Mathematisches Forschungsinstitut Oberwolfach

Report No. 21/2018

DOI: 10.4171/OWR/2018/21

Quadratic Forms and Related Structures over Fields

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29 April – 5 May 2018

Abstract. The range of topics discussed at the workshop "Quadratic Forms and Related Structures over Fields" included core themes from the algebraic theory of quadratic and hermitian forms and their Witt groups, several aspects of the theory of linear algebraic groups and homogeneous varieties, cohomological invariants as well as some arithmetic aspects pertaining to the theory of quadratic forms over certain types of ground fields, e.g., function fields.

Mathematics Subject Classification (2010): 06F25, 11Exx, 11H56, 12G05, 14Cxx, 14F42, 14Gxx, 14Hxx, 14L15, 14T05, 15Axx, 16H05, 16K20, 16W10, 17D05, 19D45, 20Gxx.

Introduction by the Organisers

The workshop was organized by Karim Johannes Becher (Antwerp), Detlev Hoffmann (Dortmund), and Anne Quéguiner-Mathieu (Paris), and was attended by 53 participants. Funding from the Leibniz Association within the grant 'Oberwolfach Leibniz Graduate Students' (OWLG) provided support toward the participation of seven young researchers. Additionally, the "US Junior Oberwolfach Fellows" program of the US National Science Foundation funded travel expenses for one post doc from the USA. Finally, Parimala, from Emory University, was supported by the Simons Foundation: she benefited from an extended stay in Europe, spending a month in Paris 13, several days in Lyon, and a week in Dortmund, the first two weeks being partially supported by the 'Simons Visiting Professors' program.

The workshop was the thirteenth Oberwolfach meeting on the algebraic theory of quadratic forms and related structures, following a tradition initiated by

Manfred Knebusch, Albrecht Pfister, and Winfried Scharlau in 1975. Throughout the years, the theme of quadratic forms has consistently provided a meeting ground where methods from various areas of mathematics successfully cross-breed. Frequently, results on quadratic and hermitian forms served as test case for far-reaching generalizations. While research emphases have often shifted reflecting current trends, the theory of quadratic forms has absorbed these developments ensuring that its study has stayed timely over the years. Its scope now includes aspects of the theories of algebras with involutions and of linear algebraic groups and their homogeneous spaces over arbitrary fields as well as geometric methods stemming from homotopy and cobordism theories. In addition, the study of quadratic and hermitian forms over specific fields, such as function fields over arithmetic base fields, formally real fields and fields of characteristic 2, has seen quite a resurgence over the last two decades or so and was also the focus of discussions.

The program consisted of 23 talks, including a number of remarkable talks by young participants, who presented impressive results. With the exception of two 30 minute talks, all lectures were scheduled to last 45 minutes. This allowed ample time for questions after each talk. The schedule also included generous recess periods meant to provide more time for less formal research interaction. The participants made full use of this offer by engaging actively in various smaller and often spontaneously formed discussion groups exchanging ideas and knowledge on pertinent workshop related topics.

Whenever possible, an attempt was made to group the talks thematically within a morning or afternoon session. The talks provided an excellent overview of the many exciting developments, new results and current trends in and around the workshop themes and they covered a wide range of topics including, among others, cohomological invariants, local-global principles in various guises, field invariants pertaining to quadratic and hermitian forms, to central simple algebras or to cohomology groups, questions concerning isotropy of quadratic forms or of linear algebraic groups under field extensions, the Grothendieck-Serre conjecture for reductive group schemes over semi-local Dedekind domains, as well as rather novel topics such as alternative Clifford algebras or supertropical quadratic forms.

Acknowledgment: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Raman Parimala in the "Simons Visiting Professors" program at the MFO.

Workshop: Quadratic Forms and Related Structures over Fields

Table of Contents

Abstracts

Automorphisms of even, unimodular lattices

Eva Bayer-Fluckiger
(joint work with L. Taelman)

A *lattice* is a pair (L, b), where L is a free $\mathbf{Z}$-module of finite rank, and $b : L \times L \to \mathbf{Z}$ is a symmetric, bilinear form, with $\det(b) \neq 0$. The lattice is said to be *even* if $b(x, x) \in 2\mathbf{Z}$ for all $x \in L$, and *unimodular* if $\det(b) = \pm 1$. It is well-known that if (r, s) is the signature of an even, unimodular lattice, then we have $r \equiv s \bmod 8$. An *automorphism* is an element of $\mathrm{SO}(L, b)$, in other words an isomorphism of $\mathbf{Z}$-modules $t : L \to L$ such that $b(tx, ty) = b(x, y)$ for all $x, y \in L$, and that $\det(t) = 1$.

In [5], Gross and McMullen raise the question of characterizing the *irreducible* polynomials that can arise as characteristic polynomials of an automorphism of an even, unimodular lattice. For *definite* lattices this question was already settled in [1]. Note that the orthogonal group of a definite lattice is finite, hence only products of cyclotomic polynomials can occur as characteristic polynomials of automorphisms. Let us denote by ϕ_m the cyclotomic polynomials of the m-th roots of unity. We have

Theorem ([1])**.** *Let $F = \phi_m^n$, and assume that m is not a power of 2. Then there exists a definite, even, unimodular lattice having an automorphism with characteristic polynomial F if an only if $F(1)F(-1)$ is a square.*

The case where m is a power of 2 is also settled in [1], and the problem is solved for some other products of cyclotomic polynomials in [2].

For any irreducible polynomial $F \in \mathbf{Z}[X]$, let us denote by $m(F)$ be the number of roots $z \in \mathbf{C}$ of F such that $|z| > 1$. Let (r, s) be such that $r \equiv s \bmod 8$, and set $2n = r + s$, and let $F \in \mathbf{Z}[X]$ be an irreducible polynomial. Gross and McMullen prove in [5] that if there exists an even, unimodular lattice of signature (r, s) having an automorphism with characteristic polynomial F, then the following conditions hold :

(C1) $F(X) = X^{2n}F(X^{-1})$;
(C2) $(-1)^n F(1)F(-1)$ is a square;
(C3) $r \geq m(F)$, $s \geq m(F)$, and $m(F) \equiv r \equiv s \pmod 2$;
(C4) $|F(1)|$ and $|F(-1)|$ are squares.

They also prove the following :

Theorem ([5])**.** *Let (r, s) be such that $r \equiv s \bmod 8$, and let $F \in \mathbf{Z}[X]$ be an irreducible polynomial of degree $2n = r+s$ such that $|F(1) = |F(-1)| = 1$. Assume moreover that conditions (C1), (C2) and (C3) hold. Then there exists an even, unimodular lattice of signature (r, s) having an automorphism with characteristic polynomial F.*

Another proof of this result is given in [3]. Gross and McMullen speculate that conditions (C1)-(C4) may be sufficient for the existence of an even, unimodular lattice as in the theorem; this was recently proved in collaboration with Lenny Taelman.

Theorem ([4]). *Let (r, s) be such that $r \equiv s \bmod 8$, and let $F \in \mathbf{Z}[X]$ be a power of an irreducible polynomial with $\deg(F) = 2n = r + s$. Assume that F satisfies conditions* (C1)-(C4)*. Then there exists an even, unimodular lattice of signature (r, s) having an automorphism with characteristic polynomial F.*

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Uniform bounds on sums of squares modulo sums of 2 squares in function fields of curves

David Grimm

(joint work with K. J. Becher)

Witt's theorem that every sum of squares in a function fields in one variable $F/\mathbb{R}$ is already a sum of 2 squares generalizes to function fields in one variable F/k where k is a hereditarily Euclidean field (i.e. uniquely ordered Pythagorean and so are all its finite real extensions). It is a natural question to ask whether k needs to be uniquely ordered in order for the statement to hold, or whether it is sufficient to assume that k is hereditarily Pythagorean (i.e. real Pythagorean and so are all its finite real extensions).

Becker showed that this is indeed the case for $k(X)$, the rational function field in one variable over a hereditarily Pythagorean field k (see [3]), and his result was extended to arbitrary real function fields F/k of curves of genus zero in [7]. However, the result does not extend to function fields of higher genus, as Tikhonov gave the example of the elliptic curve $Y^2 = (X^2 + 1)(tX - 1)$ over the real power series field $\mathbb{R}(\!(t)\!)$ (a typical example of a hereditarily pythagorean field), which admits a sum of 3 squares of rational functions that cannot be written as a sum of 2 squares. These previous results inspired subsequent works, where sums of squares in function fields of hyperelliptic (and elliptic) curves over $\mathbb{R}(\!(t)\!)$ were studied. In [6] the case of good reduction was studied, and it was shown that every sum of squares is indeed a sum of 2 squares in this case. In the case of bad reduction (like the example given by Tikhonov), it was shown in [2] that on the other hand the discrepancy between sums of squares and sums of 2 squares is not too big. More precisely, they showed that the quotient of multiplicative groups of nonzero sums of squares modulo nonzero sums of 2 squares $(\sum F^2)^\times/(F^2+F^2)^\times$ is finite. In fact, they give the upper bound $2^{\mathfrak{g}}$, where $\mathfrak{g}$ is roughly half the degree of a planar model of the curve $Y^2 = f(X)$ (which also corresponds roughly to the genus of the desingularized curve), or more generally $2^{n\mathfrak{g}}$ when the hyperelliptic curve is defined over $\mathbb{R}(\!(t_1)\!)\dots(\!(t_n)\!)$.

In [1], we extended the finiteness result for $(\sum F^2)^\times/(F^2+F^2)^\times$ to function fields of *arbitrary* (i.e. not necessarily hyperelliptic) curves over $\mathbb{R}(\!(t_1)\!)\dots(\!(t_n)\!)$. Extending ideas developed in the research for the previously mentioned article (for the case $n = 1$), we are now also able to show the existence of a uniform bound in terms of the genus of the curve and of the number of "Laurent variables" n:

Theorem. *For every $n, \mathfrak{g} \in \mathbb{N}$, there exists $N_{n,\mathfrak{g}} \in \mathbb{N}$ such that for every smooth geometrically integral projective curve C over $\mathbb{R}(\!(t_1)\!)\dots(\!(t_n)\!)$ of genus $\mathfrak{g}$, we have*

$$\left|\frac{\left(\sum F^2\right)^\times}{(F^2+F^2)^\times}\right| \leq 2^{N_{n,\mathfrak{g}}}$$

for the function field F of C. Furthermore, in the case $\mathfrak{g} = 1$, the optimal value is $N_{n,1} = 2$ for all $n \in \mathbb{N}$. When only considering elliptic curves (curves of genus $\mathfrak{g} = 1$ with a rational point), the optimal value is $N_{n,elliptic} = 1$ for all $n \in \mathbb{N}$.

The main ingredient of the proof are on the one hand a local-global principle for isotropy of quadratic forms of dimension at least 3 over function fields of curves over a complete discretely valued (see [4]). On the other hand, we rely on combinatorial descriptions of the special fiber of a minimal arithmetic model over $\mathbb{R}(\!(t_1)\!)\dots(\!(t_{n-1})\!)[\![t_n]\!]$ of a curve of genus ≥ 2 due to Artin-Winters, the classification of minimal elliptic arithmetic surfaces by Kodaira-Néron, as well as a generalization [5] to minimal arithmetic surfaces whose generic fiber has genus 1 but no rational point. We observe that symmetries in the corresponding reduction graphs restrict the number of components of the special fiber of an arithmetic model that are non-split smooth conics over some finite extension of $\mathbb{R}(\!(t_1)\!)\dots(\!(t_{n-1})\!)$. A general result from intersection theory of arithmetic surfaces shows that the number of components of the special fiber that are not conics over a finite extension $\mathbb{R}(\!(t_1)\!)\dots(\!(t_{n-1})\!)$ is automatically bounded by $2\mathfrak{g} - 2$. Due to Becker's earlier mentioned result, the components of the special fiber that either are not conics or are non-split smooth conics over a finite extension of $\mathbb{R}(\!(t_1)\!)\dots(\!(t_{n-1})\!)$ turn out to correspond to a finite set of discrete valuations S such that the diagonal map

$$\frac{\left(\sum F^2\right)^\times}{(F^2+F^2)^\times} \longrightarrow \prod_{v\in S} \frac{\left(\sum F_v^2\right)^\times}{(F_v^2+F_v^2)^\times}$$

is already an embedding, by virtue of the earlier mentioned local-global principle. Since the factors on the right-hand side are bounded by twice the order of the corresponding term for the respective residue field, which is the function field of a curve over $\mathbb{R}((t_1))\dots((t_{n-1}))$ of genus at most $\mathfrak{g}$, one is now in the position to prove the result by induction on n.

Although this process does yield effective bounds, we believe the bounds obtained this way to be far from being optimal (except in the case $\mathfrak{g} = 1$).

Support by the Chilean national research grant Fondecyt Iniciacion 11150956 is gratefully acknowledged.

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Local-global Principles for Zero-Cycles

Julia Hartmann

(joint work with J.-L. Colliot-Thélène, D. Harbater, D. Krashen, R. Parimala, V. Suresh)

Local-global principles study properties of algebraic structures over a field by studying the behaviour of those structures over completions of the field. Famous examples over number fields include the theorem of Hasse-Minkowski on isotropy of quadratic forms and the Albert-Brauer-Hasse-Noether theorem on splitness of central simple algebras. These and many other local-global principles for algebraic structures can be rephrased as local-global principles for rational points on varieties. More abstractly, one can consider local-global principles with respect to arbitrary collections of overfields of the given field.

In recent years, the authors and other researchers have studied such local-global principles over semi-global fields, by which we mean function fields of curves over complete discretely valued fields. Basic examples of such fields are $\mathbb{Q}_p(x)$ for p a prime or $k((t))(x)$ for any field k. Semi-global fields are important and interesting partially because of their connection to finitely generated fields (the hope is that understanding $\mathbb{Q}_p(x)$ very well for all p will lead to a better understanding of $\mathbb{Q}(x)$), but also because they are amenable to methods such as patching. Moreover, semi-global fields admit several natural collections of overfields which are motivated geometrically. Local-global principles with respect to those may be of independent interest but often also give information about the more classical case of overfields which are completions.

In this note, we consider local-global principles for zero-cycles on varieties over semi-global fields. In fact, we will deduce such local-global principles from corresponding principles for rational points.

We first take a look at various local-global setups over semi-global fields: Given a semi-global field F over a complete discretely valued field with valuation ring T, we may pick a *normal model* $\mathcal{X}$ of F. By this we mean a flat projective T-scheme with function field F which is normal as a variety. We let k denote the residue field of T and write X for the closed fiber $\mathcal{X} \times_T k$ of $\mathcal{X}$. For every point $P \in X$ (not necessarily closed) we let F_P denote the fraction field of the complete local ring of $\mathcal{X}$ at P. This gives a collection $\Omega_{\mathcal{X}}$ of overfields of F. Let Ω_F be the collection of completions of F with respect to discrete valuations.

In [3], the following is shown: Every element of Ω_F contains an element of $\Omega_{\mathcal{X}}$. This means that if a variety Z/F has rational points over all fields F_P for $P \in X$, then it also has points over all completions, i.e., over all elements of Ω_F. Hence if Z satisfies a local-global principle for rational points with respect to Ω_F, then it also satisfies a local-global principle for rational points with respect to $\Omega_{\mathcal{X}}$.

There is a third collection of overfields which plays a role over semi-global fields, coming from patching. Let $\mathcal{P}$ be a nonempty finite subset of closed points of X containing all points where X is not unibranched. Let $\mathcal{U}$ be the set of components of the complement $X \setminus \mathcal{P}$. For each $U \in \mathcal{U}$, we may consider the fraction field of the t-adic completion of the subring of F consisting of functions which are regular along U, where $t \in T$ is a uniformizer. We let $\Omega_{\mathcal{P}}$ denote the set $\{F_P |\ P \in \mathcal{P}\} \cup \{F_U |\ U \in \mathcal{U}\}$.

It is shown in [3] that a local-global principle for rational points with respect to $\Omega_{\mathcal{X}}$ implies a corresponding local-global principle with respect to $\Omega_{\mathcal{P}}$ for any $\mathcal{P}$ as above, and in fact $\Omega_{\mathcal{X}}$ can be viewed as a limit over all $\Omega_{\mathcal{P}}$.

