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Cohomological and Metric Properties of Groups of Homeomorphisms of R

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Abstract. In recent years, the family of groups sharing features or design principles with classical Thompson groups has grown considerably. The workshop highlights new developments in this field with special emphasis on algorithmic questions, cohomological properties, and smoothability of actions.

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Introduction by the Organisers

About 50 years ago, Richard Thompson discovered three finitely presented groups F, T, and V. They are subgroups of the homeomorphism groups of the interval, the circle, and the Cantor set, respectively; and they can serve as discrete approximations of their ambient groups. The groups T and V were the first known infinite, finitely presented simple groups. Classical results about F show that it features a somewhat surprising combination of properties: it is torsion-free, of infinite cohomological dimension, and of type F_∞; similarly, it does not contain non-abelian free groups and is not elementary amenable (whether it is amenable is a famous open problem). Investigation of the classical Thompson groups thrives through to this day. Recently, new constructions of groups, based on Thompson groups, have provided us with more exciting groups, some of which especially crafted to exhibit prescribed properties.

The birthday conference *Thompson's group at 40 Years* in 2004 (at AIM) established Thompson groups as a viable conference topic. Since then, the community has organized conferences with this focus about every three to four years, and this workshop *Cohomological and Metric Properties of Groups of Homeomorphisms of* $\mathbf{R}$, organized by José Burillo (Barcelona), Kai-Uwe Bux (Bielefeld), and Brita Nucinkis (London), belongs to this sequence. More than 20 participants from eleven nations on four continents discussed recent developments regarding the three classical Thompson groups as well as their more recent offspring.

The topics of our 17 talks included decision problems, finiteness properties, smoothability of actions, and the structure of the class of all finitely generated homeomorphisms groups of the interval. In addition, we had two discussion sessions. The first concerned the broken Baumslag–Solitar groups. The other was a session about open problems. In particular, we discussed progress that has been made since the last Thompson family meeting at St Andrews, May 2014.

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Groups of homeomorphisms of R and their relatives

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Abstracts

Closed subgroups of F

Mark Sapir

The R. Thompson group F can be described as the group of all increasing homeomorphisms $f(x)$ of $[0,1]\cap \mathbf{Q}_2$ (where $\mathbf{Q}_2$ is the set of all binary fractions) which are locally affine of the form $a+2^n x$ for some a and $n\in \mathbf{Z}$. So it is the "topological full group" of the affine group of maps of the form $f(x)=a+2^n x$ acting on $[0,1]\cap \mathbf{Q}_2$.

Now given any subgroup H of F we can define its *closure* $\bar{H}$ as the group of all homeomorphisms of F which are locally H. A subgroup H is called *c*losed if $H=\bar{H}$. In the middle of the 90s Guba and I conjectured that $\bar{H}$ can be described in a way similar to the Stallings description of subgroups of free groups, using Stallings cores.

Recall that if a subgroup H of a free group $F_n=\langle a_1,...,a_n\rangle$ is generated by words $w_1,..,w_m$ then the Stallings core $C(H)$ is the graph labeled by $\{a_1,...,a_n\}$ obtained as follows. Consider m paths $p_1,...,p_m$ labeled by the words $w_1,...,w_m$, identify the start and end points of these paths to obtain a bouquet of m circles with a common vertex e. After that do the Stallings foldings: if there is a vertex o with two edges with the same labels going out of o, we identify the edges of their end-vertices. The Stallings core $C(H)$ is an automaton (with input=output vertex e) having the property that it accepts a word w if and only if $w\in H$. Essentially the whole Nielsen–Schreier–Hall theory of subgroups of free groups follows from that observation.

Now if H is a subgroup of the R. Thompson group F generated by tree diagrams $(T_1^+,T_1^-),...,(T_m^+,T_m^-)$ we can obtain the 2-core $Core(H)$ by first identifying the top vertices of all trees $T_i^{\pm}$ with a vertex e, and then doing 2-foldings, that is if we have two carets $f\to f_1, f\to f_2$ and $f\to f_1', f\to f_2'$, we identify vertices $f_1\equiv f_1', f_2\equiv f_2'$ (and the carets), and if we have two carets $f\to f_1, f\to f_2$ and $g\to f_1, g\to f_2$ we identify the vertices f and g (and the carets). As a result, we get an "automaton" with input=output=e and it is easy to define the meaning of when the 2-core accepts a tree diagram (T^+,T^-). Guba and I conjectured that the subgroup $\bar{H}$ is precisely the subgroup of all elements of F accepted by $Core(H)$. That was proved by Gili Golan [18].

There are several unexpected corollaries (the first one was observed by Guba and myself).

Theorem 1. *If H is finitely generated then $\bar{H}$ has membership problem solvable in linear time.*

Many known subgroups of F are (isomorphic to) closed subgroups: all cyclic subgroups, $\mathbf{Z}\wr\mathbf{Z}$, $(...(\mathbf{Z}\wr\mathbf{Z})\wr\mathbf{Z})...)\wr\mathbf{Z}$, $F\wr\mathbf{Z}$, F_p $(p\geq 2)$, the Brin–Navas subgroup B, etc.

Theorem 2 (Golan [18]). *The generation problem for F is decidable, that is given a finite number of elements of F, one can decide if these elements generate F.*

Theorem 3 (Golan). *For every $n \geq 2$, the probability that n elements of F generate F is > 0.*

Theorem 4 (Golan [108]). *The generating set $\{x_0, x_1, x_0x_1\}$ of F has the property that every set of conjugates of these elements $\{x_0^{g_1}, x_1^{g_2}, (x_0x_1)^{g_3}\}$ generates F.*

Theorem 5 (Golan [18]). *The closure of every solvable subgroup of F is solvable of the same degree.*

If $e_1, ..., e_s$ are edges of $Core(H)$, then $\mathcal{P}(H) = \langle e_1, ..., e_s \mid e_ie_j = e_k$, where $e_k \to e_i, e_k \to e_j$ is a caret$\rangle$ is a semigroup presentation, and $\bar{H}$ is the diagram group of $\mathcal{P}$ with the base word e. Thus all properties of diagram groups [93] hold for closed subgroups of H. The presentation $\mathcal{P}$ has the following tree-like property: If $x = yz, x = y'z'$ are relations in $\mathcal{P}$, then $y = y', z = z'$ and if $x = yz, x' = yz$ are relations of $\mathcal{P}$, then $x = x'$. Conversely, every diagram group of a tree-like semigroup presentation is a subgroup of F [18].

There are several open problems related to closed subgroups of F.

By our result with Guba [94], a diagram group $DG(\mathcal{P}, w)$ contains a copy of F if and only if the semigroup given by the presentation $\mathcal{P}$ contains an idempotent dividing w.

Problem 6. *If it is decidable whether a semigroup given by a finite tree-like presentation contains an idempotent. Equivalenly, is it decidable whether the closure of a finitely generated subgroup of F contains a copy of F?*

Problem 7. *Is the closure of every finitely generated subgroup of F finitely generated?*

Problem 8. *Is every finitely generated closed subgroup of F undistorted?*

Golan and I can prove that if a tree-like presentation has a finite confluent and terminating Knuth–Bendix completion and the completion is "terminating in linear time", then the corresponding closed subgroup of F is undistorted. That implies all known results about undistorted subgroups of F.

Groups of piecewise isometric rearrangements of tessellations

Robert Bieri

(joint work with Heike Sach)

This is recent joint work with Heike Sach and grew out of her excellent Diploma Thesis [49] which sadly didn't make it to a Ph.D. project, as I failed to convince her that creating and nursing a blooming garden of higher dimensional Houghton groups is all you need for a happy life.

We consider a tessellated piece S of either Euclidean or hyperbolic n-space X and are interested in rearranging the tiles of this tessellation by cutting S along tile-boundaries into finitely many essentially convex rigid pieces and rearranging them to a new tessellation of S. Each such rearrangement defines a permutation of the tile-centers, and we call this a *piecewise Euclidean isometric (pei) permutation* (resp. a *piecewise hyperbolic isometric (phi) permutation*). We are interested in the corresponding permutation groups: pei(S) and phi(S), and also in subgroups like pet(S) $\leq$ pei(S) where the isometries are restricted to translations. All phi-, pei-, and pet-groups contain the normal subgroup S_∞ consisting of all finite permutations.

1. The case when X is the hyperbolic plane $\mathbb{H}^2$

The case when X is the hyperbolic plane $\mathbb{H}^2$ endowed with the tessellation Δ by ideal triangles relates our groups with highly interesting recent developments in contemporary geometric group theory: the retraction of the hyperbolic plane through the horoballs at the vertices of Δ onto the Farey tree T retracts the tessellation Δ of the hyperbolic plane to a tessellation $\overline{\Delta}$ of the Farey tree T by isometric Y-shaped "tree-tiles". We observe that the vertices ver(T) of T can be used as the mid-points of the tiles of both Δ and $\overline{\Delta}$. As rearranging Δ corresponds precisely to rearranging $\overline{\Delta}$ it follows that phi(Δ) is isomorphic to what could be called the group of all piecewise planar tree-isometric (ppti) permutations of ver(T). The latter group is obviously the group of all almost isomorphisms of T (the permutations ver(T) that tear only finitey many edges apart).

Now it remains to observe that rearranging the infinite pieces of the Farey tree are precisely the tree-moves that describe the elements of Richard Thompson's group V (I am told that Thurston observed that already in the mid sixties). Hence we have:

Theorem 9. phi(Δ)$/S_\infty \cong V$*, where* S_∞ *is the finitary symmetric group of infinite degree.*

Question 10. *Which Thompson-like groups are of the form* phi(Δ)$/S_\infty$ *for some tessellation* Δ *of the hyperbolic plane? And what is the group theoretic property that characterizes them?*

Question 11. *Is it the case that* phi(Δ)$/S_\infty$ *and* phi(Δ) *have the same finiteness properties?*

2. The Euclidean orthogonal case

Our main concern is the case when X is a Euclidean space $\mathbb{E}^n$, tessellated by unit cubes, and $S \subseteq \mathbb{E}^n$ is an *orthohedral* subset (i.e., a polyhedral union of tiles - see below under B). We obtain rather detailed insight into the structure of the groups pei(S) and pet(S) and results on their *finiteness lengths*[1].

[1]The *finiteness length* of a group G, denoted fl(G), is the maximum n (possibly ∞) with the property that G is of type F_n but not of type F_{n+1}; *to be of type* F_n means that G admits a finite n-dimensional $(n-1)$-connected free G-CW-complex.

A) Results on finiteness properties. In the special case when S is the union of the n canonical positive axes, the index of $\mathrm{pet}(S)$ in $\mathrm{pei}(S)$ is finite and $\mathrm{pet}(S)$ is the Houghton group H_n on n rays. In that case we know by [36] that $\mathrm{fl}(H_n) = n-1$.

Heike Sach's Diploma thesis [49] was a first step towards our generalization: she examined the possibilities to extend Brown's proof to case when S is the union of n quadrants and proved $\mathrm{fl}(\mathrm{pet}(S)) \geq n-1$ in that case. Now we consider arbitrary k-dimensional orthohedral sets and prove the following.

Theorem 12 ([35])**.** $\mathrm{fl}(\mathrm{pei}(S)) \geq h(S) - 1$, *where $h(S)$ is the maximum number of pairwise disjoint isometric copies of $\mathbb{N}^k$ that fit into S.*

In particular, $\mathrm{fl}(\mathbb{Z}^n) \geq 2^n - 1$. And if S consists of $h(S)$ parallel copies of $\mathbb{N}^k$, then we have:

Theorem 13 ([35])**.** $\mathrm{fl}(\mathrm{pet}(S)) = h(S) - 1$.

B) The structure of $\mathrm{pei}(S)$**.** In order to investigate the structure of $\mathrm{pei}(S)$ we need more vocabulary on orthohedral sets $S \subseteq \mathbb{Z}^n$.

Let $0 \leq k \in \mathbb{Z}$. By a rank-k *orthant* $L \subseteq \mathbb{Z}^n$ we mean a subset L isometric to $\mathbb{N}^k$. An *orthohedral* set S is a disjoint union of finitely many orthants. The rank $\mathrm{rk}(S)$ is the maximum rank of the orthants in S. Two orthants L, L' are *commensurable* if $\mathrm{rk}(L) = \mathrm{rk}(L \cap L') = \mathrm{rk}(L')$, and the commensurability class of the orthant L is the *germ* of L, denoted $\gamma(L)$. Every element $g \in \mathrm{pei}(S)$ has a orthohedral support $\mathrm{supp}(g)$ and its rank, denoted $\mathrm{rk}(g)$, is the *rank of g*. An injective map $f : S \to S'$ between orthohedral sets is *piecewise isometric* if every orthant L of S contains a commensurable suborthant $K \subseteq L$ on which f restricts to an isometric embedding $f\,|_K \colon K \to S'$ (equivalently: S is the union of a finite number of orthants on which f restricts to isometric embeddings).

Bijective pei-injections are pei-*isomorphisms*, and it is not hard to prove: *Every orthohedral set S is* pei-*isomorphic to* $\{n \mid 1 \leq n \leq h(S)\} \times \mathbb{N}^{\mathrm{rk}(S)}$.

The finiteness properties that we establish for $\mathrm{pei}(S)$ are easier appreciated in view of the following result on $\mathrm{pei}(S)$ for an arbitrary orthohedral set S.

Theorem 14 ([35])**.** *For $0 \leq k \leq h(S)$ the subgroups $G_k := \{g \mid \mathrm{rk}(g) \leq k\}$ are normal in* $\mathrm{pei}(S)$ *and form a series*

$$1 \leq G_0 \leq G_1 \leq \ldots \leq G_{\mathrm{rk}(S)} = \mathrm{pei}(S)$$

whose sections G_k/G_{k-1} fit in a short exact sequence

$$1 \to A^k(S) \to G_k/G_{k-1} \to S_{h(S)} \wr S_\infty \to 1,$$

where $A^k(S)$ is free abelian (of infinite rank when $1 \leq k \leq h(S)$).

Smoothability and Property (*T*) for group actions on 1-manifolds by countably singular diffeomorphisms

Yash Lodha
(joint work with Nicolás Matte Bon, Michele Triestino)

We start with two broad questions, which we shall later specialise.

Question 15. *Let M be a connected 1-manifold. Let $G < \mathrm{Homeo}^+(M)$ be a group action.*

(1) *Does the underlying group admit a faithful action by C^r-diffeomorphisms on M?*
(2) *Is the original group action smoothable? In other words, does there exist a homeomorphism $\phi : M \to M$ such that the action $\phi^{-1}G\phi$ is an action by C^r diffeomorphisms?*
(3) *What are the obstructions to admitting an action by C^r-diffeomorphisms on a 1-manifold?*

Question 16. *Does there exist an infinite Kazhdan group that admit a faithful action by homeomorphisms on the circle?*

Remark 17. Here by a C^r diffeomorphism ν of M we mean that the $\lfloor r \rfloor$-th derivative of ν is $r - \lfloor r \rfloor$-Holder continuous.

