrm{Sp}).$$

where the tensor product denotes the symmetric monoidal structure on the ∞-category of stable presentable ∞-categories for which Sp is the unit. $\qquad\square$

Corollary 5. *For an $\mathbb{E}_1$-group M, we have*
$$\mathrm{THH}(\mathbb{S}[M]) \simeq \Sigma_+^\infty L(BM)$$
as cyclotomic spectra.

Proof. Since M is an $\mathbb{E}_1$-group, the category $\mathbb{B}M$ is in fact an ∞-groupoid and as such equivalent to BM, the classifying space of the grouplike $\mathbb{E}_1$-space M. Thus we obtain
$$\Sigma_+^\infty L(BM) \simeq \Sigma_+^\infty \mathrm{uTHH}(BM) \simeq \mathrm{THH}(\mathrm{Fun}(BM, \mathrm{Sp})^\omega).$$
It hence suffices to recall from the Schwede-Shipley theorem that
$$\mathrm{Fun}(BM, \mathrm{Sp}) \simeq \mathrm{Mod}_{\mathbb{S}[M]}$$
as the former is compactly generated and $\mathbb{S}[M]$ is the $\mathbb{E}_1$-spectrum of endomorphisms of a generator. $\qquad\square$

Making the cyclotomic Frobenius explicit gives a commutative diagram

$$
\begin{array}{ccc}
\mathrm{THH}(\mathbb{S}[M]) & \longrightarrow & \Sigma_+^\infty L(BM) \\
\downarrow & & \downarrow \\
\mathrm{THH}(\mathbb{S}[M])^{tC_p} \leftarrow \left(\Sigma_+^\infty L(BM)\right)^{tC_p} \leftarrow \left(\Sigma_+^\infty L(BM)\right)^{hC_p} \leftarrow \Sigma_+^\infty L(BM)^{hC_p}
\end{array}
$$

In other words, $\mathrm{THH}(\mathbb{S}[M])$ is an example of a p-cyclotomic spectrum with Frobenius lift:

Definition 6. A lift of the Frobenius on a p-cyclotomic spectrum (X, φ_p) is a 2-simplex in the ∞-category $\mathrm{Sp}^{B\mathbb{T}}$ whose boundary looks as follows

$$
\begin{array}{ccc}
X & \xrightarrow{\tilde{\varphi}_p} & X^{hC_p} \\
 & \searrow_{\varphi_p} & \downarrow \mathrm{can} \\
 & & X^{tC_p}.
\end{array}
$$

In particular, a lift of Frobenius $\tilde{\varphi}_p$ comes equipped with a homotopy between $\mathrm{can} \circ \tilde{\varphi}_p$ and φ_p.

We will now discuss two examples.

Example 7. *We aim to describe the spaces* $\mathrm{uTHH}(B\mathbb{N}) \simeq \mathrm{B}^{\mathrm{cyc}}(\mathbb{N})$ *and* $\mathrm{uTHH}(B\mathbb{Z}) \simeq \mathrm{B}^{\mathrm{cyc}}(\mathbb{Z})$. *As $\mathbb{Z}$ is a group, for the latter, we can use the equivalence*
$$\mathrm{uTHH}(B\mathbb{Z}) \simeq L(B\mathbb{Z}) \simeq L(S^1).$$
The map
$$\mathbb{Z} \times S^1 \longrightarrow L(S^1)$$
given by $(n,t) \mapsto (s \mapsto t \cdot s^n)$ is a homotopy equivalence and it is S^1-equivariant if we endow the space $\mathbb{Z} \times S^1$ with the action $\lambda \cdot (n,t) = (n, \lambda^n \cdot t)$.

In order to calculate $\mathrm{uTHH}(B\mathbb{N})$ we really use the equivalence to the cyclic bar construction and calculate the cyclic bar construction explicitly. It is easy to see

that $\mathrm{B}^{\mathrm{cyc}}(\mathbb{N})$ has components indexed by $\mathbb{N}$ and that the space indexed by 0 is simply a point. However, the space indexed by $n \geq 1$ turns out to be equivalent to S^1, and the map induced by the inclusion $\mathbb{N} \to \mathbb{Z}$ induces a homotopy equivalence on cyclic bar constructions when restricted to components indexed by $n \geq 1$. In pictures we obtain
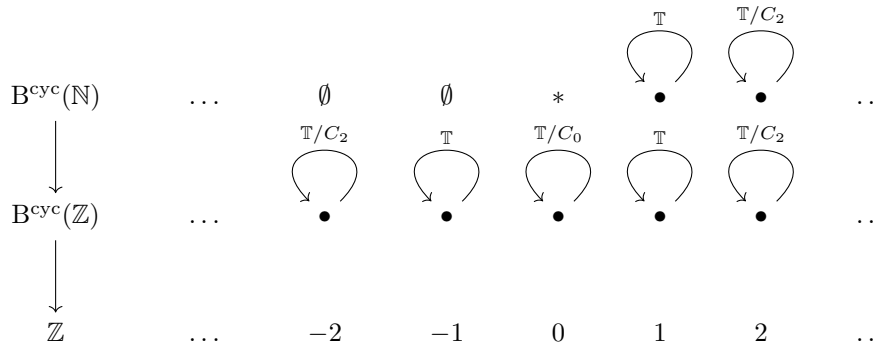
where $\bullet$ denotes the space S^1, and the $\mathbb{T}/C_n$ indicates that $\mathbb{T}$ acts on S^1 via the nth-power map. In short, we can summarise this as to read that

$$\mathrm{B}^{\mathrm{cyc}}(\mathbb{N}) = * \amalg \coprod_{n \geq 1} S^1/C_{|n|}$$

and likewise that

$$\mathrm{B}^{\mathrm{cyc}}(\mathbb{Z}) = \coprod_{n \in \mathbb{Z}} S^1/C_{|n|}$$

where S^1/C_0 is the space S^1 with trivial $\mathbb{T}$-action.

We will now prove the following version of the main theorem, compare [1, Proposition IV.3.4].

Theorem 8. *Let X be a bounded below, p-complete, p-cyclotomic spectrum equipped with a Frobenius lift $\tilde{\varphi}_p \colon X \to X^{hC_p}$ and let $\bar{\varphi}_p$ be the composite $X \to X^{hC_p} \to X$. Then there is a pullback diagram of the form*

$$
\begin{array}{ccc}
\mathrm{TC}(X) & \longrightarrow & \Sigma X_{h\mathbb{T}} \\
\downarrow & & \downarrow{\scriptstyle \mathrm{tr}} \\
X & \xrightarrow{\mathrm{id}-\bar{\varphi}_p} & X.
\end{array}
$$

Here, the map tr *is the $\mathbb{T}$-transfer and will be explained in the course of the proof.*

Notice that this theorem implies the main theorem, since the p-completion of $\mathrm{THH}(\mathbb{S}[M])$ is bounded below, p-complete, p-cyclotomic and equipped with a Frobenius lift by our previous results.

Proof of the Theorem. By definition of topological cyclic homology, we have a pull-back diagram

$$
\begin{array}{ccc}
\mathrm{TC}(X) & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
X^{h\mathbb{T}} \xrightarrow{\ \mathrm{id}-\tilde\varphi_p^{h\mathbb{T}}\ } X^{h\mathbb{T}} & \xrightarrow{\ \mathrm{can}^{h\mathbb{T}}\ } & (X^{tC_p})^{h\mathbb{T}}
\end{array}
$$

where we use the Frobenius lift. By patching of pullback diagrams, we may therefore first calculate the fibre of the map

$$
X^{h\mathbb{T}} \xrightarrow{\ \mathrm{can}^{h\mathbb{T}}\ } (X^{tC_p})^{h\mathbb{T}}.
$$

This will require some steps.

First, we claim that for any bounded below spectrum X with $\mathbb{T}$-action, the canonical map

$$
X^{t\mathbb{T}} \longrightarrow \lim_n X^{tC_{p^n}}
$$

is a p-completion map, compare [1, Lemma II.4.2]. Furthermore, if X is p-complete, then so is $X^{t\mathbb{T}}$ and thus this map is an equivalence. To prove this, we can reduce, by an induction over the Postnikov tower of X, to the case where $X = HM$, for an abelian group M on which $\mathbb{T}$ necessarily acts trivially. By a length two resolution of M we may further reduce to the case where M is torsion-free. Then we obtain that

$$
\pi_*(HM^{t\mathbb{T}}) = M[u^{\pm}]
$$

for $|u| = 2$. Similarly,

$$
\pi_*(HM^{tC_{p^n}}) = M/p^n[u^{\pm}]
$$

and the restriction map is the canonical projection. The claim then follows.

Next, as we have already seen in an earlier lecture, an iterative application of the Tate-orbit lemma shows that there is a canonical equivalence

$$
X^{tC_{p^n}} \xrightarrow{\ \simeq\ } (X^{tC_p})^{hC_{p^n}/C_p}
$$

see [1, Lemma II.4.1].

Combining with the previous equivalence, for a bounded below and p-complete spectrum X, we obtain an equivalence

$$
X^{t\mathbb{T}} \xrightarrow{\ \simeq\ } \lim_n (X^{tC_p})^{hC_{p^n}/C_p} \simeq (X^{tC_p})^{hC_{p^\infty}}\ .
$$

Recall that $C_{p^\infty} \subseteq \mathbb{T}$ is the subgroup of elements that are annihilated by a power of p. Lastly, we note that the canonical map

$$
BC_{p^\infty} \longrightarrow B\mathbb{T}
$$

is a p-adic equivalence: It is a nice exercise to see that this map induces an equivalence on mod p homology. Putting all together, it follows that for a bounded below and p-complete spectrum X, we obtain an equivalence

$$
X^{t\mathbb{T}} \xrightarrow{\ \simeq\ } (X^{tC_p})^{h\mathbb{T}}
$$

which makes the diagram

$$X^{h\mathbb{T}} \xrightarrow{\text{can}} X^{t\mathbb{T}}$$
$$\searrow_{\text{can}^{h\mathbb{T}}} \quad \downarrow{\simeq}$$
$$(X^{tC_p})^{h\mathbb{T}}$$

commute. Now we use that there is a natural fibre sequence

$$\Sigma X_{h\mathbb{T}} \xrightarrow{N} X^{h\mathbb{T}} \xrightarrow{\text{can}} X^{t\mathbb{T}}$$

The map N is the appropriate version of the norm map for compact Lie groups that are not necessarily finite: Recall that we have defined a norm map for finite groups in the first lecture. For compact Lie groups, the adjoint representation of the group enters to "twist" the homotopy orbits. In the case of the circle group $\mathbb{T}$, this adjoint representation is a trivial 1-dimensional representation, and so the domain of the norm map is given by $\Sigma X_{h\mathbb{T}}$. We refer to [1, Corollary I.4.3] for details and a proof of the above fibre sequence.

Summarizing the above, we obtain a pullback square as follows

$$\begin{array}{ccc} \mathrm{TC}(X) & \longrightarrow & \Sigma X_{h\mathbb{T}} \\ \downarrow & & \downarrow{N} \\ X^{h\mathbb{T}} & \xrightarrow{\text{id}-\tilde{\varphi}_p^{h\mathbb{T}}} & X^{h\mathbb{T}}. \end{array}$$

For a spectrum with $\mathbb{T}$ action X, we shall define the $\mathbb{T}$-transfer as the composite

$$\Sigma X_{h\mathbb{T}} \xrightarrow{N} X^{h\mathbb{T}} \longrightarrow X$$

where the latter map is induced by restricting homotopy orbits along the trivial subgroup of $\mathbb{T}$. Thus, to finish the proof of the theorem it suffices to argue that the diagram

$$\begin{array}{ccc} X^{h\mathbb{T}} & \xrightarrow{\text{id}-\tilde{\varphi}_p^{h\mathbb{T}}} & X^{h\mathbb{T}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\text{id}-\bar{\varphi}_p} & X \end{array}$$

is also a pullback.

We recall that X is a functor $B\mathbb{T} \to \mathrm{Sp}$ and that $\tilde{\varphi}_p$ can be viewed as a natural transformation

$$X \circ f_p \xrightarrow{\psi_p} X$$

where $f_p \colon B\mathbb{T} \to B\mathbb{T}$ is the map induced by the pth-power map on $\mathbb{T}$. In this language, the map $\tilde{\varphi}_p^{h\mathbb{T}}$ is given by

$$\lim_{B\mathbb{T}} X \xrightarrow{f_p^*} \lim_{B\mathbb{T}}(X \circ f_p) \xrightarrow{\psi_p} \lim_{B\mathbb{T}} X \ .$$

We will need a general principle for calculating limits indexed over spaces. For sake of clarity we formulate it as a lemma; its proof follows from a simple calculation in mapping spaces using the Yoneda lemma.

Lemma 9. *Let* $F\colon I \to \mathcal{S}$ *be a functor, and let* $X = \operatorname*{colim}_I F$ *be its colimit. Then,*

$$\operatorname{Fun}(X, \operatorname{Sp}) \simeq \lim_I \operatorname{Fun}(F(i), \operatorname{Sp})$$

and for each X*-parametrized spectrum* $\mathscr{F}\colon X \to \operatorname{Sp}$ *we have*

$$\lim_X \mathscr{F} \simeq \lim_{i \in I} \lim_{F(i)} \mathscr{F}_{|F(i)}.$$

Now we observe that $B\mathbb{T} \simeq \mathbb{CP}^\infty$, and that f_p can be modelled by raising all projective coordinates to the pth-power. In particular, f_p restricts to self-maps of $\mathbb{CP}^n$ for all $n \geq 0$. We use this to deduce that the map $\tilde{\varphi}_p^{h\mathbb{T}}$ is also given by inverse limit (over n) of the composite

$$\lim_{\mathbb{CP}^n} X \xrightarrow{\ f_p\ } \lim_{\mathbb{CP}^n}(X \circ f_p) \xrightarrow{\ \psi_p^n\ } \lim_{\mathbb{CP}^n} X$$

which we call $\tilde{\varphi}_p^n$. Thus in order to show that the diagram

$$
\begin{array}{ccc}
\displaystyle\lim_n \lim_{\mathbb{CP}^n} X & \xrightarrow{\ \mathrm{id} - \lim_n \tilde{\varphi}_p^n\ } & \displaystyle\lim_n \lim_{\mathbb{CP}^n} X \\
\downarrow & & \downarrow \\
X & \xrightarrow{\ \mathrm{id} - \bar{\varphi}_p\ } & X
\end{array}
$$

is a pullback it suffices to show that for each $n \geq 0$, the diagram

$$
\begin{array}{ccc}
\displaystyle\lim_{\mathbb{CP}^n} X & \xrightarrow{\ \mathrm{id} - \tilde{\varphi}_p^n\ } & \displaystyle\lim_{\mathbb{CP}^n} X \\
\downarrow & & \downarrow \\
X & \xrightarrow{\ \mathrm{id} - \bar{\varphi}_p\ } & X
\end{array}
$$

is a pullback. Performing an induction over n (the induction start $n = 0$ being trivial as $\tilde{\varphi}_p^0 = \bar{\varphi}_p$), it then suffices to show that the diagram

$$
\begin{array}{ccc}
\displaystyle\lim_{\mathbb{CP}^n} X & \xrightarrow{\ \mathrm{id} - \tilde{\varphi}_p^n\ } & \displaystyle\lim_{\mathbb{CP}^n} X \\
\downarrow & & \downarrow \\
\displaystyle\lim_{\mathbb{CP}^{n-1}} X & \xrightarrow{\ \mathrm{id} - \tilde{\varphi}_p^{n-1}\ } & \displaystyle\lim_{\mathbb{CP}^{n-1}} X
\end{array}
$$

is a pullback diagram. For this we observe that by the above lemma, the pushout of spaces

$$
\begin{array}{ccc}
S^{2n-1} & \longrightarrow & \mathbb{CP}^{n-1} \\
\downarrow & & \downarrow \\
* & \longrightarrow & \mathbb{CP}^n
\end{array}
$$

induces a pullback diagram of spectra

$$
\begin{array}{ccc}
\lim\limits_{\mathbb{CP}^n} X & \longrightarrow & \lim\limits_{\mathbb{CP}^{n-1}} X \\
\downarrow & & \downarrow \\
X & \longrightarrow & \mathrm{map}(S^{2n-1}_+, X)
\end{array}
$$

as X restricted to S^{2n-1} is trivial. It follows that there is a fibre sequence

$$
\Omega^{2n} X \longrightarrow \lim\limits_{\mathbb{CP}^n} X \longrightarrow \lim\limits_{\mathbb{CP}^{n-1}} X.
$$

We thus have a diagram

$$
\begin{array}{ccc}
\Omega^{2n} X & \longrightarrow \lim\limits_{\mathbb{CP}^n} X \longrightarrow & \lim\limits_{\mathbb{CP}^{n-1}} X \\
\downarrow{\scriptstyle \mathrm{id}-p^n\cdot f} & \downarrow{\scriptstyle \mathrm{id}-\tilde\varphi_p^n} & \downarrow{\scriptstyle \mathrm{id}-\tilde\varphi_p^{n-1}} \\
\Omega^{2n} X & \longrightarrow \lim\limits_{\mathbb{CP}^n} X \longrightarrow & \lim\limits_{\mathbb{CP}^{n-1}} X
\end{array}
$$

The fact that the induced map on fibres is of the claimed form follows from the fact that the map $f_p \colon \mathbb{CP}^n \to \mathbb{CP}^n$ has degree p^n. To prove the theorem it thus suffices to show that for each $n \geq 1$, the map

$$
\Omega^{2n} X \xrightarrow{\ \mathrm{id}-p^n\cdot f\ } \Omega^{2n} X
$$

is an equivalence. Now we claim in general, that a map of the form

$$
Z \xrightarrow{\ \mathrm{id}-p\alpha\ } Z
$$

for some self map α is a p-adic equivalence; from this the theorem follows as we assumed X to be p-complete. To show the claim, we need to show that it induces an equivalence after tensoring with $\mathbb{S}/p$, the mod p Moore spectrum. For this, consider the fibre sequence

$$
Z \xrightarrow{\ \cdot p\ } Z \longrightarrow Z/p \simeq Z \otimes \mathbb{S}/p
$$

and deduce from the long exact sequence in homotopy groups the short exact sequence

$$
0 \longrightarrow \mathrm{coker}(\cdot p_{|\pi_n(Z)}) \longrightarrow \pi_n(Z/p) \longrightarrow \ker(\cdot p_{|\pi_{n-1}(Z)}) \longrightarrow 0
$$

where the left hand and right hand side are p-torsion groups. It follows that $\mathrm{id} - p\alpha$ induces isomorphisms on $\mathrm{coker}(\cdot p_{|\pi_n(Z)})$ and $\ker(\cdot p_{|\pi_{n-1}(Z)})$ and thus by the 5-lemma also on $\pi_n(Z/p)$. $\qquad\square$

To finish, let us make the main theorem explicit for the trivial $\mathbb{E}_1$-group. We obtain that there is a pullback diagram of p-complete spectra

$$
\begin{array}{ccc}
\mathrm{TC}(\mathbb{S}) & \longrightarrow & \Sigma\mathbb{S}_{h\mathbb{T}} \\
\downarrow & & \downarrow{\scriptstyle\mathrm{tr}} \\
\mathbb{S} & \xrightarrow{\ 0\ } & \mathbb{S}.
\end{array}
$$

The fibre of the (shifted) $\mathbb{T}$-transfer $\mathrm{tr}\colon \mathbb{S}_{h\mathbb{T}} \to \Sigma^{-1}\mathbb{S}$ is often referred to as $\mathbb{P}^\infty_{-1}(\mathbb{C})$. It is a spectrum with one cell in each even dimension ≥ -2. Putting this all together, we obtain the famous p-adic equivalence

$$
\mathrm{TC}(\mathbb{S}) \simeq \mathbb{S} \oplus \Sigma\mathbb{P}^\infty_{-1}(\mathbb{C}).
$$

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The Cyclotomic Trace

DAVID GEPNER

Although it can be defined more generally,[1] the algebraic K-theory functor is quite naturally regarded as a lax symmetric monoidal functor

$$
K : \mathrm{Cat}^{\mathrm{st}}_\infty \to \mathrm{Sp}
$$

from the ∞-category of small stable ∞-categories and exact functors to the ∞-category of spectra. While there are a number of constructions which produce this and related functors, the two that will be most relevant to this talk are Waldhausen's $S_\bullet$ construction, which directly constructs a spectrum from a stable ∞-category $\mathcal{C}$ by splitting sequences of morphisms in $\mathcal{C}$, and the theory of noncommutative motives over the sphere, which constructs a large symmetric monoidal stable ∞-category such that the spectrum of maps from the unit (the motive associated to the sphere) to the motive associated to an arbitrary small stable ∞-category $\mathcal{C}$ recovers the algebraic K-theory of $\mathcal{C}$.

We begin by recalling the ∞-categorical version of Waldhausen's $S_\bullet$-construction. Given an arbitrary ∞-category $\mathcal{C}$, we have the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$ of arrows of $\mathcal{C}$. In particular, $\mathrm{Fun}(\Delta^1, \Delta^n)$ is equivalent to the nerve of the partially ordered set $\{i,j\}_{0 \leq i \leq j \leq n}$. Hence an object of $\mathrm{Fun}(\mathrm{Fun}(\Delta^1, \Delta^n), \mathcal{C})$ consists of objects $A_{i,j}$ of $\mathcal{C}$ for each $0 \leq i \leq j \leq n$, maps $A_{i,j} \to A_{i',j'}$ for $i \leq i'$ and $j \leq j'$, etc. If $\mathcal{C}$ is a stable ∞-category, we write $S_n(\mathcal{C}) \subset \mathrm{Fun}(\mathrm{Fun}(\Delta^1, \Delta^n), \mathcal{C})$ for the

---

[1] For instance, in the setting of exact ∞-categories or even Waldhausen ∞-categories.

full subcategory consisting of those objects $A_{i,j}$ such that $A_{i,j} \simeq 0$ whenever $i = j$ and each square

$$
\begin{array}{ccc}
A_{i,j} & \longrightarrow & A_{i,j+1} \\
\downarrow & & \downarrow \\
A_{i+1,j} & \longrightarrow & A_{i+1,j+1}
\end{array}
$$

is (co)cartesian. Note that $S_0(\mathcal{C}) \simeq \Delta^0$, $S_1(\mathcal{C}) \simeq \mathcal{C}$, and more generally $S_n(\mathcal{C}) \simeq \mathrm{Fun}(\Delta^{n-1}, \mathcal{C})$. Also note that, letting n vary, we obtain a simplicial stable ∞-category $S_\bullet(\mathcal{C})$, and that the three maps $\mathrm{Fun}(\Delta^1, \mathcal{C}) \simeq S_2(\mathcal{C}) \to S_1(\mathcal{C}) \simeq \mathcal{C}$ send a morphism $f : A \to B$ to A, B, and the cofiber C of f, respectively.

Passing the to maximal subgroupoid $(S_\bullet(\mathcal{C}))^\sim$ yields a simplicial space whose geometric realization $|(S_\bullet(\mathcal{C}))^\sim|$ comes equipped with a natural map $\Sigma((\mathcal{C})^\sim) \to |(S_\bullet(\mathcal{C}))^\sim|$ of spaces obtained by restriction to the 1-skeleton. Finally, observe that each $S_n(\mathcal{C})$ is itself a stable ∞-category, so that we may iterate the $S_\bullet$ construction to form a multisimplicial space $(S_\bullet^n(\mathcal{C}))^\sim$, and we will[2] write $|(S_\bullet^n(\mathcal{C}))^\sim|$ for its (iterated) geometric realization.

Definition 1. The algebraic K-theory of $\mathcal{C}$ is the spectrum associated to the pre-spectrum $\{K(\mathcal{C})^n\}_{n \in \mathbb{N}}$ given by $K(\mathcal{C})^n = |(S_\bullet^n(\mathcal{C}))^\sim|$, with structure maps induced by the natural transformation $\Sigma((-)^\sim) \to |(S_\bullet(-))^\sim|$ described above.

Remark 2. It turns out that this is surprisingly close to a spectrum: the maps $K(\mathcal{C})^n \to \Omega K(\mathcal{C})^{n+1}$ are already equivalences for $n > 0$. In particular, $\Omega^\infty K(\mathcal{C}) \simeq \Omega |S_\bullet(\mathcal{C})^\sim|$. This follows from Waldhausen's Additivity Theorem.

The aforementioned map $\mathcal{C}^\sim \to \Omega^\infty K(\mathcal{C}) \simeq \Omega |(S_\bullet(\mathcal{C}))^\sim|$ induces a map on π_0 such that, if $A \to B \to C$ is a (co)fiber sequence in $\mathcal{C}$, then $[B] = [A] + [C] \in \pi_0 K(\mathcal{C})$. This, together with the higher homotopy coherences which occur as a result of forming loop space of the geometric realization, make rigorous the intuitive notion that the algebraic K-theory "splits exact sequences". There is a categorification of this notion which leads to the theory of noncommutative motives.

Definition 3. A functor $F : \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Sp}$ is said to be localizing (respectively, additive) if F preserves filtered colimits and sends localization sequences (respectively, additivity sequences) in $\mathrm{Cat}_\infty^{\mathrm{st}}$ to (co)fiber sequences of spectra.

Let $\mathrm{Pre}_{\mathrm{Sp}}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{st}})$ denote the additive localization of the presentably symmetric monoidal stable ∞-category of spectral presheaves on small stable ∞-categories; that is,

$$\mathrm{Pre}_{\mathrm{Sp}}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{st}}) \subset \mathrm{Pre}_{\mathrm{Sp}}((\mathrm{Cat}_\infty^{\mathrm{st}})^\omega)$$

is the full subcategory of those functors $X : (\mathrm{Cat}_\infty^{\mathrm{st}})^\omega)^{\mathrm{op}} \to \mathrm{Sp}$ such that, for any additivity sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ in $\mathrm{Cat}_\infty^{\mathrm{st}}$, $X(\mathcal{C}) \to X(\mathcal{B}) \to X(\mathcal{A})$ is a (co)fiber

---

[2]Unambiguously, as the n-fold product $(\Delta^{\mathrm{op}})^{\times n}$ of the simplicial indexing category with itself is sifted, by [6, Lemma 5.5.8.4].

sequence. The stable Yoneda embedding

$$\Sigma_+^\infty \mathrm{Map}(-,-) : \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Pre}_{\mathrm{Sp}}((\mathrm{Cat}_\infty^{\mathrm{st}})^\omega)$$

followed by the additive localization determines a functor

$$M_{\mathrm{add}} : \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Pre}_{\mathrm{Sp}}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{st}})$$

which sends $\mathcal{C}$ to its additive noncommutative motive $M_{\mathrm{add}}(\mathcal{C})$. When $\mathcal{C} = \mathrm{Perf}_R$ is the stable ∞-category of perfect R-modules for some ring spectrum R, we will often write $M_{\mathrm{add}}(R)$ in place of $M_{\mathrm{add}}(\mathrm{Perf_R})$.

Theorem 4. [2] *For any small stable ∞-category $\mathcal{C}$, the spectrum of maps from $M_{\mathrm{add}}(\mathbb{S})$ to $M_{\mathrm{add}}(\mathcal{C})$ in $\mathrm{Pre}_{\mathrm{Sp}}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{st}})$ is canonically equivalent to $K(\mathcal{C})$. Moreover, if $F : \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Sp}$ is any additive functor then the spectrum of natural transformations of additive functors $K \to F$ is equivalent to $F(\mathbb{S})$.*

Corollary 5. [3] *The set of equivalence classes of maps from K to THH is equivalent to $\pi_0 \mathrm{THH}(\mathbb{S}) \cong \pi_0 \mathbb{S} \cong \mathbb{Z}$. The set of equivlaence classes of lax symmetric monoidal maps from K to THH is a singleton set (corresponding to the unit $1 \in \mathbb{Z}$). In particular, the Dennis trace map is the unique lax symmetric monoidal functor $K \to \mathrm{THH}$.*

Unfortunately, TC is neither additive nor localizing; rather, it is pro-additive and pro-localizing, in the sense that $\mathrm{TC} \simeq \lim_n \mathrm{TC}^n$ and each TC^n is localizing and therefore additive as well. Similar calculations apply to calculating spaces of natural transformations from K-theory to each TC^n, which therefore assemble to the cyclotomic trace map $K \to \mathrm{TC}$, which again is the unique lax symmetric monoidal natural transformation. Our goal for the remainder of this talk is to construct a cyclotomic analogue of K-theory, called cyclic K-theory, which comes equipped with a map of cyclotomic spectra to THH, and therefore refines the cyclotomic trace map. For this, we need the notion of an additive trace theory.

The ∞-category Λ^{st} is obtained as a hybrid of $\mathrm{Cat}_\infty^{\mathrm{st}}$ and Λ, Connes' cyclic category, by decorating $\mathbb{T}_n$, the circle with n marked points, $n > 0$, with stable ∞-categories X_i, $1 \le i \le n$, and the edges connecting these marked points with bimodules (a.k.a. correspondences, distributors, profunctors, etc.)

$$F_i \in \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}(X_{i+1}), \mathrm{Ind}(X_i)).$$

As the geometry suggests, we work mod n, so in particular $X_0 = X_n$ and $X_1 = X_{n+1}$, etc., by definition. It sits over Λ^{op} as a cocartesian fibration such that the fiber of Λ^{st} over $\mathbb{T}_n$ is the ∞-category of stable flat fibrations over $\mathbb{T}_n$.