The collection $\Omega_{\mathcal{P}}$ (for a fixed $\mathcal{P}$) is finite and more accessible than $\Omega_{\mathcal{X}}$ or Ω_F. In particular, it is often possible to give a combinatorial description of the obstruction to a local-global principle with respect to $\Omega_{\mathcal{P}}$, which can lead to criteria for when such local-global principles hold. In particular if G is a connected linear algebraic group over a semi-global field F which is rational as a variety, and Z is a G-torsor, then Z satisfies a local-global principles for rational points with respect to $\Omega_{\mathcal{P}}$ (for any $\mathcal{P} \subseteq X$ as above), see [2].

A *zero-cycle* on an F-variety Z is a formal $\mathbb{Z}$-linear combination $\sum n_i P_i$ of closed points P_i on Z; its *degree* is $\sum n_i \deg(P_i)$. Zero-cycles of degree one may be considered as a generalization of rational points: It is obvious that if Z has an F-rational point then it has a zero-cycle of degree one. The converse is false, e.g., it fails for curves over finite fields, which always have a zero-cycle of degree one by a theorem of F. K. Schmidt (but need not have a rational point).

We are interested in local-global principles for zero-cycles of degree one with respect to collections of overfields as considered above. If E is a finite separable field extension of a semi-global field F with normal model $\mathcal{X}$, we let $\mathcal{X}_E$ denote the normalization of $\mathcal{X}$ in E (this is a normal model for E), and let $\mathcal{P}_E$ denote the preimage of $\mathcal{P}$. This gives a collection of overfields $\Omega_{\mathcal{P}_E}$ of the semi-global field E as explained above.

Theorem 1 ([1]). *Let Z be a smooth variety and fix $\mathcal{P} \subseteq X$ as before. Suppose Z_E satisfies a local-global principle for rational points with respect to $\Omega_{\mathcal{P}_E}$ for all finite separable field extensions E/F. Then Z satisfies a local-global principle for zero-cycles of degree one.*

The theorem is analogous to a (special case of) a theorem of Liang for varieties over number fields, see [4]. It may be worth noting that the proof here is much more involved. A main ingredient is the study of whether finite separable field extensions of fields of the type F_U and F_P are induced from extensions of F. In [1], we prove further local-global principles for zero-cycles, also with respect to Ω_F.

Acknowledgement: The speaker was supported in part by NSF Grant DMS-1463733.

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Duality and local-global principle over two-dimensional henselian local rings

Diego Izquierdo

According to Sansuc, over number fields, Brauer-Manin obstruction is the only obstruction to the local-global principle for torsors under linear connected algebraic groups. In a recent article ([1]), Colliot-Thélène, Parimala and Suresh introduce a new kind of obstruction to the local-global principle over function fields of regular integral schemes of any dimension, and they ask whether it is the only obstruction to the local-global principle for torsors under linear connected algebraic groups over the Laurent series field $\mathbb{C}((x, y))$. In this talk, I will explain why this question has a positive answer.

1. Fields of interest

In this report, we are interested in finite extensions of the Laurent series field in two variables $\mathbb{C}((x,y))$. More generally, we adopt the following notations:

k: algebraically closed field of characteristic 0.
R: integral, local, normal, henselian, 2-dimensional k-algebra with residue field k.
$\mathcal{X} := \operatorname{Spec} R$.
$X := \mathcal{X} \setminus \{s\}$ where s is the closed point of $\mathcal{X}$.
$X^{(1)}$: set of codimension 1 points in X.
K: the fraction field of R.

We are interested in the field K.

2. Brauer-Hasse-Noether exact sequence

In this paragraph, we want to understand the Brauer group of K. To do so, consider a desingularization $f : \tilde{\mathcal{X}} \to \mathcal{X}$ of $\mathcal{X}$ such that:

- f is projective and $\tilde{\mathcal{X}}$ is an integral, regular, 2-dimensional scheme;
- $f : f^{-1}(X) \to X$ is an isomorphism;
- the special fiber $Y := f^{-1}(s)$ is a strict normal crossing divisor of $\tilde{\mathcal{X}}$.

Such a desingularization always exists.

Now observe that Y is a projective k-curve whose irreducible components are smooth. Let $g_1, ..., g_n$ be the genera of the irreducible components of Y. Also let Γ be the graph attached to Y: by definition, this is the graph whose vertices are the irreducible components of Y and whose edges connect two vertices if, and only if, the corresponding irreducible components intersect. Denote by c the first Betti number of Γ.

Theorem 1. *There is an exact sequence:*

$$0 \to (\mathbb{Q}/\mathbb{Z})^{c+2\sum g_i} \to Br\, K \to \bigoplus_{v \in X^{(1)}} Br\, K_v \to \mathbb{Q}/\mathbb{Z} \to 0$$

where K_v is the completion of K at v for $v \in X^{(1)}$ and the middle map is the restriction map.

The proof uses the Gersten conjecture for the regular scheme $\tilde{\mathcal{X}}$ (such a result is due to Panin) and requires to carry out a geometrical and combinatorial study of the desingularization $\tilde{\mathcal{X}}$.

3. Duality theorems

3.1. **Duality in étale cohomology.** Let $j : U \hookrightarrow X$ be an open immersion, with U non-empty. Let F be a finite étale group scheme over U. By using the Brauer-Hasse-Noether exact sequence of the previous paragraph, one can define a natural pairing:

$$AV : H^r(U,F) \times H^{3-r}(X, j_! F') \to H^3(X, j_! \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z},$$

where $F' = \underline{Hom}(F, \mathbb{G}_m)$ is the Cartier dual of F.

Theorem 2. *The pairing AV is a perfect pairing of finite groups for each integer $r \in \{0, 1, 2, 3\}$.*

There are two proofs for this theorem: one can proceed "by hand" by making quite subtle dévissages to reduce to the case when F is constant and then use the Brauer-Hasse-Noether exact sequence of the previous paragraph, or one can use Gabber's general results on the existence of dualizing complexes ([3]).

3.2. **Duality in Galois cohomology.** For each Galois module M over K, we define its Tate-Shafarevich groups by:

$$Ш^r(K, M) := \mathrm{Ker}\left(H^r(K, M) \to \prod_{v \in X^{(1)}} H^r(K_v, M)\right).$$

By using extensively Theorem 2, one can prove the following duality theorem:

Theorem 3. *Let T be a K-torus. Let $\hat{T}$ be its module of characters. Then there is a natural pairing:*

$$PT : Ш^1(K, T) \times Ш^2(K, \hat{T}) \to \mathbb{Q}/\mathbb{Z}$$

which is non-degenerate on the left, and whose right-kernel is the maximal divisible subgroup of $Ш^2(K, \hat{T})$.

4. Obstructions to the local-global principle

Recall the Brauer-Hasse-Noether exact sequence:

$$\mathrm{Br}\, K \to \bigoplus_{v \in X^{(1)}} \mathrm{Br}\, K_v \xrightarrow{\theta} \mathbb{Q}/\mathbb{Z} \to 0.$$

When Z is a smooth K-variety, one can introduce the set of adelic points $Z(\mathbb{A}_K)$ of Z and then define a pairing:

$$BM : Z(\mathbb{A}_K) \times \mathrm{Br}\, Z \to \mathbb{Q}/\mathbb{Z}, ((p_v)_{v \in X^{(1)}}, \alpha) \mapsto \theta((p_v^* \alpha)_v).$$

By using Theorem 3 and by comparing the pairings PT and BM, it is possible to describe the obstructions to local-global principle for torsors under linear connected algebraic groups over K:

Theorem 4. *Let G be a linear connected algebraic group over K. Let Z be a K-torsor under G. If the orthogonal of $Br Z$ in $Z(\mathbb{A}_K)$ for the pairing BM is non-empty, then Z has a rational point.*

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Local-global principle for constant tori over semi-global fields

Raman Parimala

(joint work with J.-L. Colliot-Thélène, D. Harbater, J. Hartmann, D. Krashen, V. Suresh)

Let K be a complete discrete valued field, R the valuation ring of K and k the residue field of R. Let X be a smooth geometrically integral projective curve over K and $F = K(X)$. Let Ω be the set of all discrete valuations of F and for $\nu \in \Omega$, let F_ν denote the completion of F at ν. Let G be a connected linear algebraic group defined over F. We say that *Hasse principle* holds for G if a G-torsor over F admitting an F_ν - rational point for all $\nu \in \Omega$ admits an F-rational point.

Let

$$Ш^1(F, G) = \ker(H^1(F, G) \to \prod_{\nu \in \Omega} H^1(F_\nu, G)).$$

The set $Ш^1(F, G)$ measures failure of Hasse principle for G. In particular Hasse principle holds for G if and only if $Ш^1(F, G)$ is trivial.

If G is F-rational and k algebraically closed, Harbater-Hartman-Krashen proved that $Ш^1(F, G)$ is trivial ([4]). The first counter examples to failure of Hasse principle for (non rational) tori were due to Colliot-Thélène, Parimala and Suresh ([2]) and the obstruction to the Hasse principle was detected by Brauer - Manin like obstruction. We discuss here Hasse principle for connected smooth groups defined over R which we call the *constant groups*. We explain examples of constant groups G for which Hasse principle fails and also show that under some additional hypothesis on the special fibre of a regular proper model over R of X, $Ш^1(F, T)$ is trivial for constant tori.

Let X be a smooth geometrically integral curve over K and $\mathscr{X}$ a regular proper model of X over R. Let X_0 be the reduced special fibre of $\mathscr{X}$. We define the notion of a *special tree* for X_0. This notion of special tree is equivalent to X_0 being geometrically a tree if $\mathrm{char}(k) = 0$. We show that if X_0 is a special tree and T is a smooth torus over R, then $Ш^1(F, T)$ is trivial.

Let k be a field with $\mathrm{char}(k) = 0$, $R = k[[t]]$ and $K = k((t))$. Suppose G is a connected linear algebraic group over k which admits an element in $G(k)$ which is not R-trivial. Let X/K be a smooth geometrically integral curve over K with a regular proper model $\mathscr{X}$ over R whose special fibre X_0 admits a rational triangle with rational nodes. Then we show that $Ш^1(F, G)$ is not trivial. Using this method one produces semisimple simply connected linear algebraic groups for which Hasse principle fails. There is a conjecture (cf. [1]) which asserts that $Ш^1(F,G)$ is trivial if G is semisimple simply connected and k is a finite field. This conjecture has been proved in several cases for classical groups (cf. [7], [5], [6]). The methods of proof used tools from class field theory and the question remained open whether one could have a Hasse principle for semisimple simply connected groups for semiglobal fields without the assumption that k is a finite field. We now have a negative answer to this question.

The method of proof is via identifying $Ш^1(F,G)$ for constant groups with the patching Sha's as described in the work of Harbater-Hartmann-Krashen. Given a regular proper model $\mathscr{X}$ of X over K and the reduced special fiber X_0 a union of regular curves with normal crossings, let $\mathscr{P}$ be a finite set of closed points of X_0 which contains the nodal points of X_0 as well as at least one point on each component and let $\mathscr{U}$ denote the irreducible components of $X_0 \setminus \mathscr{P}$. Then the overfields $\{F_P, F_U, F_b\}$ are defined ([3]) where b denotes the branch at a pair (P,U) if the point P belongs to the closure of U. The double cosets $\prod_P G(F_P)\backslash\prod_b G(F_b)/\prod_U G(F_U)$ define elements in $Ш^1(F,G)$ ([3]) and for constant groups, by varying patches on $\mathscr{X}$, one obtains the entire set $Ш^1(F,G)$.

The examples of the failure of Hasse principle for constant tori lead to examples where $Ш^1(F,G)$ is infinite, thereby clarifying speculations about the finiteness of $Ш^1(F,G)$.

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Systems of quadratic forms over complete discretely valued fields

David B. Leep

For an arbitrary field F, let $u_r(F)$ denote the supremum of the set of positive integers n such that there exists an anisotropic system of r quadratic forms defined over F in n variables. Note that $u_1(F) = u(F)$, the classical u-invariant of the field F.

There has been great interest for a long time to compute $u_r(F)$ for all $r \geq 1$. Particular interest has been paid to the cases where F is a field that arises in number theory or algebraic geometry. If F is a finitely generated function field of transcendence degree m over an algebraically closed field, then $u_r(F) = 2^m r$ for all $r \geq 1$. This follows from the well known theory of C_i-fields as developed by Lang [8], and described in [6]. If F is a finite field, then results of Chevalley [4] and Warning [12] imply that $u_r(F) = 2r$ for all $r \geq 1$. This result also holds because a finite field is a C_1-field (see [8] or [6]).

Let K be a p-adic field, which means that K is a finite extension of the field of p-adic numbers $\mathbb{Q}_p$ for some prime number p. A consequence of a conjecture of E. Artin states that $u_r(K) = 4r$ for all $r \geq 1$. It should be noted that Artin's conjecture is false in general. See [6] for a good treatment of this conjecture. However, we show in this paper that Artin's conjecture for systems of quadratic forms over p-adic fields K is valid. Results of Brauer [3] imply that $u_r(K)$ is always finite. Hasse proved that $u_1(K) = 4$ and Birch-Lewis-Murphy [2] proved that $u_2(K) = 8$. Birch and Lewis [1] proved that $u_3(K) = 12$ as long as the residue field k of K has odd characteristic and is sufficiently large. Their bound was $|k| > 49$. S. Schuur [10] corrected some errors and supposedly lowered the bound to 11, although some details were left out in the paper. In [11], W. Schmidt proved that $u_r(K) \leq 4r^2 + 4r$.

For arbitrary fields F, Elman and Lam proved in [5, p.299], that $u_r(F) \leq (2^r - 1)u(F)$ for all $r \geq 1$. Then Leep proved in [9] that $u_r(F) \leq \frac{r^2+r}{2}u(F)$ for all $r \geq 1$ and that this bound is optimal for $r = 1, 2, 3$. Since $u(K) = 4$ for a p-adic field K, it follows that for a p-adic field K, we have $u_r(K) \leq 2r^2 + 2r$. Heath-Brown proved in [7] that $u_r(K) = 4r$ for all $r \geq 1$ if the residue field k satisfies $|k| > (2r)^r$.

In this paper we prove that $u_r(K) = 4r$ for all $r \geq 1$ and all p-adic fields K. Thus the hypothesis on the size of the residue field k is not needed. This result is a special case of our main theorem stated in Theorem 1.

Theorem 1. *Assume that K is a complete discretely valued field with residue field k. Assume that $u_r(k) \leq Ar$ for all $r \geq 1$ and some positive real number A.*

(1) Then $u_r(K) \leq 2Ar$ for all $r \geq 1$.
(2) Assume that $u_r(k) = Ar$ for all $r \geq 1$ and some positive real number A. Then $A = u(k)$ and $u_r(K) = 2Ar = 2u(k)r$ for all $r \geq 1$.
(3) $u_r(K(x_1, \ldots, x_m)) \leq 2^m \cdot 2Ar$ for all $r \geq 1$.
(4) If k is a finite field and thus K is a p-adic field, then $u_r(K(x_1, \ldots, x_m)) = 2^{m+2}r$ for all $r \geq 1$ and all $m \geq 0$. In particular, $u(K(x_1, \ldots, x_m)) = 2^{m+2}$ for all $m \geq 0$.