A classical obstruction to C^1-smoothability is the Thurston stability theorem.

Theorem 18 (Thurston Stability [117])**.** *A group of C^1 diffeomorphisms of an interval $[a, b]$ is* locally indicable*, which means that every finitely generated subgroup admits a homomorphism onto* $\mathbf{Z}$.

An obstruction to admitting C^2-actions on the circle is the following result of Navas.

Theorem 19 (Navas [72])**.** *Let G be an infinite Kazhdan group. Then G does not admit a faithful action by C^2 diffeomorphisms on $\mathbf{S}^1$.*

Recall that the group $\mathrm{PSL}_2(\mathbf{R})$ acts in the projective line $\mathbf{R} \cup \{\infty\}$ by projective transformations.

Example 20. (Thompson's group F) The group of piecewise $\mathrm{PSL}_2(\mathbf{Z})$ homeomorphisms of $\mathbf{R}$ with breakpoints in $\mathbf{Q}$.
(Thompson's group T) The group of piecewise $\mathrm{PSL}_2(\mathbf{Z})$ homeomorphisms of $\mathbf{R} \cup \{\infty\}$ with breakpoints in $\mathbf{Q} \cup \{\infty\}$.

Theorem 21 (Ghys–Sergiescu [67])**.** *The standard actions of F and T are topologically conjugate to an action by C^∞ diffeomorphisms of the real line.*

Theorem 22 (Thompson)**.** *T is a finitely presented, infinite, simple group.*

Whether Thompson's group F is amenable is a well known open problem. A group G is amenable if it admits a finitely additive, left translation invariant probability measure. The original interest in the amenability of F arose due to the interest in finding examples of groups which are non amenable, despite the fact that they do not contain non abelian free subgroups. Recently, Monod and L.–Moore constructed some generalisations of F that are non amenable, despite the fact that they do not contain non abelian free subgroups.

Example 23. Let $A < \mathbf{R}$ be a subring.
(Monod's groups) Define $H(A)$ as the group of piecewise $\mathrm{PSL}_2(A)$-projective homeomorphisms of the real line with breakpoints in the set Q_A of fixed points of hyperbolic elements of $\mathrm{PSL}_2(A)$.
(L.–Moore) Define

$$c(t) = t \text{ if } t \notin [0,1] \qquad c(t) = \frac{2t}{1+t} \text{ if } t \in [0,1]$$

Define the group $G_0 = \langle F, c\rangle$.

The following holds for these groups.

Theorem 24 (Monod [80])**.** *If A is dense in $\mathbf{R}$, $H(A)$ is non amenable and does not contain free subgroups.*

Theorem 25 (L.–Moore [98])**.** *G_0 is nonamenable and does not contain free subgroups. It is finitely presented, and admits a presentation with 3 generators and 9 relations.*

Recall that a group G is said to be of type F_∞ if there is a connected, aspherical CW complex X such that $\pi_1(X) = G$. The following holds for G_0, thereby making it the first example of a type F_∞ group which is non amenable and does not contain non abelian free subgroups.

Theorem 26 (L. [97])**.** *G_0 is of type F_∞.*

It is natural to inquire what the *Tarski numbers* of these groups are. Recall that the Tarski number of a group is the smallest number of pieces required in a paradoxical decomposition of the group.

Theorem 27 (L. [68])**.** *G_0 and $H(A)$ (for each subring $A < \mathbf{R}, A \neq \mathbf{Z}$) admit a paradoxical decomposition with 25 pieces. It follows that their Tarski numbers lie in the range $[5, 25] \cap \mathbf{N}$.*

Next, we address the following question.

Question 28 (Navas)**.** *Are these group smoothable?*

We resolve this question in the following manner.

Theorem 29 (Bonatti, L., Triestino [66])**.**
(1) *$H(A)$ is not C^1-smoothable for any subring A of $\mathbf{Z}$.*
(2) *If A contains non-trivial units, then $H(A)$ does not admit a faithful action by C^1 diffeomorphisms on the real line or a closed interval.*
(3) *The group G_0 does not admit a faithful action by C^1 diffeomorphisms on the real line or a closed interval.*

One of the obstructions to smoothability that we discovered is the following family of abelian-by-cyclic groups.

Example 30. (Broken Baumslag–Solitar groups) For any $\lambda \in \mathbf{Q}_{>0}$, define the group $G_\lambda = \langle a, b_-, b_+ \rangle$ where

$$a(t) = t + 1 \quad b_-(t) = \lambda t \text{ if } t \leq 0 \text{ and } t \text{ if } t \geq 0 \quad b_+(t) = \lambda t \text{ if } t \geq 0 \text{ and } t \text{ if } t \leq 0$$

Theorem 31 (L. [70]). *Let $S = \langle T, s \rangle$ where*

$$s(t) = \begin{cases} t & \text{if } t \leq 0 \\ \frac{2t}{1+t} & \text{if } 0 \leq t \leq 1 \\ \frac{2}{3-t} & \text{if } 1 \leq t \leq 2 \\ t & \text{if } t \geq 2 \end{cases}$$

Then S is a finitely presented, infinite, simple group of homeomorphisms of the circle. However, S does not admit a non-trivial action by C^1-diffeomorphisms on the circle.

Here is a simple argument that shows that S is not Kazhdan. Define a map

$$\phi : S \to l^2(\mathbf{R} \cup \{\infty\}) \qquad \phi(g)(r) = Log \frac{g'_+(r)}{g'_-(r)}$$

This map is easily verified to be a cocycle under the natural group action. This provides an affine isometric action on $l^2(\mathbf{R} \cup \{\infty\})$ without a fixed point.

Definition 32. Let M be a closed manifold. We define $\Omega\text{Diff}^r(M)$ as the group of homeomorphisms of M that are C^r in the complement of a countable closed set of singularities (which depend on the homeomorphism).

Definition 33. Let G be a countable group that acts on a set X. A subset $A \subset X$ is said to be *commensurate* if for each $g \in G$

$$|A \triangle (A \cdot g)| < \infty$$

A commensurate set $A \subset X$ is said to be *transfixed* if there a G-invariant set $B \subset X$ such that

$$|A \triangle B| < \infty$$

A countable group has Property (FW) if for every G-action, each commensurate set is transfixed.

Property FW is a consequence of Kazhdan's property (but is strictly weaker). FW implies Serre's property FA, hence groups with FW are finitely generatable.

Definition 34. Let G be a countable group. G is said to have property (T) if every affine isometric action of G on a real Hilbert space has a fixed point.

To see that (T) implies (FW), we shall show the contrapositive. Let G act on a set X and let $A \subset X$ be a commensurated set that is not transfixed. Consider the real Hilbert space $l^2(X)$ and let $\pi : G \to O(l^2(X))$ be the left regular representation. Let

$$b : G \to l^2(X) \qquad g \mapsto 1_{B \cdot g \setminus B}$$

This is a cocycle but not a coboundary, since B is not transfixed. It follows that the affine isometric action

$$\rho : G \to Isom(l^2(X)) \qquad \rho(g) \cdot \psi = \pi(g) \cdot \psi + b(g)$$

does not have a fixed point.

Theorem 35 (L.–Matte Bon–Triestino [71])**.** *Let G be an aperiodic action of a property FW group on a closed manifold M by countably singular C^r-diffeomorphisms. Then the action is topologically conjugate to an action by C^r-diffeomorphisms on a homeomorphic (but not necesssarily diffeomorphic) manifold N.*

Corollary 36 (L.–Matte Bon–Triestino [71])**.** *Let G be an infinite group of piecewise C^2 diffeomorphisms on $\mathbf{S}^1$. Then G does not have (T).*

For higher-rank lattices some of the most interesting (conjectural) rigidity properties are described by the so-called Zimmer's program. An important conjecture in this program states that a lattice in a higher rank simple Lie group has only (virtually) trivial actions on closed manifolds of dimension $< d$, where d is an explicit constant depending on the ambient Lie group (bounded below by its real rank). This conjecture has been (partially) solved recently with the breakthrough work of Brown, Fisher and Hurtado. It is a well known open problem whether Zimmer's conjecture holds for action by homeomorphisms that are not diffeomorphisms. In combination with the results of Brown–Fisher–Hurtado, our work yields the following.

Theorem 37 (L.–Matte Bon–Triestino [71])**.** *Let M^d be a closed manifold of dimension d. Let G be a connected Lie group, whose Lie algebra is simple and with finite centre. Assume that the real rank of G is $r > d$ and let $\Gamma \subset G$ be a cocompact lattice, or $\Gamma = \mathrm{SL}(r+1, \mathbf{Z})$. For any morphism $\rho : \Gamma \to \Omega\mathrm{Diff}^2(M)$, the action of $\rho(\Gamma)$ on M has a finite orbit.*

Is F automatic?

Murray Elder

Let G be a group with finite symmetric generating set $X = X^{-1}$. An *automatic structure* for (G, X) is the following collection of finite state automata (FSA):

- an FSA M accepting $L \subseteq X^*$ in bijection[2] with G
- for each $x \in X \cup \{\epsilon\}$ an FSA M_x accepting $\{u \otimes v \mid u, v \in L, v =_G ux\}$

where the notation $u \otimes v$ means words of the form

$$\binom{u_1}{v_1} \cdots \binom{u_s}{v_s} \binom{\$}{v_{s+1}} \cdots \binom{\$}{v_t}$$

if $u = u_1 \dots u_s, v = v_1 \dots v_t$ with $t \geqslant s$,

$$\binom{u_1}{v_1} \cdots \binom{u_t}{v_t} \binom{u_{t+1}}{\$} \cdots \binom{u_s}{\$}$$

[2]Equivalently, L surjects to G.

if $s > t$, and $ is a padding symbol[3]. If such a structure exists then (G, X) is *automatic.*

An equivalent, more geometric definition is (G, X) is automatic if there is:

- a regular language $L \subseteq X^*$ in bijection with G
- a constant $k \in \mathbf{N}$ such that for each $u, v \in L$ with $v =_G ux$ for some $x \in X \cup \{\epsilon\}$

$$d_X(u(t), v(t)) \leqslant k.$$

That is, in the Cayley graph for (G, X) L-words which start at the identity and end distance at most 1 apart must *synchronously k-fellow travel.*

Example 38. $\mathbf{Z}^2 = \langle a, b \mid ab = ba \rangle$, $L = \{a^i b^j \mid i, j \in \mathbf{Z}\}$. Figure 1 shows the automaton M_a.

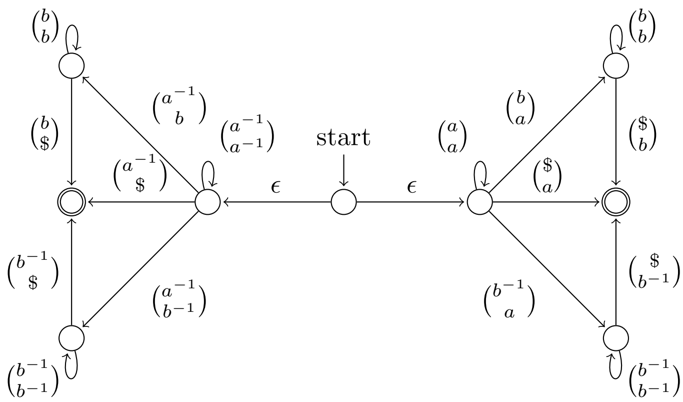


FIGURE 1. The FSA M_a for $\mathbf{Z}^2$.

Example 39. If G is any δ-hyperbolic group with finite generating set $X = X^{-1}$, the set of all shortlex geodesics is regular and satisfies the synchronous fellow travelling condition for a constant depending on δ. In fact, the set of all geodesics also gives an automatic structure (replacing bijection by surjection in the definition), as does the set of all (λ, μ)-quasigeodesics provided $\lambda \in \mathbf{Q}$ and some mild extra conditions [22].

Here are some facts [16]:

- being automatic is independent of the choice of finite generating set
- L-words are quasi-geodesics; this follows easily from the pumping lemma for regular languages as follows. Let $u \in L$ be the L-word for the identity, $|u| = c$, m the maximum number of states in any M_x, and consider a geodesic $v = a_1 \dots a_n \in X^*$. Define a sequence of L-words recursively by $v_0 = u, v_i =_G v_{i-1}a_i$ Then $|v_i| \leqslant |v_{i-1}| + m$ since otherwise one could pump

[3]Equivalently, (u, v) are accepted by a *synchronous 2-tape automaton.*

the suffix containing $\binom{\$}{x}$ symbols and obtain infinitely many L-words for v. Then $|v_n| \leqslant mn + c$.
- the word problem for automatic groups can be solved in at most quadratic time and linear space (use the previous argument to compute the L-words v_i for a given input word $v = a_1 \dots a_n$)
- automatic implies G has a Dehn function that is at most quadratic
- automatic implies G is type FP_∞ [42, 1].

So, is F automatic? Recall that Thompson's group F has the finite presentation

$$\langle x_0, x_1 \mid [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] \rangle.$$

It is known that F has quadratic Dehn function [20], is type FP_∞ [37], has a quasi-linear ($n \log n$) time word problem (algorithm: draw the tree pair diagram). So none of the obvious properties rule F out from being automatic.

Guba and Sapir give the following regular normal form for elements of F: $L =$ all freely reduced words which avoid factors ($i > 0$):
- $x_1^{\pm 1} x_0^i x_1$
- $x_1^{\pm 1} x_0^{i+1} x_1^{-1}$.

The comparison automaton M_{x_0} is easy to construct, since multiplying a word in L on the right by x_0 changes the suffix by at most one letter. However multiplication by x_1 can cause word length to explode: consider $w_i = x_1 x_0^i$ with $i > 0$. Then

$$x_1 x_0^i x_1 \to x_0^i x_1 x_0^{-i-1} x_1 x_0^{i+1}.$$

Then the L-words for $w_i, w_i x_1$ have length difference $2i + 3$ so when i is greater than then number of states of M_{x_1} we can apply the pumping lemma to obtain infinitely many words u with $w_i \otimes u$ accepted, which is a contradiction.

Note that a weaker version of automatic is to allow words that end at most an edge apart to *asynchronously* fellow travel, or equivalently the comparator automata M_x to read words asynchronously. Consider $w_{m,i} = x_1^m x_0^i$ with $m, i > 0$. The L-word for $w_{m,i} x_1$ is

$$x_0^i x_1 x_0^{-i-1} x_1^m x_0^{i+1}$$

and a careful pumping lemma argument also leads to a contradiction showing that the language also fails to give an asynchronous automatic structure for F.