The (co)cartesian morphisms of Λ^{st} are of three types, corresponding to the generators of Λ^{op}: rotation, contraction, and insertions. The rotation morphisms simply reindex the indexing ∞-categories X_i and bimodules F_i, mod n, while the contraction morphisms act by tensor product of bimodules (equivalently, composition of functors after applying Ind), and the insertion morphisms act by repeating an object X_i and inserting the identity functor of X_i, viewed as a bimodule.

Definition 6. A trace theory is a functor $T : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ which sends cocartesian morphisms in Λ^{st} to equivalences.

As might be expected, THH refines to a trace theory THH : $\mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Sp}$ via the formula
$$\mathrm{THH}(F_1, \ldots, F_n) := |B_\bullet^{\mathrm{cyc}}(F_1, \ldots, F_n)|,$$
where the simplicial structure comes from the functor $\Delta \to \Lambda$ given on objects by $[n] \mapsto \mathbb{T}_{n+1}$. However, there is a still more universal example, namely cyclic K-theory. In order to construct cyclic K-theory, we must first consider the K-theory of endomorphisms.

Definition 7. Let End : $\mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Cat}_\infty^{\mathrm{st}}$ be the endofunctor of $\mathrm{Cat}_\infty^{\mathrm{st}}$ given by $\mathrm{End}(X) = \mathrm{Fun}(\Delta^1/\partial\Delta^1, X)$. The endomorphism K-theory functor
$$K^{\mathrm{end}} = K \circ \mathrm{End} : \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Sp}$$
is the composite of the endomorphism functor followed by the K-theory functor.

If $X = \mathrm{Perf}_R$ for R a connective ring spectrum, then
$$\pi_0 K^{\mathrm{end}}(R) \simeq \pi_0 K(R) \oplus \mathbb{W}_{\pi_0 R}^{\mathrm{rat}},$$
where $\mathbb{W}_{\pi_0 R}^{\mathrm{rat}} \subset \mathbb{W}_{\pi_0 R}$, the so-called rational Witt ring, is the dense subring of the Witt vectors $\mathbb{W}_{\pi_0 R}$ of $\pi_0 R$ consisting, roughly, of those power series which arise as rational functions (see [1] and [4] for details). More or less by construction, K^{end} refines to a functor $K^{\mathrm{end}} : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$.

Recall that we have an equivalence
$$\mathrm{End}(F_1, \ldots, F_n)^\sim \simeq \operatorname*{colim}_{(x_1, \ldots, x_n) \in (X_1 \times \cdots \times X_n)^\sim} \prod_{i=1}^n \mathrm{Map}(x_i, F_i x_{i+1}),$$
so that $E(F_1, \ldots, F_n) := \Sigma_+^\infty \mathrm{End}(F_1, \ldots, F_n)^\sim$ is given by the formula
$$E(F_1, \ldots, F_n) \simeq \operatorname*{colim}_{(x_1, \ldots, x_n) \in (X_1 \times \cdots \times X_n)^\sim} \bigotimes_{i=1}^n \Sigma_+^\infty \mathrm{Map}(x_i, F_i x_{i+1}).$$

By construction, E is a functor from Λ^{st} to spectra; moreover, it is universal in the following precise sense.

Theorem 8. *E is the initial fiberwise lax symmetric monoidal functor $\Lambda^{\mathrm{st}} \to \mathrm{Sp}$.*

Theorem 9. *There is a canonical equivalence $E^{\mathrm{add}} \simeq K^{\mathrm{end}}$ between the additivization of E and endomorphism K-theory.*

Theorem 10. *K^{end} is the initial fiberwise lax symmetric monoidal additive functor $\Lambda^{\mathrm{st}} \to \mathrm{Sp}$.*

The set of equivalence classes of natural transformations of additive functors $K \to K^{\mathrm{end}}$ is canonically isomorphic to $\pi_0 K^{\mathrm{end}}(\mathbb{S}) \cong \mathbb{Z} \oplus \mathbb{W}_{\mathbb{Z}}^{\mathrm{rat}}$. As a product of rings, we have units $1 \in \mathbb{Z}$ and $1 \in \mathbb{W}_{\mathbb{Z}}^{\mathrm{rat}}$, and one checks fairly easily that the natural transformation corresponding to the unit $1 \in \mathbb{Z}$ is induced by the natural transformation Id $\to$ End : $\mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Cat}_\infty^{\mathrm{st}}$ given by the zero endomorphism,

whereas the natural transformation corresponding to the unit $1 \in \mathbb{W}_{\mathbb{Z}}^{\mathrm{rat}}$ is induced by the natural transformation $\mathrm{Id} \to \mathrm{End} : \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Cat}_\infty^{\mathrm{st}}$ given by the identity endomorphism.

Definition 11. The cyclic K-theory functor $K^{\mathrm{cyc}} : \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Sp}$ is the cofiber of the zero endomorphism inclusion $K \to K^{\mathrm{end}}$.

Proposition 12. K^{cyc} *extends to a functor* $\Lambda^{\mathrm{st}} \to \mathrm{Sp}$ *equipped with a natural transformation* $K^{\mathrm{end}} \to K^{\mathrm{cyc}}$ *which exhibits* K^{cyc} *as the initial reduced (a.k.a. pointed) functor under* K^{end}.

Corollary 13. K^{cyc} *is the initial fiberwise lax symmetric monoidal reduced additive functor* $\Lambda^{\mathrm{st}} \to \mathrm{Sp}$.

Proposition 14. K^{cyc} *is a trace theory.*

The proof is an elaboration of the following argument. Let X_1 and X_2 be small stable ∞-categories equipped with colimit preserving functors $F_1 : X_2 \to X_1$ and $F_2 : X_1 \to X_2$, and consider the following commutative diagram

$$
\begin{array}{ccccc}
K(X_2) & \longrightarrow & K(X_1 \times X_2) & \longrightarrow & K(X_1) \\
\downarrow & & \downarrow & & \downarrow \\
K(X_2) & \longrightarrow & K^{\mathrm{end}}(F_1, F_2) & \longrightarrow & K^{\mathrm{end}}(F_1 F_2) \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K^{\mathrm{cyc}}(F_1, F_2) & \longrightarrow & K^{\mathrm{cyc}}(F_1 F_2)
\end{array}
$$

in which the top left vertical map is an equivalence. It follows that the top right square is (co)cartesian, and therefore that the bottom right horizontal map is an equivalence, so that contraction cocartesian morphisms are sent to equivalences.

Proposition 15. $\mathrm{THH} : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ *is an additive trace theory.*

Corollary 16. *There is a morphism of additive trace theories* $K^{\mathrm{cyc}} \to \mathrm{THH}$.

This is the highly structured version of the Dennis trace map, and we recover the cyclotomic trace by applying TC. Of course, TC(THH) recovers ordinary topological cyclic homology, but in the case of K^{cyc}, which admits Frobenius lifts, we can apply $\widetilde{\mathrm{TC}}$ instead, as defined and studied in [5]. That is,

$$
\widetilde{\mathrm{TC}}_p(K^{\mathrm{cyc}}(X)) \simeq \mathrm{fib}\left(K^{\mathrm{cyc}}(X)^{h\mathbb{T}} \xrightarrow{1 - \tilde{\varphi}_p} K^{\mathrm{cyc}}(X)^{h\mathbb{T}} \right).
$$

Since the algebraic K-theory functor factors through this fiber, we obtain the cyclotomic trace

$$
K(X) \to \widetilde{\mathrm{TC}}_p(K^{\mathrm{cyc}}(X)) \to \mathrm{TC}(\mathrm{THH}(X)) \simeq \mathrm{TC}(X)
$$

as the composite. This is a natural transformation of functors from small stable ∞-categories to spectra.

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The theorem of Dundas-Goodwillie-McCarthy

Sam Raskin

The purpose of this talk was to outline the proof of the following theorem, following [1].

Theorem 1 (Dundas-Goodwillie-McCarthy). *For $B \to A$ a morphism of connective $\mathbb{E}_1$-rings such that $\pi_0(B) \to \pi_0(A)$ is surjective with kernel a nilpotent ideal, the diagram of spectra induced by the cyclotomic trace map:*

$$
\begin{array}{ccc}
K(B) & \longrightarrow & \mathrm{TC}(B) \\
\downarrow & & \downarrow \\
K(A) & \longrightarrow & \mathrm{TC}(A)
\end{array}
$$

is Cartesian.

(This result is true with K-theory indicating either connective or non-connective algebraic K-theory. In what follows, we use K-theory to denote connective K-theory.)

Roughly, the proof of Theorem 1 proceeds as follows. First, one shows that the cyclotomic trace map induces an isomorphism on *Goodwillie derivatives*. In fact, the Goodwillie derivatives of both sides identify with the suspension of topological Hochschild homology.

Then one reduces the general case of Theorem 1 to this "linearized" version, and certain *structural features* (meaning commutation with certain colimits, etc.) of K-theory and TC.

One remarkable feature is how asymmetrically K-theory and TC are studied here. K-theory is treated fairly abstractly, while problems about TC are essentially reduced to an explicit calculation in the split square-zero case (Theorem 16).

Below, we will explain the reduction to the calculation of derivatives plus structural features. Then we formulate what is needed about K-theory and TC and indicate the key ideas.

Throughout this note, all language should be understood in the homotopical sense. So *category* means ∞-category, limits and colimits are homotopy limits and colimits, etc. We use cohomological indexing throughout.

<center>GOODWILLIE CALCULUS</center>

The Goodwillie derivative. Suppose $\psi : \mathscr{C} \to \mathscr{D}$ is a functor between cocomplete stable categories commuting with sifted colimits (i.e., colimits of filtered and simplicial diagrams). The *Goodwillie derivative* $\partial \psi : \mathscr{C} \to \mathscr{D}$ of ψ is initial among continuous exact functors receiving a natural transformation from ψ.

Explicitly, $\partial \psi$ is calculated as follows.

First, observe that for any $\mathscr{F} \in \mathscr{C}$, the morphisms $0 \to \mathscr{F} \to 0$ give a functorial direct sum decomposition $\psi(\mathscr{F}) = \psi_{red}(\mathscr{F}) \oplus \psi(0)$. Then ψ_{red} is *reduced*, i.e., it takes $0 \in \mathscr{C}$ to $0 \in \mathscr{D}$.

Then there is a canonical natural transformation $\Sigma \circ \psi_{red} \to \psi_{red} \circ \Sigma$, or equivalently, $\psi_{red} \to \Omega \psi_{red} \Sigma$. Finally, we have:

$$\partial \psi = \operatorname{colim}\left(\psi_{red} \to \Omega \psi_{red} \Sigma \to \Omega^2 \psi_{red} \Sigma^2 \to \dots \right)$$

We will actually use this construction in slightly more generality. Suppose $\mathscr{C}$ is equipped with a t-structure compatible with filtered colimits and we are given $\psi : \mathscr{C}^{\leq 0} \to \mathscr{D}$ commuting with sifted colimits. Then there is again a functor $\partial \psi : \mathscr{C} \to \mathscr{D} \in \mathsf{StCat}_{cont}$ initial among functors commuting with colimits and receiving a natural transformation $\psi \to (\partial \psi)|_{\mathscr{C}^{\leq 0}}$.

To construct $\partial \psi$ in this setup, note that ψ_{red} makes sense as before, and then one has:

$$\partial \psi(\mathscr{F}) = \operatorname*{colim}_{m} \operatorname*{colim}_{n \geq m} \Omega^n \psi_{red}(\Sigma^n \tau^{\leq m} \mathscr{F})$$

If the t-structure on $\mathscr{C}$ is right complete and $\psi : \mathscr{C} \to \mathscr{D}$ commutes with sifted colimits, then $\partial \psi$ in the previous sense coincides with $\partial(\psi|_{\mathscr{C}^{\leq 0}})$.

Notation. We let Alg denote the category of $\mathbb{E}_1$-algebras. We let $\mathsf{Alg}_{conn} \subset \mathsf{Alg}$ denote the subcategory of connective $\mathbb{E}_1$-algebras. For $A \in \mathsf{Alg}$ and M an A-bimodule, we let $A \oplus M \in \mathsf{Alg}$ denote the *split square-zero extension* of A by M (whose underlying spectrum is $A \oplus M$).

The following result is a first approximation to the main result of this section.

Theorem 2. *Let $\Psi : \mathsf{Alg}_{conn} \to \mathsf{Sp}$ be a functor. Suppose that for every $A \in \mathsf{Alg}_{conn}$, the functor:*

$$\Psi_A : A\text{--}\mathsf{bimod}^{\leq 0} \to \mathsf{Sp}$$
$$M \mapsto \Psi(A \oplus M)$$

commutes with sifted colimits, has vanishing Goodwillie derivative, and has underlying reduced functor $\Psi_{A,red}$ mapping A--$\mathsf{bimod}^{\leq 0}$ to $\mathsf{Sp}^{\leq 0}$.

Then for every $A \in \mathsf{Alg}_{conn}$ and $M \in A$--$\mathsf{bimod}^{\leq -1}$, the map $\Psi(A \oplus M) \to \Psi(A)$ is an isomorphism.

A categorical variant. We will deduce Theorem 2 from the following result.

Theorem 3. *Let $\mathscr{C}$ and $\mathscr{D}$ be cocomplete stable categories equipped with t-structures compatible with filtered colimits, and suppose the t-structure on $\mathscr{D}$ is left separated. Let $\psi : \mathscr{C}^{\leq 0} \to \mathscr{D}^{\leq 0}$ be a reduced functor that commutes with sifted colimits.*

Suppose that $\partial(\psi(\mathscr{F} \oplus -)) = 0$ for every $\mathscr{F} \in \mathscr{C}^{\leq -1}$. Then $\psi(\mathscr{F}) = 0$ for every $\mathscr{F} \in \mathscr{C}^{\leq -1}$.

Proof that Theorem 3 implies Theorem 2. Fix $A \in \mathsf{Alg}_{conn}$ and define $\psi : A\text{--}\mathsf{bimod}^{\leq 0} \to \mathsf{Sp}$ as $\Psi_{A,red}$. We claim that the hypotheses of Theorem 3 are satisfied.

The only non-tautological point is that for $M \in A\text{--}\mathsf{bimod}^{\leq -1}$ (in fact, even $M \in A\text{--}\mathsf{bimod}^{\leq 0}$), the Goodwillie derivative of $\psi(M \oplus -)$ is zero. For $M = 0$, this is an assumption. In general, we have:

$$\psi(M \oplus -) \oplus \Psi(A) = \Psi(A \oplus (M \oplus -)) = \Psi((A \oplus M) \oplus -).$$

The latter functor has vanishing derivative by the hypothesis that the Goodwillie derivative of $\Psi_{A \oplus M}$ vanishes.

$\square$

The bilinear obstruction to linearity. To prove Theorem 3, it is convenient to use the following construction.

For $\mathscr{F}, \mathscr{G} \in \mathscr{C}^{\leq 0}$, define $B_\psi(\mathscr{F}, \mathscr{G})$ to be the natural summand $\psi(\mathscr{F} \oplus \mathscr{G}) = \psi(\mathscr{F}) \oplus \psi(\mathscr{G}) \oplus B_\psi(\mathscr{F}, \mathscr{G})$.

Note that ψ commutes with all colimits if and only if it commutes with pairwise direct sums (because ψ is reduced and commutes with sifted colimits). Therefore, B_ψ may be understood as the obstruction to ψ commuting with all colimits.

Simplicial review. Suppose $\mathscr{F}_\bullet$ is a simplicial object in $\mathscr{C}$. We let $|\mathscr{F}_\bullet|$ denote the *geometric realization* of this simplicial diagram, i.e., the colimit. Similarly, let $|\mathscr{F}_\bullet|_{\leq n}$ denote the partial geometric realization $\operatorname*{colim}_{\mathbf{\Delta}^{op}_{\leq n}} \mathscr{F}_\bullet$, were $\mathbf{\Delta}_{\leq n} \subset \mathbf{\Delta}$ is the full subcategory of simplices of order $\leq n$.

We recall:

Lemma 4. *For $n \geq 0$, $\operatorname{Coker}(|\mathscr{F}_\bullet|_{\leq n} \to |\mathscr{F}_\bullet|_{\leq n+1})$ is a direct summand of $\mathscr{F}_{n+1}[n+1]$.*

Our main technique is now the following.

Lemma 5. *Suppose $\psi : \mathscr{C}^{\leq 0} \to \mathscr{D}$ commutes with sifted colimits. For every $\mathscr{F} \in \mathscr{C}$, $\psi(\Sigma\mathscr{F})$ admits an increasing filtration $\mathrm{fil}_\bullet \psi(\Sigma\mathscr{F})$ such that:*

- *$\mathrm{fil}_i \psi(\Sigma\mathscr{F}) = 0$ for $i < 0$.*
- *For $i \geq 0$, $\mathrm{gr}_i \psi(\Sigma\mathscr{F})$ is a direct summand of $\psi(\mathscr{F}^{\oplus i})[i]$.*
- *More precisely, $\mathrm{gr}_0 \psi(\Sigma\mathscr{F}) = \psi(0)$, $\mathrm{gr}_1(\psi(\Sigma\mathscr{F})) = \psi_{red}(\mathscr{F})[1]$, and $\mathrm{gr}_2(\psi(\Sigma\mathscr{F})) = B_{\psi_{red}}(\mathscr{F}, \mathscr{F})[2]$.*

Proof. There is a canonical simplicial diagram:

$$\ldots \mathscr{F} \oplus \mathscr{F} \underset{\Rightarrow}{\overset{\Rightarrow}{\Rightarrow}} \mathscr{F} \rightrightarrows 0$$

with geometric realization $\Sigma \mathscr{F}$. (For example, this simplicial diagram is the Cech construction for $0 \to \Sigma \mathscr{F}$.)

Because ψ commutes with geometric realizations, we have:

$$\psi(\Sigma \mathscr{F}) = |\psi(\mathscr{F}^{\oplus \bullet})|.$$

We then set $\mathrm{fil}_i \, \psi(\Sigma \mathscr{F}) = |\psi(\mathscr{F}^{\oplus \bullet})|_{\leq i}$. This filtration tautologically satisfies the first property, and it satisfies the second property by Lemma 4. The third property follows by refining Lemma 4 to identify exactly which summand occurs, which we omit here.

$\square$

By a clear inductive argument, we obtain:

Corollary 6. *Suppose that $\psi : \mathscr{C}^{\leq 0} \to \mathscr{D}^{\leq 0}$ commutes with sifted colimits. Then for every $n \geq 0$, $\psi(\mathscr{C}^{\leq -n}) \subset \mathscr{D}^{\leq -n}$.*

Proof of Theorem 3. We will show by induction on n that these hypotheses on ψ force $\psi(\mathscr{C}^{\leq -1}) \subset \mathscr{D}^{\leq -n}$. The case $n = 1$ is given by Corollary 6. In what follows, we assume the inductive hypothesis for n and deduce it for $n + 1$.

First, we claim that for $\mathscr{F} \in \mathscr{C}^{\leq -1}$, the functor $B_\psi(\mathscr{F}, -)[-1] : \mathscr{C}^{\leq 0} \to \mathscr{D}$ satisfies the hypotheses of Theorem 3. Clearly this functor is reduced and commutes with sifted colimits.

Let us show that this functor maps $\mathscr{C}^{\leq 0}$ into $\mathscr{D}^{\leq 0}$. Fix $\mathscr{G} \in \mathscr{C}^{\leq 0}$. The reduced functor $B_\psi(-, \mathscr{G})$ commutes with sifted colimits and maps $\mathscr{C}^{\leq 0}$ into $\mathscr{D}^{\leq 0}$. By Corollary 6, $B_\psi(\mathscr{F}, \mathscr{G}) \in \mathscr{D}^{\leq -1}$, so $B_\psi(\mathscr{F}, \mathscr{G})[-1] \in \mathscr{D}^{\leq 0}$ as desired.

Finally, note that for any $\mathscr{F}' \in \mathscr{C}^{\leq -1}$, the functor $\Omega B_\psi(\mathscr{F}, \mathscr{F}' \oplus -) : \mathscr{C}^{\leq 0} \to \mathscr{D}$ has vanishing Goodwillie derivative, as it is a summand of the functor $\psi(\mathscr{F} \oplus \mathscr{F}' \oplus -)[-1]$.

Therefore, we may apply the inductive hypothesis to this functor. We obtain that $B_\psi(\mathscr{F}, -)$ maps $\mathscr{C}^{\leq -1}$ to $\mathscr{D}^{\leq -n-1}$. In particular, $B_\psi(\mathscr{F}, \mathscr{F}) \in \mathscr{D}^{\leq -n-1}$.

Next, we claim that $\mathrm{Coker}(\Sigma \psi(\mathscr{F}) \to \psi(\Sigma \mathscr{F})) \in \mathscr{D}^{\leq -n-3}$. Note that by the construction of Lemma 5, the map:

$$\Sigma \psi(\mathscr{F}) = \mathrm{gr}_1 \, \psi(\Sigma \mathscr{F}) = \mathrm{fil}_1 \, \psi(\Sigma \mathscr{F}) \to \psi(\Sigma \mathscr{F})$$

is the canonical map used in the definition of the Goodwillie derivative. Therefore, it suffices to show that $\mathrm{gr}_i \, \psi(\Sigma \mathscr{F}) \in \mathscr{D}^{\leq -n-3}$ for $i \geq 2$.

By induction, $\psi(\mathscr{F}^i) \in \mathscr{D}^{\leq -n}$ by induction. Therefore, $\mathrm{gr}_i \, \psi(\Sigma \mathscr{F}) \in \mathscr{D}^{\leq -n-i}$. This gives the claim for $i \geq 3$. If $i = 2$, then $\mathrm{gr}_i \, \psi(\Sigma \mathscr{F}) = B_\psi(\mathscr{F}, \mathscr{F})[2]$, and the claim follows from the above. We deduce that for $\mathscr{F} \in \mathscr{C}^{\leq -1}$, the map $\psi(\mathscr{F}) \to \Omega \psi(\Sigma \mathscr{F})$ is an isomorphism on H^{-n}.

We obtain $H^{-n}(\psi(\mathscr{F})) \xrightarrow{\simeq} H^{-n}(\partial \psi(\mathscr{F}))$. But of course, $\partial \psi = 0$, so we obtain $\psi(\mathscr{F}) \in \mathscr{D}^{\leq -n-1}$, providing the inductive step.

Further vanishing results. We now wish to extend the above results to give vanishing for connective objects. In what follows, $\mathscr{C}, \mathscr{D} \in \mathsf{StCat}_{cont}$ are equipped with t-structures compatible with filtered colimits.

Definition 7. (1) A functor $\psi : \mathscr{C}^{\leq 0} \to \mathscr{D}^{\leq 0}$ is *extensible* if there exists $\widetilde{\psi} : \mathscr{C}^{\leq 1} \to \mathscr{D}^{\leq 1}$ commuting with sifted colimits with $\widetilde{\psi}|_{\mathscr{C}^{\leq 0}} = \psi$.

(2) A functor $\psi : \mathscr{C}^{\leq 0} \to \mathscr{D}^{\leq 0}$ is *pseudo-extensible* if:
- ψ is reduced and commutes with sifted colimits.
- Every φ in $\mathscr{S}_\psi$ maps $\mathscr{C}^{\leq 0} \to \mathscr{D}^{\leq 0}$, where $\mathscr{S}_\psi \subset \mathsf{Hom}(\mathscr{C}, \mathscr{D})$ is the minimal subgroupoid such that $\psi \in \mathscr{S}_\psi$ and such that for every $\mathscr{F} \in \mathscr{C}^{\leq 0}$ and $\varphi \in \mathscr{S}_\psi$, $B_\varphi(\mathscr{F}, -)[-1] \in \mathscr{S}_\psi$.

The following lemma follows from Corollary 6.

Lemma 8. *If $\psi : \mathscr{C}^{\leq 0} \to \mathscr{D}^{\leq 0}$ is extensible, then it is pseudo-extensible.*

We now have the following result, which in the extensible case is just a rephrasing of Theorem 3.

Theorem 9. *In the setting of Theorem 3, suppose ψ is pseudo-extensible and $\partial(\psi(\mathscr{F} \oplus -)) = 0$ for every $\mathscr{F} \in \mathscr{C}^{\leq 0}$. Then ψ maps $\mathscr{C}^{\leq 0}$ to $\cap \mathscr{D}^{\leq -n}$.*

Proof. First, we claim $\psi(\mathscr{F}) \in \mathscr{D}^{\leq -1}$ for all $\mathscr{F} \in \mathscr{C}^{\leq 0}$. As in the proof of Theorem 3, it suffices to show $\mathrm{gr}_i \psi(\Sigma \mathscr{F}) \in \mathscr{D}^{\leq -3}$ for all i, and this is automatic for $i \geq 3$. For $i = 2$, we have $\mathrm{gr}_2 \psi(\Sigma \mathscr{F}) = B_\psi(\mathscr{F}, \mathscr{F})[2]$, and $B_\psi(\mathscr{F}, \mathscr{F}) \in \mathscr{D}^{\leq -1}$ by pseudo-extensibility.

Next, observe that any $\varphi \in \mathscr{S}_\psi$ is pseudo-extensible, and by induction the Goodwillie derivatives of the functors $\varphi(\mathscr{F} \oplus -)$ vanish for any $\mathscr{F} \in \mathscr{C}^{\leq 0}$. Therefore, by the above argument, every $\varphi \in \mathscr{S}_\psi$ maps $\mathscr{C}^{\leq 0}$ into $\mathscr{D}^{\leq -1}$. This shows that $\psi[-1]$ is pseudo-extensible, so we may apply induction to obtain the theorem. $\square$

We immediately deduce the following.

Corollary 10. *In the setting of Theorem 2, suppose that the functors $\Psi_{A,red} : A\text{–}\mathsf{bimod}^{\leq 0} \to \mathsf{Sp}$ are pseudo-extensible.*

Then $\Psi(A \oplus M) \xrightarrow{\cong} \Psi(A)$ for any $M \in A\text{–}\mathsf{bimod}^{\leq 0}$.

Infinitesimal extensions. We now extend the above to arbitrary nilpotent extensions, as in the statement of Theorem 1.

Let $\mathsf{Alg}_{conn}^{\mathrm{SqZero}}$ be the category whose objects are square-zero extensions $B \to A$ of connective $\mathbb{E}_1$-algebras and whose morphisms preserve this structure in the natural sense.

We now introduce the following hypotheses on a functor $\Psi : \mathsf{Alg}_{conn} \to \mathsf{Sp}$.

Definition 11. (1) Ψ is *convergent* if for any $A \in \mathsf{Alg}_{conn}$, the natural morphism $\Psi(A) \to \lim_n \Psi(\tau^{\geq -n} A)$ is an isomorphism.

(2) Ψ *infinitesimally commutes with sifted colimits* if the functor:

$$\mathsf{Alg}_{conn}^{\mathrm{SqZero}} \to \mathsf{Sp}$$

$$(B \to A) \mapsto \mathrm{Ker}\left(\Psi(B) \to \Psi(A)\right)$$

commutes with sifted colimits.

Proposition 12. *Suppose that* Ψ *is convergent and infinitesimally commutes with sifted colimits. Suppose moreover that* Ψ *is constant on split square-zero extensions, i.e., for every* $A \in \mathsf{Alg}_{conn}$ *and* $M \in A\text{--}\mathsf{bimod}^{\leq 0}$, *the morphism* $\Psi(A \oplus M/A) = 0$.

Then Ψ *is infinitesimally constant, that is, for every* $f : B \to A \in \mathsf{Alg}_{conn}$ *with* $H^0(B) \to H^0(A)$ *surjective with nilpotent kernel, the map* $\Psi(B) \to \Psi(A)$ *is an isomorphism.*

Proof. A standard argument using convergence of Ψ reduces to showing that Ψ is constant along square-zero extensions. Then using that Ψ infinitesimally commutes with sifted colimits, we reduce to A being a free $\mathbb{E}_1$-algebra on generators in degree 0, for which every square-zero extension splits. $\square$

<div align="center">OUTLINE OF THE PROOF</div>

First, one has the following two results, about K-theory and TC respectively.

Theorem 13.   *(1) For* $A \in \mathsf{Alg}_{conn}$, *the functor:*

$$A\text{--}\mathsf{bimod}^{\leq 0} \to \mathsf{Sp}^{\leq 0}$$

$$M \mapsto K(A \oplus M)$$

commutes with sifted colimits. Moreover, the underlying reduced functor is extensible in the sense of §.
(2) For $A \in \mathsf{Alg}_{conn}$, *the Goodwillie derivative of the functor:*

$$A\text{--}\mathsf{bimod}^{\leq 0} \to \mathsf{Sp}$$

$$M \mapsto K(A \oplus M)$$

is canonically isomorphic to the functor $M \mapsto THH(A, M)[1]$.

Theorem 14.   *(1) For* $A \in \mathsf{Alg}_{conn}$, *the functor:*

$$A\text{--}\mathsf{bimod}^{\leq 0} \to \mathsf{Sp}$$

$$M \mapsto \mathrm{TC}(A \oplus M)$$

is pseudo-extensible in the sense of §.
(2) For A *as above and* $M \in A\text{--}\mathsf{bimod}^{\leq 0}$:

$$\mathrm{TC}_{red}(A \oplus M) \coloneqq \mathrm{Ker}(\mathrm{TC}(A \oplus M) \to \mathrm{TC}(A)) \in \mathsf{Sp}^{\leq -1}.$$

(3) The Goodwillie derivative of the above functor is canonically isomorphic to $THH(A, -)[1]$. *Moreover, this isomorphism is compatible with the cyclotomic trace and the isomorphism of Theorem 13.*

Applying Corollary 10 to the cokernel of the cyclotomic trace map, these two results clearly imply Theorem 1 in the split square-zero case.

