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Interactions between Operator Space Theory and Quantum Probability with Applications to Quantum Information

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Abstract. Operator space theory was the joining element of our workshop. This theory is a quantisation of the theory of Banach spaces. The talks at our meeting investigated the interactions of operator space theory with operator algebras and other areas of functional analysis, with classical and quantum probability, with noncommutative harmonic analysis, and, in particular, with quantum information theory.

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Introduction by the Organisers

The interplay between operator spaces and quantum probability has started more than a decade ago, but applications of these theories to quantum information began to appear only more recently. Noncommutative L_p-spaces are at the intersection of these areas and play a crucial role in recent mathematical research motivated by concepts and problems from quantum physics. These "quantum mathematics" are at the frontiers between theoretical physics and mathematics.

The key object in quantum information theory (QIT) is the notion of quantum channel, which describes quantum communication protocols. It is a linear map (trace-preserving and completely positive) between state spaces, which are themselves embedded into matrix algebras. Therefore, typical problems in QIT can be reformulated as problems in matrix theory, and, via a probabilistic approach to the study of spaces of matrices, benefit very much from RMT and free probability. One of the main advantages offered by QIT is the possibility for a quantum transmission channel to exceed the known (and theoretically proven) bounds of classical transmission channels. Tensor products play a fundamental rôle and this opens a door for applications of operator space techniques and noncommutative L_p-spaces.

The talks by Ion Nechita, Hun Hee Lee, Cécilia Lancien, Andreas Winter, and Ken Dykema presented interactions between QIT and operator space theory, and the first Open Problem Session on Tuesday evening was dedicated to QIT. Tsirelson's Problem concerns a family of questions that ask roughly whether the descriptions of quantum correlations via commuting operators and via tensor products are equivalent. Ken Dykema reported on recent progress by Slofstra, Dykema, and Paulsen, who proved that the sets of quantum correlations $C_q(100, 8)$ and $C_q(5, 2)$ on tensor products of two Hilbert spaces with dimensions 100 and 8, or 5 and 2, resp., are not closed, see also his extended abstract in this report. It is still open, whether $C_q(4, 2)$ is closed.

It is well-known that Connes' Embedding Problem and Kirchberg's Conjecture are equivalent to Tsirelson's Problem. Many other properties are now known to be equivalent to these three open problems, and many interesting questions arise. Does Kirchberg's conjecture

$$C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) \cong C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty)$$

for the full group-C*-algebra $C^*(\mathbb{F}_\infty)$ of the free group with countably many generators imply that a similar relation for triple tensor products

$$C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) \cong C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty)$$

also holds? Does the Tsirelson problem for bipartite systems imply the Tsirelson problem for tripartite systems? Are the third-order Kirchberg and Tsirelson problem equivalent? Does there exist a version of Connes' embedding problem that is equivalent third-order Connes or Kirchberg?

Narutaka Ozawa presented a proof based on a delicate interplay between pure mathematics and computer science that proves that the automorphism group $\mathrm{Aut}(\mathbb{F}_5)$ of the free group with five generators has Kazhdan's property (T). The pure mathematics part consists in an algebraic characterization of property (T), which says that group Γ has property (T), if certain polynomial of its non-normalized Laplacian belongs to the positive cone of hermitian squares in the full group C*-algebra. In the case of $\mathrm{Aut}(\mathbb{F}_d)$, it was possible to verify this for $d = 5$ using semidefinite programming, see Ozawa's extended abstract in this report and the preprint cited there.

Mikael de la Salle showed that the groups $SL_n(\mathbb{Z})$ have strong property (T) for $n \geq 3$, by studing "representations" where one is only allowed to compose once.

Compact and locally compact quantum groups were succesfully axiomatized by Woronowicz, Kustermans, Vaes, and their collaborators in the 1980's and 1990, giving rise to a very vibrant theory. Martijn Caspers, Anna Wysoczańska-Kula, Amaury Freslon, Moritz Weber, and Piotr Sołtan presented recent progress in harmonic analysis and probability theory on quantum groups. Yulia Kuznetsova looked at the next step and gave survey over possible approaches to a theory of topological quantum semigroups.

In total, 51 mathematicians participated in this meeting. The aim of the workshop was to bring together a mixed group of experts and young researchers from different areas in functional analysis, quantum probability, and quantum information theory. Topics ranging from operators spaces and Fourier multipliers, compact and locally quantum groups, to random walks and quantum correlations were presented and eagerly discussed not only during the 30 talks, but also during the breaks and the excursion. The diversity of the topics and participants stimulated very fruitful discussions and gave rise to new collaborations.

The organizers and participants thank the Mathematisches Forschungsinstitut Oberwolfach for giving us the opportunity to hold this meeting in such an inspiring setting.

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Workshop: Interactions between Operator Space Theory and Quantum Probability with Applications to Quantum Information

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Abstracts

Return time and entropy estimates

MARIUS JUNGE
(joint work with Gao Li and N. LaRacuente, consultation with I. Badet)

There is a deep connection between isoperimetric inequalities and functional inequalities, in particular hyper-contractivity estimates, see [1, 11, 10] for more information. In quantum information theory interest in hyper-contractivity has recently gained a lot of interest because of the connection to multi-body systems, see [9, 13].

The idea is behind this connection is quite simple. On a grid of quantum systems a Hamiltonian acts locally, i.e. a small amount of interaction between neighboring configurations. However, the system as a whole now undergoes the action of a large Hamiltonian with many terms. Of course, one should expect to be able to make some global assertion starting from local properties of the small actions. For example it is expected that after some interaction the global system is back, or close to an equilibrium state.

In order to formalize this problem we start with a so-called quantum Markov system on a finite dimensional matrix algebra. For simplicity, and trough-out this abtract, we will assume that the Markov system is selfadjoint preserving, i.e. the Lindblatt generator is given by

$$\mathcal{L}(\rho) = \sum_{k=1}^{m} (a_k^2\rho + \rho a_k - 2a_k\rho a_k).$$

Here $a_1, \cdots, a_m$ are selfadjiont matrices and the commutator $N = \{a_1, \cdots, a_m\}'$ is exactly the fixpoint algebra of this semigroup. Return to equilibrium can be captured with different expression. Inspired by the work of Otto-Villani's work on log-Sobolev inequalities, see [12], we will use the *relative entropy*

$$D(\rho|\sigma) \;=\; \tau(\rho \log \rho) - \tau(\rho \log \sigma)$$

where $\tau(x) = \frac{1}{m}\sum_{j=1}^{m} x_{jj}$ is the normalized trace on our m-dimensional matrix algebra. The Fischer Information, appears as the derivative

$$I_{\mathcal{L}}(\rho) = \tau(\mathcal{L}(\rho)\log\rho) = \frac{d}{dt} D(T_t\rho|E_N(T_t\rho))$$

of the relative entropy. Here EN is the unique trace preserving conditional expectation. Following Otto-Villani [12], we say that $\mathcal{L}$ satisfies λ-LSI (logarithmic Sobolev inequality) if

$$\lambda D(\rho|E(\rho)) \le I_{\mathcal{L}}(\rho).$$

This inequality implies

$$D(T_t(\rho)|E(T_t\rho)) \le e^{-\lambda t} D(\rho|E(\rho)),$$

i.e. exponential loss for large time. We say that $\mathcal{L}$ satisfies the complete *Logarithmic Soboloev Inequality* λ-CLSI if $\mathcal{L} \otimes id_{M_k}$ satisfies λ-LSI for all k. The good news here is that CLSI is stable under tensorization, which is not known for LSI. However, in order to pass from small to many body systems, we *need* stability under tensorization. Tensor stability is considerably more difficult in the noncommutative setting.

Theorem 1 (GJLR)**.** *The set of selfadjoint Lindblatt generators with CLSI is dense.*

In proving this result we have to overcome a number of obstacles. First the standard route via hyper-contractivity has to be abandoned, because no viable notion of complete hyper-contractivity seems to be valid, see also [2]. So far there a very few tools to show CLSI, and all of these tools are based on some gaussian matrix model, which is to special for arbitrary Lindblatt generators. Our tool to overcome this problem stems from a very basic, but crucial observation. The generator $\mathcal{L}_N = id?E_N$ has 1-CLSI, because $\mathcal{L}_N \otimes id = id \otimes id_{M_k} - E_N \otimes id_{M_k}$ is of the same form, and for *any* conditional expectation

$$I_{id-E_N}(\rho) = D(\rho|E_N(\rho)) + D(E_N(\rho)|\rho) \geq D(\rho|E_N(\rho))$$

reflects the symmetric definition of divergence [8]. Our second ingredient is the gradient form

$$2\Gamma_{\mathcal{L}}(x,y) = \mathcal{L}(x^*)y + x^*\mathcal{L}(y) - \mathcal{L}(x^*y).$$

Using derivation techniques from [6], we show that

Lemma 2. $\lambda\Gamma_{I-E_N}(x,x) \leq \Gamma_{\mathcal{L}}(x,x)$ implies λ-CLSI.

Therefore it suffices to verify the stronger, but not tensor-stable property $\Gamma\mathcal{E}$ for a large class of selfadjoint generators.

Theorem 3.
(1) *Let Δ_X be the sublaplacian on a compact manifold given by a Hörmander system. Then Δ_X^θ has $\Gamma\mathcal{E}$ for every $0 < \theta < 1$.*
(2) *Let $\mathcal{L}$ be a selfadjoint Lindblatt generator. Then $\mathcal{L}^\theta$ has $\Gamma\mathcal{E}$ for all $0 < \theta < 1$.*

It is now clear that our density results is obtained by approximating $\mathcal{L}$ by $\mathcal{L}^\theta$ for θ close to one and hence derive λ-CLSI from our derivation Lemma. Let us note that there are selfadjoint generators $\mathcal{L}$ which do not satisfy $\Gamma\mathcal{E}$ itself, hence the approximation is indeed necessary. An additional feature of λ-$\Gamma\mathcal{E}$ is the direct decay estimate

$$\|T_t(\rho) - E(\rho)\|_p \leq e^{-\lambda}\|\rho - E(\rho)\|_p$$

which is, surprisingly, also true for the endpoints $p = 1$ and $p = \infty$.

Problem 4. Does every selfadjoint Lindlblatt generator satisfy λ-CLSI for some $\lambda(\mathcal{L})$?

Our results use heavily the theory of amalgamated and conditional L_p spaces from [5] and are closely connected to the work of [7] on concentration inequalities and [4] on the cb-version of Varapoulos' theorem, see [14]. Once CLSI is established Carlen-Maas? adaptation of the work of Otto-Villani can be used to prove Talagrand type concentration inequalities with respect to the Rieffel type distance

$$d(\rho,\sigma) = \sup\{|\tau(\rho f) - \tau(\sigma f)| : E_N(f) = 0, f = f^*, \Gamma(f,f) \leq 1\}.$$

According to Connes [3] this induced the Riemanian distance on manifold if we replace $\gamma(f,f) \leq 1$ by the equivalent condition $\|[D,f]\| \leq 1$, D the Dirac operator.

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Noncommutative Mazur maps

Éric Ricard

In Banach space and integration theory, Mazur maps $M_{p,q} : L_p \to L_q; f \mapsto f|f|^{\frac{p-q}{q}}$ are convenient tools to transfer results from one L_p-space to another. When the measure space is commutative, the behavior of its modulus of continuity is well known (see [4]); $M_{p,q}$ is Lipschitz on balls if $p > q$ and $\frac{p}{q}$-Hölder if $p < q$. This has many applications, such as studying group actions on L_p-spaces. The paper [8] gives some illustrations for noncommutative L_p.

When dealing with noncommutative integration on von Neumann algebras, the inequalities for the modulus of continuity are much more involved. It was first proved by Raynaud [10] that $M_{p,q}$ is indeed uniformly continuous on balls with an ultraproduct argument that do not give a precise estimate. This presentation aims to give some answers, this is of course related to the perturbation theory for the functional calculus.

Usually the results are often first stated for the von Neumann $B(\ell_2)$ with its usual trace so that the L_p-spaces correspond the Schatten classes S^p. When dealing with positive elements in S^p, an old inequality by Birman Koplienko and Solomjak [3] also obtained by Ando [2] states that with $p = \theta q$, $0 < \theta < 1$

$$\left\|x^\theta - y^\theta\right\|_{p/\theta} \leq \left\| x - y \right\|_p^\theta.$$

This was extended to all semi-finite von Neumann algebras in [5]. For general von Neumann algebras, Kosaki got the case $p = \theta$ in [6] with an extra factor.

Davies proved that for the Schatten classes S^p the map $x \mapsto |x|$ is Lipschitz when $1 < p < \infty$. This is also a particular case of a more remarkable result by Potapov and Sukochev [9], if $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz then $x \mapsto f(x)$ is also Lipschitz on all the self-adjoint part of L_p-spaces associated to semi-finite von Neumann algebras. Combining that fact with the previous inequality, one gets that $M_{p,q}$, $1 < p < q$ is $\frac{p}{q}$-Hölder. A result of Aleksandrov and Peller [1] extends it (at least on the selfadjoint part of Schatten classes); if $f : \mathbb{R} \to \mathbb{R}$ is $\frac{p}{q}$-Hölder, then so does $x \mapsto f(x)$ as a map $S^p \to S^q$.

Very little was known when $p \leq 1$, in [7], for type II von Neumann algebras a strange quantitative estimate was obtained for the modulus of continuity of $M_{p,q}$. The general result we present is

Theorem 1. *For all von Neumann algebras, $0 < p < q < \infty$, the map $M_{p,q}$: $L_p \to L_q; f \mapsto f|f|^{\frac{p-q}{q}}$ is $\frac{p}{q}$-Hölder.*

For all von Neumann algebras, $1 \leq q < p < \infty$, the map $M_{p,q} : L_p \to L_q; f \mapsto f|f|^{\frac{p-q}{q}}$ is Lipschitz on balls.

The techniques involved are those of perturbation theory of the functional calculus. First one needs precise estimates for the norm of so called Schur multipliers (very usual in the Banach space case), next the new ingredient is to use a suitable dyadic decomposition of elements in L_p. The reduction from semi-finite algebras to general ones relies on the Haagerup reduction principle if $p, q \geq 1$. Whereas when $p < 1$, one has first to prove a similar result for the more general Lorentz spaces $L_{p,q}$.

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Fourier L_p-multipliers in $SL_n(\mathbf{R})$

Javier Parcet
(joint work with Éric Ricard and Mikael de la Salle)

Let (G,μ) be $SL_n(\mathbf{R})$ with its Haar measure and let us write $\widehat{\mathrm{G}}$ to denote the group von Neumann algebra of G. If $\lambda : \mathrm{G} \to \mathcal{U}(L_2(\mathrm{G}))$ stands for the left regular representation, noncommutative L_p spaces over this algebra are defined with respect to the natural trace determined by

$$\tau(f) = \tau\Big(\int_{\mathrm{G}} \widehat{f}(g)\lambda(g)d\mu(g)\Big) = \widehat{f}(e)$$

for smooth enough $\widehat{f} \in \mathcal{C}_c(\mathrm{G})$. Given $m : \mathrm{G} \to \mathbf{C}$, let

$$T_m(f) \, = \, \int_{\mathrm{G}} m(g)\widehat{f}(g)\lambda(g)\, d\mu(g),$$

the Fourier multiplier associated to m. Our main results are sufficient and rigidity conditions for L_p-boundedness of these maps, in terms of the regularity of the symbol $m : \mathrm{G} \to \mathbf{C}$. The relation between smoothness and L_p-boundedness of Fourier multipliers is central in classical harmonic analysis and orbits around the Hörmander-Mikhlin criterion [4, 9] for $1 < p < \infty$

$$\text{(HM)} \qquad \big\| T_m : L_p(\mathbf{R}^d) \to L_p(\mathbf{R}^d) \big\| \, \le \, C_{p,d} \sup_{0 \le |\gamma| \le [\frac{d}{2}]+1} \big\| |\xi|^{|\gamma|} \partial^\gamma m(\xi) \big\|_\infty.$$

Our sufficient conditions require to introduce some notation:

- $\sigma_n = \big[\frac{n^2}{2}\big]$,
- $\lceil g \rfloor = \min\big\{1, \mathrm{dist}(g,e)\big\}$ for $g \in SL_n(\mathbf{R})$,
- $L(g) = \max\big\{\|g\|, \|g^{-1}\|\big\} \rightsquigarrow \Theta_\varepsilon(g) = L(g)^{-[\frac{n^2}{2}]-\varepsilon}$.

Given $g = \exp(s\mathrm{X})$ with X a norm 1 element in the Lie algebra $\mathfrak{sl}_n(\mathbf{R})$, the function $g \mapsto e^{|s|} - 1$ behaves like $\lceil g \rfloor$ for small $|s|$ and like $L(g)$ for large $|s|$. Next, given

$\gamma = (\mathrm{X}_{j_1}, \mathrm{X}_{j_2}, \ldots, \mathrm{X}_{j_{|\gamma|}})$ in $\mathfrak{sl}_n(\mathbf{R})$, we shall be working with the standard Lie differential operators

$$\partial_{\mathrm{X}_j} f(g) \, = \, \frac{d}{ds}\Big|_{s=0} f\big(g \exp(s\mathrm{X}_j)\big),$$

$$d_g^\gamma f(g) \, = \, \partial_{\mathrm{X}_{j_1}} \partial_{\mathrm{X}_{j_2}} \cdots \partial_{\mathrm{X}_{j_{|\gamma|}}} f(g) \, = \, \Big(\prod_{1 \le k \le |\gamma|}^{\rightarrow} \partial_{\mathrm{X}_{j_k}} \Big) f(g).$$

Theorem 1. *Let $m \in \mathcal{C}^{\sigma_n + 1}(\mathrm{G} \setminus \{e\})$ such that*

$$(\star) \qquad \mathopen{|\!|} g \mathclose{|\!|}^{|\gamma|} \big| d_g^\gamma m(g) \big| \, \le \, \Theta_\varepsilon(g) \quad \textit{for all} \quad |\gamma| \le \sigma_n + 1$$

and some $\varepsilon > 0$. Then $\big\| T_m \colon L_p(\widehat{\mathrm{G}}) \to L_p(\widehat{\mathrm{G}}) \big\|_{\mathrm{cb}} \, \le \, C_{p,n}^\varepsilon$ for $1 < p < \infty$.

We refer to [1, 11] for somehow related work on Riemannian symmetric spaces.

Theorem A is clearly inspired by the Hörmander-Mikhlin criterion, although it is substantially and necessarily different. Locally, we find sharp growth rates of derivatives around the singularity and nearly optimal regularity order. Note that $\dim SL_n(\mathbf{R}) = n^2 - 1$. Asymptotically, our condition admits a rearrangement which looks like (HM) for the highest order derivatives in the L-metric, which grows exponentially with respect to the word length. The decay of lower order terms comes imposed by this growth and our highest order condition. It is worth mentioning that Lafforgue/de la Salle's rigidity theorem [8] fits under this decay assumptions. The proof of Theorem A includes a new relation between Fourier and Schur L_p-multipliers for nonamenable groups. In $SL_n(\mathbf{R})$, this holds in terms of Harish-Chandra's almost L_2 matrix coefficients, which give a local measurement of nonamenability of $SL_n(\mathbf{R})$. By transference, matters are then reduced to a rather nontrivial RC_p-inequality for $SL_n(\mathbf{R})$-twisted forms of Riesz transforms associated to fractional laplacians. By Kazhdan's property (T) for high ranks or the lack of finite-dimensional representations of $SL_2(\mathbf{R})$, we known that Euclidean geometry mirrors $SL_n(\mathbf{R})$ only via nonorthogonal actions. This is an important difficulty which escapes the recent methods by Junge/Mei/Parcet [5, 6].

Given an open interval $J \subset \mathbf{R}$ and $\alpha > 0$, let $\mathcal{C}^\alpha(J)$ be the space of functions which admit $[\alpha]$ continuous derivatives in J and such that the $[\alpha]$-th derivative of φ is Hölder continuous of order $\alpha - [\alpha]$ on every compact subset of J. We shall also write $\mathcal{C}^{\alpha-}(J)$ for the space

$$\mathcal{C}^{\alpha-}(J) = \bigcap_{\beta < \alpha} \mathcal{C}^\beta(J).$$

Given $g \in SL_n(\mathbf{R})$, we use normalized Hilbert-Schmidt norms $|g|^2 = \frac{1}{n}\mathrm{tr}(g^* g)$.

Theorem 2. *Let $\varphi \colon (1, \infty) \to \mathbf{C}$ be such that the $SL_n(\mathbf{R})$-symbol $m(g) = \varphi(|g|)$ defines an S_p-bounded Schur multiplier $S_m(g,h) = m(g^{-1}h)$ for some $p > 2 + \frac{2}{n-2}$*

so that $\alpha_0 = (n-2)/2 - (n-1)/p > 0$. *Then* φ *is of class* $\mathcal{C}^{\alpha_0}$ *when* $\alpha_0 \notin \mathbf{Z}$ *and of class* $\mathcal{C}^{\alpha_0 -}$ *otherwise. Moreover, if*

$$\alpha = \alpha_0 \delta_{\alpha_0 \notin \mathbf{Z}} + (\alpha_0 - \varepsilon)\delta_{\alpha_0 \in \mathbf{Z}},$$

the following local/asymptotic estimates hold for the function φ:

i) φ *has a limit* φ_∞ *at* ∞ *and*

$$\big|\varphi(x) - \varphi_\infty\big| \le C_{p,n}^{\varepsilon} \frac{\|S_m\|_{\mathcal{B}(S_p(L_2(\mathrm{G})))}}{x^{c_0}},$$

where $c_0 = n / \big[\frac{3}{1-\frac{2}{p}}\big]$ *for* $\alpha > 1$ *and* $c_0 = \alpha \frac{n}{n-2}$ *for* $\alpha < 1$.

ii) *Given* $\xi > 1$ *and an integer* $1 \le k \le [\alpha]$

$$\big|\partial^k \varphi(x)\big| \le C_{p,n}^{\varepsilon} \frac{\|S_m\|_{\mathcal{B}(S_p(L_2(\mathrm{G})))}}{(x-1)^k x^{c_k}} \quad \textit{where} \quad c_k = \frac{n}{\big[\frac{2k+1}{1-\frac{2}{p}}\big]}.$$

Theorem B gives a major strengthening of the rigidity theorems in this context [7, 8]. It is also valid for radial Fourier multipliers in the group algebra, using the corresponding cb-norm instead. Its Euclidean form for radial Fourier multipliers $m(\xi) = \varphi(|\xi|)$ in dimension $d \ge 2$ is the regularity condition

$$\text{(RC)} \qquad p > 2 + \frac{2}{d-1} \ \Rightarrow \ |\xi|^k \big|\partial^k \varphi(\xi)\big| \ \le \ C_{p,d} \big\|T_m : L_p(\mathbf{R}^d) \to L_p(\mathbf{R}^d)\big\|$$

for $k < (d-1)/2 - d/p$. A comparison of (RC) with (HM) enlightens to certain point the structure of radial Fourier multipliers. Taking p arbitrarily large, Mikhlin conditions are necessary up to order $[d/2] - 1$ and sufficient from $[d/2] + 1$. The condition (RC) has it roots in the study of regularity properties for the Hankel transform of radial L_p functions, which goes back to Schoenberg [10]. This is among the most satisfactory results for radial multipliers before the celebrated characterization [2, 3]. Although our argument is very different from the Euclidean one, we discovered a posteriori that Theorem B ii) exactly reproduces the above condition (RC) around the singularity ($x = 1$ and $\xi = 0$ respectively) when we replace the Euclidean dimension d by the rank $n - 1$. In addition, Theorem B confirms that the growth rate around the singularity of low order derivatives given in Theorem A is best possible. Asymptotical rigidity arises from the extra decay provided by the exponents $c_0, c_1, \ldots, c_{[\alpha]}$. This rigidity increases with the rank and there exist radial multipliers which satisfy ($\star$) in rank n and fail to be in $\mathcal{C}^\alpha$ for ranks $m >> n$. The gap we find between necessary and sufficient conditions is of course larger than in the Euclidean setting.

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Gradient forms and strong solidity of free quantum groups

Martijn Caspers

The area of quantum groups is concerned with non-commutative versions of groups on the following sense. The celebrated Gelfand-Naimark theorem states that a topological compact Hausdorff space X can fully be understood in terms of the commutative C*-algebra of continuous functions $X \to \mathbb{C}$. This gives a categorical contragredient duality between compact Hausdorff spaces and commutative C*-algebras. Therefore C*-algebras are often considered as non-commutative spaces, eventhough there might not be a single point visible in such C*-algebras.

In the same spirit a compact quantum group is the non-commutative geometric analogue of a compact group (or even locally compact groups are part of the theory). A suitable framework of C*-algebraic compact quantum groups was found by Woronowicz in the late 1980's. In this framework all examples that were known at that time could be incorporated. Most notably $SU_q(2)$, the quantum $SU(2)$ group is a compact quantum group.

Two important classes of quantum groups were found by Wang and van Daele [14]. They arise as libertations of the group of orthogonal matrices, or unitary matrices. Their quantum versions are well-studied nowadays and are called the *free orthogonal quantum groups* and the *free unitary quantum groups*. As a C*-algebra they are generated by the operators $u_{i,j}, 1 \leq i,j \leq N$ such that the matrix $u = (u_{i,j})_{1 \leq i,j \leq N}$ is unitary and such that $\overline{u} = u$, where the bar denotes the entrywise adjoint. A comultiplication (the analogue of group multiplication) is given by $\Delta(u_{i,j}) = \sum_k u_{i,k} \otimes u_{k,j}$. We refer to this quantum group as O_N^+. In fact more general Q-deformations of this quantum group can be defined, but we do not give details here. Through a canonical GNS-construction the C*-algebra of O_N^+ generates a von Neumann algebra (a non-commutative measure space) which we call $L_\infty(O_N^+)$ (the reader should keep in mind that these are thus non-commutative algebras).

