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Algebraische Zahlentheorie

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Abstract. The origins of Algebraic Number Theory can be traced to over two centuries ago, wherein algebraic techniques are used to glean information about integers and rational numbers. It continues to be at the forefront of modern research as it evolves to straddle wider areas of Mathematics.

Mathematics Subject Classification (2010): 11Dxx, 11Exx, 11Fxx, 11Gxx 11Mxx, 11Rxx und 11Sxx.

Introduction by the Organisers

The Oberwolfach workshop 'Algebraic Number Theory' is among the oldest workshops that is marked by its continuity, quality and breadth, The present workshop continued in this tradition. It was well-attended with 48 participants, with wide ranging diversity in lectures and participation. The range of topics of the lectures is testimony to the evolution of this subject. The quality of talks was exceptional both in depth and breadth and it is worth noting that around half a dozen talks at the workshop were in areas related to the work of Fields Medallists in 2018, as well as invited talks in ICM 2018. The workshop also nicely captured the emerging areas within this topic while establishing connections to other areas within mathematics. It also ensured a lively discussion among the speakers and participants during their stay at Oberwolfach.
A central theme in modern Algebraic Number Theory today goes under the heading of Arithmetic Geometry. As the title suggests, it is an area that brings connections between Algebraic Geomtry and Arithmetic to the fore. In Algebraic

Geomtery, the different cohomology theories such as Betti, de Rham play an important role and many of these theories have echoes in Arithmetic. The construction, properties and connections between these different cohomology theories need p-adic techniques, where p is a prime number. In the last decade, work of Peter Scholze has formalized an algebraic bridge that connects algberaic geometry to arithmetic. This area is poised to play a dominant role in Number Theory for the next decades. Many talks in the workshop exposed aspects of such connections.

David Hilbert posed a set of twenty three problems in 1900. Two talks in the workshop demonstrated the durability and depth of these problems. These talks explained how Hilbert's 12th problem and 13th problem can be viewed through the lens of modern mathematics and what light current research methods shed on these problems. Another classical conjecture is the celebrated Birch and Swinnerton-Dyer conjecture, one of the Clay millennial problems. There were talks in the workshop that reported on important recent advances made in this conjecture for elliptic curves. Iwasawa theory is an important tool that has helped advance our current state of knowledge in the Birch and Swinnerton-Dyer conjecture and a few talks reported on important recent results in this area, as well as other related conjectures in Number Theory.

The theory of L-functions is an important area of study in Automorphic forms, which in turn sheds light on representations of the Galois group of rational numbers and number fields. It is also the central subject of investigation in other important conjectures in Arithmetc from the last century, such as Langlands program, Beilinson's conjectures, etc. In the explicit linkage between automorphic forms and Galois representations predicted by the Langlands program, the theory of algebraic cycles intervenes. This theme was highlighted in some talks where it was explained how the theory of algebraic cycles can be used to reprove some recent results of Peter Scholze. Closely associated with this framework and that of Motives is the theory of Periods. These are transcendental objects and play a vital role since a precise understanding of periods is necessary in the study of certain explicit numerical formulae that are part of the full Birch and Swinnerton-Dyer conjecture and L-values. One of the talks in the workshop reported on Hilbert's 7th problem, revisited through the topic of periods.

Representation Theory arises in modern Arithmetic Geometry in different guises. They range from the classical areas of automorphic forms to the recent area of p-adic Langlands program, and mod-p Langalands program. There were several talks that reported on new results in these areas and the interconnections of these with more classical areas of Algebraic Number theory. Thus the workshop week was vigorous and exciting in presenting the broad vistas of Algebraic Number Theory.

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Workshop: Algebraische Zahlentheorie

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Abstracts

P-adic Riemann-Hilbert correspondence, de Rham comparison and periods on Shimura varieties

Ruochuan Liu

(joint work with Hansheng Diao, Kai-Wen Lan, Xinwen Zhu)

Let X be a connected smooth complex algebraic variety, X^{an} the associated analytic space. Recall that the classical Riemann–Hilbert correspondence establishes tensor equivalences among the following:

- the tensor category of local systems of finite dimensional $\mathbb{C}$-vector spaces on the underlying topological space X^{top} of X^{an};
- the tensor category of vector bundles of finite rank with integrable connections on X^{an}; and
- the tensor category of vector bundles of finite rank with integrable connections on X, with regular singularities at infinity.

The equivalence of the first and second categories is a simple consequence of the Frobenius theorem: Let $\mathbb{L}$ be a local system on X^{top}. Then the associated vector bundle with an integrable connection is $(\mathbb{L} \otimes_{\underline{\mathbb{C}}} \mathcal{O}_{X^{\mathrm{an}}}, 1 \otimes d)$; and conversely, for a vector bundle with an integrable connection on X^{an}, its sheaf of horizontal sections is the desired local system. The equivalence of the second and third categories is a deep theorem due to Deligne.

One of the main goals of this work is to prove the following result (as one step towards the p-adic Riemann–Hilbert correspondence):

Theorem 1. [2, Theorem 1] *Let X be a smooth algebraic variety over a finite extension k of $\mathbb{Q}_p$. Then there is a tensor functor $D_{\mathrm{dR}}^{\mathrm{alg}}$ from the category of de Rham p-adic étale local systems $\mathbb{L}$ on X to the category of algebraic vector bundles on X with regular integrable connections and decreasing filtrations satisfying the Griffiths transversality. In addition, there is a canonical comparison isomorphism*

$$H^i_{\text{ét}}\big(X_{\overline{k}}, \mathbb{L}\big) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \cong H^i_{\mathrm{dR}}\big(X, D^{\mathrm{alg}}_{\mathrm{dR}}(\mathbb{L})\big) \otimes_k B_{\mathrm{dR}}$$

compatible with the canonical filtrations and the actions of $\mathrm{Gal}(\overline{k}/k)$ *on both sides.*

Here B_{dR} is Fontaine's p-adic period ring, and H_{dR} is the algebraic de Rham cohomology. The notion of *de Rham p-adic étale local systems* was first introduced by Scholze [4] for rigid varieties (generalizing earlier work of Brinon [1]) using some relative de Rham period sheaf. However, it turns out that this notion satisfies a rather surprising rigidity property: a p-adic étale local system $\mathbb{L}$ on X^{an} is de Rham if and only if, on each connected component of X, there exists some classical point x such that the p-adic representation $\mathbb{L}_x$ of the absolute Galois group of the residue field of x is de Rham [3]. In this situation, it follows that the same is also true at every classical point x of X. A p-adic étale local system $\mathbb{L}$ on X is called de Rham if $\mathbb{L}^{\mathrm{an}}$ is a de Rham local system on X^{an}. Then the key step of proving Theorem 1 is to show the algebraicity of $D_{\mathrm{dR}}(\mathbb{L}^{\mathrm{an}})$ (to obtain $D^{\mathrm{alg}}_{\mathrm{dR}}(\mathbb{L})$).

It is natural to ask whether this functor is compatible with Deligne's classical Riemann–Hilbert correspondence. Let X be a smooth algebraic variety over a number field E. We fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ and a field homomorphism $\sigma : E \to \mathbb{C}$, and write $\sigma X = X \otimes_{E,\sigma} \mathbb{C}$. Let $\mathbb{L}$ be a p-adic étale local systems on X. It follows that $\mathbb{L}|_{\sigma X} \otimes_{\mathbb{Q}_p,\iota} \mathbb{C}$ can be regarded as a complex local system on $(\sigma X)^{\mathrm{top}}$, denoted by $\iota\mathbb{L}_\sigma$. Then the classical Riemann–Hilbert correspondence produces a regular integrable connection on σX.

On the other hand, the composition $E \xrightarrow{\sigma} \mathbb{C} \xrightarrow[\sim]{\iota^{-1}} \overline{\mathbb{Q}}_p$ determines a p-adic place v. Let E_v be the completion of E with respect to v, and assume that $\mathbb{L}|_{X_{E_v}}$ is de Rham. Then we obtain $D^{\mathrm{alg}}_{\mathrm{dR}}(\mathbb{L}|_{X_{E_v}}) \otimes_{E_v,\iota} \mathbb{C}$, which is another regular integrable connection, with an additional decreasing filtration $\mathrm{Fil}^\bullet$ satisfying the Griffiths transversality. In order to compare the above two constructions, we need to impose a further restriction on $\mathbb{L}$.

We say that $\mathbb{L}$ is *geometric* if, for every closed point x of X, the corresponding p-adic representation $\mathbb{L}_x$ is *geometric in the sense of Fontaine–Mazur*. Note that geometric p-adic étale local systems on X form a full tensor subcategory of the category of all étale local systems. If $\mathbb{L}$ is geometric, then $\mathbb{L}|_{X_{E_v}}$ is de Rham. In addition, it turns out that $\mathbb{L}$ is geometric if and only if, on each connected component, there is a closed point such that the stalk of $\mathbb{L}$ at this point is a geometric p-adic representation [3].

Conjecture 2. [2, Conjecture 1.2] *The above two tensor functors from the category of geometric p-adic étale local systems on X to the category of regular integrable connections on σX are canonically isomorphic. In addition, $(D^{\mathrm{alg}}_{\mathrm{dR}}(\mathbb{L}|_{X_{E_v}}) \otimes_{E_v,\iota} \mathbb{C}, \mathrm{Fil}^\bullet)$ is a complex variation of Hodge structures.*

Unsurprisingly, the motivation behind this conjecture is the theory of motives. Note that this conjecture is closely related to a relative version of the Fontaine–Mazur conjecture proposed in the introduction of [3], but it might be more accessible because it is stated purely in terms of sheaves Even so, it seems to be out of reach at the moment. Nevertheless, in the case of Shimura varieties, we can partially verify this conjecture. Let (G, X) be a Shimura datum, $K \subset \mathrm{G}(\mathbb{A}_f)$ a neat open compact subgroup, and $\mathrm{Sh}_K = \mathrm{Sh}_K(\mathrm{G}, \mathsf{X})$ the corresponding Shimura variety, defined over the reflex field $E = E(\mathrm{G}, \mathsf{X})$. Let G^c be the quotient of G by the maximal subtorus of the center of G that is $\mathbb{Q}$-anisotropic but $\mathbb{R}$-split. Recall that there is a tensor functor from the category $\mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{G}^c)$ of algebraic representations of G^c over $\mathbb{Q}_p$ to the category of p-adic étale local systems on Sh_K, whose essential image consists of only geometric p-adic étale local systems.

Theorem 3. [2, Theorem 1.3] *The conjecture holds for the (p-adic) étale local systems on Sh_K coming from $\mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{G}^c)$ as above.*

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On some consequences of a theorem of J. Ludwig

Vytautas Paškūnas

Let D_0 and D be quaternion algebras over $\mathbb{Q}$ with the same ramification outside p and ∞, and such that D_0 is split at p and ramified at ∞ and D is ramified at p and split at ∞. In the talk we have presented a theorem which we think of as a global Jacquet–Langlands correspondence between the p-adic automorphic forms on $D_0^\times$ and $D^\times$.

We let our coefficient field L be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field k.

We identify $(D_0 \otimes \mathbb{A}^{p,\infty})^\times$ with $(D \otimes \mathbb{A}^{p,\infty})^\times$ and let $U^p = \prod_\ell U_\ell$ be an open compact subgroup - the tame level, which will remain fixed throughout. In this setting one may consider the tower of Shimura varieties by varying the level at p. In the case of D_0 the Shimura varieties are just finitely many points and the tower is a profinite set. In the case of D the Shimura varieties are Shimura curves. One may then consider the completed cohomology of these towers following Emerton [1]. In the case of D_0 only the 0-th cohomology is interesting, and it is essentially the continuous functions $\mathcal{O}$-valued functions on the tower, which we denote by $S(U^p, \mathcal{O})$. This space has a continuous action by $(D_0 \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times = \mathrm{GL}_2(\mathbb{Q}_p)$. In the case of D it is the 1-st cohomology that it is most interesting and we denote it by $\widehat{H}^1(U^p, \mathcal{O})$. It has a continuous action by $(D \otimes \mathbb{Q}_p)^\times = D_p^\times$, where D_p is the non-split quaternion algebra over $\mathbb{Q}_p$.