The general case of Theorem 1 now follows from Proposition 12 and:

Theorem 15. *The functors* $K, \mathrm{TC} : \mathsf{Alg}_{conn} \to \mathsf{Sp}$ *are convergent and infinitesimally commute with sifted colimits.*

Unfortunately, the proofs of the above results are outside the scope of this summary. Let us give some indications though.

The results on TC all essentially reduce to the following calculation.

Theorem 16. *For* $A \in \mathsf{Alg}_{conn}$ *and* $M \in A\text{–}\mathsf{bimod}^{\leq 0}$, *there is a natural isomorphism:*

$$\mathrm{TC}(A \oplus M) \xrightarrow{\simeq} \mathrm{TC}(A) \oplus \lim_n \mathrm{THH}(A, M[1]_A^{\otimes n})^{\mathbb{Z}/n}$$

where the limit is over positive integers ordered under divisibility. The notation indicates genuine $\mathbb{Z}/n$-invariants; *implicitly, we are using a natural genuine* $\mathbb{Z}/n$-action on $\mathrm{THH}(A, M_A^{\otimes n})$ *constructed using a generalization of Nikolaus-Scholze's Tate Frobenius.*

In particular, $\mathrm{TC}_{red}(A \oplus M)$ *has a complete decreasing filtration indexed by positive integers under divisibility, and there is a canonical isomorphism:*

$$\mathrm{gr}_n \mathrm{TC}_{red}(A \oplus M) \xrightarrow{\simeq} \mathrm{THH}(A, M[1]_A^{\otimes n})_{h\mathbb{Z}/n}.$$

The extension of K-theory from Theorem 13 (1) is given by so-called *parametrized* K-theory. The calculation of its derivative is a theorem of Dundas-McCarthy [2], [3], and may be proved by comparing universal properties for K-theory and Goodwillie derivatives. Convergence of K-theory is straightforward. To show infinitesimal commutation with colimits, one notes that $(B \to A) \mapsto \mathrm{Ker}(GL_\infty(B) \to GL_\infty(A))$ commutes with sifted colimits by noting that this kernel is $M_\infty(I)$ for $I = \mathrm{Ker}(B \to A) \in A\text{–}\mathsf{bimod}$, and then uses Volodin's construction of K-theory to compare $\mathbb{B}GL_\infty$ with K-theory itself.

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THH of log rings

Malte Leip

Fix throughout K, a complete discrete valuation field of characteristic 0, with perfect residue field k of characteristic $p > 2$ and valuation ring A. Examples are $K = \mathbb{Q}_p$ with $A = \mathbb{Z}_p$ and $k = \mathbb{F}_p$ or more generally p-adic fields, i.e. finite extensions of $\mathbb{Q}_p$.

1. Motivation

This was the first in a series of four talks with the goal of understanding Hesselholt and Madsen's calculation [HM03, Theorem A] of the K-theory groups $K_n(K, \mathbb{Z}/p^v)$.

There is a localization sequence due to Quillen [Qui73, Corollary after Theorem 5 in §5] relating the K-theory of k, A and K:

$$(1) \qquad\qquad K(k) \xrightarrow{i_*} K(A) \xrightarrow{j^*} K(K)$$

In [HM03, Addendum 1.5.7], Hesselholt and Madsen show that it takes part in a morphism of cofiber sequences

$$(2) \qquad
\begin{array}{ccccc}
K(k) & \xrightarrow{\ i_*\ } & K(A) & \xrightarrow{\ \ j^*\ \ } & K(K) \\
\text{tr}\downarrow & & \downarrow\text{tr} & & \downarrow\text{tr} \\
\mathrm{TC}(k;p) & \xrightarrow{\ i_*\ } & \mathrm{TC}(A;p) & \xrightarrow{\ j^*\ } & \mathrm{TC}(A, A \cap K^\times; p)
\end{array}$$

where the left-hand and middle vertical maps are the usual cyclotomic trace maps, the vertical maps as well as both maps j^* are maps of $\mathbb{E}_\infty$-ring spectra, and the maps i_* are maps of $K(A)$- and $\mathrm{TC}(A;p)$-module spectra, respectively. The object fitting in the lower right turns out not to be $\mathrm{TC}(K;p)$, but something different, namely TC of the *log ring* $(A, A \cap K^\times)$, denoted by $\mathrm{TC}(A, A \cap K^\times; p)$. The lower cofiber sequence arises from an analogous cofiber sequence of cyclotomic spectra involving $\mathrm{THH}(A, A \cap K^\times)$, the topological Hochschild homology of the log ring[1] $(A, A \cap K^\times)$. The left-hand and middle vertical morphisms in diagram (2) induce isomorphisms on $\pi_m(-, \mathbb{Z}/p^n)$ for $m \geq 0$ by a result of Hesselholt and Madsen [HM97, Theorem D], so in order to calculate $K_m(K, \mathbb{Z}/p^n)$ for $m \geq 1$, we can instead study $\mathrm{TC}_m(A, A \cap K^\times; p, \mathbb{Z}/p^n)$.

2. THH of log rings

In this section we will define THH of log rings (R, M). The definition used here [HS17, Description of talk 11], given in the framework of Nikolaus and Scholze [NS17], differs from the one used in [HM03], where in order to define THH of the log ring $(A, A \cap K^\times)$, Hesselholt and Madsen start with the category of bounded complexes of finitely generated projective A-modules with weak equivalences those

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[1]The program used the term *logarithmic* THH where we use THH *of log rings*.

morphisms that become quasi-isomorphisms after tensoring with K [HM03, Definition 1.5.5]. All statements discussed below for $\mathrm{THH}(A, A \cap K^{\times})$ hold for both definitions, though some statements are significantly easier to prove in the new setting.

Definition 1 ([Kat89, 1.1])**.** Let R be a commutative ring. A pre-log structure on R consists of a commutative monoid M together with a morphism of monoids $\alpha \colon M \to R$, where R is a monoid under multiplication.

A log ring is a commutative ring together with a pre-log structure.[2]

From now, R will always denote a commutative ring and (R, M) a log ring.

Examples 2 ([Kat89, 1.5])**.**
  (a) The inclusion of the units $R^{\times} \to R$ is called the *trivial* (pre-)log structure.
  (b) The *canonical* (pre-)log structure on A is $A \cap K^{\times} = A \setminus \{0\} \to A$.
  (c) If we choose a uniformizer $\pi \in A$, then we obtain a pre-log structure $\alpha \colon \mathbb{N} \to A$ defined by $n \mapsto \pi^n$.
  (d) Note that the pre-log structure in example (b) is the monoidal product of the ones in examples (a) and (c), where the morphisms of pre-log structures on A from the trivial one to the canonical one and from the one in example (c) to the canonical one are the unique ones.

Recall from talk 8 by Markus Land that $\mathbb{S}[B^{\mathrm{cyc}}M]$ denotes the spherical group ring of the cyclic bar construction (see [NS17, Section IV.3]) of the $\mathbb{E}_{\infty}$-monoid in spaces M. The pre-log structure $\alpha \colon M \to R$ induces a morphism $\mathbb{S}[M] \to R$ of $\mathbb{E}_{\infty}$-ring spectra, which in turn induces a morphism of $\mathbb{E}_{\infty}$-algebras in cyclotomic spectra

$$(3) \qquad \mathbb{S}[B^{\mathrm{cyc}}M] \simeq \mathrm{THH}(\mathbb{S}[M]) \to \mathrm{THH}(R)$$

where the equivalence is the one discussed in talk 8.

For any $\mathbb{E}_{\infty}$-monoid N in spaces, the universal property of $N \to B^{\mathrm{cyc}}N$ as the inital $\mathbb{E}_{\infty}$-morphism from N into an $\mathbb{E}_{\infty}$-monoid in spaces with circle action (like in [NS17, Proposition IV.2.2]) furnishes us with a dashed $\mathbb{T}$-equivariant $\mathbb{E}_{\infty}$-morphism $B^{\mathrm{cyc}}N \to N$ (where N carries the trivial $\mathbb{T}$-action) as in the following diagram.

$$(4) \qquad \begin{array}{ccc} N & \longrightarrow & B^{\mathrm{cyc}}N \\ & {\scriptstyle \mathrm{id}} \searrow & \downarrow \\ & & N \end{array}$$

Applying this to the group completion of M, denoted by M^{gp}, we can form the pullback $M \times_{M^{\mathrm{gp}}} B^{\mathrm{cyc}}M^{\mathrm{gp}}$, where the other morphism is the canonical map from M into its group completion. This is again an $\mathbb{E}_{\infty}$-monoid in spaces with $\mathbb{T}$-action,

---

[2]It would be more correct to call this a pre-log ring. A pre-log structure (M, α) is a log structure if α induces an isomorphism $M^{\times} \to R^{\times}$. For our purposes, the distinction is not relevant, so we follow the terminology of [HM03] and call any ring with a pre-log structure a log ring.

and it can be given a space level Frobenius lift using the one for $B^{\mathrm{cyc}}M^{\mathrm{gp}}$ (see [NS17, Lemma IV.3.1]). Taking the spherical group ring we obtain an $\mathbb{E}_\infty$-algebra in cyclotomic spectra $\mathbb{S}[M \times_{M^{\mathrm{gp}}} B^{\mathrm{cyc}}M^{\mathrm{gp}}]$.

Finally, we can also define a morphism of $\mathbb{E}_\infty$-algebras in cyclotomic spectra $\mathbb{S}[B^{\mathrm{cyc}}M] \to \mathbb{S}[M \times_{M^{\mathrm{gp}}} B^{\mathrm{cyc}}M^{\mathrm{gp}}]$, induced from a morphism $B^{\mathrm{cyc}}M \to M$ as in diagram (4) and a morphism $B^{\mathrm{cyc}}M \to B^{\mathrm{cyc}}M^{\mathrm{gp}}$ that is in turn induced by the canonical map $M \to M^{\mathrm{gp}}$.

We can now define THH of a log ring:

Definition 3 ([HS17])**.** Let (R, M) be a log ring. Then let

$$(5) \qquad \mathrm{THH}(R, M) = \mathrm{THH}(R) \underset{\mathbb{S}[B^{\mathrm{cyc}}M]}{\otimes} \mathbb{S}[M \underset{M^{\mathrm{gp}}}{\times} B^{\mathrm{cyc}}M^{\mathrm{gp}}]$$

where the tensor product is taken as $\mathbb{E}_\infty$-algebras in cyclotomic spectra, and the maps are the ones discussed above.

3. Log differential graded rings

In the previous section we defined $\mathrm{THH}(R, M)$ as an $\mathbb{E}_\infty$-algebra in cyclotomic spectra. In this section we will discuss how $\pi_*(\mathrm{THH}(R, M))$ carries the structure of a log differential graded ring with underlying log ring (R, M).

Definition 4 ([HM03, Section 2.2])**.** A *log differential graded ring* consists of a graded commutative differential graded ring E^*, together with a pre-log structure $M \to E^0$, where E^0 is a monoid under multiplication, and a morphism of monoids $d\log\colon M \to E^1$, where E^1 is a monoid under addition. This data has to satisfy two properties: $d \circ d\log = 0$ and $\alpha(x)d\log(x) = d(\alpha(x))$ for all $x \in M$.

The log ring (E^0, M) is called the *underlying* log ring of E^*.

Remark 5 ([Kat89, Section 1.7 and 1.9])**.** There is a universal example of a log differential graded ring with underlying log ring (R, M), namely, the de Rham complex with log poles, given by:

$$(6) \qquad\qquad\qquad \Omega^*_{(R,M)} = \Lambda^*_R \Omega^1_{(R,M)}$$

where

$$(7) \qquad \Omega^1_{(R,M)} = \left(\Omega^1_R \oplus \left(R \underset{\mathbb{Z}}{\otimes} M^{\mathrm{gp}} \right) \right) \big/ \langle (d(\alpha(x)), 0) - (0, \alpha(x) \otimes x) \mid x \in M \rangle$$

and $d\log(x) = (0, 1 \otimes x)$.

In order to explain how $\pi_*(\mathrm{THH}(R, M)$ carries the structure of a log differential graded ring, we first need to define the graded ring structure and differential on $\pi_*(\mathrm{THH}(R, M)$, and construct the maps $\alpha\colon M \to \pi_0(\mathrm{THH}(R, M))$ as well as $d\log\colon M \to \pi_1(\mathrm{THH}(R, M))$. This is what we do next.

First, as $\mathrm{THH}(R, M)$ is an $\mathbb{E}_\infty$-ring spectrum, the homotopy groups obtain a graded commutative graded ring structure. It was explained in talk 6 by Eva Höning how one obtains out of the $\mathbb{T}$-action on $\mathrm{THH}(R, M)$ a morphism of spectra $\Sigma \mathrm{THH}(R, M) \to \mathrm{THH}(R, M)$. The induced morphism on homotopy groups, which increases degree by one, will be denoted by d.

Next, we have to define $\alpha\colon M \to \pi_0\left(\mathrm{THH}(R, M)\right)$. This is to be the composite from the top left to the bottom right in the following commutative diagram:

$$
(8) \quad
\begin{array}{ccccccc}
M & \longrightarrow & \pi_0\left(\mathbb{S}[M]\right) & \longrightarrow & \pi_0\left(\mathbb{S}[B^{\mathrm{cyc}}M]\right) & \longrightarrow & \pi_0\left(\mathbb{S}[M \underset{M^{\mathrm{gp}}}{\times} B^{\mathrm{cyc}}M^{\mathrm{gp}}]\right) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
R & \longrightarrow & \pi_0(R) & \longrightarrow & \pi_0\left(\mathrm{THH}(R)\right) & \longrightarrow & \pi_0\left(\mathrm{THH}(R, M)\right)
\end{array}
$$

Note that we can also define a morphism analogous to the composite of the top row, but with M replaced by M^{gp}:

$$
(9) \qquad \alpha'\colon M^{\mathrm{gp}} \to \pi_0\left(\mathbb{S}[M^{\mathrm{gp}} \underset{M^{\mathrm{gp}}}{\times} B^{\mathrm{cyc}}M^{\mathrm{gp}}]\right)
$$

After composing this with the morphism into the tensor product with $\mathrm{THH}(R)$ over $\mathbb{S}[B^{\mathrm{cyc}}M]$ this corresponds to $M^{\mathrm{gp}} \to \pi_0\,\mathrm{THH}(R[\alpha(M)^{-1}])$.

As M is discrete, $\mathbb{S}[M \times_{M^{\mathrm{gp}}} B^{\mathrm{cyc}}M^{\mathrm{gp}}]$ splits $\mathbb{T}$-equivariantly into a coproduct $\coprod_{m \in M} \mathbb{S}[\{m\} \times_{M^{\mathrm{gp}}} B^{\mathrm{cyc}}M^{\mathrm{gp}}]$, making $\pi_*\left(\mathbb{S}[M \times_{M^{\mathrm{gp}}} B^{\mathrm{cyc}}M^{\mathrm{gp}}]\right)$ into an M-graded ring, and similarly for M^{gp} instead of M. In the defining identity for $d\log$, the equation $\alpha(x)d\log(x) = d(\alpha(x))$, $\alpha(x)$ has M-degree x and d does not change the degree, so we expect $d\log(x)$ to have M-degree 1_M. Let

$$
(10) \qquad d\log'\colon M^{\mathrm{gp}} \to \pi_1\left(\mathbb{S}[M^{\mathrm{gp}} \underset{M^{\mathrm{gp}}}{\times} B^{\mathrm{cyc}}M^{\mathrm{gp}}]\right)
$$

be defined by $d\log'(x) = \alpha'(x^{-1})d(\alpha'(x))$. As this map has image in the degree $1_{M^{\mathrm{gp}}}$ part, and

$$
(11) \qquad \pi_*\left(\mathbb{S}[M \underset{M^{\mathrm{gp}}}{\times} B^{\mathrm{cyc}}M^{\mathrm{gp}}]\right) \to \pi_*\left(\mathbb{S}[M^{\mathrm{gp}} \underset{M^{\mathrm{gp}}}{\times} B^{\mathrm{cyc}}M^{\mathrm{gp}}]\right)
$$

maps the degree 1_M part isomorphically to the degree $1_{M^{\mathrm{gp}}}$ part, we may define $d\log$ as the dashed arrow in the following diagram, where the dotted arrow is the unique one making the diagram commute and such that its image is in the degree 1_M part.

$$
(12) \qquad
\begin{array}{ccc}
 & & \pi_1\left(\mathrm{THH}(R, M)\right) \\
 & \overset{d\log}{\nearrow} & \nearrow \\
M & \cdots\cdots\to & \pi_1\left(\mathbb{S}[M \underset{M^{\mathrm{gp}}}{\times} B^{\mathrm{cyc}}M^{\mathrm{gp}}]\right) \\
\downarrow & & \downarrow \\
M^{\mathrm{gp}} & \underset{d\log'}{\longrightarrow} & \pi_1\left(\mathbb{S}[M^{\mathrm{gp}} \underset{M^{\mathrm{gp}}}{\times} B^{\mathrm{cyc}}M^{\mathrm{gp}}]\right)
\end{array}
$$

Proposition 6 ([HM03, Proposition 2.3.1]). π_* (THH(R, M) *together with* d, α *and* $d \log$ *as defined above is a log differential graded ring with underlying log ring (canonically isomorphic to)* (R, M).

Proof. It was already remarked above that the graded ring structure is graded commutative. One can check that d satisfies the identities $d^2 = 0$ as well as[3] $d(xy) = (dx)y + (-1)^{\deg(x)}x(dy)$.

That α is a morphism of monoids is immediate from the definition, as all morphisms in diagram (8) are morphisms of monoids. That $d \log$ is a morphism of monoids and that $\alpha(x)d\log(x) = d(\alpha(x))$ holds can immediately be checked in $\pi_*(\mathbb{S}[M^{\mathrm{gp}} \times_{M^{\mathrm{gp}}} B^{\mathrm{cyc}} M^{\mathrm{gp}}])$. For $d \circ d\log = 0$ we have to additionally use that $(d\alpha(x)) \cdot (d\alpha(x)) = 0$ in THH(R) (see for example [KN18, Lemma 2.3 and Prop 3.11]).

As the lower horizontal morphisms in diagram (8) are isomorphisms, the underlying log ring can be identified with (R, M). $\qquad\square$

4. The cofiber sequence for THH$(A, A \cap K^\times)$

In this section we move on from the case of a general log ring (R, M) and consider instead example (b), $(A, A \cap K^\times)$.

Recall from example (d) that $(A, A \cap K^\times)$ is the monoidal product of the trivial pre-log structure on A and the pre-log structure given by $\mathbb{N} \to A, n \mapsto \pi^n$, after choosing a uniformizer $\pi \in A$. By compatibility of B^{cyc} etc. with the monoidal product we obtain

(13)   THH$(A, A \cap K^\times)$

$$\simeq \mathrm{THH}(A) \underset{\mathbb{S}[B^{\mathrm{cyc}}(A^\times) \times B^{\mathrm{cyc}}(\mathbb{N})]}{\otimes} \mathbb{S}\left[\left(A^\times \underset{A^\times}{\times} B^{\mathrm{cyc}} A^\times\right) \times \left(\mathbb{N} \underset{\mathbb{Z}}{\times} B^{\mathrm{cyc}}\mathbb{Z}\right)\right]$$

$$\simeq \mathrm{THH}(A) \underset{\mathbb{S}[B^{\mathrm{cyc}}(A^\times)] \otimes \mathbb{S}[B^{\mathrm{cyc}}(\mathbb{N})]}{\otimes} \left(\mathbb{S}\left[A^\times \underset{A^\times}{\times} B^{\mathrm{cyc}} A^\times\right] \otimes \mathbb{S}\left[\mathbb{N} \underset{\mathbb{Z}}{\times} B^{\mathrm{cyc}}\mathbb{Z}\right]\right)$$

$$\simeq \left(\mathrm{THH}(A) \underset{\mathbb{S}[B^{\mathrm{cyc}} A^\times]}{\otimes} \mathbb{S}\left[B^{\mathrm{cyc}} A^\times\right]\right) \underset{\mathbb{S}[B^{\mathrm{cyc}}(\mathbb{N})]}{\otimes} \mathbb{S}\left[\mathbb{N} \underset{\mathbb{Z}}{\times} B^{\mathrm{cyc}}\mathbb{Z}\right] \simeq \mathrm{THH}(A, \mathbb{N})$$

where the first equivalence uses compatibility of B^{cyc} with products and that products of pullbacks are pullbacks, and the second that $\mathbb{S}[-]$ is monoidal.

In talk 8, Markus Land identified the $\mathbb{T}$-equivariant map of $\mathbb{E}_\infty$-monoids in spaces $B^{\mathrm{cyc}}\mathbb{N} \to B^{\mathrm{cyc}}\mathbb{Z}$ with the inclusion of the submonoid $\{(1, 0)\} \cup \mathbb{T} \times \mathbb{Z}_{>0}$ into $\mathbb{T} \times \mathbb{Z}$, where $t \in \mathbb{T}$ acts on (s, n) by $t \cdot (s, n) = (t^n \cdot s, n)$.

Considering the induced map $B^{\mathrm{cyc}}\mathbb{N} \to \mathbb{N} \times_{\mathbb{Z}} B^{\mathrm{cyc}}\mathbb{Z}$, and adding basepoints, we can thus identify the cofiber with S^1, with trivial $\mathbb{T}$-action, and taking suspension spectra yields

(14) $\mathbb{S}[B^{\mathrm{cyc}}\mathbb{N}] \to \mathbb{S}[\mathbb{N} \underset{\mathbb{Z}}{\times} B^{\mathrm{cyc}}\mathbb{Z}] \to \Sigma\mathbb{S}$

---

[3]This boils down to the fact that the diagonal map considered as an element of $\pi_1(\mathbb{T} \times \mathbb{T})$ is $[\mathrm{id} \times \mathrm{id}] = [\mathrm{id} \times \mathrm{const}_1] + [\mathrm{const}_1 \times \mathrm{id}]$, where const_x is the constant map with image x.

which is a cofiber sequence of $\mathbb{S}[B^{\mathrm{cyc}}\mathbb{N}]$-module cyclotomic spectra. Tensoring with $\mathrm{THH}(A)$ over $\mathbb{S}[B^{\mathrm{cyc}}\mathbb{N}]$, we obtain the following:

Proposition 7 ([HM03, Theorem 1.5.6])**.** *There is a natural cofiber sequence*

$$\tag{15} \mathrm{THH}(k) \xrightarrow{i_*} \mathrm{THH}(A) \xrightarrow{j^*} \mathrm{THH}(A, A \cap K^\times)$$

of $\mathrm{THH}(A)$*-module cyclotomic spectra, where the* $\mathrm{THH}(A)$*-module structure on* $\mathrm{THH}(k)$ *comes from the morphism of* $\mathbb{E}_\infty$*-ring spectra* $\mathrm{THH}(A) \to \mathrm{THH}(k)$ *induced by* $A \to A/\pi A = k$*, and the* $\mathrm{THH}(A)$*-module structure on* $\mathrm{THH}(A, A \cap K^\times)$ *comes from the morphism* $\mathrm{THH}(A) \to \mathrm{THH}(A, A \cap K^\times)$ *appearing in the cofiber sequence, which is a morphism of* $\mathbb{E}_\infty$*-ring spectra.*

Proof. The only thing left to note is that

$$\tag{16} \mathrm{THH}(A) \underset{\mathrm{THH}(\mathbb{S}[\mathbb{N}])}{\otimes} \mathbb{S} \simeq \mathrm{THH}(A) \underset{\mathrm{THH}(\mathbb{S}[\mathbb{N}])}{\otimes} \mathrm{THH}(\mathbb{S})$$

$$\simeq \mathrm{THH}\left(A \underset{\mathbb{S}[\mathbb{N}]}{\otimes} \mathbb{S}\right) \simeq \mathrm{THH}(A/\pi A) \simeq \mathrm{THH}(k)$$

$\square$

5. The log differential graded ring $\pi_*\left(\mathrm{THH}(A, A \cap K^\times), \mathbb{Z}/p\right)$

In this section we discuss Hesselholt and Madsen's calculation of the homotopy groups with $\mathbb{Z}/p$-coefficients of $\mathrm{THH}(A, A \cap K^\times)$ as a log differential graded ring.

5.1. **Definition of the morphism.** The result identifies the log differential graded ring $\pi_*\left(\mathrm{THH}(A, A \cap K^\times), \mathbb{Z}/p\right)$ with $\Omega^*_{(A, A \cap K^\times)} \otimes_\mathbb{Z} (\mathbb{Z}/p)[\kappa]$, where κ has degree 2 and $d\kappa = \kappa d\log(-p)$. We start by defining a morphism from the latter log differential graded ring to the former.

First, note that the morphism $\mathbb{S} \xrightarrow{\simeq} \mathrm{THH}(\mathbb{S}) \to \mathrm{THH}(A, A \cap K^\times)$ is $\mathbb{T}$-equivariant where $\mathbb{S}$ carries the trivial $\mathbb{T}$-action, and thus $\mathrm{THH}(A, A \cap K^\times)/p$ also carries a $\mathbb{T}$-action, acting through morphisms of $\mathbb{E}_\infty$-ring spectra, and the morphism from $\mathrm{THH}(A, A \cap K^\times)$ is compatible with that structure. Hence, as in section 3, $\pi_*\left(\mathrm{THH}(A, A \cap K^\times), \mathbb{Z}/p\right)$ obtains the structure of a graded commutative differential graded ring, which can be upgraded to a log differential graded ring using the morphism of differential graded rings

$$\tag{17} \pi_*(\mathrm{THH}(A, A \cap K^\times)) \to \pi_*(\mathrm{THH}(A, A \cap K^\times), \mathbb{Z}/p)$$

Here, the log differential graded structure on the source is the one from Proposition 6, which has underlying log ring (canonically isomorphic to) $(A, A \cap K^\times)$. In Remark 5 it was noted that $\Omega^*_{(A, A \cap K^\times)}$ is the universal such thing. Hence we obtain a morphism of log differential graded rings

$$\tag{18} \Omega^*_{(A, A \cap K^\times)} \to \pi_*\left(\mathrm{THH}(A, A \cap K^\times)\right)$$

Composed with morphism (17) we finally obtain a morphism

$$\tag{19} \Omega^*_{(A, A \cap K^\times)} \to \pi_*\left(\mathrm{THH}(A, A \cap K^\times)\right) \to \pi_*\left(\mathrm{THH}(A, A \cap K^\times), \mathbb{Z}/p\right)$$

As κ has degree 2, the statement we are aiming at in particular claims that in degrees 0 and 1, morphism (19) is an isomorphism after tensoring the source with $\mathbb{Z}/p$. In fact, this is already true integrally, i.e. morphism (18) is already an isomorphism in degrees 0 and 1. We record this now together with some other facts which will use below without proof[4]:

Facts 8.

(A) [HM03, Proposition 2.3.4] Morphism (18)
$$\Omega^n_{(A,A\cap K^\times)} \to \pi_n\left(\mathrm{THH}(A, A\cap K^\times)\right)$$
is an isomorphism for $n = 0, 1$.

(B) [HM03, Proof of proposition 2.3.4] The abelian group
$$\pi_2\left(\mathrm{THH}(A, A\cap K^\times)\right)$$
is uniquely divisible.

(C) [HM03, Corollary 2.2.5] $\Omega^1_{(A,A\cap K^\times)}[p] = (A/p) \cdot d\log(-p)$

What is still missing from the definition of a morphism

$$(20) \qquad \Omega^*_{(A,A\cap K^\times)} \otimes_{\mathbb{Z}} (\mathbb{Z}/p)[\kappa] \to \pi_*\left(\mathrm{THH}(A, A\cap K^\times), \mathbb{Z}/p\right)$$

is the image of κ. To define $\kappa \in \pi_2\left(\mathrm{THH}(A, A\cap K^\times), \mathbb{Z}/p\right)$, we use the Bockstein sequence:

$$(21) \quad 0 \to \pi_2\left(\mathrm{THH}(A, A\cap K^\times)\right)/p \to \pi_2\left(\mathrm{THH}(A, A\cap K^\times), \mathbb{Z}/p\right)$$
$$\xrightarrow{\beta} \pi_1\left(\mathrm{THH}(A, A\cap K^\times)\right)[p] \to 0$$

By fact (B), the left hand group is 0. Hence β is an isomorphism. Furthermore, for the right hand group we obtain

$$(22) \qquad \pi_1\left(\mathrm{THH}(A, A\cap K^\times)\right)[p] \xleftarrow{\cong} \Omega^1_{(A,A\cap K^\times)}[p] = (A/p) \cdot d\log(-p)$$

where the isomorphism is the one induced by the isomorphism from (A), and the equality is the one from (C).

We can now define κ as the element mapped to $d\log(-p)$ under this composition of isomorphisms. In particular, as an A/p-module we have

$$(23) \qquad \pi_2\left(\mathrm{THH}(A, A\cap K^\times), \mathbb{Z}/p\right) = A/p \cdot \kappa$$

5.2. **Statement of the Theorem.**

Theorem 9 ([HM03, Theorem B=2.4.1]). *The morphism of log differential graded rings constructed in subsection 5.1*

$$(24) \qquad \Omega^*_{(A,A\cap K^\times)} \underset{\mathbb{Z}}{\otimes} (\mathbb{Z}/p)[\kappa] \to \pi_*\left(\mathrm{THH}(A, A\cap K^\times), \mathbb{Z}/p\right)$$

is a natural isomorphism. Here, κ has degree 2 and $d\kappa = \kappa d\log(-p)$.