Ever since its introduction the interest in O_N^+ has been quite large, especially because of its remarkable analogues with the free group factors. Many results for its C*-algebra and von Neumann algebra have been obtained, in particular in the past 5 years. There are factoriality results, approximation properties, Cartan algebras, results on the Baum-Connes conjecture, the Connes embedding problem, etc. Though that $L_\infty(O_N^+)$ shares many properties with the free group factors, by recent results of Brannan-Vergnioux [3] we know that they are non-isomorphic. In particular, this renders the investigation of the above properties non-trivial.

In [9] and [10] it was proved that $L_\infty(O_N^+)$ is strongly solid. This means that if $A \subseteq L_\infty(O_N^+)$ is an amenable diffuse von Neumann subalgebra then the normalizing algebra generated by all unitaries $u \in L_\infty(O_N^+)$ such that $uAu^* = A$ is amenable again. Amenability is a very strong property for von Neumann algebras: Connes [8] showed that amenable von Neumann algebras can be approximated by matrix algebras in a strong sense and based on this can be classified. Much later Ozawa and Popa [11] introduced this notion of strong solidity to find a new proof that free group factors do not have Cartan subalgebras. Strong solidity then became a standard tool in the theory and many techniques have been introduced to show that specific classes of von Neumann algebras possess this property.

The strong solidity results by Fima-Vergnioux [9] and Isono [10] hold for the tracial versions of $L_\infty(O_N^+)$. But as mentionned above these algebras have Q-deformations that renders them into non-tracial algebras; in fact in many cases we know that they are non-amenable type III factors [15]. The strong solidity question there remained open.

What we show in this talk is that also the type III deformations of our algebras have the property of strong solidity. This is our main result.

Let us comment here on some ingredients of the proof. All approaches to strong solidity are based on a two step strategy (as in [11]): (1) one needs a notion of 'weak compactness' of actions of the (stable) normalizing algebra and (2) one needs to construct a 'deformation' of the algebra. Point (1) of the proof we can overcome through techniques recently introduced in [1]. For (2) we introduce a new deformation based on earlier results by Peterson [13] and again Ozawa-Popa [12]. The deformation is based on the construction of a derivation from results of Cipriani-Sauvageot [6] in combination with very recent results on the Haagerup property of $L_\infty(O_N^+)$ [2], [5], [4].

The result completes the strong solidity question for O_N^+. In fact we show that our results also apply to some other quantum groups. This involves U_N^+, the free unitary quantum groups, as well as some free products of these algebras.

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Lévy processes and Hochschild cohomology on universal quantum groups

Anna Wysoczańska-Kula

(joint work with Biswarup Das, Uwe Franz, Adam Skalski)

Let (A, ε) be an augmented algebra, i.e. a unital $*$-algebra with a character (unital $*$-homomorphism) ε. A linear mapping $\psi : A \to \mathbb{C}$ is called *generating functional* (abreviated to GF) if it is normalized: $\psi(1) = 0$, hermitian: $\psi(a^*) = \overline{\psi(a)}$ for any $a \in A$, and conditionally positive: $\psi(a^*a) \geq 0$ for all $a \in \ker \varepsilon$. A GF ψ is called *Gaussian* if it vanishes on triple products from the kernel of the counit, i.e. $\psi(abc) = 0$ whenever $a, b, c \in \ker \varepsilon$. The problem we are interested in can roughly be formulated in the following way:

Problem 1. Given an augmented algebra (A, ε) and a generating functional ψ on A, is it always possible to write ψ as a sum of two generating functionals ψ_G and ψ_R such that ψ_G is maximally Gaussian? (The precise meaning of maximality will be made clear below.)

The interest in this problem comes from noncommutative stochastic processes. Namely, if we assume (which we do in the sequel) that A is not only an augmented algebra, but also has the $*$-bialgebra structure, then – following M. Schürmann (see [10]) – one can associate to any GF a (unique up to isomorphism) Lévy process on A. The latter is a family of $*$-homomorphisms from A to some noncommutative probability space (B, Φ), which has independent and stationary increments (see for instance [6] for a detailed definition). Conversely, any Lévy process gives rise to a generating functional.

Furthermore, GFs associated with Lévy processes on $*$-bialgebras are analogues of characteristic functions for classical real-valued Lévy processes, and then a GF is Gaussian if it is (the exponent of) the characteristic function associated to the Brownian motion (with drift). In the classical case, characteristic functions are classified by the Lévy-Khintchine formula, which roughly states that the generators of Lévy processes are sums of continuous (or Gaussian) parts and jump parts. Hence, if A is a $*$-bialgebra, Problem 1 is concerns the existence of an analoguous Lévy-Khintchine type decomposition for Lévy processes on A.

The precise formulation of Problem 1 requires introducing some (cohomological) objects related to generating functionals, the so-called *Schürmann triples.* Given a GF ψ we can construct (a GNS-type construction described by Schürmann) a unital *-representation $\rho : A \to B(H)$ on a (pre-)Hilbert space H and a ρ-ε-cocycle η, i.e. a linear map $\eta : A \to H$ satisfying $\eta(ab) = \rho(a)\eta(b) + \eta(a)\varepsilon(b)$, $a, b \in A$, such that $\psi : A \to H$ satisfies the relation

$$\psi(ab) = \psi(a)\varepsilon(b) + \langle \eta(a^*), \eta(b)\rangle + \varepsilon(a)\psi(b), \quad a, b \in A. \tag{1}$$

The triple (ρ, η, ψ) is unique, up to a unitary isomorphism, provided $\eta(A)$ is dense in H (which we always assume). Moreover, a GF ψ is Gaussian if and only if the related representation is $\pi(.) = \varepsilon(.)I_H$ or, equivalently, if $\eta(a) = \varepsilon(a)\eta(b)+\eta(a)\varepsilon(b)$ for any $a, b \in A$.

It is easy to see that given a GF ψ and the associated Schürmann triple (ρ, η, ψ), with the representation ρ on a Hilbert space H, we can always extract from H the maximal Gaussian subspace H_G; this is the maximal space on which ρ acts as $\varepsilon(\cdot)I_H$, reducing for π. Then the representation and the cocycle split into

$$\rho_G = \rho|_{H_G},\ \rho_N = \rho|_{H_G^\perp}, \quad \text{and} \quad \eta_G = P_G\eta,\ \eta_N = (I - P_G)\eta,$$

where P_g denotes the projection onto H_G. Furthermore, η_G is a Gaussian cocycle and η_N is a purely non-Gaussian cocycle (i.e. $(H_N)_G = \{0\}$). If there exist GFs ψ_G and ψ_N such that (ρ_G, η_G, ψ_G) and (ρ_N, η_N, ψ_N) are Schürmann triples, then we say that ψ admits a *Lévy-Khintchine decomposition.* Note that ψ_N is associated to the cocycle which has no Gaussian part (in this sense ψ_G is maximal). In general, a pair (ρ, η), consisting of a representation and a ρ-ε-cocycle need not admit a generating functional (see [12, Example 2.1]). If any GF on A admits a Lévy-Khintchine decomposition, then we say that A has *the property (LK).*

We can now rephrase Problem 1: does a given $*$-(bi)algebra A have the (LK) property? The known results are the following: M. Schürmann [9] proved that the decomposition exists on any commutative $*$-bialgebra and on the Brown-Glockner-von Waldenfels algebra $K\langle d\rangle$. The latter is the universal unital *-algebra generated by d^2 noncommuting indeterminates x_{jk} $(j,k=1,2,...,d)$ such that the matrix $x:=(x_{jk})_{j,k=1}^d$ is unitary. M. Schürmann and M. Skeide [11] proved that any GF on the quantum group $SU_q(2)$ $(q\in(-1,1)\setminus\{0\})$ admits a Lévy-Khintchine type decomposition. For more than ten years the problem saw no progress, in particular the case of $SU_q(d)$ with $d\geq 3$ is still open. Then, in 2015, we [4] showed that generating functionals on Hopf $*$-algebras satisfying some symmetry condition always have the decomposition into the maximal Gaussian part and the rest. The first example of a $*$-bialgebra which does **not** have the property (LK) was found by U. Franz, M. Gerhold and A. Thom [7]. Still, the example they found was of a very specific nature: this is the ring algebra of a discrete group, hence a cocommutative object. In 2016, it was proved [8] that the quantum permutation group S_d^+ has no nontrivial Gaussian generators (only the so-called *drifts*), hence it has the property (LK). This result was further generalized by J. Bichon, U. Franz and M. Gerhold [2] to the algebra $S_N^+/\langle uD=Du\rangle$, where D is a complex matrix. This includes the quantum reflexion groups and the quantum automorphism groups of graphs.

The main result of [5] solves Problem 1 in the two extreme cases of universal compact quantum groups U_F^+ and O_F^+. Let us recall that the $*$-bialgebra of U_F^+ is the universal unital $*$-algebra generated by d^2 elements u_{jk} $(j,k=1,2,...,d)$ such that the matrices $u:=(u_{jk})_{j,k=1}^d$ and $F\bar{u}F^{-1}$ are unitary, and the character is the counit $\varepsilon(u_{jk})=\delta_{jk}$. The $*$-bialgebra of O_F^+ is the algebra of U_F^+ divided by the additional relations $v=F\bar{v}F^{-1}$.

Theorem 2. (a) *If a matrix $F\in GL_d(\mathbb{C})$ is such that F^*F has pairwise distinct eigenvalues, then both U_F^+ and O_F^+ have the (LK) property.*

(b) *The quantum groups U_d^+ $(d\geq 2)$ and O_d^+ $(d\geq 3)$ do not have the (LK) property.*

This result provides the first (family of) example(s) of a non-cocommutative quantum group without the LK-decomposition. The proof of part (b) requires finding a characterization of cocycles on U_d^+ and O_d^+ which admit GFs. This condition gives a link to computations of second Hochschild cohomology via the following remark: in terms of Hochschild cohomology, the π-ε-cocycle η of a Schürmann triple is an element of $Z^1(A,{}_\pi H_\varepsilon)$ (a 1-cocycle). Then always $c_\eta(a\otimes b)=\langle\eta(a^*),\eta(b)\rangle$ belongs to $Z^2(A,{}_\varepsilon\mathbb{C}_\varepsilon)$ (is a 2-cocycle). Saying that η admits a GF ψ is equivalent to say that $c_\eta\in B^2(A,{}_\varepsilon\mathbb{C}_\varepsilon)$ is a 2-coboundary. We prove the following.

Theorem 3. *Define $\Delta: Z^2(U_d^+)\to M_d(C)$ by the formula*

$$\Delta(c)=\left(\sum_{p=1}^d\left(c(u_{pj}^*\otimes u_{pk})-c(u_{kp}^*\otimes u_{jp})\right)\right)_{j,k=1}^d.$$

Then $\ker\Delta = B^2(U_d^+, {}_\varepsilon\mathbb{C}_\varepsilon)$ *and* $\mathrm{im}\ \Delta = sl(d,\mathbb{C})$*, space of* $d \times d$ *complex matrices with trace zero. Hence* $H^2(U_d^+, {}_\varepsilon\mathbb{C}_\varepsilon) \cong sl(d,\mathbb{C})$ *and* $\dim\ H^2(U_d^+, {}_\varepsilon\mathbb{C}_\varepsilon) = d^2 - 1$*.*

A similar method allows us to find that $\dim\ H^2(O_d^+, {}_\varepsilon\mathbb{C}_\varepsilon) = \frac{d(d-1)}{2}$, the fact already known from [3] and [1]. In a forthcoming paper we show that by a generalization of this approach an estimate of the size $H^2(U_F^+, {}_\varepsilon\mathbb{C}_\varepsilon)$ for arbitrary F can be obtained: namely, if $F^*F = \sum_i \lambda_i P_{d_i}$, λ_i pairwise distinct, then $\sum_i {d_i}^2 - n \leq \dim\ H^2(U_F^+) \leq \sum_i {d_i}^2 - 2$.

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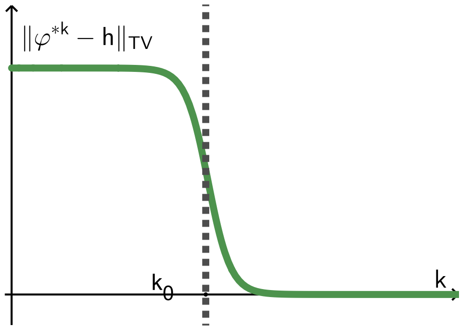
Cut-off for quantum random walks

Amaury Freslon

Motivated by the mathematical study of card shuffles, P. Diaconis and his coauthors discovered in the 80's a surprising behaviour of certain random walks on finite groups : for a number of steps, the random walks stays far apart from the Haar measure and then it suddenly drops exponentially close to it. Here is a standard picture of this so-called *cut-off phenomenon*, the distance being measure using the total variation distance :



These results triggered numerous works providing both examples and counter-examples. It was then natural to turn to the case of infinite compact groups and J. Rosenthal obtained the first result in that direction. He considered a modification of a random walk defined by Kac on the special orthogonal group $SO(N)$: Consider a fixed angle θ and a matrix R_θ of rotation by θ in a fixed plane. One step of the random walk is obtained by picking a special orthogonal matrix g at random according to the Haar measure and conjugating R_θ by it. Applying the random walk to the north pole of the $(N-1)$-dimensional sphere yields the *Uniform plane Kac random walk*. The combination of results of J. Rosenthal [4] and Y. Jiang and B. Hough [3] shows that this random walk has a cut-off at $N\ln(N)/2(1-\cos(\theta))$ steps.

We explain how to define an analogue of the uniform plane Kac random walk on the free orthogonal quantum group O_N^+ of S. Wang [5]. It turns out to be given, on coefficients of irreducible representations, by

$$\varphi_\theta : u_{ij}^n \mapsto \delta_{ij}\frac{T_n(N-2+2\cos(\theta))}{T_n(N)},$$

where $T_0(X) = 1$, $T_1(X) = X$ and $XT_n(X) = T_{n+1}(X) + T_{n-1}(X)$. We prove in [1] that this random walk has a cut-off at $N\ln(N)/2(1-\cos(\theta))$ steps, exactly as in the classical case. This is all the more surprising that the computations have nothing to do with one another. More results in this direction can be found in [2] involving free unitary quantum groups and free wreath products.

If we consider now the quantum permutation group S_N^+ of [6], we can consider the generalisation of the following random permutation walk : spread a deck of N cards on a table and independently pick two of them, then swap them if they were different. This is linked to random permutation generation in computers. On S_N^+ the corresponding state is

$$\varphi : u_{ij}^n \mapsto \delta_{ij}\frac{N-1}{N}\frac{T_{2n}(\sqrt{N-2})}{T_{2n}(\sqrt{N})} + \frac{1}{N}\delta_{ij}.$$

and it follows from the non-coamenability of S_N^+ that no convolution power of this is bounded on the von Neumann algebra $L^\infty(S_N^+)$. Thus, the total variation distance does not make sense and we have to use another norm to compare it to the

Haar state. We conjecture that the completely bounded norm of the corresponding transition operators $P_\varphi = (\mathrm{id} \otimes \varphi) \circ \Delta$ yields the correct cut-off parameter.

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Quantum automorphism groups of graphs

Moritz Weber

Let $\Gamma = (V, E)$ be a finite graph, i.e. $|V| = n$. Furthermore, assume that Γ has no multiple edges, i.e. $E \subset V \times V$. Let $\epsilon \in M_n(\{0,1\})$ be the adjacency matrix of Γ. We are going to explain the following picture.

$$
\begin{array}{llll}
C(S_n)/\langle u\epsilon = \epsilon u\rangle & \longleftrightarrow \mathrm{Aut}(\Gamma) \subset S_n & & \\
 & \quad \cap & & \\
C(S_n^+)/R_\epsilon^* & & & \\
=C(S_n^+)/R_\epsilon^{**} & \longleftrightarrow G^*_{\mathrm{aut}}(\Gamma) \subset O_n^\epsilon \overset{\text{THM B}}{=} \mathrm{QSym}(S^{n-1}_{\mathbb{R},\epsilon}) & \longleftrightarrow & C(O_n^+)/R_\epsilon^{**} \\
 & \quad \cap & & \\
C(S_n^+)/R_\epsilon^+ & & & \\
=C(S_n^+)/\langle u\epsilon = \epsilon u\rangle & \longleftrightarrow G^+_{\mathrm{aut}}(\Gamma) \overset{\text{THM A}}{=} \mathrm{QSym}(C^*(\Gamma)) & & \\
 & \quad \cap & & \\
 & \quad S_n^+ & &
\end{array}
$$

Here, the relations R_ϵ^+, R_ϵ^* and R_ϵ^{**} are defined as follows.

R_ϵ^+	R_ϵ^*	R_ϵ^{**}	
$u_{ik}u_{jl} = 0$	$u_{ik}u_{jl} = 0$	$u_{ik}u_{jl} = u_{jk}u_{il}$	if $\epsilon_{ij} = 1, \epsilon_{kl} = 0$
$u_{ik}u_{jl} = 0$	$u_{ik}u_{jl} = 0$	$u_{ik}u_{jl} = u_{il}u_{jk}$	if $\epsilon_{ij} = 0, \epsilon_{kl} = 1$
	$u_{ik}u_{jl} = u_{jl}u_{ik}$	$u_{ik}u_{jl} = u_{jl}u_{ik}$	if $\epsilon_{ij} = 1, \epsilon_{kl} = 1$

Compact Matrix Quantum Groups. Let us briefly recall some facts about Woronowicz's quantum groups. A tuple (A, u) is a compact matrix quantum group, if for some $n \in \mathbb{N}$

- A is a unital C^*-algebra generated by elements u_{ij} for $i, j = 1, \dots, n$,
- $u = (u_{ij})$ and $\bar{u} = (u_{ij}^*)$ are invertible matrices,
- and $\Delta : A \to A \otimes_{\min} A$, $u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}$ is a *-homomorphism.

Due to a fundamental theorem by Gelfand-Naimark (1940's) and Woronowicz (1980's) we know that for a given compact matrix quantum group (A, u):

$$A \text{ is commutative} \iff A \cong C(G), G \subset \mathrm{GL}_n(\mathbb{C}) \text{ is a compact group}$$

An example of a compact matrix quantum group is given via the following universal C^*-algebra:

$$C(O_n^+) := C^*(u_{ij}, i, j = 1, \dots, n \mid u_{ij} = u_{ij}^*, u = (u_{ij}) \text{ is orthogonal})$$

Indeed, $O_n^+ = (C(O_n^+), u)$ satisfies the axioms of a compact matrix quantum group. Moreover, since the C^*-algebra surjects canonically onto the algebra $C(O_n) \cong C(O_n^+)/\langle u_{ij} \text{ commute}\rangle$ of functions over the orthogonal group O_n, we may write $O_n \subset O_n^+$ and view O_n^+ as a quantum version of the orthogonal group. Roughly speaking, we may think of O_n^+ as orthogonal matrices with operator-valued entries. This example was given by Sh. Wang (1990's). In the same spirit, he defined (also in the 1990's) the quantum permutation group S_n^+ via:

$$C(S_n^+) := C^*(u_{ij}, i, j = 1, \dots, n \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_k u_{ik} = \sum_k u_{kj} = 1)$$

As $C(S_n) \cong C(S_n^+)/\langle u_{ij} \text{ commute}\rangle$, we may again write $S_n \subset S_n^+$ and think of S_n^+ as permutation matrices with operator-valued entries. In a way, there are more quantum permutations than permutations.

Quantum automorphism groups of finite graphs. Coming back to a finite graph $\Gamma = (V, E)$, $|V| = n$ without multiple edges, we may express its automorphism group (i.e. the set of all bijections $\sigma : V \to V$ with the property $(i, j) \in E$ if and only if $(\sigma(i), \sigma(j)) \in E$) as:

$$\mathrm{Aut}(\Gamma) = \{\sigma \in S_n \mid \sigma\epsilon = \epsilon\sigma\} \subset S_n$$

We infer that:

$$C(\mathrm{Aut}(\Gamma)) = C(S_n)/\langle u\epsilon = \epsilon u\rangle$$

Now, as S_n^+ is a quantum analogue of S_n, the following gives rise to a natural definition of a quantum automorphism group of Γ:

$$C(G_{\mathrm{aut}}^+(\Gamma)) := C(S_n^+)/\langle u\epsilon = \epsilon u\rangle$$

It is easy to check that $G_{\mathrm{aut}}^+(\Gamma) = (C(G_{\mathrm{aut}}^+(\Gamma)), u)$ is a compact matrix quantum group. Moreover, a direct computation yields that the relations $u\epsilon = \epsilon u$ are equivalent to the above relations R_ϵ^+. The definition of $G_{\mathrm{aut}}^+(\Gamma)$ has been given by Banica in 2005 [1]. Two years earlier, Bichon [2] defined another quantum automorphism group of Γ by:

$$C(G^*_{\text{aut}}(\Gamma)) := C(S_n^+)/R^*_\epsilon$$

We immediately see that:

$$\text{Aut}(\Gamma) \subset G^*_{\text{aut}}(\Gamma) \subset G^+_{\text{aut}}(\Gamma)$$

Observe that if Γ is the complete graph (i.e. $E = V \times V$), then:

$$\text{Aut}(\Gamma) = G^*_{\text{aut}}(\Gamma) = S_n \quad \neq \quad S_n^+ = G^+_{\text{aut}}(\Gamma)$$

However, for its complement Γ^c (i.e. $E = \emptyset$), we have:

$$\text{Aut}(\Gamma^c) = S_n \quad \neq \quad S_n^+ = G^*_{\text{aut}}(\Gamma^c) = G^+_{\text{aut}}(\Gamma^c)$$

See [6] for an overview of properties and recent results on these two versions of quantum automorphism groups of graphs. Besides the references therein, we want to highlight the very recent articles by Schmidt [5] on the quantum automorphism group of the Petersen graph (we have $\text{Aut}(\Gamma) = G^*_{\text{aut}}(\Gamma) = G^+_{\text{aut}}(\Gamma) = S_5$ in that case), as well as the article by Lupini, Mancinska and Roberson [4] on links to quantum information theory.

Theorem A. In joint work with Schmidt [6], we recently showed that the graph C^*-algebra $C^*(\Gamma)$ preserves the quantum symmetry (in the sense of Banica) of the graph Γ: We showed that $G^+_{\text{aut}}(\Gamma)$ is the universal compact matrix quantum group acting on $C^*(\Gamma)$ linearly. Recall that $C^*(\Gamma)$ is the universal C^*-algebra generated by mutually orthogonal projections p_v for $v \in V$ together with partial isometries s_e for $e \in E$ and the relations:

$$s_e^* s_e = p_{r(e)}, e \in E, \qquad \sum_{e \in E:\, s(e)=v} s_e s_e^* = p_v, v \in V$$

Here, $r(e)$ and $s(e)$ denote the range and the source of the edge $e \in E$ respectively. More precisely, our theorem with Schmidt states:

(a) The maps α and β are left resp. right actions of $G^+_{\text{aut}}(\Gamma)$ on $C^*(\Gamma)$.

$$\alpha : C^*(\Gamma) \to C(G^+_{\text{aut}}(\Gamma)) \otimes C^*(\Gamma) \qquad \text{and} \qquad \beta : C^*(\Gamma) \to C(G^+_{\text{aut}}(\Gamma)) \otimes C^*(\Gamma)$$

$$p_i \mapsto \sum_{k=1}^n u_{ik} \otimes p_k \qquad\qquad p_i \mapsto \sum_{k=1}^n u_{ki} \otimes p_k$$

$$s_{e_j} \mapsto \sum_{l=1}^m u_{s(e_j)s(e_l)} u_{r(e_j)r(e_l)} \otimes s_{e_l} \qquad\qquad s_{e_j} \mapsto \sum_{l=1}^m u_{s(e_l)s(e_i)} u_{r(e_l)r(e_i)} \otimes s_{e_l}$$

(b) Whenever G is a compact matrix quantum group acting on $C^*(\Gamma)$ from the left and right as in (a), then $G \subset G^+_{\text{aut}}(\Gamma)$ in the sense that we have a canonical surjection from $C(G^+_{\text{aut}}(\Gamma))$ to $C(G)$ mapping generators to generators.

The above actions α and β are inspired from Bichon's article [2]. Let us mention that the above result has been put into the framework of quantum isometry groups by Joardar and Mandal [3].

Theorem B. While Theorem A identifies Banica's quantum automorphism group $G^+_{\text{aut}}(\Gamma)$ as the right quantum symmetry group of some object, let us now come to a case in which Bichon's version $G^*_{\text{aut}}(\Gamma)$ is the correct one, referring to a joint article with Speicher [7]. We define the partially quantized sphere $S^{n-1}_{\mathbb{R},\epsilon}$ via the following universal C^*-algebra:

$$C(S^{n-1}_{\mathbb{R},\epsilon}) := C^*(x_1, \dots, x_n \mid x_i = x_i^*, \sum_i x_i^2 = 1, x_i x_j = x_j x_i \text{ if } \epsilon_{ij} = 1)$$

For the complete graph (i.e. $\epsilon_{ij} = 1$ for all i, j), we obtain the classical sphere $S^{n-1}_{\mathbb{R},\epsilon} = S^{n-1}_{\mathbb{R}} \subset \mathbb{R}^n$. For the empty graph however (i.e. $\epsilon_{ij} = 0$ for all i, j), we get Banica and Goswami's free sphere $S^{n-1}_{\mathbb{R},\epsilon} = S^{n-1}_{\mathbb{R},+}$.