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Linkage in Kato-Milne Cohomology

Adam Chapman

1. Kato-Milne Cohomology

Suppose $\text{char}(F) = p$ for some prime integer p. There is the Artin-Schreier map

$$\wp : \Omega^n F \to \Omega^n F / d\Omega^{n-1} F$$

$$\alpha \frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_n}{\beta_n} \mapsto (\alpha^p - \alpha) \frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_n}{\beta_n}.$$

Then $\nu_F(n) = \ker(\wp)$ and $H_p^{n+1}(F) = \text{coker}(\wp)$. By [3], $\nu_F(n) \cong K_n F / p K_n F$ and as a result ([1] for $p = 2$, 1992; [7, Page 152] for any p) $H_p^{n+1}(F) \cong H^1(\Gamma_F, K_n F^{sep} / p K_n F^{sep}) \cong {}_p H^2(\Gamma_F, K_n F^{sep})$, the last isomorphism being a result of the exact sequence

$$K_n F^{sep} \overset{x \mapsto p \cdot x}{\longrightarrow} K_n F^{sep} \longrightarrow K_n F^{sep} / p K_n F^{sep}.$$

In particular, we have $H_p^1(F) = F/\wp(F)$ and $H_p^2(F) \cong H^1(\Gamma_F, F^\times / (F^\times)^p) \cong {}_2 H^2(\Gamma_F, (F^{sep})^\times) \cong {}_p Br(F)$ given by

$$\alpha \frac{d\beta}{\beta} \mapsto [\alpha, \beta)_{p,F} = F\langle x, y : x^2 + x = \alpha, y^2 = \beta, yxy^{-1} = x + 1 \rangle.$$

Theorem 1 ([8]). *When $p = 2$,*

$$H_2^n(F) \cong I_q^n F / I_q^{n+1} F$$

$$\alpha \frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_{n-1}}{\beta_{n-1}} \mapsto \langle\langle b_{n-1}, \ldots, b_1, \alpha]].$$

2. Linkage

We say that $I_q^n F$ is linked if every two quadratic n-fold Pfister forms have a common $(n-1)$-fold factor. Linked fields have interesting properties. One of them is the following:

Theorem 2 ([6], $\mathrm{char}(F) \neq 2$; [4], $\mathrm{char}(F) = 2$)**.** *If $I_q^2 F$ is linked then $u(F)$ is either 1,2,4, or 8.*

The crucial part of the theorem in the case of $\mathrm{char}(F) = 2$ was to show that $I_q^4 F = 0$ (which also means that $H_2^4(F) = 0$). Similarly one can prove that if $I_q^n F$ is linked and F is nonreal then $I_q^{n+2} F = 0$.

3. Triple Linkage

We say that $I_q^n F$ is triple linked if every three n-fold Pfister forms share a common $(n-1)$-fold factor. The following was recently proved:

Theorem 3 ([2], $\mathrm{char}(F) \neq 2$; [5], $\mathrm{char}(F) = 2$)**.** *If $I_q^n F$ is triple linked and F is non-real, then $I_q^{n+1} F = 0$.*

4. Questions

One can define linkage and triple linkage for the groups $H_p^n(F)$ based on the generating symbols $\alpha \frac{d\beta_1}{\beta 1} \wedge \cdots \wedge \frac{d\beta_{n-1}}{\beta_{n-1}}$. The following questions arise naturally:

Question 4. *If $H_p^n(F)$ is linked, does it follow that $H_p^{n+2}(F) = 0$?*

Question 5. *If $H_p^n(F)$ is triple linked, does it follow that $H_p^{n+1}(F) = 0$?*

In [4] some efforts were made to answer Question 4, but a positive answer was obtained only when $p = 2$.

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Supertropical quadratic forms

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(joint work with Z. Izhakian, L. Rowen)

We initiate the theory of a quadratic form on a module V over a semiring R. As customary, one can write

$$q(x+y) = q(x) + q(y) + b(x,y),$$

where b is a **companion bilinear form**. In contrast to the classical theory of quadratic forms over a field, the companion bilinear form does not need to be uniquely defined. Nevertheless, if V is free, q can always be written as a sum of quadratic forms $q = q_{QL} + \rho$, where q_{QL} is quasilinear in the sense that $q_{QL}(x+y) = q_{QL}(x) + q_{QL}(y)$, and ρ is rigid in the sense that it has a unique companion. In the case that the semiring R is supertropical (see [1]), we obtain an explicit classification of these decomposition $q = q_{QL} + \rho$ and of all companions b of q, and see how this relates to the tropicalization procedure (see [2]).

All this is of interest for a quadratic form q on a vector space V over a field F of any characteristic, since after we choose a base $\mathcal{L} = (v_i \mid i \in I)$ of V and a so called supervaluation $\varphi : F \to R$, we obtain a quadratic form $\varphi_*(q)$ on the free module $R^{(I)}$, a supertropicalization of q, which is a very rigid object, amenable to combinatorics, strange at first glance, but easy to handle. The main point here is that $R^{(I)}$, up to multiplication by units has only one base, the standard one. Different bases on V may give different isometry classes $[\varphi_*(q)]$. Our philosophy is, that $[\varphi_*(q)]$ is an invariant of the pair $(q, \mathcal{L})$, perhaps too clumsy for computations when I is big. But by analysing $\varphi_*(q)$, it is possible to detect more amenable invariants and features of $(q, \mathcal{L})$. For example if F is a field and q is anisotropic, we can ask, for which non-zero $x, y \in R^{(I)}$ the quadratic form $\varphi_*(q)_{|Rx+Ry}$ is quasilinear. This depends on the the value of the "Cauchy-Schwartz ratio"

$$CS(x,y) = \frac{eb(x,y)^2}{eq(x)q(y)}$$

in the ghost ideal eR, $e = 1_R + 1_R$, and leads to a kind of supertropical trigonometry [3].

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Positive cones and gauges on algebras with involution

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(joint work with V. Astier)

The connections between quadratic forms, orderings and valuations on fields are well-known [8]. Building on our work on signatures of hermitian forms over algebras with an involution [1, 2], positivity, and our answer to a question of Procesi and Schacher analogous to Hilbert's 17th problem [3], we developed a theory of positive cones on algebras with involution [4].

The canonical "valuations" associated to positive cones turn out to be Tignol-Wadsworth gauges [9, 10, 11]. There is a natural notion of compatibility between positive cones and gauges, that can be described in several equivalent ways, reminiscent of the field case, and which also gives rise to a theorem in the style of Baer-Krull about lifting positive cones from the residue algebra [5].

We present some of our main results on these topics in this note. We refer to [4], [5] for the details. Let F be a field of characteristic not 2 and let A be an F-algebra, equipped with an F-linear involution σ.

1. Positive cones

Definition 1. A set $\mathscr{P} \subseteq \mathrm{Sym}(A,\sigma)$ is a prepositive cone on (A,σ) if

(P1) $\mathscr{P} \neq \emptyset$;
(P2) $\mathscr{P} + \mathscr{P} \subseteq \mathscr{P}$;
(P3) $\sigma(a)\mathscr{P}a \subseteq \mathscr{P}$, for all $a \in A$;
(P4) $\mathscr{P}_F := \{u \in F \mid u\mathscr{P} \subseteq \mathscr{P}\}$ belongs to X_F, the space of orderings of F;
(P5) $\mathscr{P} \cap -\mathscr{P} = \{0\}$.

A maximal prepositive cone is called a positive cone. We denote the set of positive cones on (A,σ) by $X_{(A,\sigma)}$. For $\mathscr{P} \in X_{(A,\sigma)}$, $\leq_{\mathscr{P}}$ denotes the partial ordering on A induced by $\mathscr{P}$.

From now on we assume that (A,σ) is a central simple F-algebra with involution in the sense of the Book of Involutions [7].

Examples 2. (1) The only two positive cones on $(M_n(\mathbb{R}), t)$ are the set of positive semidefinite matrices and the set of negative semidefinite matrices.
(2) For $P \in X_F$, let $\mathscr{M}_P = \{a \in \mathrm{Sym}(A,\sigma) \cap A^\times$ of maximal signature at $P\} \cup \{0\}$. If A is a division algebra, then $\mathscr{M}_P \in X_{(A,\sigma)}$ if and only if $\mathscr{M}_P \neq \mathrm{Sym}(A,\sigma)$.

Theorem 3. *If A is division, then $X_{(A,\sigma)} = \{\mathscr{M}_P, -\mathscr{M}_P \mid \mathscr{M}_P \neq \mathrm{Sym}(A,\sigma),\ P \in X_F\}$. In general, $X_{(A,\sigma)} = \{\mathscr{C}_P(\mathscr{M}_P), -\mathscr{C}_P(\mathscr{M}_P) \mid \mathscr{M}_P \neq \mathrm{Sym}(A,\sigma),\ P \in X_F\}$, where $\mathscr{C}_P$ denotes the closure under* (P2), (P3) *and* (P4) *with $\mathscr{P}_F = P$.*

Theorem 4 ("Artin-Schreier"). *The following are equivalent:*

(1) *(A,σ) is formally real, i.e., $X_{(A,\sigma)} \neq \emptyset$;*
(2) *There exists $a \in \mathrm{Sym}(A,\sigma) \cap A^\times$ such that $\langle a\rangle_\sigma$ is strongly anisotropic;*
(3) *The Witt group $W(A,\sigma)$ is not torsion.*

Theorem 5 ("Artin", simplified version)**.** *Assume that for every $\mathscr{P} \in X_{(A,\sigma)}$ we have $1 \in \mathscr{P} \cup -\mathscr{P}$. Then*

$$\bigcap\{\mathscr{P} \in X_{(A,\sigma)} \mid 1 \in \mathscr{P}\} = \{\sum_{i=1}^{s} \sigma(x_i)x_i \mid s \in \mathbb{N},\ x_i \in A\}.$$

We also use the techniques developed for the proofs of the above theorems to give a Sylvester decomposition of hermitian forms over (A, σ) with respect to a positive cone and obtain in this way another description of signatures of hermitian forms.

Theorem 6. *$X_{(A,\sigma)}$ is a spectral space with respect to the "Harrison" topology with basis $H_\sigma(a_1, \ldots, a_\ell) := \{\mathscr{P} \in X_{(A,\sigma)} \mid a_1, \ldots, a_\ell \in \mathscr{P}\}$.*

2. Gauges from positive cones

Gauges were defined by Tignol and Wadsworth, cf. [9, 10, 11]:

Definition 7. Let $v : F \to \Gamma_v \cup \{\infty\}$ be a valuation of F and let Γ be a totally ordered abelian group. A map $w : A \to \Gamma \cup \{\infty\}$ is a v-gauge if

(1) w is a v-value function on A, i.e. for all $x, y \in A$ and $\lambda \in F$, we have
$w(x) = \infty \Leftrightarrow x = 0;\ w(x + y) \geq \min\{w(x), w(y)\};\ w(\lambda x) = v(\lambda) + w(x);$

(2) w is surmultiplicative, i.e., $w(1) = 0$ and $w(xy) \geq w(x) + w(y)$, for all $x, y \in A$.

(3) w is a v-norm, i.e., A has a "splitting basis" $\{e_1, \ldots, e_m\}$ such that
$$w(\sum_{i=1}^{m} \lambda_i e_i) = \min_{1\leq i\leq m}(v(\lambda_i) + w(e_i)), \quad \forall \lambda_1, \ldots, \lambda_m \in F.$$

(4) the graded algebra $\mathrm{gr}_w(A)$ (with grading determined by w) is a graded semisimple $\mathrm{gr}_v(F)$-algebra.

A gauge w is σ-special if $w(\sigma(x)x) = 2w(x)$ for all $x \in A$. If w is a gauge on A, we define $R_w := \{a \in A \mid w(a) \geq 0\}$ and $I_w := \{a \in A \mid w(a) > 0\}$.

Let $\mathscr{P} \in X_{(A,\sigma)}$ such that $1 \in \mathscr{P}$ (this is always possible after scaling), and let $P = \mathscr{P}_F$. Following the standard definition in the field case, and inspired by Holland [6], we define for a subfield k of F,

$$R_{k,\mathscr{P}} := \{x \in A \mid \exists m \in k \quad \sigma(x)x \leq_{\mathscr{P}} m\},$$
$$I_{k,\mathscr{P}} := \{x \in A \mid \forall \varepsilon \in k^\times \cap P \quad \sigma(x)x \leq_{\mathscr{P}} \varepsilon\}.$$

It is not difficult to see that $R_{k,\mathscr{P}}$ is a subring of A and that $I_{k,\mathscr{P}}$ is a two-sided ideal of $R_{k,\mathscr{P}}$. Both are stable under σ. Note that $R_{k,\mathscr{P}}$ is in general not a total valuation ring, nor a Dubrovin valuation ring.

Theorem 8. *Let $v_{k,P}$ be the valuation on F whose valuation ring is $\{x \in F \mid \exists m \in k : \quad -m \leq_P x \leq_P m\}$. There exists a $v_{k,P}$-gauge $w_{k,\mathscr{P}}$ on A such that $R_{k,\mathscr{P}} = R_{w_{k,\mathscr{P}}}$ and $I_{k,\mathscr{P}} = I_{w_{k,\mathscr{P}}}$. Moreover, $w_{k,\mathscr{P}}$ is the unique σ-special $v_{k,P}$-gauge on A.*

3. Compatibility between gauges and positive cones

Let w be a σ-special v-gauge on A, let σ_0 be the induced involution on the residue algebra $A_0 := R_w/I_w$, and let $\pi_w : R_w \to A_0$ be the canonical projection.

Theorem 9. *Let $\mathscr{P} \in X_{(A,\sigma)}$ such that $1 \in \mathscr{P}$. The following are equivalent:*

(1) $0 \leq_{\mathscr{P}} a \leq_{\mathscr{P}} b \Rightarrow w(b) \leq w(a)$*, for all* $a, b \in A$*;*
(2) R_w *is* $\mathscr{P}$*-convex;*
(3) $1 + \mathrm{Sym}(I_w, \sigma) \subseteq \mathscr{P}$.

The above statements imply that $\pi_w(\mathscr{P} \cap R_w)$ is a positive cone on (A_0, σ_0).

Definition 10. We say that w and $\mathscr{P}$ (with $1 \in \mathscr{P}$) are compatible if one of the above equivalent statements holds.

Theorem 11 ("Baer-Krull")**.** *If $\mathscr{Q} \in X_{(A_0,\sigma_0)}$, then there exists $\mathscr{P} \in X_{(A,\sigma)}$ such that $\mathscr{P}$ is compatible with w, $\pi_w(\mathscr{P} \cap R_w) = \mathscr{Q}$ and $w = w_{\mathscr{P}}$. If $r := \dim \Gamma_v/2\Gamma_v$ is finite, then there are 2^r such liftings of $\mathscr{Q}$.*

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Residue maps for hermitian Witt groups of Azumaya algebras and/or maximal orders

Stefan Gille

Let R be a discrete valuation ring with 2 being a unit in R. Denote by K the fraction- and by k the residue field of R. Then it is a classical result that there is a short exact sequence of Witt groups of quadratic forms

$$0 \longrightarrow W(R) \longrightarrow W(K) \xrightarrow{\partial_\pi} W(k) \longrightarrow 0\,,$$

where the homomorphism ∂_π, the so called second residue map, depends on the choice of an uniformizer π.

Using derived or coherent Witt theory one can construct the following hermitian analog of this complex, and show that it is exact as well. For this set $A_S := S \otimes_R A$ for an R-algebra S.

Theorem. *Let R be a DVR with 2 invertible and (A, τ) an R-Azumaya algebra with involution. Then there is a short exact sequence of ϵ-hermitian Witt groups ($\epsilon \in \{\pm 1\}$):*

$$0 \longrightarrow W_\epsilon(A, \tau) \longrightarrow W_\epsilon(A_K, \tau_K) \xrightarrow{\partial} W_\epsilon(A_k, \tau_k) \longrightarrow 0\,,$$

where τ_K and τ_k denote the by τ induced involutions.

In the case R is complete and A_K is a division algebra one can describe the "residue map" ∂, however in general it seems there is no formula similar to the one for the second residue map for Witt groups of quadratic forms.

A corollary of this theorem is the following purity result.

Corollary. *Let R be a regular local ring of dimension ≤ 2 and (A, τ) an R-Azumaya algebra with involution. Assume that R is quadratic over the fix ring of the involution if τ is of the second kind. Then the ϵ-hermitian Gersten-Witt complex of (A, τ) is exact in degree 0, i.e.*

$$W_\epsilon(A, \tau) \simeq W_{\epsilon, unr}(A, \tau)$$

for $\epsilon = \pm 1$.

Remark. In the equicharacteristic case, i.e. R contains a field, and if the involution τ is of the first kind both results are well known. Actually in this situation the hermitian Gersten-Witt complex of (A, τ) is exact even if $\dim R \geq 2$.

There are analogous exact sequences for R-orders in central simple algebras. Let B be a central simple algebra over the fraction field K of the discrete valuation ring R, and let Δ be a maximal R-order in B. Assume that $\frac{1}{2} \in R$, and that B has an involution of the first kind τ which maps Δ into itself. Then there is an exact sequence of ϵ-hermitian Witt groups ($\epsilon \in \{\pm 1\}$)

$$0 \longrightarrow W_\epsilon(\Delta, \tau) \longrightarrow W_\epsilon(B, \tau) \xrightarrow{\partial} W_\epsilon(\Delta/\mathrm{rad}\Delta, \bar{\tau})\,,$$

where $\bar{\tau}$ is the by τ induced involution on $\Delta/\mathrm{rad}\Delta$. In case R is complete the "residue map" ∂ is onto, and this should hold more general for arbitrary discrete valuations rings R.