---

[4]For G an abelian group, $G[p]$ denotes the kernel of multiplication by p on G.

The proof of the theorem can be divided into two parts: Showing that the morphism is an isomorphism of graded A-algebras, and showing that $d\kappa = \kappa d\log(-p)$ holds in $\pi_* \left(\mathrm{THH}(A, A \cap K^\times), \mathbb{Z}/p\right)$.

In order to show the first part, Hesselholt and Madsen distinguish the wildly ramified and tamely ramified cases, where the latter uses the former. We will only discuss the wildly ramified case, and give a brief sketch of the proof that morphism (24) is an isomorphism of graded A-algebras in that case in subsection 5.3.

Assuming that morphism (24) is an isomorphism of graded A-algebras in all cases, subsection 5.4 will then sketch the main argument to obtain the formula for $d\kappa$.

5.3. Isomorphism of graded A-algebras (in the wildly ramified case).
Assume for this entire subsection that we are in the wildly ramified case.

From (A) and multiplication by p on A being injective we directly obtain the isomorphism in degrees 0 and 1. One can check that $\Omega^n_{(A, A\cap K^\times)}/p \cong 0$ for $n > 1$, so that multiplication by κ is an isomorphism from degree $n \geq 0$ to degree $n + 2$ on the source. Hence it suffices to show that the same is true in the target. The case $n = 0$ can be handled by the description $\pi_2 \left(\mathrm{THH}(A, A \cap K^\times), \mathbb{Z}/p\right) = A/p \cdot \kappa$ obtained in (23). Thus only the case $n \geq 1$ remains. Using the Bockstein sequence as well as facts (B) and (C) again one can show that

$$(25) \qquad \pi_2 \left(\mathrm{THH}(A), \mathbb{Z}/p\right) \to \pi_2 \left(\mathrm{THH}(A, A \cap K^\times), \mathbb{Z}/p\right)$$

is an isomorphism. Hence the source is also isomorphic to A/p as an A-module, with generator $\tilde{\kappa}$ that is defined as the unique element that is sent to κ under isomorphism (25).

As the cofiber sequence from Proposition 7 is a sequence of $\mathrm{THH}(A)$-modules, we obtain the following morphism of long exact sequences of A/p-modules (here, $\overline{\pi}_n(T(-))$ is short-hand notation for $\pi_n \left(\mathrm{THH}(-), \mathbb{Z}/p\right)$ and M_A for $A \cap K^\times$):

$$
\begin{array}{ccccccc}
\overline{\pi}_n\left(T(k)\right) & \xrightarrow{i_*} & \overline{\pi}_n\left(T(A)\right) & \xrightarrow{j^*} & \overline{\pi}_n\left(T(A, M_A)\right) & \xrightarrow{\partial} & \overline{\pi}_{n-1}\left(T(k)\right) \\
\downarrow{\scriptstyle \tilde{\kappa}\cdot} & & \downarrow{\scriptstyle \tilde{\kappa}\cdot} & & \downarrow{\scriptstyle \kappa\cdot} & & \downarrow{\scriptstyle \tilde{\kappa}\cdot} \\
\overline{\pi}_{n+2}\left(T(k)\right) & \xrightarrow{i_*} & \overline{\pi}_{n+2}\left(T(A)\right) & \xrightarrow{j^*} & \overline{\pi}_{n+2}\left(T(A, M_A)\right) & \xrightarrow{\partial} & \overline{\pi}_{n+1}\left(T(k)\right)
\end{array}
$$

The claim now follows from the five lemma if multiplication by $\tilde{\kappa}$ is an isomorphism on $\pi_n\left(\mathrm{THH}(k), \mathbb{Z}/p\right)$ and $\pi_n\left(\mathrm{THH}(A), \mathbb{Z}/p\right)$ for $n \geq 0$. But this follows from the explicit calculation of those rings by Lindenstrauss and Madsen for A in [LM00] and Hesselholt and Madsen for k in [HM97], see the discussion in [HM03, Section 2.4, page 40].

This shows that morphism (24) is an isomorphism of graded A-algebras in the wildly ramified case.

5.4. Compatibility with log differential graded ring structure; formula for $d\kappa$.
As morphism (24) is by construction compatible with $d\log$ and the differential on $\Omega^*_{(A, A\cap K^\times)}$, what is left after having shown that the morphism is an isomorphism of graded A-algebras is just the relation $d\kappa = \kappa d\log(-p)$.

To obtain this identity up to a unit u in $\mathbb{F}_p$ one can use naturality and check it on $A = \mathbb{Z}_p$, where d is an isomorphism from degree 2 to degree 3, the source being isomorphic to $\mathbb{Z}/p$ and generated by κ, the target isomorphic to $\mathbb{Z}/p$ and generated by $\kappa d \log(-p)$.

As we already know that morphism (24) is an isomorphism of A-algebras, we know that $\kappa d \log(-p)$ is nonzero. Hence it suffices to check that $u = 1$ in a single example. We can thus just consider the wildly ramified case.

In this case, $\tilde{\kappa}$ can be checked to be the unique element of $\pi_2\left(\mathrm{THH}(A)\right)$ such that

$$(26) \qquad \beta(\tilde{\kappa}) = -\left((e_K/p)\pi^{e_K-1} + \theta'(\pi)\right)\theta(\pi)^{-1}d\pi$$

where β is the Bockstein morphism, e_K is the ramification index and $\theta \in W(k)[x]$ such that $A = W(k)[\pi]/\left(\pi^{e_K} + p\theta(\pi)\right)$. In particular, this description of $\tilde{\kappa}$ is independent of the definition of $\mathrm{THH}(A, A \cap K^\times)$. See [Ser79, §5, Theorem 4] for the structure of A and [HM03, Section 5.4, page 78] for the formula for $\beta(\tilde{\kappa})$. Hesselholt and Madsen obtain in [HM03, Remark 5.3.3] that $d\tilde{\kappa} = -\theta'(\pi)\theta(\pi)^{-1}d\pi \cdot \tilde{\kappa}$, from which the required identity for $d\kappa$ follows by a simple calculation using that $\pi_*\left(\mathrm{THH}(A), \mathbb{Z}/p\right) \to \pi_*\left(\mathrm{THH}(A, A \cap K^\times), \mathbb{Z}/p\right)$ is a morphism of differential graded A-algebras that sends $\tilde{\kappa}$ to κ.

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The logarithmic de Rham–Witt complex
Piotr Achinger

Let K be a complete discrete valuation field of characteristic zero with perfect residue field k of characteristic $p > 2$. We denote by $A = \mathcal{O}_K$ the integral subring of K. In [HM03], Hesselholt and Madsen compute the mod p^v K-theory groups of K:

$$
\begin{aligned}
K_{2s}(K, \mathbb{Z}/p^v) &\cong H^0(K, \mu_{p^v}^{\otimes s}) \oplus H^2(K, \mu_{p^v}^{\otimes (s+1)}), \\
K_{2s-1}(K, \mathbb{Z}/p^v) &\cong H^1(K, \mu_{p^v}^{\otimes s}).
\end{aligned}
$$

Let us briefly outline their strategy.

For any category with fibrations and weak equivalences $\mathcal{C}$ (for example, the exact category of finite projective modules over a ring), we have the *cyclotomic trace* from K-theory to topological cyclic homology (see Lecture 9):

$$
\mathrm{tr}\colon K(\mathcal{C}) \longrightarrow \mathrm{TC}(\mathcal{C}; p).
$$

In an earlier paper [HM97], Hesselholt and Madsen prove that for every finite $W(k)$-algebra R, this map induces an isomorphism on non-negative homotopy groups with mod p^v coefficients $K_i(R, \mathbb{Z}/p^v) \cong \mathrm{TC}_i(R; p, \mathbb{Z}/p^v)$. We apply this with $R = A$ and $R = k$, and then the localization sequence for K-theory and Waldhausen's approximation theorem can be used to show that

$$
\mathrm{tr}\colon K_i(K, \mathbb{Z}/p^v) \longrightarrow \mathrm{TC}_i(A|K; p, \mathbb{Z}/p^v)
$$

is an isomorphism for $i \geq 1$. Here $A|K$ is the category of bounded complexes of finite projective A-modules, with levelwise monomorphisms as cofibrations, but with maps inducing a quasi-isomorphism *after inverting p* as weak equivalences. The idea is that $A|K$ is sufficiently close to the category of finite projective K-modules to yield $K_i(K) \cong K_i(A|K)$, but at the same time it can be studied p-adically since A is a p-adic ring. One can think of $A|K$ as a relative p-adic compactification (or: a model) of K. This idea is very close to the philosophy of logarithmic structures, which partially explains the appearance of log structures in what follows.

(In fact, the construction of topological Hochschild homology (and consequently of topological cyclic homology TC) has been generalized to log rings (R, M), i.e. rings R endowed with a multiplicative monoid homomorphism $M \to R$ (see §1.2 below). In our situation, there are natural maps

$$
K(K) \xrightarrow{\mathrm{tr}} \mathrm{THH}(A|K) \leftarrow \mathrm{THH}(A, M),
$$

the latter being an equivalence of cyclotomic spectra. Following this recent change of perspective, and for notational uniformity, we shall use $\mathrm{TC}((A, M); p)$ in place of $\mathrm{TC}(A|K; p)$ in these notes. The reader should keep this in mind when comparing with the original paper [HM03][1].)

In the next step, one realizes the spectrum $\mathrm{TC}((A, M); p)$ as the homotopy fixed points of "Frobenius" on another spectrum $\mathrm{TR}(A, M)$ (denoted $\mathrm{TR}(A|K; p)$

---

[1]Other changes in notation: we shall denote the log de Rham–Witt complex by $W_\bullet \Omega^*_{(A,M)}$ instead of $W_\bullet \omega^*_{(A,M)}$, and write $[a]_n$ instead of $\underline{a}_n$ for the Teichmüller character.

in [HM03]). This way we can reduce the computation of $K_i(K, \mathbb{Z}/p^v)$ further to the computation of $\mathrm{TR}_*^\bullet((A, M), \mathbb{Z}/p^v)$.

The spectra $\mathrm{TR}^n(A, M)$ themselves are $C_{p^{n-1}}$-fixed points of the topological Hochschild homology spectrum $\mathrm{THH}(A, M)$. This construction endows them with a lot of extra structure:

(1) a commutative ring structure,
(2) the Connes differential $d\colon \mathrm{TR}_q^n(A, M) \to \mathrm{TR}_{q+1}^n(A, M)$,
(3) $F\colon \mathrm{TR}^n(A, M) \to \mathrm{TR}^{n-1}(A, M)$ the natural inclusion,
(4) $V\colon \mathrm{TR}^{n-1}(A, M) \to \mathrm{TR}^n(A, M)$, the transfer map, ,
(5) the restriction maps $R\colon \mathrm{TR}^n(A, M) \to \mathrm{TR}^{n-1}(A, M)$.

The operators d, F, and V are induced by the circle action on THH, while the definition of R requires the cyclotomic structure. In addition, one can construct the Teichmüller lift (a multiplicative map)

$$A \to \mathrm{TR}_0^n(A, M)$$

and the log differential

$$d\log\colon M := A \cap K^\times \to \mathrm{TR}_1^n(A, M).$$

(In fact, $d\log$ naturally factors through $K_1(K) = K^\times$ via the cyclotomic trace). This data makes $\mathrm{TR}_*^\bullet(A, M)$ into a *log Witt complex* over (A, M) (see Definition 2 and Theorem 3). The category of log Witt complexes over any log ring (A, M) has an initial object, the *log de Rham–Witt complex* $W_\bullet\Omega_{(A,M)}^*$ and hence we obtain a natural map

$$(1) \qquad\qquad W_\bullet\Omega_{(A,M)}^* \longrightarrow \mathrm{TR}_*^\bullet(A, M).$$

It is shown that this map is an isomorphism in degres $* \leq 2$.

Finally, if $\mu_{p^v} \subseteq K$, then one can construct a natural map

$$(2) \qquad\qquad \mu_{p^v} \to \mathrm{TR}_2^\bullet((A, M), \mathbb{Z}/p^v)$$

as follows. By definition of mod p^v homotopy groups, we have the exact sequence

$$0 \to \mathrm{TR}_*^\bullet(A, M)/p^v \to \mathrm{TR}_*^\bullet((A, M), \mathbb{Z}/p^v) \to \mathrm{TR}_{*-1}^\bullet(A, M)[p^v] \to 0$$

Moreover, it is shown that $\mathrm{TR}_*^\bullet(A, M)$ is p-divisible for $* > 1$. It follows that $\mathrm{TR}_2^\bullet((A, M), \mathbb{Z}/p^v)$ is the p^v-torsion subgroup of $\mathrm{TR}_1^\bullet(A, M) \cong W_\bullet\Omega_{(A,M)}^1$. The map (2) desired above sends $\zeta \in \mu_{p^v}(K)$ to $d\log\zeta$. The main result of [HM03] is then the following.

Theorem 1 ([HM03, Theorem C]). *The map induced by* (1) *and* (2)

$$W_*\Omega_{(A,M)}^\bullet \otimes_\mathbb{Z} S(\mu_{p^v}) \longrightarrow \mathrm{TR}_*^\bullet((A, M), \mathbb{Z}/p^v)$$

is an isomorphism of log Witt complexes over (A, M). *Here* $S(\mu_{p^v})$ *is the symmetric algebra of* $\mu_{p^v}(K)$, *considered as a differential graded algebra with* μ_{p^v} *in degree two, with differential on the tensor product defined by*

$$d(1 \otimes \zeta) = d\log[\zeta] \otimes \zeta \quad (\zeta \in \mu_{p^v}).$$

By taking Frobenius invariants, one obtains a formula for $\mathrm{TC}_i((A, M); p, \mathbb{Z}/p^v)$ and consequently $K_i(K, \mathbb{Z}/p^v)$. The proof of Theorem C is quite easily reduced to the case $v = 1$, in which case the left hand side is a (twisted) symmetric algebra over the mod p reduction $\overline{W}_\bullet \Omega^*_{(A,M)}$ of the log de Rham–Witt complex. The proof in this case relies on an explicit calculation of this complex as well as the computation of the Tate spectral sequence.

The goal of this talk is to explain some details going into the proof of Theorem 1 pertaining to the log de Rham–Witt complex $W_* \Omega^\bullet_{(A,M)}$. More precisely, we are going to:

(1) Define the notion of a log Witt complex over a log ring (A, M) and the universal example, the log de Rham–Witt complex $W_* \Omega^\bullet_{(A,M)}$, and state the result that $\mathrm{TR}^\bullet_*((A, M), \mathbb{Z}/p^v)$ is a log Witt complex.

(2) Explicitly compute the mod p reduction

$$\overline{W}_* \Omega^\bullet_{(A,M)} := W_* \Omega^\bullet_{(A,M)} / p W_* \Omega^\bullet_{(A,M)}.$$

Part (2) will be self-contained except for a dimension computation (Proposition 11) which relies on the computation of the Tate spectral sequence (see Lectures 13 and 14).

Let us remark here that the assumption $p > 2$ is used many times in the proof. In fact, for $p = 2$ the very definition of a Witt complex needs to be changed; this has been carried out in V. Costeanu's thesis [Cos08]. The analogs of the results of [HM03] for $p = 2$ have not yet been obtained.

1. The log de Rham–Witt complex

1.1. **Witt vectors.** We briefly review some relevant facts about Witt vectors; the reader is advised to consult e.g. [Hes15, §1] for a complete discussion.

To any ring R, one functorially attaches rings $W_n(R)$ ($n \geq 1$) of *Witt vectors of length n*, with a functorial bijection of sets $W_n(R) = R^{\times n}$. The ring structure is uniquely determined by the requirement that the *ghost map* $w \colon W_n(R) \to R^{\times n}$

$$w(a_0, \ldots, a_{n-1}) = (a_0, a_0^p + p a_1, \ldots, a_0^{p^{n-1}} + p a_1^{p^{n-2}} + \ldots + p^{n-1} a_{n-1}),$$

is a natural transformation of ring-valued functors. One has $W_1(R) = R$ as rings, and the *restriction map* (forgetting the last coordinate)

$$R \colon W_n(R) \to W_{n-1}(R)$$

is a ring homomorphism. The composition $R^{n-1} \colon W_n(R) \to W_1(R) = R$ has a natural multiplicative section

$$[-]_n \colon R \to W_n(R), \quad [a]_n = (a, 0, \ldots, 0)$$

called the *Teichmüller character*. The *Verschiebung* (shift) map

$$V \colon W_{n-1}(R) \to W_n(R), \quad V(a_0, \ldots, a_{n-2}) = (0, a_0, \ldots, a_{n-2})$$

is additive and every element in $W_n(R)$ can be represented as

$$(a_0, \ldots, a_{n-1}) = [a_0]_n + V[a_1]_{n-1} + \ldots + V^{n-1}[a_{n-1}]_1.$$

The *Frobenius* $F \colon W_n(R) \to W_{n-1}(R)$ is a ring homomorphism satisfying $F[a] = [a^p]$. If $pR = 0$, then F is the map induced by the Frobenius on R by functoriality followed by the restriction map $R \colon W_n(R) \to W_{n-1}(R)$. In general, one has

$$F(a_0, \ldots, a_{n-1}) \equiv (a_0^p, \ldots, a_{n-2}^p) \mod pW_n(R).$$

The map V is F-linear in the sense that

$$V(F(x)y) = xV(y).$$

Moreover, $FV = p$ and $VF = V(1)$. It follows that

(3) $$V(x)V(y) = V(FV(x) \cdot y) = V(px \cdot y) = pV(xy).$$

1.2. **Log rings.** A *log ring* is a pair (R, M) where R is a ring and M is a commutative unital monoid endowed with a homomorphism $\alpha \colon M \to R$ to the multiplicative monoid underlying R (called here a log structure, though technically speaking maybe it should rather be called a pre-log structure). In our situation, we will endow $A = \mathcal{O}_K$ with the standard log structure $M = A \cap K^* \hookrightarrow A$. Alternatively, we could pick a uniformizer π of A and set $M = \mathbb{N}$ with the map $M \to A$ sending k to π^k, and get the same results in what follows, but then some functoriality is lost. We shall think of (A, M) as a compactification of K (relative to $\mathbb{Z}_p$), where the log structure is supposed to indicate that it is K, not A, that we care about. In particular, we expect certain invariants of (A, M) to reflect properties of K rather than A.

The module of differentials Ω_R^1 has a logarithmic variant $\Omega_{(R,M)}^1$, the module of *log differentials*: it is the initial target for a *log derivation*, i.e. a pair of maps

$$d \colon R \to \Omega_{(R,M)}^1 \quad \text{and} \quad d\log \colon M \to \Omega_{(R,M)}^1$$

where d is a derivation and $d\log$ is a homomorphism satisfying

$$\alpha(m) \cdot d\log m = d\alpha(m).$$

Explicitly, one can express $\Omega_{(R,M)}^1$ in terms of Ω_R^1 and M as follows:

$$\Omega_{(R,M)}^1 \cong \Omega_R^1 \oplus (R \otimes_{\mathbb{Z}} M^{\mathrm{gp}})/\langle (d\alpha(m), 0) - (0, \alpha(m) \otimes m) \rangle, \quad d\log m = (0, 1 \otimes m).$$

A *log differential graded algebra* over (R, M) is a differential graded algebra (E^*, d) over R endowed with a map $d\log \colon M \to E^1$ satisfying

$$\alpha(m) \cdot d\log m = d\alpha(m) \quad \text{and} \quad d \circ d\log = 0.$$

The universal (initial) such object is the *log de Rham complex* $\Omega_{(R,M)}^*$. The underlying graded algebra is the exterior algebra on $\Omega_{(R,M)}^1$.

Finally, let us note that the construction of Witt vectors $W_n(R)$ is easily adapted to log rings: given a log ring (R, M), we make $W_n(R)$ into a log ring using the composition

$$M \longrightarrow R \xrightarrow{[-]_n} W_n(R)$$

with the Teichmüller character. We shall write $(W_n(R), M)$ for this log ring.

1.3. Log Witt complexes and log de Rham–Witt. The classical de Rham–Witt complex $W\Omega_X^\bullet$ of a smooth scheme over a perfect field k of characteristic $p > 0$, defined by Deligne and Illusie [Ill79] inspired by ideas of Bloch, is a functorial complex computing the crystalline cohomology of X over $W(k)$. It has been generalized to log schemes by Hyodo and Kato [HK94].

Hesselholt and Madsen defined the absolute de Rham–Witt complex [HM04] of a scheme X over $\mathbb{Z}_{(p)}$ and its log variant in [HM03]. Their motivation is very different from the original one, as they are not interested in the hypercohomology of these complexes, but rather the structure of the spectra TR.

Let us cut to the chase and give the definition we will need.

Definition 2 ([HM03, Definition 3.2.1]). Let (R, M) be a log ring.

(1) A *log Witt complex* over (R, M) consists of:
  (i) a pro-log differential graded ring $(E_\bullet^*, M_E)$ together with a map of pro-log rings $\lambda \colon (W_\bullet(R), M) \to (E_\bullet^0, M_E)$;
  (ii) a map of pro-log graded rings

  $$F \colon E_n^* \to E_{n-1}^*$$

  such that $\lambda F = F\lambda$ and such that

  $$F d\log_n a = d\log_{n-1} a, \qquad\qquad \text{for all } a \in M$$
  $$F d[a]_n = [a]_{n-1}^{p-1} d[a]_{n-1}, \qquad\qquad \text{for all } a \in R;$$

  (iii) a map of pro-graded modules over the pro-graded ring $E_\bullet^*$,

  $$V \colon F^* E_n^* \to E_{n+1}^*$$

  such that $\lambda V = V\lambda$, $FV = p$ and $FdV = d$.
  A map of log Witt complexes over (R, M) is a map of pro-log differential graded rings which commutes with the maps λ, F and V.
(2) The *log de Rham–Witt complex* of (R, M) is the initial object of the category of Witt complexes over (R, M), denoted by $W_\bullet\Omega^*_{(R,M)}$.

The map F is a "divided Frobenius:" it does not commute with d, but it satisfies $dF = pFd$ (dually, we have $Vd = pdV$; both formulas easily follow from $d = FdV$). If R is an $\mathbf{F}_p$-algebra, then $F = p^i F_R \colon W_n\Omega_R^i \to W_{n-1}\Omega_R^i$ where F_R is the map of Witt complexes induced by the absolute Frobenius of R. We shall also use the following formula

$$V(x \cdot dy) = V(x \cdot FdVy) = V(x) \cdot dV(y).$$

The proofs of the existence of the initial objects $W_\bullet\Omega_R^*$ and $W_\bullet\Omega_{(R,M)}^*$ is a rather formal application of the adjoint functor theorem. By construction, the natural maps

$$\Omega_R^\bullet \to W_\bullet\Omega_R^* \quad \text{and} \quad \Omega_{(R,M)}^\bullet \to W_\bullet\Omega_{(R,M)}^*$$

are surjective.

Theorem 3 ([HM03, Proposition 3.3.1]). *The homotopy groups* $\mathrm{TR}_*^\bullet(A, M)$, *endowed with the maps* d, F, R, V, $[-]_n$ *and* $d\log_n$ *form a log Witt complex over* (A, M).

2. The log de Rham–Witt complex of (A, M) modulo p

Let A again be the ring of integers of K, endowed with the standard log structure $M = A \cap K^\times \hookrightarrow A$. The goal of this section is to study the mod p reduction $\overline{W}_\bullet\Omega_{(A,M)}^*$ of the log de Rham–Witt complex $W_\bullet\Omega_{(A,M)}^*$ of (A, M).

The ring A itself admits the following description: there is a unique ring homomorphism

$$f\colon W(k) \to A$$

inducing the identity on k. This extension of complete discrete valuation rings is totally ramified, thus if π is a uniformizer of A (which we fix from now on), then it generates A as a $W(k)$-algebra and its minimal polynomial $\phi(x)$ is an Eisenstein polynomial. In other words, A admits the following presentation

$$A \cong W(k)[\pi]/(\pi^e + p \cdot \theta(\pi))$$

with $\theta(x) \in W(k)[x]$ of degree $< e$ and invertible constant term. The integer e is called the absolute ramification index of K. In particular, $\overline{W}_1(A) = A/p = k[x]/(x^e)$ is a truncated polynomial ring.

It follows from this presentation that the module of relative log differentials $\Omega_{(A,M)/W(k)}^1$ (where $W(k)$ is given the trivial log structure) can be written as

$$(4) \qquad\qquad \Omega_{(A,M)/W(k)}^1 \cong A/(\pi\phi'(\pi)) \cdot d\log\pi.$$

Later on, we shall need the following:

Lemma 4. *The i-th module of log differentials* $\Omega_{(A,M)}^i = \bigwedge^i \Omega_{(A,M)}^1$ *is zero modulo* p *for* $i \geq 2$.

In fact, the module is even uniquely divisible [HM03, Lemma 2.2.4], but we shall not need this. Note that when we write Ω_A^1, $\Omega_{(A,M)}^1$, we treat A as an abstract ring (with no topology), which is not the same as considering continuous differentials. For example, $\Omega_{\mathbb{Z}_p}^1$ is quite large while the corresponding continuous differentials vanish.

Proof. First we show that $\Omega_{W(k)}^1$ is zero modulo p: we have

$$\Omega_{W(k)}^1/p = \Omega_{W(k)}^1/(p, dp) = \Omega_k^1 = 0.$$

Next, $\Omega^1_{(A,M)}$ sits in an exact sequence

$$\Omega^1_{W(k)} \otimes_{W(k)} A \to \Omega^1_{(A,M)} \to \Omega^1_{(A,M)/W(k)} \to 0,$$

where the module on the right is a cyclic torsion A-module (4). When we take exterior powers $\bigwedge^i$ for $i \geq 2$, this torsion module disappears, and therefore the result $\Omega^i_{(A,M)}$ is uniquely divisible. $\qquad\square$

2.1. **Degree zero.** We shall now discuss $\overline{W}_n(A) := W_n(A)/pW_n(A)$. For starters, let us deal with $A = \mathbb{Z}_p$.

Lemma 5. *The $\mathbb{F}_p$-algebra $\overline{W}_n(\mathbb{Z}_p)$ admits the elements*

$$1, V(1), V^2(1), \ldots, V^{n-1}(1)$$

as a basis, and

$$V^s(1) \cdot V^t(1) = 0$$

if $s, t > 0$.

Proof. For any p-torsion free ring R, the sequences

(5) $$0 \to R \xrightarrow{V^{n-1}} \overline{W}_n(R) \to \overline{W}_{n-1}(R) \to 0$$

are exact. In particular,

$$0 \to \mathbb{F}_p \xrightarrow{V^{n-1}} \overline{W}_n(\mathbb{Z}_p) \to \overline{W}_{n-1}(\mathbb{Z}_p) \to 0$$

are exact, and it follows by induction that the elements $V^s(1)$ form an $\mathbb{F}_p$-basis. To prove the second assertion, note that (3) shows that

$$V(x) \cdot V(y) = 0$$

in $\overline{W}_n(R)$ for any ring R. $\qquad\square$

The above proof shows that for an arbitrary ring R, the image of $V(W_{n-1}(R))$ in $\overline{W}_n(R)$ is an ideal of square zero. The short exact sequence (exact also on the left if R has no p-torsion)

$$\overline{W}_{n-1}(R) \xrightarrow{V} \overline{W}_n(R) \to R/p \to 0$$

shows that $\overline{W}_n(R)$ is a square-zero extension of R/p. This extension does not split in general.

The k-algebra structure on $\overline{W}_n(A)$. Passing to $\overline{W}_n(A)$, we note first that the p-th power of the Teichmüller character

$$[-]^p_n \colon A \to W_n(A),$$

which is multiplicative, becomes also additive modulo p, that is

$$[x]^p_n + [y]^p_n \equiv [x+y]^p_n.$$

This way, we can consider $\overline{W}_n(A)$ as an algebra over A/p, which itself is a k-algebra via $k = W(k)/p \to A/p$. Since k is perfect, we can undo the Frobenius twist we just made, by considering $\overline{W}_n(A)$ as a k-algebra via

$$\rho_n\colon k \xrightarrow{F_k^{-1}} k \to A/p \xrightarrow{[-]_n \mod p} \overline{W}_n(A) \xrightarrow{F_{\overline{W}_n(A)}} \overline{W}_n(A).$$

Explicitly, this map sends $a \in k$ to the class of $[b]_n^p$ modulo p, where $b \in W(k) \subseteq A$ is any lift of $a^{1/p}$. Our goal is to obtain a presentation of $\overline{W}_n(A)$ as a k-algebra via the map ρ_n. We will make frequent use of the filtration induced by the V-filtration on $W_n(A)$.