Non-automatic groups with quadratic Dehn function. Stallings' group

$$\left\langle a, b, c, d, s \;\middle|\; \begin{array}{l} [a,c] = [a,d] = [b,c] = [b,d] = 1, \\ (a^{-1}b)^s = a^{-1}b, (a^{-1}c)^s = a^{-1}c, (a^{-1}d)^s = a^{-1}d \end{array} \right\rangle$$

is not type FP_3 [52] and has quadratic Dehn function [64]. It can be seen as the kernel of the map $F_2 \times F_2 \times F_2 \to \mathbf{Z}$ which sends words to their exponent sum; taking n copies of F_2 gives the n-th Bieri–Stallings group which is type FP_{n-1} but not type FP_n [34], and these (for $n > 3$) were also shown to have quadratic Dehn function [62].

Another interesting example is

$$\langle a, b, s, t \mid ab = ba, a^s = ab, a^t = ab^{-1} \rangle$$

which is type FP_∞, not CAT(0) [65], has a quadratic Dehn function [58], has an asynchronously automatic structure [14], but does not admit an automatic structure [101]. The proof of non-automatic relies on a direct argument that, if it were, the set of *slopes* you would expect to see in the embedded $\mathbf{Z}^2$ planes in the Cayley graph should be finite, which leads to a contradiction. It is possible that some similar direct argument can be constructed to rule out the possibility that F is automatic.

Why should F not be automatic? None of the following facts prove that F cannot have an automatic structure, but they do not bode well.

- F has many "bad" subgroups such as $\mathbf{Z}^d$ for any $d \in \mathbf{N} \cup \{\infty\}$, and arbitrary iterated wreath products of $\mathbf{Z}$.
- Cleary, the author and Taback [13] showed that for the standard generating set, any set of words that contains at least one geodesic for each element cannot be regular, so $(F, \{x_0, x_1\})$ has no geodesic automatic structure.
- Jeremy Hauze [21] strengthened this to: languages that have at least one representative of each element of F of word length that is within a *fixed constant* of the geodesic length cannot be part of an automatic structure.

Is F graph automatic? Weakening the notion of automatic further we arrive at the following. A *graph automatic structure* [24] for (G, X) is:

- a finite *symbol alphabet* S (not necessarily corresponding to group elements)
- an FSA M accepting $L \subseteq S^*$ in bijection[4] with G
- for each $x \in X \cup \{\epsilon\}$ an FSA M_x accepting $\{u \otimes v \mid u, v \in L, v =_G ux\}$.

Example 40. The 3-dimensional Heisenberg group consisting of matrices

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

which correspond to triples (a, b, c) of integers. Writing a, b, c in binary we can use an alphabet $S =$ consisting of symbols (i, j, k) with $i, j, k \in \{0, 1, +, -\}$. For example

$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

is represented as $(-, +, +)(1, 0, 0)(1, 0, 1)(0, 1, 0)$. It is easy to check that multiplication by generators $(1, 0, 0), (0, 1, 0)$ simply adds 1 in one position. Berdinsky and Trakuldit [8] attribute this observation to Sénizergues.

Other examples of graph automatic groups include include all Baumslag–Solitar groups, various wreath products, all finitely generated nilpotent groups of nilpotency class at most two [24, 7, 6]. As for automatic groups we have [24]:

- L-words (over symbols) have quasi-geodesic length

[4] Equivalently, L surjects to G

– at most a quadratic time word problem
– being graph automatic is invariant under change of finite generating set
– can assume without loss of generality that S is a subset of the generating set. However, paths in the Cayley graph labeled by S-edges do not necessarily end anywhere near the group element represented by the label of the path. See [8].

Thompson's group F seems like a natural candidate for graph automaticity, since we have many nice ways to represent elements, for example as tree pair diagrams. However, any encoding of a tree pair using a finite alphabet will require some memory. This leads to the notion of a $\mathscr{C}$-graph automatic structure where we replace regular languages by languages in the class $\mathscr{C}$ in the definition. This even weaker notion still implies some nice properties: for counter-graph automatic with a quasigeodesic normal form we still have a polynomial time algorithm to compute L-words, which means a polynomial time word problem [15]. In [29] Taback and Younes constructs a (3-counter)-graph automatic structure based on tree pair diagrams for F.

Encoding the infinite normal form in a certain way, the author and Taback were able to lower the complexity to (1-counter)-graph automatic. We write words

$$x_0^{i_0}x_1^{i_1}\dots x_r^{i_r}x_s^{-j_s}\dots x_0^{-j_0}$$

as strings over an alphabet $\{\#, a, b\}$ in such a way that the conditions required to have unique representatives are regular to check. The single counter is needed to check multiplication by x_1. Specifically we represent $x_0^{i_0}\dots x_r^{i_r}x_s^{-j_s}\dots x_0^{-j_0}$ as

$$a^{i_0}b^{j_0}\#\dots\#a^{i_m}b^{j_m}$$

where $m = \max\{r, s\}$. The words obtained are quasigeodesic [60].

Final remarks. Another extension of the notion of automatic which I did not discuss in the talk is *autostackable* [9] and the weaker notion of *algorithmically stackable* [59]. Brittenham, Hermiller and Holt introduced these notions, showing that they also imply some nice computation properties. Cleary, Hermiller, Stein and Taback prove that F is algorithmically stackable with respect to a deterministic context-free language of normal forms [63, 59].

Whether F is another example of a group with quadratic Dehn function that is not automatic, or if in fact it admits some nice automatic or graph automatic structure remains open. Once again F proves itself to be an enigma.

Spraiges, 3-manifolds, and conjugacy for a braided Thompson group

Yuri Santos Rego
(joint work with Kai-Uwe Bux)

The braided version of Thompson's group V, introduced by M. G. Brin and P. Dehornoy [88, 92] and denoted V_{br} here, is a certain subgroup of the braid group on a Cantor set of strands. We presented a strategy to solve the conjugacy problem for V_{br} following ideas previously applied to braid groups (by E. Artin [82]) as well as to diagram groups (by V. Guba and M. Sapir [93]) and to Thompson's classical groups $F \subset T \subset V$ (by J. M. Belk and F. Matucci [5]). Our methods employ "split-braid-merge band diagrams" (spraiges [39]) and reformulate the problem into a question about the algorithmic recognition of certain 3-manifolds.

1. The problem

Since the original triad of groups $F \subset T \subset V$ was introduced in the sixties by Richard Thompson [102], one is usually confronted with three basic questions about a Thompson-like group when it is born. Specifically: what can be said about its normal subgroup structure, its decision problems, and its homological and homotopical finiteness properties?

Regarding decision problems for Thompson-like groups, the conjugacy problem is in general quite challenging; see, for instance, [77, 23, 17, 27, 11, 2, 4], to name a few. Even for the classical groups $F \subset T \subset V$ we have a gap of many years between the first known solutions. Recall the following.

Theorem 41. *The conjugacy problem is decidable for F (Guba–Sapir* [93]*), for T (Belk–Matucci* [5]*), and for V (Higman* [111] *and Barker–Duncan–Robertson* [2]*).*

Inspired by the ideas of Guba–Sapir in the context of diagram groups [93], and using strand diagrams popularized by Jim Belk in his thesis [57], Belk and Matucci developed in [5] a strategy that actually works for F, T and V. This provided in particular a unified proof of the results of Guba–Sapir and Higman. The key ingredient in Belk–Matucci's proof is to produce a geometric conjugacy invariant obtained by "closing" in an annulus the diagrams used to represent elements of the given groups. This lead Matucci to conjecture in his Ph.D. thesis that their ideas might work for other Thompson-like groups. In particular, he asked the following.

Question 42 (Matucci [25])**.** *Does a similar strategy work for the group V_{br}?*

It turns out that Belk and Matucci's ideas are very similar to what has been observed over eight decades ago in the theory of knots, links and braids. Recall the following classical result.

Theorem 43 (Artin [82, 115])**.** *Two braids are conjugate if and only if their closures in the solid torus are ambient isotopic.*

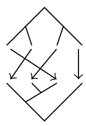
Making use of the strong similarities between knots and links and Belk–Matucci's arguments as well as the natural connection between V_{br} and braid groups, we answer Matucci's question in the afirmative.

Theorem 44. *The conjugacy problem for V_{br} is decidable.*

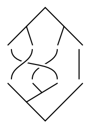
2. Spraiges

Recall that V is the group given by equivalence classes (under "expansion" and "reduction") of tree-paired diagrams with permutations between the leaves of the given trees [102]. To define the braided variant of V, one allows for braidings (instead of just permutations) in the diagrams that depict typical elements of V;

see Figure 2. We refer the reader to [86] for more on such *braided tree-paired diagrams.*



(a)



(b)

FIGURE 2. (a) An element of V; (b) An element of V_{br}.

However, we shall need a concrete geometric version of such diagrams. To define them, we simply combine the strand diagrams of Belk–Matucci with the braided band diagrams considered by Bux–Sonkin [12]. Loosely speaking, a *split-braid-merge band diagram*, nicknamed *spraige* by Zaremsky in [39], is a surface in the solid cylinder which "flows" from top to bottom and is made up of a strip which (can) split and than merges back before reaching the bottom. Moreover, the surface keeps track of the relative order in which the splits and merges occur, and braidings between the strips can occur "in the middle"; see Figure 3 for an example.

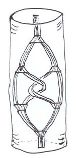


FIGURE 3. A spraige...

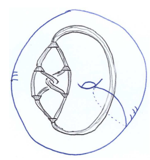


FIGURE 4. ...and its closure.

Spraiges are considered up to ambient isotopy fixing the boundary of the ambient cylinder. Defining reduction moves on spraiges similar to "reduction and expansion" [12, 39], it follows that V_{br} is the group of equivalence classes of (isotopy classes of) spraiges with operation given by concatenation, i.e. gluing spraiges on top of each other.

3. Closures and 3-manifolds

Following Artin and Belk–Matucci, we define the *closure* of a spraige to be the surface, embedded in the solid torus, obtained by identifying the top and bottom of the ambient cylinder; see Figure 4. (Here, one has to be a little bit more careful when defining equivalence of closed spraiges since ambient isotopies in the torus can produce undesired equivalent surfaces, e.g. by "flipping" a closed spraige.) We establish a notion of *admissible* isotopy and reduction moves, and obtain the desired conjugacy invariants.

Proposition 45. *Two (classes of) spraiges in V_{br} are conjugate if and only if their closures in the solid torus are admissibly equivalent.*

This raises the question of how to algorithmically compare two given closed spraiges. The main challenge in following this line of thought of Belk–Matucci is the nature of the diagrams employed. In their case, solutions to the graph isotopy problem in surfaces apply, whereas closed spraiges live in 3-manifolds and thus comparing them algorithmically is trickier. We then appeal to the similarities with knot theory as a second step towards our solution. Recall the following.

Theorem 46 (Gordon–Luecke [110])**.** *Two knots in the* 3*-sphere S^3 are ambient isotopic if and only if their complement spaces are homeomorphic.*

The complement of a knot is obtained by removing from S^3 an open tubular neighborhood containing the knot. If we imagine a closed spraige as a flat hose and fill it up with water, we obtain a handlebody whose boundary contains the boundary of the underlying closed spraige. The *complement of a closed spraige* is the 3-manifold obtained from the solid torus by removing the interior of such a handlebody. These 3-manifolds already come with nice features, namely, they are compact (with boundary), oriented, and sufficiently large.

In contrast with Gordon–Luecke's famous result, however, it is not hard to see that the homeomorphism type of the complement of a closed spraige does not give complete information about it. Furthermore, knot complements are "even better" spaces, namely, they are Haken manifolds. Nevertheless, we can still improve the complement as to encode more information on the isotopy class of the given closed spraige and become a "better" space.

More precisely, by removing from the complement small tori around its already missing tubes yields a Haken manifold. Moreover, we draw certain canonical graphs on the boundary of the resulting space to obtain a *boundary pattern* which essentially determines the underlying closed spraige.

Finally, to obtain Theorem 44, we can make use of the following program to distinguish Haken manifolds, initiated by Wolfgang Haken in the sixties and completed by Sergey Matveev in the last decade after fundamental works of Johannson, Hemion, Bestvina and Handel.

Theorem 47 (Haken–Hemion–Matveev–Thurston [26])**.** *There exists and algorithm that, given two Haken manifolds with boundary patterns, decides whether there exists a homeomorphism between the manifolds which takes one boundary pattern isomorphically onto the other.*

4. Follow-up questions

After Belk–Matucci and as pointed out by Bleak and Zaremsky, one can state two natural problems.

Question 48. *Is the conjugacy problem decidable for the groups F_{br}* [86] *and T_{br}* [55]*? (These are the braided variants of F and T, respectively.)*

It is unclear how to adapt the strategy presented here to the subgroups $F_{br} \subset T_{br}$ of V_{br}—the former is constructed using pure braids, and the latter might need more complicated diagrams (or surfaces) to produce the desired conjugacy invariants; compare [5].

Question 49. *What other kinds of information do closed spraiges give for the group V_{br}?*

In [85, 5], for example, similar diagrams are used to obtain information on the dynamics of Thompson-like groups.

Lastly, one may also repeat Matucci's question in broad generality: for which Thompson-like groups does a similar strategy to check decidability of the conjugacy problem work?

Groups of fast homeomorphisms of the interval and the ping-pong argument

Collin Bleak

(joint work with Matthew G. Brin, Martin Kassabov, Justin T. Moore, Matthew C. B. Zaremsky)

This extended abstract of the talk of C. Bleak in the 2018 Oberwolfach workshop "Cohomological and Metric Properties of Groups of Homeomorphisms of $\mathbf{R}$" details work on a large set of subgroups of the group $\mathrm{Homeo}_+(I)$; the group of the orientation preserving homeomorphisms of the unit interval. It represents work in an article to appear in the International Journal of Combinatorial Algebra (and in some places, shamelessly steals the text of that article) which is joint with Matthew G. Brin, Martin Kassabov, Justin Tatch Moore, and Matthew C. B. Zaremsky. There will be two further extended abstracts of talks representing more of our work in the area (the second by Moore will more specifically be on the complexity classes of the elementary amenable subgroups of F, representing a joint paper under review of Bleak, Brin, and Moore, and the final extended abstract will be by Brin, where he will discuss the limits of our current knowledge on the subject).

1. Introduction

The ping-pong argument was first used in [112, §16] and [107, §II,3.8] to analyze the actions of certain groups of linear fractional transformations on the Riemann sphere. Later distillations and generalizations of the arguments (e.g., [114, Theorem 1]) were used to establish that a given group is a free product. We adapt the ping-pong argument to the setting of subgroups of $\mathrm{Homeo}_+(I)$ with the motivation of developing a better understanding of the finitely generated subgroups of the group $\mathrm{PL}_+(I)$ of piecewise linear order-preserving homeomorphisms of the unit interval. The analysis described resembles the original ping-pong argument in that it establishes a forest structure (instead of a tree structure) on certain orbits of a group action, and uses this to determine algebraic properties of the acting group.

The focus of this extended abstract is on subgroups of $\mathrm{Homeo}_+(I)$ which are specified by what we term geometrically fast generating sets. A geometrically fast set X of generators admits a set M of markings (finite in the subcase of piecewise-linear generators), and the set $M\langle X\rangle$ is the orbit upon which we establish a forest structure. Our main result shows that the isomorphism type of a group specified by a "geometrically fast" generating set is determined by its "dynamical diagram," a finite directed labelled graph which encodes the qualitative relative dynamics of the generators.