Isotropy indices of quadratic forms over function fields of quadrics

Stephen Scully

Let F be a field, let p and q be anisotropic quadratic forms of dimension ≥ 2 over F, and let $F(p)$ denote the function field of the (integral) projective F-quadric of equation $p = 0$. A central problem in the algebraic theory of quadratic forms (upon which many others rest) is that of understanding the extent to which q can become isotropic after scalar extension to $F(p)$. Let $\mathfrak{i}_0(q_{F(p)})$ denote the *isotropy index* (i.e., the maximum dimension of a totally isotropic subspace) of q extended to $F(p)$. Since the determination of the precise conditions under which this integer assumes a given value seems to be rather intractable in general, the main thrust of the recent research in this direction has focused on identifying general constraints imposed on $\mathfrak{i}_0(q_{F(p)})$ by some of the basic invariants of p and q. Despite the substantial progress which has been made here, particularly with the advent of strong methods from the theory of algebraic cycles and motives, many basic questions remain unresolved. In particular, it is already a non-trivial open problem to determine the constraints on $\mathfrak{i}_0(q_{F(p)})$ coming from the simplest invariants of all – the dimensions of p and q. With this in mind, we propose the following conjecture which predicts a very precise general relationship between $\mathfrak{i}_0(q_{F(p)})$, $\dim(p)$ and $\dim(q)$:

Conjecture 1 ([9, Conj. 1.1])**.** *With notation as above, let s be the unique non-negative integer such that $2^s < \dim(p) \leq 2^{s+1}$, and let $k = \dim(q) - 2\mathfrak{i}_0(q_{F(p)})$. Then $k \geq 0$, and we have $\dim(q) = a2^{s+1} + \epsilon$ for some non-negative integer a and some integer $-k \leq \epsilon \leq k$.*

When $\dim(q) \leq 2^s$, Conjecture 1 simply asserts that q remains anisotropic over $F(p)$, which is precisely the content of the well-known *Separation Theorem* originally discovered by Hoffmann (see [2], [3]). This is merely one extreme, however. For instance, Corollary 4 below exhibits another special case of the conjecture where the Separation Theorem (as well as existing extensions of it; see [5, 7]) gives essentially no information.

Conjecture 1 is optimal, in the sense that there are simple examples showing that there are no further gaps in the possible values of $\dim(q)$ determined by s and $\mathfrak{i}_0(q_{F(p)})$ alone (see [8, Ex. 1.5], [9, Ex. 4.5]). Although it is not yet proved in general, we can nevertheless show that the statement holds in a large number of cases (see [8, 9] for a complete list). In particular, we have the following result:

Theorem 2 ([8, Thm. 1.3], [9, Thm. 1.3])**.** *Conjecture 1 holds if $k < 2^{s-1}$ and either*

(1) $\mathrm{char}(F) \neq 2$, *or*
(2) $\mathrm{char}(F) = 2$ *and q is quasilinear* (*i.e., diagonalizable*).

Remark 3. Note that Conjecture 1 is trivially true if $k \geq 2^s - 1$, so it suffices to consider the case where $k \leq 2^s - 2$.

The proofs of the two cases of Theorem 2 are entirely separate from one another: In characteristic $\neq 2$, the approach is algebraic-geometric, and makes use of results of Vishik on the descent of mod-2 algebraic cycles on algebraic varieties over function fields of quadrics ([10]). The action of the Steenrod operations of Brosnan-Voevodsky on Chow groups modulo 2 is essential here, so this approach cannot be extended to characteristic 2 at the present time.[1] It is for this reason that we must limit our considerations to the quasilinear case in characteristic 2. Here we build upon the previous works [6, 7] in which quasilinear analogues of some of the major results in the characteristic $\neq 2$ theory were established using more direct algebraic arguments.[2]

Assume now that $\mathrm{char}(F) \neq 2$. In this case, q is non-degenerate, and so $\mathfrak{i}_0(q_{F(p)})$ coincides with the *Witt index* of $q_{F(p)}$. In particular, in the statement of Conjecture 1, we have $k = 0$ if and only if q becomes *hyperbolic* over $F(p)$, i.e., if and only if q represents an element in the kernel $W(F(p)/F)$ of the scalar extension map $W(F) \to W\big(F(p)\big)$ on the Witt ring. Theorem 2 therefore yields the following discrete information pertaining to arbitrary elements of this Witt kernel:

Corollary 4 ([8, Cor. 1.4]). *Suppose that* $\mathrm{char}(F) \neq 2$*, and let s be as in Conjecture 1. If q becomes hyperbolic over $F(p)$, then* $\dim(q)$ *is divisible by* 2^{s+1}.

A few low-dimensional cases aside, this seems to have been unknown, even conjecturally. In fact, we can say rather more here:

Theorem 5 ([8, Thm. 3.4]). *Suppose that* $\mathrm{char}(F) \neq 2$*, and let s be as in Conjecture 1. If q becomes hyperbolic over $F(p)$, then all higher Witt indices of q, with the possible exception of the last, are divisible by* 2^{s+1}.

Theorem 5 refines Corollary 4 and gives a new lower bound for the dimension of an anisotropic element of the Witt kernel $W(F(p)/F)$ in terms of its height (as defined by Knebusch). This, in particular, gives a satisfying explanation of a classic result of Fitzgerald ([1]) asserting that the "low-dimensional" part of $W(F(p)/F)$ consists of scalar multiples of Pfister forms (see [8, Cor. 3.10, Cor. 3.11]).

Finally, it is also shown in [8] that the characteristic $\neq 2$ case of Conjecture 1 may be deduced from Corollary 4 modulo a deep conjecture of B. Kahn on the descent of quadratic forms over function fields of quadrics ([4]). Kahn's conjecture remains wide open, but the known partial results permit to establish Conjecture 1 in the case where $\mathrm{char}(F) \neq 2$ and $k \leq 7$ (see [8, §5]).

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Does isomorphism over a field imply isomorphism over its valuation rings?

Uriya A. First

Let R be a discrete valuation ring with fraction field F and $2 \in R^{\times}$. The following results, all sharing a common flavour, are well known:

(1) Two nondegenerate quadratic forms over R which become isomorphic over F are already isomorphic over R.
(2) Two Azumaya R-algebras which become isomorphic over F are already isomorphic over R.
(3) Two Azumaya algebras with involution over R which become isomorphic over F are already isomorphic over R.

These results are in fact special cases of the famous Gorthendieck–Serre conjecture: For every regular local ring R and every (connected) reductive group scheme $G \to \operatorname{Spec} R$, the restriction map $\mathrm{H}^1_{ét}(R, G) \to \mathrm{H}^1_{ét}(F, G)$ has trivial kernel. This conjecture is now a theorem when R contains a field thanks to Panin and Fedorov ([5], [6]), and many other cases are known.

In two recent works with Eva Bayer-Fluckiger and Mathieu Huruguen ([2], [3]), we consider "nearly nondegenerate" versions of (1)–(3) above in the case R is a DVR, or more generally, a semilocal Dedekind domain.

To make this precise, recall that an R-order A in F-algebra E is called *hereditary* if its one sided ideals are projective. If $\tau : A \to A$ is an involution, and $h : P \times P \to A$ is a hermitian form, then h is said to be *nondegnerate* or *unimodular* if the induced map $x \mapsto h(x, -) : P \to \mathrm{Hom}_A(P, A)$ is an isomorphism. We call h *nearly nondegenerate* if the cokernel of the latter map is a semisimple A-module. Informally, nearly nondegenerate hermitian forms (resp. hereditary R-orders, hereditary R-orders with involution) can be regarded as degenerate versions of nondegenerate hermitian forms (resp. Azumaya R-algebras, Azumaya algebras with involution over R) in which the degeneration is "very small". Alternatively, the group R-schemes of automorphisms of these objects are "very close" to being reductive, while not being reductive in general.

The motivation for considering nearly degenerate objects as above comes from the fact that given a corresponding object over F, one can always extend it to a *nearly* nondegenerate object defined over R, while this may be impossible to accomplish with a nondegenerate object. Specifically:

(1′) (Exercise) For every nondegenerate quadratic form Q over F, there exists a nearly nondegenerate quadratic form q over R with $Q \cong q_F$.

(2′) (Auslander, Goldman) Every central simple F-algebra E contains a hereditary R-order A (e.g. take any maximal order) and $E \cong A \otimes_R F$.

(3′) (W. Scharlau) For every central simple algebra with involution (E, σ), there exists a hereditary R-order $A \subseteq E$ with $\sigma(A) = A$. In particular, $(E, \sigma) = (A, \tau)_F$, where $\tau := \sigma|_F$.

(4′) (Bayer-Fluckiger, F.) In the notation of (3′), for every nondegenerate hermitian form H over (E, σ), there exists a nearly unimodular hermitian form h over (A, τ) such that $H \cong h_F$.

When considering degenerate objects, one should also take into account the "type of degeneration". In the context of nearly nondegenerate hermitian forms $h : P \times P \to A$, this datum is the isomorphism classes of the A-modules P and $\mathrm{coker}(x \mapsto h(x, -) : P \to \mathrm{Hom}_A(P, A))$. In general, though, it is more convenient to say that two nearly degenerate objects defined over R have the *same degeneration* if they become isomorphic over some faithfully flat étale algebra.

We now come to the **main results**. Firstly, Auel, Parimala and Suresh [1, §3] showed that (1) above remains true in a nearly nondegenerate context. Rephrased and slightly strengthened, their result reads as:

(1-d) If q and q' are nearly nondegenerate quadratic forms over R with same degeneration, then $q_F \cong q'_F$ implies $q \cong q'$.

We stress that this statement is false for arbitrary degenerate forms.

We strengthened (1-d) to hermitian forms and also established a version for hereditary orders:

(2-d) (Bayer-Fluckiger, F., Huruguen) Let A and A' be two hereditary R-orders in central simple F-algebras such that A and A' become isomorphic over some faithfully flat étale R-algebra. Then $A_F \cong A'_F$ implies $A \cong A'$.

(4-d) (Bayer-Fluckiger, F.) Let (A, τ) be a hereditary R-order with involution. If h and h' are nearly nondegenerate hermitian forms over (A, τ) with the same degeneration, then $h_F \cong h'_F$ implies $h \cong h'$.

However, in contrast to these positive results, we showed with Bayer-Fluckiger and Huruguen that the analogous statement (3-d) is false. That is, there exist a DVR R and two hereditary R-orders with involution (A, τ), (A', τ') which become isomorphic over F and over a finite étale R-algebra, but which are nevertheless nonisomorphic over R.

The reason for the failure of (3-d) seems to be the fact that the involutions τ and τ' become *isotropic* modulo the radical of A. That is, writing $\overline{A}$ to denote the quotient of A by its Jacobson radical and $\overline{\tau} : \overline{A} \to \overline{A}$ for the involution induced by τ, there exists $0 \neq x \in \overline{A}$ with $\overline{\tau}(x)x = 0$. We conjecture that if $\overline{\tau}$ and $\overline{\tau'}$

are assumed to be anisotropic, then (3-d) should hold. More generally, we believe that this holds under the milder assumption that each closed fiber of the group scheme $U(A, \tau) \to \operatorname{Spec} R$ is almost anisotropic in the sense that it has no proper parabolic subgroups. We also believe that every central simple F-algebra with involution (E, σ) is extended from an order with involution (A, τ) satisfying the latter condition, or in other words, an appropriate modification of $(3')$ holds.

We finally comment that the group R-schemes of automorphisms of the nearly degenerate objects considered above are strongly related to group schemes defined by Bruhat and Tits in [4] when R is henselian. The previous conjectures are inspired from this connection. Furthermore, the relation with Bruhat–Tits theory shows that, loosely speaking, the nearly nondegenerate objects considered above are different manifestations of a single construction, and not just handy picks for which theorems happen to work. This construction can be used to associate with any connected reductive algebraic group over F a family of "nearly reductive" extensions over R. More details about this are given in [2, §6], [3, §5].

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On the Grothendieck–Serre conjecture for semi-local Dedekind domains

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(joint work with A. Stavrova)

Let R be a commutative unital ring. Recall that an R-group scheme $\mathbf{G}$ is called *reductive*, if it is affine and smooth as an R-scheme and if, moreover, for each algebraically closed field Ω and for each ring homomorphism $R \to \Omega$ the scalar extension $\mathbf{G}_\Omega$ is a connected reductive algebraic group over Ω. This definition of a reductive R-group scheme coincides with [2, Exp. XIX, Definition 2.7].

A well-known conjecture due to J.-P. Serre and A. Grothendieck [12, Remarque, p.31], [5, Remarque 3, p.26-27], and [6, Remarque 1.11.a] asserts that given a regular local ring R and its field of fractions K and given a reductive group scheme

$\mathbf{G}$ over R the map

$$H^1_{\text{ét}}(R, \mathbf{G}) \to H^1_{\text{ét}}(K, \mathbf{G}),$$

induced by the inclusion of R into K, has trivial kernel.

The Grothendieck–Serre conjecture holds for semi-local regular rings containing a field. That is proved in [3] and in [8]. The first of these two papers is heavily based on results of [11] and [9]. For the detailed history of the topic see, for instance, [3]. Assuming that R is not equicharacteristic, the conjecture has been established only in the case where $\mathbf{G}$ is an R-torus [1] and in the case where $\mathbf{G}$ is a reductive group scheme over a discrete valuation ring R [7, Theorem 4.2]. In [10] the latter result is extended partially to the case of semi-local Dedekind domains. Namely, the following theorem is proved there.

Theorem 1. *Let R be a semi-local Dedekind domain and let K be the field of fractions of R. Let $\mathbf{G}$ be a reductive simple simply connected R-group scheme containing a split R-torus $\mathbb{G}_{m,R}$. Then the kernel of the map*

$$H^1_{ét}(R, \mathbf{G}) \to H^1_{ét}(K, \mathbf{G})$$

induced by the inclusion of R into K, is trivial.

Question. Is the kernel of the map $H^1_{\text{ét}}(R, \mathbf{G}) \to H^1_{\text{ét}}(K, \mathbf{G})$ trivial for any semi-local Dedekind domain R and any reductive R-group scheme $\mathbf{G}$?

Firstly, the positive answer on this *Question* will extend the Nisnevich's result [7, Theorem 4.2] to semi-local Dedekind domain case. Secondly, the positive answer on this *Question* will be a first step towards a proof of the following new

Assertion. Let A be a discrete valuation ring and $p : X \to \text{Spec}(A)$ be a smooth projective morphism with an irreducible X. Let $\mathbf{G}$ be a reductive A-group scheme and E be a principal $\mathbf{G}$-bundle on X. Suppose E is trivial over the generic point of X. Then E is Zarisky locally trivial.

Let X_0 be the closed fibre of p and $\{\eta_1, ..., \eta_n\}$ be the set of all generic points of X_0. If there is a closed subset Z in X such that Z does not contain any of the points η_i's and the restriction of E to $X - Z$ is trivial, then one could try to prove the *Assertion* following the strategy from [8].

The first step in an approach to prove the Assertion. Let L be the field of rational functions on X. Regarding points $\{\eta_1, ..., \eta_n\}$ as points of X consider the semi-local ring $\mathcal{O} := \mathcal{O}_{X,\{\eta_1,...,\eta_n\}}$. Clearly, it is a semi-local Dedekind domain. If the *Question* has the positive answer, then the restriction of E to $\text{Spec}(\mathcal{O})$ is trivial. Hence there is a closed subset Z in X such that Z does not contain any of the points η_i's and the restriction of E to $X - Z$ is trivial. *Now one could try to prove the Assertion* using the strategy from [8].

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Spinor Groups with good reduction

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(joint work with A. Rapinchuk, I. Rapinchuk)

1. The main conjecture

Let K be a field and v a discrete valuation of K. If G is an absolutely almost simple linear algebraic group defined over K, we say that G has *good* (or smooth) reduction at v if there exists a reductive group scheme $\mathfrak{G}_v$ over the valuation ring O_v of the completion K_v of K whose generic fiber $\mathfrak{G}_v \otimes_{O_v} K_v$ is isomorphic to $G_{K_v} = G \otimes_K K_v$. One can show that this is equivalent to saying that there exists a reductive group scheme $\mathfrak{G}$ over the valuation ring $\mathcal{O} \subset K$ of v whose generic fiber is isomorphic to G.