The structure of $\overline{W}_n(A)$ as a k-algebra. Using the fact that

$$V^s[-]_n\colon A \to W_n(A)$$

is additive modulo V^{s+1}, it is easy to see that the elements $V^s[\pi^i]_n$ with $s < n$ and $i < e$, form a k-basis of $\overline{W}_n(A)$. The multiplication table is not too complicated. Let us try to compute $V^s[\pi^i] \cdot V^t[\pi^j]$: using the relation (3), we have

$$V^s[\pi^i] \cdot V^t[\pi^j] = 0 \quad \text{if } s,t > 0.$$

For $s = 0$, we compute:

$$[\pi^i] \cdot V^t[\pi^j] = V^t(F^t[\pi^i] \cdot [\pi^j]) = V^t([\pi^{p^t i + j}]).$$

Here we also used the relation $F[a] = [a^p]$ modulo p. Since the resulting exponent $p^t i + j$ might be $\geq e$, we need to express $V^s([\pi^i])$ with $i \geq e$ in the chosen basis.

To this end, let us first deal with $[\pi^e]$. Here is the key computation:

$$[\pi^e] = [-p\theta_K(\pi)] = [-p] \cdot [\theta_K(\pi)] = V(1) \cdot [\theta_K(\pi)]$$
$$= V(F(\theta_K(\pi))) = V([\theta_K(\pi)^p]) = V(\theta_K^{(1)}([\pi]^p)) = V(1)\theta_K([\pi]).$$

Here we used the relation $[-p] = V(1)$ modulo p, and $\theta_K^{(1)}$ is obtained from the image of θ_K in $k[x]$ by raising the coefficients to the p-th power. For example, if $K = W(k)[p^{1/e}]$, so $\theta_K = 1$, then we obtain $[\pi^e] = V(1)$, and in general

$$V^s[\pi^{e+i}] = V^s(V(1)[\pi^i]) = V^s(V(F[\pi^i])) = V^{s+1}([\pi^{pi}]).$$

This calculation suggests that we should adapt our basis to θ_K. This is done as follows: introduce the *modified Verschiebung*

$$V_\pi\colon \overline{W}_{n-1}(A) \to \overline{W}_n(A), \quad V_\pi(a) = \theta_K([\pi])V(a),$$

then we still have $FV_\pi = 0 \mod p$. And since $\theta_K(0)$ is a unit, the elements

$$V_\pi^i([\pi^j]) \quad \text{with } i = 0, \dots, n-1, \quad j = 0, \dots, e-1$$

also form a basis of $\overline{W}_n(A)$. We have now seen all ingredients of the proof of:

Proposition 6 ([HM03, Proposition 3.1.5]). *In the algebra $\overline{W}_n(A)$, the elements $V_\pi^s([\pi^i])$ ($s < n$, $i < e$) form a k-basis. Moreover*

$$V_\pi^s([\pi^i]) \cdot V_\pi^t([\pi^j]) = \begin{cases} 0 & s,t > 0 \\ V_\pi^t([\pi^{p^t i + j}]) & s = 0. \end{cases}$$

and

$$V_\pi^s([\pi]^{e+i}) = V_\pi^{s+1}([\pi]^{pi}).$$

In particular, the product of two basis elements is either zero or another basis element.

An interesting feature of this description is that the resulting k-algebra $\overline{W}_n(A)$ depends only on the absolute ramification index e (up to isomorphism depending on the choice of the uniformizer).

Addendum. For future reference, we note that repeated application of the second relation yields the following. Let $r(i)$ be the p-adic valuation of $i - pe/(p-1)$, and let $s \leq r(i)$, so that p^s divides the number

$$(i - pe/(p-1)) + p^{s+1}e/(p-1) = i + pe + p^2e + \ldots + p^s e.$$

Then

$$(6) \quad [\pi^{p^{-s}(i+pe+p^2e+\ldots+p^s e)}] = V([\pi^{p^{-s+1}(i+pe+p^2e+\ldots+p^s e)-e}])$$
$$= V([\pi^{p^{-s+1}(i+pe+p^2e+\ldots+p^{s-1}e)}]) = \ldots = V^s([\pi^i]).$$

2.2. **Degrees** ≥ 2. We will now show that $\overline{W}_n\Omega^i_{(A,M)}$ is zero for $i \geq 2$. In fact, $W_n\Omega^i_{(A,M)}$ is uniquely divisible for $i \geq 2$. In Lemma 4, we have shown this for $\Omega^*_{(A,M)}$, which is the case $n = 1$. To deduce the divisibility of $W_n\Omega^i_{(A,M)}$ for larger n by induction, we need a devissage argument relating the kernel of the restriction map

$$R\colon W_n\Omega^i_{(A,M)} \to W_{n-1}\Omega^i_{(A,M)}$$

to the modules $\Omega^*_{(A,M)}$. Let us pause to think of what an element in the kernel of R should look like. First, an element of the form $V^n(x)$ should be killed, and for $i = 0$ this describes the kernel completely. Second, it may happen that $dV^n(x)$ is not of this form, since $Vd = pdV$, while still $dV^n(x)$ should be killed by R. This motivates the definition of the *standard filtration* (which makes functorial sense for any Witt complex):

$$\mathrm{Fil}^s W_n\Omega^i_{(A,M)} = V^s W_{n-s}\Omega^i_{(A,M)} + dV^s W_{n-s}\Omega^{i-1}_{(A,M)}.$$

Lemma 7 ([HM03, Lemma 3.2.4]). *The kernel of*

$$R^s\colon W_n\Omega^i_{(A,M)} \to W_{n-s}\Omega^i_{(A,M)}$$

equals $\mathrm{Fil}^s W_n\Omega^i_{(A,M)}$.

Proof. For fixed value of $n - s$, the system $W_n\Omega^i_{(A,M)}/\mathrm{Fil}^s$ forms a log Witt complex, and can be shown to satisfy the universal property. $\qquad\square$

This implies that we have a surjection

$$N\colon \Omega^i_{(A,M)} \oplus \Omega^{i-1}_{(A,M)} \to \mathrm{Fil}^{n-1} W_n\Omega^i_{(A,M)}, \quad N(\omega_1,\omega_2) = dV^{n-1}\lambda(\omega_1) + V^{n-1}\lambda(\omega_2).$$

By induction, Lemma 4 implies then that $W_n\Omega^i_{(A,M)}$ is uniquely divisible for $i \geq 3$. It remains to analyze the structure of $\overline{W}_n\Omega^2_{(A,M)}$. To show that $\overline{W}_n\Omega^2_{(A,M)} = 0$, by induction it suffices to show the following lemma:

Lemma 8. *The map*

$$dV^{n-1} \colon \Omega^1_{(A,M)} = W_1\Omega^1_{(A,M)} \to W_n\Omega^2_{(A,M)}$$

is zero modulo p.

Proof. The source is generated by the elements $\pi^i d\log\pi$ with $i < e$. Since in any Witt complex we have

$$V(x)d\log m = V(x \cdot F(d\log m)) = V(x \cdot d\log m),$$

we see that

$$V^{n-1}(\pi^i d\log\pi) = V^{n-1}[\pi^i] \cdot d\log\pi.$$

We will show by induction on s that the elements $V^s(\pi^i d\log\pi) = V^s[\pi^i] \cdot d\log\pi$ are closed. For $s = 0$, this element is either $d\log\pi$ (and we require that $d \circ d\log = 0$), or $i > 0$ and

$$[\pi^i]d\log\pi = [\pi^{i-1}]d[\pi]$$

which is closed. For the induction step, suppose that $s > 0$ and that i is prime to p. Then

$$[\pi^i]d\log\pi = \frac{1}{i}d[\pi^i],$$

which is exact, while $Vd = pdV = 0 \bmod p$, so in fact $V^s([\pi^i]d\log\pi) = 0$. On the other hand, if $i = pi'$ then the second relation in Proposition 6 shows that

$$V^s[\pi^i] = V^s[\pi^{pi'}] = V^{s-1}[\pi^{e+i'}],$$

which is closed by the induction assumption. $\square$

2.3. **Degree 1.** From what we have learned so far, we see that $\overline{W}_n\Omega^1_{(A,M)}$ is spanned by monomials in the elements

$$V^s_\pi([\pi^i]), \quad dV^s_\pi([\pi^i]) \quad \text{and} \quad V^s_\pi([\pi^i]d\log\pi) = V^s_\pi([\pi^i])d\log\pi$$

with $s < n$ and $i < e$. By an explicit calculation of the $\overline{W}_n(A)$-module structure on $\overline{W}_n\Omega^1_{(A,M)}$ using Proposition 6, we can see that in fact the elements of the latter two types span $\overline{W}_n\Omega^1_{(A,M)}$ as a k-vector space.

Proposition 9 ([HM03, Proposition 3.4.1(ii)])**.** *We have the following formulas in $\overline{W}_n\Omega^1_{(A,M)}$:*

$$V^s_\pi([\pi^i]) \cdot dV^t_\pi([\pi^j]) = \begin{cases} dV^t_\pi([\pi^{p^t i + j}]) - iV^t_\pi([\pi^{p^t i + j}]d\log\pi) & \text{if } 0 = s \leq t, \\ -iV^t_\pi(\theta_K(\pi)^{p^{t-s}\left(\frac{p^{s+1}-1}{p-1}\right)}[\pi^{p^{t-s}i+j}]d\log\pi) & \text{if } 0 < s \leq t, \\ jV^s_\pi(\theta_K(\pi)^{p^{s-t}\left(\frac{p^{t+1}-1}{p-1}\right)}[\pi^{i+p^{s-t}j}]d\log\pi) & \text{if } s \geq t, \end{cases}$$

$$V_\pi^s([\pi^i]) \cdot V_\pi^t([\pi^j]d\log\pi) = \begin{cases} V_\pi^t([\pi^{p^t i + j}]d\log\pi) & \text{if } s = 0, \\ V_\pi^s([\pi^{i + p^s j}]d\log\pi) & \text{if } t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the 2ne elements $dV_\pi^s([\pi^i])$, $V_\pi^s([\pi^i])d\log\pi$ *($s < n$, $i < e$) span* $\overline{W}_n\Omega^1_{(A,M)}$.

Proof. These formulas follow from differentiating the formulas in Proposition 6 and the formula

$$FdV_\pi(x) = \theta([\pi])^p dx. \qquad \square$$

So we have $2ne$ elements spanning a vector space. It turns out that almost half of them vanish, as the following proposition shows.

Proposition 10 ([HM03, Proposition 3.4.1(i)]). *Let $r(i)$ be the p-adic valuation of $i - pe/(p-1)$ and let $\alpha = p^{-r(i)}(i - pe/(p-1)) \in \mathbf{F}_p^\times$. Then for every $i < e$*

$$dV_\pi^0([\pi^i]) = dV_\pi([\pi^i]) = \ldots = dV_\pi^{r(i)-1}([\pi^i]) = 0,$$
$$dV_\pi^{r(i)}([\pi^i]) = \alpha \cdot V_\pi^{r(i)}([\pi^i]d\log\pi),$$
$$V_\pi^{r(i)+1}([\pi^i]d\log\pi) = V_\pi^{r(i)+2}([\pi^i]d\log\pi) = \ldots = V_\pi^{n-1}([\pi^i]d\log\pi) = 0.$$

In particular, the ne elements

(7) $\qquad \left\{ dV_\pi^s([\pi^i]d\log\pi) \,|\, i < e, \, s < r(i) \right\} \cup \left\{ dV_\pi^s([\pi^i]) \,|\, i < e, \, r(i) \leq s < n \right\}$

span $\overline{W}_n\Omega^1_{(A,M)}$.

Proof. If $s \leq r$, then (6) shows that

$$V^s([\pi^i]) = [\pi^{p^{r(i)-s}\alpha}],$$

so

$$dV^s([\pi^i]) = d[\pi^{p^{r(i)-s}\alpha}] = p^{r(i)-s}\alpha[\pi^{p^{r(i)-s}\alpha - 1}]d[\pi],$$

which is zero mod p if $r(i) - s > 0$, showing the first relation, and for $s = r(i)$ it equals

$$\alpha[\pi^{\alpha-1}]d[\pi] = \alpha[\pi^\alpha]d\log[\pi] = \alpha V_\pi^{r(i)}([\pi^i]d\log\pi).$$

The last set of relations follows from

$$V_\pi^{r(i)+j}([\pi^i]d\log\pi) = V_\pi^j(V_\pi^{r(i)}([\pi^i]d\log\pi)) = V_\pi^j(\alpha^{-1}dV_\pi^{r(i)}([\pi^i]))$$

which vanishes for $j > 0$ since dV_π is zero modulo p. $\qquad \square$

Theorem 11 ([HM03, Proposition 6.1.1]). $\dim_k \overline{W}_n\Omega^1_{(A,M)} = ne$. *In particular, the elements* (7) *form a basis of* $\overline{W}_n\Omega^1_{(A,M)}$.

About the proof. We know already that the dimension is at most ne. To show that it is $\geq ne$, it is enough to exhibit a log Witt complex $E_\bullet^*$ with $\dim \overline{E}_n^1 \geq ne$ and $W_\bullet\Omega^*_{(A,M)} \to E_\bullet^*$ surjective in degree 1. Such an example is provided by the log Witt complex $\mathrm{TR}_\bullet^*(A, M)$; the dimension of $\mathrm{TR}_n^1((A, M), \mathbb{Z}/p)$ is computed to be ne in Section 6, using the computation of the Tate spectral sequence in Section 5 (see Lecture 13). $\qquad \square$

2.4. **A generalization to smooth schemes over** A**.** In the paper [GH06], Geisser and Hesselholt generalize the above computation to smooth schemes X over A. As can be seen from [HM03], in the case of $\operatorname{Spec} A$, the cohomology groups $H^q(K, \mu_{p^v}^{\otimes s})$ can be identified as the Frobenius invariants inside log de Rham–Witt (at least if K contains μ_{p^v}). The analog of these cohomology groups are the *sheaves of nearby cycles* $i^* R^q j_* \mu_{p^v}$ where $i \colon X_k \to X$ resp. $j \colon X_K \to X$ is the inclusion of the special resp. general fiber. We endow X with the log structure M_X induced by the open immersion j.

Theorem 12 ([GH06, Theorem A]). *Suppose that K contains p^v-th roots of unity. Then for every $q \geq 0$ there is a natural exact sequence*

$$0 \to i^* R^q j_* \mu_{p^v}^{\otimes q} \to i^* W\Omega_{(X,M_X)}^q / p^v \xrightarrow{1-F} i^* W\Omega_{(X,M_X)}^q / p^v \to 0.$$

As in [HM03], one ingredient of the proof is the computation of the mod p reduction $\overline{W}\Omega_{(X,M_X)}^*$ of the log de Rham–Witt complex of (X, M_X).

Theorem 13 ([GH06, Proposition 1.3.2]). *Let $d = \dim X_k = \dim X - 1$. Locally on X_k, the sheaf $i^* \overline{W}_n \Omega_{(X,M_X)}^*$ on X_k can be given a non-canonical structure of a vector bundle of rank*

$$\binom{d+1}{q} e \sum_{s=0}^{n-1} p^{ds}.$$

Note that for $d = 0$ we obtain ne for $q = 0, 1$ and 0 otherwise, while for $n = e = 1$ the rank is the rank of $\bigwedge^q \Omega_{(X,M_X)}^1$.

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The Tate spectral sequence for THH of log rings
ACHIM KRAUSE

This talk is concerned with the computation of

$$\mathrm{TR}(A, M; p) = \lim \left(\cdots \xrightarrow{R} \mathrm{THH}(A, M)^{C_{p^2}} \xrightarrow{R} \mathrm{THH}(A, M)^{C_p} \xrightarrow{R} \mathrm{THH}(A, M) \right),$$

following [1]. Here p is an odd prime, and A is a complete discrete valuation ring of mixed characteristic $(0, p)$, with perfect residue field k and field of fractions K, endowed with the canonical pre-log structure $M = A \cap K^\times$.

THH$(A, M)^{C_{p^n}}$ denotes "genuine fixed points", which can be defined without appealing to genuine equivariant homotopy theory as the iterated pullback

(1) $\quad \mathrm{THH}^{C_{p^n}} := \mathrm{THH}^{hC_{p^n}} \times_{\mathrm{THH}^{tC_{p^n}}} \mathrm{THH}^{hC_{p^{n-1}}} \times_{\mathrm{THH}^{tC_{p^{n-1}}}} \cdots \times_{\mathrm{THH}^{tC_p}} \mathrm{THH} .$

(Here we have abbreviated THH(A, M) by THH for reasons of space.)

In particular, the inverse limit $\mathrm{TR}(A, M; p)$ can be thought of as the infinite iterated pullback of this form.

The computation proceeds by means of the Tate spectral sequence

$$\widehat{H}^{-i}(C_{p^n}; \pi_j(\mathrm{THH}(A, M)/p)) \Rightarrow \pi_{i+j}(\mathrm{THH}(A, M)^{tC_{p^n}}/p).$$

Here the grading convention is homological Serre grading, so the differential d^r goes from bidegree $(i, j) \to (i - r, j + r - 1)$, and the abutment is such that the homotopy group $\pi_m(\mathrm{THH}(A, M)^{tC_{p^n}}/p)$ is an iterated extension of the E^∞-terms along the codiagonal $i + j = m$.

One has an explicit description of the E^2-page. This depends on conventions for generators of the group cohomology of C_{p^n}. In [1], Lemma 4.2.1, concrete generators $t \in H^2(C_{p^n}; \mathbb{F}_p)$ and $u_n \in H^1(C_{p^n}; \mathbb{F}_p)$ are constructed.

Lemma 1. *For A a complete discrete valuation ring of mixed characteristic $(0, p)$ with residue field k and field of fractions K, let e_K be the ramification degree of K. Choose a generator π_K of the maximal ideal of A. Then*

$$\pi_*(\mathrm{THH}(A, M)/p) = k[\pi_K, \kappa]/\pi_K^{e_K} \otimes_k \Lambda_k(\mathrm{dlog}\,\pi_K)$$

where π_K is the image of $\pi_K \in A$ in $\pi_0(\mathrm{THH}(A, M))$, $\mathrm{dlog}\,\pi_K$ is an element in degree 1 coming from the log ring structure, and κ is the unique element in degree 2 sent to the p-torsion element $\mathrm{dlog}(-p)$ under the connecting homomorphism $\pi_2(\mathrm{THH}(A, M)/p) \to \pi_1(\mathrm{THH}(A, M))$.

With this description,

$$\widehat{H}^*(C_{p^n}; \pi_*(\mathrm{THH}(A, M)/p)) = k[\pi_K, \kappa, t^{\pm 1}]/\pi_K^{e_K} \otimes_k \Lambda_k(\mathrm{dlog}\,\pi_K, u_n),$$

where the bidegrees of generators are given by $|\pi_K| = (0,0)$, $|\kappa| = (0,2)$, $|\mathrm{dlog}\,\pi_K| = (0,1)$, $|u_n| = (-1,0)$, $|t| = (-2,0)$.

The d^2-differentials in the Tate spectral sequence are determined by the Connes operator, and can be seen to satisfy

$$d^2 \pi_K = t \pi_K \,\mathrm{dlog}\,\pi_K$$
$$d^2 \kappa = t \kappa \,\mathrm{dlog}(-p)$$

One can show the following two facts about the spectral sequence:

(1) The elements $\mathrm{dlog}\, x$ are always permanent cycles, coming from $x \in K^\times$ under the composition

$$K^\times \to K_1(K) \xrightarrow{\mathrm{tr}} \pi_1 \,\mathrm{TR}(A, M; p)$$

(2) The elements $-t\kappa^p$ and $(-t\kappa)^{p^n}$ are permanent cycles in the Tate spectral sequence for $(\mathrm{THH}(A, M)/p)^{tC_{p^n}}$, representing the image of $v_1 \in \pi_{2(p-1)}(\mathbb{S}/p)$ in $\pi_*(\mathrm{THH}(A, M)^{tC_{p^n}}/p)$ and $V^{n-1}(1) \in \pi_*(\mathrm{THH}(A, M)^{tC_{p^n}}/p)$, i.e. the image of $1 \in \pi_*(\mathrm{THH}(A, M)^{tC_p}/p)$ under the transfer

$$\mathrm{THH}^{tC_p} \to \mathrm{THH}^{tC_{p^n}}$$

associated to the inclusion $C_p \subset C_{p^n}$.

For $n = 1$, THH^{tC_p} is a module over THH via the Frobenius map

$$\mathrm{THH} \xrightarrow{\varphi_p} \mathrm{THH}^{tC_p}.$$

Since $\mathrm{THH}(A, M)$ is a module over the Eilenberg-MacLane spectrum HA, this implies that both $\mathrm{THH}(A, M)$ and $\mathrm{THH}(A, M)^{tC_p}$ are generalized Eilenberg-Mac-Lane spectra. In particular, for $n = 1$, the action of $v_1 \in \pi_*(\mathbb{S}/p)$ on $\pi_*(\mathrm{THH}(A, M)^{tC_p}/p)$ has to be zero.

It follows that the permanent cycle $-t\kappa^p$ has to be hit by a differential.

For simplicity, let us look at the unramified case $K = W(k)$, so $e_K = 1$. In that situation, the E^2-page is given by $k[\kappa, t^{\pm 1}] \otimes_k \Lambda_k(\mathrm{dlog}(-p), u_n)$. After the d^2-differential, which is determined by $d^2\kappa = t\kappa\,\mathrm{dlog}(-p)$, the E^3-page takes the form $k[(\kappa^p), t^{\pm 1}] \otimes_k \Lambda_k(\mathrm{dlog}(-p), u_n)$.

For degree reasons, the only differential that can hit $t\kappa^p$ is $d^{2p+1}(t^{-p}u_1) \doteq t\kappa^p$. Since this implies that $d^{2p+1}(u_1) \doteq t^{p+1}\kappa^p = t(t\kappa)^p$, and $(t\kappa)^p$ is a permanent cycle, it also follows that t is a permanent cycle. Thus, we have a periodic family of differentials, and the E^{2p+2}-page consists only of $k[t^{\pm 1}] \otimes \Lambda(\mathrm{dlog}(-p))$. For degree reasons, the spectral sequence degenerates at this stage, and $\pi_*(\mathrm{THH}^{tC_p}/p)$ is just a single copy of k in each degree (positive and negative).
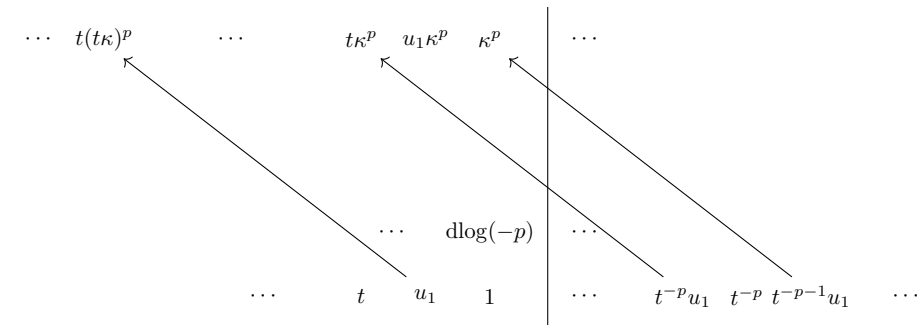
It turns out that the Frobenius map $\mathrm{THH} \xrightarrow{\varphi_p} \mathrm{THH}^{tC_p}$ is actually an equivalence on p-completed connective covers. One can see this for $\mathrm{THH}(A, M)$ for

general A, too: Using the fact that the spectral sequence is natural in A one can see that all differentials d_r for $2 < r < p+1$ vanish, and by comparison with the unramified case one sees d_{2p+1} to be nontrivial.

Lemma 2. *For A a complete discrete valuation ring of mixed characteristic as above, the Frobenius map*

$$\mathrm{THH}(A, M) \xrightarrow{\varphi_p} \mathrm{THH}(A, M)^{tC_p}$$

is an equivalence on p-completed connective covers.

Theorem 3. *For all $n \geq 1$, the maps*

$$\mathrm{THH}(A, M)^{C_{p^n}} \to \mathrm{THH}(A, M)^{hC_{p^n}} \to \mathrm{THH}(A, M)^{tC_{p^{n+1}}}$$

are equivalences after passing to p-completed connective covers.

Proof. From the Tate-Orbit Lemma of [2] and Lemma 2, one obtains that the map

$$\mathrm{THH}(A, M)^{hC_{p^n}} \xrightarrow{\varphi_p} (\mathrm{THH}(A, M)^{tC_p})^{hC_{p^n}} \simeq \mathrm{THH}(A, M)^{tC_{p^{n+1}}}$$

is an equivalence on p-completed connective covers. Combining this with the definition of genuine fixed points (1), one sees that the projection map

$$\mathrm{THH}(A, M)^{C_{p^n}} \to \mathrm{THH}(A, M)^{hC_{p^n}}$$

is an equivalence on p-completed connective covers, too. $\square$

Having determined the structure of the Tate spectral sequence for $n = 1$, one can also compute $\mathrm{THH}(A, M)^{hC_p}$: One compares the Tate spectral sequence to the homotopy fixed point spectral sequence

$$H^{-i}(C_p; \pi_j(\mathrm{THH}(A, M)/p)) \Rightarrow \pi_{i+j}(\mathrm{THH}(A, M)^{hC_p}/p)$$

whose E^2-page agrees with the second-quadrant truncation of the Tate spectral sequence. The differentials in the homotopy fixed point spectral sequence are completely determined by the differentials in the corresponding Tate spectral sequence.

As a result of the truncation, terms along the right edge that were hit in the Tate spectral sequence are not hit in the homotopy fixed point spectral sequence. For example, the element $(-t\kappa^p)$, detecting v_1, cannot be hit anymore for degree reasons (it was hit by $t^{-p}u_1$ via d^{2p+1} in the Tate spectral sequence, which has origin in the truncated area).

Instead, some higher power of $(-t\kappa^p)$ is being hit in the homotopy fixed point spectral sequence, namely

$$(-t\kappa^p)^{p+1} = t^{p+1}\kappa^p \kappa^{p^2} = d^{2p+1}(u_1 \kappa^{p^2}),$$

and for degree reasons, this is the first power of $(-t\kappa^p)$ that can be hit by a differential. One sees that in $\pi_*(\mathrm{THH}(A, M)^{hC_p}/p)$, $v_1^{p+1} = 0$, and $v_1^j \neq 0$ for $j < p+1$.

Through the equivalence on positive-degree homotopy groups given by the map $\mathrm{THH}(A, M)^{hC_p}/p \to \mathrm{THH}(A, M)^{tC_{p^2}}/p$ from Theorem 3, one thereby infers information on the Tate spectral sequence for $\mathrm{THH}(A, M)^{tC_{p^2}}/p$. For example, we can see that v_1^{p+1} acts trivially, so the Tate spectral sequence for C_{p^2} will have a

horizontal vanishing line again, but of height about $2p^2$. This is caused by a longer differential than in the C_p case, namely a non-zero d^{2p^2+2p+1}-differential on u_2.

By truncating this Tate spectral sequence for C_{p^2}, one obtains the homotopy spectral sequence for C_{p^2}, which in turn yields information on the Tate spectral sequence for C_{p^3} by Theorem 3. Through a complicated induction, one can completely determine the structure of the Tate and homotopy fixed point spectral sequences for all K and C_{p^n} with $v_p(e_K) \geq n$, i.e. for sufficiently wildly ramified K. The case of general K is then decided by passing to a suitable extension L/K with $v_p(e_L) \geq n$.

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Proving the main results in Hesselholt-Madsen
Vigleik Angeltveit

Recall that K is a complete discrete valuation field of mixed characteristic $(0, p)$ with p odd. Associated to K we have its ring of integers $A = \mathcal{O}_K$ and a log structure given by the inclusion $\alpha : M = A \cap K^\times \to A$. We choose a uniformizer π, and we assume that the quotient field $k = A/(\pi)$ is perfect. Also recall that $\Omega^*_{(A,M)}$ denotes the universal log differential graded ring, and that $W_\bullet \Omega^*_{(A,M)}$ is the log de Rham Witt complex, which is defined as the initial log Witt complex over (A, M). Our first goal is to prove the following:

Theorem 1 (Hesselholt-Madsen [1]). *Suppose $\mu_{p^v} \subset K$. Then*

$$W_\bullet \Omega^*_{(A,M)} \otimes S_{\mathbb{Z}/p^v}(\mu_{p^v}) \to \mathrm{TR}^\bullet_*(A, M, \mathbb{Z}/p^v)$$

is an isomorphism of pro-abelian groups.

Here $S_{\mathbb{Z}/p^v}(\mu_{p^v}) \cong \mathbb{Z}/p^v[x]$ with $|x| = 2$ is a polynomial ring on a generator in degree 2. It is convenient to not having to explicitly choose a generator. The group denoted $\mathrm{TR}^n(-)$ usually has an extra p in the notation to remind us that we are taking $C_{p^{n-1}}$ fixed points of THH, but because p is fixed we omit it from the notation.

We interpret this as saying that $\mathrm{TR}^n_*(A, M, \mathbb{Z}/p^v)$ is close to $W_n \Omega^*_{(A,M)} \otimes \mathbb{Z}/p^v[x]$. Hence we can compute $\mathrm{TC}_*(A, M)$ by understanding $W_\bullet \Omega^*_{(A,M)}$ and the x just comes along. Hence $\mathrm{TC}_*(A, M, \mathbb{Z}/p^v)$ is 2-periodic whenever $\mu_{p^v} \subset K$. We also know that the cyclotomic trace map $\mathrm{trc} : K_*(K) \to \mathrm{TC}_*(K)$ is an equivalence after p-completing, so the above result shows that algebraic K-theory is also 2-periodic when $\mu_{p^v} \subset K$.