Theorem 50. *If two geometrically fast sets $X, Y \subseteq \mathrm{Homeo}_+(I)$ have only finitely many transition points and have isomorphic dynamical diagrams, then the induced bijection between X and Y extends to an isomorphism of $\langle X\rangle$ and $\langle Y\rangle$ (i.e. $\langle X\rangle$ is* marked isomorphic *to $\langle Y\rangle$). Moreover, there is an order preserving bijection $\theta : M\langle X\rangle \to M\langle Y\rangle$ such that $f \mapsto f^\theta$ induces the isomorphism $\langle X\rangle \cong \langle Y\rangle$.*

We will give the definitions of the various undefined terms from the statement of Theorem 50 later. For now, we simply mention that a dynamical diagram is small a combinatorial object describing the relative dynamics of a set of elements of $\mathrm{Homeo}_+(I)$, and that any set of elements of $\mathrm{Homeo}_+(I)$ satisfying some mild technical conditions can be made into a "geometrically fast" set by raising the individual generators to powers, without changing the related dynamical diagram. Thus, our theorem represents a form of stable rigidity for subgroups of $\mathrm{Homeo}_+(I)$. We also mention that all groups satisfying the conditions of the theorem can actually be embedded in R. Thompson's group F.

2. Drawing comparison with the classical Ping-Pong Lemma

First recall the classical *Ping-Pong Lemma* (see [118, Prop. 1.1]):

Lemma 51 (Ping Pong). *Let S be a set and A be a set of permutations of S such that $a^{-1} \notin A$ for all $a \in A$. Suppose there is an assignment $a \mapsto D_a \subseteq S$ of pairwise disjoint sets to each $a \in A^\pm := A \cup A^{-1}$ and an $x \in S \setminus \bigcup_{a\in A^\pm} D_a$ such that if $a \neq b^{-1}$ are in $A^\pm$, then*

$$\big(D_b \cup \{x\}\big)a \subseteq D_a.$$

Then A freely generates $\langle A\rangle$.

(We adopt the convention of writing permutations to the right of their arguments.)

In the current discussion, we will relax the hypothesis so that the containment $D_b a \subseteq D_a$ is required only when D_b is contained in the *support* of a; similarly $xa \in D_a$ is only required when $xa \neq x$.

Consider the three functions $(b_i \mid i < 3)$ in $\mathrm{Homeo}_+(I)$ whose graphs are shown in Figure 5.

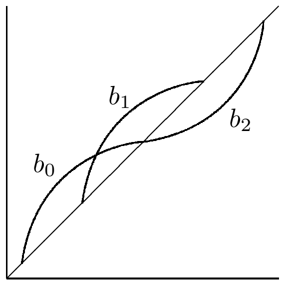
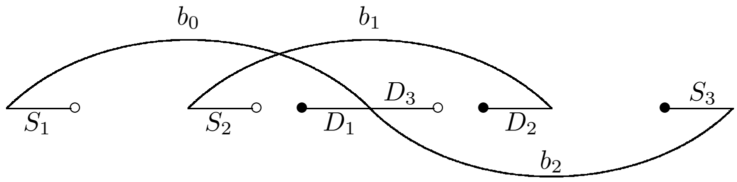


Figure 5. Three homeomorphisms

A schematic diagram (think of the line $y = x$ as drawn horizontally) of these functions might be:



In this diagram, we have assigned intervals S_i and D_i to the ends of the support to each b_i so that the entire collection of intervals is pairwise disjoint. Our system of homeomorphisms is assumed to have an additional dynamical property reminiscent of the hypothesis of the Ping-Pong Lemma:

- $S_i b_i \cap D_i = \emptyset$;
- b_i carries $\operatorname{supt}(b_i) \setminus S_i$ into D_i;
- b_i^{-1} carries $\operatorname{supt}(b_i) \setminus D_i$ into S_i for each i.

A special case of our main result is that these dynamical requirements on the b_i's are sufficient to characterize the isomorphism type of the group $\langle b_i \mid i < 3\rangle$: any triple $(c_i \mid i < 3)$ which produces this same dynamical diagram and satisfies these dynamical requirements will generate a group isomorphic to $\langle b_i \mid i < 3\rangle$. In fact the map $b_i \mapsto c_i$ will extend to an isomorphism. In particular,

$$\langle b_i \mid i < 3\rangle \cong \langle b_i^{k_i} \mid i < 3\rangle$$

for any choice of $k_i \geq 1$ for each $i < 3$.

3. Definitons and further results

We will now formalise the general discussion above.

In this abstract, recall that we use right actions. For instance, tg will denote the result of applying a homeomorphism g to a point t. Recall that from the introduction that if f is in $\mathrm{Homeo}_+(I)$, then its *support* is defined to be $\operatorname{supt}(f) := \{t \in I \mid t \neq tf\}$. The support of a subset of $\mathrm{Homeo}_+(I)$ is the union of the supports

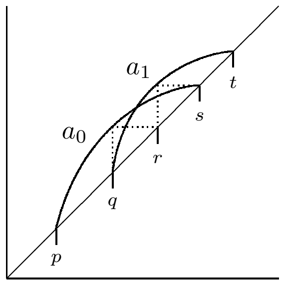


FIGURE 6. A geometrically fast set of bumps

of its elements. A left (right) *transition point* of f is a $t \in I \setminus \operatorname{supt}(f)$ such that for every $\epsilon > 0$, $(t, t + \epsilon) \cap \operatorname{supt}(f) \neq \emptyset$ (respectively, $t - \epsilon, t) \cap \operatorname{supt}(f) \neq \emptyset$. An *orbital* of f is a component its support. An orbital of f is *positive* if f moves elements of the orbital to the right; otherwise it is *negative.* If f has only finitely many orbitals, then the left (right) transition points of f are precisely the left (right) end points of its orbitals. An *orbital* of a subset of $\operatorname{Homeo}_+(I)$ is a component of its support.

An element of $\operatorname{Homeo}_+(I)$ with one orbital will be referred to as a *bump function* (or simply a *bump*). If a bump a satisfies that $ta > t$ on its support, then we say that a is *positive*; otherwise we say that a is *negative.* If $f \in \operatorname{Homeo}_+(I)$, then $b \in \operatorname{Homeo}_+(I)$ is a *signed bump* of f if b is a bump which agrees with f on its support. If X is a subset of $\operatorname{Homeo}_+(I)$, then a bump a is *used in* X if a is positive and there is an f in X such that f coincides with either a or a^{-1} on the support of a. A bump a is used in f if it is used in $\{f\}$. We adhere to the convention that only positive bumps are used by functions to avoid ambiguities in some statements. Observe that if $X \subseteq \operatorname{Homeo}_+(I)$ is such that the set A of bumps used in X is finite, then $\langle X \rangle$ is a subgroup of $\langle A \rangle$.

If $(g_i \mid i < n)$ and $(h_i \mid i < n)$ are two generating sequences for groups, then we will say that $\langle g_i \mid i < n \rangle$ is *marked isomorphic* to $\langle h_i \mid i < n \rangle$ if the map $g_i \mapsto h_i$ extends to an isomorphism of the respective groups.

A precursor to the notion of a geometrically fast generating set is that of a geometrically proper generating set. A set $X \subseteq \operatorname{Homeo}_+(I)$ is *geometrically proper* if there is no element of I which is a left transition point of more than one element of X or a right transition point of more than one element of X. Observe that any geometrically proper generating set with only finitely many transition points is itself finite.

If X is a finite geometrically proper subset of $\operatorname{Homeo}_+(I)$, then we will often identify X with its enumeration in which the minimum transition points of its elements occur in increasing order. When we write $\langle X \rangle$ is marked isomorphic to $\langle Y \rangle$, we are making implicit reference to these canonical enumerations of X and Y.

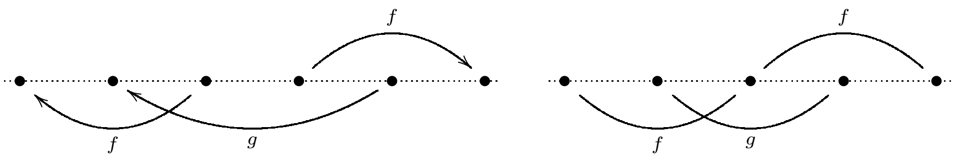


FIGURE 7. The dynamical diagram for the Brin–Navas generators, with an illustration of the contraction convention

Before turning to the definition of *geometrically fast* in the context of finite subsets of $\mathrm{Homeo}_+(I)$, we first need to develop some further terminology. A marking of a geometrically proper collection of bumps A is an assignment of a marker $t \in \mathrm{supt}(a)$ to each $a \in A$. If a is a positive bump with orbital (x, y) and marker t, then we define its *source* to be the interval $\mathrm{src}(a) := (x, t)$ and its *destination* to be the interval $\mathrm{dest}(a) := [ta, y)$. We also set $\mathrm{src}(a^{-1}) := \mathrm{dest}(a)$ and $\mathrm{dest}(a^{-1}) := \mathrm{src}(a)$. The source and destination of a bump are collectively called its *feet*.

A collection A of bumps is *geometrically fast* if it there is a marking of A for which its feet form a pairwise disjoint family (in particular we require that A is geometrically proper). This is illustrated in Figure 6, where the feet of a_0 are (p, q) and $[r, s)$, and the feet of a_1 are (q, r) and $[s, t)$. Being geometrically fast is precisely the set of dynamical requirements made on the set $\{a_i \mid i < 3\}$ of homeomorphisms mentioned earlier.

Notice that, since pairwise disjoint families of intervals in I are at most countable, any geometrically fast set of bumps is at most countable.

We now generalise further. A set $X \subseteq \mathrm{Homeo}_+(I)$ is *geometrically fast* if it is geometrically proper and the set of bumps used in X is geometrically fast.

Observe that if X is geometrically proper, each of its elements uses only finitely many bumps, and the set of transition points of X is discrete, then there is a map $f \mapsto k(f)$ of X into the positive integers such that $\{f^{k(f)} \mid f \in X\}$ is geometrically fast, as raising elements to large powers reduces the size of the feet appearing in each bump. Also notice that if $\{f^{k(f)} \mid f \in X\}$ is geometrically fast and if $k(f) \le l(f)$ for $f \in X$, then $\{fl^{l(f)} \mid f \in X\}$ is geometrically fast as well.

If X is a geometrically fast generating set with only finitely many transition points, then the *dynamical diagram* D_X of X is the edge labeled vertex ordered directed graph defined as follows:

- the vertices of D_X are the feet of X with the order induced from the order of the unit interval;
- the edges of D_X are the signed bumps of X directed so that the source (destination) of the edge is the source (destination) of the bump;
- the edges are labeled by the elements of X that they come from.

The dynamical diagram of a generating set for the Brin–Navas group B of [76, 81] and first called B in [73] is illustrated in the left half of Figure 7; the

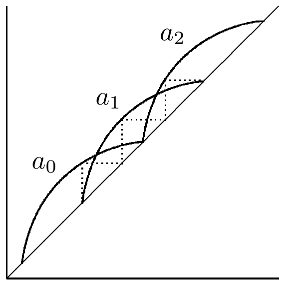


FIGURE 8. A point is tracked through a fast transition chain

generators are $f = a_0^{-1}a_2$ and $g = a_1^{-1}$, where the $(a_i \mid i < 3)$ is the geometrically fast generating sequence illustrated in Figure 8. We have found that when drawing dynamical diagram D_X of a given X, it is more aesthetic whilst being unambiguous to collapse pairs of vertices u and v of D_X such that:

- v is the immediate successor of u in the order on D_X,
- u's neighbor is below u, and v's neighbor is above v.

Additionally, arcs can be drawn as over or under arcs to indicate their direction, eliminating the need for arrows. This is illustrated in the right half of Figure 7. The result qualitatively resembles the graphs of the homeomorphisms rotated so that the line $y = x$ is horizontal.

An isomorphism between dynamical diagrams is a directed graph isomorphism which preserves the order of the vertices and induces a bijection between the edge labels (i.e. two directed edges have equal labels before applying the isomorphism if and only if they have equal labels after applying the isomorphism). Notice that such an isomorphism is unique if it exists.

Supposing $X \subseteq \mathrm{Homeo}_+(I)$ is geometrically fast and has a finite dynamical diagram. Then, we can find ε-close elements of R. Thompson's group F, so that the components of support of these new elements overlap each other with the same relative structure as the components of support of the elements of X. Raising these elements to sufficiently high powers results in a geometrically fast set in f with a dynamical diagram isomorphic to that of X. Thus, we have the following.

Theorem 52. *For each finite dynamical diagram D, there is a geometrically fast $X_D \subseteq F$ such that if $X \subseteq \mathrm{Homeo}_+(I)$ is geometrically fast and has dynamical diagram D, then there is a marked isomorphism $\phi : \langle X\rangle \to \langle X_D\rangle$ and a continuous order preserving surjection $\hat\theta : I \to I$ such that $f\hat\theta = \hat\theta\phi(f)$ for all $f \in \langle X\rangle$.*

Since by [77] F does not contain nontrivial free produces of groups, subgroups of $\mathrm{Homeo}_+(I)$ which admit geometrically fast generating sets are not free products. It should also be remarked that while our motivation comes from studying the groups F and $\mathrm{PL}_+(I)$, the results here are much broader: for instance, one can re-state the Theorem 52 using $\mathrm{Diff}_+^\infty(I)$ instead of F.

It is natural to ask how restrictive having a geometrically fast or geometrically proper generating set is. The next theorem makes use of the main result of [1] to show that many finitely generated subgroups of $\mathrm{PL}_+(I)$ have at least a geometrically proper generating set.

Theorem 53. *Every n-generated one orbital subgroup of $\mathrm{PL}_+(I)$ either contains an isomorphic copy of F or else admits an n-element geometrically proper generating set.*

Notice that every subgroup of $\mathrm{Homeo}_+(I)$ is contained in a direct product of one- orbital subgroups of $\mathrm{Homeo}_+(I)$. Thus if one's interest lies in studying the structure of subgroups of $\mathrm{PL}_+(I)$ which do not contain copies of F, then it is typically possible to restrict one's attention to groups admitting geometrically proper generating sets.

Finitely generated elementary amenable subgroups of F

Justin Tatch Moore
(joint work with Collin Bleak, Matthew G. Brin)

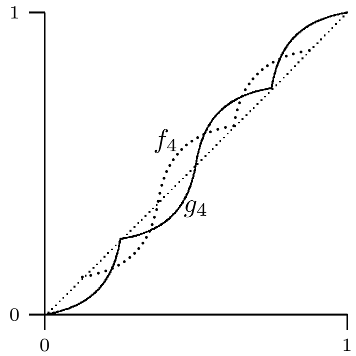
This extended abstract is essentially part of the introduction of [75] *which is currently submitted for publication. It has been lightly edited and truncated to fit the present context.*

Our aim is to initiate a program to classify the finitely generated subgroups of Richard Thompson's group F. While we are far from completing this program, we isolate a class $\mathfrak{S}$ of finitely generated subgroups of F which exhibits a high degree of complexity, but which admits a complete structural analysis and seems likely to play a central role in the classification of all finitely generated subgroups of F.