Let S be a finite number of places containing p and ∞ such that $U_\ell = \mathrm{GL}_2(\mathbb{Z}_\ell)$ for all $\ell \notin S$. Both $S(U^p, \mathcal{O})$ and $\widehat{H}^1(U^p, \mathcal{O})$ are modules over the algebra $\mathbb{T}^{\mathrm{univ}}_{\mathcal{O},S} := \mathcal{O}[T_\ell, S_\ell : \ell \notin S]$, where the formal variables T_ℓ and S_ℓ act as Hecke operators. The action of $\mathbb{T}^{\mathrm{univ}}_{\mathcal{O},S}$ commutes with the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ and $D_p^\times$ respectively.

Let $G_{\mathbb{Q},S}$ be the Galois group of a maximal extension of $\mathbb{Q}$ unramified outside S. To a continuous representation $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathcal{O})$ we associate a maximal ideal $\mathfrak{m}_\rho$ of $\mathbb{T}^{\mathrm{univ}}_{\mathcal{O},S}[1/p]$ by letting $\mathfrak{m}_\rho$ be generated by $T_\ell - \mathrm{tr}\,\rho(\mathrm{Frob}_\ell), \ell S_\ell - \det \rho(\mathrm{Frob}_\ell)$ for all $\ell \notin S$, where Frob_ℓ denotes the geometric Frobenius at ℓ. Now the spaces $S(U^p, \mathcal{O}) \otimes_{\mathcal{O}} L$, $\widehat{H}^1(U^p, \mathcal{O}) \otimes_{\mathcal{O}} L$ are L-Banach spaces with a unitary $\mathrm{GL}_2(\mathbb{Q}_p)$ and $D_p^\times$-actions, respectively. We denote by $[\mathfrak{m}_\rho]$ the subspaces annihilated by $\mathfrak{m}_\rho$; these are closed and also carry a unitary action by the respective group. Let $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(k)$ be the reduction of ρ modulo p.

Theorem 1. *Assume $p \geq 5$, $\bar{\rho}$ is absolutely irreducible, $\rho|_{G_{\mathbb{Q}_p}}$ is absolutely irreducible and $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is reducible and generic then $(S(U^p, \mathcal{O}) \otimes_{\mathcal{O}} L)[\mathfrak{m}_\rho]$ is non-zero if and only if $(\widehat{H}^1(U^p, \mathcal{O}) \otimes_{\mathcal{O}} L)[\mathfrak{m}_\rho]$ is non-zero.*

If ρ is the Galois representation associated to a classical automorphic form on $D_0^\times$ with tame level U^p, which is discrete series at p, then by taking locally algebraic vectors for the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $(S(U^p, \mathcal{O}) \otimes_{\mathcal{O}} L)[\mathfrak{m}_\rho]$ and for the action of $D_p^\times$ on $(\widehat{H}^1(U^p, \mathcal{O}) \otimes_{\mathcal{O}} L)[\mathfrak{m}_\rho]$ one may recover the classical Jacquet–Langlands correspondence. If the classical automorphic form is a principal series representation at p then the classical Jacquet–Langlands correspondence cannot deal with it – indeed one may calculate that the subspace of locally algebraic vectors for the action of $D_p^\times$ on $(\widehat{H}^1(U^p, \mathcal{O}) \otimes_{\mathcal{O}} L)[\mathfrak{m}_\rho]$ is zero in this case, however our theorem can be interpreted that there is a genuine p-adic automorphic form on $D^\times$ corresponding to the classical automorphic form on $D_0^\times$.

We expect that $(\widehat{H}^1(U^p, \mathcal{O}) \otimes_{\mathcal{O}} L)[\mathfrak{m}_\rho]$ is of finite length as an admissible unitary Banach space representation of $D_p^\times$ and we prove some results towards this, which are too technical to explain here. Our results suggest that there should be a local correspondence between admissible unitary L-Banach space representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and $D_p^\times$.

The main ingredient in the proof are results of Scholze in [5], which allow us to pass from $\mathrm{GL}_2(\mathbb{Q}_p)$ to $D_p^\times$ representations. In fact Scholze's construction is available in more general setting, but to get some control over it a result of Ludwig [2], who shows that a certain cohomology group vanishes, is used in an essential way. Ludwig's result is available only for $\mathrm{GL}_2(\mathbb{Q}_p)$ and further her techniques work only with sheaves associated to mod p principal series representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. For this reason we are forced to make the assumption that $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is reducible in the Theorem above. A further ingredient are results of [3] on the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. For further details please consult [4].

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Hilbert's 13th problem and moduli of abelian varieties

Mark Kisin
(joint work with Benson Farb and Jesse Wolfson)

1. History

In the sixteenth century Tartaglia, Cardano and Ferrari proved that cubic and quartic polynomials could be solved by radicals. Although this is not possible for quintic polynomials a 1786 result of Bring says that a quintic can be solved using radicals and solutions of polynomials of the form $X^5+aX+1=0$. Thus a general polynomial of degree at most 5 can be solved by admitting solutions to $f(X)=0$ where f is an algebraic function of one variable. Hilbert conjectured that this was not possible for general polynomials of degree ≥ 6, and that for polynomials of degree $6,7,8,9$ the minimal number of variables was $2,3,4,4$ respectively. One can generalize and formalize this problem as follows:

Let X be a variety over an algebraically closed field K of characteristic 0, and let $Y\to X$ be a connected, finite étale cover. We are interested in the geometric complexity of $Y\to X$

Definition 1. The *essential dimension* $\mathrm{ED}(Y/X)$ of $Y\to X$ is the smallest integer e such that there exists a dense open $U\subset X$, and a map $U\to U'$ such that

- $\dim U'=e$
- $Y|_U=Y'\times_{U'}U$ for some finite étale $Y'\to U'$.

We can also allow composites of such coverings:

Definition 2. The *resolvent degree* $\mathrm{RD}(Y/X)$ of $Y\to X$ is the smallest integer r such that there exists $U\subset X$ dense open, and a tower of finite étale coverings

$$U_n\to U_{n-1}\to\cdots\to U_0=U$$

such that

- $\mathrm{ED}(U_i/U_{i-1})\leq r$.
- The composite $U_n\to U$ factors through Y.

Definition 3. For G a finite group, we define $\mathrm{ED}(G)$ and (resp. $\mathrm{RD}(G)$) as the sup of $\mathrm{ED}(Y/X)$ (resp. $\mathrm{RD}(Y/X)$) taken over all covers $Y\to X$ whose Galois closure has group G.

These definitions are due to Brauer, Shimura-Arnold, and Buhler-Reichstein, but they formalize a circle of questions going back to Hamilton, Hilbert and others.

Let $\mathbb{P}_n=\mathring{A}^n-V(\mathrm{disc})$ be the space of monic, separable polynomials of degree n; and

$$\tilde{\mathbb{P}}_n=\{(f,z):f\in\mathbb{P}_n,z\in K,f(z)=0\}\to\mathbb{P}_n$$

the solution of the universal polynomial. Then $\mathrm{ED}(\tilde{\mathbb{P}}_n/\mathbb{P}_n)=\mathrm{ED}(S_n)$ and $\mathrm{RD}(\tilde{\mathbb{P}}_n/\mathbb{P}_n)=\mathrm{RD}(S_n)=\mathrm{RD}(A_n)$ and Hilbert's conjecture can be reformulated as

Conjecture 4. *(Hilbert)* $\mathrm{RD}(\tilde{\mathbb{P}}_n \to \mathbb{P}_n) \geq 2$ *for* $n \geq 6$. *For* $n = 6, 7, 8, 9$ *this is equal to* $2, 3, 4, 4$ *respectively.*

There are no examples of covers where $\mathrm{RD}(Y \to X) > 1$ is known ! There are non-trivial *upper* bounds for RD.

Theorem 5. *(Hamilton (1836), Brauer (1975), Conjectured by Segre (1947)) For all* $r \in \mathbf{N}^+$ *there exists* $H(r)$ *such that for* $n > H(r)$,

$$\mathrm{RD}(\tilde{\mathbb{P}}_n \to \mathbb{P}_n) \leq n - r.$$

2. Examples from geometry

Let $\mathcal{H}_{3,3}$ = the moduli space of smooth cubic surfaces in $\mathbb{P}^3$ modulo the action of PGL(4). This is 4-dimensional. Let

$$\mathcal{H}_{3,3}(1) = \{(S, L),\ S \in \mathcal{H}_{3,3},\ L \subset S \text{ a line}\} \to \mathcal{H}_{3,3}$$

Then $\mathcal{H}_{3,3}(1) \to \mathcal{H}_{3,3}$ is a 27-sheeted cover, with group $W(E_6) \subset S_{27}$. A theorem of Burkhart-Klein says we have $\mathrm{RD}(\mathcal{H}_{3,3}(1) \to \mathcal{H}_{3,3}) \leq 3$.

Conjecture 6. $\mathrm{RD}(\mathcal{H}_{3,3}(1) \to \mathcal{H}_{3,3}) = 3$.

Let $\mathcal{H}_{4,2}$ be the space of smooth quartics in $\mathbb{P}^2$ modulo the action of PGL(3). This has dimension 6. Let

$$\mathcal{H}_{4,2}(1) = \{(S, L),\ S \in \mathcal{H}_{4,2},\ L \subset \mathbb{P}^2 \text{ a line bitangent to } S\}.$$

$\mathcal{H}_{4,2}(1) \to \mathcal{H}_{4,2}$ is a 28-sheeted cover, with group $W(E_7) \subset S_{28}$. It is conjectured that $\mathrm{RD}(\mathcal{H}_{4,2}(1) \to \mathcal{H}_{4,2}) = \dim \mathcal{H}_{4,2} = 6$.

3. Moduli of PPAV's:

Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties. For a prime p let $\mathcal{A}_{g,p} \to \mathcal{A}_g$ be the p-torsion of the universal family over $\mathcal{A}_g$.

Then (the Galois closure of) $\mathcal{A}_{g,p} \to \mathcal{A}_g$ has group $\mathrm{Sp}_{2g}(\mathbb{F}_p)$.

Proposition 7. *We have*

- $\mathrm{RD}(\mathcal{H}_{3,3}(1) \to \mathcal{H}_{3,3}) = \mathrm{RD}(\mathcal{A}_{2,3} \to \mathcal{A}_2)$.
- $\mathrm{RD}(\mathcal{H}_{4,2}(1) \to \mathcal{H}_{4,2}) = \mathrm{RD}(\mathcal{A}_{3,2} \to \mathcal{A}_3)$.

Proof. Use that

$$W(E_6) \supset W(E_6)^+ \xrightarrow{\sim} \mathrm{PSp}_4(\mathbb{F}_3)$$

together with the versality of the two families, which shows the first equality is equivalent to $\mathrm{RD}(W(E_6)) = \mathrm{RD}(\mathrm{Sp}_4(\mathbb{F}_3))$. Similarly for the second equality use

$$W(E_7) \supset W(E_7)^+ \xrightarrow{\sim} \mathrm{Sp}_6(\mathbb{F}_2).$$

□

Conjecture 8. *If* $(g, p) \neq (2, 2)$ *then*

$$\mathrm{RD}(\mathcal{A}_{g,p} \to \mathcal{A}_g) = \dim \mathcal{A}_g = \frac{g(g+1)}{2}$$

The case $(g,p) = (2,2)$ should have RD $= 2$ as $\mathrm{Sp}_4(\mathbb{F}_2)$ has a normal index 2 subgroup. This agrees with the conjectures for $\mathrm{RD}(\mathcal{H}_{3,3}(1) \to \mathcal{H}_{3,3})$ and $\mathrm{RD}(\mathcal{H}_{4,2}(1) \to \mathcal{H}_{4,2})$. We prove the analogous statements for essential dimension.

Proposition 9. $\mathrm{ED}(\mathcal{A}_{g,p} \to \mathcal{A}_g) = \dim \mathcal{A}_g$.

More generally, let $P = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{F}_p)\}$ be the points of the Siegel parabolic, and $U \subset P$ the unipotent radical. So $U = (\mathbb{F}_p)^{g(g+1)/2}$.

For any subgroup $G \subset \mathrm{Sp}_{2g}(\mathbb{F}_p)$, let $G_U = G \cap U$ and $e_G = \mathrm{rk}_{\mathbb{F}_p} G_U$.