Let us define the partially quantized orthogonal group O^ϵ_n via

$$C(O^\epsilon_n) := C(O^+_n)/R^{**}_\epsilon,$$

where R^{**}_ϵ are the relations as defined above. For the complete graph, we obtain $O^\epsilon_n = O_n$ whereas the empty graph yields $O^\epsilon_n = O^+_n$. Our theorem with Speicher states

$$\mathrm{QSym}(S^{n-1}_{\mathbb{R},\epsilon}) = O^\epsilon_n,$$

Where again $\mathrm{QSym}(S^{n-1}_{\mathbb{R},\epsilon})$ is defined in a similar way to Theorem A. For the case of the empty graph, this theorem recovers a result by Banica and Goswami from 2010. Let us mention that a unitary version of this theorem has been proven by Simeng Wang recently (yet unpublished). Now, as

$$C(S^+_n)/R^{**}_\epsilon = C(S^+_n)/R^*_\epsilon$$

We conclude that O^ϵ_n is a kind of an orthogonal version of $G^*_{\text{aut}}(\Gamma)$. In other words, we may define S^ϵ_n analogously to O^ϵ_n via $C(S^\epsilon_n) := C(S^+_n)/R^{**}_\epsilon$ and we obtain $G^*_{\text{aut}}(\Gamma) = S^\epsilon_n \subset O^\epsilon_n$. In the same spirit, we could define partially quantized versions of any easy quantum group by quotienting with respect to the relations R^{**}_ϵ, R^*_ϵ or R^+_ϵ. This general approach still awaits its systematic study.

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Aut($\mathbb{F}_5$) has property (T)

NARUTAKA OZAWA
(joint work with Marek Kaluba, Piotr W. Nowak)

Introduction. Amenability and Kazhdan's property (T) are the most important properties in analytic group theory. They are generalized notions of being finite (but into the opposite directions). Groups with property (T) have a number of applications in pure and applied mathematics. It has long been thought that groups with property (T) are rare among the "naturally-occurring" groups, but it may not be so and it may be possible to observe this by extensive computer calculations. I will present a computer-assisted (but mathematically rigorous) method of confirming property (T) based on semidefinite programming with some operator algebraic input. I will report the progress recently made in collaboration with M. Kaluba and P. Nowak ([4]), which confirms property (T) of Aut($\mathbb{F}_5$). This solves a well-known problem ([5, 10.3], [6], [1, 7.1]) in geometric group theory, at least partially, leaving the tantalizing question in the case of Aut($\mathbb{F}_d$), $d = 4$ and $d > 5$, unsettled. Since the four color theorem was proved by a computer in 1976, some people have purported that computer-assisted research would dominate mathematics. This didn't happen, at least around me so far, but I finally found one. Computer capacity grows in an exponential manner and a Japanese saying holds that if you see one cockroach in your house, there are thirty more around you. So, it may be time to start looking for them.

Kazhdan's property (T). A (discrete) group Γ is said to have Kazhdan's property (T) if for any orthogonal representation (π, H), any almost Γ-invariant vector is close to a Γ-invariant vector: $\exists S \subset \Gamma$ finite and $\exists \kappa = \kappa(S) > 0$ which satisfy

$$\forall(\pi, H)\ \forall v \in H \text{ one has } \mathrm{dist}(v, H^\Gamma) \leq \kappa^{-1} \max_{s\in S} \|v - \pi(s)v\|.$$

Here H^Γ denotes the subspace of Γ-invariant vectors. If Γ has property (T), then S as above has to be a generating subset of Γ and so Γ is finitely generated; Moreover, for any finite generating subset S, there is a Kazhdan constant $\kappa = \kappa(S)$ that satisfies the above condition. Property (T) inherits to quotient groups and finite-index subgroups. Property (T) is similarly defined for a locally compact groups and a lattice Γ in a locally compact group G has property (T) if and only if G has it.

A group Γ is amenable if there are almost invariant vectors in $\ell_2\Gamma$: $\exists v_n \in \ell_2\Gamma$ such that $\|v_n\| = 1$ and $\|v_n - sv_n\| \to 0$ for every $s \in \Gamma$. Abelian groups (or more generally groups with subexponential growth) are amenable. Since $(\ell_2\Gamma)^\Gamma \neq 0$ only if Γ is finite, any group that satisfy both amenability and property (T) is finite. D. Kazhdan (1967) defined property (T) and proved that every simple connected Lie group with real rank ≥ 2 (e.g., $\mathrm{SL}(d \geq 3, \mathbb{R})$) has property (T) and so every lattice of it is finitely generated and has finite abelianization.

Algebraic characterization of property (T). Noncommutative real algebraic geometry is a subject that deals with equations and inequalities in noncommutative algebras (over real or complex). Recall Artin's theorem (Hilbert's 17th problem) from the classical real algebraic geometry: If a polynomial f in $\mathbb{R}[x_1,\dots,x_d]$ satisfies $f \geq 0$ on $\mathbb{R}^d$, then there are rational polynomials $g_1,\dots,g_n$ in $\mathbb{R}(x_1,\dots,x_d)$ such that $f=\sum_i g_i^2$. This theorem becomes trivial if one passes to the completion of the polynomial algebra, which is the continuous function algebra. Likewise in noncommutative real algebraic geometry, we solve an inequality in the completion (which is a C*-algebra) and bring down the solution to the original algebra.

Let a group Γ be given. We consider the real group algebra $\mathbb{R}[\Gamma]$ with the involution $(\sum_t \alpha_t t)^* = \sum_t \alpha_t t^{-1}$, where $\alpha_t \in \mathbb{R}$ are all zero but finitely many. The positive cone of hermitian squares is given by

$$\Sigma^2\mathbb{R}[\Gamma] := \{\sum_i g_i^* g_i : g_i \in \mathbb{R}[\Gamma]\} = \{\sum_{x,y} P_{x,y} x^{-1}y : P \in \mathbb{M}_\Gamma^+(\mathbb{R})\}.$$

Here $\mathbb{M}_\Gamma^+(\mathbb{R})$ denotes the set of finitely supported positive definite matrices indexed by Γ. The full group C*-algebra C$^*\Gamma$ is the universal enveloping completion of $\mathbb{R}[\Gamma]$ with respect to orthogonal representations of Γ (= $*$-representations of $\mathbb{R}[\Gamma]$ on Hilbert spaces). We assume Γ is generated by a finite symmetric subset S and consider the non-normalized Laplacian

$$\Delta := \tfrac{1}{2}\sum_{s\in S}(1-s)^*(1-s) = |S| - \sum_{s\in S} s \in \Sigma^2\mathbb{R}[\Gamma].$$

For any orthogonal representation (π,H) and $v \in H$, one has $\pi(\Delta)v = 0$ iff v is Γ-invariant, and $\pi(\Delta)v \approx 0$ iff v is almost Γ-invariant. Hence it follows from the spectral theory that Γ has property (T) iff Δ has a spectral gap: $\exists\varepsilon > 0$ such that $\mathrm{Sp}(\Delta) \subset \{0\} \cup [\varepsilon,\infty)$ in C$^*\Gamma$. On the other hand, by the spectral mapping theorem, one has $\mathrm{Sp}(\Delta) \subset \{0\}\cup[\varepsilon,\infty)$ iff $\Delta^2 - \varepsilon\Delta \geq 0$ in C$^*\Gamma$.

Theorem 1 ([8]). *A finitely generated group $\Gamma = \langle S\rangle$ has property (T) iff $\exists\varepsilon > 0$ such that $\Delta^2 - \varepsilon\Delta \in \Sigma^2\mathbb{R}[\Gamma]$. If this is the case, one has $\kappa(S)^2 \geq 2|S|^{-1}\varepsilon$.*

Semidefinite programming. For the computer verification of property (T) of a given group Γ, we fix a finite subset $E \subset \Gamma$ and restrict the search area from $\mathbb{M}_\Gamma^+(\mathbb{R})$ to $\mathbb{M}_E^+(\mathbb{R})$. This results in the semidefinite programming (SDP):

$$\begin{array}{l}\texttt{minimize } -\varepsilon \\ \texttt{subj. to } \exists P \in \mathbb{M}_E^+(\mathbb{R}) \texttt{ such that } \Delta^2 - \varepsilon\Delta = \displaystyle\sum_{x,y\in E} P_{x,y}x^{-1}y \texttt{ in } \mathbb{R}[\Gamma]\end{array}$$

If $\varepsilon > 0$, then we conclude that Γ has property (T), but since we have restricted the search area, the converse need not hold. By the way, we will ignore in this text the word problem of identifying elements in Γ.

Suppose that a hypothetical solution (ε_0, P_0) to the above SDP is given. We describe here how to ensure existence of a rigorous solution to the inequality $\Delta^2 - \varepsilon\Delta \in \Sigma^2\mathbb{R}[\Gamma]$ out of it. We factorize P_0 as $P_0 \approx Q^TQ$ for some Q with $Q\mathbf{1} = 0$. We utilize the fact that $\|f\|\Delta - f \in \Sigma^2\mathbb{R}[\Gamma]$ for every $f \in \mathbb{R}[\Gamma]$, $f = f^*$ and $\sum_x f(x) = 0$. Here $\|f\|$ is a weighted ℓ_1-norm which is explicitly calculable. Thus property (T) of Γ is guaranteed if one sees

$$\|\Delta^2 - \varepsilon_0\Delta - \sum_{x,y}(Q^TQ)_{x,y}x^{-1}y\| < \varepsilon_0$$

by a computer calculation with guaranteed accuracy (rational arithmetic or interval arithmetic). We remark that finding a solution is practically difficult but verifying a given solution is relatively easy.

The size of SDP. Due to computer capacity limitation, we almost always take E to be the ball Ball(2) of radius 2. So the dimension of SDP is $\dim \mathbb{M}_E = |\mathrm{Ball}(2)|^2 \approx |S|^4$ and the number of constraints is $|E^{-1}E| = |\mathrm{Ball}(4)| \approx |S|^4$. The ball of radius 2 may appear too small, but property (T) has been be confirmed on Ball(2) in many cases, by Netzer–Thom ([7]), Fujiwara–Kabaya ([2]), and Kaluba–Nowak ([3]). We were a lot encouraged by these success. People often complain that we do not learn anything (besides it is true) from a computer-assisted proof, and indeed we do not learn why it is true, but in fact we can learn how the truth can be verified.

The group $\mathrm{SAut}(\mathbb{F}_d)$ is an index-two subgroup of $\mathrm{Aut}(\mathbb{F}_d)$ and is generated by left and right transvections $S = \{L^{\pm}_{i,j},\ R^{\pm}_{i,j}\}$. One has $|S| = 4d(d-1) = 48, 80, 120$ for $d = 4, 5, 6$. It was too large for currently existing computers to run the above algorithm. So, we divided the problem by the symmetry group $\Sigma := \{\alpha \in \mathrm{Aut}(\Gamma) : \alpha(S) = S\}$ and carried out the invariant SDP. In the case of $\mathrm{SAut}(\mathbb{F}_d)$, since $\Sigma = (\bigoplus_{i=1}^d \mathbb{Z}/2) \rtimes \mathfrak{S}_d$ is quite large, $|\Sigma| = 384, 3840, 46080$ for $d = 4, 5, 6$, this greatly facilitates the SDP. We have used a Polish supercomputer for the symmetrization process.

Results. We were able to verify property (T) of $\mathrm{SAut}(\mathbb{F}_d)$ for $d = 5$. One can verify our solution with a reasonably good desktop computer (with 32GB RAM). It is known $\mathrm{SAut}(\mathbb{F}_d)$ does not have (T) for $d \leq 3$. For $d = 4$, we did not find a solution in Ball(2). I think we can have a definitive result/conjecture (depending on the outcome) if we are able to run the algorithm on Ball(3) (this could become feasible in a decade or so). We were not able to run the algorithm for $d = 6$ because the symmetrization process was beyond the computer's capacity. Still, it is very natural to conjecture that $\mathrm{Aut}(\mathbb{F}_d)$ has property (T) for all $d \geq 5$.

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Free Fourier multipliers associated with the first segment of reduced words

Tao Mei
(joint work with Quanhua Xu)

The Fourier multipliers on free groups have been widely studied, but the study is often restricted on the "radial? ones. This talk will introduce the recent join works (with Q. Xu) for a type of Fourier multipliers associated with the first segment of reduced words. The main result is that they are Lp-completely bounded iff their restriction on the integer group is Lp-completely bounded. This recovers the previous work of Mei and Ricard on the Free Hilbert transforms answering positively a question raised by P. Biane and G. Pisier.

Quantum de Finetti theorems and Reznick's Positivstellensatz

Ion Nechita
(joint work with Alexander Müller-Hermes and David Reeb)

Consider Hermitian forms on the k-th symmetric power $\vee^k\mathbb{C}^d$, where k and d are positive integers. These forms can be defined by a Hermitian matrix W acting on the k-th symmetric tensor power of $\mathbb{C}^d$ by

$$p_W(y) = \langle y^{\otimes k}|W|y^{\otimes k}\rangle, \qquad \forall y \in \mathbb{C}^d.$$

We are interested in certifying positivity of such a Hermitian form, i.e. *when is it true that $p_W(y) \geq 0$ for all $y \in \mathbb{C}^d$?* Obviously, this is the case when p_W is a *sum of squares* (SOS):

$$p_W(y) = \sum_i |q_i(y)|^2, \qquad \forall y \in \mathbb{C}^d.$$

However, there are examples of homogeneous, non-negative polynomials which cannot be written as a sum of squares; a famous such example is the *Motzkin polynomial*

$$p_{\text{Motzkin}}(x, y, z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2.$$

The relation between non-negative polynomials and sums of squares was recognized by Hilbert as an important one (number 17 on his list of 23 problems presented at the ICM 1900 in Paris) and a complete solution was obtained by Artin in 1927, who showed that any non-negative real polynomial can be written as a fraction of a sum of squares and of a square of a polynomial.

Our work follows closely a refinement of Artin's result by Reznick [3], who found a simple form for the denominator (as powers of $\|y\|^2$). Note that the problem of determining a sum-of-square decomposition of a non-negative polynomial is a semidefinite program (SDP). Therefore, Reznick's result suggests a hierarchy of SDPs for testing positivity of a non-negative real polynomial. It furthermore quantifies the run-time of this method in terms of quantities related to the polynomial. Our work is in the same line, extending Reznick's result from real polynomials to complex Hermitian forms and improving several aspects of it. Our method of proof is also quite original, since it borrows from the newly established field of quantum information theory.

In order to assert the positivity of the Hermitian form p_W, we introduce the quantities $m(W) := \min_{\|y\|=1} p_W(y)$ and $M(W) := \max_{\|y\|=1} p_W(y)$, where we denote by $\|y\|$ the ℓ^2 norm of the vector y, $\|y\|^2 := |y_1|^2 + |y_2|^2 + \cdots + |y_d|^2$. We have shown the following result, a Positivstellensatz for complex variables, improving on similar results by To and Yeung [4].

Theorem 1. *Let W be a hermitian operator acting on $\vee^k\mathbb{C}^d$, and assume $m(W) > 0$. Then, for all*

$$n \geq \frac{dk(2k-1)}{\ln\left(1 + \frac{m(W)}{M(W)}\right)} - d - k + 1 \tag{1}$$

such that $n \geq k$, there exists a non-negative *function c_φ such that*

$$\|y\|^{2(n-k)} p_W(y) = \int_{\|\varphi\|=1} c_\varphi |\langle \varphi, y\rangle|^{2n} d\varphi,$$

where the integration is with respect to the Haar measure on the unit sphere of $\mathbb{C}^d$, or any discrete n-spherical design [1]. *In particular, the polynomial $\|y\|^{2(n-k)} p_W(y)$ is a sum of squares.*

Our proof of the above theorem uses some techniques from Gaussian probability theory (e.g. Wick's formula) and some elementary combinatorics. We also use an identity known as the Chiribella formula (first established in [2] in the complex case). In the real case this formula is known under the guise of Hobson's lemma, and using our methods we can give a more transparent proof of this lemma and of Reznick's original result. Moreover, by improving some estimates we obtain better constants in a bound similar to (1) and our methods allow us to obtain improved bounds for specific values of d, k, which are much better than the general bound.

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Interacting Fock Spaces, Subproduct Systems, and Their Operator Algebras

Michael Skeide
(joint work with Malte Gerhold)

So-called interacting Fock spaces (IFS) capture, in a sense, the most general Fock-like structure, that is, a Hilbert space with creation operators that generate it out of a single vacuum vector. Important examples of operator algebras (selfadjoint or not) emerge by looking at the algebras ($*$ or not) generated by these creators. In examples (for instance from algebraic geometry), frequently it turns out that the non-selfadjoint versions capture better the structure of the underlying IFS: The classification of the IFS by their non-selfadjoint operator algebras is strictly finer than the classification by their self-adjoint operator algebras. When the creators are bounded, all operator algebras embed naturally into Cuntz-Pimsner-Toeplitz algebras acting on full Fock modules.

Every irreversible dynamics (that is, a semigroup of completely positive maps) gives rise to a subproduct system, and every subproduct system arises this way. The subproduct system is proper (that is, not a product system) if and only if the semigroup is properly irreversible (that is, not homomorphic). Subproduct systems play a crucial role in constructing dilations of CP-semigroups. Every subproduct system over discrete time $n = 0, 1, \ldots$ gives rise to its associated Fock space, in fact, an IFS of a special type. (This is, how the aforementioned examples from algebraic geometry due to Davidson, Ramsey, and Shalit arise.)

In this talk we present our latest results with Malte Gerhold on the structure of IFS. The original definition from 1997 by Accardi, Lu, and Volovich and the definition we proposed in 2008 with Accardi are equivalent, but the latter is easier to work with. The former is suitable, in particular, when the IFS arises from a usual full Fock space by imposing a new semiinner product via an (even, vacuum preserving) positive operator, so-called positive operator induced (POI) IFS. We show the latter a strictly fewer than the former. We show that every IFS can be embedded (vacuum preservingly and respecting the natural grading) into a usual full Fock space (identifying this way the creators of the former with a sort of “squeezed” creators of the latter, and allowing to parametrize efficiently IFS by even vacuum preserving operators on usual full Fock spaces; the solution of a long standing open problem). We show precisely when the creators of an IFS are bounded (that is, which of them have operator algebras that embed into a Cuntz-Pimsner-Toeplitz algebra). We find out precisely when an IFS with bounded creators is an POI-IFS; also this is the solution of a long standing open problem. We find out precisely when, in our parametrization, an IFS is the Fock space of a subproduct system. Looking at those that are not, promotes a new generalization of the notion of product system, which now includes all predecessors (that is, subproduct systems and super product systems). Analyzing how general IFS can be related to subproduct systems, raises several natural questions for a possible theory of such generalized product systems.

Details on the open problem: What are CPH-maps, CPH-semigroup, and CPH-dilations possibly good for?

In a(n erroneous) paper, Asadi [1] proposed (a special class of) τ-maps between Hilbert modules as maps T from a Hilbert $\mathcal{B}$-module E to a Hilbert $\mathcal{C}$-module F that fufill

$$\langle T(x), T(y)\rangle = \tau(\langle x, y\rangle)$$

for a CP-map $\tau\colon \mathcal{B} \to \mathcal{C}$. For reasons of space, we dispense with giving a full account of the super-well-know history of τ-maps, and refer the reader to Skeide and Sumesh [2], where also a quite comprehensive discussion of their characterizations and properties can be found. (For instance a bounded map τ fulfilling the equation for some map T has no choice but being CP on $\mathcal{B}$ if E is full; so, in a sense, the CP-condition is superfluous.)

τ-Maps have been introduced without any motivation, nor any (interesting and non-obvious) example, nor any application. Despite that lack of evidence why they should be considered interesting, many people (including ourselves) have written papers on them. In [2], we have proposed, as something that smells at least a bit like a possible application, the notion of ***CPH-dilation***. (Motivated by the fact the τ-maps admit an extension to a CP-map between the linking algebras which is partially a homomorphism, we started calling them ***CPH-maps***.) For full E, a CPH-dilation of a CP-semigroup τ_t to an endomorphism semigroup ϑ_t via a CPH-semigroup T_t is captured in the following diagram.

$$\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\ \tau_t\ } & \mathcal{B} \\
{\scriptstyle \langle x, \bullet x'\rangle}\uparrow & & \uparrow{\scriptstyle \langle T_t(x), \bullet T_t(x')\rangle} \\
B^a(E) & \xrightarrow[\ \vartheta_t\ ]{} & B^a(E)
\end{array}$$

The terminology is motivated by the following fact: If ξ is a unit vector in E and a fixed point of T_t, then for $x = x' = \xi$ the diagram transforms into the usual dilation diagram (dilation of the corner $\mathcal{B} \cong \xi\mathcal{B}\xi^*$ to $B^a(E)$).

While dilations of nonunital CP-semigroups are nonunital, their CPH-dilation may be unital. There are powerful existence theorems for CPH-dilations to a unital endomorphism semigroup (based on the so-called GNS-product system of a CP-semigroup and our existence result of unital endomorphism semigroups for such product systems); see [2] for details and more references.

We repeat our question: What might such an, obviously quite rich, structure be good for? Answers are invited to `skeide@unimol.it`

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Temperley-Lieb quantum channels

Hun Hee Lee
(joint work with Michael Brannan, Benoît Collins, Sang-Gyun Youn)

A fundamental problem in classical information theory is to understand capacity of a channel. The same priciple applies to its quantum counter-part, namely quantum information theory (QIT). However, in the quantum world, several additional problems arose. First of all, there are many capacities according to the scenarios including classical capacity and quantum capacity. Secondly, these quantities can be expressed as the regularizations of their "one-shot" versions, which are known not to be additive. This phenomenon of additivity violation imposes a serious difficulties of computing the values of various capacities of quantum channels. The most important recent breakthrough in this direction of research was the proof of additivity violation of Holevo capacity (which is the "one-shot" version of the classical capacity) by Hastings. His proof was based on the close relationship of Holevo capacity with a different information quantity called minimum output entropy (MOE). More precisely, Hastings proved that MOE does not satisfy additivity using a probabilistic approach, which leaves the question of a deterministic example of additivity violation. This motivates us to collect more models of quantum channels in the hope to find such an example.

In this talk we will focus on a class of deterministic quantum channels arising from the representation theory of compact quantum groups under the name of Clebesch-Gordan channels. After a collection of general observations we will quickly move to the case of free orthogonal quantum groups O^+_N, $N \geq 2$ as well as its classical counter-part $SU(2)$. This class of channels have been already investigated in [1] for example. Since the representation category of O^+_N, $N \geq 2$ and $SU(2)$ can be understood as Temperley-Lieb algebras, we call the associated quantum channels as Temperley-Lieb channels. The first highlight of this project is to use diagrammatic of Temperley-Lieb algebras to get an explicit form of Choi matrices associated to Temperley-Lieb channels and obtain an asymptotically sharp estimates for MOE, "one-shot" quantum capacity and Holevo capacity. Secondly, we turn our attention to the problem of estimating the MOE of various tensor products of the Temperley-Lieb channels. In this setting, the standard upper bound for the MOE comes from estimating the output of a product channel under a maximally entangled input state (Bell). In our particular situation, it turns out that we are able to exploit some basic algebraic tools from Temperley-Lieb recoupling theory to obtain exact formulas (involving the quantum-6j symbols) for the outputs of Bell states and more general highly-entangled states. The key observation here is that a suitably chosen Bell state can be taken to be a fixed vector for a tensor product of O^+_N-representations, and this allows one to interpret the output of this Bell state as an intertwiner in the representation category, which then yields exact formulas. Finally, we note that we were able to check that the class of Temperley-Lieb channels is not included in the class of modified TRO-channels, which was recently introduced and investigated by Gao/Junge/LaRacuente [2] focusing on estimating their classical and quantum capacities.

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Approximating quantum channels by completely positive maps with small Kraus rank

Cécilia Lancien

(joint work with Andreas Winter)

We study the problem of approximating a quantum channel by one with as few Kraus operators as possible (in the sense that, for any input state, the output states of the two channels should be close to one another). Our main result is that any quantum channel mapping states on some input Hilbert space A to states on some output Hilbert space B can be compressed into one with order $d \log d$ Kraus operators, where $d = \max(|\mathrm{A}|, |\mathrm{B}|)$, hence much less than $|\mathrm{A}||\mathrm{B}|$. In the case where the channel's outputs are all very mixed, this can be improved to order d, which can be shown to be optimal.

1. Presentation of the problem

Quantum channels are the most general framework in which the transformations that a quantum system may undergo are described. These are defined as completely positive and trace preserving (CPTP) maps from the set of bounded operators on some input Hilbert space A to the set of bounded operators on some output Hilbert space B. Indeed, to be a physically valid evolution, a linear map $\mathcal{N}$ has to preserve quantum states (i.e. positive semi-definiteness and unit-trace conditions) even when tensorized with the identity map $\mathcal{I}\!\lceil$ on an auxiliary system.

Let us fix some notation: Given a Hilbert space H, we denote by $\mathcal{L}(\mathrm{H})$ the set of linear operators on H and by $\mathcal{D}(\mathrm{H})$ the set of density operators (i.e. positive semi-definite and trace 1 operators) on H. Whenever H is finite dimensional (which is the case of all the Hilbert spaces we deal with) we denote by $|\mathrm{H}|$ its dimension.