The groups in $\mathfrak{S}$ all have generating sets with simple descriptions in the language of [74]. Using [74] one can specify certain subgroups of $\mathrm{Homeo}_+(I)$, the group of orientation preserving self homeomorphisms of the unit interval I, by the qualitative dynamics of their generating sets. All groups given in this way embed homomorphically into F. The groups in $\mathfrak{S}$ provide simple and natural examples of elementary amenable groups of high EA-class that we feel are of interest independent of being subgroups of F.

There are two main features of our work. The first is the shift of attention away from the usual "isomorphism type and containment relation" (the Hasse diagram) of subgroups, and toward the coarser "biembeddability class and embeddability relation" where two groups are biembeddable if each embeds in the other. A finer analysis of the isomorphism types of subgroups of F does not seem feasible at this time.

The second feature is the discovery of a rich arithmetic that lives on $\mathfrak{S}$ that greatly facilitates transfinite induction and recursion. The usual ingredients of transfinite recursion are base, successor, and limit stage: a base object A_0 must be built, an object $A_{\alpha+1}$ must be built from the object A_α, and for a limit α, an object A_α must be built from the objects A_β with $\beta < \alpha$. We show that $\mathfrak{S}$



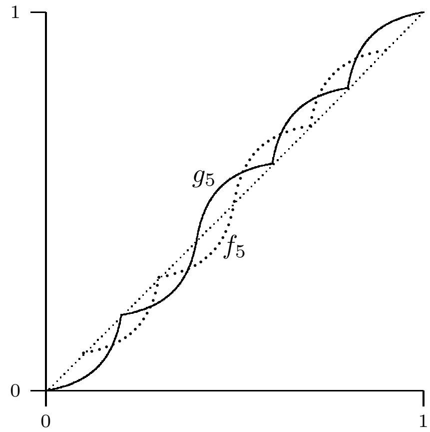


FIGURE 9. $G_{\tau_4} := \langle f_4, g_4 \rangle$ and $G_{\tau_5} := \langle f_5, g_5 \rangle$. The EA-classes of these groups are $\omega^\omega + 2$ and $\omega^{\omega^\omega} + 2$, respectively.

can be equipped with arithmetic operations that allow us to easily build from $B_\alpha \in \mathfrak{S}$ not only $B_{\alpha+1}$, but also $B_{\alpha\cdot\omega}$ and even B_{ω^α} with equal ease. This has two consequences. First, our groups are remarkably easy to "write down." Second, the bulk of the work in the paper is shifted from construction to analysis. In fact, it is still a wonder to the authors that these groups can be analyzed at all.

Elementary amenable groups form a class EG and are those groups that can be built from finite and abelian groups by a (possibly transfinite) process using extension and directed union. The EA-class of a group G in EG is a measure of the complexity of the construction process for G. Thompson's group F is not *elementary amenable (EA)* — it cannot be built from the class of amenable and finite groups by using the operations of extensions and direct limits. Matt Brin and Mark Sapir have made the following conjecture, to the effect that F is the only obstruction the elementary amenability among its subgroups:

Conjecture 54 ([76, 116])**.** *If G is a subgroup of F, then either G is elementarily amenable or else G contains a copy of F.*

Our basic thesis is that this conjecture will eventually be a corollary of a more complete understanding of the partial order $(\mathfrak{F}, \hookrightarrow)$ where $\mathfrak{F}$ is the set of biembeddability classes of finitely generated subgroups of F and $A \hookrightarrow B$ asserts that members of the class A embed into members of the class B.

The complex nature of $(\mathfrak{F}, \hookrightarrow)$ is demonstrated by our main result:

Theorem 55 ([75])**.** *There is a transfinite sequence $(G_\xi \mid \xi < \epsilon_0)$ of finitely generated elementary amenable subgroups of F such that:*

- *G_0 is the trivial group and $G_{\xi+1} \cong G_\xi + \mathbf{Z}$;*
- *G_ξ embeds into G_η if and only if $\xi \leq \eta$;*

- *Given $0 \leq \alpha < \epsilon_0$ and $n < \omega$, let $\xi = \omega^{(\omega^\alpha)\cdot(2^n)}$. If $\alpha > 0$, then the EA-class of G_ξ is $\omega \cdot \alpha + n + 2$. If $\alpha = 0$, then the EA-class of G_ξ is $n+1$.*

In particular, for each $\alpha < \epsilon_0$, there is a ξ such that the EA-class of G_ξ is $\alpha + 2$. (If the EA-class of a finitely generated group is infinite, it is always of the form $\alpha + 2$.) Thus Theorem 55 improves previous work of Brin [76], who demonstrated that there are finitely generated subgroups of F in EG of class $\xi+2$ for each $\xi < \omega^2$. With ω the smallest infinite ordinal, the ordinal ϵ_0 is the smallest ordinal solution to the equation $\omega^x = x$. If we define a sequence $(\tau_k)_{k\in\omega}$ of ordinals recursively by $\tau_0 := 2$, $\tau_1 := \omega$ and $\tau_{k+1} := \omega^{\tau_k}$ for $k > 1$, then ϵ_0 can be described as

$$\epsilon_0 = \sup\{\tau_k \mid k \in \omega\} = \omega^{\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}}.$$

The groups in $\mathfrak{G} := \{G_\xi \mid \xi < \epsilon_0\}$ are built from $\mathbf{Z}$ using certain familiar group-theoretic operations — direct sums and wreath products — as well as a new operation which is analogous to ordinal exponentiation base ω. Whether this new operation is meaningful in a broader setting is unclear but even in our rather restrictive setting, it already yields a wealth of examples. The operations also make the construction of the groups in $\mathfrak{G}$ straightforward and highly analogous to the construction of ordinals below ϵ_0 from 0 using exponentiation base ω and addition. Specifically, given the Cantor normal form for an ordinal $\xi < \epsilon_0$ there is an efficient algorithm that lets one write down a finite number of generators (explicitly as words in the generators of F if desired) for a group with EA-class $\omega \cdot \xi + 2$.

While the results of this paper concern groups, the focus of the analysis is on generating sets. The groups in $\mathfrak{G}$ are specified by a family of generating sets $\mathcal{S}$. This collection has the property that A is in $\mathcal{S}$ if and only if each of its two element subsets is in $\mathcal{S}$. The 2-element sets in $\mathcal{S}$ generate precisely the groups G_{τ_k} in the family $\mathfrak{G} = \{G_\xi \mid \xi < \epsilon_0\}$; this is the reason for setting $\tau_0 := 2$. Theorem 55 implies, in particular, that the G_{τ_k} are an infinite family of elementary amenable 2-generated subgroups of F which are not pairwise biembeddable. Two of these generating pairs are illustrated in Figure 9.

The isomorphism types of the G_{τ_k} are parametrized by the nonnegative integer k which we refer to as the *oscillation* of the generating pair from $\mathcal{S}$. Figure 9 illustrates pairs with oscillation 4 and 5. The function giving the oscillations of the pairs from an $A \in \mathcal{S}$ is the *signature* of A. Each generating set in $\mathcal{S}$ is equipped with a total order, and the signature serves as a complete invariant for all of $\mathcal{S}$.

Theorem 56 ([75]). *If $A, B \in \mathcal{S}$ have the same signature, then the order preserving bijection from A and B extends to an isomorphism from $\langle A\rangle$ to $\langle B\rangle$.*

Thus one may analyze $\mathfrak{G}$ by analyzing the set $\mathscr{S}$ of all signatures of $\mathcal{S}$. We also algebraically characterize the relation $\leq$ on $\mathscr{S}$ which comes from the embeddability relation on $\mathfrak{G}$.

The family $\mathcal{S}$ is robust at a group-theoretic level: if $A \in \mathcal{S}$, then $\langle A\rangle$ is an HNN extension of a group which is itself an increasing union subgroups of the form $\langle B\rangle$ for $B \in \mathcal{S}$. On the other hand, while the closure properties of $\mathcal{S}$ — and thus of $\mathscr{S}$ — are important in the group-theoretic analysis of $\mathfrak{S}$, they introduce redundancies which obscure the structure of the order on these classes. This is resolved by introducing algebraic operations $+$, $*$, and exp on $\mathscr{S}$ and using them to define a subclass $\mathscr{R}$ of $\mathscr{S}$. The next theorem is at the core of the proof of Theorem 55. It shows that $\mathscr{R}$ provides a notion of "normal form" for $\mathscr{S}$ and consequently for $\mathcal{S}$ (here $\mathsf{A} \equiv \mathsf{B}$ denotes $\mathsf{A} \leq \mathsf{B} \leq \mathsf{A}$).

Theorem 57 ([75]). *For each* A *is in* $\mathscr{S}$ *there is a unique* B *in* $\mathscr{R}$ *such that* $\mathsf{A} \equiv \mathsf{B}$. *Moreover there is a natural isomorphism*

$$(\mathscr{R}, <, +, \exp) \cong (\epsilon_0, \in, +, \zeta \mapsto \omega^{-1+\zeta})$$

provided $+$ *is restricted to those pairs in* $\mathscr{R}$ *for which the sum remains in* $\mathscr{R}$.

Thus each biembeddability class in $\mathfrak{S}$ has a distinguished representative — unique up to marked isomorphism — identified by the form of its signature. Moreover, this representative can be viewed as being built up from $\mathbf{Z}$ using simple arithmetic operations which are analogs of the fundamental operations of ordinal arithmetic.

Is Thompson's group F the only interesting subgroup of $PL_o(I)$?

Matthew G. Brin

(joint work with Collin Bleak, Justin Tatch Moore)

We review what is known and not known about the subgroup structure of $PL_o(I)$. The summary below uses vocabulary from the talks of Collin Bleak and Justin Moore that were given at this workshop.

1. Introduction

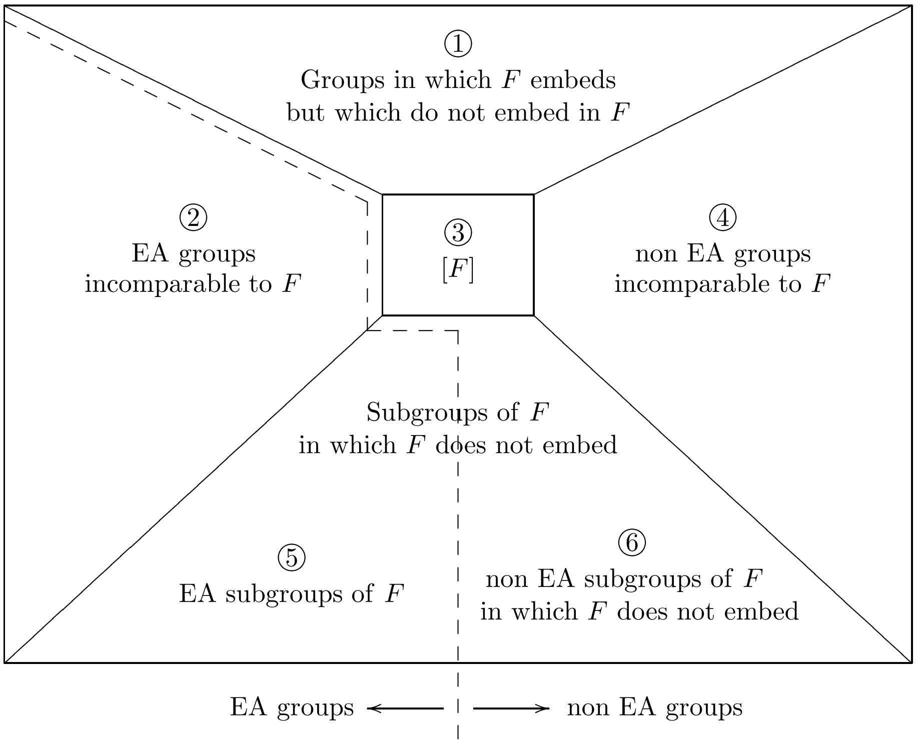
The diagram below is a picture of the universe under discussion.

Our universe is the set X of all finitely generated subgroups of $PL_o(I)$, and the ambition is to understand its structure. As a limit on the ambition, we look for structure using the relation $\hookrightarrow$ on X, where $G \hookrightarrow H$ means that there is a homomorphic embedding of G into H. The biembeddability class $[G]$ of G consists of all those H for which $G \hookrightarrow H$ and $H \hookrightarrow G$.

The relation $\hookrightarrow$ induces partial order (that we also denote by $\hookrightarrow$) on the biembeddability classes of X and we use $[X]$ to denote the set of such classes. The central class that others are compared to is the class $[F]$ of Thompson's group F. To facilitate the discussion, the set $[X]$ is broken down as shown in the diagram below. Whether a group is EA (elementary amenable) or not will figure prominently in the discussion.

The breakdown creates six regions and we discuss which groups might belong in each region. We talk of individual groups belonging to a region even though technically, the region is a set of classes. We will not agonize over this inconsistency.

Finitely generated subgroups of $PL_o(I)$



In the following, we list some facts that are known and raise questions whose answers are not yet known.

2. On the title

We place some groups in the diagram above.

Fact 58. *(Lodha) The Stein groups $F_{p_1,\dots,p_n}$ with $n > 1$ are in Region 1, and $F_{p_1,\dots,p_m}$ cannot embed in $F_{p_1,\dots,p_n}$ if $m > n$.*

Fact 59. *For each integer $n \geq 2$, the Thompson group F_n that uses slopes integral powers of n with breaks in $\mathbf{Z}[1/n]$ is in Region 3.*

Fact 60. *The solvable subgroups as analyzed by Bleak, and the chain of elementary amenable subgroups G_ξ, $\xi < \epsilon_0$ introduced by Bleak–Brin–Moore are in Region 5.*

Our main question and the reason for the title is the following.

Question 61. *Are Regions 2, 4, 6 empty?*

Our impression of the relation $\hookrightarrow$ is that it has behavior similar to the relation $|$ (divides) on the positive integers. We have $G \hookrightarrow H$ and $m|n$ if H has all the complexity of G (and possibly more) and if n has all the complexity of m (and possibly more). In the case of the integers the complexity refers to the details of the prime factorization, while in the case of groups we have no strict definition.

With this view in mind, a group in Region ① would be elementary amenable but of a complexity not reflected in the complexity of any subgroup of F. This would be an interesting group.