Proposition 10. *Let $\tilde{\mathcal{A}}_{g,p} \to \mathcal{A}_g$ be the Galois closure of $\mathcal{A}_{g,p} \to \mathcal{A}_g$, and $\mathcal{A}_G = \tilde{\mathcal{A}}_{g,p}/G$. Then* $\mathrm{ED}(\tilde{\mathcal{A}}_{g,p} \underset{G}{\to} \mathcal{A}_G) \geq e_G$.

Proposition 11. *For $n \geq 1$ there is an $A_n \hookrightarrow \mathrm{Sp}_{2g}(\mathbb{F}_2)$ such that $(A_n)_U$ is the maximal elementary abelian 2-subgroup of A_n. Then $e_{A_n} = 2[\frac{n}{4}]$, so*

$$\mathrm{ED}(\tilde{\mathcal{A}}_{g,2} \underset{A_n}{\to} \mathcal{A}_{A_n}) \geq 2[\frac{n}{4}].$$

The coverings $\tilde{\mathcal{A}}_{g,2} \underset{A_n}{\to} \mathcal{A}_{A_n}$ are versal when $n = 5, 6, 7$ but probably not in general. Here are the first few values of e_{A_n}.

n	4	5	6	7	8	9
e_{A_n}	2	2	2	2	4	4

Note that for $n = 6, 8, 9$ they coincide with with Hilbert's conjectured value for $\mathrm{RD}(A_n) = \mathrm{RD}(S_n) = \mathrm{RD}(\tilde{\mathbb{P}}_n \to \mathbb{P}_n)$. For 7 we get $e_{A_n} = 2$ rather than 3.

Conjecture 12. *For $n \geq 6$ we have*

$$\mathrm{RD}(\tilde{A}_{g,2} \to A_{A_n}) \geq e_{A_n}.$$

This would imply Hilbert's conjecture on $\mathrm{RD}(\tilde{\mathbb{P}}_n \to \mathbb{P}_n)$ for $n = 6, 8, 9$ and the conjectures on RD of lines on cubics and bitangents on quartics.

4. Sketch of proof:

The proof that $\mathrm{ED}(\tilde{A}_{g,p} \to \mathcal{A}_G) \geq e_G$ uses Serre-Tate theory. Consider the case $G = \mathrm{Sp}_{2g}(\mathbb{F}_p)$. We have to show $\mathrm{ED}(\mathcal{A}_{g,p} \to \mathcal{A}_g) = \mathrm{rk}_{\mathbb{F}_p} U = \dim \mathcal{A}_g$. If this covering comes from something of smaller dimension $\mathcal{A}_g \to V$, we can extend it to a map of integral models $\mathcal{A}_g^\circ \to V^\circ$ over $\mathcal{O}_K$, where now $K/\mathbb{Q}_p$ is finite.

Then one shows that after passing to p-adic completions, $\hat{\mathcal{A}}_g^\circ \to \hat{V}^\circ$, the finite flat group scheme $\tilde{\mathcal{A}}_{g,p}^\circ$ descends to $\hat{V}^\circ$, at least over a formal open in the ordinary locus $\hat{\mathcal{A}}_g^{\mathrm{ord}}$.

But for any closed point $x \in \hat{\mathcal{A}}_g^{\mathrm{ord}}$, the deformation of $(\tilde{\mathcal{A}}_{g,p})_x$ corresponding to any tangent direction is non-trivial. This follows from Serre-Tate theory or Grothendieck-Messing theory. Now we get a contradiction, as the pullback of a finite flat group scheme on $\hat{V}^\circ$ cannot give enough deformations.

Geometry and analysis of regulator formulae

Shrenik Shah
(joint work with Aaron Pollack)

1. Overview

Recently there has been a tremendous effort towards extending the ideas of Kato [Kat04] to prove more cases of conjectures on special values of L-functions of motives, including the Beilinson, Bloch-Kato, and Iwasawa conjectures. Kato studied modular curves; generalizations pursue the cohomology of Shimura varieties using the following strategy.

(1) Construct classes in the motivic cohomology of smooth toroidal compactifications of Shimura varieties. (These generalize the *Beilinson–Flach elements* of Bertolini-Darmon-Rotger [BDR15a, BDR15b].)
(2) Prove that the regulator of these classes is non-vanishing and connected to an L-value, establishing (part of) Beilinson's conjecture in this setting.
(3) Develop relations between the classes to form an Euler system.
(4) Use the p-adic realization of the Euler system to establish a case of the Iwasawa main conjecture.

A great deal of progress has been made recently on (3) and (4) in works of Kings, Lei, Loeffler, Skinner, and Zerbes [LLZ14, KLZ17, LSZ17]. Lemma recently studied (1) and (2) for the case of Siegel modular threefolds in [Lem15, Lem17].

The talk reported on some work towards (1) and (2) in two new cases: Picard modular surfaces and certain compactified Shimura varieties associated to unitary groups over $\mathbf{Q}$ of signature $(2, 2)$. It also mentioned a result towards (1) and (2) in the case of the Siegel modular sixfold $\mathcal{A}_3$ [PS18], which establishes the link to the L-value at almost all places using the Rankin-Selberg method. Cauchi and Rodrigues Jacinto have established some partial results towards (3) on $\mathcal{A}_3$ using this construction [CR18].

2. Beilinson's conjecture

For a variety X over $\mathbf{Q}$, Beilinson constructed a map

$$\mathrm{Reg}_X : H^{i+1}_{\mathcal{M}}(X, \mathbf{Q}(i + 1 - m)) \to H^{i+1}_{\mathcal{D}}(X_{/\mathbf{R}}, \mathbf{R}(i + 1 - m))$$

from motivic cohomology (defined using K-theory) to absolute Hodge or Deligne cohomology. Roughly, he conjectured that if $m < \frac{i}{2}$ is an integer, an "integral subspace" of motivic cohomology provides a $\mathbf{Q}$-structure on Deligne cohomology via Reg_X, and he predicted the value of the determinant of the regulator map.

Beilinson–Flach elements can be constructed inside $H^{i+1}_{\mathcal{M}}(X, \mathbf{Q}(i + 1 - m))$ for some Shimura varieties X. In [PS17b, PS17a], we construct new such elements for smooth compactifications of Shimura varieties associated to $\mathrm{GU}(2, 1)$ and $\mathrm{GU}(2, 2)$. We also study the near-central value of the L-function of the middle-degree cohomology, a point at which non-vanishing becomes a delicate issue that must be addressed both locally and globally.

3. Non-unique models

For the groups $\mathrm{GU}(2,2)$ and GSp_6, we have constructed new Rankin–Selberg integrals in [PS17a, PS18] that are on the one hand connected to a regulator pairing and on the other are connected to motivic L-functions. (The former is used in the regulator computation above.) These integrals are very unusual in that they employ the seldom-used technique of Piatetski-Shapiro and Rallis [PSR88] to work with *non-unique models*. For instance, with respect to the standard anti-diagonal symplectic form, the GSp_6 construction uses a Fourier coefficient with a transformation property with respect to the group

$$\begin{pmatrix} 1 & * & * & * \\ & 1_2 & * & * \\ & & 1_2 & * \\ & & & 1 \end{pmatrix},$$

where 1_2 is the 2×2 identity matrix. These models complicate our analysis of the local properties of the regulator pairing, and we are developing some new techniques to address this issue.

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p-converse to a Theorem of Gross–Zagier, Kolyvagin and Rubin

Ashay A. Burungale

Let E be an elliptic curve over the rationals. A fundamental arithmetic invariant is the Mordell–Weil rank given by the rank of the finitely generated abelian group $E(\mathbb{Q})$. As E varies, the rank is typically expected to be 0 or 1. Let $\text{Ш}(E_{/\mathbb{Q}})$ be the conjecturally finite Tate–Shafarevich group. For a prime p, the p^∞-Selmer group $\mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}})$ encodes arithmetic of the elliptic curve via the exact sequence

$$0 \to E(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) \to \text{Ш}(E_{/\mathbb{Q}})[p^\infty] \to 0.$$

A fundamental analytic invariant is the analytic rank given by the vanishing order $\mathrm{ord}_{s=1}L(s, E_{/\mathbb{Q}})$ for the complex L-function $L(s, E_{/\mathbb{Q}})$.

The BSD conjecture predicts a deep relation among the arithmetic and analytic facets.

Conjecture 1. *Let E be an elliptic curve over the rationals. For $r = 0, 1$, the following are equivalent.*

(1). $\mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q}) = r$ *and* $\text{Ш}(E_{/\mathbb{Q}})$ *is finite.*
(2). $\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) = r$ *for a prime* p.
(3). $\mathrm{ord}_{s=1}L(s, E_{/\mathbb{Q}}) = r$.

Part (2) evidently follows from part (1). On the other hand, the implication

$$\mathrm{ord}_{s=1}L(s, E_{/\mathbb{Q}}) = r \implies \mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q}) = r, \#\text{Ш}(E_{/\mathbb{Q}}) < \infty$$

is a fundamental result on the BSD conjecture due to Coates–Wiles ([3]), Gross–Zagier ([4]), Kolyvagin ([6]) and Rubin ([7]). This passage to arithmetic side from the analytic one goes back to mid 70's and mid 80's. In the case $r = 0$, the finiteness of the Mordell–Weil group for CM elliptic curves is due to Coates–Wiles around mid 70's. In the case $r \leq 1$, the implication is due to Gross–Zagier, Kolyvagin and Rubin for non-CM and CM elliptic curves, respectively around mid 80's. This is the theorem alluded to in the title. It is one of the rare instances where results were almost simultaneously obtained for non-CM and CM elliptic curves.

We refer to the implication

$$\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) = r \implies \mathrm{ord}_{s=1}L(s, E_{/\mathbb{Q}}) = r$$

as p-converse. Visibly, this is a p-adic criteria for an elliptic curve to have analytic rank r. From now, we suppose that p is a good ordinary prime for E. In the case $r = 0$, the p-converse is well-known to follow from a divisibility in an Iwasawa main conjecture (IMC) for the elliptic curve. Here divisibility refers to a lower bound for Iwasawa-theoretic Selmer group of E along an Iwasawa-extension of $\mathbb{Q}$ in terms of a p-adic L-function. For CM elliptic curves, the rank zero p-converse thus follows from IMC for $\mathrm{GL}_{1/K}$ due to Rubin ([8]) around early 90's for $p > 2$. Here K is the underlying CM field. For non-CM elliptic curves, the rank zero p-converse follows from a divisibility in an IMC for $\mathrm{GL}_{2/\mathbb{Q}}$ due to Skinner–Urban ([10]) around late 00's under certain hypotheses. In the case $r = 1$, the p-converse appeared out of reach until recently. For non-CM elliptic curves, first general results towards such a p-converse are independently due to Skinner ([11]) and Zhang ([13]) a few years ago. The results were obtained almost simultaneously under different hypotheses. The striking approaches due to Skinner and Zhang appear markedly distant. We only mention that both crucially rely on an auxiliary IMC over an imaginary quadratic field. The results and subsequent developments ([2], [12]) exclude the case of CM elliptic curves.

Joint with Tian, our main result in [1] concerns such a p-converse in the case of CM elliptic curves.

Theorem 2. *Let E be a CM elliptic curve over the rationals. Let $p > 3$ be a good ordinary prime for E. Then,*

$$\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) = 1 \implies \mathrm{ord}_{s=1} L(s, E_{/\mathbb{Q}}) = 1.$$

In particular, $\mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q}) = 1$ *and* $Ш(E_{/\mathbb{Q}})$ *is finite whenever* $\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) = 1$.

Note that, “In particular” part follows from the work of Gross–Zagier, Kolyvagin and Rubin. We would like to emphasise that finiteness of the Tate–Shafarevich group $Ш(E_{/\mathbb{Q}})$ is not our hypothesis but in fact a consequence. For CM elliptic curves, the p-converse under finiteness of $Ш(E_{/\mathbb{Q}})[p^\infty]$ is indeed due to Rubin ([9]) around early 90’s.

The following gives a mod p criteria for a CM elliptic curve to have analytic rank one and the p-part of the corresponding Tate–Shafarevich group to be trivial.