So assume from now on that the Hilbert spaces A and B are finite dimensional. Then, we know by Choi's representation theorem [2] that a CPTP map $\mathcal{N} : \mathcal{L}(\mathrm{A}) \to \mathcal{L}(\mathrm{B})$ can always be written as

$$\mathcal{N} : X \in \mathcal{L}(\mathrm{A}) \mapsto \sum_{i=1}^{s} K_i X K_i^\dagger \in \mathcal{L}(\mathrm{B}), \tag{1}$$

where the operators $K_i : \mathrm{A} \to \mathrm{B}$, $1 \leqslant i \leqslant s$, are called the Kraus operators of $\mathcal{N}$ and satisfy the normalization relation $\sum_{i=1}^{s} K_i^\dagger K_i = \mathbb{1}_{\mathrm{A}}$. The minimal $s \in \mathbf{N}$ such that $\mathcal{N}$ can be decomposed in the Kraus form (1) is called the Kraus rank of $\mathcal{N}$, which we shall denote by $r_K(\mathcal{N})$. By Stinespring's dilatation theorem [4], another alternative way of characterizing a CPTP map $\mathcal{N}:\mathcal{L}(\mathrm{A})\to\mathcal{L}(\mathrm{B})$ is as follows

$$\mathcal{N}: X\in\mathcal{L}(\mathrm{A})\mapsto \mathrm{Tr}_{\mathrm{E}}\left(VXV^\dagger\right)\in\mathcal{L}(\mathrm{B}), \tag{2}$$

for some environment Hilbert space E and some isometry $V:\mathrm{A}\hookrightarrow\mathrm{B}\otimes\mathrm{E}$ (i.e. $V^\dagger V=\mathbb{1}_{\mathrm{A}}$). In such picture, $r_K(\mathcal{N})$ is the minimal environment dimension $|\mathrm{E}|\in\mathbf{N}$ such that $\mathcal{N}$ may be expressed in the Stinespring form (2).

Yet another way of viewing the Kraus rank of a CPTP map $\mathcal{N}:\mathcal{L}(\mathrm{A})\to\mathcal{L}(\mathrm{B})$ is as the rank of its associated Choi-Jamiołkowski state. Denoting by ψ the maximally entangled state on $\mathrm{A}\otimes\mathrm{A}$, the latter is defined as the state $\tau(\mathcal{N})=\mathcal{I}\!\!\lceil\otimes\mathcal{N}(\psi)$ on $\mathrm{A}\otimes\mathrm{B}$. Consequently, it holds that any quantum channel from A to B has Kraus rank at most $|\mathrm{A}||\mathrm{B}|$. The case $r_K(\mathcal{N})=1$ corresponds to $\mathcal{N}$ being a unitary evolution, whereas whenever $r_K(\mathcal{N})>1$, $\mathcal{N}$ is a noisy summary of a unitary evolution on a larger system. The Kraus rank of a quantum channel is thus a measure of its "complexity": it quantifies the minimal amount of ancillary resources needed to implement it. Hence, a natural question in this context is: given any quantum channel, is it possible to reduce its complexity while not affecting too much its action, or in other words to find a channel with much smaller Kraus rank which approximates it?

We now need to specify what we mean by "approximating a quantum channel", since indeed, several definitions of approximation may be considered. In our setting, the most natural one is probably that of approximation in (1→1)-norm: given CPTP maps $\mathcal{N},\widehat{\mathcal{N}}:\mathcal{L}(\mathrm{A})\to\mathcal{L}(\mathrm{B})$, we say that $\widehat{\mathcal{N}}$ is an ε-approximation of $\mathcal{N}$ in (1→1)-norm, where $\varepsilon>0$ is some fixed parameter, if

$$\forall\ \varrho\in\mathcal{D}(\mathrm{A}),\ \left\|\widehat{\mathcal{N}}(\varrho)-\mathcal{N}(\varrho)\right\|_1\leqslant\varepsilon. \tag{3}$$

It might appear at first that an even more natural error quantification would be in terms of the completely-bounded (1→1)-norm. That is, we would call $\widehat{\mathcal{N}}$ an ε-approximation of $\mathcal{N}$ if, for any Hilbert space A',

$$\forall\ \varrho\in\mathcal{D}(\mathrm{A}\otimes\mathrm{A}'),\ \left\|\widehat{\mathcal{N}}\otimes\mathcal{I}\!\!\lceil(\varrho)-\mathcal{N}\otimes\mathcal{I}\!\!\lceil(\varrho)\right\|_1\leqslant\varepsilon. \tag{4}$$

However, this notion of approximation is too strong for our purposes. Indeed, if $\mathcal{N}$ and $\widehat{\mathcal{N}}$ satisfy (4), it implies that their associated Choi-Jamiołkowski states have to be ε-close in 1-norm. And this, in general, is possible only if $\mathcal{N}$ and $\widehat{\mathcal{N}}$ have a comparable number of Kraus operators, so that no reduction can be achieved.

The question of quantum channel compression has already been studied in one specific case, which is the one of the fully randomizing channel $\mathcal{R}$ defined by

$$\mathcal{R}: X\in\mathcal{L}(\mathrm{A})\mapsto(\mathrm{Tr}\,X)\frac{\mathbb{1}}{|\mathrm{A}|}\in\mathcal{L}(\mathrm{A}),$$

so that, in particular, all input states $\varrho\in\mathcal{D}(\mathrm{A})$ are sent to the maximally mixed state $\mathbb{1}/|\mathrm{A}|\in\mathcal{D}(\mathrm{A})$. $\mathcal{R}$ has maximal Kraus rank $|\mathrm{A}|^2$ (because $\tau(\mathcal{R})$ is simply $\mathbb{1}/|\mathrm{A}|^2$, and hence has rank $|\mathrm{A}|^2$), which was to be expected from the intuitive idea that the bigger its Kraus rank the noisier the channel. It was initially established in [3] and later improved in [1] that there exist almost randomizing channels with drastically smaller Kraus rank. More specifically: for any $0 < \varepsilon < 1$, the CPTP map $\mathcal{R}$ can be ε-approximated in (1→1)-norm by a CPTP map $\widehat{\mathcal{R}}$ with Kraus rank at most $C|\mathrm{A}|/\varepsilon^2$, where $C > 0$ is a universal constant. The question we investigate here is whether such kind of statement actually holds true for any channel.

2. Statement of the main results

Theorem 1. *Fix $0 < \varepsilon < 1$ and let $\mathcal{N} : \mathcal{L}(\mathrm{A}) \to \mathcal{L}(\mathrm{B})$ be a CPTP map with Kraus rank $|\mathrm{E}| \geq |\mathrm{A}|, |\mathrm{B}|$. Then, there exists a CP map $\widehat{\mathcal{N}} : \mathcal{L}(\mathrm{A}) \to \mathcal{L}(\mathrm{B})$ with Kraus rank at most $C \max(|\mathrm{A}|, |\mathrm{B}|) \log(|\mathrm{E}|/\varepsilon)/\varepsilon^2$ (where $C > 0$ is a universal constant) and such that*

$$\forall\ \varrho \in \mathcal{D}(\mathrm{A}),\ -\varepsilon \left(\mathcal{N}(\varrho) + \frac{\mathbb{1}}{|\mathrm{B}|} \right) \leqslant \widehat{\mathcal{N}}(\varrho) - \mathcal{N}(\varrho) \leqslant \varepsilon \left(\mathcal{N}(\varrho) + \frac{\mathbb{1}}{|\mathrm{B}|} \right). \tag{5}$$

Note that if $\widehat{\mathcal{N}}$ satisfies (5), then it approximates $\mathcal{N}$ in the sense of (3)

$$\forall\ \varrho \in \mathcal{D}(\mathrm{A}),\ \left\| \widehat{\mathcal{N}}(\varrho) - \mathcal{N}(\varrho) \right\|_1 \leqslant 2\varepsilon,$$

And for such (1→1)-norm approximation, we can further impose that $\widehat{\mathcal{N}}$ is strictly, and not just up to an error 2ε, trace preserving.

An important question at this point is that of optimality in Theorem 1. It can be shown that $n \geq \max(|\mathrm{A}|, |\mathrm{B}|)$ is necessary in Theorem 1. But it is not clear whether the $\log |\mathrm{E}|$ factor can be removed. In the case of channels whose range is only composed of sufficiently mixed states we can answer affirmatively, which is the content of Theorem 2 below. However, we leave the question open in general.

Theorem 2. *Fix $0 < \varepsilon < 1$ and let $\mathcal{N} : \mathcal{L}(\mathrm{A}) \to \mathcal{L}(\mathrm{B})$ be a CPTP map with Kraus rank $|\mathrm{E}| \geq |\mathrm{A}|, |\mathrm{B}|$. Then, there exists a CP map $\widehat{\mathcal{N}} : \mathcal{L}(\mathrm{A}) \to \mathcal{L}(\mathrm{B})$ with Kraus rank at most $C \max(|\mathrm{A}|, |\mathrm{B}|)/\varepsilon^2$ (where $C > 0$ is a universal constant) and such that*

$$\sup_{\varrho \in \mathcal{D}(\mathrm{A})} \left\| \widehat{\mathcal{N}}(\varrho) - \mathcal{N}(\varrho) \right\|_\infty \leqslant \varepsilon \sup_{\varrho \in \mathcal{D}(\mathrm{A})} \| \mathcal{N}(\varrho) \|_\infty .$$

Let us briefly explain the main ideas in the proofs of Theorems 1 and 2. These existence results of CPTP maps having some desired properties stem from proving that suitably constructed random ones have them with high probability. Showing that for a random CPTP map $\widehat{\mathcal{N}}$ the probability is high that, for every input state ϱ, $\widehat{\mathcal{N}}(\varrho)$ is close to $\mathcal{N}(\varrho)$ is done in two steps: proving first that it holds for a given input state and second that it in fact holds for all of them simultaneously. The fact that the individual deviation probability from average is small is a consequence of the concentration of measure phenomenon in high dimensions. Deriving then that the global deviation probability is also small is done by discretizing the input set and using a union bound. This line of proof is extremely standard in asymptotic geometric analysis. In our case though, the first step requires a careful analysis of the sub-exponential behavior of a certain random variable.

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Energy-constrained diamond norms on quantum channels

Andreas Winter

Let A and B be separable Hilbert space of two quantum systems whose states are described by the trace class operators $\mathcal{T}(A)$ and $\mathcal{T}(B)$, respectively:

$$\mathcal{T}(A) = \{\xi : A \to A \text{ s.t. } \|\xi\|_1 < \infty\},$$

where $\|\xi\|_1 = \operatorname{Tr}\sqrt{\xi^\dagger\xi}$ is the trace norm, i.e. the sum of all the singular values of ξ. Quantum channels, or in physics language, open system state evolutions, are modelled as completely positive and trace preserving (cptp) maps $\mathcal{N} : \mathcal{T}(A) \to \mathcal{T}(B)$. As these, and more generally their real linear combinations, which are Hermitian-preserving superoperators $\Delta : \mathcal{T}(A) \longrightarrow \mathcal{T}(B)$, are linear maps between Banach spaces, they inherit a natural norm, often called the *trace norm*:

$$\begin{aligned}\|\Delta\|_{1\to1} :&= \sup \|\Delta\xi\|_1 \text{ s.t. } \|\xi\|_1 \leq 1 \\ &= \sup \|\Delta\rho\|_1 \text{ s.t. } \rho \text{ state on } A.\end{aligned}$$

As is well-known, this norm induces a topology on maps that is often too strong for quantum mechanical applications – we will discuss such instances below. However, as a norm it is actually too weak, since it is not stable under tensor products with the identity map. Indeed, the natural norm on superoperators is the of the *completely bounded trace norm*, also known as *diamond norm* [1]:

$$\|\Delta\|_\diamond := \sup \|(\Delta \otimes \mathrm{id}_C)\rho\|_1 \text{ s.t. } \rho \text{ state on } A \otimes C. \tag{1}$$

From the definition, it is easy to see that the supremum may be restricted to pure states (namely by purification and the contractivity of the trace norm under partial traces), and that w.l.o.g. $C = A' \simeq A$. In particular, in finite dimension, the supremum is always attained on a pure state on $A \otimes A'$. As a matter of terminology, in the present article, superoperators are generally assumed to be Hermitian-preserving, which means that they are differences of completely positive (cp) maps; and *channels* are those maps that are cptp.

The diamond norm has an operational interpretation for $\Delta = p\mathcal{N}_1 - (1-p)\mathcal{N}_2$, in a Helstrom context of binary hypothesis testing between to channels $\mathcal{N}_i$, as follows: $\frac{1}{2}\big(1 - \|p\mathcal{N}_1 - (1-p)\mathcal{N}_2\|_\diamond\big)$ equals the minimum error probability of distinguishing $\mathcal{N}_1$ from $\mathcal{N}_2$, which come with prior probabilities p and $1-p$, respectively, when we are allowed preparation of a probe state ρ^{AC}, one application of the unknown channel, and an arbitrary measurement on the system BC.

Shirokov has proposed a modification of the diamond norm to take into account an energy limit at the input of the channel:

Definition 1 (Shirokov [2])**.** For a Hermitian-preserving map Δ, define the *E-constrained diamond norm*

$$\|\Delta\|_{\diamond E} := \sup \|(\Delta \otimes \mathrm{id}_C)\rho\|_1 \text{ s.t. } \rho \text{ state on } A \otimes C,\ \operatorname{Tr} \rho^A H_A \leq E. \tag{2}$$

By the same reasoning as for the diamond norm, the supremum can be restricted to pure states, and w.l.o.g. $C = A' \simeq A$. A related definition was proposed by Pirandola *et al.* [3], for the special case of quantum harmonic oscillators, and with the slight difference that the energy (photon number) bound was applied to both system A and reference C. The resulting norm is equivalent to the one of Shirokov, but has not the same ideal mathematical properties.

In this contribution, based on [4], we explore this energy-constrained diamond norm on superoperators. Our main motivation is the continuity of capacities and other entropic quantities of quantum channels, but we also present an application to the continuity of one-parameter unitary groups and certain one-parameter semigroups of quantum channels. As an example, we can prove the following variant and indeed refinement of the Margolus-Levitin "quantum speed limit" [5]:

Theorem 2. *The unitary time evolution $\mathcal{U}(t)\rho = e^{-itH}\rho\, e^{itH}$ generated by a grounded Hamiltonian H is uniformly continuous in $\|\cdot\|_{\diamond E}$ w.r.t. the same H:*

$$\frac{1}{2}\|\mathcal{U}(t) - \mathcal{U}(0)\|_{\diamond E} \leq \sqrt[3]{16tE}. \tag{3}$$

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Quantum Relative entropy and Quantum Optimal Transport

Eric Carlen

(joint work with Jan Maas and Anna Vershynina)

For density matrices ρ and σ on a Hilbert space $\mathcal{H}$, the Umegaki relative entropy $D(\rho||\sigma)$ is defined by

$$D(\rho||\sigma) = \operatorname{Tr}[\rho(\log \rho - \log \sigma)]\ .$$

In 1975, building on work of Lieb and Ruskai, Lindblad proved the monotonicity of $D(\rho||\sigma)$ under completely positive trace preserving (CPTP) maps: If Φ is such a map, then

$$D(\rho||\sigma) \geq D(\Phi\rho||\Phi\sigma) \ . \tag{1}$$

This inequality is now known as the *Data Processing Inequality* (DPI). Roughly speaking, the relative entropy has the following operational meaning: If someone is sending you quantum states, and you know that the state is either ρ or σ, but want to determine, by repeated measurements, which one is being sent, and want to do this with only an ϵ error of probability, you will need to conduct a sequence of approximately n measurements on n copies of the system where n is such that $e^{-nD(\rho||\sigma)} = \epsilon$. The Data processing inequality says that applying a CPTP map (also known as a quantum operation) can only make it more difficult to distinguish between the states.

Around the same time, Araki extended the definition of the Umegaki relative entropy to a rather general von Neumann algebra setting, and proved, in broad generality, the key convexity property that is the source of the DPI. His definition involved the *relative modular operator*, which in the case that that Hilbert space $\mathcal{H}$ is finite dimensional and ρ is invertible, is the operator on the space of operators on $\mathcal{H}$ equipped with the Hilbert-Schmidt norm given by

$$\Delta_{\sigma,\rho}A = \sigma A \rho^{-1} \ ,$$

which is easily seen to be a positive operator. Using $\Delta_{\sigma,\rho}$, one can write the relative entropy as

$$D(\rho||\sigma) = \langle -\sqrt{\rho}, \log\left(\Delta_{\sigma,\rho}\right)\sqrt{\rho}\rangle_{\mathrm{HS}} \ . \tag{2}$$

Everyone who is familiar with the Tomita-Takesaki theory will see immediately how to generalize this definition to the setting of a general von Neumann algebra equipped with two faithful normal states. The relative modular operator is at the center of the two problems discussed in this talk. The first concerns stability in the DPI. Petz had determined the conditions for equality in the DPI, as we explain below, and the question we consider is: If there is nearly equality in (1), then is the condition of Petz nearly satisfied? Consider the case in which Φ is the restriction of a state to a subalgebra $\mathcal{N}$, and write $\rho_{\mathcal{N}}$ to denote $\Phi\rho$, etc. The Petz recovery map, acting on any state γ, is given by

$$\mathcal{R}_\rho(\gamma) = \rho^{1/2}(\rho_{\mathcal{N}}^{-1/2}\gamma\rho_{\mathcal{N}}^{-1/2})\rho^{1/2} \ .$$

It is evident that $\mathcal{R}_\rho\rho_{\mathcal{N}} = \rho$, but it is not in general the case that $\mathcal{R}_\rho\sigma_{\mathcal{N}} = \sigma$, but since $\mathcal{R}_\rho$ is a CPTP map, when this *is* the case

$$D(\rho||\sigma) = D(\mathcal{R}_\rho\rho_{\mathcal{N}}||\mathcal{R}_\rho\sigma_{\mathcal{N}}) \leq D(\rho_{\mathcal{N}}||\sigma_{\mathcal{N}}) \leq D(\rho||\sigma) \ ,$$

and there is equality in the DPI. The result of Petz is that this is the only case in which there is equality in the DPI. The result of myself and A. Vershynina [1] is that

$$D(\rho||\sigma) - D(\rho_{\mathcal{N}}||\sigma_{\mathcal{N}}) \geq \left(\frac{1}{8\pi}\right)^4 \|\Delta_{\sigma,\rho}\|^{-2}\|\mathcal{R}_\rho(\sigma_{\mathcal{N}}) - \sigma\|_1^4 \ ,$$

which is a quantitive version of Petz's Theorem.

The second problem concerns quantum Markov semigroups, by which we mean semigroups $\{\mathcal{P}_t\}_{t\geq 0}$ of completely positive unital maps. Let $\{\mathcal{P}_t^\dagger\}_{t\geq 0}$ denote the dual CPTP semigroup. We suppose that σ is the unique invariant state for $\{\mathcal{P}_t^\dagger\}_{t\geq 0}$; i.e., $\mathcal{P}_t^\dagger\sigma = \sigma$ for all t. It follows from the DPI that for all t and all ρ,

$$D(\mathcal{P}_t^\dagger\rho, \sigma) \leq D(\rho||\sigma) \ .$$

Thus the relative entropy with respect to the steady state is monotone decreasing along such a flow. This suggests that the flow might be the gradient flow for the relative entropy with respect to σ in an appropriate metric, which based on analogy with the classical case, one would expect to be an optimal mass transport metric. This was proved in joint work with Jan Maas [2, 3] under the assumption of a certain *detailed balance* condition, which by a result of Alicki, means that the generator of the semigroup commutes with the modular operator $\Delta_{\sigma,\sigma}$. This gradient flow structure is then shown to be the source of sharp entropy production inequalities [3].

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Strong convergence for random permutations

Benoît Collins

(joint work with Charles Bordenave)

This presentation is a report on the preprint [1]. Let $(\sigma_1, \ldots, \sigma_d)$ be a finite sequence of independent random permutations, chosen uniformly either among all permutations or among all matchings on n points. We show that, in probability, as $n \to \infty$, these permutations, viewed as operators on the $n-1$ dimensional vector space $\{(x_1, \ldots, x_n) \in \mathbb{C}^n, \sum x_i = 0\}$, are asymptotically strongly free. Our proof relies on the development of a matrix version of the non-backtracking operator theory and a refined trace method.

As a byproduct, we show that the non-trivial eigenvalues of random n-lifts of a fixed based graphs approximately achieve the Alon-Boppana bound with high probability in the large n limit. This result generalizes Friedman's Theorem stating that with high probability, the Schreier graph generated by a finite number of independent random permutations is close to Ramanujan.

Finally, we extend our results to tensor products of random permutation matrices. This extension is especially relevant in the context of quantum expanders.

We also present a version of our result where part of the permutations involved in $(\sigma_1, \dots, \sigma_d)$ can be chosen to be random pairings (involutions with no fixed point, for n even).

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Non-closure of a set of quantum correlations

Ken Dykema
(joint work with Vern Paulsen, Jitendra Prakash)

We consider various sets of quantum correlations (used, for example, in the theory of non-local quantum games). For a fixed finite input set $\mathcal{I}$, of cardinality n, and finite output set $\mathcal{O}$, of cardinality k, the set $C_{qc}(n,k)$ of *quantum commuting corelations* is the set of all tuples

$$p = \big(p(i,j|x,y)\big)_{i,j\in\mathcal{I},\, x,y\in\mathcal{O}} \in [0,1]^{n^2k^2}$$

arising as

$$p(i,j|x,y) = \langle E_{i,x}F_{j,y}\psi, \psi\rangle$$

where

- ψ is a unit vector in a Hilbert space H,
- for each $i \in \mathcal{I}$, $(E_{i,x})_{x\in\mathcal{O}}$ is a projection-valued measure, namely, a family of self-adjoint projections in $B(H)$ such that
$$\sum_{x\in\mathcal{O}} E_{i,x} = I_H,$$
- for each $j \in \mathcal{I}$, $(F_{j,y})_{y\in\mathcal{O}}$ is a projection-valued measure,
- for all $i, j \in \mathcal{I}$ and $x, y \in \mathcal{O}$,
$$E_{i,x}F_{j,y} = F_{j,y}E_{i,x}.$$

Other sets of quantum correlations that we consider are: the set

$$C_{qs}(n,k)$$

where we require in addition $H = H_A \otimes H_B$ and

$$\begin{aligned} \forall i,x \quad & E_{i,x} \in B(H_A)\otimes I_{H_B} \\ \forall j,y \quad & F_{j,y} \in I_{H_A}\otimes B(H_B); \end{aligned}$$

the set

$$C_q(n,k)$$

which is like $C_{qs}(n,k)$ but where we require in addition that H_A and H_B be finite dimensional; the set

$$C_{qa}(n,k) = \overline{C_q(n,k)},$$

where the closure is taken in $[0,1]^{n^2k^2}$; finally, we have also the set

$$C_{vect}(n,k)$$

consisting of all $p \in [0,1]^{n^2k^2}$ arising as

$$p(i,j|x,y) = \langle v_{i,x}, w_{j,y} \rangle$$

where there exist vectors $v_{i,x}$, $w_{j,y}$ and ψ in a Hilbert space H such that

- $\|\psi\| = 1$ and, for all $i,j \in \mathcal{I}$ and $x,y \in \mathcal{O}$, $\langle v_{i,x}, w_{j,y} \rangle \geq 0$,
- for all $i \in \mathcal{I}$, $\sum_{x\in\mathcal{O}} v_{i,x} = \psi$ and $v_{i,x} \perp v_{i,y}$ whenever $x \neq y$,
- for all $j \in \mathcal{I}$, $\sum_{x\in\mathcal{O}} w_{j,x} = \psi$ and $w_{j,x} \perp w_{j,y}$ whenever $x \neq y$.

Then for each fixed n and k, we have the inclusions

$$C_q \subseteq C_{qs} \subseteq C_{qa} \subseteq C_{qc} \subseteq C_{vect}.$$

It is clear that each of these sets is convex. The famous *Tsirelson Conjectures* [10] relate to equality in these inclusions. The Strong Tsirelson Conjecture is that $C_q = C_{qc}$. The Weak Tsirelson Conjecture is that $C_{qa} = C_{qc}$. Due to work of several authors [2], [3], [7], Connes' Embedding Conjecture is known to be equivalent to the equality $C_{qa}(n,k) = C_{qc}(n,k)$ holding for all $n,k \in \mathbb{N}$, namely, to the Weak Tsirelson Conjecture. W. Slofstra [9] proved that, for n approximately 100 and $k = 8$,

$$C_{qs}(n,k) \neq C_{qa}(n,k).$$

In particular, $C_q(n,k)$ is not closed, and the Strong Tsirelson Conjecture is false. Slofstra's clever proof uses group theory, including a classical result of Mal'cev [6].

In [1], we used more elementary techniques to prove the following.

Theorem 1. *$C_q(5,2)$ and $C_{qs}(5,2)$ are not closed.*

Our proof uses synchronous correlations, which were introduced in [8]. A correlation $p = (p(i,j|x,y))_{i,j\in\mathcal{I},\, x,y\in\mathcal{O}}$ is said to be *synchronous* if $p(i,i|x,y) = 0$ whenever $x \neq y$. For $r \in \{q, qs, qa, qc, vect\}$, $C_r^{(s)}(n,k)$ denotes the set of all correlations $p \in C_r(n,k)$ that are synchronous.

Theorem 2 ([8])**.** *Let $n,k \in \mathbb{N}$.*

(i) *If $p \in C_{vect}^{(s)}(n,k)$ has a realization with vectors $v_{i,x}$, $w_{j,y}$ and ψ in a Hilbert space H, then*

$$\forall i \in \mathcal{I}\, \forall x \in \mathcal{O} \quad v_{i,x} = w_{i,x}.$$

(ii) *If $p \in C_{qc}^{(s)}(n,k)$ has a realization from projections $E_{i,x}$ and $F_{j,y}$ and unit vecotr ψ in a Hilbert space H, then*

$$\forall i \in \mathcal{I}\, \forall x \in \mathcal{O} \quad E_{i,x}\psi = F_{i,x}\psi$$

and the restriction of the vector state $\langle \cdot\psi, \psi \rangle$ to the unital algebra generated by $\{E_{i,x} \mid i \in \mathcal{I},\, x \in \mathcal{O}\}$ is a trace.

Thus, part (ii) relates synchronous correlations to traces on certain C*-algebras.