A group G in Region ④ or ⑥ would be interesting for a different reason. Such a group would have no subgroup isomorphic to F (would be "F-less") and as such would have finite generating set where the interior of the closure of the support of one generator (the "top" generator) contains the closures of support of all other generators (the "bottom" generators). With N the normal closure in G of the bottom generators, this gives G a preferred structure as $G = N \rtimes \mathbf{Z}$. Writing $G \to H$ if H is a finitely generated subgroup of N with connected support defines a "simplification" relation $\to$. If this relation is well founded, then G will be elementary amenable, so this relation cannot be well founded for a G in Region ④ or ⑥. An infinite descending chain under $\to$ starting from G would have to rely on a complexity of G that does not arise from the presence of a copy of F.

3. On a conjecture of Brin and Sapir

The conjecture states that every subgroup of $PL_o(I)$ either contains a copy of F or is elementary amenable. This is equivalent to asking if Regions ④ and ⑥ are empty. A positive answer to the next two questions would imply the conjecture. These questions are likely to be harder then the conjecture itself.

Question 62. *If the above relation $\to$ is restricted to groups G with connected support and with finite, fast generating set with a "top" element, then is the relation well founded?*

Question 63. *Given a finitely generated, F-less group H in $PL_o(I)$, is there a finite, fast generating set A with $\langle A\rangle \in [H]$?*

Related to Questions 62 and 63 is the following.

Question 64. *Can one usefully characterize fast, F-less subgroups of $PL_o(I)$.*

4. Order

Question 65. *Are the biembeddability classes in $[F]$ well quasi-ordered?*

A partial order $<$ is well quasi-ordered (WQO) if there is no infinite descending (under $>$) sequence and no infinite antichain. Equivalently, every sequence $(a_i \mid i > 0)$ has some $a_i \leq a_j$ with $i < j$.

WQO is likely to fail when not restricted to subgroups of F, so we ask:

Question 66. *Are there examples that violate WQO in $PL_o(I)$?*

5. Subgroups of direct products

The introduction of subgroups of direct products gives behavior that might be unexpected. To give the examples, we say that $a_1 < a_2 < \cdots a_n$ generates the n-fold wreath product of $\mathbf{Z}$ to mean that each a_i is a 1-bump function, the support of a_i has closure in the support of a_{i+1} and the set of the a_i is fast. For convenience our examples are given the unified setting $(\mathbf{Z} \wr \mathbf{Z} \wr \mathbf{Z})^3$ where the three factors are generated by $a < b < c$, $a' < b' < c'$, and $a'' < b'' < c''$, respectively.

First we have that while $\{[G_\xi] \mid \xi < \epsilon_0\} \cup \{[F]\}$ is a chain, it is not a maximal chain.

Fact 67. *The interval* $\Big[[\mathbf{Z} \wr \mathbf{Z}], [(\mathbf{Z} \wr \mathbf{Z})^2] \Big]$ *of the above chain consists of all* $(\mathbf{Z} \wr \mathbf{Z}) + \mathbf{Z}^n$*,* $n \geq 0$*, and* $(\mathbf{Z} \wr \mathbf{Z})^2$*. With* $H = \langle ab', ba' \rangle$*, then for all* $n \geq 0$*, we have* $(\mathbf{Z} \wr \mathbf{Z}) + \mathbf{Z}^n \hookrightarrow H \hookrightarrow (\mathbf{Z} \wr \mathbf{Z})^2$ *and the reverse embeddings are not possible.*

Even among the solvable subroups of $PL_o(I)$, the order $\hookrightarrow$ is not a linear order.

Fact 68. *The groups* $H = \langle ab'c'', a'b''c, a''bc' \rangle$ *and* $G = (\mathbf{Z} \wr \mathbf{Z} \wr \mathbf{Z})^2$ *are not comparable under* $\hookrightarrow$*.*

There is an inherently slow group.

Fact 69. *The group* $H = \langle c, ba^c \rangle$ *is not isomorphic to any* $\langle A \rangle$ *with* A *fast and finite.*

6. Isomorphisms

For each $n > 1$ let $\mathfrak{C}_n$ be the set of groups G generated by n fast, 1-bump functions so that G is not a non-trivial direct sum and G is not a non-trivial wreath product.

Fact 70. *For each* $n > 1$*, the elements of* $\mathfrak{C}_n$ *are easily detected from the graphs of their generators; all elements of* $\mathfrak{C}_n$ *are finitely presented; and* F_n *is an element of* $\mathfrak{C}_n$*.*

Question 71. *For each* $n > 1$*, are all the groups in* $\mathfrak{C}_n$ *of type* F_∞*?*

Question 71 will have a positive answer if the next question does.

Question 72. *For each* $n > 1$ *are all the groups in* $\mathfrak{C}_n$ *isomorphic?*

The answer to Question 72 is "yes" for $n = 2$ and $n = 3$ and is suspected to be "no" for $n > 3$. Assuming fast sets of bumps, the following shows two generating sets in $\mathfrak{C}_4$. The set on the left generates a group isomorphic to F_4.

The next question depends on a coloring of the infinite binary tree. The nodes of the tree are colored from $\{R, G, B\}$ so that the root of the tree is colored R and every caret is colored so that as one goes around the three nodes of the caret counterclockwise from its root, one reads the colors as one of RGB, GBR or

BRG. That is, one gets one of the three cyclic rotations of RGB. We consider the subgroup H of F represented by pairs of trees (S,T) so that the colors of the leaves of S read left-to-right are the same as the colors of the leaves of T read left-to-right. There are non-trivial elements in H and we ask the following.

Question 73. *What is the structure of the group H?*

It is known that H is a diagram group that is closed from the point of view of Golan–Guba–Sapir. Calculations indicate that it has 8 orbits in $\mathbf{Z}[1/2]$ and it is reasonable to guess that H is isomorphic to[5] F_9.

7. General and specific

One can ask the following general question.

Question 74. *Can one add to the knowledge of the structure of $[X]$?*

It has been suggested that Cleary's golden ratio group F_τ, discussed in the talk by Lawrence Reeves, belongs in Region 1. The group F_τ has torsion in its abelianization. It was suggested at the workshop that perhaps every finitely generated subgroup of F has torsion free abelianization, but evidence was later produced that this is not the case. We are left with the following specific question.

Question 75. *Does F_τ embed in F?*

Simple groups separated by finiteness properties

Stefan Witzel, Matthew C. B. Zaremsky
(joint work with Rachel Skipper)

We gave a pair of talks, about our positive answer in [50] to a question asked by Bertrand Rémy during a workshop on Thompson groups in St. Andrews in 2014:

Question 76. *Do there exist infinitely many quasi-isometry classes of finitely presented simple groups within the realm of Thompson groups?*

This question implicitly refers to a result of Rémy together with Pierre-Emmanuel Caprace, which gives a positive answer to the question in the realm of Kac–Moody groups:

Theorem 77 (Caprace–Rémy [103, 104])**.** *If $\mathbf{G}$ is an irreducible, 2-spherical, non-affine Kac–Moody functor and $\mathbb{F}_q$ is a sufficiently large finite field then $\mathbf{G}(\mathbb{F}_q)$ is finitely presented and virtually simple. There are infinitely many quasi-isometry classes among these groups.*

[5]**Added in proof:** Question 73 above has since been answered by Victor Guba with the result that the group described is, in fact, isomorphic to F_9.

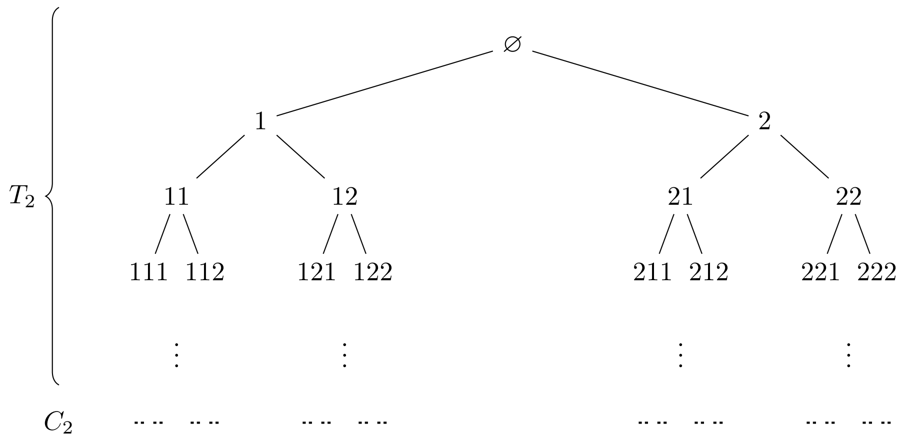


FIGURE 10. The tree T_2 and its boundary C_2.

In answering Question 76 we use topological finiteness properties as our quasi-isometry invariants. A group is of type F_n if it acts freely and cocompactly on an $(n-1)$-connected CW-complex, and of type F_∞ if it is of type F_n for all n. Each of the properties of being of type F_n is a quasi-isometry invariant. Using these invariants for groups in the Thompson realm did not seem like a promising idea at first because of the following:

Meta Theorem 78. *Relatives of Thompson groups are of type F_∞.*

Instances where Meta Theorem 78 holds can be found for example in [37, 36, 53, 40, 41, 32, 39, 47, 54, 56, 51, 83, 55].

Despite this we use Thompson groups to prove:

Theorem 79. [50] *For every $n \in \mathbf{N}_{>0}$ there exists a simple group of type F_{n-1} but not F_n.*

Corollary 80. *The answer to Question 76 is "yes".*

The basic idea is to blend the Higman–Thompson group V_d, which is simple and of type F_∞, with a self-similar group G that is not of type F_n, to obtain a Röver–Nekrashevych group $V_d(G)$ that is simple but not of type F_n.

1. Ingredients

Higman–Thompson groups. Write $[d] = \{1, \ldots, d\}$ and let T_d be the infinite rooted d-ary tree whose vertex set is the set of finite words $[d]^*$ and whose boundary is the Cantor space $C_d = \partial T_d = [d]^\omega$ of infinite words (see Figure 10).

If F_+ and F_- are finite rooted subtrees of T_d with (the same number of) leaves $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ and if $\sigma \in S_k$ is a permutation there is a homeomorphism of C_d given by

$$\begin{aligned}[F_+, \sigma, F_-]\colon C_d &\to C_d\\ u_i w &\mapsto v_{\sigma(i)} w\end{aligned}$$

for $w \in [d]^\omega$. The d-ary Higman–Thompson group is the group

$$V_d = \{[F_+, \sigma, F_-] \mid F_+, F_- < T_d, \sigma \in S_n\}$$

consisting of such homeomorphisms.

Self-similar groups. If $g \in \mathrm{Aut}(T_d)$ is a tree automorphism and $v \in [d]^*$ is a vertex, the *state* g_v of g at v is defined by the equation $g(vw) = g(v)g_v(w)$. A group $G < \mathrm{Aut}(T_d)$ is *self-similar* if every state of every element of G is itself in G. It is *finite state* if every element has only finitely many states. Every $g \in \mathrm{Aut}(T_d)$ is determined by the permutation $g(\varnothing) \in S_d$ it induces on the set of level-1 vertices of T_d, together with the tuple $(g_1, \ldots, g_d)$ of its level-1 states. With this in mind we will often write things like $g = \sigma(g_1, \ldots, g_d)$ with $\sigma = g(\varnothing)$.

Example 81 (Grigorchuk's group)**.** The self-similar group G generated by the recursively defined tree-automorphisms $a = (1\ 2)(1, 1)$, $b = ()(a, c)$, $c = ()(a, d)$, $d = ()(1, b)$ is *Grigorchuk's group.*

Röver–Nekrashevych. Given a self-similar group $G < \mathrm{Aut}(T_d)$ one can form the *Röver–Nekrashevych* group

$$V_d(G) = \{[F_+, \sigma(g_1, \ldots, g_k), F_-] \mid [F_+, \sigma, F_-] \in V_d, (g_1, \ldots, g_k) \in G^d\}$$

where the homeomorphism is given by

$$\begin{aligned}[F_+, \sigma(g_1, \ldots, g_k), F_-]\colon C_d &\to C_d\\ u_i w &\mapsto v_{\sigma(i)} g_i(w)\end{aligned}$$

The first example, developed by Röver in [100] and now called the *Röver group*, was $V_2(G)$ for G Grigorchuk's group. In [99] Nekrashevych developed the general theory for groups of the form $V_d(G)$, and proved a variety of results, in particular that the commutator subgroup $V_d(G)'$ is simple.

2. Proof sketch

The proof of Theorem 79 is modular and decomposes into four parts. Let $G < \mathrm{Aut}(T_d)$ be a self-similar group.

Proposition 82. *The commutator subgroup $V_d(G)'$ is simple, and if G is coarsely diagonal then $V_d(G)'$ has finite index in $V_d(G)$.*

Proposition 83. *If G is of type F_{n-1} then $V_d(G)$ is of type F_{n-1}.*

Proposition 84. *If G is finite state, persistent and not of type F_n then $V_d(G)$ is not of type F_n.*

Proposition 85. *For every $n \in \mathbf{N}_{>0}$ and every $d \geq 3$ there exists a coarsely diagonal, persistent, finite-state self-similar group $G < \mathrm{Aut}(T_d)$ that is of type F_{n-1} but not of type F_n.*

Here G is *coarsely diagonal* if for any $g \in G$ and any vertex v the element $(g_v)^{-1}g$ has finite order. It is *persistent* if for any $g \in G$ the wreath recursion $g = \sigma(g_1, \ldots, g_d)$ satisfies $g_d = g$.

The proof of Proposition 82 is straightforward; we know $V_d(G)'$ is simple, so it is just a computation showing the abelianization is finite.

The proof of Proposition 83 is done using an established strategy involving Brown's Criterion [36] and Bestvina–Brady Morse theory [33]. This strategy builds on work of Brown [36], Stein [53] and Farley [40], and more recently has been used in a variety of contexts to prove various Thompson-like groups satisfy Meta Theorem 78. A streamlined framework is given in [55].

The most novel proof is the one for Proposition 84. The fact that G is persistent ensures that the map

$$\begin{aligned} \rho\colon V_d(G) &\to G \\ [F_+, \sigma(g_1, \ldots, g_k), F_-] &\mapsto g_k \end{aligned}$$

is well-defined and the fact that G is finite state implies ρ is a quasi-retract. Then a result of Alonso [31] implies the proposition. This type of approach is quite different from previously utilized tools in the world of Thompson groups, and our hope is that it finds further uses in the future.

To provide the groups requested in Proposition 85 we use groups of the form $\mathcal{O}_S^\times \ltimes \mathcal{O}_S$ where $\mathcal{O}_S$ is the ring of S-integers in a rational function field of positive characteristic. These are known to be of type $F_{|S|-1}$ but not of type $F_{|S|}$ by work of Bux [38] and Kochloukova [44] using the action on Bruhat–Tits trees corresponding to places in S. We show that the action on a Bruhat–Tits tree corresponding to a place not in S gives rise to a coarsely diagonal, persistent, finite-state and self-similar action. Finite-state self-similar actions of these and related groups were also recently produced by Kochloukova and Sidki [46] and our investigation of the groups was inspired by theirs.