Corollary 3. *Let E be a CM elliptic curve over the rationals. Let $p > 3$ be an ordinary prime for E. Suppose that the following holds.*

(i). *The mod p Galois representation $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_p)$ arising from the p-torsion $E[p]$ is absolutely irreducible for the absolute Galois group $G_{\mathbb{Q}}$.*

(ii). *We have $\mathrm{Sel}_p(E_{/\mathbb{Q}}) \simeq \mathbb{Z}/p\mathbb{Z}$ for the p-Selmer group $\mathrm{Sel}_p(E_{/\mathbb{Q}})$ arising from $\overline{\rho}$.*

Then,

$$\mathrm{ord}_{s=1} L(s, E_{/\mathbb{Q}}) = 1, Ш(E_{/\mathbb{Q}})[p^\infty] = 0.$$

Remark 4. (1). The approach is Iwasawa-theoretic and involves an auxiliary Rankin–Selberg setup over the CM field of the CM elliptic curve. It is partly based on an interaction among the auxiliary Heegner points and elliptic units.

(2). The approach leads to a construction of an anticyclotomic Euler system for self-dual Hecke characters over a CM field. This maybe viewed as an analogue of anticyclotomic Euler system arising from elliptic units over a rather general CM field.

(3). In an ongoing joint work in progress with Skinner and Tian, we consider p-converse for CM elliptic curves with p a good supersingular prime.

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Cohomological dimension in pro-p towers

Hélène Esnault

If X is a proper variety of finite type of dimension d defined over an algebraically closed field k of characteristic $p > 0$, Artin-Schreier theory implies that the cohomological dimension of étale cohomology of X with $\mathbb{F}_p$-coefficients is d, i.e. $H^i(X, \mathbb{F}_p) = 0$ for $i > d$. If k has characteristic not equal to p, the cohomological dimension of étale cohomology of X with $\mathbb{F}_p$-coefficients is $2d$ if X is proper, and d if X is affine by Artin's vanishing theorem. However, when X is projective, Peter Scholze showed that there is a specific tower of degree p-power covers of X which makes its cohomological dimension equal to d in the limit.

Let $X \subset \mathbb{P}^n$ be a projective variety of dimension d. We choose coordinates $(x_0 : \ldots : x_n)$ on $\mathbb{P}^n$. With this choice of coordinates, we define the covers

$$\phi_r^n : \mathbb{P}^n \to \mathbb{P}^n,\ (x_0 : \ldots : x_n) \mapsto (x_0^{p^r} : \ldots : x_n^{p^r}).$$

We define X_r as the inverse image of X by ϕ_r^n.

Theorem 1 (See Scholze, [Sch14], Theorem 17.3.)**.** *If k is an algebraically closed of characteristic* 0*, for $i > d$, one has*

$$\varinjlim_r H^i(X_r, \mathbb{F}_p) = 0.$$

Scholze obtains the theorem as a corollary of his theory of perfectoid spaces. He does not detail the proof in *loc. cit.*, but his argument is documented in [Sch15]. By classical base change, we may assume that $k = \bar{\mathbb{Q}}_p$. By the comparison theorem [Sch15, Thm.IV.2.1], $\varinjlim_r H^r(X_r, \mathbb{F}_p) \otimes \mathcal{O}_C/p$ is 'almost' equal to $H^i(\mathcal{X}, \mathcal{O}^+_{\mathcal{X}}/p)$ where $\mathcal{X}$ is a perfectoid space he constructs, associated to $\varprojlim_r X_r$, and $C = \hat{\bar{\mathbb{Q}}}_p$. By [Sche92, Thm. 4.5], the spectral space $\mathcal{X}$ has cohomological dimension at most the Krull dimension of X.

We give in [Esn18, Thm. 1.2] a direct proof, as was asked for over $\mathbb{C}$ in [Sch14, Section 17]. It turns out that the proof holds in characteristic not equal to p as well. One obtains

Theorem 2. *If k is an algebraically closed of characteristic not equal to p, for $i > d$, one has*

$$\varinjlim_r H^i(X_r, \mathbb{F}_p) = 0.$$

The ingredients are constructibility and base change properties for relative étale cohomology with compact supports, functoriality, and some easy fact of representation theory of a cyclic group of p-power order.

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Arithmetic of triple product p-adic L-functions

Massimo Bertolini

(joint work with Marco A. Seveso, Rodolfo Venerucci)

This abstract reports on recent work on the arithmetic properties of triple product p-adic L-functions, in collaboration with M.A. Seveso and R. Venerucci.

Let $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ denote a triple of cuspidal Hida families of modular forms of tame levels (N_f, N_g, N_h) and tame characters (χ_f, χ_g, χ_h). By definition $\mathbf{f}$ is given by a formal q-expansion $\sum_{n\geq 1} a_n(\mathbf{f}, \cdot) \cdot q^n$, whose coefficients $a_n(\mathbf{f}, \cdot)$ are rigid-analytic functions on an open p-adic disc U_f centered at a positive integer $k_0 \geq 2$ in the so-called weight space. When k is a classical weight in $U_f \cap \mathbf{Z}_{\geq 2}$, satisfying $k \equiv k_0 \pmod{2(p-1)}$, the weight k specialisation $\mathbf{f}_k = \sum_{n\geq 1} a_n(\mathbf{f}, k) \cdot q^n$ of $\mathbf{f}$ is the p-stabilisation of a classical ordinary cuspidal eigenform f_k on $\Gamma_1(N_f)$ (unless $k = 2$ and $\mathbf{f}_2$ is a classical eigenform on $\Gamma_1(N_f) \cap \Gamma_0(p)$ which new at p). Define the set of classical weights U_f^{cl} to be $U_f \cap \mathbf{Z}_{\geq 2}$, with the addition of $k = 1$ if $\mathbf{f}_1$ is a classical weight one form (in this case $k_0 = 1$ is allowed). Similar conventions are in force for the families $\mathbf{g}$ and $\mathbf{h}$.

The fundamental *self-duality* hypothesis $\chi_f \chi_g \chi_h = 1$ is assumed throughout. For (k, ℓ, m) in $U_f^{\mathrm{cl}} \times U_g^{\mathrm{cl}} \times U_h^{\mathrm{cl}}$, this implies that $k + \ell + m$ is even and the degree-eight triple product complex L-function $L(f_k \times g_\ell \times h_m, s)$ satisfies a functional equation relative to $s \mapsto k + \ell + m - 2 - s$ with associated sign $\epsilon(f_k, g_\ell, h_m) = \pm 1$ and central critical point $c(k, \ell, m) = (k + \ell + m - 2)/2$.

A basic principle in the work presented here consists in investigating the arithmetic meaning of $L(f_{k_0} \times g_{\ell_0} \times h_{m_0}, s)$ at the central critical point $c(k_0, \ell_0, m_0)$ via *p-adic deformations*, i.e., by studying the p-adic variation of the central critical values $L(f_k \times g_\ell \times h_m, c(k, \ell, m))$ and their associated arithmetic invariants for (k, ℓ, m) varying in a p-adic neighbourhood of (k_0, ℓ_0, m_0). For example, suppose that $(k_0, \ell_0, m_0) = (2, 1, 1)$ is a classical weight and that f_2 is the modular form attached to an elliptic curve E over $\mathbf{Q}$. In this setting, the above-mentioned work leads to a p-adic approach to the study of the arithmetic of E over the extension cut out by the tensor product of the Artin representations associated to the weight one forms g_1 and h_1.

A first step in carrying out of the above program consists in the p-adic interpolation of the central values $L(f_k \times g_\ell \times h_m, c(k, \ell, m))$. A remarkable fact is that the sign of the functional equation of $L(f_k \times g_\ell \times h_m, s)$ is constant over certain regions of classical weights. More precisely, set $\Sigma = U_f^{\mathrm{cl}} \times U_g^{\mathrm{cl}} \times U_h^{\mathrm{cl}}$ and define the *unbalanced regions* $\Sigma_f = \{(k, \ell, m) \in \Sigma : k \geq \ell + m\}$, $\Sigma_g = \{(k, \ell, m) \in \Sigma : \ell \geq k + m\}$, $\Sigma_h = \{(k, \ell, m) \in \Sigma : m \geq k + \ell\}$. Finally, define the *balanced region* Σ_{bal} to be $\Sigma - (\Sigma_f \cup \Sigma_g \cup \Sigma_h)$. We distinguish the following two cases: the *indefinite case*, in which the sign $\epsilon(f_k, g_\ell, h_m)$ is equal to $+1$ on the three unbalanced regions, and is equal to -1 on the balanced region (with a possible exception); the *definite case*, where the signs are opposite to those occurring in the indefinite case. The above terminology is motivated by the fact that, in the region where the sign $\epsilon(f_k, g_\ell, h_m)$ is $+1$, the corresponding central values are described in terms of trilinear forms arising from definite, resp. indefinite quaternion algebras over $\mathbf{Q}$ in the definite, resp. indefinite case. This description of special values stems from the work of several authors, including Gross–Kudla, Harris–Kudla and Ichino. In the definite case, it leads to the definition of a "square-root" triple product p-adic L-function $L_p(\mathbf{f}, \mathbf{g}, \mathbf{h})$ in the p-adic variables (k, ℓ, m), which interpolates p-adically the square-roots of suitable algebraic normalisations of the central values $L(f_k \times g_\ell \times h_m, c(k, \ell, m))$ for (k, ℓ, m) in Σ_{bal}. See [3] and [4] for details. In the indefinite case, for any choice of unbalanced region – say Σ_f to fix notations – as the region of classical interpolation, one defines (see [4]) a triple product p-adic L-function $L_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$ interpolating the square-roots of the algebraic parts of

$L(f_k \times g_\ell \times h_m, c(k,\ell,m))$ for (k,ℓ,m) in Σ_f. (It should be noted that the p-adic L-functions considered above depend on a choice of "test vector" of a common level $N = lcm(N_f, N_g, N_h)$ for the triple $(\mathbf{f}, \mathbf{g}, \mathbf{h})$, and not only on the triple itself as our notation somewhat abusively suggests.)

Motivated by applications to elliptic curves, it is of interest to study the specialisation of the p-adic L-function $L_p(\mathbf{f}, \mathbf{g}, \mathbf{h})$ or $L_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$ – whichever is defined – at the point $(2,1,1) \in \Sigma_f$. Assume that f_2 is the modular form of an elliptic curve E and, in the indefinite case, that $L(f_2 \times g_1, \times h_1, 1) = 0$ (the latter condition being automatic in the definite case by sign reasons). In both cases, we expect these values to be p-adic avatars of appropriate derivatives of $L(f_2 \times g_1 \times h_1, s)$ at $(2,1,1)$, and in fact to be related to formal group logarithms of rational points on E defined over the extension of $\mathbf{Q}$ cut out by the Galois representation of $g_1 \times h_1$. We focus in the sequel on the indefinite case, as our findings in this setting are at the moment more definitive. Details on these results are contained in [1]; the reader may wish to compare them with some related results of [2].

Let $V(\mathbf{f}, \mathbf{g}, \mathbf{h})$ denote the big Galois representation interpolating the self-dual twists $V(f_k, g_\ell, h_m)$ of the tensor product of the Deligne representations attached to (f_k, g_ℓ, h_m). There is a *big Selmer group* $\mathrm{Sel}(\mathbf{Q}, V(\mathbf{f}, \mathbf{g}, \mathbf{h}))$ which interpolates over Σ_{bal} the Bloch–Kato Selmer groups attached to $V(f_k, g_\ell, h_m)$. Furthermore, one has (a suitable branch of) a *big Perrin-Riou logarithm* $\mathcal{L}_p^f$ mapping the above Selmer group to the 3-variable Iwasawa algebra to which $L_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$ belongs. This big logarithm interpolates at the balanced points the Bloch–Kato logarithms.

Theorem A (cf. [1]) *There exists a canonical diagonal class* $\kappa(\mathbf{f}, \mathbf{g}, \mathbf{h})$ *in the big Selmer group* $\mathrm{Sel}(\mathbf{Q}, V(\mathbf{f}, \mathbf{g}, \mathbf{h}))$ *whose image by* $\mathcal{L}_p^f$ *is equal to* $L_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$.

Since $\mathcal{L}_p^f$ factors through the restriction at p, the statement of Theorem A does not determine the diagonal class $\kappa(\mathbf{f}, \mathbf{g}, \mathbf{h})$. Our construction is obtained from the push-forward of a canonical generator of an invariant space of locally analytic functions along the diagonal embedding of a modular curve into its triple product.