Given $p \in C^{(s)}_{vect}$, we consider the marginal distributions which are given by, for all $i \in \mathcal{I}$ and $x \in \mathcal{O}$,

$$p_A(i|x) = \sum_{y \in \mathcal{O}} p(i,j|x,y),$$

which is indpendent of j, and similaraly, for all $j \in \mathcal{I}$ and $y \in \mathcal{O}$, we let

$$p_B(j|y) = \sum_{x \in \mathcal{O}} p(i,j|x,y),$$

which is indpendent of i.

Henceforth, let $k = 2$ and $\mathcal{O} = \{0,1\}$. For $r \in \{q, qs, qa, qc, vect\}$ and $t \in [0,1]$, we let

$$\Gamma_r(t) = \{p \in C^{(s)}_r(n,2) \mid \forall i,j \in \mathcal{I},\, p_A(i|0) = p_B(j|0) = t\}.$$

It is not difficult to see that $\Gamma_r(t)$ is nonempty and, being a slice of $C_r(n,2)$, it is convex. We consider the functional F on $C^{(s)}_r(n,2)$ defined by

$$F(p) = \sum_{i,j \in \mathcal{I},\, i \neq j} p(i,j|0,0)$$

and we let

$$f_r(t) = \inf\{F(p) \mid p \in \Gamma_r(t)\}. \tag{1}$$

Clearly, we have

$$f_q \geq f_{qs} \geq f_{qa} \geq f_{qc} \geq f_{vect}.$$

Moreover, it is not difficult to show that each f_r is a convex function. A key question is: when is the infimum in (1) attained? Clearly, if $C_r(n,k)$ is closed, then each $\Gamma_r(t)$ is closed and this infimum is attained.

The following is a key result in the proof of Theorem 1.

Proposition 3. *If $t \in [0,1]\backslash\mathbb{Q}$ and if the value of $f_q(t)$ is attained in the corresponding infimum* (1)*, then there exist $a, b \in \mathbb{Q}$ satisfying $0 \leq a < t < b \leq 1$, such that the restriction of the function f_q to the interval $[a,b]$ is linear.*

The proof of this proposition relies on Theorem 2 and the nature of traces on finite dimensional C*-algebras.

The following is not difficult to show, using Gram matrices of vectors.

Proposition 4.

$$f_{vect}(t) = \begin{cases} 0, & 0 \leq t \leq \frac{1}{n} \\ nt(nt-1), & \frac{1}{n} \leq t \leq \frac{n-1}{n} \\ (n^2-n)(2t-1), & \frac{n-1}{n} \leq t \leq 1. \end{cases}$$

The final ingredient in the proof of Theorem 1 is the following. Its proof is an application of a beautiful result [5] of Kruglyak, Rabanovich, and Samoĭlenko, that characterizes the scalar multiples of the identity that arise as sums of projections.

Proposition 5. *If* $n = 5$ *and if*

$$t \in \left[\frac{\sqrt{5}-1}{2\sqrt{5}}, \frac{\sqrt{5}+1}{2\sqrt{5}} \right], \tag{2}$$

then $f_q(t) = f_{vect}(t)$.

The proof of Theorem 1 now follows, because we see that the function f_q is not linear on any subinterval of the interval in (2), hence, by Proposition 3, the infimum $f_q(t)$ cannot be attained for any rational t in that interval. We also need the result from [4], that asserts $C_q^{(s)}(n,k) = C_{qs}^{(s)}(n,k)$.

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Cwikel estimates revisited

Fedor Sukochev

(joint work with G. Levitina, D. Zanin)

One of the most beautiful results in operator theory are the so-called Cwikel estimates concerning the singular values of the operator $M_f g(-i\nabla)$ on $L_2(\mathbb{R}^d)$ in weak Schatten ideals $\mathcal{L}_{p,\infty}(\mathcal{H})$, $2 < p < \infty$. Here, ∇ denotes the gradient on $L_2(\mathbb{R}^d)$, that is ∇ the d-tuple $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_d})$ of operators $\frac{\partial}{\partial x_j}$ of partial differentiation on $L_2(\mathbb{R}^d)$. For a compact operator x on a Hilbert space $\mathcal{H}$ we denote by $\mu(x) = \{\mu(k,x)\}$ the sequence of singular values of x, that is the sequence of eigenvalues of $|x| = (x^*x)^{1/2}$ taken in decreasing order counting multiplicities.

For $p>0$, let $\mathcal{L}_{p,\infty}(\mathcal{H})$ be defined by

$$\mathcal{L}_{p,\infty}(\mathcal{H})=\{x:\ \sup_{k\geq 0}(k+1)^{\frac{1}{p}}\mu(k,x)<\infty\}.$$

Its natural quasi-norm is given by the formula

$$\|x\|_{p,\infty}=\sup_{k\geq 0}(k+1)^{\frac{1}{p}}\mu(k,x).$$

For $p>0$, this is a quasi-Banach ideal. For $p>1$, this is a Banach ideal (when equipped with equivalent norm).

There exists also substantial literature concerning the estimates of f and g so that $M_f g(-i\nabla)$ belongs to Schatten ideals $\mathcal{L}_p(\mathcal{H})$, $1\leq p<\infty$ and, other ideals in the algebra $\mathcal{L}(L_2(\mathbb{R}^d))$ of all bounded operators on $L_2(\mathbb{R}^d)$.

It is well known that

$$(M_f g(-i\nabla))\xi)(s)=\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} f(s)(\mathcal{F}^{-1}g)(s-t)\xi(t)dt,\quad \xi\in L_2(\mathbb{R}^d)$$

Hence, the operator $f(t)g(-i\nabla)$ is in the Hilbert-Schmidt class if and only if $f,g\in L_2(\mathbb{R}^d)$; moreover, $\|f(t)g(-i\nabla)\|_2=(2\pi)^{-n/2}\|f\|_2\|g\|_2$. It was conjectured by Simon [1] and proved by Cwikel [2], that conditions $f\in L_p(\mathbb{R}^d)$ and $g\in L_{p,\infty}(\mathbb{R}^d)$, $p>2$, ensure that the operator $M_f g(-i\nabla)$ belongs to the ideal $\mathcal{L}_{p,\infty}(L_2(\mathbb{R}^2))$ and

$$\|M_f g(-i\nabla))\|_{p,\infty}\leq \text{const }\|f\|_p\|g\|_{p,\infty}. \tag{1}$$

Via the Fourier transform $\mathcal{F}$ on $L_2(\mathbb{R}^d)$, we have

$$M_f g(-i\nabla)=M_f\mathcal{F}M_g\mathcal{F}^{-1}=\mathcal{F}\big(\mathcal{F}^{-1}M_f\mathcal{F}M_g\big)\mathcal{F}^{-1}.$$

Hence, the operators $M_f g(-i\nabla)$ and $f(i\nabla)M_g$ are unitarily equivalent, and therefore, their singular numbers coincide. However, the Cwikel estimate (1) does not reflect the fact that the functions f and g are interchangeable. Firstly, we present a corollary of our main result: a symmetric form of Cwikel estimates:

Corollary 1. *Suppose that $f\otimes g\in L_{p,\infty}(\mathbb{R}^{2d})$, then the operator $M_f g(-i\nabla)\in\mathcal{L}_{p,\infty}(L_2(\mathbb{R}^d))$ and*

$$\|M_f g(-i\nabla))\|_{p,\infty}\leq \text{const }\|f\otimes g\|_{p,\infty}.$$

Note that if $f\in L_p(\mathbb{R}^d)$ and $g\in L_{p,\infty}(\mathbb{R}^d)$, then $f\otimes g$ (as well as $g\otimes f$) is a function in $L_{p,\infty}(\mathbb{R}^{2d})$. However, the converse implication does not hold. Thus, a corollary of our main result provides a more general version of original Cwikel estimates.

We now present the abstract Cwikel estimates.

Let $\mathcal{A}_1$ (respectively $\mathcal{A}_2$) be a semifinite von Neumann algebra represented via π_1 (respectively, π_2) on the same (separable) Hilbert space $\mathcal{H}$ and let τ_1 (respectively, τ_2) be a faithful normal semifinite trace on $\mathcal{A}_1$ (on $\mathcal{A}_2$).

Theorem 2. *Assume that for the representations of the algebras* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *we have* $\pi_1(x)\pi_2(y) \in \mathcal{L}_2(\mathcal{H})$ *provided that* $x \in \mathcal{L}_2(\mathcal{A}_1, \tau_1), y \in \mathcal{L}_2(\mathcal{A}_2, \tau_2)$*, and*

$$\|\pi_1(x)\pi_2(y)\|_2 \leq \text{const } \|x\|_2\|y\|_2.$$

Assume, in addition, that $(E(0,\infty), \|\cdot\|_E)$ *is an interpolation space for* (L_2, L_∞)*. If* $x \otimes y \in E(\mathcal{A}_1 \otimes \mathcal{A}_2)$ *for some* $x \in S(\mathcal{A}_1, \tau_1)$*,* $y \in S(\mathcal{A}_2, \tau_2)$*, then* $\pi_1(x)\pi_2(y) \in E(\mathcal{H})$ *and*

$$\|\pi_1(x)\pi_2(y)\|_{E(\mathcal{H})} \leq C_E \|x \otimes y\|_{E(\mathcal{A}_1 \otimes \mathcal{A}_2)}.$$

To explain properly the notations used in the statement of Theorem 2 we need to recall fe notions from the theory of noncommutative integration.

Let $\mathcal{M}$ be a semifinite von Neumann algebra on a Hilbert space H and let τ be a faithful normal semifinite trace on $\mathcal{M}$.

An operator x affiliated to $\mathcal{M}$ is called $\tau-$measurable if $\tau(E_{|x|}(s,\infty)) < \infty$ for sufficiently large s. We denote the set of all $\tau-$measurable operators by $S(\mathcal{M}, \tau)$.

For every $x \in S(\mathcal{M}, \tau)$, we define its singular value function $\mu(A)$ by setting

$$\mu_{\mathcal{M}}(t, x) = \inf\{\|x(1-p)\|_\infty : \quad \tau(p) \leq t\}.$$

In the case, when $\mathcal{M} = \mathcal{L}_\infty(H)$ and the trace τ is the classical trace tr, the singular value function of $x \in S(L_\infty(H), \tau) = \mathcal{L}_\infty(H)$ has the form

$$\mu_{\mathcal{B}(\mathcal{H})}(t, x) = \sum_{k=0}^{\infty} \mu(k, x)\chi_{[k,k+1)},$$

where $\{\mu(k, x)\}$ is the sequence of singular values of x.

In the case, when $\mathcal{M} = L_\infty(\mathbb{R}^d)$, the singular value function $\mu_{L_\infty(\mathbb{R}^d}(f), f \in S(L_\infty(\mathbb{R}^d))$ coincides with the decreasing rearrangement f^* of f.

A linear subspace $\mathcal{E}(\mathcal{M})$ of $S(\mathcal{M}, \tau)$ equipped with a complete (quasi-)norm $\|\cdot\|_{\mathcal{E}}$, is called *symmetric space* if $x \in S(\mathcal{M}, \tau)$, $y \in \mathcal{E}(\mathcal{M})$ and $\mu_{\mathcal{M}}(x) \leq \mu_{\mathcal{M}}(y)$ imply that $x \in \mathcal{E}(\mathcal{M})$ and $\|x\|_{\mathcal{E}} \leq \|y\|_{\mathcal{E}}$.

The so-called Calkin correspondence provides a construction of symmetric operator spaces associated with the von Neumann algebra $\mathcal{M}$ from concrete symmetric function spaces on $(0,\infty)$. Namely, let $(E(0,\infty), \|\cdot\|_{E(0,\infty)})$ be a symmetric function space on the semi-axis $(0,\infty)$. Then the pair

$$E(\mathcal{M}) = \{x \in S(\mathcal{M}, \tau) : \mu(x) \in E(0,\infty)\}, \quad \|x\|_{E(\mathcal{M})} := \|\mu(x)\|_{E(0,\infty)}$$

is a symmetric space on $\mathcal{M}$ [3]. For convenience, we denote $\|\cdot\|_{E(\mathcal{M})}$ by $\|\cdot\|_E$.

Applications of Theorem 2 are given for noncommutative Euclidean spaces.

Furthermore, we show that our assumption that the symmetric function space $E(0,\infty)$ is an interpolation space for (L_2, L_∞) is optimal. That is, if we omit the assumption that $E(0,\infty)$ is an (L_2, L_∞)-interpolation space, then the corresponding Cwikel estimates fail. The counterexample is yielded by the space $L_{2,\infty}(0,\infty)$, which is not an (L_2, L_∞)-interpolation space and uses the operator $M_f g(-i\nabla)$ in the one-dimensional setting.

The case, when $p < 2$.

The case of Cwikel estimates the weak Schatten ideal $\mathcal{L}_{p,\infty}(L_2(\mathbb{R}^d))$ when $0 < p < 2$ is quite different and the best result to date is due to Birman and Solomyak.

Let K be the unit cube in $\mathbb{R}^d$ and let $E \subset \ell_2$ be a symmetric quasi-Banach sequence space. Let $E(L_2)(\mathbb{R}^d)$ be the space of all (measurable) functions such that the sequence $\{\|f\chi_{K+m}\|_2\}_{m\in\mathbb{Z}^d}$ is in E. For $f \in E(L_2)(\mathbb{R}^d)$ set

$$\|f\|_{E(L_2)(\mathbb{R}^d)} = \left\|\{\|f\chi_{K+m}\|_2\}_{m\in\mathbb{Z}^d}\right\|_E.$$

Birman and Solomyak showed that if $f \in \ell_p(L_2)(\mathbb{R}^d)$, $g \in \ell_{p,\infty}(L_2)(\mathbb{R}^d)$ guarantee that $M_f g(-i\nabla) \in \mathcal{L}_{p,\infty}(L_2(\mathbb{R}^d))$ and

$$\|M_f g(-i\nabla)\|_{p,\infty} \leq \text{const } \|f\|_{\ell_p(L_2)(\mathbb{R}^d)}\|g\|_{\ell_{p,\infty}(L_2)(\mathbb{R}^d)}.$$

Observe that this result has the same asymmetry as Cwikel's result for $p > 2$.

We say that ξ is majorised by η (notation $\xi \prec \eta$), if $\xi \prec\prec \eta$ and

$$\int_0^\infty \mu(s,\xi)ds = \int_0^\infty \mu(s,\eta)ds,$$

assuming that both integrals are finite.

Theorem 3. *Let $E \subset \ell_2$ be a symmetric quasi-Banach sequence space such that $\mu^2(x) \prec \mu^2(y)$, $x \in E$ implies that $y \in E$ and $\|y\|_E \leq c_E\|x\|_E$. If $f \otimes g \in E(L_2)(\mathbb{R}^{2d})$, then $M_f g(-i\nabla) \in E(L_2(\mathbb{R}^d))$ and*

$$\|M_f g(-i\nabla)\|_{E(L_2(\mathbb{R}^d))} \leq c_{E,p}\|f \otimes g\|_{E(L_2)(\mathbb{R}^{2d})}.$$

Theorem 4. *The space $\ell_{p,\infty}$, $0 < p < 2$ satisfies the assumption of Theorem 3. Hence, if $f \otimes g \in \ell_{p,\infty}(L_2)(\mathbb{R}^{2d})$, then $M_f g(-i\nabla) \in \mathcal{L}_{p,\infty}(L_2(\mathbb{R}^d))$ and*

$$\|M_f g(-i\nabla)\|_{p,\infty} \leq C_p\|f \otimes g\|_{\ell_{p,\infty}(L_2)(\mathbb{R}^{2d})}.$$

Conjecture 5. The following conditions for a quasi-Banach sequence space $E \subset \ell_2$ are equivalent:

(1) Condition $\mu^2(x) \prec \mu^2(y)$, $x \in E$, implies that $y \in E$ and $\|y\|_E \leq c_E\|x\|_E$.
(2) E is an interpolation space for (ℓ_p, ℓ_2) for some $0 < p < 2$.

This conjecture has been confirmed for the case when $1 \leq p < 2$ by Michael Cwikel and Per Nilsson [4].

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Invariant subspaces for H^2 spaces of σ-finite algebras

Louis Labuschagne

In the late 50's and early 60's of the previous century, it became apparent that many famous theorems about $H^\infty(\mathbb{D})$, could be generalized to the setting of abstract function algebras. The paper [8] of Srinivasan and Wang, dating back to the mid-1960s, organized and summarized much of this 'generalized H^∞-theory'.

Shortly afterwards Arveson introduced his notion of subdiagonal subalgebras of von Neumann algebras as a possible context for extending this cycle of results to the noncommutative context [1]. Given a σ-finite von Neumann algebra M equipped with a faithful normal state φ, a weak*-closed unital subalgebra A of M is called subdiagonal, if on the one hand $A + A^*$ is weak* dense in M (where $A^* = \{a : a^* \in A\}$), and if on the other there exists a faithful normal conditional expectation $\mathcal{E}$ onto the subalgebra $\mathcal{D} = A \cap A^*$, which leaves ν invariant and which is also multiplicative on A.

In a sequence of papers (cf. [3]), complemented by important contributions from Ueda [9] and Bekjan and Xu [2], Blecher and Labuschagne introduced the notion of *tracial* subalgebras in the context of finite von Neumann algebras equipped with faithful normal tracial state, defining these to be weak*-closed unital subalgebras A of M for which the canonical τ-invariant faithful normal conditional expectation $\mathcal{E}$ onto $\mathcal{D} = A \cap A^*$, is multiplicative on A. By making use of the noncommutative Szegö formula of Arveson [1, Eqn γ, p 611], they then went went on to show that the *entire* cycle of results established by Srinivasan and Wang, extends to the setting of tracial subalgebras.

The success of this theory raised the question of whether any of this theory will survive the passage to the case of type III σ-finite algebras. However the transition from finite to σ-finite von Neumann algebras comes at the cost of losing the theory of the Fuglede-Kadison determinant, and hence the concomitant Szegö formula. We show that despite this drawback a substantial portion of the theory survives the transition to the type III case. The first step in achieving this extension, is to identify a suitable type III analog of a tracial subalgebra.

Theorem([10], [6]) *Let A be a weak* closed unital subalgebra of M with $\mathcal{D}$ and $\mathcal{E}$ as before, with in addition $A + A^*$ weak*-dense in M. Then A is a maximal subdiagonal subalgebra with respect to $\mathcal{D}$ if and only if $\sigma_t^\varphi(A) = A$ for all $t \in \mathbb{R}$.*

Let A, $\mathcal{D}$ and $\mathcal{E}$ be as before. On the basis of the above result we will call any weak* closed unital subalgebra A of M for which

(a) $\sigma_t^\varphi(A) = A$ for all $t \in \mathbb{R}$,
(b) and for which the faithful normal conditional expectation $\mathcal{E} : M \to \mathcal{D} = A \cap A^*$ satisfying $\varphi \circ \mathcal{E} = \varphi$ (ensured by (a)), is multiplicative on A

an *analytically conditioned* subalgebra of M. It is these algebras that are the type III analogs of tracial algebras. Xu's proof of the "if" part of the above theorem relies on a very elegant extension of Haagerup's reduction theorem [5] to subdiagonal subalgebras. It is this self-same reduction theorem that is the key ingredient in achieving the main result. Specifically, one may show that Xu's construction goes through for analytically conditioned algebras, and then use that construction to prove the following deep lemma. (Here R is the crossed product $R = M \rtimes_\nu \mathbb{Q}_d$ of M with the diadic rational $\mathbb{Q}_d$, $\{R_n\}$ a sequence of von Neumann subalgebras increasing to R, $\widehat{A} = A \rtimes_\nu \mathbb{Q}_d$, and $\widehat{A}_n = \widehat{A} \cap R_n$.)

Lemma *Let A be an analytically conditioned algebra. If A satisfies the criterion that any $f \in L^1(M)^+$ which is in the annihilator of $A_0 = \ker(\mathcal{E}) \cap A$ must belong to $L^1(\mathcal{D})$, then also*

- *any $f \in L^1(R)^+$ which is in the annihilator of $\widehat{A}_0$ must belong to $L^1(\widehat{\mathcal{D}})$,*
- *and for any n, any $f \in L^1(R_n)^+$ which is in the annihilator of $(\widehat{A}_n)_0$, must belong to $L^1(\mathcal{D}_n)$.*

The above lemma combined with the noncommutative peak set theorem of Blecher and Labuschagne [4], are they key tools in establishing our main result:

Theorem *For any analytically conditioned algebra A, the following are equivalent:*

(i) $\overline{A + A^*}^{w*} = M$,

(ii) *For every right A-invariant subspace X of $L^2(M)$, the right wandering subspace $W = X \ominus [XA_0]_2$ satisfies $W^*W \subset L^1(\mathcal{D})$, and $W^*(X \ominus [WA]_2) = (0)$.*

(iii) *The canonical embedding of $A + A_0^*$ into $L^2(M)$ is dense, and any $f \in L^1(M)^+$ which is in the annihilator of A_0 belongs to $L^1(\mathcal{D})$.*

(iv) *There is a unique Hahn-Banach extension to M of any weak* continuous functional on A, and this extension is weak* continuous.*

A fuller statement of the theory outlined above, may be found in [7].

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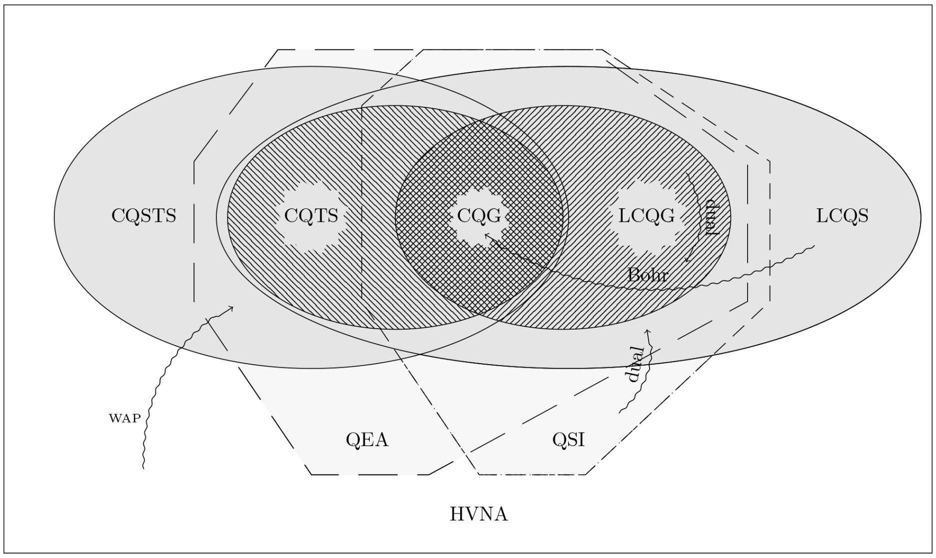
A small survey of quantum semigroups

Yulia Kuznetsova

Quantum groups in their topological setting are noncommutative analogues of function algebras on topological groups. Every class of quantum (semi)groups corresponds to a classical family of (semi)groups, varying by the degree of continuity of the multipication, absence or presence of compactness, presence of additional structures. In my talk, I list known families of quantum (semi)groups and certain maps between them, encoded in the following diagram:



The letters appearing on this scheme are explained below.

In every class of quantum (semi)groups the commutative algebras correspond to classical functional algebras. We will pay special attention to these commutative representatives, since they serve as motivating examples and allow to trace the logic of the definitions.

1. **CQG** is the class of compact quantum groups, introduced by S. L. Woronowicz [7]. A CQG is a pair of a unital C^*-algebra A and a comultiplication Δ which maps A to the space $M(A \otimes A)$ of multipliers on the minimal tensor product of A with itself. Additional density conditions correspond to the cancellation law and in the commutative case imply the existence of the inverse on the underlying group. The classical example of a CQG is the algebra $C(G)$ of continuous functions on a compact topological group G.
2. **LCQG** is the second best known class and stands for locally compact quantum groups, as defined by J. Kustermans and S. Vaes [4]. A LCQG is a C^*-algebra A with a comultiplication $\Delta : A \to M(A \otimes A)$ and a pair of Δ-invariant weights ϕ, ψ on A which correspond to the left and right

Haar measure. The classical example is given by the algebra $C_0(G)$ of continuous functions vanishing at infinity on a locally compact group G.

3. **LCQS** means localy compact semigroups and includes just C^*-algebras with comultiplication, without any additional structure. In the commutative case, we get the algebras isomorphic to $C_0(S)$ on a locally compact space S, and the comultiplication induces multiplication on S so that it becomes a topological semigroup.
4. **CQTS** includes most known quantum semigroups which are not groups. A compact quantum topological semmigroup (CQTS) is a unital LCQS, with the canonical example of the algebra $C(S)$ of continuous functions on a compact topological semigroup. The word "topological" in the term CQTS is often omitted, however it is essential: if $A \simeq C(S)$ is a commutative CQTS, then the multiplication on S induced by the comultiplication of A is jointly continuous.
5. Weakening conditions on the mutiplication, we get the class **CQSTS** of compact quantum semitopological semigroups defined by M. Daws [3]. The classical example is the algebra $A = C(S)$ where S is a compact semitopological semigroup; multiplication on S is just separately continuous, which means that $\Delta(A)$ is not in $C(S \times S) \simeq A \otimes A$ anymore. In the noncommutative case, one can define the space of separately continuous elements $A \overset{sc}{\otimes} A$ in $A^{**} \otimes A^{**}$, where A^{**} is identified with the enveloping von Neumann algebra of A. Now, a CQSTS is a unital C^*-algebra A with a comultipication $\Delta : A \to A \overset{sc}{\otimes} A$.
6. A quantum Eberlein algebra (**QEA**) [1] is a C^*-algebra A with a comultiplication $\Delta : A \to A^{**} \otimes A^{**}$ equipped with a canonical corepresentation on a Hilbert space H: as it is often done in the theory of quantum groups, this corepresentation is encoded by its "generator" $V \in B(H) \otimes A^{**}$ and is such that the map $\mu \mapsto (\mathrm{id} \otimes \mu)(V)$ from A^* to $B(H)$ is a homomorphism. It is known that every unital QEA is a CQSTS [2].
7. **QSI** is the class of quantum semigroups with involution [5]. An involution on a semigroup is a map $^* : S \to S$ such that $x^{**} = x$ and $(xy)^* = y^* x^*$ for all $x, y \in S$; if S is a group, then an involution is given by the group inverse. A classical example of a QSI is the algebra $C(S)^{**}$, where S stands for a compact semigroup with involution (there is no assumption of compactness in a general QSI however). In general, a QSI is a von Neumann algebra M with a comultiplication $\Delta : M \to M \otimes M$ and a (possibly unbounded) coinvolution $\varkappa$ on M. In the commutative case, $\varkappa$ sends a function to its composition with the involution.
8. Finally, all these classes are embedded in the class **HVNA** of Hopf–von Neumann algebras. A HVNA is a von Neumann algebra M with a comultiplication $\Delta : M \to M \otimes M$; for every C^*-algebra A in one the classes defined above (except QSI which is already defined for von Neumann algebras), its enveloping von Neumann algebra A^{**} is a HVNA.