Rational Embeddings of Hyperbolic Groups

Francesco Matucci
(joint work with James Belk, Collin Bleak)

Let $\{0,1\}^\omega$ denote the Cantor set of all infinite binary sequences. A homeomorphism of $\{0,1\}^\omega$ is said to be **rational** if there exists an asynchronous transducer that implements the homeomorphism on infinite binary strings. In [19], Grigorchuk, Nekrashevych and Sushchanskiĭ observe that the set of all rational homeomorphisms of $\{0,1\}^\omega$ forms a group $\mathcal{R}$ under composition, which they refer to as the **rational group**. They also observe that the group of rational homeomorphisms of A^ω is isomorphic to $\mathcal{R}$ for any finite alphabet A with at least two elements.

The word **asynchronous** refers to transducers that can output a finite binary sequence of any length each time they take a digit as input. This is a generalization of **synchronous** transducers, which are required to output a single binary digit each time they take a digit of input. The asynchronous rational group $\mathcal{R}$ contains the group of synchronous rational homeomorphisms corresponding to any finite alphabet.

Groups of synchronous transducers have received much attention in the literature, primarily as this class of groups contain numerous 'exotic' groups providing examples of unusual or unexpected behaviour. While these groups do provide counterexamples to various forms of the Burnside conjecture and Milnor's conjecture, they also remain natural in many ways. Indeed, this class houses well known foundational groups which arise in other circumstances, including free groups [30], $\mathrm{GL}_n(\mathbf{Z})$ and its subgroups [10], the solvable Baumslag–Solitar groups $BS(1,m)$ [3], and the generalized lamplighter groups $(\mathbf{Z}/n\mathbf{Z}) \wr \mathbf{Z}$ [28].

On the other hand, less is known about the more complex class of groups generated by asynchronous transducers, and the full asynchronous rational group $\mathcal{R}$ of Grigorchuk, Nekrashevych, and Sushchanskiĭ. It is known that $\mathcal{R}$ is simple and not finitely generated [84]. Also, while the word problem is solvable in finitely generated subgroups of $\mathcal{R}$ [19], the periodicity problem for elements of $\mathcal{R}$ has no solution [4]. Finally, the group $\mathcal{R}$ houses 'exotic' groups of another type: the R. Thompson groups F, T, and V all embed into $\mathcal{R}$ [19], as do the Brin–Thompson groups nV (see [4] for the embedding of the group $2V$) and groups such as the Röver group V_Γ. Any group of synchronous automata embeds into $\mathcal{R}$, so $\mathcal{R}$ also contains the groups mentioned earlier.

Our main focus is on embedding questions for a different class of groups.

Theorem 86. *Any δ-hyperbolic group that acts faithfully on its Gromov boundary embeds into $\mathcal{R}$.*

A ***δ*-hyperbolic group** is a finitely generated group G whose Cayley graph satisfies Gromov's thin triangles condition. This is a vast class of finitely presented groups: in a precise sense, "generic" finitely presented groups are hyperbolic.

Every hyperbolic group G has a **horofunction boundary** ∂G, which is a compact metrizable space. Such a group G acts on ∂G by homeomorphisms, and the kernel of the action is always a finite normal subgroup of G. This gives us the following result.

Corollary 87. *Every torsion-free hyperbolic group embeds into $\mathcal{R}$.*

The proof of Theorem 86 is dynamical as opposed to algebraic. Indeed, there is a general dynamical procedure for showing that a group embeds into $\mathcal{R}$. For each natural number n, we construct a partition of the Cayley graph Γ in a union of a finite set and a finite number of infinite sectors $S_{i,n}$. Each of these infinite sectors is a disjoint union of a finite set and a finite number of infinite sectors $S_{i,n+1}$. This construction effectively builds an infinite rooted tree T_Γ whose nodes represent such sectors. The tree T_Γ is **self-similar** in the following sense:

(1) It has a partition of $\mathrm{Vert}(T_\Gamma)$ into finitely many types.
(2) For every $u, v \in \mathrm{Vert}(T_\Gamma)$ of the same type, a nonempty, finite set $\mathrm{Mor}(u,v)$ of isomorphisms $T_u \to T_v$, each of which maps every vertex of T_u to a vertex of T_v of the same type.

(3) The collection of all morphisms forms an inverse semigroup of isomorphisms between subtrees of T_Γ

We recall that an **inverse semigroup** is a semigroup S in which for every $x \in S$ there is a unique $y \in S$ such that $x = xyx$ and $y = yxy$. It can be shown that the action of a hyperbolic group on the leaves of T_Γ is rational in a sense similar to the one defined above and that allows one to embed any hyperbolic group in the rational group $\mathcal{R}$.

We observe that the realizations arising in this study of embedded copies of hyperbolic groups in the rational group are generally not synchronous. Because of this, we obtain no immediate information towards the old question of whether all hyperbolic groups are residually finite.

It is of interest to determine whether one can see structural features of the Gromov boundary (such as local cut points) reflected in the structure of transducers representing these group elements: we expect future work may explore these interactions with the theory of the structure of the Gromov boundary.

Decision Problems on Homeomorphism Groups

Altair Santos de Oliveira Tosti
(joint work with Francesco Matucci)

We consider the group of piecewise projective orientation-preserving homeomorphisms of $\mathbf{R} \cup \{\infty\}$ that stabilize infinity with a finite number of breakpoints which are fixed points of hyperbolic elements of $\mathrm{PSL}_2(\mathbf{R})$, introduced in [80]. This group is called *Monod's group* and is denoted by $H := H(\mathbf{R})$. We also consider the subgroups $H(A)$ of H which consist of all elements that are piecewise in $\mathrm{PSL}_2(A)$ with breakpoints in $\mathcal{P}_A$, the set of fixed points of hyperbolic elements of $\mathrm{PSL}_2(A)$, where A is any subring of $\mathbf{R}$.

These groups are interesting because H does not contain non-abelian free subgroups and, for any non-trivial subring $A \neq \mathbf{Z}$ of $\mathbf{R}$, $H(A)$ is non-amenable. These properties provide us a collection of counterexamples to the von Neumann–Day conjecture, stating that a group is non-amenable if and only if it contains a non-abelian free subgrop. The group H also contains a finitely presented subgroup, called the Lodha–Moore group. See [79] for more properties.

We regard H as the group of homeomorphisms of $\mathbf{R}$: an element f is in H if there are finitely many points $t_1, t_2, \ldots, t_n$ such that, on each interval $[t_i, t_{i+1}]$,

$$f\colon t \mapsto \frac{a_i t + b_i}{c_i t + d_i}, \text{ where } a_i d_i - c_i b_i = 1, \text{ for suitable } a_i, b_i, c_i, d_i \in \mathbf{R}$$

and $f\colon t \mapsto (a_0 t + b_0)/d_0$ on $(-\infty, t_1]$ and $f\colon t \mapsto (a_n t + b_n)/d_n$ on $[t_n, +\infty)$, where $a_0 d_0 = a_n d_n = 1$, for $a_0, a_n, b_0, b_n \in \mathbf{R}$.

We discuss the current progress on the study of the conjugacy problem and centralizers in H and its subgroups by generalizing techniques developed in [23, 11]. In this direction we present the *Stair Algorithm* developed by Kassabov and Matucci in [23] and our basics definitions and results.

We start by defining what we call the *affine group of* **R** as follows.

$$\mathcal{A}(\mathbf{R}) \coloneqq (\mathbf{R}_{>0}, \cdot) \ltimes (\mathbf{R}, +),$$

where the operation is given by $(a,b)\cdot(c,d) \coloneqq (ac, b+ad)$ for all $(a,b),(c,d) \in \mathcal{A}(\mathbf{R})$. With this group in mind we define the *initial germ* of an element $g \in H$. If $g(t) = a_0^2 t + a_0 b_0$ on its first piece, then its initial germ is defined as $g_{-\infty} \coloneqq (a_0^2, a_0 b_0)$. Similarly we define the *final germ* $g_{+\infty}$ of g. Notice that the initial and final germs of all element of H lie in $\mathcal{A}(\mathbf{R})$. We also define the conjugacy class of an initial germ:

$$y_{-\infty}^{\mathcal{A}(\mathbf{R})_{-\infty}} = \left\{ \left(a_0^2, a_0 b_0 a^{-2} + \left(a_0^2 - 1\right) a^{-1} b\right) \mid g_{-\infty} = (a^2, ab) \right\}.$$

An analogous definition for conjugacy classes of final germs $y_{+\infty}^{\mathcal{A}(\mathbf{R})_{+\infty}}$ exists. Finally, we present the following partial results towards the construction of a conjugacy invariant.

Lemma 88. *For any $y, z \in H$ such that $y^g = z$ for some $g \in H$, it holds that*

$$y_{-\infty}^{\mathcal{A}(\mathbf{R})_{-\infty}} = z_{-\infty}^{\mathcal{A}(\mathbf{R})_{-\infty}} \qquad \text{and} \qquad y_{+\infty}^{\mathcal{A}(\mathbf{R})_{+\infty}} = z_{+\infty}^{\mathcal{A}(\mathbf{R})_{+\infty}}.$$

Lemma 89 (Initial and Final Boxes)**.** *For any $y, z \in H$ such that $y^g = z$ for some $g \in H$, there exists a constant $L \in \mathbf{R}$ such that g is affine on the initial box $(-\infty, L]^2$. Similarly for a final box $[R, +\infty)^2$.*

Lemma 90 (Identification Lemma)**.** *Let $y, z \in H^<$ and $g \in H$ be maps such that $y^g = z$ and g defined on $(-\infty, L]$. Then g is determined on $\left(-\infty, z^{-1}(L)\right]$.*

Proposition 91. *Let $y, z \in H^<$ and $g \in H$ be functions such that $y^g = z$. Then the conjugator g is uniquely determined by its initial germ.*

An irrational-slope Thompson group

Lawrence Reeves
(joint work with José Burillo, Brita E. A. Nucinkis)

We present some results on F_τ, the irrational slope Thompson group introduced by Cleary [91]. We give presentations (both infinite and finite), show that its commutator subgroup is simple and show that several natural embeddings of F in F_τ are undistorted [90].

On presentations and cohomological finiteness properties of generalizations of Thompson–Higman groups

Conchita Martínez-Pérez
(joint work with Francesco Matucci, Brita E. A. Nucinkis)

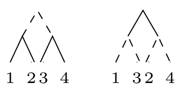
In this note we consider a family of generalizations of Higman-Thompson group V that can be defined using Universal Algebra and which share the main cohomological finiteness properties of V such as being of type F_∞.

To define these groups, consider first a finite set of colours $S = \{1, \dots, s\}$ and associate to each $i \in S$ an integer $n_i > 1$, called arity of the colour i. An Ω-algebra is a set U together with an n_i-ary operation called contraction $\lambda_i : U^{n_i} \to U$ and an operation called expansion $\alpha_i : U \to U^{n_i}$ for each $1 \leq i \leq s$ such that $\lambda_i \alpha_i = 1_U$ and $\alpha_i \lambda_i = 1_{U^{n_i}}$ (we use right notation). We also refer to the λ_i's as ascending operations and to the α_i's as descending operations and in the case when a set B is obtained from a set A by descending operations only we put $A \leq B$. Moreover, we consider a series of laws which are identifications between sets which are obtained from compositions of two different descending operations. We say that Σ is *complete* if these identifications occur between pairs so that the composition $\lambda_i \lambda_j$ is identified in certain way with the composition $\lambda_j \lambda_i$ for all the possible pairs $i \neq j$. (For details, see [48], [47].)

We consider the free object $U_r(\Sigma)$ on a set X of r elements where Σ refers to all the identifications or laws above. Moreover, we say that Σ is *valid* if for any set Y that can be obtained from X by a series of ascending or descending operations there are no identifications between elements of Y. In this case one can show that the sets Y as before are precisely the (free) basis of the algebra $U_r(\Sigma)$ and we define:

Definition 92. Assume Σ is valid. The group $V_r(\Sigma)$ is the group of algebra automorphisms of $U_r(\Sigma)$, and each element $g \in U_r(\Sigma)$ is given by a bijection $g : Y_1 \to Y_2$ between two basis.

For example, if we have two colours both of arity 2 and the laws in Σ are represented as follows:



where one colour corresponds to dashed and the other to bold lines, then the associated group $V_1(\Sigma)$ is precisely the two-dimensional Brin-Higman group $2V$.

We also say that Σ is *bounded* if given basis A, B, C such that $A \leq B, C$ there is a least upper bound for B and C, i.e., there is some basis D such that $B, C \leq D$ and for any other basis Z with $B, C \leq Z$ we have $D \leq Z$.

As examples of algebras for which Σ is valid, bounded and complete, we have all the Higman groups $V_{n,r}$, Stein groups and also Brin–Higman groups sV. All these groups are known to be of type F_∞ ([36], [53], [45], [41]) and there are known explicit presentations for them ([36], [61], [43]). Here, we propose a common framework to prove these results and others at once for all the groups in this family. For example

Theorem 93 ([47]). *Let Σ be valid, bounded and complete. Then the group $V_r(\Sigma)$ is of type FP_∞.*

The proof uses the action of the group on certain complexes associated to the set of basis in $U_r(\Sigma)$. Observe first that this set is a poset with the order $\leq$ defined above. Associated to this poset there is a simplicial complex on which the group $V_r(\Sigma)$ acts with finite stabilizers. However, it is convenient to work with a reduced version of this complex, the so called Stein complex $\mathrm{St}_r(\Sigma)$ first defined in [53]. This complex has the set of basis as 0-skeleton and a k-simplex of vertices $A_0, \ldots, A_k$ whenever we have $A_0 < \ldots < A_k$ and in the expansion $A_0 < A_k$ there is no element of A_0 such that in the process to obtain A_k we apply twice the same colour. One shows that $\mathrm{St}_r(\Sigma)$ is contractible too and admits an action of $V_r(\Sigma)$ with finite stabilizers. This action is not cocompact but filtering $\mathrm{St}_r(\Sigma)$ according to the cardinality of the basis one gets a sequence of spaces whose connectivity tends to infinity all of them with a cocompact action of $V_r(\Sigma)$ with finite stabilizers. Then, using a criterium by Brown, one deduces that $V_r(\Sigma)$ is in fact of type F_∞.

Recall that for a (discrete) group G, a G-CW-complex Z is a *classifying space for proper actions* if for any subgroup H of G, the fixed points subcomplex Z^H is contractible if H is finite and empty otherwise. We have

Theorem 94 ([48, 47]). *The Stein complex $\mathrm{St}_r(\Sigma)$ is a model for $\underline{\mathrm{E}}V_r(\Sigma)$.*

There is an analogous of the property FP_∞ for proper actions: namely, we say that a group G is of type $\underline{FP}_\infty$ if it has a model for $\underline{\mathrm{E}}G$ with cocompact k-skeleton for any k. By a Theorem by Lück a group has type $\underline{FP}_\infty$ if and only if the two following properties hold:

i) G has finitely many conjugacy classes of finite subgroups,
ii) for any finite subgroup $H \leq G$, the centralizer $C_G(H)$ is of type FP_∞.