The next result (which holds under suitable technical assumptions) deals with the specialisation $\kappa(f_k, g_\ell, h_m)$ of the diagonal class at an unbalanced point (k,ℓ,m) in Σ_f.

Theorem B (cf. [1]) *The class* $\kappa(f_k, g_\ell, h_m)$ *is crystalline at* p *if and only if the central value* $L(f_k \times g_\ell \times h_m, c(k,\ell,m))$ *is zero.*

Theorem B follows from Theorem A in light of the properties of the Perrin-Riou logarithm, except when $L_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$ has an exceptional zero. This exceptional zero scenario arises precisely in the setting where: $(k,\ell,m) = (2,1,1)$, p divides exactly the conductor of $f_2 = \mathbf{f}_2$ and does not divide the conductors of g_1 and h_1, and the equality $\chi_f(p) \cdot a_p(\mathbf{g}, 1) \cdot a_p(\mathbf{h}, 1) = a_p(\mathbf{f}, 2)$ holds.

Finally, we place ourselves in a setting where the triple (f_2, g_1, h_1) is exceptional in the above sense. Moreover assume that f_2 is the modular form of an elliptic curve E, that g_1 and h_1 are theta-series attached to the same quadratic extension $K/\mathbf{Q}$ in which p is inert, and that the special value $L(f_2 \times g_1 \times h_1, 1)$ vanishes as a consequence of a suitable generalised Heegner hypothesis. By Theorem B, the class $\kappa(f_2, g_1, h_1)$ is crystalline at p. It is natural to ask whether this class bears any relation to the rational points of E over anticyclotomic extensions of K.

In this setting, the Bloch-Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f_2, g_1, h_1))$ decomposes as a direct sum $\mathrm{Sel}(K_\varphi, E)^\varphi \oplus \mathrm{Sel}(K_\psi, E)^\psi$, where φ and ψ are anticyclotomic characters attached to the pair (g_1, h_1), K_φ, resp. K_ψ is the ring class field of conductor equal to that of φ, resp. ψ, and $\mathrm{Sel}(K_\varphi, E)^\varphi$, resp. $\mathrm{Sel}(K_\psi, E)^\psi$ denotes the φ-, resp. ψ-component of the Selmer group of E over K_φ, resp. K_ψ. When K is imaginary quadratic, Heegner points $P_\varphi \in E(K_\varphi)^\varphi$, $P_\psi \in E(K_\psi)^\psi$ are defined. Similarly, if K is real quadratic, there are so-called Stark–Heegner points $P_\varphi \in E(K_\varphi \otimes \mathbf{Q}_p)^\varphi$, $P_\psi \in E(K_\psi \otimes \mathbf{Q}_p)^\psi$. These are local points, which are expected to be global points and to share some of the properties of Heegner points.

Theorem C (cf. [1]) *The equality* $\log \kappa(f_2, g_1, h_1) = \log_E(P_\varphi) \cdot \log_E(P_\psi) \pmod{\bar{\mathbf{Q}}^\times}$ *holds up to a non-zero algebraic constant, where* $\log$ *is a suitable component of the Bloch–Kato logarithm and* $\log_E$ *is the formal group logarithm of* E.

In the imaginary quadratic case, Theorem C combined with S-W. Zhang's Gross–Zagier formula implies that $\log \kappa(f_2, g_1, h_1) \neq 0$ if and only if the second derivative $L''(f_2 \times g_1 \times h_1, 1) \neq 0$. In the real quadratic case, the non-triviality of the Stark–Heegner points P_φ and P_ψ implies that $\mathrm{Sel}(\mathbf{Q}, V(f_2, g_1, h_1))$ has dimension at least two; moreover, these points can be realised as the restriction at p of Selmer classes.

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Nearly overconvergent forms and p-adic L-functions

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(joint work with Adrian Iovita)

The p-adic L-functions of the title are those associated to a quadratic imaginary field K and an eigenform f. More precisely, let us fix:

- $\iota \colon K \subset \mathbb{C} \cong \mathbb{C}_p$: an imaginary quadratic field, with trivial class group, $\mathcal{O}_K^* = \{\pm 1\}$ (for simplicity of exposition);
- $N \geq 4$ an integer;
- $\mathfrak{N} \subset \mathcal{O}_K$ an ideal such that $\mathcal{O}_K/\mathfrak{N} \cong \mathbb{Z}/N\mathbb{Z}$ (Heegner hyp.);
- χ a character of $(\mathbb{Z}/N\mathbb{Z})^*$, and hence of $(\mathcal{O}_K/\mathfrak{N})^*$;
- $p \geq 5$ a prime, not dividing N;
- f an eigenform, of weight $k \geq 2$, level $\Gamma_1(N)$, nebentype χ of parity k.

For example we have the Eisenstein series

$$E_{k,\chi} = L(1-k,\chi) + 2\sum_{n\geq 1} \sigma_{k-1,\chi}(n)q^n, \quad \sigma_{k-1,\chi}(n) = \sum_{d|n} \chi(d)d^{k-1}.$$

We will consider two types of L-functions:

$$L\big(K,\chi^{-1},k_1,k_2\big) = \Big[\sum_{0\neq\alpha\in\mathcal{O}_K} \frac{\chi^{-1}(\alpha)}{\alpha^{k_1}\overline{\alpha}^{k_2}\mathrm{Nm}(\alpha)^s}\Big]|_{s=0}, \quad k_1,k_2\in\mathbb{Z}$$

and

$$L\big(f,\chi^{-1},k+j,-j\big) = \prod_{\mathfrak{P},\gamma=\alpha,\beta} \big[1-\gamma_{\mathrm{Nm}(\mathfrak{P})}\tilde{\chi}^{-1}(\mathfrak{P})\mathrm{Nm}(\mathfrak{P})^{-s}\big]^{-1}|_{s=0}$$

with $\tilde{\chi}^{-1}((\alpha)) = \frac{\chi^{-1}(\alpha)}{\alpha^{k+j}\overline{\alpha}^{-j}}$. The first L-function has been considered by N. Katz [Ka], the second by M. Bertolini, H. Darmon and K. Prasanna in [BDP] (see also [BCDDPR] for an overview). Notice that the L-function considered by Katz is a factor of the one associated to $f = E_{k,\chi}$. The special values of the L-functions above above are not algebraic numbers but if we fix

- A the elliptic curve with CM by $\mathcal{O}_K$,
- t_A a generator of $A[\mathfrak{N}]$
- ω_A a Néron differential, that we express as $\omega_A = \Omega_\infty dz$ (here z is a variable for a complex uniformization of $A(\mathbb{C})$)

Then we have the following algebraicity result due to Shimura

$$L_{\mathrm{alg}}\big(f,\chi^{-1},k+j,-j\big) := \delta^j(f)\big(A,t_A,\omega_A\big) \in \overline{\mathbb{Q}}$$

for every non-negative integer j. Here

$$\delta(f) = \frac{1}{2\pi i}\Big(\frac{\partial(f)}{\partial\tau} + \frac{kf}{\tau-\overline{\tau}}\Big)$$

is the Shimura-Maas operator. Moreover,

Theorem We have

$$L_{\mathrm{alg}}\big(E_{k,\chi},\chi^{-1},k+j,-j\big) = \frac{(k+j-1)!N^{k+2j+1}\pi^j}{s(\chi)a(O_K)^j\Omega_\infty^{k+2j}}L\big(K,\chi^{-1},k+j,-j\big)$$

$$L_{\mathrm{alg}}\big(f,\chi^{-1},k+j,-j\big)^2 = \frac{*}{\Omega_\infty^{2(k+2j)}}L\big(f,\chi^{-1},k+j,-j\big).$$

The first equality is provided in [Ka] and the second is an explicit version of a Waldspurger's formula worked out in [BDP]. One then has the following result (see [Ka] and [BDP]) *if* p splits as $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$. Take ω^{can} to be a suitable p-adic differential of the formal group of A at $\mathfrak{p}$ and write $\omega_A = \Omega_p\omega^{\mathrm{can}}$. Set

$$L_p\big(k+j,-j\big) := \big(1-\frac{\chi(\mathfrak{p})\mathfrak{p}^{k+j}}{p\overline{\mathfrak{p}}^j}\big)\big(1-\frac{\chi^{-1}(\overline{\mathfrak{p}})\mathfrak{p}^j}{p\overline{\mathfrak{p}}^{k+j}}\big)\Omega_p^{k+2j}\times L_{\mathrm{alg}}\big(?,k+j,-j\big).$$

Then,

Theorem There exists a continuous function in one variable $L_p\big(k+\nu,-\nu\big)$ whose values at $\nu=j$ at a non-negative integer is the L-value above.

The key fact is that

$$\delta_k^j(f)\big(A,t_A,\omega_A\big)=\theta^j(f)\big(A,t_A,\omega_A\big)$$

where θ is the theta operator $\theta(\sum_n a_nq^n) = \sum_n na_nq^n$ on p-adic modular forms. In fact, in order to have a p-adic interpolation of θ one needs to take the *p-depletion* $f^{[p]}$ of f: If $f(q)=\sum_n a_nq^n$, write $f^{[p]}(q)=\sum_{(n,p)=1}a_nq^n$. It is the q-expansion of $\big(1-(V\circ U)\big)(f)=\big(1-a_pV+\chi(p)p^{k-1}V^2\big)(f)$, a modular form of level $\Gamma_0(p^2)\cap\Gamma_1(N)$.

If X is the modular curve of level $\Gamma_1(N)$ and $X^{\mathrm{ord}}\subset X$ is its ordinary locus (at p), then $\theta^j(f^{[p]})$ is a p-adic modular form and as such it can be evaluated *only* at points of X^{ord}. As $(A,t_A)\in X^{\mathrm{ord}}$ if and only of p splits in K, this explains the assumptions in the results of [Ka] and [BDP].

We prove the following:

Theorem Assume p non-split. We have an analytic function $L_p\big(k+\nu,-\nu\big)$ in ν such that for $\nu=j\geq 0$ an integer

$$L_p\big(k+j,-j\big) := \mathcal{E}_p(f)\Omega_p^{k+2j}L_{\mathrm{alg}}\big(?,k+j,-j\big)$$

where $\mathcal{E}_p(f)=1-\frac{(p-1)a_p^2}{\chi(p)(p+1)}-\frac{1}{p^2}$ if p inert and $\mathcal{E}_p(f):=1-\frac{a_p}{\chi(\mathfrak{p})}\big(1-\frac{1}{p^2}\big)-\frac{1}{p^3}$ if $p\mathcal{O}_K=\mathfrak{p}^2$ is ramified.

As in [Ka] and [BDP] we also have relations of the special values of L_p at integers, which are not in the region of interpolation, with Abel-Jacobi of Heegner cycles. A weaker form of these results has been first obtained by Daniel Kriz [Kr] using perfectoid techniques and it was attending his lecture that we understood that using the techniques of [AI] could be used to reprove and strengthen his results.

The Key idea is to interpolate powers of the Gauss-Manin connection on the relative de Rham cohomology H of the universal elliptic curve over X, instead of powers of θ. More precisely, for every analytic character ε of $(\mathbb{Z}_p)^*$ we define

- the space of nearly overconvergent forms $\mathbb{W}_\varepsilon$ over a strict neighborhood $X^{\mathrm{ord}}\subset X^\dagger$ such that for $\varepsilon=k$ integer we have $\mathrm{Sym}^kH\subset\mathbb{W}_k$;

- an interpolated connection $\nabla^\varepsilon \colon \mathrm{H}^0\big(X^\dagger, \omega^k\big)^{U_p=0} \to \mathrm{H}^0\big(X^\dagger, \mathbb{W}_{k+2\varepsilon}\big)$ compatible with the Gauss-Manin connection on $\mathrm{Sym}^k H$ for $\varepsilon = k$;
- a projection $\mathrm{pr}_\varepsilon \colon \mathbb{W}_\varepsilon|_{(A,t_A)} \to \omega_A^\varepsilon$.

The first two constructions are contained in [AI] already. With these tools in hand we can define

$$\delta^\varepsilon\big(f^{[p]}\big)\big(A, t_A, \omega^{\mathrm{can}}\big) := \mathrm{pr}_{k+2\varepsilon}\left(\nabla^\varepsilon\big(f^{[p]}\big)|_{(A,t_A)}\right)\big(\omega^{\mathrm{can},\varepsilon}\big).$$

This provides the sought for interpolation.