Example. Let $G = \mathrm{Spin}(f)$. Then G has good reduction if and only if f has a diagonalization $f = a\langle a_1, \ldots, a_n\rangle$ over K where $a, a_1, \ldots, a_n \in K$ and all the a_i are v-adic units.

Our goal is to discuss the following general problem.

Question 1. *Let K be a field. (When) can one equip K with a natural set V of discrete valuations such that for a given absolutely almost simple simply connected (or adjoint) algebraic K-group G, the set of K-isomorphism classes of (inner) K-forms of G having good reduction at all $v \in V \setminus S$ where $S \subset V$ is an arbitrary finite subset is finite?*

We conjecture that Question 1 has the positive answer for all finitely generated fields of good characteristic.

2. Motivation

2.1. **Borel-Serre theorem.** We first mention the following result which is due to A. Borel and J.-P. Serre [1].

Theorem. *Let k be a number field and G be an affine (not necessary connected) algebraic group over k. Let V be the set of all places of k. Then for an arbitrary finite set $S \subset V$ the canonical map*

$$\omega_{S,G} : H^1(k, G) \longrightarrow \prod_{v \notin S} H^1(k_v, G)$$

is proper, i.e. the pre-image of any finite set is finite. In particular, the corresponding Tate-Shafarevich set $Ш(G)_S := \operatorname{Ker} \omega_{S,G}$ is finite.

It is natural to ask if this result can be generalized to the case of arbitrary fields.

Question 2. *Let K be a field. (When) can one equip K with a natural set V of discrete valuations such that for a given absolutely almost simple adjoint algebraic K-group G, the natural map $\omega_{S,G}$ is proper for any finite set $S \subset V$?*

We conjecture that Question 2 has the positive answer for all finitely generated fields of good characteristic; in particular the corresponding Tate-Shafarevich set $Ш(G)_S$ is finite.

To provide a different perspective on the Borel-Serre theorem, we recall that if an absolutely simple k-group G where k is a number field has good reduction at a non-archimedean place v, then it becomes quasi-split over the completion k_v. Combining this with the properness of $\omega_{S,G}$ one concludes that for an absolutely almost simple simply connected k-group G, the set of k-isomorphism classes of k-forms of G having good reduction at all $v \in V^K \setminus S$ is finite, for any finite subset $S \subset V^k$ containing all archimedean places. (With some additional efforts, this result can be extended to all reductive groups.) *Conversely*, this property *implies* the properness of $\omega_{S,G}$ for adjoint semi-simple groups defined over number fields.

Thus we arrive in a natural way to the question on finiteness of the set of groups defined over an arbitrary field K and having good reduction. It is worth noticing that the affirmative answer on Question 1 immediately implies the affirmative answer on Question 2.

2.2. **Genus of a simple group.** Recall that given an absolutely almost simple simply connected algebraic K-group G, its genus $\mathbf{gen}_K(G)$ is defined to be the set of K-isomorphism classes of K- forms G' of G that have the same isomorphism classes of maximal K-tori as G (the latter means that every maximal K-torus T of G is K-isomorphic to some maximal K-torus T' of G', and vice versa). It was proved by A. Rapinchuk and G. Prasad that if K is a number field, then $\mathbf{gen}_K(G)$ is finite for any G. For applications in geometry it is important to consider groups defined arbitrary fields and we put forward the following question.

Question 3. *When* $\mathbf{gen}_K(G)$ *is finite?*

The case of number fields was studied by G. Prasad and A. Rapinchuk in [2]. We conjecture that $\mathbf{gen}_K(G)$ is finite for any absolutely simple simply connected group G and any finitely generated field K of good characteristic.

To connect Question 3 with Question 1 we mention the following result which is due to V. Chernousov, A. Rapinchuk and I. Rapinchuk.

Theorem. *Let G be an absolutely simple simply connected algebraic group defined over a finitely generated field K of good characteristic. Let $G' \in \mathbf{gen}_K(G)$ and let v be a discrete valuation of K. If G has a good reduction with respect to v then so is G'. In particular, the affirmative answer on Question 1 implies the affirmative answer on Question 3.*

3. Inner forms of type A_n and Spinor groups

3.1. **Choice of V.** We now come back to Question 1. Assume that K is a finitely generated field. Then one can present K as the function field $K = k(C)$ of a smooth geometrically integral projective curve C over a field k where $k = l(x_1, \dots, x_m)$ is a pure transcendental extension of either a number field or a finite field l. Let V_0 be the set of discrete valuations on K associated to closed points of C. Next, we take the set of discrete valuations on k corresponding to all divisors on the corresponding projective space $\mathbb{P}^m$ together with all non-archimedean places on l and take their extensions on K. If V_1 is the resulting set of such discrete valuations on K we let $V = V_0 \cup V_1$.

3.2. **Inner forms of type A_n.** An evidence that Question 1 can have the positive answer for all finitely generated fields is provided by the Borel-Serre theorem in the case of number fields and the following result.

Theorem. *Let G be an inner form of type A_n defined over a finitely generated field K of good characteristic and let V be as above. Then Question 1 has the affirmative answer. In particular, for an arbitrary central simple algebra A over K the Tate-Shafarevich set $Ш(\mathrm{PGL(A)})_S$ is finite.*

3.3. **Spinor groups.** The case of the function field $K = k(C)$ of a smooth geometrically connected projective curve C over a number field k is widely open and not much is known. The following result deals with the case of spinor groups defined over such fields. As above denote by V_0 the set of discrete valuations of K

associated to closed points of C. Furthermore, pick a finite subset $S \subset V^k$ that contains all archimedean places and all places of bad reduction for a certain model of C. Then every $v \in V^k \setminus S$ has a canonical extension to a discrete valuation $\tilde{v}$ of K. We set $V_1 = \{ \tilde{v} \mid v \in V^k \setminus S \}$, and $V = V_0 \cup V_1$.

Theorem. *Keep the above notation. The number of K-isomorphism classes of spinor groups $G = \mathrm{Spin}_n(q)$ of nondegenerate quadratic forms in $n \geq 5$ variables over K that have good reduction at all $v \in V$ is finite.*

Corollary 1. *Let K, V be as above and $S \subset V$ be any finite subset. Let q be a quadratic form in $n \geq 5$ variables over K. Then the Tate-Shafarevich set $Ш(\mathrm{SO}(q))_S$ is finite.*

Remark. The similar result holds for groups of type G_2 and special unitary groups $\mathrm{SU}(L/K, f)$ where L/K is a quadratic extension and f is a hermitian form over L.

Corollary 2. *Let K, V be as above. Then Question 1 has the affirmative answer for all groups of type B_n.*

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Degree three invariants for semisimple groups of types B, C, and D

Sanghoon Baek

Let G be a (split) semisimple group over an algebraically closed field F. Consider the degree 3 cohomological invariant of G with values in $\mathbb{Q}/\mathbb{Z}(2)$, i.e., a natural transformation G-*torsors* $\to H^3$, where G-*torsors*(K) is the set of isomorphism classes of G-torsors over a field extension K/F and $H^3(K) = H^3(K, \mathbb{Q}/\mathbb{Z}(2))$. Let T^* be the character group of a maximal torus T of G and let W be the Weyl group of G. Then, by [13] the group of normalized invariants of G, denoted by $\mathrm{Inv}^3(G)$, can be identified with the factor group $S^2(T^*)^W / \langle \frac{1}{2} \sum_{\lambda \in W(\chi)} \lambda^2 \mid \chi \in T^* \rangle$, where $W(\chi)$ denotes the W-orbit of χ.

Let G_{red} be a reductive group over F such that the derived subgroup of G_{red} is G and the center $Z(G)$ is a torus. Then by [12] the restriction map $\mathrm{Inv}^3(G_{\mathrm{red}}) \to \mathrm{Inv}^3(G)$ is injective and its image is independent of the choice of G_{red}. This image is called the subgroup of *reductive* invariants of G and is denoted by $\mathrm{Inv}^3(G)_{\mathrm{red}}$. For instance, for $G = \mathbf{Spin}_n$ ($n \geq 7$), we have $G_{\mathrm{red}} = \mathbf{\Gamma}_n$, where $\mathbf{\Gamma}_n$ is the even Clifford group, and $\mathrm{Inv}^3(G) = \mathrm{Inv}^3(G)_{\mathrm{red}} = \mathbb{Z}/2\mathbb{Z}$. This invariant is induced by the Arason invariant $\mathbf{e}_3 : \mathbf{Spin}_n$-*torsors* $\to \mathbf{\Gamma}_n$-*torsors* $\stackrel{\mathbf{e}_3}{\to} H^3$.

The group $\mathrm{Inv}^3(G)$ has been completely determined for all simple groups in [13, 4, 8] and for some semisimple groups in [11, 2, 3]. In particular, in [11] the group $\mathrm{Inv}^3(G)_{\mathrm{red}}$ was obtained by Merkurjev for all semisimple groups of type A. The purpose of this talk is to classify the degree 3 (reductive) invariants of all semisimple groups of types B, C, D, which completes the invariants of classical groups. For simplicity we only present the case of type B. Similar results hold for types C and D [1]. A basic question motivating the work is as follows:

Question 1. *Let* $G = \left(\prod_{i=1}^{3} \mathbf{Spin}_{2n_i+1}\right)/\boldsymbol{\mu}$ *with* $n_i \geq 2$*, where we say* $\boldsymbol{\mu} = \langle(-1,-1,-1)\rangle \subset \boldsymbol{\mu}_2^3$*. What would be the degree* 3 *invariants of* G*?*

By a simple calculation, we see that $\mathrm{Inv}^3(G) = \mathrm{Inv}^3(G)_{\mathrm{red}} = (\mathbb{Z}/2\mathbb{Z})^2$. In general, we have the following result for type B:

Theorem 1 ([1]). *Let* $G = (\prod_{i=1}^{m} \mathbf{Spin}_{2n_i+1})/\boldsymbol{\mu}$*,* $m, n_i \geq 1$*, where* $\boldsymbol{\mu} \simeq (\boldsymbol{\mu}_2)^k$ *is a central subgroup. Let* R *be the subgroup of* $(\boldsymbol{\mu}_2^m)^* = \bigoplus_{i=1}^{m}(\mathbb{Z}/2\mathbb{Z})e_i$ *whose quotient is the character group* $\boldsymbol{\mu}^*$*. Let* $R_1' = \langle e_i \in R \,|\, n_i \leq 2\rangle$ *with* $l_1 = \dim R_1'$ *and* $R_2' = \langle e_i + e_j \in R \,|\, e_i, e_j \notin R,\, n_i = n_j = 1\rangle$ *with* $l_2 = \dim R_2'$ *be subspaces of* R*. Then, we have* $\mathrm{Inv}^3(G)_{\mathrm{red}} = (\mathbb{Z}/2\mathbb{Z})^{m-k-l_1-l_2}$*. In particular, if* $n_i \geq 2$ *for all* i*, then* $\mathrm{Inv}^3(G) = \mathrm{Inv}^3(G)_{\mathrm{red}}$*.*

To describe the invariants of G, let us consider the corresponding reductive group $G_{\mathrm{red}} = (\prod_{i=1}^{m} \mathbf{\Gamma}_{2n_i+1})/\boldsymbol{\mu}$. Then, we have the following description of torsors in terms of m-tuples of quadratic forms:

$$H^1(K, G_{\mathrm{red}}) \simeq \{\phi := (\phi_1, \ldots, \phi_m) \,|\, \dim \phi_i = 2n_i + 1, \operatorname{disc} \phi_i = 1, \phi[r] \in I^3(K)\}$$

for all $r = (r_i) \in R$, where

$$\phi[r] := \begin{cases} \perp_{i=1}^{m} r_i\phi_i & \text{if } \sum_{i=1}^{m} r_i \equiv 0 \mod 2, \\ (\perp_{i=1}^{m} r_i\phi_i) \perp \langle 1\rangle & \text{otherwise.} \end{cases}$$

For each $r \in R$, we define the invariant $\mathbf{e}_3[r] : G_{\mathrm{red}}$*-torsors* $\to H^3$ by $\phi \mapsto \mathbf{e}_3(\phi[r])$. In [1] it is shown that the invariant $\mathbf{e}_3[r]$ is nontrivial for any $r \in R \backslash (R_1' + R_2')$. Hence, Theorem 1 can be restated as follows: let $R \to \mathrm{Inv}^3(G)_{\mathrm{red}}$ be the homomorphism given by $r \to \mathbf{e}_3[r]$. Then, this morphism is surjective and its kernel is the subspace $R_1' + R_2'$.

The main result in [1] tells us that for all semisimple groups of types B, C, D there are essentially two types of degree three reductive invariants given by the Arason invariant and the Garibaldi-Parimala-Tignol invariant [7] and no other invariants exist.

Let V be a generically free representation of G, i.e., there exists a G-torsor $U \to U/G$ for some G-invariant open subset $U \subseteq V$. We write BG for U/G. A generalized Noether's problem asks whether the classifying space BG is stably rational or retract rational and it is still open for a connected algebraic group G over an algebraically closed field. By [6, 10] the retract rationality of BG implies the triviality of unramified cohomology group $H^d_{\mathrm{nr}}(F(BG))$ for any degree d. For $d \leq 2$, we have $H^d_{\mathrm{nr}}(F(BG)) = 0$ [5]. Recently, it was shown that the group $H^3_{\mathrm{nr}}(F(BG))$ is trivial if G is a simple group [12] or a semisimple group of type A

[9, 11]. Note that BG is stably birational to BG_{red}. Using a complete description of the invariants obtained in Theorem 1 we show (see [1] for details):

Theorem 2. *Let* $G = (\prod_{i=1}^{m} \mathbf{Spin}_{2n_i+1})/\boldsymbol{\mu}$ $(n_i \geq 1)$ *or* $(\prod_{i=1}^{m} \mathbf{Sp}_{2n_i})/\boldsymbol{\mu}$ $(n_i \geq 1)$ *or* $(\prod_{i=1}^{m} \mathbf{Spin}_{2n_i})/\boldsymbol{\mu}$ $(n_i \geq 3)$ *defined over an algebraically closed field* F *of characteristic* 0*,* $m \geq 1$*, where* $\boldsymbol{\mu}$ *is an arbitrary central subgroup. Then, there are no nontrivial unramified degree* 3 *invariants for* G*, i.e.,* $H^3_{\mathrm{nr}}(F(BG)) = 0$.

Using a similar method developed in [1] we expect that the same result as in the Theorem 2 holds for all exceptional types and we propose the following question:

Question 2. *Let* G *be a semisimple group of mixed Dynkin types (i.e., each component of the Dynkin diagram is of an arbitrary type) over an algebraically closed field. Is* $H^3_{\mathrm{nr}}(F(BG))$ *trivial?*

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Critical varieties for higher isotropy of semi-simple groups

Charles De Clercq

Deep conjectures in the algebraic theory of quadratic forms lie in the study of *isotropy* and *higher isotropy* of quadratic forms, i.e. the splitting behaviour of quadratic forms over field extensions. The recent introduction of Chow-motivic techniques lead to important progresses in this study: the proof of Hoffmann's conjecture by Karpenko [6] and the construction of fields of odd u-invariant by Vishik [10] rely on computations of Chow motives of projective quadrics.

Let us get back to the beginning of the story: in his work towards a proof of Milnor's conjecture, Rost studied in [8] Chow motives of Pfister quadrics, introducing the celebrated Rost motive. Rost also shows that the isotropy of a quadratic form (namely its Witt index) can be directly read on the motivic decomposition of the associated projective quadric. The uniform study of motivic properties of projective quadrics was then carried on wonderfully in [9]. One of the most beautiful properties envisioned by Vishik relies on the fact that motives of projective quadrics actually control *higher isotropy* of quadratic forms. More precisely the following result is obtained.

Theorem 1. *Let q and q' be two quadratic forms of the same dimension over a field F, and denote by Q and Q' the associated projective quadrics. The (integral) Chow motives of Q and Q' are isomorphic if and only if for any field extension E/F, the Witt indices $i_w(q_E)$ and $i_w(q'_E)$ are equal.*

This results raises several questions:

(1) How restrictive the condition on Witt indices of Theorem 1 is? Is this condition equivalent to the fact that q and q' are similar?
(2) Can this result be also obtained replacing projective quadrics by, say, Grassmannians of q-isotropic subspaces?
(3) Can this result be extended to all semi-simple groups, i.e. can we enclose (higher) isotropy of semi-simple groups in the motives of projective homogeneous varieties?