Observing this panorama, one can notice that there is no notion of topological quantum group as such, which is not supposed to be locally compact. To work in this setting, one would need to take a quite different approach, since every commutative C^*-algebra is described as the algebra $C(X)$ on a locally compact space X.

In conclusion, let us list a few maps between the classes above.

- The duality is a functor on the category of locally compact quantum groups [4], the second dual of a LCQG $\mathbb{G}$ being isomorphic to $\mathbb{G}$.
- The quantum Bohr compactification [6] sends any locally compact semigroup to a compact quantum group and has the universal property of factoring every morphism to a CQG.
- The weakly almost periodic (WAP) compactification [3] sends any locally compact quantum group to a compact quantum semitopological semigroup, having the respective universal property.
- The duality of quantum semigroups with involution [5] sends a QSI to another QSI, which is also a LCQS. The composition of the square of this map with WAP, applied to the algebra $C_0(G)$ on a locally compact group G, equals to the enveloping von Neumann algebra of the Bohr compactification.

There are certainly other relations between topological quantum groups and semigroups, for today unexplored.

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On some recent applications of quantum Markov semigroups

Adam Skalski

(joint work with Ami Viselter)

Approximation or rigidity properties of objects such as groups, quantum groups or operator algebras are sometimes characterised in terms of the existence (or non-existence) of discrete families of maps with particular features. In favourable circumstances these discrete families can be replaced via continuous evolutions; here we report on recent work motivated by this observation.

For the purposes of this report a *quantum Markov semigroup* on a von Neumann algebra M equipped with a faithful normal semifinite weight ϕ is a pointwise weak*-continuous semigroup of unital completely positive ϕ-reducing maps $(P_t)_{t\geq 0}$ acting on M, with $P_0 = \mathrm{id}_{\mathsf{M}}$.

As an example of the situation mentioned in the first paragraph we have the following classical result (essentially due to Schönberg): a countable discrete group Γ has the Haagerup property if and only if the von Neumann algebra $\mathrm{VN}(\Gamma)$ admits a symmetric quantum Markov semigroup consisting of L^2-compact Herz-Schur multipliers. Here the desired semigroup is constructed via its *generating functional*, i.e. a conditionally positive definite function. Article [1] contains an analogous result describing the Haagerup property of an arbitrary von Neumann algebra equipped with a faithful normal state as the existence of an immediately L^2-compact KMS-symmetric quantum Markov semigroup. There however the semigroup is constructed directly, using the earlier ideas of Sauvageot, Jolissaint and Martin.

The symmetry condition appearing above is to be understood in the sense of the self-adjointness of the associated operators induced on the L^2-space; as we need to work in the non-tracial context, we have to consider how the L^2-operators arise from their von Neumann algebra counterparts; this is related to the phrase 'KMS-symmetry' appearing above and in what follows.

Let now $\mathbb{G}$ be a locally compact quantum group in the sense of Kustermans and Vaes. A *convolution semigroup of states* on $\mathbb{G}$ is a family $(\mu_t)_{t\geq 0}$ of states on the universal C*-algebra $\mathrm{C}_0^u(\mathbb{G})$ such that

(i) $\mu_{t+s} = \mu_t \star \mu_s := (\mu_t \otimes \mu_s) \circ \Delta, \quad t, s \geq 0;$

(ii) $\mu_t(a) \xrightarrow{t\to 0^+} \mu_0(a) := \epsilon(a), \quad a \in \mathrm{C}_0^u(\mathbb{G}),$

where ϵ denotes the counit of $\mathrm{C}_0^u(\mathbb{G})$.

The following theorem is one of the main results of [7]; its compact predecessor may be found in [2].

Theorem 1. *Let $\mathbb{G}$ be a locally compact quantum group, let ϕ be its left Haar weight and R^{u} denote its universal unitary antipode. There exist $1-1$ correspondences between:*

(i) *convolution semigroups $(\mu_t)_{t\geq 0}$ of R^{u}-invariant states of $\mathrm{C}_0^u(\mathbb{G})$;*

(ii) *quantum Markov semigroups $(T_t)_{t\geq 0}$ on $\mathsf{L}^\infty(\mathbb{G})$ that are KMS-symmetric with respect to ϕ and are translation invariant, i.e. $\Delta \circ T_t = (T_t \otimes \mathrm{id}) \circ \Delta$ for every $t \geq 0$;*

(iii) *completely Dirichlet forms Q on $\mathsf{L}^2(\mathbb{G})$ with respect to ϕ that are invariant under $\mathcal{U}(\mathsf{L}^\infty(\hat{\mathbb{G}})')$ (modulo multiplication of forms by a positive number).*

This characterisation was used in [7] to establish the following two theorems characterising approximation properties of locally compact quantum groups via existence (or non-existence) of quantum convolution semigroups of states – or equivalently 'translation invariant' quantum Markov semigroups on $\mathsf{L}^\infty(\mathbb{G})$ – with particular properties. Note that in the first case the suitable semigroup is obtained via constructing its Dirichlet form, whereas in the second it is built directly.

Theorem 2. *Let $\mathbb{G}$ be a second countable locally compact quantum group. Then $\hat{\mathbb{G}}$ has Property (T) of Kazhdan if and only if every convolution semigroup of R^u-invariant states on $\mathrm{C}_0^u(\mathbb{G})$ has a bounded generator.*

Theorem 3. *Let $\mathbb{G}$ be a second countable locally compact quantum group. Then $\hat{\mathbb{G}}$ has the Haagerup property if and only if there exists a convolution semigroup of R^u-invariant states on $\mathrm{C}_0^u(\mathbb{G})$ such that the KMS-implementations of the associated convolution operators (acting) on $\mathsf{L}^2(\mathbb{G})$ belong to $\mathrm{C}_0(\hat{\mathbb{G}})$.*

We should note that in the case of compact $\mathbb{G}$ variants of these results were shown respectively in [3] and in [4].

In Theorem 2 the quantum Dirichlet form Q encodes (in the semigroup theory sense) the generator of the L^2-version of the quantum Markov semigroup $(T_t)_{t\geq 0}$. On the other hand, given a quantum convolution semigroup of states we associate to it its *generating functional*, defined by the formula

$$\gamma(a) = \lim_{t\to 0^+} \frac{\mu_t(a) - \epsilon(a)}{t},$$

naturally declaring the domain of γ to be the set of these $a \in \mathrm{C}_0^u(\mathbb{G})$ for which the limit above exists. Such a generating functional is hermitian, takes non-negative values on the kernel of the counit, and vanishes at 1 (if we extend the definition to the minimal unitization of $\mathrm{C}_0^u(\mathbb{G})$). If $\mathbb{G}$ happens to be compact, generating functionals are always defined on a natural dense *-subalgebra, $\mathrm{Pol}(\mathbb{G})$ – the algebra spanned by coefficients of finite-dimensional unitary representations of $\mathbb{G}$. In the general locally compact case there is no such natural candidate for the domain of γ.

We have however the following recent result (the first statement dates back to [5]).

Theorem 4 ([8])**.** *Let $(\mu_t)_{t\geq 0_+}$ be a convolution semigroup of states on $\mathbb{G}$. Then*

(i) *its generating functional determines $(\mu_t)_{t\geq 0}$ uniquely;*
(ii) *the domain of γ contains a dense *-subalgebra of $\mathrm{C}_0^u(\mathbb{G})$.*

Conversely, assume we are given a dense unital *-subalgebra A inside the minimal unitization of $\mathrm{C}_0^u(\mathbb{G})$ and a functional $\gamma : A \to \mathbb{C}$ which is hermitian, positive on the kernel of ϵ and vanishes at 1. If A satisfies certain technical conditions, then there is a convolution semigroup of states on $\mathbb{G}$ whose generating functional extends γ.

Note that item (ii) in the above theorem in particular opens a possibility of classifying quantum Lévy processes on a locally compact quantum group $\mathbb{G}$ in the spirit of Schürmann (see [6]), and in particular of defining Gaussian processes on $\mathbb{G}$.

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Noncommutative maximal ergodic inequalities associated with doubling conditions

Simeng Wang

(joint work with Guixiang Hong, Benben Liao)

This talk focus on the maximal inequalities and ergodic theorems for group actions on noncommutative L_p-spaces. The study of individual ergodic theorems in the noncommutative setting took off with Lance's pioneering work [17] in 1976. The topic was then extensively investigated in a series of works of Conze, Dang-Ngoc, Kümmerer, Yeadon and others (see [7, 16, 28] and the references therein). Among them, Yeadon [28] obtained a maximal ergodic theorem in the preduals of semifinite von Neumann algebras. But the corresponding maximal inequalities in L_p-spaces remained open until the celebrated work of Junge and Xu [14], which established the noncommutative analogue of the Dunford-Schwartz maximal ergodic theorem. This breakthrough motivates further research on noncommutative ergodic theorems in recent years. We remark that all these works essentially remain in the class of Dunford-Schwartz operators, that is, do not go beyond Junge-Xu's setting.

On the other hand, in the classical ergodic theory, a number of significant developments related to individual ergodic theorems for group actions have been established in the past years. In particular, Breuillard [4] and Tessera [27] studied the balls in groups of polynomial growth; they proved that for any invariant metric quasi-isometric to a word metric (such as invariant Riemannian metrics on connected nilpotent Lie groups), the balls are asymptotically invariant and satisfy the doubling condition, and hence satisfy the individual ergodic theorem. This settled a long-standing problem in ergodic theory since Calderón's classical paper

[5] in 1953. Also, Lindenstrauss [18] proved the individual ergodic theorem for a tempered Følner sequences, which resolves the problem of constructing pointwise ergodic sequences on an arbitrary amenable group. We refer to the survey paper [22] for more details.

Thus it is natural to extend Junge-Xu's work to actions of more general amenable groups rather than the integer group. As in the classical case, the first natural step would be to establish the maximal ergodic theorems for doubling conditions. However, since we do not have an appropriate analogue of covering lemmas in the noncommutative setting, no significant progress has been made in this direction.

In our recent work [11] we provide a new approach to this problem. This approach is based on both classical and quantum probabilistic methods, and allows us to go beyond the class of Dunford-Schwartz operators considered by Junge and Xu.

Our main results establish the noncommutative maximal and individual ergodic theorems for ball averages under the doubling condition. Let G be a locally compact group equipped with a right Haar measure m. Let d be an invariant metric on G (we always assume that d is a measurable function on $G \times G$ and m is a Radon Borel measure with respect to (G, d)). Assume that the balls $B_r = \{g \in G : d(g, e) \leq r\}$ satisfy the *doubling condition*

$$m(B_{2r}) \leq Cm(B_r), \quad r > 0,$$

where C is a constant independent of r, and satisfy the *asymptotically invariance*, that is, for every $g \in G$,

$$\lim_{r \to \infty} \frac{m((B_r g) \triangle B_r)}{m(B_r)} = 0,$$

where $\triangle$ denotes the usual symmetric difference of subsets. We consider a von Neumann algebra $\mathcal{M}$ equipped with a normal semifinite trace τ. Let α be a continuous action of G on $\mathcal{M}$ by τ-preserving automorphisms. Note that then α naturally extends to actions on the associated *noncommutative L_p-spaces* $L_p(\mathcal{M})$ (see [25]). Let A_r be the averaging operators

$$A_r x = \frac{1}{m(B_r)} \int_{B_r} \alpha_g x dm(g), \quad x \in \mathcal{M}, r > 0.$$

Then $(A_r)_{r>0}$ is of weak type $(1, 1)$ and of strong type (p, p) for $1 < p < \infty$. Moreover for all $1 \leq p < \infty$, the sequence $(A_r x)_{r>0}$ converges almost uniformly for $x \in L_p(\mathcal{M})$. Here the notion of weak and strong type (p, p) inequalities and the notion of almost uniform convergence in the noncommutative setting has been already explained detailedly in [17, 14].

There exist a number of examples satisfying the assumptions of the above result, for which we refer to [4, 27, 22] as is quoted before. In particular, assume that G is generated by a symmetric compact subset V and is of polynomial growth. Let α be a strongly continuous action of G on $\mathcal{M}$ by τ-preserving automorphisms. Then the operators defined by

$$A_n x = \frac{1}{m(V^n)} \int_{V^n} \alpha_g x dm(g), \quad x \in \mathcal{M}, n \in \mathbb{N}$$

is of weak type $(1,1)$ and of strong type (p,p) for all $1 < p < \infty$. The sequence $(A_n x)_{n \geq 1}$ converges almost uniformly for $x \in L_p(\mathcal{M})$ for all $1 \leq p < \infty$.

The results rely on several key results obtained in our paper [11]. The subjects that we address are as follows:

i) *Noncommutative transference principles.* Our first key ingredient is a noncommutative variant of Calderón's transference principle [6, 8, 9]. More precisely, we prove that for actions by an amenable group, in order to establish the noncommutative maximal ergodic inequalities, it suffices to show the inequalities for translation actions on operator-valued functions.

ii) *Noncommutative Hardy-Littlewood maximal inequalities on metric measure spaces.* For a doubling metric measure space (X, d, μ), denote by $B(x,r)$ the ball with center x and radius r with respect to the metric d. We prove that the Hardy-Littlewood averaging operators on the $L_p(\mathcal{M})$-valued functions

$$A_r f(x) = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f d\mu, \quad f \in L_p(X; L_p(\mathcal{M})), x \in X, r > 0$$

satisfy the weak type $(1,1)$ and strong type (p,p) inequalities. We remark that the classical argument via covering lemmas does not seem to fit into this operator-valued setting. Instead, our approach is based on the study of random dyadic systems by Naor, Tao [24] and Hytonen, Kairema [10], and is inspired by Mei's famous work [19, 20] which asserts that the usual continuous BMO space is the intersection of several dyadic BMO spaces.

iii) *Domination by Markov operators.* For a group G of polynomial growth with a symmetric compact generating subset $V \subset G$, and for an action α of G, there exists a constant c such that

$$\frac{1}{m(V^n)} \int_{V^n} \alpha_g x dm(g) \leq \frac{c}{n^2} \sum_{k=1}^{2n^2} T^k x, \quad x \geq 0,$$

where $T = \frac{1}{m(V)} \int_V \alpha_g dm(g)$. Our approach is new and the construction follows easily from some typical Markov chains on these groups studied by Hebisch and Saloff-Coste [12].

iv) *Individual ergodic theorems for L_p representations.* In the classical setting, the individual ergodic theorem holds for positive contractions on L_p-spaces with one fixed $p \in (1, \infty)$ ([13, 2]). The results can be also generalized for positive power-bounded operators and more general Lamperti operators (see for example [21, 15, 26]). However in the noncommutative setting, the individual ergodic theorems on L_p-spaces were only known for operators which can be extended to $L_1 + L_\infty$. In our work we will develop some new methods to prove the individual ergodic theorems for operators on one fixed L_p-space.

We remark that apart from the above approach, we also have an alternative proof of the main result for discrete groups of polynomial growth. Compared to the previous approach, this proof is much more group-theoretical and has its own interests. It relies essentially on the concrete structure of groups of polynomial growth discovered by Bass, Gromov and Wolf.

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About noncommutaive Khintchine inequalities in $L_{1,\infty}$ and $L_{2,\infty}$

Léonard Cadilhac

Noncommutative Khintchine inequalities were introduced by F. Lust-Piquard in [2]. They give an equivalent of the p-norm of the sum of Rademacher variables with coeficients in a noncommutative integration space. The precise statement is as follows. Let $\mathcal{M}$ be a von Neumann algebra equipped with a semifinite positive faithful normal trace τ. Let $(\varepsilon_i)_{i\geq 0}$ be a sequence of rademacher variables and $x = (x_n)_{n\geq 0}$ a finite sequence of elements in $L_p(\mathcal{M})$. Then:

$$\Big\| \sum_{i\geq 0} x_i \otimes \varepsilon_i \Big\|_p \approx \begin{cases} \|x\|_{R+C(p)} & \text{if } p \leq 2 \\ \|x\|_{R\cap C(p)} & \text{if } p \geq 2 \end{cases}$$

where:

$$\|x\|_{R+C(p)} = \inf\{\Big\| \sum_{i\geq 0} y_i^* y_i \Big\|_p + \Big\| \sum_{i\geq 0} z_i z_i^* \Big\|_p : y + z = x\},$$

and:

$$\|x\|_{R\cap C(p)} = \max(\Big\| \sum_{i\geq 0} x_i^* x_i \Big\|_p, \Big\| \sum_{i\geq 0} x_i x_i^* \Big\|_p).$$

It was proven by Lust-Piquard for $1 < p < \infty$ and then by Lust-Piquard and Pisier for $p = 1$ ([2],[4]).

We can go further and ask, for any symmetric quasi-norm $\|.\|_X$ and any sequence of random variables $(\xi_i)_{i\geq 0}$, whether $\| \sum_{i\geq 0} x_i \otimes \xi_i \|_X \approx \|x\|_{R+C(X)}$ or $\| \sum_{i\geq 0} x_i \otimes \xi_i \|_X \approx \|x\|_{R\cap C(X)}$. This problem has been studied for more than three decades under different forms and is motivated by operator space theory, noncommutative harmonic analysis and free probrability. One of the most striking results that falls into that category is Haagerup's inequality which answers the case $X = \infty$ when the ξ_i are random Haar unitaries. Articles by Pisier and then Pisier and Ricard answer the case $X = p < 1$ with few conditions on the variables (see [3]). However $X = 1, \infty$ and $X = 2, \infty$ remained open. Indeed, known interpolation techniques could not be used to deal with them due to their particular positions in the interpolation scale.

Our first result is a negative one. Indeed, by exhibiting two counterexamples we obtain the following proposition.

Proposition 1. *Neither:*

$$\Big\|\sum_{i\geq 0} x_i \otimes \xi_i\Big\|_{2,\infty} \approx \|x\|_{R+C(2,\infty)}$$

nor:

$$\Big\|\sum_{i\geq 0} x_i \otimes \xi_i\Big\|_{2,\infty} \approx \|x\|_{R\cap C(2,\infty)}.$$

Then, we introduce the notion of an optimal decomposition. An optimal decomposition of x for p is a couple y, z such that $x = y+z$ and $\|\sum_{i\geq 0} y_i^* y_i\|_p^p + \|\sum_{i\geq 0} z_i z_i^*\|_p^p$ is minimal. For the remainder of this summary, the ξ_i are assumed to be free Haar unitaries.

Theorem 2. *For any $p \leq 2$, there exists an absolute constant c_p such that if y, z is an optimal decomposition of x for p then for any $q \in (p, \infty)$ and $r \in (0, \infty]$:*

$$\Big\|\sum_{i\geq 0} y_i^* y_i\Big\|_{q,r} + \Big\|\sum_{i\geq 0} z_i z_i^*\Big\|_{q,r} \leq c_p \Big\|\sum_{i\geq 0} x_i \otimes \xi_i\Big\|_{q,r}.$$

The proof of this theorem is short and based on the idea already present in [1] of using commuting projections together with the Khintchine inequality for some fixed p to obtain some more powerful information. In [1], it is done for $p = \infty$ and gives a control of the generalised singular value, here we do it for $p \leq 2$ and get an estimate of the K-functional from which the theorem follows directly. As a consequence, we can prove the Khintchine inequalities in $L_{1,\infty}$:

Corollary 3. *We have:*

$$\Big\|\sum_{i\geq 0} x_i \otimes \xi_i\Big\|_{1,\infty} \approx \|x\|_{R+C(1,\infty)}.$$

One inequality was already known by a conditional expextation argument and for the other one, it suffices to apply the theorem above whith $p < 1$. This implies to use the main result of [3].

In the case $p = 1$, we can go further.

Theorem 4. *Assume that $\mathcal{M}$ is a matrix algebra and that the x_i are self-adjoint. Then there exists $\alpha \in \mathcal{M}^+$ and $u = (u_i)_{i\geq 0}$ such that:*

- *for all $i \geq 0$, $u_i = u_i^*$,*
- $\sum_{i\geq 0} u_i^2 = 1$,
- *for all $i \geq 0$, $x_i = u_i \alpha + \alpha u_i$.*

Furthermore, for such α and u, $u\alpha, \alpha u$ is an optimal decomposition of x for $p = 1$ and for any $q \in (0, \infty)$ and $r \in 0, \infty$:

$$\|\alpha\|_{q,r} \approx \Big\|\sum_{i\geq 0} x_i \otimes \xi_i\Big\|_{q,r}.$$

In particular, this yields an equivalent of $\big\| \sum_{i\geq 0} x_i \otimes \xi_i \big\|_{2,\infty}$ which is, however, not explicit. It gives a compact proof of Khinctchine inequalities for at least all Lorentz spaces. It is based on an inequality following from the main result of [6] and on a short differentiation argument.

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Bimonotone quantum Lévy processes

Malte Gerhold

Inspired by Voiculescu's bifreeness [9], we define bimonotone independence, which is another example of a universal independence for pairs of noncommutative random variables. By universal independence we mean that there is an underlying product construction for distributions of noncommutative random variables fulfilling the axioms of [7] for a unital associative universal product such that independence is equivalent to factorization of the joint distribution as the product of its marginals. Recently, more such independences for pairs have been studied, e.g. by Gu, Skoufranis and Hasebe [5] or Liu [6].

In the talk, which is based on [3], we define the *bimonotone product* of states on augmented $*$-algebras A with a given free product decomposition $A = A^\ell \sqcup A^r$ with $*$-subalgebras A^ℓ, A^r referred to as left face and right face of A respectively. In fact, we first define a corresponding product of representations, which simplifies the proof of associativity (this was observed by Franz for the monotone product in [2]). This general setting in particular allows us to define bimonotone independence for pairs of noncommutative random variables. We present a combinatorial rule how to calculate mixed moments of such bimonotonely independent pairs using the notion of *bimonotone partitions*. By an application of a general central limit theorem of Accardi, Hashimoto, and Obata [1, Lemma 2.4] to a bimonotone-iid sequence, we derive a bimonotone central limit theorem and see that the moments of the central limit distribution can be described by bimonotone pair partitions.

Monotone and antimonotone Brownian motion have a standard realization as sums of creation and annihiliation operators $\lambda_t^* + \lambda_t, \rho_t^* + \rho_t$ on *monotone Fock space* [8]. Using the fact that these operator processes, considered as a process of pairs, have bi-monotonely independent and stationary increments in the vacuum state, their joint distribution is seen to be a bimonotone central limit distribution. In particular, we can thus determine the distribution of the selfadjoint operators $\lambda_t^* + \lambda_t + \rho_t^* + \rho_t$ and see that they are not arcsine distributed, in contrast to the monotone or antimonotone operators alone. It is, however, an open problem to give a concrete description of the corresponding probability measures.

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Product estimates in function spaces on quantum tori and quantum Euclidean spaces

Guixiang Hong

Starting with a general question: Given a metric measure space, for which kinds of function spaces X, Y, Z there holds

$$X \cdot Y \hookrightarrow Z?$$

On a measure space, the Hölder inequalities say that $L_p \cdot L_q \hookrightarrow L_r$ whenever $0 < p, q, r \leq \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, which are very useful in analysis. On some "nice" measure spaces, for instance Euclidean spaces or domains on Euclidean spaces, product estimates in function spaces with regularity data such as Sobolev spaces, Besov spaces, Triebel-Lizorkin spaces etc, play important roles in perturbation theory, the theory of partial differential equations and geometric analysis. We refer the reader to [5] and references therein for the product estimates and their applications in the commutative setting.

It is well-known that on a noncommutative measure space $(\mathcal{M}, \tau)$—a von Neumann algebra equipped with a normal semi-finite faithful (tracial) state, the Hölder inequalities are also true, whose proof require completely new idea and techniques. And this report is devoted to the presentation of the product estimates in functions spaces involving regularity information on the “nice” noncommutative measure spaces–quantum Euclidean spaces and quantum tori.

Since 2013, Xu et al initiated the research of noncommutative harmonic analysis on quantum tori: Chen, Xu and Yin [1] first studied various approximation properties in terms of different means of summation; Xia, Xiong, Xu and Yin [8] [9] introduced Sobolev spaces, Besov spaces and Triebel-Lizorkin spaces on quantum tori and investigated the embedding properties and interpolation properties of these spaces etc; along this research line, Xia and Xiong [6] [7] introduced local Hardy space on quantum tori and showed the mapping properties of some pseudo-differential operators. On the other hand, very recently motivated by quantum groups, Gonzalez, Junge and Parcet [2] cleverly introduced the diagonal on quantum Euclidean spaces, provided an appropriate notion of regularity condition of kernels of singular integrals, and gave a criteria of L_p-boundedness of Calderón-Zygmund operators. This theory of Calderón-Zygmund on quantum Euclidean spaces finds applications to pseudo-differential calculus in the same paper [2].