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BSD Formula for Elliptic Curves and Modular Forms

Xin Wan

Let $E/\mathbb{Q}$ be an elliptic curve over $\mathbb{Q}$. By the Taniyama-Shimura conjecture proved by the historic work of Wiles and Breuil-Conrad-Diamond-Taylor, we know it is modular, and thus the associated L-series $L(E, s)$ is an entire function for $s \in \mathbb{C}$. The famous Birch and Swinnerton-Dyer conjecture states that

- The vanishing order r of $L(E, s)$ at $s = 1$ is equal to the rank of the Mordell-Weil group $E(\mathbb{Q})$ of E. This is called the rank part of the conjecture.
- A formula for the leading Taylor coefficient $L^r(E, 1)$ in terms of arithmetic invariants. This part is called full BSD.

If $r = 0$ or 1 then the rank part is known by work of Gross-Zagier and Kolyvagin. If $r >\geq 2$ then very little is known. (The main reason is when $r \leq 1$, there is an explicit way of constructing rational points on the elliptic curve, namely Heegner points). Recently Bhargava-Skinner-Zhang proved at least 66 percent of elliptic curves have $r = 0$ or 1, and satisfying the rank part of BSD conjecture. Our work is on the full BSD in the $r = 0$ and 1 cases.

In the CM cases a lot is known by the work of Coates-Wiles, K.Rubin, Ye Tian, etc. We focus on non-CM cases in this talk. In this case, previously people only know full BSD for a finite number of non-CM elliptic curves by computer check. Recently we have made progresses on Iwasawa theory, making it possible to prove the following [3]

Theorem 1. *For many explicitly described infinite families of non-CM elliptic curves, the full BSD conjecture is true.*

There is large room of possible improvement for this theorem, which is work in progress. A key ingredient for this theorem is the following

Theorem 2. *Let $E/\mathbb{Q}$ be an elliptic curve of conductor N and p an odd prime not dividing N. Suppose $E[p]$ is an absolutely irreducible representation of $G_{\mathbb{Q}}$. Suppose there is an $\ell||N$ such that $E[p]|_{G_{\mathbb{Q}_\ell}}$ is a ramified representation. If $r = 0$ or 1 then the p-part of full BSD for E is true.*

The theorem is due to many people, including Kato, Skinner, Urban, Kolyvagin, Jetchev, Sprung, and myself [2], [5], [3], [1].

In fact the above formulation has been vastly generalized by Bloch-Kato to general motives. Now under certain assumptions we are also able to prove the BSD formula for motives associated to general weight GL_2 modular forms, when $r = 0$ and 1. The $r = 0$ case follows from the corresponding Iwasawa main conjecture we proved in [4]. The $r = 1$ case is an ongoing joint work with Jetchev and Skinner (also uses Iwasawa theory as a main tool).

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A geometric construction of some irreducible admissible locally analytic representations

Konstantin Ardakov

1. Intersection theory

Fix fields $L \subset k$, with k algebraically closed, and let $\mathbb{G}$ be a connected affine algebraic group of finite type over L. Let $G = \mathbb{G}(k)$ denote its group of k-points, and let X be a projective algebraic variety over k equipped with a morphic action of G. Finally, let Y be a Zariski closed subset of X.

Definition 1. We say that the $\mathbb{G}(L)$-orbit of Y in X is *regular* if for all $g \in \mathbb{G}(L)$, $gY \cap Y \neq \emptyset$ implies that $gY = Y$: *distinct translates of* Y *are disjoint.*

Of course every closed point $Y = \{y\}$ of X satisfies this regularity condition. For a non-example, note that if Y is a curve in $X = \mathbb{P}^2(k)$, then any two distinct translates of Y under an automorphism of X will intersect by Bezout's Theorem, so the orbit of Y can only be regular if Y is already stabilised by all of $\mathbb{G}(L)$.

Question 2. Can we find some examples of subvarieties Y with a regular $\mathbb{G}(L)$-orbit with $\dim(Y) > 0$?

Definition 3. We say that $h \in GL_n(k)$ is a *transvection* if $\mathrm{rk}(h-1) \leq 1$.

Because of the following Lemma, it turns out that it is not so easy to do this.

Lemma 4. *Suppose that* $\mathbb{G} = GL_n$ *and* $X = \mathbb{P}^n(k)$. *If* $\dim(Y) > 0$ *and* $h \in G$ *is a transvection, then* $Y \cap hY \neq \emptyset$.

Proof. Let $A = k[x_0, \ldots, x_n] = \mathrm{Sym}\, V$, so that $\mathbb{P}^n = \mathrm{Proj}(A)$. Without loss of generality Y is irreducible, so that $Y = V(I)$ for some prime ideal I of A with $\mathrm{Kdim}(A/I) \geq 2$. Choose some non-zero $f \in V$ such that $(h-1)(V) = kf$. Then $(h-1)(A) \subseteq Af$, so $I + hI = I + (h-1)I \subseteq I + Af$. Now $\mathrm{Kdim}(A/(I + Af)) \geq 1$ and hence $\emptyset \neq V(I + Af) \subseteq V(I + hI) = Y \cap hY$. $\square$

Thus, for example, the $GL_n(L)$-orbit of Y is never regular — transvections are somehow too lazy for the purpose of producing regular orbits. So in order for $\mathbb{G}(L)$ to admit regular orbits, our group $\mathbb{G}(L)$ cannot contain any transvections. Since transvections frequently give examples of unipotent elements in $GL_n(k)$, we observe that at least when the characteristic of k is zero, the anisotropic semisimple algebraic k-groups are good candidates for avoiding transvections, as these groups do not contain any non-trivial unipotent elements whatsoever.

Definition 5. Let $Y \subset X$ and $G = \mathbb{G}(k) \supset \mathbb{G}(L)$ be as above.

(1) The *intersection obstruction* is $\mathcal{Z}_Y := \{g \in G : Y \cap gY = \emptyset\}$.
(2) The *core obstruction* is $\mathcal{Z}_Y^\circ := \bigcap_{g \in G} g\mathcal{Z}_Y g^{-1}$.
(3) $x \in G$ is *generic* if the canonical map $x : \mathcal{O}(\mathbb{G}) \to k$ is injective.

It is not hard to see that $\mathcal{Z}_Y$ and $\mathcal{Z}_Y^\circ$ are Zariski closed subsets of G. Using these concepts, we can give some good news for the Question above.

Lemma 6. *Suppose that $\mathcal{Z}_Y^\circ \cap \mathbb{G}(L) = \{1\}$. Then the $\mathbb{G}(L)$-orbit of xY is regular in X whenever $x \in G$ is generic.*

Sketch proof. Let $h \in \mathbb{G}(L)$. Then $xY \cap hxY \neq \emptyset$ implies that $x^{-1}hxY \cap Y \neq \emptyset$, i.e. $x^{-1}hx \in \mathcal{Z}_Y$ and $h \in x\mathcal{Z}_Y x^{-1}$. Because x is *generic*, by analysing the conjugation representation of $\mathbb{G}$ on $\mathcal{O}(\mathbb{G})$, we can show that

$$x\mathcal{Z}_Y x^{-1} \cap \mathbb{G}(L) \subset \mathcal{Z}_Y^\circ.$$

So, $h \in x\mathcal{Z}_Y x^{-1} \cap \mathbb{G}(L) \subseteq \mathcal{Z}_Y^\circ \cap \mathbb{G}(L)$ which is trivial by assumption. □

It turns out to be possible to calculate the core of the intersection obstruction in the following example.

Proposition 7. *Let $\mathbb{G} = GL_4$, $X = \mathbb{P}^3(k)$ and let Y be the* twisted cubic curve*: $Y = V(x_0x_2 - x_1^2, x_1x_3 - x_2^2, x_0x_3 - x_1x_2) \subset X$. Then $\mathcal{Z}_Y^\circ = \{transvections\} \cdot k^\times$.*

In order to satisfy $\mathcal{Z}_Y^\circ \cap \mathbb{G}(L) = \{1\}$, we turn to p-adic division algebras.

Lemma 8. *Suppose that L is a finite extension of $\mathbb{Q}_p$ and that $\mathbb{G}(L) = D^\times$ where D is a division algebra of degree 4 over L. Then $D^\times$ contains no transvections.*

We can now give a positive answer to the above Question.

Corollary 9. *The $D^\times$-orbit of any generic twisted cubic curve in $\mathbb{P}^3(k)$ is regular.*

We can use this example to construct subvarieties of flag varieties having a regular orbit, just taking the preimage of any generic cubic curve in $\mathbb{P}^3$ under any one of the two projection maps from the flag variety of GL_4 to $\mathbb{P}^3$.

2. Main result

Let $\mathbb{Q}_p \subset L \subset K$ be non-Archimedean fields, with L finite over $\mathbb{Q}_p$, and let $\mathbb{G}$ be a connected affine algebraic group, of finite type over L, such that $\mathbb{G}_K := \mathbb{G} \otimes_L K$ is split semisimple. We fix an open subgroup G of $\mathbb{G}(L)$. Let $X = (\mathbb{G}_K/\mathbb{B})^{\mathrm{an}}$ be the rigid-analytic flag variety over K and let $Y \subset X$ be a connected, smooth, Zariski closed subset. We denote its stabiliser in G by G_Y.

Definition 10. $G_Y - \mathrm{MIC}(Y) := \mathrm{coh}(\mathcal{O}_Y) \cap \mathrm{mod}\ (\mathcal{D}_Y, G_Y)$ denotes the category of G_Y-equivariant vector bundles on Y equipped with an integrable connection.

$G_Y - \mathrm{MIC}(Y)$ is in fact an abelian tensor category. Here is our main result.

Theorem 11. *Suppose that G_Y is co-compact in G and the G-orbit of Y is regular in X. Then*

1. *For every irreducible object $\mathcal{F} \in G_Y - \mathrm{MIC}(Y)$, there is an irreducible co-admissible $D(G,K)$-module $M_{\mathcal{F}}$.*
2. *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are not isomorphic, then $M_{\mathcal{F}_1}$ and $M_{\mathcal{F}_2}$ are not isomorphic.*

Here $D(G,K)$ is a suitable generalisation to arbitrary non-Archimedean coefficient fields K of the algebra of locally L-analytic K-valued distributions on the group G, introduced by Schneider and Teitelbaum in [4].

Remark 12.

(1) If K is discretely valued and $\mathcal{F} \in G_Y - \mathrm{MIC}(Y)$ is irreducible, then $M^*_{\mathcal{F}}$ is an irreducible admissible locally analytic K-representation of G.
(2) Theorem 11 applies if $G = D^\times \leq GL_4(K)$ is the group of units of a central division algebra D of degree 4 over L, and Y is the full preimage in the rigid analytic flag variety of a generic twisted cubic curve in $\mathbb{P}^3(K)$.

3. The construction

In our earlier work [2], we introduced the abelian category $\mathcal{C}_{X/G}$ of *co-admissible G-equivariant $\mathcal{D}$-modules on X*: here G is a p-adic Lie group acting continuously on a smooth rigid analytic space X. The construction of the $D(G,K)$-module $M_{\mathcal{F}}$ appearing in Theorem 11 proceeds in three steps. First, we have the following equivariant version of the Kashiwara equivalence from [1].

Theorem 13. *Let $Y \subset X$ be a smooth, Zariski closed subset of the smooth rigid analytic space X. Then there is an equivalence of categories*

$$i_+ : \mathcal{C}_{Y/G_Y} \xrightarrow{\cong} \mathcal{C}^Y_{X/G_Y}.$$

Here $\mathcal{C}^Y_{X/G_Y}$ denotes the full subcategory of $\mathcal{C}_{X/G_Y}$ consisting of sheaves whose support is wholly contained in Y. Next, we have an Induction Equivalence, which tells us how to study objects in $\mathcal{C}^{GY}_{X/G}$, provided the G-orbit of Y in X is regular, and certain other somewhat technical conditions hold.