Question (1) was addressed in [4], where Izhboldin shows that motivic equivalent quadratic forms of odd dimension are similar and produces counterexamples for even-dimensional quadratic forms. It was also shown in [3] that motivic equivalent quadratic forms are similar over local and global fields in. The answer of Question (2) is "no" in general - considering for instance Borel varieties associated to quadratic forms - but requires further developments of Question (3) to get finer results.

Question (3) is at the heart of the notion of motivic equivalence for semi-simple groups, which was introduced in [1]. Note that isotropy of semi-simple algebraic groups over fields is controlled by quite classical invariants: the Tits indices. More precisely Borel-Tits classification asserts that to any subset Θ of the Dynkin diagram of G can be associated a (isomorphism class of) twisted flag G-variety $X_{\Theta,G}$. The Tits index of a semisimple group G is then the data of its Dynkin diagram, on which a vertex i is colored if the G-variety $X_{\{i\},G}$ is rational. The main result of [1] asserts that higher isotropy of semi-simple groups is controlled by all motives of twisted flag G-varieties:

Theorem 2. *Two semi-simple groups G and G' of the same type are motivic equivalent (modulo a prime p) if and only if their Tits (p-)indices coincide over any extension of the base field.*

Note that in the case of orthogonal groups $O^+(q)$ and $O^+(q')$ associated to quadratic forms of the same dimension, one then sees that the isotropy condition appearing in Vishik's theorem is in fact equivalent to the fact that the motives twisted flag varieties for G and G' of fixed type are *all* isomorphic.

The notion *critical variety* is introduced in [2]. A twisted flag G-variety $X_{\Theta,G}$ is critical if $X_{\Theta,G}$ is a test-variety for motivic equivalence, i.e. if motivic equivalence between G and another group G' can be checked only through the motives $X_{\Theta,G}$ and $X_{\Theta,G'}$. We construct in [2] critical varieties for all classical groups, as well as in many exceptional cases.

Theorem 3. *All semi-simple groups of classical type admits a critical variety modulo* 2*. More precisely the following twisted flag G-varieties are critical:*

(1) If G is of type A_n (with $n \geq 2$), $X_{\{1,n\},G}$ is critical.
(2) If G is of type B_n, $X_{\{1\},G}$ is critical.
(3) If G is of type C_n (with $n \geq 2$), $X_{\{2\},G}$ is critical.
(4) If G is of type D_n (with $n \geq 3$), $X_{\{1\},G}$ is critical.

The proof of this result relies on the main theorems of [1] and makes an essential use of the anisotropy theorems for algebras with involutions produced by Karpenko and Karpenko-Zhykhovich [5],[7]. One may deduce from Theorem 3 complete generalizations of Vishik's criterion to all classical groups.

Theorem 4. *If (A, σ) and (B, τ) are of the same orthogonal or symplectic type over F, the involutions σ and τ are motivic equivalent if and only if we have*

$$A \simeq B \quad \text{and} \quad i_{w,2}(\sigma_M) = i_{w,2}(\tau_M) \text{ for all field extensions } M/F.$$

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The alternative Clifford algebra

Uzi Vishne

(joint work with A. Chapman, I. Rosenbaum)

This talk is based on a paper titled "The Alternative Clifford Algebra of a Ternary Quadratic Form", which was written with ADAM CHAPMAN and is submitted for publication elsewhere; and on joint work with my Master's student ITAY ROSENBAUM.

1. The problem

The classical Clifford algebra of a quadratic form $q : V \to F$ is the free associative algebra generated by V, modulo the relations $x^2 = q(x)$ for $x \in V$. It is an important invariant of quadratic forms, essentially the second cohomological invariant.

In her recent PhD thesis, Stacy Marie Musgrave, working under Danny Krashen, suggested that one should consider the Clifford algebra in other varieties of algebras, in particular alternative algebras. By definition, $\mathrm{C}^{\mathrm{alt}}(q)$ is the free alternative algebra generated by V, modulo the same relations as in the associative case. This is a new algebraic invariant of quadratic forms.

Recall that the associator in a nonassociative algebra is defined by $(a, b, c) = (ab)c - a(bc)$; and the algebra is alternative if the form (a_1, a_2, a_3) alternates. We denote $a \circ b = ab + ba$.

We give a complete description of $\mathrm{C}^{\mathrm{alt}}(q)$ when $\dim(q) = 3$, and give some details on the structure in higher dimension.

2. Ternary forms

Theorem 1. *Let q be a nondegenerate ternary quadratic form over an arbitrary field F. Then $\mathrm{C}^{\mathrm{alt}}(q)$ is an octonion algebra over a ring of polynomials in one variables.*

In fact:

(1) When $\operatorname{char} F \neq 2$, $\mathrm{C}^{\mathrm{alt}}(\langle \alpha_1, \alpha_2, \alpha_3 \rangle) \cong (\alpha_1, \alpha_2, \lambda^2 + 4\alpha_1\alpha_2\alpha_3)_{F[\lambda]}$.

(2) When $\operatorname{char} F = 2$, $\mathrm{C}^{\mathrm{alt}}([\alpha_1, \alpha_2] \perp \langle \alpha_3 \rangle) \cong [\alpha_1\alpha_2, \alpha_2, \lambda^2 + \alpha_3)_{F[\lambda]}$.

The proof when $\operatorname{char} F \neq 2$ is based on properties of the element $x_i \circ (x_j x_k)$ where $\{x_1, x_2, x_3\}$ form an orthogonal basis; which is well-defined up to scalars. Similar ideas, somewhat more elaborate, work when $\operatorname{char} F = 2$.

Clearly, the only simple associative quotient of $\mathrm{C}^{\mathrm{alt}}(q)$ is the (simple quotients of the) associative Clifford algebra.

Let $\mathfrak{a} = (x_1, x_2, x_3)$ denote the associator of basis elements. It is known that the square of associators is central in the free alternative algebra on 3 generators, hence $\mathfrak{a}^2$ is central in $\mathrm{C}^{\mathrm{alt}}(V)$.

Proposition 2. *The algebra* $\mathrm{C}^{\mathrm{alt}}(q)[\mathfrak{a}^{-2}]$ *is "alternative Azumaya", in the sense that all the simple quotients are octonion algebras.*

3. Four dimensional forms

Our observation on the alternative Clifford algebra for a nondegenerate form $q = \langle\alpha_1, \alpha_2, \alpha_3, \alpha_4\rangle$ is based on computer computation.

Let x_1, x_2, x_3, x_4 be an orthogonal basis for which $x_i^2 = \alpha_i$.

Lemma 3. *(1) The element*

$$c' = x_1(x_2(x_3x_4)) + 2[x_1x_2, x_3x_4] - 3\left[(x_1x_2)(x_3x_4) + x_3((x_1x_2)x_4) - x_4((x_1x_2)x_3)\right]$$

is central in $\mathrm{C}^{\mathrm{alt}}(q)$*;*

(2) Let

$$c''_1 = \sum_{(ijk)=(234)} x_i[x_1(x_jx_k)x_1] + x_1((x_jx_k)(x_1x_i)) - 3\alpha_1(x_2 \circ (x_3x_4));$$

then $c''_{\sigma(i)} = \mathrm{sgn}(\sigma)c''_i$ *defines a 4-dimensional representation of* S_4*, which is contained in the center of* $\mathrm{C}^{\mathrm{alt}}(q)$*.*

Proposition 4. *The nonzero element* $g = \sum \alpha_i^{-1}{c''}_i^2 - {c'}^2$ *is central, and satisfies* $g^2 = 0$*.*

Since g generates a nilpotent ideal, we have that:

Corollary 5. *The alternative Clifford algebra in dimension* ≥ 4 *is not semiprime.*

4. The octonionic quotient

We assume $\operatorname{char} F \neq 2$. Let $q = \langle\alpha_1, \dots, \alpha_n\rangle$ be a nondegenerate form in dimension $n \geq 4$. Let $x_1, \dots, x_n$ form an orthogonal basis with $x_i^2 = \alpha_i$. The elements

$$w_{ijk} = x_i \circ (x_jx_k)$$

are not central in $\mathrm{C}^{\mathrm{alt}}(q)$, but w_{ijk}^2 is central; furthermore, w_{ijk} is central in any octonion quotient, but in no associative quotient.

Let $\mathrm{C}^{\mathrm{oct}}(q)$ be the maximal quotient of $\mathrm{C}^{\mathrm{alt}}(q)$ all of whose simple quotients are octonion algebras.

Theorem 6. *When* $\dim(q) > 3$*,* $\mathrm{C}^{\mathrm{oct}}(q)$ *is the quotient of* $\mathrm{C}^{\mathrm{alt}}(q)$ *obtained by forcing the* w_{ijk} *to be central.*

Let $Z = \mathrm{Z}(\mathrm{C}^{\mathrm{oct}}(q))$.

Proposition 7. $\mathrm{C}^{\mathrm{oct}}(q)$ *is a finite module over* Z*, contained in a standard octonion algebra over* Z*.*

Theorem 8. *The transcendence degree of Z is as in the following table:*

$\dim q$	3	4	5	6	7	...
$\operatorname{tr}\deg Z$	1	4	6	7	7	0

Indeed, when $\dim(q) > 7$ we have that $\mathrm{C}^{\mathrm{oct}}(q) = 0$.

Octonion Algebras with Isometric Quadratic Forms over Rings, via Triality

Seidon Alsaody
(joint work with P. Gille)

This is a report on the paper [1], where we give a description, over any commutative unital ring, of all octonion algebras having isometric quadratic forms.

1. Introduction

A *composition algebra* over a field k is a k-algebra C endowed with a non-degenerate quadratic form $q : C \to k$ that is multiplicative, i.e. permits composition in the sense that

$$q(x \cdot y) = q(x)q(y)$$

for all $x, y \in C$, whence the name. In the definition of an algebra we neither require associativity nor commutativity, but we will here always assume the existence of a unity, although various classes of non-unital composition algebras have been studied by several authors.

It is known that composition algebras only exist in dimensions 1, 2, 4 and 8; those of dimension 4 are the *quaternion algebras* and those of dimension 8 are known as *octonion algebras*. The quadratic form of a composition algebra over a field is a Pfister form, and is determined by the algebra structure via its minimal polynomial. Conversely, it determines the algebra in the sense that two algebras are isomorphic if and only if their quadratic forms are isometric. This reduces the classification of composition algebras over fields to a question of quadratic forms.

The definition of a composition algebra can be generalized, replacing the field by a ring. In this setting, the quadratic form is still determined by the algebra, but the converse fails for octonion algebras, as was proved by P. Gille in [3] using cohomological arguments.

2. Torsors, Twists and Triality

An octonion algebra over a (unital, commutative) ring R is an R-algebra C, the underlying module of which is projective of constant rank 8, endowed with a non-degenerate, multiplicative quadratic form $q = q_C : C \to R$, known as the norm. Our aim is to understand the class K_C of all octonion algebras C' with $q_C \sim q_{C'}$. The automorphism group $\mathbf{Aut}(C)$ of C is an affine group of type G_2, which is a closed subgroup of the orthogonal group $\mathbf{O}(q)$. The set K_C is in bijection with the kernel of the cohomology map

$$H^1(R, \mathbf{Aut}(C)) \to H^1(R, \mathbf{O}(q_C))$$

induced by the inclusion $\mathbf{Aut}(C) \to \mathbf{O}(q_C)$. (All sheaves and cohomology sets in this talk are with respect to the fppf-topology.) As detailed in [4], this implies that the algebras in K_C are obtained as twists of C by the $\mathbf{Aut}(C)$-torsor

$$\mathbf{O}(q_C) \to \mathbf{O}(q_C)/\mathbf{Aut}(C),$$

and twists corresponding to two fibres are isomorphic precisely when the fibres are in the same orbit of the natural $\mathbf{O}(q_C)$-action. In order to understand these twists explicitly, we proceed in several steps, in which triality plays a key role. This phenomenon, going back to the work of É. Cartan on isometries of real octonion algebras, has been generalized to arbitrary fields (see e.g. [2] and [5]). We generalize this over rings in the following way (a different genralization was given in [6]): define the closed subgroup scheme $\mathbf{RT}(C)$ of $\mathbf{SO}(q_C)^3$ of *related triples* by setting, for each commutative, unital R-algebra S,

$$\mathbf{RT}(C)(S) = \{(t_1, t_2, t_3) \in \mathbf{SO}(q_C)(S)^3 | t_1(\overline{x} \cdot \overline{y}) = \overline{t_2(x)} \cdot \overline{t_3(y)}\}$$

where x and y run through $C \otimes_R S$. The octonionic involution $z \mapsto \overline{z}$ serves the purpose of simplifying the following result, which also shows that these triples are essentially determined by either of their components.

Proposition 1 ([1], 3.6 and 3.9). *Let C be an octonion algebra over R.*

(1) The group $\mathbf{RT}(C)$ is a semisimple simply connected group of type D_4, and the assignment $(t_1, t_2, t_3) \mapsto t_1$ is a universal cover of $\mathbf{SO}(q_C)$.
(2) The cyclic group C_3 acts on $\mathbf{RT}(C)$ by automorphisms of order 3, and $\mathbf{Aut}(C)$ embeds in $\mathbf{RT}(C)$ as the fixed subgroup of this action.

The first step of our simplification of the torsor is the following.

Proposition 2 ([1], 6.6). *The set K_C is in bijection with*

$$H^1(R, \mathbf{Aut}(C)) \to H^1(R, \mathbf{RT}(C)).$$

Thus we are led to studying the torsor

$$\mathbf{RT}(C) \to \mathbf{RT}(C)/\mathbf{Aut}(C),$$

where we are able to describe the quotient using two copies of the octonionic unit sphere $\mathbb{S}_C$ (defined by the equation $q_C(x) = 1$ in C).

Proposition 3 ([1], 4.1). *The map $\mathbf{RT}(C) \to \mathbb{S}_C \times \mathbb{S}_C$ sending (t_1, t_2, t_3) to $(t_3(1), t_2(1))$ identifies the quotient $\mathbf{RT}(C)/\mathbf{Aut}(C)$ with $\mathbb{S}_C \times \mathbb{S}_C$, on which $\mathbf{RT}(C)$ acts by $(t_1, t_2, t_3) \cdot (a, b) = (t_3(a), t_2(b))$.*

This defines an $\mathbf{Aut}(C)$-torsor over $\mathbb{S}_C \times \mathbb{S}_C$. We denote the fibre over the point (a, b) by $\mathbf{E}^{a,b}$

Having described the torsor explicitly, we can now determine the twists themselves. These turn out to be given by a classical construction, namely that of isotopes. For our purposes, if $a, b \in C$ are elements of norm 1, the *isotope* $C^{a,b}$ is the algebra with underlying module C, and with multiplication $x \circ y = (x \cdot a) \cdot (b \cdot y)$. It is an octonion algebra with the same norm as C.

Theorem 4 ([1], 4.6)**.** *The twist of* C *by* $\mathbf{E}^{a,b}$ *is naturally isomorphic to* $C^{a,b}$.

Thus all octonion algebras with norms isometric to q_C are isotopes of C.

3. Final Remarks

The key idea of the proof is that if (t_1, t_2, t_3) is a related triple, then t_1 is an isomorphism $C \to C^{a,b}$ with $a = t_3(1)$ and $b = t_2(1)$. This illustrates in a precise way how the pair of spheres (each of dimension 7) bridges the gap between the automorphism group of C (of dimension 14) and the orthogonal group of q_C (of dimension $28 = 14 + 7 + 7$). The advantage of the explicit description is that under certain circumstances (for example, when the ring is a (Laurent) polynomial ring over a field), one can in many cases determine when two isotopes are isomorphic, using algebraic manipulations. While such manipulations in general provide sufficient conditions for isotopes to be isomorphic, the problem of finding explicit conditions that are both necessary and sufficient seems still open. A wider open problem is that of describing the behaviour, over rings, of certain non-unital composition algebras. Most prominent among these are the so-called symmetric ones, some of which were first studied by Okubo in work on particle physics.