Theorem 2 (Hesselholt-Madsen [1]). *For $s \geq 1$, and K not necessarily containing μ_{p^v}, there are canonical isomorphisms*

$$K_{2s}(K, \mathbb{Z}/p^v) \cong H^0(K, \mu_{p^v}^{\otimes s}) \oplus H^2(K, \mu_{p^v}^{\otimes s+1})$$

$$K_{2s-1}(K, \mathbb{Z}/p^v) \cong H^1(K, \mu_{p^v}^{\otimes s})$$

If $\mu_{p^v} \subset K$ then $\mathrm{Gal}(K)$ acts trivially on μ_{p^v}, so we get the same answer for all $s \geq 1$.

In general we have a Galois cohomology spectral sequence converging to the étale K-theory $K_*^{\text{ét}}(K, \mathbb{Z}/p^v)$ with the same E_2-term as in Theorem 2. For $v = 1$ and $\mu_p \subset K$ we can compute both sides and prove directly that the canonical map

$$K_*(K, \mathbb{Z}/p) \to K_*^{\text{ét}}(K, \mathbb{Z}/p)$$

is an isomorphism for $* \geq 1$. If K does not contain μ_p we can pass to a field extension that does, and the passage from $\mathbb{Z}/p$ coefficients to $\mathbb{Z}/p^v$ coefficients is formal. So in the end Theorem 2 follows by a comparison with étale K-theory.

We return to the proof of Theorem 1. Recall that if $\mu_p \subset K$ a choice of generator of μ_p determines a Bott class $b_n \in \mathrm{TR}_2^n(A, M, \mathbb{Z}/p)$. The class b_n^{p-1} is independent of this choice, and is the image of $v_1 \in \pi_{2p-2}(\mathbb{S}/p)$. Similarly, if $\mu_{p^v} \subset K$ we have such a class in $\mathrm{TR}_2^n(A, M, \mathbb{Z}/p^v)$. We can use that to build a map $S_{\mathbb{Z}/p^v}(\mu_{p^v}) \to \mathrm{TR}_*^n(A, M)$ that is independent of choices for each n.

Recall that $W_\bullet \Omega_{(A,M)}^* \to \mathrm{TR}_*^\bullet(A, M)$ is an isomorphism in degree $* \leq 2$. For $\mu_{p^v} \subset K$ we then get a map

$$(1) \qquad W_\bullet \Omega_{(A,M)}^* \otimes S_{\mathbb{Z}/p^v}(\mu_{p^v}) \to \mathrm{TR}_*^\bullet(A, M, \mathbb{Z}/p^v).$$

Here R, F and V act as the identity on $S_{\mathbb{Z}/p^v}(\mu_{p^v})$ while d acts by 0.

Now the plan is to prove that Equation 1 is a pro-isomorphism by studying $\pi_*(\mathrm{THH}(A, M)^{tC_{p^n}}, \mathbb{Z}/p^v)$. For $v = 1$ we will do this explicitly, while the passage to $v > 1$ is mostly formal.

So we start by considering $v = 1$. From the previous lecture we know that the map $\mathrm{TR}^n(A, M) \to \mathrm{THH}(A, M)^{tC_{p^n}}$ to the Tate spectrum is an isomorphism on mod p homotopy groups in degree $* \geq 0$ so we study $\mathrm{TR}_*^n(A, M, \mathbb{Z}/p)$ through its map to $\pi_*(\mathrm{THH}(A, M)^{tC_{p^n}}, \mathbb{Z}/p)$. Calculating the Tate spectral sequence is difficult but doable, and we saw some of the techniques that go into the calculation in the last lecture. Assuming either that $A = W(k)$ or that $\mu_p \subset K$, the E^2-term of the Tate spectral sequence converging to $\pi_*(\mathrm{THH}(A, M)^{tC_{p^n}}, \mathbb{Z}/p)$ is given by

$$E^2 = \Lambda\{u_n, \mathrm{dlog}\,\pi_K\} \otimes S\{\pi_K, \alpha_K, \tau_K^{\pm 1}\}/(\pi_K^{e_K}).$$

Here $|u_n| = -1$, $|\tau_K| = -2$, $|\mathrm{dlog}\,\pi_K| = 1$, $|\pi_K| = 0$, and $|\alpha_K| = 2$. The variable α_K is a slight modification of the variable κ from before and τ_K is a slight modification of t. These are not quite canonical, as they depend on a choice of generator of μ_p and uniformizer π_K. If $A = W(k)$ then $\alpha_K = \kappa$ and $\tau_K = t$, otherwise we need $\mu_p \subset K$ to make the change of variables.

It's helpful to organize this as a module over $k[v_1]$ where $v_1 = \tau_K \alpha_K^p$ represents the usual v_1:

$$E^2 = k[v_1]\{u_n^\epsilon (\mathrm{dlog}\,\pi_K)^\delta \tau_K^a \pi_K^r \alpha_K^d\}$$

where $\epsilon, \delta \in \{0, 1\}$, $0 \le r \le e_K - 1$, $0 \le d \le p - 2$, and $a \in \mathbb{Z}$. Define

$$\{a, r, d\}_K = \frac{(pa - d)e_K}{p - 1} + r.$$

This is going to act as a sort of π_K-adic valuation on $\pi_*(\mathrm{THH}(A, M)^{tC_{p^n}}, \mathbb{Z}/p)$. After some work, which we omit, Hesselholt and Madsen found that

$$E^\infty = \bigoplus_{v=1}^{n-1} k[v_1]/v_1^{p^{v-1}+\cdots+1}\big\{u_n^\epsilon \, \mathrm{dlog}\, \pi_K \tau_K \pi_K \alpha_K^d \quad | \quad \nu_p\{a, r, d\}_K = v\big\}$$

$$\oplus \bigoplus k[v_1]/v_1^{p^{n-1}+\cdots+1}\big\{(\mathrm{dlog}\,\pi_K)^\delta \tau_K^a \pi_K^r \alpha_K^d \quad | \quad \nu_p\{a, r, d\} \ge n\big\}$$

Proposition 3. *If $\mu_p \subset K$ or $A = W(k)$ then*

$$\dim_k \mathrm{TR}_q^n(A, M, \mathbb{Z}/p) = ne_K$$

for all $q \ge 0$.

Proof. This is a counting argument using that $\mathrm{TR}_q^n(A, M, \mathbb{Z}/p)$ is isomorphic to $\pi_q(\mathrm{THH}(A, M)^{tC_{p^n}}, \mathbb{Z}/p)$ for $q \ge 0$ together with the above calculation of E^∞. $\square$

If $\mu_p \subset K$ then the Bott class b_n is represented by

$$-\pi_K^{e_K/(p-1)}\alpha_K,$$

and the multiplicative extensions in passing from E^∞ to the abutment are generated by

$$\pi_K^{e_K} = -\tau_K\alpha_K.$$

(This class is typically not an infinite cycle.) Using this we can rewrite E^∞ as a direct sum of truncated b_n-towers instead.

Lemma 4. *Suppose $\mu_p \subset K$. An element in E^∞ represents a class in the image of the composite*

$$W_n\Omega_{(A,M)}^* \otimes S_{\mathbb{Z}/p}(\mu_p) \to \mathrm{TR}_*^n(A, M, \mathbb{Z}/p) \to \pi_*(\mathrm{THH}(A, M)^{tC_{p^n}}, \mathbb{Z}/p)$$

if and only if $\{a, r, d\}_K \ge 0$.

Proof. Earlier in the day we calculated an upper bound on $\mathrm{TR}_q^n(A, M, \mathbb{Z}/p)$ for $q \le 1$. That there are no further relations follows from $\mathrm{TR}_q^n(A, M, \mathbb{Z}/p) = \mathrm{TR}_1^n(A, M)/p$ having the size given by this upper bound and it being a (part of a log) Witt complex over (A, M). Thus, the upper bound is attained.

So it suffices to show that

$$b_n^q : \bigoplus E_{s,\epsilon-s}^\infty \to \bigoplus E_{s,2q+\epsilon-s}^\infty$$

is an isomorphism for $q = 0, 1$. Using a combinatorial argument one can see that this holds because all b_n-towers either start in degree 0 or 1 or are in negative degree, and because for the class $-\pi_K^{e_K/(p-1)}\alpha_K$ that represents b_n in the spectral sequence, the number $\{a, r, d\}_K$ is zero. $\square$

We think of $\{a, r, d\}_K$ as a valuation, $\pi_*(\mathrm{THH}(A, M)^{tC_{p^n}}, \mathbb{Z}/p)$ as the field, and $\mathrm{TR}_*^n(A, M, \mathbb{Z}/p)$ as the ring of integers.

Now we can prove the $v = 1$ case of Theorem 1.

Proof. For $\bullet = n$, the left hand side consists of infinite b_n-towers starting in degree 0 and 1. The right hand side consists of finite b_n-towers as described by E^∞.

In fixed degree $2q + \epsilon$ this is an isomorphism if we exclude the b_n-towers of height $\leq q$. Let $E_\bullet^*$ denote either side, and define a filtration by

$$\mathrm{Fil}^s E_n^* = V^s E_{n-s}^* + dV^s E_{n-s}^*.$$

Then $\mathrm{Fil}^0 E_n^* = E_n^* \supset \mathrm{Fil}^1 E_n^* \supset \ldots \supset \mathrm{Fil}^n E_n^* = 0$. The map respects the filtration, so it induces a map

$$Gr^s \big(W_n \Omega_{(A,M)}^* \otimes S_{\mathbb{Z}/p}(\mu_p)\big)_{2q+\epsilon} \to Gr^s \mathrm{TR}_{2q+\epsilon}^n(A, M, \mathbb{Z}/p).$$

The point is that Fil^s for large s contains all the short b_n-towers, so in fixed degree $2q + \epsilon$ the map is an isomorphism on Gr^{s-N} for $s < n - N$ with N depending only on q.

This uses that

$$\begin{aligned}
V^s(u_{n-s} \operatorname{dlog} \pi_K \tau_K^a \pi_K^r \alpha_K^d) &= u_n \operatorname{dlog} \pi_K \tau_K^a \pi_K^r \alpha_K^d \\
d(u_n \operatorname{dlog} \pi_K \tau_K^a \pi_K^r \alpha_K^d) &= \operatorname{dlog} \pi_K \tau_K^a \pi_K^r \alpha_K^d.
\end{aligned}$$

$\square$

To prove the general case of Theorem 1 we use the long exact sequence associated to the short exact sequence $0 \to \mathbb{Z}/p^{v-1} \to \mathbb{Z}/p^v \to \mathbb{Z}/p \to 0$ of coefficients, and show that it breaks up into short exact sequences. We will omit the details, but simply remark that the key point is that given a generator ζ of μ_p, $\operatorname{dlog} \zeta$ is zero mod p^{v-1} because ζ has a p^{v-1}'st root.

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Flat descent for THH

Wiesława Nizioł

The goal of my talk was to prove the following theorem:

Theorem 1. ([2, Cor. 3.3]) *The functors*

$$\mathrm{THH}(-), \quad \mathrm{TC}^-(-), \quad \mathrm{THH}(-)_{h\mathbb{T}}, \quad \mathrm{TP}(-)$$

on the category of commutative rings are fpqc sheaves.

Proof. It suffices to prove the theorem for $\mathrm{THH}(-)$ and $\mathrm{THH}(-)_{h\mathbb{T}}$. This is because the two remaining functors are obtained from these by taking limits:

(1) $\mathrm{TC}^-(-) = \mathrm{THH}(-)^{h\mathbb{T}}$,
(2) $\mathrm{TP}(-) = \mathrm{THH}(-)^{t\mathbb{T}} = \mathrm{cofib}(\mathrm{Nm} : \mathrm{THH}(-)_{h\mathbb{T}}[1] \to \mathrm{THH}(-)^{h\mathbb{T}})$.

Case of THH$(-)$.

Step 1: reduction to HH$(-)$.

We start with recalling the following fact:

Fact 2. Let S be a connective $\mathbb{E}_1$-ring spectrum. Let $\{M_n\}_{n\in\mathbb{N}}$ be a weak Postnikov tower of connective S-module spectra[1]. Let $M := \mathrm{proj}\lim_n M_n$. Then, for any right t-exact[2] functor $F : D(S) \to \mathrm{Sp}$, the tower $\{F(M_n)\}_{n\in\mathbb{N}}$ is a weak Postnikov tower with limit $F(M)$.

Let A be a commutative ring. Apply the above fact with the tower $\{\tau_{\leq n}\,\mathrm{THH}(\mathbb{Z})\}_n$ with limit $\mathrm{THH}(\mathbb{Z})$, spectrum $S = \mathrm{THH}(\mathbb{Z})$, and $F(-) = \mathrm{THH}(A) \otimes_{\mathrm{THH}(\mathbb{Z})} -$. From Fact 2 we get the weak Postnikow tower $\{\mathrm{THH}(A) \otimes_{\mathrm{THH}(\mathbb{Z})} \tau_{\leq n}\,\mathrm{THH}(\mathbb{Z})\}_n$ that converges to $\mathrm{THH}(A)$.

It suffices now to show that, for every n, $\mathrm{THH}(-) \otimes_{\mathrm{THH}(\mathbb{Z})} \tau_{\leq n}\,\mathrm{THH}(\mathbb{Z})$ is a sheaf. From the exact triangle

$$\pi_{n+1}\,\mathrm{THH}(\mathbb{Z})[n+1] \to \tau_{\leq n+1}\,\mathrm{THH}(\mathbb{Z}) \to \tau_{\leq n}\,\mathrm{THH}(\mathbb{Z})$$

we see that it suffices to check that, for all n, $\mathrm{THH}(-) \otimes_{\mathrm{THH}(\mathbb{Z})} \pi_n\,\mathrm{THH}(\mathbb{Z})$ is a sheaf. But we have

$$\mathrm{THH}(-) \otimes_{\mathrm{THH}(\mathbb{Z})} \pi_n\,\mathrm{THH}(\mathbb{Z}) \simeq (\mathrm{THH}(-) \otimes_{\mathrm{THH}(\mathbb{Z})} \mathbb{Z}) \otimes_{\mathbb{Z}} \pi_n\,\mathrm{THH}(\mathbb{Z}).$$

Hence, by the lemma below, it suffices to show that $\mathrm{THH}(-) \otimes_{\mathrm{THH}(\mathbb{Z})} \mathbb{Z}$ is a sheaf. But, by the same lemma, this is quasi-isomorphic to HH(A).

Lemma 3. *We have*

    (1) $\pi_n\,\mathrm{THH}(\mathbb{Z})$ is a finitely generated abelian group

    (2) $\mathrm{THH}(A) \otimes_{\mathrm{THH}(\mathbb{Z})} \mathbb{Z} \simeq \mathrm{HH}(A)$.

Proof. The first claim follows from the fact that the stable homotopy groups of a sphere are finitely generated abelian groups. The second claim is immediate from the universal properties of HH$(-)$ and THH$(-)$. Let us recall what they are. For HH$(-)$ we have:

    (1) HH(A/R) is a $\mathbb{E}_\infty$-R-algebra with $\mathbf{T}$-action,

    (2) there exist a (non-equivariant) $\mathbb{E}_\infty$-R-algebra map $A \to \mathrm{HH}(A/R)$,

    (3) HH(A) is initial with respect to (1) and (2).

For THH$(-)$:

    (1) THH(A) is a $\mathbb{E}_\infty$-ring spectrum $\mathbf{T}$-action,

    (2) there exist a (non-equivariant) map $A \to \mathrm{THH}(A)$ of $\mathbb{E}_\infty$-ring spectra,

    (3) THH(A) is initial with respect to (1) and (2).

$\square$

Remark 4. The homotopy groups of THH$(\mathbb{Z})$ were computed by Bökstedt. More generally we have:

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[1]This means that the fiber of $M_{n+1} \to M_n$ is n-connected.

[2]This means that $F(D(S)^{\leq 0}) \subset \mathrm{Sp}^{\leq 0}$.

Theorem 5. (Lindenstrauss-Madsen) *Let A be a number ring. Then (noncanonically)*

$$\pi_i \operatorname{THH}(A) \simeq \begin{cases} A & \text{if } i = 0, \\ \mathcal{D}_A^{-1}/nA & \text{if } i = 2n - 1. \end{cases}$$

Here $\mathcal{D}_A$ is the different of A (recall that it can be defined as the annihilator of $\Omega_{A/\mathbb{Z}}$).

In particular, we have $\pi_1 \operatorname{THH}(A) = \pi_1 \operatorname{HH}(A) = \mathcal{D}_A^{-1}/A$.

Example 6. *If $A = \mathbb{Z}$ then*

$$\pi_i \operatorname{THH}(\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ \mathbb{Z}/n\mathbb{Z} & \text{if } i = 2n - 1. \end{cases}$$

This is a theorem of Bökstedt.

Step 2: the case of $\operatorname{HH}(-)$. Let R be a commutative ring. We have shown that it suffices to prove that $\operatorname{HH}(-/R)$ is an fpqc sheaf.
Digression 1: Cotangent complex.
Let A/R be a commutative algebra. Recall how the cotangent complex $L_{A/R}$ can be defined: take $P. \to A$ - a simplicial resolution by polynomial R-algebras. Set $L_{A/R} := \Omega_{P./R} \otimes_{P.} A$. The wedge powers are defined as $\wedge_A^i L_{A/R} := \Omega_{P./R}^i \otimes_{P.} A$. This is well defined as an ∞-functor: the complex $\wedge_A^i L_{A/R}$ does not depend on the choice of a simplicial resolution, up to a contractible choice.
    We list the following properties:

(1) $\pi_0(L_{A/R}) \simeq \Omega_{A/R}$;
(2) if A/R is smooth then $\wedge_A^i L_{A/R} \simeq \Omega_{A/R}^i$. If A/R is étale then $\wedge_A^i L_{A/R} \simeq 0$;
(3) (Flat base change): For an algebra B/R and a flat algebra C/R we have
    $L_{B/R} \otimes_R C \simeq L_{B \otimes_R C/C}$.

Digression 2: HKR-filtration (after Hochschild-Kostant-Rosenberg)

Proposition 7. *Let A/R be a commutative algebra. There exists a descending separated filtration F_{HKR} on $\operatorname{HH}(A/R)$ with*

$$\operatorname{gr}_{HKR}^i \operatorname{HH}(A/R) \simeq (\wedge_A^i L_{A/R})[i]$$

Proof. Let A/R be smooth. Then $\pi_* \operatorname{HH}(A/R)$ is a strictly anticommutative differential graded R-algebra equipped with a map of (commutative) R-algebras $A \to \pi_0 \operatorname{HH}(A/R)$. Recall that the de Rham complex is universal for this property, hence we get a canonical map

$$\Omega_{A/R}^* \to \pi_* \operatorname{HH}(A/R)$$

independent of any model of $\operatorname{HH}(A/R)$.

Theorem 8. (HKR) *The map $\Omega_{A/R}^* \xrightarrow{\sim} \pi_* \operatorname{HH}(A)$ is an isomorphism.*

For A/R smooth, define $F^i_{HKR} \operatorname{HH}(A/R) := \tau_{\geq i} \operatorname{HH}(A/R)$. We have

$$\operatorname{gr}^i_{HKR} \operatorname{HH}(A/R) \simeq \pi_i \operatorname{HH}(A/R)[i] \simeq \Omega^i_{A/R}[i].$$

This lifts to any algebra S: take $A. \to S$ - a simplicial polynomial resolution. Set $F^i_{HKR} \operatorname{HH}(S/R) := \tau_{\geq i} \operatorname{HH}(A./R)$. We have

$$\operatorname{gr}^i_{HKR} \operatorname{HH}(S/R) = \tau_{\geq i}/\tau_{\geq i+1} \operatorname{HH}(A./R) \simeq \pi_i \operatorname{HH}(A./R)[i] \simeq \Omega^i_{A./R}[i] \simeq \wedge^i_S L_{S/R}[i].$$

$\square$

Let us now return to Step 2. Let us use the HKR-filtration

$$F^{n+1}_{HKR} \operatorname{HH}(-/R) \to F^n_{HKR} \operatorname{HH}(-/R) \to \operatorname{gr}^n_{HKR} \operatorname{HH}(-/R) \simeq \wedge^n_- L_{-/R}[n]$$

to get the weak Postnikov tower $\{\operatorname{HH}(-/R)/F^n_{HKR}\}_n$. It suffices now to show that $\operatorname{HH}(-/R)/F^n_{HKR}$ is a sheaf. But this follows from the following theorem:

Theorem 9. (Bhatt, [1]) *Let $i \geq 0$. The functor $A \mapsto \wedge^i_A L_{A/R}$ is an fpqc sheaf with values in $D(R)$, i.e., if $A \to B$ is faithfully flat then*

$$\wedge^i_A L_{A/R} \xrightarrow{\sim} \lim(\wedge^i_B L_{B/R} \rightrightarrows \wedge^i_{B^2} L_{B^2/R} \Rrightarrow \wedge^i_{B^3} L_{B^3/R} \cdots),$$

where we wrote $B^n := B \otimes_R \cdots \otimes_R B$, the n'th tensor product of B over R.

Proof. If $i = 0$ this is just flat descent. Assume $i = 1$. Form the maps $R \to A \to B^\cdot := \operatorname{Cech}(A \to B)$ we obtain the exact triangle of cosimplicial $B^\cdot$-modules

$$L_{A/R} \otimes_A B^\cdot \to L_{B^\cdot/R} \to L_{B^\cdot/A}$$

We need to show that

(1) $L_{A/R} \xrightarrow{\sim} \lim(L_{A/R} \otimes_A B^\cdot)$
(2) $\operatorname{Tot} L_{B^\cdot/A} \simeq 0$

The first claim holds for any $M \in D(A)$ by fpqc descent. For the second claim, we use the Postnikov filtration to reduce to showing that $\pi_i L_{B^\cdot/A}$ is acyclic. By the flat base change we have

$$(\pi_i L_{B^\cdot/A}) \otimes_A B \simeq \pi_i L_{B^\cdot \otimes_A B/B}.$$

We need to show that this is zero. But this follows from the fact that the map $B \to B^\cdot \otimes_A B$ is a homotopy equivalence.

We leave the case of $i > 1$ to the reader. $\square$

Case of $\operatorname{THH}(-)_{h\mathbb{T}}$**.** The proof is similar to the case of $\operatorname{THH}(-)$. The key different fact used is that $\tau_{\leq n} R_{h\mathbb{T}}$ is a perfect R-complex. $\square$

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BMS filtrations on THH and its variants
Arthur-César Le Bras

Let X be a CW-complex. To obtain the Atiyah-Hirzebruch spectral sequence computing the topological K-theory of X :

$$E_2^{i,j} = H^{i-j}(X, \mathbf{Z}(-j)) = \pi_{j-i}[X, H\mathbb{Z}(-j)] \Longrightarrow K_{-i-j}^{\mathrm{top}}(X) = \pi_{-i-j}[X, KU],$$

one can, instead of using the skeletal filtration on X, use the double speed *Postnikov filtration* on the K-theory spectrum KU (recall that the n-th Postnikov section $\tau_{<n}T$ of a spectrum T is obtained by killing all homotopy groups of T above dimension n by attaching cells ; the n-th piece of the Postnikov filtration is the homotopy fiber of the map $T \to \tau_{<n}T$, which is the n-connective cover of T).

The goal of the talk is, following [2], to construct a similar – but more involved – filtration, the *BMS filtration* (called the *motivic filtration* by the authors of [2]), on THH, TC^-, TP and TC over *quasi-syntomic rings* over a characteristic p perfect ring. If A is such a ring, $n \in \mathbf{Z}$, and

$$\mathbf{Z}_p(n)(A) := \mathrm{gr}^n \mathrm{TC}(A)[-2n]$$

is the (shifted) n-th graded piece of $\mathrm{TC}(A)$ for the BMS filtration, one has a spectral sequence :

$$E_2^{i,j} = \pi_{j-i}(\mathbf{Z}_p(-j)(A)) \Longrightarrow \pi_{-i-j} \mathrm{TC}(A),$$

resembling the above Atiyah-Hirzebruch spectral sequence or the spectral sequence deduced from the filtration of algebraic K-theory by motivic cohomology. For the comparison with classical p-adic cohomology theories (as crystalline cohomology), the case of (quasi-)smooth rings over a perfect ring is probably the most interesting and doing the construction for general quasi-syntomic rings may seem like unnecessary generality ; it is actually a crucial feature of the argument.

The talk has two parts. We first introduce quasi-syntomic rings and state some properties of the quasi-syntomic site. Then we explain how to construct the filtrations, by combining some explicit computations with a descent argument. It follows very closely [2, §4, §6, §7].

1. The quasi-syntomic site

1.1. **Quasi-syntomic rings.** In the following, we will restrict to characteristic p, p being a fixed prime, though, as explained in [2], all the constructions extend to the mixed characteristic case.

Definition 1. Let A be a commutative ring, $M \in D(A)$ (the derived category of A-modules). If $a, b \in \mathbb{Z} \cup \{\pm\infty\}$, one says that M has *Tor-amplitude in* $[a, b]$ if for every A-module N, $N \otimes^{\mathbb{L}} M \in D^{[a,b]}(A)$. One says that M is *flat* if it has Tor-amplitude in $[0, 0]$ (by definition, this means that M is concentrated in degree 0 and flat in the usual sense).

Definition 2. Let A be an $\mathbf{F}_p$-algebra.

(1) The $\mathbf{F}_p$-algebra A is *quasi-syntomic* if the cotangent complex $L_{A/\mathbb{Z}} \in D(A)$ has Tor-amplitude in $[-1, 0]$. Let QSyn denote the category of all quasi-syntomic $\mathbf{F}_p$-algebras.

Let $A \to B$ be a morphism of $\mathbf{F}_p$-algebras.

(2) One says that $A \to B$ is a *quasi-smooth map* (resp. a *quasi-smooth cover*) if it is flat (resp. faithfully flat) and if $L_{B/A} \in D(B)$ is flat.

(3) One says that $A \to B$ is a *quasi-syntomic map* (resp. a *quasi-syntomic cover*) if it is flat (resp. faithfully flat) and if $L_{B/A} \in D(B)$ has Tor-amplitude in $[-1, 0]$.

We endow the category QSyn$^{\mathrm{op}}$ with the topology defined by quasi-syntomic covers.

Remark 3. A theorem of Avramov says that a Noetherian ring A is a local complete intersection ring if and only if $L_{A/\mathbb{Z}}$ has Tor-amplitude in $[-1, 0]$. Therefore, the above definition extends the classical definition of a syntomic ring to the non-Noetherian setting.

Example 4. *Any perfect $\mathbf{F}_p$-algebra R is a (usually non-Noetherian !) quasi-syntomic ring : the cotangent complex $L_{R/\mathbb{Z}}$ has Tor-amplitude in $[-1, -1]$ and is isomorphic to $R[1]$. Indeed, the composition $\mathbb{Z} \to \mathbf{F}_p \to R$ gives rise to a triangle*

$$R \otimes_{\mathbf{F}_p}^{\mathbb{L}} L_{\mathbf{F}_p/\mathbb{Z}} \to L_{R/\mathbb{Z}} \to L_{R/\mathbf{F}_p}.$$

Because $\mathbb{F}_p = \mathbb{Z}/p$, $L_{\mathbb{F}_p/\mathbb{Z}}$ is simply $p\mathbb{Z}/p^2\mathbb{Z}[1] \simeq \mathbb{F}_p[1]$. Hence it suffices to show that

$$L_{R/\mathbb{F}_p} = 0.$$

To see this, choose a simplicial resolution $R_\bullet$ of R by polynomial $\mathbb{F}_p$-algebras. The assumption that R is perfect implies that the Frobenius map $\Phi_{R_\bullet}$ induces an isomorphism $L_{R/\mathbb{F}_p} \simeq L_{\Phi_* R/\mathbb{F}_p}$. But for any k, if one identifies R_k with a polynomial algebra $\mathbb{F}_p[X_1, X_2, \dots]$, Φ_{R_k} sends X_i to X_i^p, thus is the zero map on differentials. This proves the claim.

Lemma 5. *The category* QSyn$^{\mathrm{op}}$ *with the quasi-syntomic topology forms a site.*

The only non trivial thing to check is that pull-backs of covers exist ; this is an easy exercise.

1.2. **Quasi-regular semi-perfect rings.** Once again, we restrict to the characteristic p setting.

Definition 6. An $\mathbb{F}_p$-algebra S is *quasi-regular semi-perfect* if $S \in$ QSyn and if there exists a surjective morphism $R \to S$, with R perfect (in particular, S is semi-perfect, i.e. Frobenius is surjective). We denote by QRSPerf the category of quasi-regular semi-perfect $\mathbb{F}_p$-algebras and endow QRSPerf$^{\mathrm{op}}$ with the topology defined by quasi-syntomic covers.

Remarks 7. (a) If $S \in$ QRSPerf, $L_{S/\mathbb{Z}}$ actually has Tor-amplitude in degrees $[-1, -1]$. Indeed, as S is semi-perfect, $L_{S/\mathbb{Z}}^0 = \Omega_{S/\mathbb{Z}}^1$ is zero.

(b) Any perfect ring lies in QRSPerf. Two other interesting examples are $S = \mathscr{O}_{\mathbf{C}_p}/p$ and $S = \mathbb{F}_p[T^{1/p^\infty}]/(T-1)$.

(c) The category QRSPerf$^{\mathrm{op}}$ with the quasi-syntomic topology forms a site (once again, only the existence of pull-backs is non obvious).

The following key result shows that quasi-regular semi-perfect rings form a basis of the quasi-syntomic topology on QSyn$^{\mathrm{op}}$.