As any $V_r(\Sigma)$ contains a copy of every finite subgroup, i) can never hold and the best cohomological finiteness properties for proper actions that these groups may have is being of type *quasi* $\underline{FP}_\infty$ which we define as having ii) above and i') instead of i) where

i') G has finitely many conjugacy classes of finite subgroups of each isomorphism type.

Theorem 95 ([47]). *Let Σ be valid, bounded and complete. Then the group $V_r(\Sigma)$ is of type quasi-$\underline{FP}_\infty$.*

As the groups $V_r(\Sigma)$ have the property i') above, the proof relies on a description of centralizers that can be found in [85] for the group V and in various degrees of detail in [48] and [47] in the general case. According to these descriptions one sees that if $H \leq V_r(\Sigma)$ is finite then $C_{V_r(\Sigma)}(H)$ is a direct product of finitely many copies of a group of the form

$$T = K \rtimes V_{r'}(\Sigma)$$

where K is certain locally finite group. Then one can use the conjugation action of $V_{r'}(\Sigma)$ on K to define for any $m > 0$ an action of T on the m-fold join of the discrete set K with itself, i.e. on

$$Y^m = K \star \overset{m}{\cdots} \star K.$$

We prove that this action is cocompact and then as the connectivity of Y^m tends to infinity as $m \to \infty$, using Brown's criterium we deduce that T is F_∞.

We also provide a common framework to obtain explicit finite presentations for our groups. To do that, we construct a model for the classifying space for each $V_r(\Sigma)$ as follows. Consider the set having as elements ordered tuples A with a basis of $U_r(\Sigma)$ as underlying set. We denote by $u(A)$ the underlying set of A. Given two tuples A and B we say that $A \preceq B$ if $u(A) \leq u(B)$ in the Stein complex. We make this set a (non simplicial) complex that we denote Z by gluing a k-simplex of vertices $A_0, \ldots, A_k$ whenever

$$A_0 \preceq \ldots \preceq A_k.$$

Taking underlying sets gives us a map

$$u : Z \to \mathrm{St}_r(\Sigma).$$

One can check that for any basis Y, $u^{-1}(Y)$ is contractible, in fact it it is the complex associated to the bar resolution of the finite symmetric group S_n where n is the order of the set Y. Using this and a version of Quillen's poset lemma one deduces that u is a homotopy equivalence and therefore that Z is contractible. On the other hand the group $V_r(\Sigma)$ acts freely on Z so we have a model for $\mathrm{E}V_r(\Sigma)$.

Using this model we show how to construct explicit presentations for $V_r(\Sigma)$. Moreover this construction works for Σ valid and bounded and produces a finite generating system without the assumption that Σ is complete. In the case when Σ is moreover complete we get a finite presentation.

The Lodha–Moore groups

José Burillo

This talk was an introduction to the Lodha–Moore groups [98] following the author's expository notes [89].

We give the definition using generators as homeomorphisms of $\mathbf{R}$, as well as the interpretation of elements using tree-pair diagrams. We detail an algorithm to multiply elements and finally give a proof of the presentation using the crucial property of potential cancellations.

Odd remarks on Thompson's groups, almost automorphisms, and braids

Vlad Sergiescu

We will be mainly concerned with Thompson's group T, acting on S^1. However, it also "acts" on a binary tree by almost (simplicial) automorphisms (closely related to partial bijections/symmetries, near bijections, quasi-automorphisms...).

This viewpoint [96] leads to a geometric connection with the infinite braid group B_∞ as an exact sequence

$$1 \to B_\infty \to A_T \to T \to 1.$$

The group A_T is homologically equivalent to $S^3 \times \mathbb{C}P^\infty$. We will give also an elementary approach to $H_2(T)$ involving a presentation of the F subgroup which stabilizes a point, and a combinatorial version of the Bott–Virasoro [109, 67, 95] class in $H^2(\mathrm{Diff}(S^1))$.

Finally, we mention that a related construction involves the pure mapping class group of genus zero and Thompson's group $V \subset T$. This, in turn, naturally connects to the Brin–Dehornoy group V_{br} [88, 92] which surjects onto V with a certain subgroup (constructed using pure braids) as kernel.

Coherent group actions on the real line or an interval

Yash Lodha

We introduce a new class of group actions on the real line (or an interval). One is able to use this framework to produce non-embeddability results for Thompson's group F that answer recently raised problems in the field.

A group action $G < \mathrm{Homeo}^+(\mathbf{R})$ is said to be *coherent* if:

(1) The action is minimal, i.e. the orbits are dense.
(2) The groups of germs at $\pm\infty$ are solvable.
(3) There exists an element that has a trivial germ at $-\infty$ and does not fix any point in some interval (r, ∞).
(4) There exists an element that has a trivial germ at $+\infty$ and does not fix any point in some interval $(-\infty, s)$.

(A similar definition is prescribed for a group action $G < \mathrm{Homeo}^+([0,1])$). These conditions are satisfied by a rich class of group actions by homeomorphisms, such as groups of piecewise linear and projective homeomorphisms. The class of groups that admit such actions is denoted by $\mathcal{C}$. The class contains continuum many isomorphism classes of finitely generated groups, and any group that admits a faithful action on the real line by homeomorphisms embeds in some group in this class. Groups in $\mathcal{C}$ have interesting algebraic and dynamical features, and I showed [69] the following:

Theorem 96. *Let $G \in \mathcal{C}$. Then G satisfies the following:*

(1) *G contains a subgroup isomorphic to Thompson's group F. Therefore G is non elementary amenable (in particular, G is non solvable).*

(2) *There exists an* $n \in \mathbf{N}$ *(which depends on* G*) such that every proper quotient of* G *is solvable of degree at most* n.
(3) *There exists an* $n \in \mathbf{N}$ *(which depends on* G*) such that the* n*-th derived subgroup* $G^{(n)}$ *is simple.*

I demonstrate [69] that coherent group actions are *rigid*:

Theorem 97. *Consider two coherent actions* $G, H < \mathrm{Homeo}^+(\mathbf{R})$ *such that the underlying groups* G, H *are isomorphic. Then for each isomorphism* $\nu : G \to H$ *there is a homeomorphism* $\phi : \mathbf{R} \to \mathbf{R}$ *such that* $\nu(f) = \phi^{-1} f \phi$ *for each* $f \in G$.

Using this framework, I am able to produce non embaddability results such as the following for Thompson's group F [69].

Theorem 98. *Let* $G < \mathrm{Homeo}^+(\mathbf{R})$ *be a coherent group action which produces a non* μ*-amenable equivalence relation (with respect to the Lebesgue measure). Then the underlying group* G *does not embed in Thompson's group* F.

Theorem 99. *Let* $G = F(2, p_1, ..., p_n)$ *for* $n \geq 1$ *and* $p_1, ..., p_n$ *distinct odd primes be a Brown-Stein-Thompson subgroup of* $\mathrm{PL}^+([0, 1])$*. Then* G *does not embed into Thompson's group* F*. The Bieri-Strebel group* $G(I; A, P)$ *does not embed into* F *provided* $P < \mathbf{R}_+^*$ *has abelian rank greater than one.*

Open problems on Thompson-like groups

The workshop also featured two discussion sessions, one on the recently introduced broken Baumslag–Solitar groups (see Example 30) and the other on open problems and further questions on Thompson-like groups.

Problem sessions are becoming traditional in the series of conferences on Thompson groups, with proposed questions stimulating a lot of research in the following years. Among important questions that circulated around the community, we highlight the problem lists from the conference *Thompson's group at 40 years* [105], held in January 2004 at the American Institute of Mathematics, and from the *Workshop on the extended family of R. Thompson groups* [113], held in May 2014 at the University of St Andrews.

Among the topics and results presented at this Oberwolfach Workshop, we observe that many questions from the St Andrews list [113] were addressed, some of the problems having in fact being solved. Many other important questions from that list, however, remain open, and other problems were introduced or brought back to light during our stay at the MFO. Besides all the open questions already stated at the extended abstracts presented in this report, we compile[6] in this section some other problems that remain unsettled. We hope that the current

[6]Thanks are due to Justin Moore, Francesco Matucci, Matt Zaremsky, Yash Lodha, Claas Röver and Yuri Santos Rego for their help on conducting the discussions or collecting some of the questions.

report as well as the old problem lists help promote further research on (and attract other mathematicians to) the realm of Thompson groups.

The names attached to the questions below might have suggested the problems or expressed interest in them.

1. Coarse geometry

Question 100 (Zaremsky). *It seems quasi-isometry questions about F are too hard, so as a first step, does F have any interesting quasi-retracts? E.g., any proper non-abelian ones? Is F a quasi-retract of T? Is $F \times F$ a quasi-retract of F?*

Question 101 (Zaremsky). *The* sprawl *of a group G with respect to a finite generating set S is*

$$E(G,S) := \lim_{n\to\infty} \frac{1}{|S_n|^2} \sum_{(x,y)\in S_n^2} \frac{d_S(x,y)}{n}.$$

Here S_n is the sphere of radius n in $Cay(G,S)$ and d_S is the word metric with respect to S. Sprawl was introduced by Duchin, Lelièvre and Mooney, who proved that hyperbolic groups have sprawl 2 for any finite generating set. A group G is said to be statistically hyperbolic *if it has sprawl 2 (this can depend on the generating set). The question then is, for some natural finite generating sets, what is the sprawl of F, and is it statistically hyperbolic?*

2. Group-theoretic questions

Question 102 (Brin). *Is V_{br}, the braided variant of V, a hopfian group? What about F_{br}?*

Question 103 (Various). *Is V_{br} acyclic? What about the Brin–Thompson groups nV?*

Question 104 (Geoghegan, probably others). *Can one use the Stein–Farley cube complex for T to recover Ghys–Sergiescu's computation of $H_*(T;\mathbf{Q})$?*

3. Amenability-related questions

Question 105 (Gromov by way of Lodha). *Does there exist a non-amenable group G (not necessarily in the realm of Thompson groups) with a finite $K(G,1)$ such that G contains no free subgroups? (Weakened version: what if we just require finite virtual cohomological dimension instead of finite $K(G,1)$?)*

Question 106 (Moore, Zaremsky). *Can the fact that the amenable Brin–Navas group B admits Wajnryb–Witowicz-regular relations be used to show that F does as well?*

Consider the following elements of Monod's group $H(\mathbf{R})$.

$$\alpha(t) = t + \frac{1}{2} \text{ and } \beta(t) = \begin{cases} t & \text{if } t \leq 0 \\ \frac{t}{t-1} & \text{if } 0 \leq t \leq \frac{1}{2} \\ 3 - (\frac{1}{t}) & \text{if } \frac{1}{2} \leq t \leq 1 \\ t+1 & \text{if } 1 \leq t. \end{cases}$$

Question 107 (Moore, broad problem)**.** *The subgroup $\langle \alpha, \beta \rangle$ of Monod's group is very close to F. Is this the simplest example that goes slightly beyond F and is non-amenable? Why is the $1/2$ important?*

Note that $\mathrm{PSL}_2(\mathbf{Z})$ is discrete, but any proper overgroup isn't (e.g. $\mathrm{PSL}_2(\mathbf{Z}[\frac{1}{2}])$). So the orbit structure changes drastically.

4. Subgroup structure

Question 108 (Brin)**.** *Among fast systems of 4 one-bump functions, there are at most two isomorphism classes of groups that do not non-trivially decompose as wreath products or direct products. One class is represented by F_4 and one by "pseudo-F_4". So, how many classes are there, one or two? That is, is pseudo-F_4 isomorphic to F_4?*

Question 109 (Brin, Zaremsky (more?))**.** *Is every finitely presented subgroup of $PL_o(I)$ with a fast set of one-bump generators of type F_∞? Much stronger, are they all just isomorphic to the F_n's? (Related to the question about pseudo-F_4.)*

Question 110 (Zaremsky)**.** *Does the Higman group*

$$\langle a, b, c, d \mid b^a = b^2, c^b = c^2, d^c = d^2, a^d = a^2 \rangle$$

embed into the Lodha–Moore groups? Or at least into Monod's $H(\mathbf{R})$? It's built out of copies of the Baumslag–Solitar $BS(1,2)$, so this seems like a natural question following the broken Baumslag–Solitar discussion.

Recall that Question 75 asks whether Cleary's group F_τ embeds into F. Regarding this problem, it was asked during the workshop whether every finitely generated subgroup of Thompson's group F has torsion-free abelianisation. The motivation was that an affirmative answer would imply that F_τ *does not* embed into F, as F_τ has non-trivial torsion in its abelianisation; see [90]. However, the answer is negative as the following example by C. E. Röver shows.

Let $W = \mathbf{Z} \wr \mathbf{Z}$, the standard restricted wreath product of two infinite cyclic groups. It is well known that W is a subgroup of F and has presentation

$$W = \langle a, t \mid [a^{t^k}, a] = 1, \quad k \geq 1 \rangle.$$

As usual we define $a_i = a^{t^i}$ for $i \in \mathbf{Z}$. Let H be the subgroup of W generated by $r = [a, t] = a_0^{-1} a_1$, $s = a_0^2$ and t. Then H' is the normal closure (in H, and even in W) of the two basic commutators $c = [r, t] = a_0 a_1^{-2} a_2$ and $d = [s, t] = a_0^{-2} a_1^2$, as $[r, s] = 1$. Since $r^2 = d$, it remains to show that $r \notin H'$. In fact, it suffices to prove this modulo N, the normal closure of a^2. Note that $d \in N$ and $cN = a_0 a_2 N$. Put $c_i = c^{t^i}$ for $i \in \mathbf{Z}$. Then, modulo N, every element of H' is of the form $h = c_{i_1} c_{i_2} \cdots c_{i_m}$ with $i_1 < i_2 < \ldots < i_m$. Since $h = a_{i_1} \ell a_{i_m+2}$ for some element ℓ whose non-trivial components lie between $i_1 + 1$ and $i_m + 1$, the assumption that $r = h$ implies $i_1 = 0$ and $i_m + 2 = 1$, i.e. $i_m < i_1$, a contradiction. Thus $r \notin H'$ and H/H' is not torsion-free.

Question 111 (Röver, Sapir)**.** *Does every finitely presented subgroup of F have torsion-free abelianisation?*

5. Finiteness properties

Question 112 (Zaremsky)**.** *The various Lodha–Moore groups each have a natural map to* $\mathbf{Z}$ *that, to use the model with tree pairs with black and white dots, can be viewed as, "total number of black minus white dots." The kernel of this map is finitely presented. Is it in fact of type* F_∞*? (If so, it would finish the computation of all the Bieri–Neumann-Renz–Strebel sigma invariants for these groups.)*

Question 113 (Zaremsky)**.** *In the family of Thompson-like groups, can one find non-finitely presentable groups of type* FP_2*?*

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