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Affine quadrics and the Picard group of the motivic category

Alexander Vishik

It was observed for a long time that quadratic forms play a fundamental role in many branches of mathematics. I would like to discuss still another point of view on this rich subject. In the last 30 years the algebraic theory of quadratic forms is more and more supplemented by geometric and motivic methods. These are visible already in the works of M. Knebusch, and were extensively developed by M. Rost, V. Voevodsky, the author, N. Karpenko, A. Merkurjev and other people.

Traditionally, one assigns to a quadratic form the associated projective homogeneous varieties (quadratic Grassmannians), in particular, the respective projective quadric. This permits to introduce many powerful invariants of quadratic forms which allow to distinguish them in many cases. At the same time, there is another geometric object one can assign to a form q. It is the affine quadric $\{q = 1\}$. The study of motivic properties of such objects was initiated by Po Hu in [3] in the context of stable homotopic category of Morel-Voevodsky with the aim of describing the Picard group of invertible objects there, and then expanded by T. Bachmann in the context of motivic category of Voevodsky. It is the latter strand which I would like to develop.

Let k be a field of $char(k) \neq 2$ and q be some n-dimensional (non-degenerate) quadratic form over k. We can assign to it the affine quadric $A_q = \{q = 1\}$. It is nothing else but the complement $Q' \backslash Q$ of a codimension one smooth subquadric in a smooth projective quadric, where $q' = \langle 1 \rangle \perp -q$ and we denote by the capital letter the projective quadric corresponding to a given quadratic form.

Consider the motive $M(A_q)$ of this affine quadric in the Voevodsky's motivic category (with $\mathbb{Z}/2$-coefficients) $DM(k, \mathbb{Z}/2)$. Affine quadric plays a role of a (not necessarily split) sphere, and over algebraic closure, its motive splits into a sum of two Tate-motives: $M(A_q|_{\overline{k}}) = T \oplus T([n/2])[n-1]$. In particular, the (shifted) "reduced motive" e^q of A_q we introduce below is a "form" of a Tate-motive $T([n/2])[n]$, and so is invertible over $\overline{k}$.

Definition 1. $e^q := \mathrm{Cone}(M(A_q) \to T) \in DM(k, \mathbb{Z}/2)$.

It was shown by T. Bachmann in [1] that e^q is $\otimes$-invertible already over the ground field. Moreover, as was proven in [2], e^q is a complete invariant of q (so, affine quadrics behave in this respect better than the projective ones, as the latter are not determined by their motives). We get an embedding

$$GW(k) \hookrightarrow Pic(DM(k, \mathbb{Z}/2))$$

where $[q] - r[\mathbb{H}] \mapsto e^q(-r)[-2r]$. Due to the famous result of F. Morel, the left group can be identified with $\pi^{st}_{(0)[0]}(\mathbb{S})$. The topological analogue

$$\pi^{st}_0(\mathbb{S}) \to Pic(D^b(\mathbb{Z}/2))$$

of this map is an isomorphism. In algebraic geometry it is not even a group homomorphism. In a sense, it is even better as we can describe more elements using the operation of Pic.

Definition 2. Pic_{qua} = subgroup of Pic generated by e^q, for all q/k.

We would like to describe the structure of this subgroup. Pic_{qua} contains the "split" subgroup $\mathbb{T} = \{T(i)[j]\}$ consisting of Tate motives, and it is enough to describe the quotient group $Pic_{qua}/\mathbb{T}$. We first observe ([4, Prop.2.1]) that our set of generators is stable under inverses: $(e^q)^{-1} = e^{q'}$ in $Pic_{qua}/\mathbb{T}$, where $q' = \langle 1 \rangle \perp -q$. It appears that our group is really large.

Theorem 1. [4, Thm 3.1] *Let $\{q_i\}$ be a set of quadratic forms s.t.: 1) q_i' is anisotropic, $\forall i$; 2) q_i' is not stably birationally equivalent to q_j', for $i \neq j$. Then the collection $\{e^{q_i}\}$ is linearly independent in $Pic_{qua}/\mathbb{T}$.*

Example. Let $\{\langle\langle\alpha\rangle\rangle\}$ be the collection of all anisotropic Pfister forms (of any foldness) over k. Then $\{e^{\langle\langle\alpha\rangle\rangle}\}$ is linearly independent in $Pic_{qua}/\mathbb{T}$.

In order to describe the group Pic_{qua}, we need to introduce the new set of generators.

Definition 3. Let Q be a smooth projective quadric with a complete flag $Q = Q_m \supset Q_{m-1} \supset \ldots \supset Q_1 \supset Q_0$ of smooth subquadrics. Define:

$$det(Q) := e^{Q_m \backslash Q_{m-1}} \cdot e^{Q_{m-1} \backslash Q_{m-2}} \cdot \ldots \cdot e^{Q_1 \backslash Q_0} \cdot e^{Q_0} \in Pic.$$

By [4, Prop.3.6], $det(Q)$ is well-defined (doesn't depend on the choice of a flag) and, actually, depends on $M(Q)$ only. Clearly, $\{det(Q)\}$ is another set of generators of Pic_{qua}. We can describe the relations among them.

Theorem 2. [4, Thm 3.11] *TFAE:*

(1) $\prod_i det(Q_i) = \prod_j det(P_j)$ *in* $Pic_{qua}/\mathbb{T}$

(2) $\oplus_i M(Q_i) \stackrel{Tates}{\sim} \oplus_j M(P_j)$, *where in the latter equivalence we ignore Tate-summands and Tate-shifts.*

Thus, the question about Voevodsky's triangulated category and $\otimes$ operation is reduced to the one about Chow motivic category and $\oplus$ operation. The above result implies that Pic_{qua} is a free abelian group (previously, it was known by T. Bachmann [1] that it has no torsion).

Example. Since all anisotropic indecomposable direct summands of real quadrics are Rost motives, for $k = \mathbb{R}$ we get:

$$(Pic_{qua}/\mathbb{T})(\mathbb{R}) = \oplus_{r \in \mathbb{N}} \mathbb{Z} \cdot e^{\langle\langle -1 \rangle\rangle^r}.$$

I'm unaware of any examples of elements of $Pic(DM(k, \mathbb{Z}/2))$ outside Pic_{qua}, so there is still hope that the group we study is the whole Picard group (note, that with the odd coefficients, such examples are known).

As a bi-product we obtain an extension of the criterion of motivic equivalence of projective quadrics.

Corollary. [4, Cor.3.12] *Let P, Q be smooth proj. quadrics. TFAE*

(1) $M(P) \cong M(Q)$;

(2) $det(P) = det(Q)$ *in* Pic_{qua}.

All the above results can be obtained either with the help of functors of Bachmann [1], or alternatively, using the projectors in Voevodsky category corresponding to Čech symplicial schemes [4]. The latter method permits to study the whole Picard group.

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On generic quadratic forms

Nikita A. Karpenko

Let k be a field of characteristic different from 2 and let $F_g = k(t_1, \dots, t_n)$ be the field of rational functions over k in variables $t_1, \dots, t_n$ for some $n \geq 2$. We call *generic* the diagonal quadratic form $q_g := \langle t_1, \dots, t_n \rangle$ over F_g. Thus q_g is the n-dimensional quadratic form $F_g^n \to F_g$ on the vector space F_g^n given by the formula

$$q_g \colon (x_1, \dots, x_n) \mapsto \sum_{1 \leq i \leq n} t_i x_i^2.$$

The Chow ring of the projective quadric defined by q_g has been computed in [2, Corollary 2.2]. The Chow ring of the highest orthogonal Grassmannian of a generic quadratic form has been computed in [5] (see also [6]), but this was done for a different notion of generic, which we call here *standard generic*. As shown in [1, §3], the n-dimensional standard generic quadratic form q lives over the field of rational functions $F = k(t_{ij})_{1 \leq i \leq j \leq n}$ in $n(n+1)/2$ variables t_{ij} and can be defined (in arbitrary characteristic including characteristic 2) by the formula

$$F^n \to F, \quad (x_1, \dots, x_n) \mapsto \sum_{1 \leq i \leq j \leq n} t_{ij} x_i x_j.$$

In the present paper we determine the Chow ring $\mathrm{CH}\, X$ of all orthogonal Grassmannians X associated with the generic and the standard generic quadratic forms. (The characteristic $\neq 2$ assumption is removed in the latter case; the characteristic 2 analog for the first case is provided in [1, §9].) Namely, our Main Theorem ([1, 6.1], see also [1, Corollary 8.2 and Proposition 9.2]) affirms that the ring $\mathrm{CH}\, X$ is generated by the Chern classes of the tautological vector bundle of X. A complete list of relations satisfied by these Chern classes (in general, not only in the generic situation) is provided in [1, Theorem 2.1]. All the (well-known) relations that hold over an algebraic closure of the base field actually already hold over the base field itself. This way we obtain a description of the ring $\mathrm{CH}\, X$ in terms of generators and relations. It also follows that the additive group of $\mathrm{CH}\, X$ is torsion-free (see [1, Corollary 6.2]).

Proving Main Theorem, we use computation of the Chow ring of classifying spaces for orthogonal groups $O(n)$ performed in [4] as well as in [7] over the field of complex numbers and later in [3] over an arbitrary field of characteristic not

2. We actually need only a piece of this computation which is made in [7] over arbitrary field (of arbitrary characteristic), see [1, §5].

Note that the algebraic group $O(n)$ over a field k is not connected if n is even or $\operatorname{char} k \neq 2$. In the remaining case (when n is odd and $\operatorname{char} k = 2$) the algebraic group $O(n)$ is not smooth. In contrast, the special orthogonal group $O^+(n)$ is always smooth and connected. But since $O(n)$-torsors correspond to all non-degenerate n-dimensional quadratic forms while $O^+(n)$-torsors correspond to quadratic forms of trivial discriminant, it is more appropriate to work with $O(n)$ for the question raised in this paper. On the other hand, since orthogonal Grassmannians depend only on the similarity class of the quadratic form in question and any odd-dimensional quadratic form is similar to that of trivial discriminant, $O(n)$ can be replaced by $O^+(n)$ for odd n.

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Cohomological invariants of Witt classes and algebras with involution

Nicolas Garrel

Context

We work over a base field k of characteristic not 2, and K denotes any field extension of k. If one wants to study cohomological invariants of algebraic groups, one is led to invariants of $\mathrm{Quad}_{2r} \cap I^n$ (meaning non-degenerate quadratic forms in dimension $2r$ whose Witt class is in I^n) for $n = 1$ (for orthogonal groups), $n = 2$ (special orthogonal groups) and $n = 3$ (spin groups). The cases $n = 1, 2$ have been treated by Serre, but the case $n = 3$ is very much open except for small r.

On the other hand, the Milnor conjecture provides a natural cohomological invariant e_n of I^n. One may then try to identify invariants of Witt classes in I^n, instead of isometry classes in fixed dimension, since I^n is much easier to describe than $\mathrm{Quad}_{2r} \cap I^n$ (which has no description already for $n = 3$).

This is related to the question of operations on cohomology: it is well-known that $\sum x_i \mapsto \sum_{i_1<\ldots<i_d} x_{i_1} \cup \cdots \cup x_{i_d}$, where x_i are Galois symbols, is not a well-defined operation. On the other hand, Rost constructed such an operation for $d=2$ on I^n, where the x_i are Pfister forms.

1. Operations on Witt classes

The basis of our work is that we may define natural operations $\pi_n^d : I^n(K) \to I^{nd}(K)$ for any $d \in \mathbb{N}$, such that:

- $\pi_n^0 = 1$, $\pi_n^1 = \mathrm{Id}$;
- $\forall q, q'$, $\pi_n^d(q+q') = \sum_{k=0}^d \pi_n^k(q)\pi_n^{d-k}(q')$;
- $\forall \varphi \in \mathrm{Pf}_n(K)$, $\forall d \geq 2$, $\pi_n^d(\varphi) = 0$.

In particular, this ensures that if $q = \sum_{i=1}^r \varphi_i$ where the φ_i are n-fold Pfister forms, then $\pi_n^d(q) = \sum_{i_1<\cdots<i_d} \varphi_{i_1} \cdots \varphi_{i_d}$.

These operations can be constructed as functions on the Grothendieck-Witt ring, and then restricted to $\hat{I}^n(K) \simeq I^n(K)$ where $\hat{I}(K)$ is the kernel of the degree map; more precisely, π_n^d can be taken of the form $\sum_{k=1}^d a_k \lambda^k$, where λ^k is the usual exterior power on quadratic forms and $a_k \in \mathbb{Z}$. The existence of appropriate a_k can be proved formally using the fact that a 1-fold Pfister form φ satisfies $\varphi^2 = 2\varphi$ and $\lambda^d(\varphi) = \varphi$ for $d \geq 1$.

Theorem 1. *Any natural operation $I^n(K) \to W(K)$ can be written as a unique combination $\alpha = \sum_{d\in\mathbb{N}} a_d \pi_n^d$ with $a_d \in W(k)$.*

Sketch of proof: Fixing $q \in I^n(K)$, we get an invariant of Pf_n over K by $\varphi \mapsto \alpha(q+\varphi)$. Using a result of Serre, it follows that there is a uniquely defined invariant α^+ such that $\alpha(q+\varphi) = \alpha(q) + \varphi \cdot \alpha^+(q)$. This operator satisfies $(\pi_n^d)^+ = \pi_n^{d-1}$, which allows us to proceed by induction.

Such an infinite combination will a priori take values only in the I-adic completion of W. If k is not formally real, no such problem arises and any such sum really takes values in W. In any case, it is actually possible to find another set of generators for which we can always take infinite combinations.

2. Cohomological invariants of Witt classes

We now define a cohomological invariant $u_{nd}^{(n)} = e_{nd} \circ \pi_n^d$ (in particular, $u_n^{(n)} = e_n$). If $q = \sum_{i=1}^r \varphi_i$ where the φ_i are n-fold Pfister forms, then $u_{nd}^{(n)}(q) = \sum_{i_1<\cdots<i_d} e_n(\varphi_{i_1}) \cup \cdots \cup e_n(\varphi_{i_d})$.

We can prove a similar theorem to that of Witt invariants, with a similar proof.

Theorem 2. *Any cohomological invariant α of I^n can be written as a unique combination $\alpha = \sum_{d\in\mathbb{N}} a_d \cup u_{nd}^{(n)}$ with $a_d \in H^*(k,\mu_2)$.*

With an eye towards algebras with involution, we are interested in the behaviour of cohomological invariants with respect to similitudes and ramification.

For any cohomological invariant α of I^n, we can define an invariant $\widetilde{\alpha}$ such that $\alpha(\langle\lambda\rangle q) = \alpha(q) + (\lambda) \cup \widetilde{\alpha}(q)$. In particular, $\widetilde{\alpha} = 0$ iff α is invariant under similitudes. We may prove that $\widetilde{\widetilde{\alpha}} = 0$.

If K is endowed with a k-valuation v, discrete of rank 1, then for any invariant α of I^n, if $q \in I^n(K)$ is non-ramified, then $\alpha(q)$ is non-ramified.

3. Cohomological invariants of algebras with involutions

3.1. **Generic splitting.** Let (A, σ) be an algebra with involution over K, of orthogonal type and with $\mathrm{ind}(A) = 2$. We set $M_A^d(K)$ to be the 2-torsion part of $H^d(K, \mu_4^{\otimes(d-1)})/[A] \cdot H^{d-2}(K, \mu_2)$. Following an idea of Berhuy, we use a theorem of Kahn, Rost and Sujatha which implies that the natural restriction map induces an isomorphism $M_A^d(K) \simeq H^d(K(A), \mu_2)$ where $K(A)$ is the generic splitting field of A.

We show that if α is an invariant of I^n such that $\widetilde{\alpha} = 0$, and if after generic splitting of A, σ becomes adjoint to a quadratic form in I^n, then there is a natural way to define an invariant $\alpha(A, \sigma) \in M_A^d(K)$. Furthermore, since $\widetilde{\widetilde{\alpha}} = 0$, we can always define $\widetilde{\alpha}(A, \sigma) \in M_A^{d-1}(K)$, and if $\widetilde{\alpha}(A, \sigma) = 0$ then we may define an appropriate $\alpha(A, \sigma) \in M_A^d(K)$.

In particular, we can define invariants e_n for algebras of index 2. The fact that they are only defined modulo $[A]$ is a familiar obstruction for e_2.