Proposition 8. *An* $\mathbb{F}_p$*-algebra* A *lies in* QSyn *if and only if there exists a quasi-syntomic cover* $A \to S$, *with* $S \in$ QRSPerf. *Moreover, if* $A \to S$ *is a quasi-syntomic cover with* $S \in$ QRSPerf, *all terms*

$$S^i := S \otimes_A S \otimes_A \cdots \otimes_A S \quad (i - \text{times})$$

of the Čech nerve $S^\bullet$ *lie in* QRSPerf.

Proof. If there exists a quasi-syntomic cover $A \to S$, with $S \in$ QRSPerf, the transitivity triangle :

$$L_{A/\mathbb{Z}} \otimes_A^{\mathbb{L}} S \to L_{S/\mathbb{Z}} \to L_{S/A}$$

shows that $L_{A/\mathbb{Z}} \otimes_A^{\mathbb{L}} S$ has Tor-amplitude in $[-1,1]$, as the other two terms have Tor-amplitude in $[-1,0]$ (because $S \in$ QSyn and because $A \to S$ is quasi-syntomic). By connectivity of the cotangent complex, this improves to $[-1,0]$. As $A \to S$ is faithfully flat, we get that $A \in$ QSyn.

Conversely, choose a surjective ring morphism :

$$F = \mathbb{F}_p[\{x_i\}_{i \in I}] \to A,$$

for some big enough index set I. Adjoining to F all p-power roots of the x_i, $i \in I$, one gets a perfect $\mathbb{F}_p$-algebra F_∞. Base changing $F \to F_\infty$ along $F \to A$ gives a map

$$A \to S := F_\infty \otimes_F A.$$

The map $A \to S$ is a quasi-syntomic cover, being the base change of the quasi-syntomic cover $F \to F_\infty$. This easily implies (using the transitivity triangle for $\mathbb{Z} \to A \to S$) that $S \in$ QSyn. Moreover F_∞ is perfect and surjects onto S.

The last assertion is left to the reader. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

The proposition implies that the restriction along the natural map

$$u : \text{QRSPerf}^{\mathrm{op}} \to \text{QSyn}^{\mathrm{op}}$$

induces an equivalence between sheaves on QRSPerf$^{\mathrm{op}}$ and sheaves on QSyn$^{\mathrm{op}}$ with values in any reasonable[1] category $\mathcal{C}$. If $\mathcal{F}$ is a $\mathcal{C}$-valued sheaf on QRSPerf$^{\mathrm{op}}$, we call the associated sheaf on QSyn$^{\mathrm{op}}$ the *unfolding* of $\mathcal{F}$. It is explicitly computed as follows : if $A \in$ QSyn, choose a quasi-syntomic cover $A \to S$, with $S \in$ QRSPerf, and compute the totalization of the cosimplicial object $\mathcal{F}(S^\bullet)$ in $\mathcal{C}$.

Remark 9. In what follows, we will work with the category QSyn$_R$, for some fixed perfect ring R, formed by maps $R \to A$, with $A \in$ QSyn, and similarly with QRSPerf$_R$. One can check that if $A \in$ QSyn$_R$, $L_{A/R}$ has Tor-amplitude in $[-1,0]$.

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[1]In technical terms : any presentable ∞-category.

2. Construction of the filtrations on THH and its variants

Let R be a characteristic p perfect ring, fixed from now on. Let $A \in \mathrm{QSyn}_R$. The goal is to endow $\mathrm{THH}(A)$, $\mathrm{TC}^-(A)$, $\mathrm{TP}(A)$ and $\mathrm{TC}(A)$ with complete exhaustive decreasing $\mathbb{Z}$-indexed multiplicative filtrations $\mathrm{Fil}^* \mathrm{THH}(A)$, $\mathrm{Fil}^* \mathrm{TC}^-(A)$, $\mathrm{Fil}^* \mathrm{TP}(A)$ and $\mathrm{Fil}^* \mathrm{TC}(A)$ (the *BMS filtrations*) such that

$$\hat{\Delta}_A := \mathrm{gr}^0 \, \mathrm{TC}^-(A) = \mathrm{gr}^0 \, \mathrm{TP}(A)$$

comes equipped with a complete decreasing $\mathbb{N}$-indexed multiplicative filtration $\mathcal{N}^{\geq *}\hat{\Delta}_A$ (the *Nygaard filtration*), with graded pieces $\mathcal{N}^*\hat{\Delta}_A$, together with natural isomorphisms :

$$\mathrm{gr}^n \, \mathrm{THH}(A) = \mathcal{N}^n \hat{\Delta}_A[2n] \; ; \; \mathrm{gr}^n \, \mathrm{TC}^-(A) = \mathcal{N}^{\geq n}\hat{\Delta}_A[2n] \; ; \; \mathrm{gr}^n \, \mathrm{TP}(A) = \hat{\Delta}_A[2n]$$

and :

$$\mathbb{Z}_p(n)(A) := \mathrm{gr}^n \, \mathrm{TC}(A) = \mathrm{hofib}(\varphi - \mathrm{can} : \mathcal{N}^{\geq n}\hat{\Delta}_A \to \hat{\Delta}_A).$$

Remarks 10. (a) The graded pieces $\mathbb{Z}_p(n)(A)$ are a priori spectra, but since $\mathbb{Z}_p(0)$ is the constant sheaf given by the Eilenberg-McLane spectrum of $\mathbb{Z}_p$ and since all these graded pieces are module spectra over $\mathbb{Z}_p(0)(A)$, these spectra can be represented, non-canonically, by chain complexes.

(b) This filtration gives rise to the spectral sequence

$$E_2^{i,j} = \pi_{j-i}(\mathbf{Z}_p(-j)(A)) \Longrightarrow \pi_{-i-j} \, \mathrm{TC}(A)$$

alluded to in the introduction. The sheaves $\mathbb{Z}_p(n)$ are related the more classical logarithmic de Rham-Witt sheaves, as will be explained in the next talks.

The strategy to construct these filtrations is quite simple : one defines them *explicitly* at the level of quasi-regular semi-perfect rings ; one then uses *quasi-syntomic descent* to treat the case of general quasi-syntomic rings. In both cases, this remarkably reduces by some dévissages to understanding properties of the cotangent complex.

2.1. **Computations for quasi-regular semi-perfect rings.** We start by analyzing things for R itself.

Proposition 11. *Let R be a perfect $\mathbb{F}_p$-algebra. Then $\pi_* \mathrm{THH}(R) \simeq R[u]$ is a polynomial algebra, with $u \in \pi_2 \mathrm{THH}(R)$.*

Proof. We first prove that for any $R \to R'$, with R, R' perfect, the natural map

$$\mathrm{THH}(R) \otimes_R^{\mathbb{L}} R' \to \mathrm{THH}(R')$$

is an isomorphism. It is enough to check this after tensoring by $\mathbb{Z}$ over $\mathrm{THH}(\mathbb{Z})$ (because one can then argue by induction for $\otimes_{\mathrm{THH}(\mathbb{Z})}\tau_{\leq n} \mathrm{THH}(\mathbb{Z})$). Thus one needs to prove that :

$$\mathrm{HH}(R) \otimes_R^{\mathbb{L}} R' \simeq \mathrm{HH}(R').$$

Using the Hochschild-Kostant-Rosenberg filtration (HKR filtration), this amounts to prove that :

$$\wedge_R^i L_{R/\mathbb{Z}} \otimes_R^{\mathbb{L}} R' \simeq \wedge_{R'}^i L_{R'/\mathbb{Z}},$$

which is easily deduced from Example 4. This base change property reduces us to prove the proposition for $R = \mathbb{F}_p$; in this case, this is the content of Bökstedt's theorem. $\qquad\square$

Proposition 12. *One can find generators* $u \in \pi_2 \mathrm{TC}^-(R)$, $v \in \pi_{-2} \mathrm{TC}^-(R)$ *and* $\sigma \in \pi_2 \mathrm{TP}(R)$ *such that :*

$$\pi_* \mathrm{TC}^-(R) \simeq W(R)[u, v]/(uv - p) \; ; \; \pi_* \mathrm{TP}(R) \simeq W(R)[\sigma, \sigma^{-1}]$$

and such that the map induced on homotopy groups by

$$\varphi^{h\mathbb{T}} : \mathrm{TC}^-(R) = \mathrm{THH}(R)^{h\mathbb{T}} \to \mathrm{TP}(R) = (\mathrm{THH}(R)^{tC_p})^{h\mathbb{T}}$$

is the $\varphi_{W(R)}$*-linear map sending* u *to* σ *and* v *to* $p\sigma^{-1}$, *and such that the map induced on homotopy groups by*

$$\mathrm{can} : \mathrm{TC}^-(R) \to \mathrm{TP}(R)$$

is the linear map sending u *to* $p\sigma$ *and* v *to* σ^{-1}.

Proof. We simply describe $\pi_0 \mathrm{TC}^-(R)$, which is the hardest part, and refer the reader to [2, §6] for the rest. Because $\pi_* \mathrm{THH}(R)$ is concentrated in even degrees and has trivial $\mathbb{T}$-action, the homotopy fixed point spectral sequence :

$$E_2^{i,j} = H^i(\mathbb{T}, \pi_{-j} \mathrm{THH}(R)) \Longrightarrow \pi_{-i-j} \mathrm{TC}^-(R)$$

degenerates. In particular, one can lift $u \in \pi_2 \mathrm{THH}(R)$ to an element (still denoted) $u \in \pi_2 \mathrm{TC}^-(R)$ and the natural generator of $H^2(\mathbb{T}, \pi_0 \mathrm{THH}(R))$ to $v \in \pi_{-2} \mathrm{TC}^-(R)$. The degeneracy of the spectral sequences also provides a descending complete $\mathbb{N}$-indexed multiplicative filtration on $\pi_0 \mathrm{TC}^-(R)$ such that

$$\mathrm{gr}^i \mathrm{TC}^-(R) = \pi_{2i} \mathrm{THH}(R) \simeq R,$$

for $i \geq 0$. In particular, the map

$$\pi_0 \mathrm{TC}^-(R) \to \pi_0 \mathrm{THH}(R) = R$$

makes $\pi_0 \mathrm{TC}^-(R)$ a pro-infinitesimal thickening of R. By the universal property of $W(R)$, this gives a unique map $W(R) \to \pi_0 \mathrm{TC}^-(R)$, with

$$\mathrm{im}(pW(R)) \subset \mathrm{Fil}^1 \pi_0 \mathrm{TC}^-(R) = \ker(\pi_0 \mathrm{TC}^-(R) \to R).$$

By multiplicativity of the filtration, one has

$$\mathrm{im}(p^i W(R)) \subset \mathrm{Fil}^i \pi_0 \mathrm{TC}^-(R),$$

for all $i \geq 1$. Proving that the map $W(R) \to \pi_0 \mathrm{TC}^-(R)$ is an isomorphism can thus be checked on graded pieces, i.e. by showing that certain maps from R to R are isomorphisms, which readily reduces by base change to the case $R = \mathbb{F}_p$. In this case, we know by [1, Lem. IV.4.7] that the images of p and uv in $H^2(\mathbb{T}, \pi_2 \mathrm{THH}(\mathbb{F}_p))$ are the same. By multiplicativity, the images of p^i and $u^i v^i$ in $H^{2i}(\mathbb{T}, \pi_{2i} \mathrm{THH}(\mathbb{F}_p))$ are the same. Hence all the graded maps are isomorphisms. Up to modifying u by a unit, we can also arrange that $uv = p$ in $\pi_0 \mathrm{TC}^-(R)$. $\quad\square$

Now we can turn to quasi-regular semi-perfect rings.

Theorem 13. *Let* $S \in \mathrm{QRSPerf}_R$. *Then :*
(1) $\pi_* \mathrm{THH}(S)$ *only lives in even degrees.*
(2) *Let* $i \in \mathbb{Z}$. *Multiplication by* $u \in \pi_2 \mathrm{THH}(R)$ *gives an injective map :*

$$\pi_{2i-2} \mathrm{THH}(S) \to \pi_{2i} \mathrm{THH}(S)$$

and this endows $\pi_{2i} \mathrm{THH}(S)$ *with a functorial finite increasing filtration with graded pieces* $\wedge_S^j L_{S/R}[-j]$, *for* $0 \le j \le i$ *in increasing order.*

Proof. We start by noting for any R-algebra A, we have a $\mathbb{T}$-equivariant fiber sequence

$$\mathrm{THH}(A)[2] \to \mathrm{THH}(A) \to \mathrm{HH}(A/R) \qquad (*)$$

(see [2, Th. 6.7]). We will first apply this when A is a quasi-smooth R-algebra. Then, by the universal property of the de Rham complex, the natural antisymmetrisation map

$$\Omega_{A/R}^1 = \Omega_{A/\mathbb{Z}}^1 \to \pi_1 \mathrm{HH}(A) = \pi_1 \mathrm{THH}(A)$$

(the first equality comes from the fact that R is perfect) extends to a map of graded A-algebras $\Omega_{A/R}^* \to \pi_* \mathrm{THH}(A)$. Using the HKR filtration, one sees that the composite of this map with the map $\pi_* \mathrm{THH}(A) \to \pi_* \mathrm{HH}(A/R)$ is an isomorphism. Thus the long exact sequence on homotopy groups induced by the fiber sequence $(*)$ splits in short exact sequences, for all i :

$$0 \to \pi_{i-2} \mathrm{THH}(A) \to \pi_i \mathrm{THH}(A) \to \pi_i \mathrm{HH}(A/R) \simeq \Omega_{A/R}^i \to 0.$$

Therefore, the natural map

$$\Omega_{A/R}^* \otimes_R \pi_* \mathrm{THH}(R) \to \pi_* \mathrm{THH}(A)$$

has to be an isomorphism. This proves that on the category of quasi-smooth algebras over R, the Postnikov filtration on THH is a complete decreasing $\mathbb{N}$-indexed multiplicative filtration Fil_P^* with graded pieces

$$\mathrm{gr}_P^n \mathrm{THH}(-) \simeq \bigoplus_{0 \le i \le n, i-n \ \mathrm{even}} \Omega_{-/R}^i[n].$$

By left Kan extension, we get a complete decreasing $\mathbb{N}$-indexed multiplicative filtration Fil_P^* on THH over the category of all R-algebras, with graded pieces

$$\mathrm{gr}_P^n \mathrm{THH}(-) \simeq \bigoplus_{0 \le i \le n, i-n \ \mathrm{even}} \wedge^i L_{-/R}[n].$$

Now we apply this to our quasi-regular semi-perfect ring S over R. By Remark 7 (a) and induction on i, $\wedge^i L_{S/R}$ has Tor-amplitude in $[-i, -i]$ and thus lives in homological degree i. Hence, for any n, $\mathrm{gr}_P^n \mathrm{THH}(S)$ only lives in even degrees. This implies (1), by completeness of the filtration.

To prove (2), we use the fiber sequence $(*)$ for $A = S$. As the homotopy groups of all terms are in even degrees (we just proved it for $\mathrm{THH}(S)$ and it is easily verified for $\mathrm{HH}(S/R)$ using the HKR filtration), the long exact sequence on homotopy groups splits in short exact sequences :

$$0 \to \pi_{2i-2} \mathrm{THH}(S) \to \pi_{2i} \mathrm{THH}(S) \to \pi_{2i} \mathrm{HH}(S/R) \to 0,$$

for all i. This provides the desired filtration on $\pi_{2i} \operatorname{THH}(S)$, as (by the HKR filtration), $\pi_{2i} \operatorname{HH}(S/R) = \wedge_S^i L_{S/R}[-i]$. $\qquad\square$

Remark 14. The filtration Fil_P^* was only introduced as an auxiliary tool ; as we will see, the interesting filtration is the one defined by (2) of the proposition, which comes from the (double speed) Postnikov filtration. That they differ is explained by the fact that the Postnikov filtration on THH over QRSPerf is not (the restriction to QRSPerf of) the left Kan extension of the Postnikov filtration on THH over quasi-smooth R-algebras.

Theorem 15. *Let $S \in \operatorname{QRSPerf}_R$.*
 (1) *The homotopy fixed point and Tate spectral sequences computing $\operatorname{TC}^-(S)$ and $\operatorname{TP}(S)$ degenerate. Both $\pi_* \operatorname{TC}^-(S)$ and $\pi_* \operatorname{TP}(S)$ live in even degrees.*
 (2) *The degenerate homotopy fixed point and Tate spectral sequences endow*

$$\hat{\Delta}_S := \pi_0 \operatorname{TC}^-(S) \underset{\operatorname{can}}{\simeq} \pi_0 \operatorname{TP}(S)$$

with the same descending complete $\mathbb{N}$-indexed filtration $\mathcal{N}^{\geq *}\hat{\Delta}_S$, with graded pieces denoted by $\mathcal{N}^*\hat{\Delta}_S$.
 (3) *One has, for any n, natural identifications :*

$$\pi_{2n} \operatorname{THH}(S) = \mathcal{N}^n\hat{\Delta}_S \; ; \; \pi_{2n} \operatorname{TC}^-(S) = \mathcal{N}^{\geq n}\hat{\Delta}_S \; ; \; \pi_{2n} \operatorname{TP}(S) = \hat{\Delta}_S.$$

 (4) *One has a natural isomorphism of R-algebras*

$$\hat{\Delta}_S/p \simeq \widehat{L\Omega}_{S/R}$$

(the right hand side is the Hodge-completed derived de Rham complex of S over R) and $\hat{\Delta}_S$ is p-torsion free.

Proof. As $\pi_* \operatorname{THH}(S)$ only lives in even degrees, the first three points are easy. The proof of (4) relies on the fiber sequence used in the proof of Theorem 13 and the identification of $\pi_0 \operatorname{HC}^-(S)$ as the Hodge-completed derived de Rham complex $\widehat{L\Omega}_{S/R}$ (whose proof uses quite subtle arguments about filtered derived categories and is given in [2, Prop. 5.14]). $\qquad\square$

2.2. **The filtrations.** We start by reminding the reader that the cotangent complex (and its wedge powers), the Hodge-completed derived de Rham complex, THH, TC^-, TP and TC are all fpqc sheaves. This was proved in the last talk by reduction to the case of the cotangent complex and will be crucial for us.

We first explain the construction of the BMS filtration for THH on quasi-syntomic rings. As promised, this is done in two steps. By Theorem 13, if $S \in \operatorname{QRSPerf}_R$ and $i \in \mathbb{Z}$, $\pi_{2i} \operatorname{THH}(S)$ has a functorial finite increasing filtration with graded pieces $\wedge_S^j L_{S/R}[-j]$, for $0 \leq j \leq i$ in increasing order. In other words, if $S \in \operatorname{QRSPerf}_R$, the double speed Postnikov filtration endows the spectrum $\operatorname{THH}(S)$ with a functorial complete descending $\mathbb{Z}$-indexed $\mathbb{T}$-equvariant filtration $\operatorname{Fil}^* \operatorname{THH}(S)$ such that $\operatorname{gr}^i \operatorname{THH}(S)$ is canonically an S-module spectrum (with trivial $\mathbb{T}$-action) admitting a finite increasing filtration with graded pieces given

by $\wedge_S^j L_{S/R}[2i-j]$, $0 \le j \le i$. This is our BMS filtration on $\mathrm{QRSPerf}_R$, and the end of the first step.

The second step is quasi-syntomic descent. We recalled that THH on $\mathrm{QSyn}_R^{\mathrm{op}}$ is the unfolding of its restriction to $\mathrm{QRSPerf}_R^{\mathrm{op}}$. The last paragraph demonstrates that the double speed Postnikov filtration on THH over $\mathrm{QRSPerf}_R^{\mathrm{op}}$ and the filtration on its graded pieces unfold to $\mathrm{QSyn}_R^{\mathrm{op}}$: indeed, wedge powers of the cotangent complex satisfy descent. Therefore, we see that for any $A \in \mathrm{QSyn}_R$, the spectrum $\mathrm{THH}(A)$ admits a functorial complete descending $\mathbb{Z}$-indexed $\mathbb{T}$-equvariant filtration $\mathrm{Fil}^* \mathrm{THH}(A)$ such that $\mathrm{gr}^i \mathrm{THH}(A)$ is canonically an A-module spectrum (with trivial $\mathbb{T}$-action) admitting a finite increasing filtration with graded pieces given by $\wedge_A^j L_{A/R}[2i-j]$, $0 \le j \le i$.

The same game can be played with Theorem 15, to construct the Nygaard filtration : the sheaf $\hat{\Delta}_-$ and its filtration $\mathcal{N}^{\ge *}\hat{\Delta}_-$ on $\mathrm{QRSPerf}_R^{\mathrm{op}}$ unfold[2] to $\mathrm{QSyn}_R^{\mathrm{op}}$, since by Theorem 15, one has, for all $S \in \mathrm{QRSPerf}_R$ and all n,

$$\mathcal{N}^n \hat{\Delta}_S \simeq \pi_{2n} \mathrm{THH}(S)[-2n],$$

and we just checked that the right hand side unfolds to a sheaf on $\mathrm{QSyn}^{\mathrm{op}}$.

This unfolding defines $(\hat{\Delta}_-, \mathcal{N}^{\ge *}\hat{\Delta}_-)$ on $\mathrm{QSyn}_R^{\mathrm{op}}$, and one has, for all $A \in \mathrm{QSyn}_R$ and all n,

$$\mathcal{N}^n \hat{\Delta}_A \simeq \pi_{2n} \mathrm{THH}(A)[-2n],$$

as well as a natural identification of E_∞-R-algebras $\hat{\Delta}_A/p \simeq \widehat{L\Omega}_{A/R}$.

The same kind of arguments apply to construct the sought after filtrations on TC^-, TP and TC on QSyn_R : cf. [2, Prop. 7.13].

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Identification of the graded pieces

KĘSTUTIS ČESNAVIČIUS

1. TP for quasiregular semiperfect algebras

We fix a prime number p, recall that an $\mathbb{F}_p$-algebra R is *perfect* if its absolute Frobenius endomorphism $x \mapsto x^p$ is an isomorphism, and consider the following class of $\mathbb{F}_p$-algebras.

Theorem 1 ([BMS18], 8.8). *An $\mathbb{F}_p$-algebra S is* quasiregular semiperfect *if it admits a surjection $R \twoheadrightarrow S$ from a perfect $\mathbb{F}_p$-algebra R such that the cotangent complex $\mathbb{L}_{S/R}$ is quasi-isomorphic to a flat S-module placed in degree -1.*

---

[2]The most economic way to do this is to see them as defining a sheaf with values in the complete filtered derived category of $W(R)$-modules.

Remark 2. The perfectness of R ensures that

$$\mathbb{L}_{S/\mathbb{F}_p} \xrightarrow{\sim} \mathbb{L}_{S/R},$$

so the condition on the cotangent complex does not depend on the choice of R. Moreover, since the absolute Frobenius of S is surjective, a canonical choice for R is

$$S^\flat := \varprojlim_{x \mapsto x^p} S.$$

Example 3. *Any quotient of a perfect $\mathbb{F}_p$-algebra by a regular sequence is quasiregular semiperfect. Concretely, S could be, for instance,*

$$\mathbb{F}_p[T^{1/p^\infty}]/(T-1).$$

Our goal is to review the following identification, established in [BMS18], §8, of the homotopy groups of the topological periodic cyclic homology of a quasiregular semiperfect S:

$$\pi_*(\mathrm{TP}(S)) \cong \widehat{\mathbb{A}}_{\mathrm{cris}}(S)[\sigma, \sigma^{-1}] \qquad \text{with} \qquad \deg(\sigma) = 2,$$

where $\mathbb{A}_{\mathrm{cris}}(S)$ is a certain Fontaine ring that will be reviewed in §2 and $\widehat{\mathbb{A}}_{\mathrm{cris}}(S)$ is its completion for the Nygaard filtration. Thus, concretely,

$$\pi_*(\mathrm{TP}(S)) \cong \begin{cases} \widehat{\mathbb{A}}_{\mathrm{cris}}(S) & \text{for even } *, \\ 0 & \text{for odd } *. \end{cases}$$

Example 4. *For perfect $\mathbb{F}_p$-algebras, such as $S^\flat$, we have the identification with the p-typical Witt ring:*

$$\widehat{\mathbb{A}}_{\mathrm{cris}}(S^\flat) \cong W(S^\flat), \qquad \text{so also} \qquad \pi_*(\mathrm{TP}(S^\flat)) \cong W(S^\flat)[\sigma, \sigma^{-1}].$$

The latter identification is already familiar from the earlier talks of the workshop: to derive it, one analyzes the Tate spectral sequence. This spectral sequence also gives the vanishing of $\pi_{odd}(\mathrm{TP}(S))$, so we will assume these facts as known.

In the view of 4, since $\mathrm{TP}(S)$ is always a module over $\mathrm{TP}(S^\flat)$, all we need to discuss is the identification

$$(1) \qquad \qquad \pi_0(\mathrm{TP}(S)) \cong \widehat{\mathbb{A}}_{\mathrm{cris}}(S).$$

For this, we will proceed in three steps:

  (1) in §2, we will review the construction of the ring $\widehat{\mathbb{A}}_{\mathrm{cris}}(S)$;
  (2) in §3, we will review the derived de Rham–Witt complex $LW\Omega_{S/\mathbb{F}_p}$ of S over $\mathbb{F}_p$ and will identify its Nygaard completion as follows:

$$\widehat{\mathbb{A}}_{\mathrm{cris}}(S) \cong \widehat{LW\Omega}_{S/\mathbb{F}_p};$$

 (3) in §4, we will conclude by reviewing the identification:

$$\pi_0(\mathrm{TP}(S)) \cong \widehat{LW\Omega}_{S/\mathbb{F}_p}.$$

2. The ring $\mathbb{A}_{\mathrm{cris}}(S)$

For a quasiregular semiperfect $\mathbb{F}_p$-algebra S, we consider the following $\mathbb{Z}_p$-algebras.

(i) The ring $\mathbb{A}_{\mathrm{cris}}^\circ(S)$ defined as the divided power envelope over $(\mathbb{Z}_p, p\mathbb{Z}_p)$ of the composite surjection $W(S^\flat) \twoheadrightarrow S^\flat \twoheadrightarrow S$.

(ii) The ring $\mathbb{A}_{\mathrm{cris}}(S)$ defined as the p-adic completion of $\mathbb{A}_{\mathrm{cris}}^\circ(S)$.

Thus, the kernel of the surjection $\mathbb{A}_{\mathrm{cris}}^\circ(S) \twoheadrightarrow S$ is equipped with a divided power structure that is compatible with the divided power structure on the ideal $p\mathbb{Z}_p \subset \mathbb{Z}_p$, and $\mathbb{A}_{\mathrm{cris}}^\circ(S)$ is the initial such $W(S^\flat)$-algebra: for every surjection $D \twoheadrightarrow T$ of $\mathbb{Z}_p$-algebras whose kernel is equipped with a divided power structure over $\mathbb{Z}_p$ and every morphisms a, b that fit into the commutative diagram
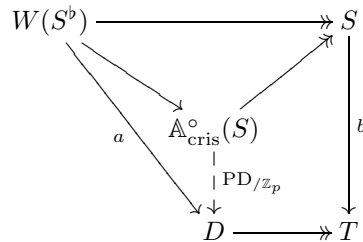
there exists a unique divided power $\mathbb{Z}_p$-morphism indicated by the dashed arrow that makes the diagram commute. The ring $\mathbb{A}_{\mathrm{cris}}(S)$ enjoys the analogous universal property among the p-adically complete D. It follows from the definitions that

$$\mathbb{A}_{\mathrm{cris}}(S)/p \cong \mathbb{A}_{\mathrm{cris}}^\circ(S)/p \cong \text{PD-envelope}_{/\mathbb{F}_p}(S^\flat \twoheadrightarrow S).$$

By functoriality, $\mathbb{A}_{\mathrm{cris}}(S)$ comes equipped with a Frobenius endomorphism φ. The resulting ideals

$$\mathcal{N}^{\geq n}(\mathbb{A}_{\mathrm{cris}}(S)) := \varphi^{-1}(p^n \mathbb{A}_{\mathrm{cris}}(S)) \subset \mathbb{A}_{\mathrm{cris}}(S) \qquad \text{for} \qquad n \geq 0$$

form an exhaustive, φ-stable filtration of $\mathbb{A}_{\mathrm{cris}}(S)$, the *Nygaard filtration*. The φ-stability implies that the *Nygaard completion*

$$\widehat{\mathbb{A}}_{\mathrm{cris}}(S) := \varprojlim_{n \geq 0} \left(\mathbb{A}_{\mathrm{cris}}(S)/\mathcal{N}^{\geq n}(\mathbb{A}_{\mathrm{cris}}(S)) \right)$$

inherits a Frobenius endomorphism from $\mathbb{A}_{\mathrm{cris}}(S)$.

In the case of a perfect ring, such as $S^\flat$, the kernel of the surjection $W(S^\flat) \twoheadrightarrow S^\flat$ carries a unique divided power structure, so

$$\mathbb{A}_{\mathrm{cris}}^\circ(S^\flat) \cong W(S^\flat) \cong \mathbb{A}_{\mathrm{cris}}(S^\flat).$$

Moreover, in this case, the Frobenius φ is an isomorphism, so

$$\mathcal{N}^{\geq n}(\mathbb{A}_{\mathrm{cris}}(S^\flat)) \cong p^n W(S^\flat), \qquad \text{and hence also} \qquad \widehat{\mathbb{A}}_{\mathrm{cris}}(S^\flat) \cong W(S^\flat).$$

3. The derived de Rham–Witt complex

The argument that relates $\pi_0(\mathrm{TP}(S))$ to $\widehat{\mathbb{A}}_{\mathrm{cris}}(S)$ uses the derived de Rham–Witt complex of S over $\mathbb{F}_p$ as an intermediary. To recall the latter, we begin by reviewing the de Rham–Witt complex using the recent approach of Bhatt–Lurie–Mathew [BLM18].