Aiming to apply the theory of noncommutative function spaces to perturbation theory, PDEs and noncommutative geometry, we deal with the product estimates in function spaces on quantum Euclidean spaces/tori. We refer the reader to the above references for better understanding what we stated below.

Let us first recall the definition of quantum Euclidean spaces or tori. Let Θ be an anti-symmetric real $n \times n$ matrix $\Theta^t = -\Theta$. Let $\{u_j(s) : \ 1 \leq j \leq n, \ s \in \mathbb{R}\}$ be a sequence of unitary groups satisfying

$$u_j(s)u_k(t) = e^{2\pi i \Theta_{jk} st} u_k(t)u_j(s).$$

Set $\lambda_\Theta(\xi) = u_1(\xi_1) \cdots u_n(\xi_n)$ for $\xi \in \mathbb{R}^n$ and for $f \in C_c(\mathbb{R}^n)$,

$$\lambda_\Theta(f) = \int_{\mathbb{R}^n} f(\xi)\lambda_\Theta(\xi) d\xi.$$

Define $\mathbb{R}_\Theta$ to be the von Neumann algebra generated by $\{\lambda_\Theta(f) : \ f \in C_c(\mathbb{R}^n)\}$. The canonical trace is defined as $\tau_\Theta(\lambda_\Theta(f)) = f(0)$ for $f \in C_c(\mathbb{R}^n)$. Then $(\mathbb{R}_\Theta, \tau_\Theta)$ is a noncommutative measure space, called a quantum Euclidean space. Replace $\mathbb{R}$ in $s \in \mathbb{R}$ with $\mathbb{Z}$, we get the definition of quantum tori $(\mathbb{T}_\Theta, \tau_\Theta)$. The product estimates hold true on both quantum Euclidean spaces and quantum tori, and we will state our results only in the setting of quantum Euclidean spaces. The whole picture of all the product estimates is quite involved, being divided into many cases according to different function spaces and the scales of parameters involved. To illustrate the main results and to explain briefly some key ideas, we will state one instance—product estimates in Triebel-Lizorkin spaces with constant p.

The Triebel-Lizorkin spaces are defined as follows. We first recall the Schwartz class and the distributions on quantum Euclidean spaces:

$$\mathcal{S}_\Theta = \lambda_\Theta(\mathcal{S}(\mathbb{R}^n)), \ \mathcal{S}'_\Theta = (\mathcal{S}_\Theta)^*$$

where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz class on $\mathbb{R}^n$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\varphi = 1$ on $\{|\xi| \le 1\}$ and $\varphi = 0$ on $\{|\xi| \ge 3\}$. Let $\psi(\cdot) = \varphi(\cdot) - \varphi(2\cdot)$. Then $\forall \xi \ne 0$,

$$\varphi(\xi) + \sum_{k=1}^{\infty} \psi(2^{-k}\xi) = 1.$$

For $x \in \mathcal{S}'(\mathbb{R}_\Theta)$, we define $S_0 x = (\varphi \hat{x})^\vee$ and $\forall k \ge 1$

$$S_k x = (\psi(2^{-k}\cdot)\hat{x})^\vee, \; S^k x = \sum_{\ell=0}^{k} S_\ell(x) \; .$$

Definition 1. Let $1 \le p < \infty$ and $\alpha \in \mathbb{R}$.

(i) The column Triebel-Lizorkin space $F_p^{\alpha,c}$ is defined by

$$F_p^{\alpha,c} \;=\; \{x \in \mathcal{S}'_\Theta : \; \|x\|_{F_p^{\alpha,c}} := \|(2^{k\alpha} S_k x)_{k \ge 0}\|_{L_p(\ell_2^c)} < \infty\}.$$

(ii) The row space $F_p^{\alpha,r} = \{x \in \mathcal{S}'_\Theta : \; \|x\|_{F_p^{\alpha,r}} := \|x^*\|_{F_p^{\alpha,c}} < \infty\}$.

(iii) The mixture space F_p^α is defined to be

$$F_p^\alpha \;=\; \begin{cases} F_p^{\alpha,c} + F_p^{\alpha,r}, & \text{if } 1 \le p < 2; \\ F_p^{\alpha,c} \cap F_p^{\alpha,r}, & \text{if } 2 \le p < \infty, \end{cases}$$

equipped with the natural norms.

As in the classical setting, the definition of these spaces are shown to be independent of the choice of φ. However, it is worth to mention that due to the noncommutativity, classical real variable approach does not work well and the proof of this independence requires new idea, see for instance [8] [9] for the case of quantum tori. A consequence of this independence is a noncommutative version of the so-called Nikol'skji representations, which is particularly useful in showing the product estimates.

The product estimates we choose to state in the present report are the following ones.

Theorem 2. *Let $2 \le p < \infty$, $\alpha \le \alpha_1 \le \alpha_2$. Then*

$$F_p^{\alpha_1} \cdot F_p^{\alpha_2} \hookrightarrow F_p^\alpha$$

provided that the parameters satisfy one of the following conditions:

(i) $\alpha < \alpha_1, \alpha_2$; $0 < \alpha_1 + \alpha_2$ and $\alpha < \alpha_1 + \alpha_2 - n/p$;

(ii) $\alpha = \alpha_1$ or α_2; $0 < \alpha_1 + \alpha_2$ and $\alpha < \alpha_1 + \alpha_2 - n/p$;

(iii) $\alpha < \alpha_1, \alpha_2$; $0 < \alpha_1 + \alpha_2$ and $\alpha = \alpha_1 + \alpha_2 - n/p$.

The proof of this result is by no means through a trivial adaption of the classical approach. Let us explain briefly. As far as the author know, in all the existed proof of the product estimates in the commutative setting, the first step is the Bony paraproduct decomposition:

$$xy = \Pi_1(x, y) + \Pi_2(x, y) + \Pi_3(x, y)$$

where

$$\Pi_1(x,y) = \sum_{k=2}^{\infty} S^{k-2}xS_k y - -\text{Low} - \text{High interaction},$$

$$\Pi_2(x,y) = \sum_{k=2}^{\infty} S_k x S^{k-2}y, - - \text{ High} - \text{Low interaction}$$

and

$$\Pi_3(x,y) = \sum_{k=0}^{\infty}\sum_{\ell=k-1}^{k+1} S_\ell x S_k y - -\text{High} - \text{High interaction}.$$

Then it suffices to deal with the three paraproducts respectively. However, in the noncommutative setting, one would be "scared" of dealing with paraproducts since the behavior of paraproduct could be quite different since it has been shown that the following operator-valued paraproduct estimate (similar to the High-Low interaction) is not true,

$$\|\sum_k d_k x E_{k-1} y\|_2 \leq \|x\|_\infty \|y|\|_2.$$

This failure is due to many authors such as Mei, Nazarov, Pisier, Treil and Volberg, see for instance [3] [4] and references therein.

On the other hand, in order to separate x and y in the paraproducts, one tends to appeal to the vector-valued Hölder inequalities. However the following classical inequalities do not admit noncommutative analogues:

$$\|(\sum_k |S^{k-2}xS_k y|^2)^{1/2}\|_p \leq \|\sup_k |S^{k-2}x|\|_{p_1} \|(\sum_k |S_k y|^2)^{1/2}\|_{p_2},$$

$$\|(\sum_k |S_k x S^{k-2} y|^2)^{1/2}\|_p \leq \|(\sum_k |S_k x|^2)^{1/2}\|_{p_1} \|\sup_k |S^{k-2}y|\|_{p_2}.$$

In order to overcome these difficulties, the first observation is that the regularity of the underlying function enables us to turn high frequency into low frequency, which is helpful to avoid the previous false paraproduct estimate. This is a new observation even in the commutative setting. The second one is some sharp embedding properties between Triebel-Lizorkin spaces and Besov spaces (where one can use the usual Hölder inequalities), which are new even in the case of quantum tori. The third one is a noncommutative version of Bernstein's inequalities which are very useful in dealing with low frequency. The proof of these sharp embedding properties is also based on Bernstein's inequalities, which is different from the approach used in [9]. It is also worth to mention that the noncommutative Bernstein's inequalities follow from a $*$-homomorphism between quantum Euclidean spaces and operator space theory, which is by no means a trivial adaption of the classical arguments.

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Jordan operator algebras, their operator space structure and noncommutative topology

David P. Blecher

(joint work with Matthew Neal, Zhenhua Wang)

The structure of this lecture, and its key points, were as follows (listed by sections of the talk):

(1) There exists a theory of Jordan operator algebras, that is norm closed subspaces of C^*-algebras which are closed under taking squares (or equivalently the Jordan product). This is joint work with Zhenhua Wang, 2016–2017.

(2) and (3) In these sections we show that there exists a fusion of the C^*-algebraic noncommutative topology of Akemann, L. G. Brown, Pedersen, etc, with the classical theory of peak sets and peak interpolation from the field of function algebras, in the setting of (associative) subalgebras of C^*-algebras. This is older work of the speaker, Charles Read, Neal, and D. Hay.

(4) (Together with M. Neal)

- We give new variants of technical C^*-algebraic approximation results of L. G. Brown related to $L + L^*$ and $L \cap L^*$ for a left ideal L in a C^*-algebra.
- This is then used in deep operator space variants of some classical peak interpolation lemmas.
- This in turn leads to a breakthrough in the noncommutative topology and hereditary subalgebra (HSA) theory of Jordan operator algebras: namely plugging the latter into the rich and important noncommutative topology and HSA theory of C^*-algebras.

- In turn this has a flood of consequences, in particular the generalization of everything in Sections (2) and (3) above to Jordan operator algebras. That is, Jordan operator algebras have a full/complete noncommutative topology theory.

The operator space aspects were emphasized.

We now amplify the above points along the lines in the lecture. In Section 1 we defined Jordan operator algebras and described a few aspects of the recent basic theory of these objects (joint with Wang). Jordan operator algebras form an interesting class of operator spaces, one which seems to have almost had no study until now (besides an old paper of Arazy and Solel, which did not address the operator space structure/matrix norms). Yet there are very many examples, some of which we exhibited in the talk. We have shown that there exists a large theory of such Jordan operator algebras; they are far more similar to associative operator algebras than was suspected. For example we mentioned two abstract operator space characterizations of Jordan operator algebras (one joint with Neal). As an immediate application of these characterizations one can show that the range of a completely contractive projection P on a Jordan operator algebra, with product $P(xy)$, is (completely isometrically Jordan isomorphic to) a Jordan operator algebra. We also discussed in this section several natural possible definitions of 'approximate identity' in a Jordan operator algebra, and showed that the existence of a net satisfying these possible definitions are all equivalent. We also discussed unitization of Jordan operator algebras, and stated some results and open problems (one of which we solved shortly afterwards, a by-product of thinking about one of the questions from the audience after the talk).

In Section 2 we described the noncommutative C^*-algebraic topology of Akemann, L. G. Brown, Pedersen, and others. In particular we discussed open, closed, and compact projections for a C^*-algebra B and their relation to hereditary subalgebras (HSA?s) in B. We also discussed some of L. G. Brown's technical results from his 124 page Canadian J paper, some of a 'noncommutative Tietze' flavor, for example lifting elements via the canonical quotient map $B \to B/J$, where J might possibly be $L, L + L^*, D = L \cap L^*$, etc, for a left ideal L. He showed for example that L and $L + L^*$ are proximinal in B, but that unfortunately the HSA $D = L \cap L^*$ is not proximinal in B. We later give a positive result in this direction: with Neal we proved that every HSA $D = L \cap L^*$ is proximinal in $L + L^*$, which will turn out to have profound consequences for our noncommutative topology. In Section 3 we discussed our fusion of Akemann's noncommutative topology with the classical theory of peak sets, generalized peak sets, peak interpolation, etc, for function algebras. The latter topics are crucial tools for studying classical algebras of functions. We defined the basic objects, surveyed some of the major results in the classical case, and showed how they generalize to associative operator algebras. Our generalization is a simultaneous generalization of the above noncommutative C^*-algebraic topology and the classical peaking theory for function algebras. A deep theorem of Hay connects noncommutative peak sets for unital subalgebras A of a C^*-algebra B, and more generally projections in A^{**} that are open relative to

A, to open projections in the C^*-algebraic sense with respect to B. This is crucial to the noncommutative topology of A since it hooks it into the powerful existing noncommutative C^*-algebraic topology for B. This theorem also allowed the speaker, Hay and Neal to develop the hereditary subalgebra theory for associative operator algebras A.

In Section 4 of our talk (longer, and joint work with Matt Neal) we generalize all of the last paragraph to a Jordan operator algebra A inside a C^*-algebra B. Our breakthrough relies on first establishing some variants of technical C^*-algebraic approximation results of Brown, many of which were mentioned above. Key is the proximinality of every HSA $D = L \cap L^*$ in $L + L^*$ mentioned above. These are used in deep operator space variants of some classical peak interpolation lemmas, some of which give for certain closed projections q with respect to B, a completely isometric isomorphism from A/J, where $J = \{x \in A : x = q^\perp x q^\perp\}$, onto a canonical subspace of B^{**} defined by q. This operator space result is used to prove (together with other arguments, including operator space arguments) the Jordan algebra variant of Hay's theorem mentioned above: A projection p in $A^{\perp\perp}$ is open with respect to B iff p is open 'relative to' A, and then $D = \{a \in A : pap = a\}$ is a hereditary subalgebra of A with support projection p. This is the breakthrough needed to enable noncommutative topology for Jordan operator algebras. And it enables a hereditary subalgebra theory for A, linking them to C^*-algebraic hereditary subalgebras in B.

Indeed there is a flood of consequences, in particular the generalization of everything in Sections (2) and (3) above to Jordan operator algebras, but for the deep reasons we mentioned (i.e. ultimately relying on our breakthrough above). One then goes through 7 or 8 of our papers on noncommutative topology in the associative case, many dozens of results, checking that they all work with suitable Jordan modifications of proofs, and using above breakthrough. The upshot is that Jordan operator algebras have a full noncommutative topology and hereditary subalgebra theory, quite as complete as for (associative) operator algebras.

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Strong property (T) for SL(n, Z)

Mikael de la Salle

The objects in the study of strong property (T) are (not necessarily isometric) representations of groups on Banach spaces, that is group morphisms $\pi\colon G \to \mathrm{GL}(X)$ from a topological group G to the group of invertible linear operators on a Banach space. One only considers locally compact compactly generated groups and actions which are strongly continuous, that is such that for every $x \in X$ the orbit map $g \mapsto \pi(g)x$ is continuous. In that case, $\|\pi(g)\|_{B(X)}$ is bounded on every compact subset of G, and if $\ell\colon G \to \mathbf{N}$ is the word-length function with respect to some compact generating set of G, there is a constant C and $s > 0$ such that $\|\pi(g)\| \leq Ce^{s\ell(g)}$ for every $g \in G$. The infimum of all such s, that we denote by s_π, is the *exponential growth rate* of π.

To motivate the study of non isometric representations, we start with a short list of interesting examples with $s_\pi = 0$:

- Assume that G acts by isometries on a (real) Banach space X. By Mazur's theorem, every isometry of X is affine, so the action can be written as $g \cdot x = \pi(g)x + b(g)$ for a linear isometric representation π of G on X and a translation part (cocycle) $b\colon G \to X$. Then the representation on $X \oplus \mathbf{C}$ given by $\widetilde{\pi}(g) = \begin{pmatrix} \pi(g) & b(g) \\ 0 & 1 \end{pmatrix}$ is a linear representation, and $\|\widetilde{\pi}(g)\| \leq 1 + \|b(g)\|$. In particular $\widetilde{\pi}$ is uniformly bounded if and only if the original action had bounded orbits.
- Let $\mathcal{G} = (V, E)$ be a hyperbolic graph with bounded degree. Then for every action of G by isometries on $\mathcal{G}$, Lafforgue [6] constructs a Hilbert space norm on $\mathbf{C}^{(V)}$, such that (1) the natural action of G on $\mathbf{C}^{(V)}$ extends to a representation π on the completion X satisfying $\|\pi(g)\| \leq C(1+\ell(g))^2$ (2) there is a nonzero invariant linear form on X and (3) there is no nonzero invariant vector in X.
- Consider an action of G by C^∞ diffeomorphisms on a on a compact Riemanian manifold M. Assume that the action satisfies the tameness condition

$$\limsup_{\ell(g)\to\infty} \sup_{x\in M} \|D_x g\| = 0. \tag{1}$$

  Then [4, 2], for every $k \geq 1, p \in [1, \infty]$ the associated representation of G on the Sobolev space $W^{k,p}(M)$ satisfies $s_\pi = 0$.

Strong property (T) is a rigidity property for representations will small enough exponential growth rate.

Definition 1. G has strong property (T) if there exists $s_0 > 0$ such that, for every representation of G on a Hilbert space satisfying $s_\pi \leq s_0$, there is a sequence μ_n of compactly supported probability measures on G such that $\pi(\mu_n) = \int \pi(g) d\mu_n(g)$ converges, for the norm of $B(X)$, to a projection on the space of invariant vectors.

By allowing representations on classes of Banach spaces, we obtain Banach space versions on strong property (T) which are also worth investigating.

By applying the definition of strong property (T) to the preceding examples, one gets the following consequences of strong property (T) (with respect to some Banach spaces $\mathcal{E}$):

- [6] Every action by isometries on a space X satisfying $X \oplus \mathbf{C} \in \mathcal{E}$ has a fixed point.
- [6] Every action by isometries on a bounded degree hyperbolic graph has a fixed point.

- [2] A C^∞ action of G on a manifold preserves a Riemanian metric if and only if it satisfies the tameness (1).

In particular, Strong property (T) is one of the key ingredients in the recent proof of (many cases of) Zimmer's conjecture for actions of higher rank lattices on small dimensional manifolds [2, 3].

So far, strong property (T) was known for connected higher rank simple Lie groups [6, 5], connected higher rank simple over non-archimedean local fields [6, 7], as well as there cocompact lattices. The case of the non cocompact lattices (for example $\mathrm{SL}_3(\mathbf{Z})$) has remained open for the following reason. The natural approach to deduce strong (T) for a lattice Γ from strong (T) for G is through *induction of representations*. That is, given a representation π of Γ on X, one considers the space $\widetilde{X}$ of functions $f\colon G \to X$ satisfying $f(g\gamma) = \pi(\gamma^{-1})f(g)$ for every $g \in G$, on which G acts by left-translation. The problem is to find a norm on $\widetilde{X}$ for which this action is by bounded operators. When the original representation is uniformly bounded or when Γ is cocompact, the norm coming from $L^2(\Omega; X)$ (for Ω a fundamental domain, taken relatively compact in the second case) is fine. This is the proof that property (T) passes to lattices and that strong (T) passes to cocompact lattices. In the general case, there is no reason why the representation would be by bounded operators.

The solution to this obstacle was found only recently [9]. The starting observation is that, in the case of $\mathrm{SL}_n(\mathbf{Z}) \subset \mathrm{SL}_n(\mathbf{R})$ (or more generally a higher rank lattice; here we use [8]), if s_π is small enough, the representation maps boundedly $L^2(\Omega; X)$ into $L^1(\Omega; X)$, and also $L^\infty(\Omega; X)$ into $L^2(\Omega; X)$. More generally, for every $\varepsilon > 0$, if s_π is small enough, the representation maps $L^p(\Omega; X)$ to $L^q(\Omega; X)$ whenever $\frac{1}{q} - \frac{1}{p} \geq \varepsilon$. So we are indeed dealing with bounded operators, but not on a fixed Banach space; each application of G decreases the integrability, which means that we are only allowed to use composition in the group a bounded $(1/\varepsilon - 1)$ number of times. It turns out that a form of strong property (T) also holds for this kind of "representations" of higher rank groups, where one is only allowed to compose once. And this property passes to lattices in higher rank groups. This leads to

Theorem 2. [9] $\mathrm{SL}_n(\mathbf{Z})$ *has strong property (T) for* $n \geq 3$.

The same theorem holds for every higher rank lattices, for example the non cocompact $\mathrm{SL}_n(\mathbf{Z}_p)$, $\mathrm{SL}_n(\mathbf{F}_p[t])$ and $\mathrm{SL}_n(\mathbf{Z}[\frac{1}{p}])$ for $n \geq 3$, or $\mathrm{Sp}_{2n}(\mathbf{Z})$, $\widetilde{\mathrm{Sp}}_{2n}(\mathbf{Z})$ (the preimage of $\mathrm{Sp}_{2n}(\mathbf{Z})$ in the universal cover of $\mathrm{Sp}_{2n}(\mathbf{R})$) and $\mathrm{Sp}_{2n}(\mathbf{F}_p[X])$ for $n \geq 2$.

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Normal idempotent states on a locally compact quantum group

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(joint work with Paweł Kasprzak)

Let G be a locally compact group. The Banach space $\mathrm{C}_0(G)^*$ is equipped with a natural structure of a Banach algebra with product given by convolution

$$\mathrm{C}_0(G)^* \times \mathrm{C}_0(G)^* \ni (\omega,\mu) \longmapsto \omega * \mu = (\omega\otimes\mu)\circ\Delta \in \mathrm{C}_0(G)^*$$

where Δ is the map dualizing the product in G:

$$\big(\Delta(f)\big)(g,h) = f(gh), \qquad f\in \mathrm{C}_0(G),\ g,h\in G.$$

By the classical result of Kawada and Itô ([5, Theorem 3]) the elements ω which are positive, normalized and idempotent (i.e. $\omega*\omega=\omega$) are given by integration with respect to the Haar measure of a compact subgroup K of G.

Now let $\mathbb{G}$ be a locally compact quantum group ([7]). Idempotent states in that context were investigated by many authors, see e.g. [3, 9]. For technical reasons instead of the C^*-algebra $\mathrm{C}_0(\mathbb{G})$ it is preferable to work with its *universal version* $\mathrm{C}_0^{\mathrm{u}}(\mathbb{G})$ (see [6]) and consider states $\omega\in\mathrm{C}_0^{\mathrm{u}}(\mathbb{G})^*$ idempotent with respect to the natural convolution. It was first pointed out by Pal in [8] that not every such ω is of the form $\omega=\psi_{\mathbb{K}}\circ\pi$, where $\mathbb{K}$ is a compact quantum subgroup of $\mathbb{G}$, $\psi_{\mathbb{K}}$ its Haar measure and $\pi:\mathrm{C}_0^{\mathrm{u}}(\mathbb{G})\twoheadrightarrow\mathrm{C}^{\mathrm{u}}(\mathbb{K})$ is the epimorphism corresponding to inclusion of $\mathbb{K}$ into $\mathbb{G}$ (cf. [1]). Nevertheless it proves a worthwhile exercise to consider an idempotent state $\omega\in\mathrm{C}_0^{\mathrm{u}}(\mathbb{G})$ as corresponding to a "subgroup-like" object of $\mathbb{G}$ which we choose to call a *compact quasi-subgroup*.

We let $\mathrm{Idem}(\mathbb{G})$ be the set of all idempotent states on $\mathrm{C}_0^{\mathrm{u}}(\mathbb{G})$ and $\mathrm{Idem}_0(\mathbb{G}) = \mathrm{Idem}(\mathbb{G})\cup\{0\}$.

1. Order and operations on $\mathrm{Idem}(\mathbb{G})$

There is a natural order on $\mathrm{Idem}(\mathbb{G})$ defined by declaring $\omega\preccurlyeq\mu$ if $\omega*\mu=\mu$. In case of states arising as Haar measures of compact quantum subgroups, this order corresponds to the natural order on the subgroups. We can also extend the order to $\mathrm{Idem}_0(\mathbb{G})$ by declaring 0 to be the largest element. We have

Theorem 1. *Given $\omega,\mu\in\mathrm{Idem}(\mathbb{G})$ there exist*

$$\sup\big\{\nu\in\mathrm{Idem}(\mathbb{G})\,\big|\,\nu\preccurlyeq\omega,\ \nu\preccurlyeq\mu\big\} \quad \textit{and} \quad \inf\big\{\nu\in\mathrm{Idem}_0(\mathbb{G})\,\big|\,\omega\preccurlyeq\nu,\ \mu\preccurlyeq\nu\big\}.$$

The proof of Theorem 1 is based on a bijective correspondence between idempotent states ω and integrable coideals $\mathsf{N} \subset L^\infty(\mathbb{G})$ invariant under the scaling group of $\mathbb{G}$ described in [2]. A state ω corresponds to

$$\mathsf{N}_\omega = \{\omega * x \,|\, x \in L^\infty\},$$

where $\omega * x$ is the convolution of $x \in L^\infty(\mathbb{G})$ with ω ($\omega * x = (\mathrm{id}\otimes\omega)\big(\mathbb{W}(x\otimes\mathbb{1})\mathbb{W}^*\big)$, cf. [6, 4]). Under this correspondence we have

$$\mathsf{N}_{\omega\wedge\mu} = \mathsf{N}_\omega \vee \mathsf{N}_\mu, \quad \mathsf{N}_{\omega\vee\mu} = \mathsf{N}_\omega \cap \mathsf{N}_\mu$$

with $\omega \vee \mu = 0$ if and only if the coideal $\mathsf{N}_\omega \cap \mathsf{N}_\mu$ is not integrable.