Theorem 14. *Suppose further that the following conditions are satisfied.*

(1) *The G-orbit of Y in X is regular,*
(2) *G/G_Y is compact,*
(3) *X is quasi-compact and separated,*
(4) *Y is connected and smooth.*

Then the functor $\mathcal{H}^0_Y$ of local cohomology gives an equivalence of categories

$$\mathcal{H}^0_Y : \mathcal{C}^{GY}_{X/G} \xrightarrow{\cong} \mathcal{C}^Y_{X/G_Y}.$$

The quasi-inverse $\mathrm{ind}^G_{G_Y}$ of $\mathcal{H}^0_Y$ is given by the following formula:

$$\mathrm{ind}^G_{G_Y}(\mathcal{N})(U) := \varprojlim_{H} \bigoplus_{Z \in H\backslash G/G_Y} \lim_{s \in Z} \widehat{\mathcal{D}}(U,H) \underset{\widehat{\mathcal{D}}(U,{}^sG_Y \cap H)}{\widehat{\otimes}} [s]\, \mathcal{N}(s^{-1}U)$$

for all affinoid subdomains U of X. We refer the reader to the forthcoming paper [3] for the details of the construction of this induction functor $\mathrm{ind}^G_{G_Y}$, as well as the proofs of Theorems 13 and 14. The third step is the locally analytic version of the Beilinson-Bernstein equivalence, which has already been established at [2, theorem c].

Theorem 15. *Let $X = (\mathbb{G}_K/\mathbb{B})^{\mathrm{an}}$ be the rigid analytic flag variety. Then the functor of global sections gives an equivalence of categories between $\mathcal{C}_{X/G}$ and the category of co-admissible $D(G,K)$-modules with trivial infinitesimal central character.*

We can now give the definition of $M_{\mathcal{F}}$ appearing in Theorem 11:

$$M_{\mathcal{F}} := \Gamma((\mathbb{G}_K/\mathbb{B})^{\mathrm{an}}, \mathrm{ind}_{G_Y}^{G}\, i_+(\mathcal{F})).$$

It is straightforward to combine Theorems 13, 14, 15 to deduce Theorem 11.

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Functoriality, theta correspondence and motivated cycles

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(joint work with Atsushi Ichino)

1. Introduction

We consider the following question: *is Langlands functoriality for cohomological automorphic forms (on Shimura varieties) realized by algebraic cycles?* We recall some examples below in §1.1-1.3. In all these cases, the Tate conjecture predicts that functoriality should be given by an algebraic cycle. However, very little is known in this direction. Mainly, one knows relations between Galois representations (eg. by using the Langlands-Kottwitz method) and between rational Hodge structures (by proving relations between periods) – however, previously one did not even know in any family of examples that the isomorphisms of Hodge structures and of Galois representations have anything to do with each other.

1.1. Let A be an elliptic curve over $\mathbf{Q}$ with CM by an imaginary quadratic field K. There is an attached Hecke character $\psi : \mathbf{A}_K^\times \to \mathbf{C}^\times$ of infinity type $(1,0)$ such that $L(A,s) = L_K(\psi, s)$. Let θ_{ψ^2} denote the theta function associated to ψ^2; it is a weight 3 form on $\Gamma_1(N)$ for suitable N. The motive "$\mathrm{Ind}_K^{\mathbf{Q}}\psi^2$" occurs in H^2 of two different varieties, namely $A \times A$ and the Kuga-Sato surface W over $X_1(N)$. The Tate conjecture then predicts the existence of a codimension two cycle on $W \times (A \times A)$, realizing this equality of motives.

1.2. Let f be a modular form of weight 4 on $\Gamma_0(N)$ and consider the Saito-Kurokawa lift $\mathrm{SK}(f)$ of f to a Siegel modular form. (The lift is actually to a packet, but we ignore this here.) Frequently, $\mathrm{SK}(f)$ contributes to $H^3(X)$ for X a Siegel threefold, with Galois representation isomorphic to the Galois representation attached to f. On the other hand, this representation also occurs in $H^3(W)$ for a Kuga-Sato threefold W, so there should be a cycle on $W \times X$ realizing this isomorphism.

1.3. Let π be a cohomological Hilbert modular form for a totally real field F, and let B_1 and B_2 be two non-isomorphic quaternion algebras over F such that π admits a Jacquet-Langlands transfer to both $B_1^\times$ and $B_2^\times$. Let X_1 and X_2 denote the associated Shimura varieties. We assume that B_1 and B_2 are split at the same set of infinite places Σ, where $\Sigma_\infty = \Sigma \sqcup \Sigma'$ is a partition of the set of infinite places of F. Then $\dim X_{B_1} = \dim X_{B_2} = d$, where $d := |\Sigma|$, and the π-isotypic component of the cohomology of X_1 and X_2 is concentrated in the middle degree d. The π-isotypic component of ℓ-adic cohomology is well understood, by work of many authors, including Langlands, Brylinski-Labesse, Carayol, Reimann and Nekovář; from this it follows in particular that there is an isomorphism of $\mathrm{Gal}(\overline{\mathbf{Q}}/F_\Sigma)$-representations:

$$H^d_{\mathrm{et}}(X_1 \otimes_{F_\Sigma} \overline{\mathbf{Q}}, \mathbf{Q}_\ell)_\pi \simeq H^d_{\mathrm{et}}(X_2 \otimes_{F_\Sigma} \overline{\mathbf{Q}}, \mathbf{Q}_\ell)_\pi,$$

where $F_\Sigma \subset \overline{\mathbf{Q}} \subset \mathbf{C}$ denotes the reflex field, which is the common number field over which the canonical models of X_1 and X_2 are defined. Thus there should be a cycle on $X_1 \times X_2$ realizing this isomorphism.

2. Main results and future directions

2.1. To state our results, we introduce the following definition:
Definition: *Let X be a variety over a number field k, given with a fixed embedding $k \subset \overline{\mathbf{Q}} \subset \mathbf{C}$. A Hodge-Tate cycle on X is a Hodge class $\xi \in H^{2j}(X(\mathbf{C}), \mathbf{Q}(j))$ whose image in ℓ-adic cohomology $H^{2j}_{\mathrm{et}}(X \otimes_k \overline{\mathbf{Q}}, \mathbf{Q}_\ell(j))$ (under the Betti-étale comparison isomorphism) is $\mathrm{Gal}(\overline{\mathbf{Q}}/k)$-invariant for all ℓ.*

Note that we have inclusions (on X):

$$\begin{aligned}\text{Algebraic cycles} &\subseteq \text{Motivated cycles} \subseteq \text{Absolute Hodge cycles}\\ &\subseteq \text{Hodge-Tate cycles} \subseteq \text{Hodge cycles},\end{aligned}$$

and the Hodge conjecture predicts that the inclusions are actually equalities. The notion of motivated cycles is due to André [1]. Roughly, these are obtained from cohomology classes of algebraic cycles by allowing all the usual operations on algebraic cycles (pull-back, pushforward etc.), but in addition allowing inverses under the Lefschetz isomorphism

$$L^{n-2j} : H^{2j}(Y) \xrightarrow{\simeq} H^{2n-2j}(Y),$$

for any choice of hyperplane section L on any auxiliary variety Y.

2.2. We now restrict to the setting of §1.3, and assume (for technical reasons) that Σ' is non-empty.
Theorem: *The Jacquet-Langlands correspondence for quaternionic Shimura varieties (and a fixed π as above) is realized by a Hodge-Tate cycle. In particular, there is an isomorphism*

$$\iota : H^{2d}(X_1(\mathbf{C}), \mathbf{Q}(d))_\pi \simeq H^{2d}(X_2(\mathbf{C}), \mathbf{Q}(d))_\pi$$

of rational Hodge structures, such that $\iota \otimes \mathbf{Q}_\ell$ is $\mathrm{Gal}(\mathbf{Q}/F_\Sigma)$-equivariant for all ℓ.

This theorem is proved in [2]. The idea is to embed $X_1 \times X_2 \hookrightarrow X$ in a larger Shimura variety X, construct a Hodge-Tate cycle ξ on X and show that its restriction to $X_1 \times X_2$ has the required properties. The variety X is attached to a unitary group $\mathrm{U}(V)$, but to construct ξ, we first relate it to a Shimura variety attached to a quaternionic unitary group $\mathrm{U}_B(\tilde{V})$ and then construct ξ by theta lifting. Here $B = B_1 B_2$ is the product of B_1 and B_2 in the Brauer group of F. The proof uses among other things, the theory of Kudla-Millson, nonvanishing of certain automorphic period integrals, and the Arthur classification of *non-tempered* automorphic representations on unitary groups in terms of local and global A-packets. This last bit is work in progress of Kaletha, Minguez, Shin and White (see [4]).

2.3. It seems tricky to show directly that the class ξ is absolutely Hodge, in particular that it is de Rham rational. On the other hand, we hope to show (work in progess [3]) the following stronger result: *The class ξ is motivated, thus a fortiori, absolutely Hodge.*

The basic idea is that if L is the canonical Kahler class on X, then $L^{2d}(\xi)$, which is a class of type $(3d, 3d)$ on X should be the class of an algebraic cycle attached to a sub-Shimura variety of unitary type. The proof is again expected to use Kudla-Millson theory and the classification mentioned above. It is likely that this sort of result is the best that one can obtain towards the Tate conjecture by purely automorphic methods. Indeed, any cycle constructed on $X_1 \times X_2$ by purely group theoretic techniques will have a dense set of CM points, hence by the André-Oort conjecture, should be the union of irreducible components of sub-Shimura varieties. But if $B_1 \neq B_2$, then $X_1 \times X_2$ has no interesting sub-Shimura varieties. In any case, knowing that there is a more or less explicit motivated cycle realizing functoriality may in some applications be more useful than knowing that there is some non-explicit algebraic cycle realizing the same.

2.4. Finally, we mention some speculative ideas generalizing this to the setting of unitary groups. Suppose that X_1 and X_2 correspond to $\mathrm{U}(\mathbf{V}_1)$ and $\mathrm{U}(\mathbf{V}_2)$ respectively, where $\mathbf{V}_i$ is a hermitian space over an imaginary quadratic extension E of $\mathbf{Q}$, with $\mathrm{U}(\mathbf{V}_i)(\mathbf{R}) \simeq \mathrm{U}(p,q)$. Suppose also that we are given a (tempered) cohomological cuspidal automorphic representation π of $\mathrm{U}(\mathbf{V}_1)$ that transfers to $\mathrm{U}(\mathbf{V}_2)$. Let $\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2$. Then we expect that there should exist a diagram as below:

$$X_1 \times X_2 \xrightarrow{i} X \xleftarrow{j} Y,$$

with X a Shimura variety attached to $\mathrm{U}(\mathbf{V})$ and Y a Shimura-subvariety of X attached to $\mathrm{U}(\mathbf{W})$, where $\mathbf{W}$ is a subspace of $\mathbf{V}$ also of signature (p,q), such that

$$i^* \circ T \circ (L^{2pq})^{-1}(\mathrm{cl}_X(Y))$$

(for a suitable Hecke operator T on X) is a motivated cycle which realizes the Jacquet-Langlands correspondence for π. (There is also an obvious generalization to totally real base fields.) The most subtle issue involved in studying this prediction seems to be establishing nonvanishing of the relevant automorphic periods.

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Mod p Hecke algebras and dual equivariant cohomology

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(joint work with Cédric Pépin)

Let $F/\mathbb{Q}_p$ be a finite extension field with residue field $\mathbb{F}_q$ and let $G = \mathrm{GL}_n(F)$. Let $I \subset G$ be the standard Iwahori subgroup of G. The classical Deligne-Langlands conjecture for Hecke modules, proved in the middle of the 1980's by Kazhdan-Lusztig [10], is an incarnation of the (tame) local Langlands correspondance for G. It predicts a parametrization of the simple modules of the complex Iwahori-Hecke algebra $\mathcal{H}_{\mathbb{C}} = \mathbb{C}[I \setminus G/I]$ in terms of certain pairs $(s,t) \in \widehat{\mathbf{G}}^2$ where $sts^{-1} = t^q$. Here, $\widehat{\mathbf{G}} = \mathrm{GL}_n(\mathbb{C})$ is the complex Langlands dual group of G. Let $\widehat{\mathcal{B}}$ be the complex flag variety of $\widehat{\mathbf{G}}$ together with its $\widehat{\mathbf{G}}$-action given by translations. One of the steps in the proof of the conjecture is the construction of a suitable $\mathcal{H}_{\mathbb{C}}$-action on the equivariant K-theory $K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_{\mathbb{C}}$ [11]. It identifies the action of the center $Z(\mathcal{H}_{\mathbb{C}})$ with the scalar multiplication by $K^{\widehat{\mathbf{G}}}(\mathrm{pt})_{\mathbb{C}} = R(\widehat{\mathbf{G}})_{\mathbb{C}}$ where $R(\widehat{\mathbf{G}})$ denotes the representation ring of the algebraic group $\widehat{\mathbf{G}}$.