For a fixed prime p, consider the commutative differential graded algebras

$$C^\bullet = (C^0 \xrightarrow{d} C^1 \xrightarrow{d^1} \dots) \qquad \text{with} \qquad C^i[p] = 0 \quad \text{for all} \quad i$$

equipped with an algebra endomorphism $F\colon C^\bullet \to C^\bullet$ such that:

(1) $F\colon C^0 \to C^0$ lifts the absolute Frobenius endomorphism of C^0/p;
(2) $dF = pFd$;
(3) $F\colon C^i \xrightarrow{\sim} d^{-1}(pC^{i+1})$ for all i;
(4) the unique additive endomorphism $V\colon C^\bullet \to C^\bullet$ such that $FV = p$ (whose existence is ensured by the previous requirement, and which necessarily also satisfies $VF = p$) is such that the following map is an isomorphism:

$$C^\bullet \xrightarrow{\sim} \varprojlim_{n>0} \left(\frac{C^\bullet}{\mathrm{Im}(V^n) + \mathrm{Im}(dV^n)} \right).$$

The last requirement implies that each C^i is an inverse limit of p^n-torsion abelian groups, and hence is p-adically complete (the unique limit of a p-adic Cauchy sequence exists already in each term of the inverse limit). The map F does not respect the differentials, but the Frobenius endomorphism

$$\varphi := (p^i F \text{ in degree } i)\colon C^\bullet \to C^\bullet$$

does. The resulting ideals

$$\mathcal{N}^{\geq n}(C^\bullet) := \varphi^{-1}(p^n C^\bullet) \subset C^\bullet \qquad \text{for} \qquad n \geq 0$$

form a separated, exhaustive, φ-stable filtration of $C^\bullet$, the *Nygaard filtration*.

Theorem 5 (Bhatt–Lurie–Mathew). *The functor*

$$\{C^\bullet \text{ as above}\} \xrightarrow{C^\bullet \mapsto C^0/VC^0} \mathbb{F}_p\text{-algebras}$$

admits a left adjoint

$$R \mapsto W\Omega^\bullet_{R/\mathbb{F}_p},$$

so that

$$\mathrm{Hom}_{F\text{-}cdga}(W\Omega^\bullet_{R/\mathbb{F}_p}, C^\bullet) \cong \mathrm{Hom}_{\mathbb{F}_p\text{-}alg.}(R, C^0/VC^0).$$

Moreover, for a regular $\mathbb{F}_p$-algebra R, the complex $W\Omega^\bullet_{R/\mathbb{F}_p}$ agrees with the de Rham–Witt complex of Deligne–Illusie that was defined and studied in [Ill79].

Remark 6. The last aspect implies that for regular R one has a quasi-isomorphism

$$\Omega^\bullet_{R/\mathbb{F}_p} \xrightarrow{\sim} W\Omega^\bullet_{R/\mathbb{F}_p}/p.$$

Definition 7. The *derived de Rham–Witt complex* $LW\Omega_{R/\mathbb{F}_p}$ of a simplicial $\mathbb{F}_p$-algebra R is the value at R of the left Kan extension along the vertical inclusion

$$
\{\text{simplicial } \mathbb{F}_p\text{-algebras}\} \xrightarrow{R \mapsto (LW\Omega_{R/\mathbb{F}_p}, \mathcal{N}^{\geq \bullet})} \left\{ \begin{array}{l} p\text{-complete } E_\infty\text{-algebras in} \\ \text{the filtered derived } \infty\text{-} \\ \text{category of } \mathbb{Z}_p\text{-modules} \end{array} \right\}
$$

$$
\left\{ \begin{array}{l} \text{polynomial } \mathbb{F}_p\text{-algebras} \\ \text{of finite type} \end{array} \right\} \xrightarrow{ R \mapsto (W\Omega^\bullet_{R/\mathbb{F}_p}, \mathcal{N}^{\geq \bullet}) }
$$

of the indicated diagonal functor, and its *Nygaard completion* $\widehat{LW\Omega}_{R/\mathbb{F}_p}$ is the completion of $LW\Omega_{R/\mathbb{F}_p}$ with respect to the filtration $\mathcal{N}^{\geq \bullet}$.

Remark 8. Using the left Kan extension, one may analogously define the derived de Rham complex $L\Omega_{R/\mathbb{F}_p}$ and its Hodge completion $\widehat{L\Omega}_{R/\mathbb{F}_p}$. 6 implies the canonical identification

$$
LW\Omega_{R/\mathbb{F}_p}/p \cong L\Omega_{R/\mathbb{F}_p}
$$

and further arguments imply that also

$$
\widehat{LW\Omega}_{R/\mathbb{F}_p}/p \cong \widehat{L\Omega}_{R/\mathbb{F}_p}.
$$

For us, the key significance of the derived de Rham–Witt complex comes from the following relation to the construction $\mathbb{A}_{\mathrm{cris}}$ discussed in §2.

Theorem 9 ([BMS18], 8.14). *For a quasiregular semiperfect $\mathbb{F}_p$-algebra S, there is a canonical identification*

$$
\mathbb{A}_{\mathrm{cris}}(S) \cong LW\Omega_{S/\mathbb{F}_p}
$$

that is compatible with the Nygaard filtrations; in particular, one also has

$$
\widehat{\mathbb{A}}_{\mathrm{cris}}(S) \cong \widehat{LW\Omega}_{S/\mathbb{F}_p}.
$$

Proof sketch. One eventually bootstraps the conclusion from the identifications

$$
\mathbb{A}_{\mathrm{cris}}(S)/p \cong L\Omega_{S/\mathbb{F}_p} \overset{8}{\cong} LW\Omega_{S/\mathbb{F}_p}/p,
$$

the first of which follows from [Bha12], 3.27. A key reduction is to the case of the $\mathbb{F}_p$-algebra $S^\flat[X_i^{1/p^\infty} \,|\, i \in I]/(X_i \,|\, i \in I)$, where $I := \mathrm{Ker}(S^\flat \twoheadrightarrow S)$. $\qquad \square$

4. The relation to $\pi_0(\mathrm{TP}(S))$

We fix a quasiregular semiperfect $\mathbb{F}_p$-algebra S and seek to review in 12 the identification (1). For this, we rely on the following lemmas.

Lemma 10 ([BMS18], 5.13). *Letting* HP *indicate periodic cyclic homology, we have a natural identification*

$$
\pi_0(\mathrm{HP}(S/\mathbb{F}_p)) \cong \widehat{L\Omega}_{S/\mathbb{F}_p}.
$$

Proof sketch. One combines:

(1) the Tate spectral sequence that relates HP to the Hochschild homology HH;

(2) the Hochschild–Kostant–Rosenberg theorem that gives the identification

$$\pi_{2i}(\mathrm{HH}(S/\mathbb{F}_p)) \simeq \left(\bigwedge^i \mathbb{L}_{S/\mathbb{F}_P}\right)[-i]. \qquad \square$$

Lemma 11 ([BMS18], 6.7). *We have a natural identification*

$$\pi_0(\mathrm{TP}(S))/p \cong \pi_0(\mathrm{HP}(S/\mathbb{F}_p)).$$

Proof sketch. By Bökstedt's computation, one has the fiber sequence

$$\mathrm{THH}(\mathbb{F}_p)[2] \to \mathrm{THH}(\mathbb{F}_p) \to \mathrm{HH}(\mathbb{F}_p/\mathbb{F}_p).$$

By base changing to $\mathrm{THH}(S)$ over $\mathrm{THH}(\mathbb{F}_p)$, one obtains the fiber sequence

$$\mathrm{THH}(S)[2] \to \mathrm{THH}(S) \to \mathrm{HH}(S/\mathbb{F}_p).$$

Upon applying the Tate construction, the latter becomes the fiber sequence

$$\mathrm{TP}(S)[2] \xrightarrow{p\cdot\sigma} \mathrm{TP}(S) \to \mathrm{HP}(S/\mathbb{F}_p).$$

Since the odd homotopy groups vanish, one concludes by applying π_0. $\qquad \square$

Theorem 12 ([BMS18], 8.15). *We have a natural identification*

$$\pi_0(\mathrm{TP}(S)) \cong \widehat{LW\Omega}_{S/\mathbb{F}_p} \overset{9}{\cong} \widehat{\mathbb{A}}_{\mathrm{cris}}(S).$$

Proof sketch. The lemmas imply the desired identification modulo p:

$$\pi_0(\mathrm{TP}(S))/p \cong \widehat{L\Omega}_{S/\mathbb{F}_p} \overset{8}{\cong} \widehat{LW\Omega}_{S/\mathbb{F}_p}/p \cong \widehat{\mathbb{A}}_{\mathrm{cris}}(S)/p.$$

To bootstrap from this, one relies on the universal property of $\mathbb{A}_{\mathrm{cris}}(S)$ via the identification $LW\Omega_{S/\mathbb{F}_p} \cong \mathbb{A}_{\mathrm{cris}}(S)$ of 9. The key intermediate case is that of

$$\mathbb{F}_p[T^{\pm 1/p^\infty}]/(T-1) \cong \mathbb{F}_p[\mathbb{Q}_p/\mathbb{Z}_p],$$

in which one uses the descent of the group algebra $\mathbb{F}_p[\mathbb{Q}_p/\mathbb{Z}_p]$ to its counterpart over the sphere spectrum in order to argue the identification

$$\mathrm{TP}(\mathbb{F}_p[\mathbb{Q}_p/\mathbb{Z}_p]) \cong \mathrm{HP}(\mathbb{Z}[\mathbb{Q}_p/\mathbb{Z}_p]). \qquad \square$$

References

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Topological periodic cyclic homology of smooth $\mathbb{F}_p$-algebras

Elden Elmanto

The goal of this talk is to use theorems from previous talks to deduce certain calculations of topological periodic cyclic homology of smooth k-algebras, where k is a perfect field of characteristic $p > 0$.

These calculations revolve around the *motivic filtration* constructed in [1] on the topological periodic cyclic homology of a *quasisyntomic ring A*, TP(A); see [1, Theorem 1.12]. In the present situation, being quasisyntomic means that the cotangent complex of $\mathbb{L}_{A/k}$ has tor-amplitude $[-1, 0]$. We denote by QSyn_k the full subcategory of k-algebras spanned by quasisyntomic k-algebras.

The motivic filtration is a descending filtration defined on the spectrum TP(A):

$$\mathrm{TP}(A) \cdots \leftarrow \mathrm{Fil}^{-1}\,\mathrm{TP}(A) \leftarrow \mathrm{Fil}^0\,\mathrm{TP}(A) \leftarrow \mathrm{Fil}^1\,\mathrm{TP}(A) \leftarrow \cdots \mathrm{Fil}^n\,\mathrm{TP}(A) \cdots .$$

By construction it agrees with the double-speed Postnikov filtration of spectra whenever A is *quasiregular semiperfect* [1, Definition 8.8] — this just means that the cotangent complex $\mathbb{L}_{A/k}$ is a flat module concentrated in homological degree 1 and the Frobenius on A is surjective. The first calculation is an identification of the associated graded of the motivic filtration.

Theorem 1. *Suppose that A is a smooth k-algebra where k is a perfect field of charecteristic $p > 0$, then there is an equivalence in, $\mathrm{D}(W(k))$, the derived category of $W(k)$-modules*

$$(1) \qquad\qquad \mathrm{gr}^n\,\mathrm{TP}(A) \simeq R\Gamma_{\mathrm{crys}}(A/W(k))[2n].$$

In fact, the associated graded $\mathrm{gr}^0\,\mathrm{TP}(A)$ identifies with the derived global sections of a certain homotopy sheaf which we now describe. We have a presheaf of commutative $W(k)$-algebras on $\mathrm{QSyn}_k^{\mathrm{op}}$

$$\pi_0\,\mathrm{TP}(-) : \mathrm{QSyn}_k \to \mathrm{CAlg}_{W(k)}.$$

We endow $\mathrm{QSyn}_k^{\mathrm{op}}$ with the *quasisyntomic topology* where the covers are faithfully flat maps $A \to B$ in QSyn_k such that the cotangent complex $\mathbb{L}_{B/A}$ has tor-amplitude in $[-1, 0]$. Suppose that $A \in \mathrm{QSyn}_k$, then we consider derived global sections of this presheaf restricted to $\mathrm{QSyn}_A := (\mathrm{QSyn}_k)_{/A}$, with respect to the quasisyntomic topology. This is an $\mathbb{E}_\infty$-$W(k)$-algebra which we denote by $R\Gamma_{\mathrm{syn}}(A; \pi_0\,\mathrm{TP}(-))$ and we have an equivalence

$$\mathrm{gr}^0\,\mathrm{TP}(-) \simeq R\Gamma_{\mathrm{syn}}(A; \pi_0\,\mathrm{TP}(-)).$$

This is a consequence of quasisyntomic descent for the presheaf of spectra TP($-$) [1, Corollary 3.3]. Specializing (1) to $n = 0$ we obtain an equivalence of $\mathbb{E}_\infty$-$W(k)$-algebras

$$R\Gamma_{\mathrm{syn}}(A; \pi_0\,\mathrm{TP}(-)) \simeq R\Gamma_{\mathrm{crys}}(A/W(k)),$$

which is [1, Theorem 1.10].

As a result of Theorem 1 the spectral sequence obtained from the motivic filtration is of the form

$$E_{i,j}^2 = \pi_{i+j}(\mathrm{gr}^{-j}\,\mathrm{TP}(A)) \cong H_{\mathrm{crys}}^{j-i}(A/W(k)) \Rightarrow \mathrm{TP}_{i+j}(A),$$

where the differentials are of the form

$$d^r : E^r_{i,j} \to E^r_{i-r,j+r-1}.$$

One can think of the graded pieces as the "weight" of the motivic filtration (see §2 for how the Adams operations sort out the weights).

Setting $h^{i,j} := \pi_{i+j}(\mathrm{gr}^{-j}\,\mathrm{TP}(A))$. The spectral sequence then displays as
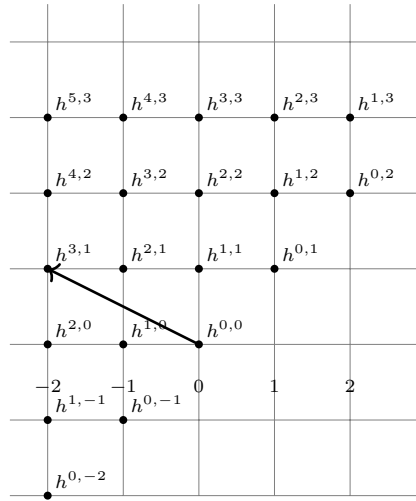
In the spectral sequence displayed above, the *divided Bott element* discussed in [2, Section 4] lies in the term $h^{0,1}$ with total degree 2. We call this element σ. The next theorem states that the motivic filtration splits after inverting p and, thus, the spectral sequence degenerates at the E_2-page. More precisely:

Theorem 2. *Suppose that A is a smooth k-algebra where k is a perfect field of charecteristic $p > 0$, then we have an equivalence of $\mathbb{E}_\infty$-$W(k)$-algebras*

$$\mathrm{TP}(A)[\tfrac{1}{p}] \simeq R\Gamma_{\mathrm{crys}}(A/W(k))[\tfrac{1}{p}][\sigma, \sigma^{-1}]$$

where $|\sigma| = 2$.

The proof of this theorem will exploit the fact that the Adams operations acts by different eigenvalues on each of the associated graded pieces.

1. Proof of Theorem 1

Recall that, by [3], the crystalline cohomology of a smooth k-algebra A can be computed as the cohomology of the de Rham-Witt complex $W\Omega_{A/k}$. We first claim that

Proposition 3. *For any smooth k-algebra A, the commutative $W(k)$-algebra $W\Omega_{A/k}$ computes the derived global sections of the presheaf $\pi_0\,\mathrm{TP}(-)|_{\mathrm{QSyn}_A}$.*

Proof. According to [1, Theorem 8.15], for any quasiregular semiperfect k-algebra A, we have an equivalence of commutative $W(k)$-algebras

$$(2) \qquad \widehat{LW\Omega}_{A/k} \simeq \pi_0 \, \mathrm{TP}(A),$$

where $\widehat{LW\Omega}_{A/k}$ is the *Nygaard completed derived de Rham-Witt complex*. By construction, this is the value on A of the left Kan extension of the de Rham-Witt complex along the inclusion of polynomial k-algebras to QSyn_A, and then completed with respect to the Nygaard filtration; see [1, Section 8.1] for details. We claim two properties about the derived de Rham-Witt complex:

(1) the presheaf on $\mathrm{QSyn}_k^{\mathrm{op}}$,

$$\widehat{LW\Omega}_{-/k} : \mathrm{QSyn}_k \to \mathrm{D}(W(k)),$$

is a sheaf for the quasisyntomic topology, and

(2) the restriction of $\widehat{LW\Omega}_{-/k}$ to SmAff_k agrees with $W\Omega_{(-)/k}$.

Let us prove the proposition assuming these two properties. Let A_{perf} be the colimit

$$A \xrightarrow{\phi} A \xrightarrow{\phi} A \cdots,$$

where ϕ is the Frobenius. Then A_{perf} is quasiregular semiperfect and furthermore the map $A \to A_{\mathrm{perf}}$ is a quasisyntomic cover; the map is faithfully flat using the characterization of regularity via the Frobenius (for a general result see [4], but this fact is an easier exercise in this setting). With this, we get the following string of equivalences in $\mathrm{CAlg}(\mathrm{D}(W(k)))$:

$$
\begin{aligned}
R\Gamma_{\mathrm{syn}}(A, \pi_0 \, \mathrm{TP}(A)) \quad &\simeq \quad \varprojlim_{\Delta} \pi_0 \, \mathrm{TP}(A_{\mathrm{perf}}^{\otimes_A \bullet}) \\
&\simeq \quad \varprojlim_{\Delta} \widehat{LW\Omega}_{A_{\mathrm{perf}}^{\otimes_A \bullet}/k} \\
&\simeq \quad \widehat{LW\Omega}_{A/k} \\
&\simeq \quad W\Omega_{A/k}.
\end{aligned}
$$

We now prove the first of the claimed properties of $\widehat{LW\Omega}_{A/k}$. Taking its mod-p reduction gives an equivalence [1, Theorem 8.14.5] in $\mathrm{D}(k)$

$$\widehat{LW\Omega}_{A/k}/p \simeq \widehat{L\Omega_{A/k}}.$$

where the right hand side is the *Hodge completed derived de Rham complex*, defined by an analogous Kan extension and completion procedure for the deRham complex. Since $\widehat{LW\Omega}_{A/k}$ is p-complete for all $A \in \mathrm{QSyn}_k$, it suffices to check descent after reduction mod-p and thus we need to check descent for the presheaf $\widehat{L\Omega_{(-)/k}}$. This is a consequence of quasisyntomic descent for the cotangent complex, and its exterior powers [1, Theorem 3.1].

To check the second property, we recall that, Zariski-locally, the structure map of a smooth k-algebra A is of the form $k \to k[x_1 \cdots, x_n] \xrightarrow{g} A$ where g is étale. Since the Nygaard completed derived deRham-Witt complex has Zariski descent and its value agrees with the de Rham-Witt complex on polynomial k-algebras, it

suffices to check that the derived deRham-Witt satisfies étale base change. This can again be checked after reduction mod p. □

Proposition 3 proves the case $n = 0$ of Theorem 1. To obtain Theorem 1, we use the periodicity of TP(A) [1, Section 6] to deduce that

$$\mathrm{gr}^n \mathrm{TP}(A) \simeq \mathrm{gr}^0 \mathrm{TP}(A)[2n],$$

where the equivalence is given by multiplication by σ^n.

2. Proof of Theorem 2

Since any k-algebra is p-complete, we have that $\mathrm{THH}(A) \simeq A^{\otimes \mathbb{T}_p^\wedge}$. Now, $\mathbb{T}_p^\wedge \simeq K(\mathbb{Z}_p, 1)$ and so its space of automorphisms identifies with the units of $\Omega K(\mathbb{Z}_p, 1)$, i.e., the group $\mathbb{Z}_p^\times$. Each $\ell \in \mathbb{Z}_p^\times$, then defines an *Adams operation*

$$(3) \qquad \psi^\ell : \mathrm{THH}(A) \simeq A^{\otimes \mathbb{T}_p^\wedge} \overset{(\mathrm{id}_A)^{\otimes \ell \cdot}}{\to} \mathrm{THH}(A) \simeq A^{\otimes \mathbb{T}_p^\wedge},$$

which is a map of $\mathbb{E}_\infty$-ring spectra, but is *not* $\mathbb{T}$-equivariant for the usual $\mathbb{T}$-action on THH(A).

We can construct a version of the Adams operation which is $\mathbb{T}$-equivariant after "speeding up" the $\mathbb{T}$-action on the target by multiplication by ℓ. Indeed, consider the self-map $m_\ell : \mathbb{T} \to \mathbb{T}; z \mapsto z^\ell$. For any $\mathbb{T}$-spectrum E we define the $\mathbb{T}$-spectrum E^{reparm_ℓ} where the underlying spectrum is E, and the $\mathbb{T}$-action is informally described by

$$\mathbb{T} \otimes E \overset{m_\ell \otimes \mathrm{id}}{\longrightarrow} \mathbb{T} \otimes E \overset{\mathrm{act}}{\longrightarrow} \mathbb{T} \otimes E.$$

More precisely, restriction along $m_\ell : \mathbb{T} \to \mathbb{T}$ induces a functor $(m_\ell)^* : \mathrm{Sp}^{B\mathbb{T}} \to \mathrm{Sp}^{B\mathbb{T}}$. The $\mathbb{T}$-spectrum E^{reparm_ℓ} is defined, as a $\mathbb{T}$-spectrum, as $(m_\ell)^* E$.

In the case of THH(A), we get the following more explicit description. we denote by $(\mathbb{T}_p^\wedge)^{\mathrm{reparm}}$ the p-complete circle equipped with an action of $\mathbb{T}$ "sped up by ℓ"; the point now is that the map $m_\ell : \mathbb{T}_p^\wedge \to (\mathbb{T}_p^\wedge)^{\mathrm{reparm}_\ell}$ is $\mathbb{T}$-equivariant and thus the map (3)

$$\psi^\ell : \mathrm{THH}(A) \overset{(\mathrm{id})^{\otimes \ell \cdot}}{\longrightarrow} \mathrm{THH}(A)^{\mathrm{reparm}_\ell} \simeq A^{\otimes (\mathbb{T}_p^\wedge)_\ell^{\mathrm{reparm}}},$$

is $\mathbb{T}$-equivariant.

We have the following observation

Lemma 4. *Let* Sp_p *denote the* ∞-*category of* p-*complete spectra and* $\ell \in \mathbb{Z}_p^\times$. *Then the functor* $(m_\ell)^* : \mathrm{Sp}_p^{B\mathbb{T}} \to \mathrm{Sp}_p^{B\mathbb{T}}$ *is an equivalence of* ∞-*categories.*

Proof. For any p-complete spectrum E, the $\mathbb{T}$-action factors uniquely through a $\mathbb{T}_p^\wedge$-action, hence we are left to prove that the induced functor $(m_\ell)^* : \mathrm{Sp}_p^{B\mathbb{T}_p^\wedge} \to \mathrm{Sp}_p^{B\mathbb{T}_p^\wedge}$ is an equivalence of ∞-categories. Since ℓ acts invertibly on $\mathbb{T}_p^\wedge$, the claim follows.

□

As a result, $\mathrm{THH}(A)^{\mathrm{reparm}_\ell} \simeq \mathrm{THH}(A)$ as $\mathbb{T}$-spectra and thus we get an induced operation

$$\psi^\ell : \mathrm{TP}(A) \xrightarrow{(\psi^\ell)^{t\mathbb{T}}} (\mathrm{THH}(A)^{\mathrm{reparm}_\ell})^{t\mathbb{T}} \simeq \mathrm{TP}(A).$$

Proposition 5. [1, Proposition 9.14] *The Adams operation ψ^ℓ acts on $\mathrm{gr}^n \mathrm{TP}(A)$ by multiplication with ℓ^n.*

In particular if we invert p, then we have an isomorphism of $\mathbb{Q}$-vector spaces $(\pi_* \mathrm{gr}^n \mathrm{TP}(A)[\frac{1}{p}])^{\psi^\ell - \ell^n} \cong \pi_* \mathrm{gr}^n \mathrm{TP}(A)[\frac{1}{p}]$. To prove Theorem 2 we consider the diagram of spectra

$$(4)$$

$$
\begin{array}{ccc}
\oplus_{n \in \mathbb{Z}}(\mathrm{TP}(A)[\frac{1}{p}])^{\psi^\ell - \ell^n} & \longrightarrow & \mathrm{TP}(A)[\frac{1}{p}] \\
\uparrow & & \\
\oplus_{n \in \mathbb{Z}}(\mathrm{Fil}^n \mathrm{TP}(A)[\frac{1}{p}])^{\psi^\ell - \ell^n} & & \\
\downarrow & & \\
\oplus_{n \in \mathbb{Z}}(\mathrm{gr}^n \mathrm{TP}(A)[\frac{1}{p}])^{\psi^\ell - \ell^n} \simeq \oplus_{n \in \mathbb{Z}}(\mathrm{gr}^n \mathrm{TP}(A)[\frac{1}{p}]) & & ,
\end{array}
$$

where, for any spectrum E with an action of ψ^ℓ, we define

$$E^{\psi^\ell - \ell^n} := \mathrm{fib}(E \xrightarrow{\psi^\ell - \ell^n} E).$$

Since $\pi_* \mathrm{TP}(A)[\frac{1}{p}]$ is a graded $\mathbb{Q}$-vector space, the top horizontal map induces an injection on homotopy groups $\oplus_{n \in \mathbb{Z}}(\pi_* \mathrm{TP}(A)[\frac{1}{p}])^{\psi^\ell - \ell^n} \hookrightarrow \pi_* \mathrm{TP}(A)[\frac{1}{p}]$. It then suffices to prove that vertical arrows are equivalences of spectra, whence we have a map of filtered $\mathbb{Q}$-vector spaces which is an isomorphism on graded pieces. The multiplicative properties of the filtration [1, Theorem 1.12.1] gives us the conclusion of Theorem 2.

To check the claimed equivalence, we consider the action of the Adams operation ψ^ℓ on $\mathrm{TP}(A)[\frac{1}{p}]$ as endowing it with the structure as a module over $\mathbb{S}[\psi^\ell]^1$. The functoriality of the Adams operations tells us that the cofiber sequence of spectra

$$\mathrm{Fil}^n \mathrm{TP}(A)[\tfrac{1}{p}] \to \mathrm{TP}(A)[\tfrac{1}{p}] \to \frac{\mathrm{TP}(A)}{\mathrm{Fil}^n \mathrm{TP}(A)}[\tfrac{1}{p}],$$

is a cofiber sequence of $\mathbb{S}[\psi^\ell]$-modules. We first claim that $(\frac{\mathrm{TP}(A)}{\mathrm{Fil}^n \mathrm{TP}(A)}[\frac{1}{p}])^{\psi^\ell - \ell^n}$ is contractible. Indeed, the we have a filtration on $\frac{\mathrm{TP}(A)}{\mathrm{Fil}^n \mathrm{TP}(A)}[\frac{1}{p}]$ given by

$$\{\frac{\mathrm{Fil}^k \mathrm{TP}(A)}{\mathrm{Fil}^n \mathrm{TP}(A)}[\tfrac{1}{p}]\}_{k > n}.$$

---

[1]This means the spherical monoid algebra of the free monoid on one generator ψ^ℓ. In other words, taking Spec of this derived ring gives us the "flat affine line" over the sphere spectrum.

Where the associated graded are $\{\operatorname{gr}^k \operatorname{TP}(A)[\frac{1}{p}]\}_{k>n}$. Proposition 5 tells us that the action of $\psi^\ell - \ell^n$ on $\operatorname{gr}^k \operatorname{TP}(A)[\frac{1}{p}]$ is homotopic to the action of $\ell^k - \ell^n$ and so is invertible. An induction argument shows that the action of $\psi^\ell - \ell^n$ on $\frac{\operatorname{TP}(A)}{\operatorname{Fil}^n \operatorname{TP}(A)}[\frac{1}{p}]$ is thus invertible and so we have an equivalence of fibers

$$\operatorname{Fil}^n \operatorname{TP}(A)[\tfrac{1}{p}])^{\psi^\ell - \ell^n} \to (\operatorname{TP}(A)[\tfrac{1}{p}])^{\psi^\ell - \ell^n},$$

which tells us that the top vertical arrow of (4) is an equivalence. A similar argument applied to the cofiber sequence

$$\operatorname{Fil}^{n+1} \operatorname{TP}(A)[\tfrac{1}{p}] \to \operatorname{Fil}^n \operatorname{TP}(A)[\tfrac{1}{p}] \to \operatorname{gr}^n \operatorname{TP}(A)[\tfrac{1}{p}]$$

tells us that the bottom vertical arrow is an equivalence.

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