As a further example, for $\alpha = u_4^{(1)}$, this is related to a result of Rost, Serre and Tignol: they had constructed what we call $\alpha(A, \sigma)$ (but with values in $H^4(K, \mu_2)$ instead of $M_A^4(K)$) when A is of degree 6 and -1 is a square in k, which actually implies that $\widetilde{\alpha} = 0$.

3.2. **Mixed Witt ring.** There are many hints that we should be able to carry out the operations π_n^d directly at the level of the base field for algebras with involution, instead of going to a splitting field. This would require some kind of λ-ring structure on hermitian forms, imitating the structure of $GW(K)$.

Indeed, we may define a graded λ-ring structure on $\widetilde{GW}^{\varepsilon}(A, \sigma) = GW(K) \oplus GW^{\varepsilon}(A, \sigma)$, as well as a filtering $I^n(A, \sigma)$ of this ring, such that $\pi_1^d(I(A, \sigma)) \subset I^m(A, \sigma)$ with $m = \lceil d/\mathrm{ind}(A) \rceil$. The quotients $I^d(A, \sigma)/I^{d+1}(A, \sigma)$ should have close links with mod 2 cohomology, allowing to define invariants of more or less cohomological nature for algebras with involution of any type in any index. We should at the very least find actual cohomological invariants in index 2, and relative invariants in any index.

Splitting Families in Galois Cohomology

Mathieu Florence
(joint work with C. Demarche)

Let k be a field, assumed to be infinite for simplicity. Let

$$S : \{\text{Fields}/k\} \longrightarrow \{* - \text{Sets}\}$$

be a functor, from the category of field extensions of k, to that of pointed sets. A typical example is the functor of isomorphism classes of algebraic structures of a given type. For instance, $S(l/k)$ can be the set of isomorphism classes of non-degenerate quadratic forms over l, of a given dimension- the distinguished element being the class of a split quadratic form. More generally, if G/k is an algebraic group, we can take $S(l/k)$ to be the pointed set $H^1(l, G_l)$, of isomorphism classes of G-torsors over l. If G is commutative, and $n \geq 2$ is an integer, the (fppf) cohomology groups $H^n(l, G_l)$ give rise to such a functor as well. Note that there is, in general, no description of this group as the set of isomorphism classes for a "nice" algebraic structure.

Given S as above, and given an element $s \in S(k)$, we address the following Problem.

$P(= P(S, s))$: Build a (smooth, geometrically integral) k-variety X, such that, for any field extension l/k, we have

$$s_l = * \in S(l)$$

if, and only if,

$$X(l) \neq \emptyset.$$

Moreover, in that case, we require that $X(l)$ is Zariski-dense in X.

In other words, for any field extension l/k, the presence of (a Zariski-dense set of) l-rational points in X is equivalent to the triviality of s over l.

Example 1. Assume that S is the set of isomorphism classes of regular quadratic forms, of given even dimension $2d$, and of trivial discriminant. Let $s \in S(k)$ be the class of such a form q. We can then take X to be the maximal orthogonal Grassmannian of q; it is a projective variety. We can also take X to be the SO_{2d}-torsor associated to q; it is an affine variety.

Example 2. Assume that S is the set of isomorphism classes of central simple algebras of given degree d. Let $s \in S(k)$ be the class of such an algebra A. We can then take X to be the Severi-Brauer variety $SB(A)$; it is a projective variety. Another available choice for X is the PGL_d-torsor associated to A; it is an affine variety.

Example 3 (more elaborate)**.** Assume that $S(l)$ is the Galois cohomology group $H^n(l, \mu_p^{\otimes n})$, where $n \geq 2$ is an integer, and where p is a prime number, invertible in k. Assume further that

$$s = (x_1) \cup (x_2) \cup \ldots \cup (x_n),$$

with $x_i \in l^*$, is a pure symbol in $S(l)$. Then, Markus Rost constructed a norm variety $X(s)$, which -almost- satisfies the requirements of Problem $P(S, s)$ (cf. [2]). More precisely, one has to replace "l-rational point" by "zero-cycle of degree prime to p, over l".
Note that these norm varieties are a main ingredient in Vladimir Voevodsky's proof of the Bloch-Kato conjecture.

In this talk, we give a complete answer to a weaker version of Problem P, as follows.

Definition 4. A (smooth, geometrically integral) ind-variety $\mathcal{X}$ over k is the data of a sequence of (smooth, geometrically integral) k-varieties $(X_i)_{i\in\mathbb{N}}$, together with closed embeddings $X_i \xrightarrow{f_i} X_{i+1}$. We write

$$\mathcal{X}(k) := \bigcup_{i\in\mathbb{N}} X_i(k).$$

The goal of my talk was to prove the following result.

Theorem 5 ([1]). *Let $n \geq 2$ be an integer. Let M be a finite étale group scheme over k. In other words, M is a finite discrete* $\mathrm{Gal}(k_{sep}/k)$*-module. Let $s \in H^n(k, M)$ be a (Galois) cohomology class. Then, there exists a (smooth, geometrically integral) ind-variety $\mathcal{X} = (X_i)_{i\in\mathbb{N}}$ over k, such that the following holds.*

- *For every field extension l/k, the restriction of s in $H^n(l, M)$ vanishes if, and only if, $\mathcal{X}(l) \neq \emptyset$.*
- *Moreover, for each index i, the set $X_i(l)$, when nonempty, is Zariski-dense in X_i.*

Note that, if $n = 2$, there is such an $\mathcal{X}$ which is actually a variety.
For $n \geq 3$, we do not know whether we can also pick $\mathcal{X}$ to be a variety. We believe this likely to be possible.
The proof we offer is a combination of two ingredients.

(a) A lifting result for the cohomology of linear algebraic groups, following from Hilbert's Theorem 90 for GL_n.
(b) Homological algebra in the category of Yoneda extensions (or in the derived category) of finite $\mathrm{Gal}(k_{sep}/k)$-modules.

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Cohomological invariants mod 2 of Weyl groups

Jean-Pierre Serre

Let G be the Weyl group of a root system, i.e., a crystallographic finite Coxeter group, cf. [1], chap.VI, §4.1. Let k_0 be a field of characteristic $\neq 2$, let $H^\bullet(k_0) = \oplus_{n\geqslant 0} H^n(k_0, \mathbf{F}_2)$ and let $I_G = \mathrm{Inv}_{k_0}(G)$ be the ring of cohomological invariants mod 2 of G, as defined in [2], §4; it is a graded $H^\bullet(k_0)$-algebra. When G is of type A, it is isomorphic to a symmetric group Sym_n, and I_G is $H^\bullet(k_0)$-free of rank $1 + [n/2]$, with an explicit basis $w_0 = 1, w_1, ..., w_{[n/2]}$, cf. [2], chap.VII.

In order to extend this description of I_G to the general case, define S_G to be the set of elements $g \in G$ with $g^2 = 1$; an element of S_G shall be called an *involution* of G. Let Σ_G be the set of conjugation classes of elements of S_G.

Theorem 1. *There exists a natural injection* $e : \Sigma_G \to I_G$ *whose image is an* $H^\bullet(k_0)$*-basis of* I_G.
[Equivalently: the $H^\bullet(k_0)$-module I_G is canonically isomorphic to the set of all maps $\Sigma_G \to H^\bullet(k_0)$.]

The map e is compatible with the grading of I_G: if $g \in S_G$, define the *degree* of g to be be the multiplicity of -1 as an eigenvalue of g in the standard linear representation of G as a Coxeter group; let $\Sigma_{G,n}$ be the set of involution classes of degree n. *If* $\sigma \in \Sigma_{G,n}$, *then* $e(\sigma)$ *belongs to the* n*-th component* I_G^n *of* I_G.

Examples.

(1) When $G = \mathrm{Sym}_n$, the elements of Σ_G are the conjugation classes of the products of i disjoint transpositions, with $2i \leqslant n$, and we recover the fact that $H^\bullet(k_0)$-free of rank $1 + [n/2]$, with a basis made up of elements of degree $0, 1, ..., [n/2]$. In that case the canonical basis is made up of the w_i^{gal}, which are closely related to the w_i mentioned above, cf. [2], §25.

(2) When $G = \mathrm{Weyl}(\mathsf{E}_8)$, we have $|\Sigma_{G,n}| = 1$ for $0 \leqslant n \leqslant 8$, with the only exception of $n = 4$ where $|\Sigma_{G,n}| = 2$; and, of course, $\Sigma_{G,n} = \varnothing$ for $n > 8$. Hence I_G is a free $H^\bullet(k_0)$-module of rank 10, with a basis made up of elements of degree $0, 1, 2, 3, 4, 4, 5, 6, 7, 8$.

(3) For E_7 and E_6, the degrees are $0, 1, 2, 3, 3, 4, 4, 5, 6, 7$ and $0, 1, 2, 3, 4$.

Definition of the map $e : \Sigma_G \to I_G$

Let a be an element of I_G and let g be an involution of G of degree n. We first define a "scalar product" $\langle a, g\rangle$, which is an element of $H^\bullet(k_0)$. To do so, choose a splitting $g = s_1 \cdots s_n$, where the s_i are commuting reflections (recall that a reflection is an involution of degree 1); such a splitting always exists. Let $C = \langle s_1, ..., s_n\rangle$ be the group generated by the s_i, and let $a_C \in I_C$ be the image of a by the restriction map $I_G \to I_C$. The algebra I_C has a natural basis (α_I) indexed by the subsets I of $[1, n]$, cf. [2], §16.4. Let $\lambda_C \in H^\bullet(k_0)$ be the coefficient of $\alpha_{[1,n]}$ in a_C ("top coefficient"). One can show that λ_C *is independent of the chosen splitting of* g, *i.e., that it only depends on* a *and* g. We then define the scalar product $\langle a, g\rangle$ as λ_C; we have $\langle a, g\rangle = \langle a, g'\rangle$ if g and g' are conjugate in G;

this allows us to define $\langle a, \sigma \rangle$ for every $\sigma \in \Sigma_G$. For a given σ, the map $a \mapsto \langle a, \sigma \rangle$ is $H^\bullet(k_0)$-linear; if a has degree m, then $\langle a, \sigma \rangle$ has degree $m - n$ (one may view $a \mapsto \langle a, \sigma \rangle$ as an n-th fold residue map).

Example. Choose for a a Stiefel-Whitney class $w_i^{\mathrm{gal}}(\mathrm{Cox})$ of the Coxeter representation of G. One has $\langle a, \sigma \rangle = 0$ if $i \neq \deg(\sigma)$ and $\langle a, \sigma \rangle = 1$ if $i = \deg(\sigma)$.

Theorem 2.

(i) If $a \in I_G$ is such that $\langle a, \sigma \rangle = 0$ for every σ, then $a = 0$.

(ii) Let n be an integer. For every $\sigma \in \Sigma_G$ of degree n, there exists $e(\sigma) \in I_G^n$ such that $\langle e(\sigma), \sigma \rangle = 1$ and $\langle e(\sigma), \sigma' \rangle = 0$ for every $\sigma' \neq \sigma$.

[Note that by (i), such an $e(\sigma)$ is unique.]

It is clear that Theorem 2 implies Theorem 1.

Indications on the proof of part (i) *of Theorem 2.*

An induction argument shows that, if $\langle a, \sigma \rangle = 0$ for every σ, then the restriction of a to every "cube" (i.e., subgroup generated by commuting reflections) is 0. In that case, if the characteristic of k_0 is good for G, the arguments of [2], §25, show that $a = 0$. This already covers the case where the irreducible components of G are of classical type, since every characteristic $\neq 2$ is good. The exceptional types can be reduced to the classical ones, thanks to the fact that, if G is such a Weyl group, there exists a subgroup G' of G, generated by a subset of S_G (hence also a Weyl group), which is of classical type, and has *odd index* in G: for G of type $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8, \mathsf{F}_4, \mathsf{G}_2$, one takes G' of type $\mathsf{D}_5, \mathsf{A}_1 \times \mathsf{D}_6, \mathsf{D}_8, \mathsf{B}_4, \mathsf{A}_1 \times \mathsf{A}_1$, respectively; one has $(G : G') = 27, 63, 135, 3, 3$. One then uses the fact that the restriction map $I_G \to I_{G'}$ is injective, cf. [2], prop. 14.4, and that every cube of G is conjugate to a cube of G'.

Indications on the proof of part (ii) *of Theorem 2.*

We need to construct enough cohomological invariants. For most Weyl groups, this is done by using Stiefel-Whitney classes. For instance, for Weyl(E_6), one takes the $w_i^{\mathrm{gal}}(\mathrm{Cox})$, $i = 0, 1, 2, 3, 4$. There are however three cases where we have to do otherwise. For each one, there are two distinct classes of involutions σ, σ' of the same degree n for which it is hard to find $a \in I_G^n$ with $\langle a, \sigma \rangle = 0$, $\langle a, \sigma' \rangle = 1$. These cases are: D_{2n}, $n \geqslant 3$; E_7, $n = 3$ and 4; E_8, $n = 4$.

For those, we use the relation given by Milnor's conjecture (now Voevodsky's theorem) between Witt invariants and cohomological invariants mod 2. The method applies to every linear group $\mathcal{G}$ over k_0. The ring $\mathrm{Inv}_{k_0}(\mathcal{G}, W)$ of Witt invariants of $\mathcal{G}$ (as defined in [2], §27.3) has a natural filtration: an invariant h has filtration $\geqslant n$ if, for every extension k/k_0 and every $\mathcal{G}$-torsor t of $\mathcal{G}$ over k, the element $h(t)$ of the Witt ring $W(k)$ belongs to the n-th power of the canonical ideal of $W(k)$; in that case, h defines (via the Milnor construction) an element a_h of $\mathrm{Inv}_{k_0}^n(\mathcal{G}, \mathbf{F}_2)$ which is 0 if and only if the filtration of h is $> n$. We thus get *an injective map* $\mathrm{gr}^n \mathrm{Inv}_{k_0}(\mathcal{G}, W) \to \mathrm{Inv}_{k_0}^n(\mathcal{G}, \mathbf{F}_2)$.

We apply this to $\mathcal{G} = G$, where G is as in the three cases above. One can find a linear orthogonal representation of G whose Brauer character χ is such that $\chi(\sigma) - \chi(\sigma') = 2^n$. This gives a G-quadratic form, hence an element of $\mathrm{Inv}_{k_0}(G, W)$; one modifies slightly that element to make it of filtration $\geqslant n$, so that it gives a cohomological invariant a of G of degree n, and one checks that $\langle a, \sigma\rangle - \langle a, \sigma'\rangle = 1$; that information is enough to conclude the proof.

Dependence of $\mathrm{Inv}_{k_0}(G)$ *on* $H^\bullet(k_0)$ *- Universal objects.*

(i) *Additive structure*

For the additive structure, $\mathrm{Inv}_{\mathbf{C}}(G)$ is a universal object, i.e., there is natural isomorphism of $\mathbf{F}_2$-vector spaces : $\mathrm{Inv}_{k_0}(G) \simeq \mathrm{Inv}_{\mathbf{C}}(G) \otimes_{\mathbf{F}_2} H^\bullet(k_0)$.

(ii) *Ring structure*

For the ring structure, it is $\mathrm{Inv}_{\mathbf{R}}(G)$ which is a universal object: there is a natural graded-$\mathbf{F}_2$-algebra isomorphism: $\mathrm{Inv}_{k_0}(G) \simeq \mathrm{Inv}_{\mathbf{R}}(G) \otimes_{H^\bullet(\mathbf{R})} H^\bullet(k_0)$.

[In this formula, $H^\bullet(k_0)$ is viewed as an $H^\bullet(\mathbf{R})$-algebra via the unique homomorphism $H^\bullet(\mathbf{R}) \to H^\bullet(k_0)$ which maps the class of -1 in $H^1(\mathbf{R}) \simeq \mathbf{R}^\times/(\mathbf{R}^\times)^2$ onto the class of -1 in $H^1(k_0) \simeq k_0^\times/(k_0^\times)^2$.]

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Note. After my lecture, Stefan Gille has pointed out to me that, using a different method (based on a theorem of Totaro, but not involving involutions), Christian Hirsch had already computed in 2009 the structure of the cohomological invariants of all the finite Coxeter groups, under some mild hypotheses on the ground field; his method also applies to other types of invariants. Reference:

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