Definition 2. For $\omega, \mu \in \mathrm{Idem}(\mathbb{G})$ we define

$$\omega \wedge \mu = \sup\{\nu \in \mathrm{Idem}(\mathbb{G}) \,|\, \nu \preccurlyeq \omega,\ \nu \preccurlyeq \mu\},$$
$$\omega \vee \mu = \inf\{\nu \in \mathrm{Idem}_0(\mathbb{G}) \,|\, \omega \preccurlyeq \nu,\ \mu \preccurlyeq \nu\}.$$

Furthermore we extend the operation $\vee$ to $\mathrm{Idem}_0(\mathbb{G})\times\mathrm{Idem}_0(\mathbb{G})$ by putting $\omega\vee\mu = 0$ if either (or both) elements are zero.

It is easy to see that $(\omega,\mu) \mapsto \omega \wedge \mu$ and $(\omega,\nu) \mapsto \omega \vee \mu$ are commutative and associative operations on $\mathrm{Idem}(\mathbb{G})$ and $\mathrm{Idem}_0(\mathbb{G})$ respectively. One can show that for states arising from Haar measures of compact quantum subgroups the operations $\wedge$ and $\vee$ correspond to the operation of intersection and the operation of passing to the subgroup generated by two subgroups (in particular $\omega \vee \mu \neq 0$ if and only if the subgroup generated by subgroups underlying ω and μ is compact). For that reason, for general ω and μ, we refer to these operations as corresponding to intersection of quasi-subgroups and passing to quasi-subgroup generated by two quasi-subgroups.

We have the following analog of modular property of subgroups:

Theorem 3. *Let $\omega, \mu, \rho \in \mathrm{Idem}(\mathbb{G})$ be such that*

(1) $\rho \preccurlyeq \omega$,
(2) $\mu * \rho = \rho * \mu$,
(3) $\mathsf{N}_{\omega\wedge\mu} = (\mathsf{N}_\omega\mathsf{N}_\mu)^{\sigma-\text{c.l.s.}}$.

Then $\omega \wedge (\mu \vee \rho) = (\omega \wedge \mu) \vee \rho$.

The assumptions of Theorem 3 have the following interpretation:

- $\rho \preccurlyeq \omega$ means that the quasi-subgroup corresponding to ρ is contained in the one for ω,
- $\mu * \rho = \rho * \mu$ means that the quasi-subgroups corresponding to μ and ρ commute,

while the last assumption is of technical nature and it is fulfilled e.g. for all idempotent states arising from Haar measures on quantum subgroups.

2. Normal idempotent states

The *reducing morphism* $\mathrm{C}_0^{\mathrm{u}}(\mathbb{G}) \twoheadrightarrow \mathrm{C}_0(\mathbb{G})$ and containment $\mathrm{C}_0(\mathbb{G}) \subset L^\infty(\mathbb{G})$ allows us to map $L^\infty(\mathbb{G})_*$ into $\mathrm{C}_0^{\mathrm{u}}(\mathbb{G})^*$. The map turns out to be injective and its image is a closed ideal in $\mathrm{C}_0^{\mathrm{u}}(\mathbb{G})^*$. We refer to elements of this ideal as *normal*. We define $\mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$ as the intersection of $\mathrm{Idem}(\mathbb{G})$ with $L^\infty(\mathbb{G})_*$.

In classical context an idempotent state $\omega \in \mathrm{C}_0(G)^*$ given by integration with respect to the Haar measure on a compact subgroup $K \subset G$ is normal if and only if K is open. Hence we may speak of "open quasi-subgroups" of a locally compact quantum group $\mathbb{G}$ as objects described by normal idempotent states on $\mathrm{C}_0^{\mathrm{u}}(\mathbb{G})$. It is easy to see that if $\omega, \mu \in \mathrm{Idem}(\mathbb{G})$, $\omega \preccurlyeq \mu$ and $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$ then $\mu \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$. Hence we obtain:

Corollary 4. *If $\mathbb{G}$ is a discrete quantum group then* $\mathrm{Idem}_{\mathrm{nor}}(\mathbb{G}) = \mathrm{Idem}(\mathbb{G})$.

The proof of Corollary 4 is a direct consequence of the fact that for $\mathbb{G}$ discrete the counit is a normal idempotent state.

Our main result about normal idempotent states is the following:

Theorem 5. *For any locally compact quantum group $\mathbb{G}$ there is a bijection*

$$\mathrm{Idem}_{\mathrm{nor}}(\mathbb{G}) \ni \omega \longmapsto \widetilde{\omega} \in \mathrm{Idem}_{\mathrm{nor}}(\widehat{\mathbb{G}})$$

reversing natural orders and such that $\widetilde{\widetilde{\omega}} = \omega$ for all $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$.

The mapping $\omega \mapsto \widetilde{\omega}$ can be thought of as a form of duality. On the level of coideals corresponding to idempotent states we have

$$\mathsf{N}_{\widetilde{\omega}} = \mathsf{N}_\omega{}' \cap L^\infty(\widehat{\mathbb{G}}).$$

This duality can be used to prove several results on normal idempotent states on compact quantum groups:

Theorem 6. *Let $\mathbb{G}$ be a compact quantum group. Then*

1. *the following conditions are equivalent for $\omega \in \mathrm{Idem}(\mathbb{G})$:*
   - $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$,
   - $\dim \mathsf{N}_\omega < +\infty$,
   - N_ω *has a finite-dimensional direct summand;*
2. *for any finite-dimensional coideal $\mathsf{N} \subset L^\infty(\mathbb{G})$ there exists $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$ such that $\mathsf{N} = \mathsf{N}_\omega$; in particular N is invariant under the scaling group.*

In fact, finite-dimensionality of the coideal N_ω defined by $\omega \in \mathrm{Idem}(\mathbb{G})$ can be interpreted as saying that the corresponding quasi-subgroup is of "finite index". This is illustrated e.g. by the following theorem:

Theorem 7. *Let $\mathbb{G}$ be a locally compact quantum group and let $\omega \in \mathrm{Idem}(\mathbb{G})$ be such that $\dim \mathsf{N}_\omega < +\infty$. Then $\mathbb{G}$ is compact and, consequently, ω is normal.*

One of the main tools used in the proofs of above-mentioned results is the bijective correspondence between idempotent states on $\mathrm{C}_0^{\mathrm{u}}(\mathbb{G})$ and *group-like projections* in $L^\infty(\widehat{\mathbb{G}})$ invariant under the scaling group proved in [2] (see [2, 4] for details). One of the consequences of studying these group-like projections is shown in the next proposition.

Proposition 8. *Let* $\mathbb{G}$ *be a discrete quantum group and let* $\omega \in \mathrm{Idem}(\mathbb{G})$*. Then* ω *has finite support in the sense that it is zero on almost all simple direct summands of* $\mathrm{c}_0(\mathbb{G})$*.*

Finally let us address the question of how the operations on $\mathrm{Idem}(\mathbb{G})$ restrict to $\mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$. Quite obviously if $\omega, \mu \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$ and $\omega \vee \mu \neq 0$ then $\omega \vee \mu \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$. On the other hand $\omega \wedge \mu$ does not have to be normal. The precise statement about this is the following:

Proposition 9. *Let* $\omega, \mu \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$*. Then* $\omega \wedge \mu \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$ *if and only if* $\widetilde{\omega} \vee \widetilde{\mu} \neq 0$*. In this case we have*

$$\omega \wedge \mu = \widetilde{\widetilde{\omega} \vee \widetilde{\mu}}.$$

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Free entropy dimension and the orthogonal free quantum groups

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(joint work with Michael Brannan)

The theory of discrete quantum groups provides a rich source of interesting examples of C*-algebras and von Neumann algebras. In addition to ordinary discrete groups, there is a wealth of examples and phenomena arising from genuinely quantum groups. Within the class of non-amenable discrete quantum groups, the so-called *free quantum groups* of Wang and Van Daele somehow form the most prominent examples.

In this work, our main focus is on the structural theory of a family of II_1-factors associated to a special family of free quantum groups, called the *orthogonal free quantum groups*. Given an integer $N \geq 2$, the orthogonal free quantum group $\mathbb{F}O_N$ is the discrete quantum group defined via the full Woronowicz C*-algebra

$$C^*_{\mathrm{f}}(\mathbb{F}O_N) = \langle u_{ij}, 1 \leq i,j \leq N \mid u = [u_{ij}] \text{ unitary}, \ u_{ij} = u^*_{ij} \ \forall i,j \rangle.$$

Using the (tracial) Haar state $h : C^*_{\mathrm{f}}(\mathbb{F}O_N) \to \mathbb{C}$, the GNS construction yields in the usual way a Hilbert space $\ell^2(\mathbb{F}O_N)$ and a corresponding von Neumann algebra $\mathcal{L}(\mathbb{F}O_N) = \pi_h(C^*_{\mathrm{f}}(\mathbb{F}O_N))'' \subseteq B(\ell^2(\mathbb{F}O_N))$, where π_h denotes the GNS representation.

Over the past two decades, the structure of the algebras $\mathcal{L}(\mathbb{F}O_N)$ has been investigated by many hands, and in many respects $\mathbb{F}O_N$ and $\mathcal{L}(\mathbb{F}O_N)$ $(N \geq 3)$ were shown to share many properties with free groups F_n and their von Neumann algebras $\mathcal{L}(F_n)$. For example, $\mathcal{L}(\mathbb{F}O_N)$ is a full type II_1-factor, it is strongly solid, and in particular prime and has no Cartan subalgebra; it has the Haagerup property (HAP), is weakly amenable with Cowling-Haagerup constant 1 (CMAP), and satisfies the Connes' Embedding conjecture. Moreover, it is known that $\mathcal{L}(\mathbb{F}O_N)$ behaves asymptotically like a free group factor in the sense that the canonical generators of $\mathcal{L}(\mathbb{F}O_N)$ become strongly asymptotically free semicircular systems as $N \to \infty$.

With these many similarities between $\mathcal{L}(\mathbb{F}O_N)$ and $\mathcal{L}(F_n)$ at hand, the following question naturally arises: *Can $\mathcal{L}(\mathbb{F}O_N)$ be isomorphic to a free group factor*?

The first evidence suggesting a negative answer to an isomorphism with a free group factor followed from work of the speaker showing the vanishing of the first L^2-cohomology group of $\mathbb{F}O_N$. Combining this result with some deep work of Connes-Shlyakhtenko, Jung and Biane-Capitaine-Guionnet on free entropy dimension, it follows that $\delta_0(u) = \delta^*(u) = 1$ $(N \geq 4)$, where u is the set of canonical self-adjoint generators of $\mathcal{L}(\mathbb{F}O_N)$, and δ_0, δ^* are Voiculescu's (modified) microstates free entropy dimension and non-microstates free entropy dimension, respectively. Note however that it is not know whether the free entropy dimensions $\delta_0(u)$, $\delta^*(u)$ are invariants of the von Neumann algebra $\mathcal{L}(\mathbb{F}O_N)$ generated by u.

In a remarkable work, Jung introduced a certain technical strengthening of the condition $\delta_0(X) \leq \alpha$, which he called *α-boundedness* of X. There, Jung proved that if (M,τ) is a finite von Neumann algebra generated by a 1-bounded set $X \subset M_{sa}$ containing at least one element with finite free entropy, then every other self-adjoint generating set X' of M has $\delta_0(X') \leq 1$. In this case, we call M a *strongly 1-bounded von Neumann algebra*, and δ_0 becomes a W^*-invariant for M. Note, in particular, that any strongly 1-bounded von Neumann algebra cannot be isomorphic to any free group factor $\mathcal{L}(F_r)$ $(r \geq 2)$. Our main result is then the following theorem:

Theorem 1. *For each $N \geq 3$, $\mathcal{L}(\mathbb{F}O_N)$ is a strongly 1-bounded von Neumann algebra. In particular, $\mathcal{L}(\mathbb{F}O_N)$ is never isomorphic to an interpolated free group factor, nor to the von Neumann algebra associated with a free product of cyclic groups.*

Note that II_1-factors which have property Gamma, or have a Cartan subalgebra, or are tensor products of infinite dimensional factors, are automatically strongly 1-bounded. This is not the case of $\mathcal{L}(\mathbb{F}O_N)$. Instead, our proof of strong 1-boundedness relies on and is heavily inspired by recent works of Jung and Shlyakhtenko.

If F is an l-tuple of non-commutative polynomials over m variables, one can compute Voiculescu's free derivative ∂F which yields by evaluation an operator $\partial F(X) \in M \otimes M^{op} \otimes B(\mathbb{C}^m, \mathbb{C}^l)$. Jung showed that if (M, τ) is a finite von Neumann algebra, $X \in M_{sa}^m$ is an m-tuple satisfying the polynomial relations $F(X) = 0$, then X is α-bounded with $\alpha = m - \mathrm{rank}(\partial F(X))$, *provided* that $\partial F(X)^* \partial F(X)$ has a non-zero modified Lück-Fuglede-Kadison determinant. Shortly after Shlyakhtenko gave another proof of this result using non-microstates free entropy techniques.

Our strategy for proving our strong 1-boundedness theorem, is to first take the canonical system of generators $X = u = (u_{ij})_{1 \leq i,j \leq N}$ and form the natural vector of quadratic relations $F(X) = 0$ associated to the defining orthogonality relations of $\mathbb{F}O_N$. Adapting results of the group case one can establish the relation $m - \mathrm{rank}(\partial F(X)) = \beta_1^{(2)}(\mathbb{F}O_N) - \beta_0^{(2)}(\mathbb{F}O_N) + 1$ and relying on the computations of the L^2-Betti numbers by Kyed and the speaker, this yields $\alpha = 1$.

On the other hand, proving that $D = \partial F(X)^* \partial F(X)$ has a non-zero modified Lück-Fuglede-Kadison determinant turns out to be much more involved and constitutes the main technical component of the paper. This amounts to proving the integrability of the function $\log_+ : [0, \infty) \to \mathbb{R}$ with respect to the spectral measure of D, where $\log_+(t) = \log(t)$ if $t > 0$ and $\log_+(0) = 0$.

This integrability condition is established by producing an identification of D, up to amplification and unitary equivalence, with the operator $2(1+\mathrm{Re}(\Theta))$, where $\Theta \in B(K)$ is the so-called *edge-reversing operator* of the quantum Cayley tree associated to the quantum group $\mathbb{F}O_N$. Here, K denotes the *edge Hilbert space* associated to the quantum Cayley tree. Quantum Cayley graphs where introduced and studied by the speaker and were in particular a key ingredient in his proof of the vanishing of the first L^2-Betti number of $\mathbb{F}O_N$. More specifically, a large part of this study was devoted to the investigation of the eigenspaces $K_g^{\pm} = \mathrm{Ker}(\Theta \pm \mathrm{id})$.

In the quantum case, Θ is not involutive and the understanding of its behavior on the orthogonal complement of $K_g^+ \oplus K_g^-$ is essential for the study of the integrability condition of D. In the present article, we unveil a shift structure for the action of $\mathrm{Re}(\Theta)$ on the orthogonal complement of $K_g^+ \oplus K_g^-$, reducing the initial problem to an integrability question for real parts of weighted shifts, which can be settled by elementary methods.

Finally, let us conclude this abstract with the following natural question: Although we now know that $\mathcal{L}(\mathbb{F}O_N)$ is not isomorphic to a free group factor, could it still be possible that $\mathcal{L}(\mathbb{F}O_N)$ is isomorphic to $\mathcal{L}(\Gamma)$ for some other discrete group Γ? In particular, what about Γ being an ICC lattice in $SL(2,\mathbb{C})$? For such Γ, it is known that $\mathcal{L}(\Gamma)$ is a full, strongly solid, strongly 1-bounded II_1-factor which has the HAP and the CMAP.

Positive definite functions on Coxeter groups with applications to operator spaces and noncommutative probability

Marek Bożejko

(joint work with Światosław R. Gal, Wojciech Młotkowski)

In 1979 Uffe Haagerup in his seminal paper [3] proved positive definiteness of the function $P_q(x) := q^{|x|}$, $-1 \le q \le 1$, on the free group $\mathbf{F}_N$ on N generators. Here $|\cdot|$ denotes the natural length function on $\mathbf{F}_N$. From this he deduced Khinchine type inequalities and showed that the regular C^*-algebra of $\mathbf{F}_N$ admits bounded approximation property and the completely bounded approximation property (CBAP), see [4]. These results had significant impact on harmonic analysis on free group and also influenced free probability as well as the operator spaces theory, see [5].

Note that the Cayley graph of $\mathbf{F}_N$ is the homogeneous tree of order $2N$, so these results can be easily translated into the free Coxeter group $W = \mathbf{Z}/2 * \cdots * \mathbf{Z}/2$. In fact, it was shown in the paper [6] that the function $P_q(x) = q^{|x|}$ is positive definite for $q \in [-1, 1]$ and for every Coxeter group, where $|\cdot|$ is now the natural word length function on a Coxeter group with respect to the set of its Coxeter generators. This implies that infinite Coxeter groups have the Haagerup property (see [7]) and do not have Kazhdan's property (T).

Later, Januszkiewicz [8] and Fendler [9] applied Haagerup's ideas to prove that the map $W \ni w \mapsto z^{|w|}$ is a coefficient of a uniformly bounded Hilbert representation of W for all $z \in \mathbf{C}$ such that $|z| < 1$. Valette [10] observed that this implies CBAP. For further extension of these Haagerup's results for a big class of groups we refer to the book [11].

Bożejko and Speicher [12] considered the free product (convolution) of classic normal distribution $N(0,1)$. They introduced a new length function on the permutation group $\mathfrak{S}_n$ which here we will call the *color-length* and denote $\|\cdot\|$. It is defined as follows: if $w = s_1 \dots s_k$ is a minimal representation of w as a product of generators $s_i \in S$ then we put $\|w\| = \#\{s_1, s_2, \dots, s_k\}$. In the case of $\mathfrak{S}_n$, S is the set of transpositions $(j, j+1)$, $1 \le j < n$. Moreover, they found formula for the free additive convolution power of the classical normal distribution $\mu_1 := N(0,1)$ and the Bernoulli distribution $\mu_{-1} := (\delta_{-1} + \delta_1)/2$, namely

$$m_{2n}\left(\mu_{\pm 1}^{\boxplus q}\right) = q^n \sum_{\pi \in \mathcal{P}_2(2n)} (\pm 1)^{|\pi|} q^{-\|\pi\|},$$

$m_{2n+1}\left(\mu_{\pm 1}^{\boxplus q}\right) = 0$, for $q \in \mathbf{N}$. These results motivated us to study the color length function $\|\cdot\|$ in more details.

For further applications we will study generalizations of the function $x \mapsto q^{\|x\|}$ on a Coxeter group (W, S), namely Riesz-Coxeter products, which are defined by $R_{\mathbf{q}}(s) := q_s$, for $s \in S$, and

$$R_{\mathbf{q}}(xy) := R_{\mathbf{q}}(x)R_{\mathbf{q}}(y) \text{ whenever } \|xy\| = \|x\| + \|y\|,$$

where $\mathbf{q} = (q_s)_{s \in S}$ is a system of real parameters. In particular, if $q_s = q$ for every $s \in S$ then $R_{\mathbf{q}}(x) = q^{\|x\|}$. In one of the most important results of our work we provide sufficient conditions for positive definiteness of the function $R_{\mathbf{q}}$:

Theorem 1. *Assume that for every $s \in S$ we are given a number q_s,*

$$\frac{-1}{d_s - 1} \leq q_s \leq 1,$$

where d_s denotes the index if the parabolic subgroup generated by $S \setminus \{s\}$ in W: $d_s = [W : W_{S\setminus\{s\}}]$. Then the Riesz–Coxeter $R_{\mathbf{q}}$ is positive definite on W.

This implies, in particular, that in an arbitrary Coxeter group (W, S) the set of generators S is a weak Sidon set, i.e. that for every $f : S \to [0, 1]$ there exist positive definite functions $\phi_+, \phi_- : W \to \mathbf{C}$ such that $f(s) = \phi_+(s) - \phi_-(s)$ for every $s \in S$. These ϕ_+, ϕ_-can be chosen as $R_{\mathbf{q}_+}, R_{\mathbf{q}_-}$ for suitable parameters $\mathbf{q}_+, \mathbf{q}_-$. This result answers a question of Pisier, who was particularly interested in the infinite permutation group $\mathfrak{S}_\infty$. As further consequence we obtain an operator version of the Khinchin inequality for arbitrary Coxeter group, which extend results of [13, 14]:

Theorem 2. *If $a_s \in M_n(\mathbf{C})$, then for all $p \geq 2$ and any Coxeter system (W, S) we have*

$$\|(a_s)_{s\in S}\|_{R\cap C} \leq \left\| \sum_{s\in S} a_s \otimes \lambda(s) \right\|_{L^p(W)} \leq 2A'\sqrt{p}\, \|(a_s)_{s\in S}\|_{R\cap C}\,.$$

Let us also mention that the color-length function on the permutation group $\mathfrak{S}_n$ was also studied in [15]. Its extension to pairpartitions was applied in the proof that classical normal law $N(0, 1)$ is free infinitely divisible under the free additive convolution $\boxplus$. We believe that positive definite functions on Coxeter groups of type B and D, especially these which are color dependent, may have applications in the development of type B and D versions of free probability (see [16, 17]).

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Open Problems

Blecher, Collins, Gerhold, Labuschagne, Ozawa, Skalski, Winter, Skeide

(1) **David Blecher:**

From B-Weaver: Given a state g on a von Neumann algebra which is countably additive on projections, prove (in ZFC) that it is sequentially weak* continuous. That is, given a sequence $x_n \to x$ weak*, prove $g(x_n) \to g(x)$. This would be a von Neumann algebraic LDCT (Lebesgue dominated convergence theorem). It is known you cannot disprove it in ZFC (maybe assuming ZFC is consistent), because it is true if you add an appropriate set theoretic axiom. This was pointed out to me by Weaver who also showed it true if g is pure. It seems quite possible that there is a ZFC proof, i.e. that this result is not a set theoretic matter, but simply needs the right von Neumann algebra argument.

Remarks: it is of course true in sigma-finite von Neumann algebras, and in commutative von Neumann algebras (by LDCT). Also I can show that the LMCT (Lebesgue monotone convergence theorem) version of this question is true in all von Neumann algebras.

(2) **Benoît Collins**

Can you find a constant $C > 0$, an integer $k > \exp(2C^2)$, and unitary matrices $U_1, \ldots, U_k$ in $M_n(\mathbb{C})$ with the property that for any n^2 matrix coefficients a_{ij},

$$|| \sum_{i \neq j} a_{ij} U_i U_j^* || \leq C \sqrt{\sum_{I \neq j} |a_{ij}|^2} \quad ?$$

Here, the norm is the operator norm on the matrices. Such matrices are known to exist with random techniques for any k integer and $C > 3 + \varepsilon$, n large enough due to a result of Collins and Male together with Haagerup inequality. The question is to find non-random examples. Any example (in particular satisfying the inequality $k > \exp(2C^2)$) is known to give a counterexample to the minimum output entropy (MOE) additivity conjecture. No random examples are known yet.

(3) **Malte Gerhold:**

Is (AC) equivalent to (H^2Z) for all CQG-algebras A?

For a unital $*$-algebra A with a character ε, we say that (H^2Z) holds if the second Hochschild cohomology with trivial coefficients $H^2(A, {}_\varepsilon\mathbb{C}_\varepsilon)$ vanishes. We say that (AC) holds if for every π-ε-cocycle η (π a $*$-representation on a pre-Hilbert space) there is a ψ such that (π, η, ψ) is a Schürman triple. In [6, 4.5] we give an example (A, ε) such that (AC) holds, but (H^2Z) does not. Can this happen for CQG-algebras?

(4) **Louis Labuschagne**

Recently the theory of Haagerup L^p spaces, has been extended to include the concept of Orlicz spaces for type III algebras. (These may be defined in one of two equivalent ways - see [5, Definition 3.4] and [4, Lemma 4.11].) An extension of the Haagerup reduction theorem [3] to this context would be an invaluable tool for refining the theory of these spaces.

(5) **Narutaka Ozawa:**

(i) Let $H(\phi)$ denote the von Neumann entropy of ϕ and

$$\delta(\phi, \psi) = H\left((\phi + \psi)/2\right) - \left(H(\phi) + H(\psi)\right)/2.$$

Is it true that $|\phi - \psi|(A) \leq (8\delta(\phi, \psi)(\phi + \psi)(A^2))^{1/2}$ for every $A \geq 0$?

(ii) Let T be a CPTP map. Is it true that

$$1 - \|T(\phi^{1/2})\|_2^2 \leq 4(H(T(\phi)) - H(\phi))$$

for every density matrix ϕ?

(6) **Adam Skalski:**

Consider the von Neumann algebra of the free group $\mathbb{F}_2$, with the generators a and b, and let $A_g = \langle a \rangle''$, $A_r = \langle a + a^{-1} + b + b^{-1} \rangle''$. Both these algebras are known to be maximal abelian in $VN(\mathbb{F}_2)$. Is there an automorphism $\alpha \in Aut(VN(\mathbb{F}_2))$ such that $\alpha(A_g) = A_r$ (it is known that no such *inner* automorphism exists)? More generally, is there a maximal abelian subalgebra of $VN(\mathbb{F}_2)$ which is free from A_r?

(7) **Michael Skeide**

What are CPH-maps, CPH-semigroup, and CPH-dilations possibly good for? More details of this open problem can be find in the abstract of Micheal Skeide, on page 1325.

(8) **Andreas Winter:**

For a semidefinite matrix ρ, thought of as a quantum state, acting on a composite system $A \otimes B \otimes C \otimes D$ (tensor product of Hilbert spaces), find the linear inequalities constraining the entropies and the log-ranks of the various marginals. Concretely:

(i). For the von Neumann entropy, do the Yeung-Zhang inequalities and their higher order extension hold? (They have been proven for the Shannon entropy.) See [1].

(ii) For the ranks, does the following inequality hold for the reduced states of a tripartite density matrix: $r_{AB} \leq r_{AC} \cdot r_{BC}$, where $r_{XY} = rank(\rho_{XY})$? See [2].

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