The idea of studying various cohomology theories of the flag variety by means of Hecke operators (nowadays called Demazure operators) goes back to earlier work of Demazure [3, 4]. Of course, much of all this extends to reductive groups more general than GL_n.

In this extended abstract we report on the very first steps towards a mod p version of Kazhdan-Lusztig theory which has the aim to understand better the geometry of the modular Hecke algebras which appear in the mod p local Langlands program. We change slightly notation and let from now on $\widehat{\mathbf{G}} = \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ be the mod p Langlands dual group of G and let $\widehat{\mathcal{B}}$ be its flag variety. Let $\mathcal{H}^{(1)}$ be the pro-p Iwahori-Hecke algebra of G, with coefficients in $\bar{\mathbb{F}}_p$ [15]. The center of $\mathcal{H}^{(1)}$ contains a subring $Z^{\circ}(\mathcal{H}^{(1)})$ which is isomorphic to the monoid algebra $\bar{\mathbb{F}}_p[\Lambda^+]$ of the dominant cocharacters Λ^+ of G [12]. In turn, the classical Steinberg isomorphism identifies the monoid algebra $\bar{\mathbb{F}}_p[\Lambda^+]$ with $R(\widehat{\mathbf{G}})_{\bar{\mathbb{F}}_p}$. An element $s \in \widehat{\mathbf{G}}$ is called supersingular, if the associated character $Z^{\circ}(\mathcal{H}^{(1)}) \to \bar{\mathbb{F}}_p$ is supersingular, i.e. sends the noninvertible fundamental dominant cocharacters to zero. An irreducible $\mathcal{H}^{(1)}$-module is called *supersingular*, if $Z^\circ(\mathcal{H}^{(1)})$ acts via a supersingular character and this notion generalizes to finite length modules [16]. The interest in this notion comes from the mod p Langlands correspondence:

Theorem (Breuil, Colmez, Grosse-Klönne, Vignéras [1, 2, 7, 8, 14]). There is an exact and fully faithful functor from the category of supersingular $\mathcal{H}^{(1)}$-modules to the category of $\mathrm{Gal}(\bar{F}/F)$-representations over $\bar{\mathbb{F}}_p$.

Let $\mathbb{T} = T(\mathbb{F}_q)$ be the finite diagonal torus of G and let $\mathbb{T}^\vee$ be its set of characters. Let $W_0 \subset W_{\text{aff}} \subset W$ be the finite, affine and Iwahori Weyl group of G respectively. The algebra $\mathcal{H}^{(1)}$ decomposes into finitely many components

$$\mathcal{H}^{(1)} = \prod_{\gamma \in \mathbb{T}^\vee / W_0} \mathcal{H}^\gamma$$

and, consequently, so does the category of $\mathcal{H}^{(1)}$-modules. If $|\gamma| = 1$ or $|\gamma| = W_0$, the γ-component is called *of Iwahori type* or *regular* respectively. The structure of a general γ-component is a 'mixture' between these two extreme cases.

Let γ be an Iwahori component. Then

$$\mathcal{H}^\gamma \simeq \mathcal{H} := \bar{\mathbb{F}}_p[I \setminus G/I]$$

is isomorphic to the Iwahori-Hecke algebra over $\bar{\mathbb{F}}_p$ (which itself is the component associated to the orbit of the trivial character of $\mathbb{T}$). Fix a set S_0 of simple reflections and a set $S_{\text{aff}} = S_0 \cup \{s_0\}$ of simple affine reflections for W_0 and W_{aff} respectively. The quadratic relations in $\mathcal{H}$ are given by $T_s^2 = -T_s$ for $s \in S_{\text{aff}}$.

Theorem 1. There exists a unique algebra homomorphism

$$\mathscr{A} : \mathcal{H} \longrightarrow \mathrm{End}_{R(\widehat{\mathbf{G}})_{\bar{\mathbb{F}}_p}}(K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_{\bar{\mathbb{F}}_p})$$

such that

- $\mathscr{A}(T_s) = -D_s \quad \forall s \in S_0$ (Demazure operator)
- $\mathscr{A}|_{Z(\mathcal{H})} : Z(\mathcal{H}) \xrightarrow{\sim} R(\widehat{\mathbf{G}})_{\bar{\mathbb{F}}_p}$ (Ollivier-Steinberg isomorphism).

Remark: Note that $K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}) = \mathbb{Z}[\Lambda]$ and $K^{\widehat{\mathbf{G}}}(\mathrm{pt}) = \mathbb{Z}[\Lambda]^{W_0}$ as abstract rings. In this setting the Demazure operator is given by

$$D_s(a) = \frac{a - s(a)}{1 - e^{\alpha^\vee}}$$

for all $a \in \mathbb{Z}[\Lambda]$ where $s = s_\alpha$ with associated simple root α [4].

Given $s \in \widehat{\mathbf{G}}$ we let $\mathcal{H}_s = \mathcal{H} \otimes_{Z(\mathcal{H}),s} \bar{\mathbb{F}}_p$. If s is semisimple, then, by localisation [13], $K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_{\bar{\mathbb{F}}_p} \otimes_{R(\widehat{\mathbf{G}})_{\bar{\mathbb{F}}_p},s} \bar{\mathbb{F}}_p = K(\widehat{\mathcal{B}}^s)_{\bar{\mathbb{F}}_p}$ where $\widehat{\mathcal{B}}^s \subset \widehat{\mathcal{B}}$ denotes the s-fixed points. Then $\mathcal{H}_s$ acts on $K(\widehat{\mathcal{B}}^s)_{\bar{\mathbb{F}}_p}$. The $\bar{\mathbb{F}}_p$-dimension of $K(\widehat{\mathcal{B}}^s)_{\bar{\mathbb{F}}_p}$ is at most $|\widehat{\mathcal{B}}^s| = |W_0|$.

Theorem 2. Let $n \leq 3$. Fix a semisimple element $s \in \widehat{\mathbf{G}}$ which is supersingular. All n-dimensional simple supersingular $\mathcal{H}_s$-modules appear as subquotients of $K(\widehat{\mathcal{B}}^s)_{\bar{\mathbb{F}}_p}$ with multiplicity one.

Let now γ be a regular orbit. Then

$$\mathcal{H}^\gamma \simeq \bar{\mathbb{F}}_p^{|\gamma|} \otimes'_{\bar{\mathbb{F}}_p} \mathcal{H}^{\text{nil}}$$

is a certain smash product between the direct product ring $\bar{\mathbb{F}}_p^{|\gamma|}$ and Kostant-Kumar's *Nil Hecke ring* $\mathcal{H}^{\text{nil}}$ with coefficients in $\bar{\mathbb{F}}_p$ [9]. The quadratic relations in $\mathcal{H}^{\text{nil}}$ are given by $T_s^2 = 0$ for $s \in S_{\text{aff}}$. Moreover, $Z^\circ(\mathcal{H}^\gamma) := Z^\circ(\mathcal{H}^{(1)}) \cap \mathcal{H}^\gamma$ is a proper subring of $Z(\mathcal{H}^\gamma)$.

We let $\widehat{\mathcal{B}}^\gamma$ be the disjoint union of $|\gamma|$ copies of $\widehat{\mathcal{B}}$ and consider its equivariant intersection theory $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^\gamma)$ [5, 6]. It is a module over $S(\widehat{\mathbf{G}}) = CH^{\widehat{\mathbf{G}}}(\text{pt})$ together with an action $\text{perm} : W \to W_0 \curvearrowright \widehat{\mathcal{B}}^\gamma$ which permutes the factors. Note that

$$CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}) = \text{Sym}(\Lambda) \qquad \text{and} \qquad CH^{\widehat{\mathbf{G}}}(\text{pt}) = \text{Sym}(\Lambda)^{W_0}$$

as abstract rings and the Demazure operator is given as

$$D_s(a) = \frac{a - s(a)}{\alpha^\vee}$$

for all $a \in \text{Sym}(\Lambda)$ where $s = s_\alpha$ [3]. On these objects, we finally invert the invariant $\chi_n = \eta_1 \cdots \eta_n$ where the $\eta_i(x) = \text{diag}(1, ..., 1, x, 1, ..., 1)$ (x in position i) are the standard basis elements for Λ.

Theorem 3. There exists a unique algebra homomorphism

$$\mathscr{A}^\gamma : \mathcal{H}^\gamma \longrightarrow \text{End}_{S(\widehat{\mathbf{G}})[\chi_n^{-1}]_{\bar{\mathbb{F}}_p}}(CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^\gamma)[\chi_n^{-1}]_{\bar{\mathbb{F}}_p})$$

such that

- $\mathscr{A}^\gamma|_{\mathcal{H}^{\text{nil}}}(T_s) = -D_s \circ \text{perm}(s) \quad \forall s \in S_0$
- $\mathscr{A}^\gamma|_{Z^\circ(\mathcal{H}^\gamma)} : Z^\circ(\mathcal{H}^\gamma) \hookrightarrow S(\widehat{\mathbf{G}})[\chi_n^{-1}]_{\bar{\mathbb{F}}_p}$.

Theorem 4. Let $n \leq 3$. Fix a semisimple element $s \in \widehat{\mathbf{G}}$ which is supersingular. All n-dimensional simple supersingular $\mathcal{H}_s^\gamma$-modules appear as subquotients of $CH(\widehat{\mathcal{B}}^{\gamma,s})[\chi_n^{-1}]_{\bar{\mathbb{F}}_p}$ with multiplicity one.

We believe that these theorems can be generalized to any n and to all γ-components of the pro-p Hecke algebra $\mathcal{H}^{(1)}$. Let $\varphi, v \in \text{Gal}(\bar{F}/F)$ be a Frobenius lift and a monodromy generator respectively. Any tame n-dimensional $\text{Gal}(\bar{F}/F)$-representation over $\bar{\mathbb{F}}_p$ leads to a couple $(s = \rho(\varphi), t = \rho(v)) \in \widehat{\mathbf{G}}^2$ with $sts^{-1} = t^q$. A natural question is then the following: Can one parametrize the supersingular $\mathcal{H}_s^{(1)}$-modules appearing in the cohomology of $\widehat{\mathcal{B}}^s$ via the parameter t? This is work in progress.

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Drinfeld's lemma for perfectoid spaces and multivariable (φ, Γ)-modules

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(joint work with Anne T. Carter, Gergely Zábrádi)

Let K be a finite extension of $\mathbb{Q}_p$ with absolute Galois group G_K. The theory of (univariate) (φ, Γ)-modules, as initiated by Fontaine [7] and extended subsequently (see [8] for an efficient exposition), gives an alternate description of the category $\mathbf{Rep}_{\mathbb{Z}_p}(G_K)$ of continuous representations of G_K on finite free $\mathbb{Z}_p$-modules. This uses a certain cartesian square of ring inclusions

$$\begin{array}{ccc} \mathbf{A}_K^\dagger & \longrightarrow & \mathbf{A}_K \\ \downarrow & & \downarrow \\ \tilde{\mathbf{A}}_K^\dagger & \longrightarrow & \tilde{\mathbf{A}}_K \end{array} \tag{1}$$

which are equivariant with respect to a certain endomorphism φ and an action of a certain one-dimensional p-adic Lie group Γ which commute with each other. A (φ, Γ)-*module* over any of the rings in (1) is a finite free module over that ring enriched with commuting semilinear actions of φ and Γ; there then exist explicit functors between $\mathbf{Rep}_{\mathbb{Z}_p}(G_K)$ and (φ, Γ)-modules (over each base ring) which define equivalences of categories.

These rings can be described explicitly for $K = \mathbb{Q}_p$ as follows: take

$$\mathbf{A}_K := \mathbb{Z}_p((\pi))^{\wedge}_{(p)}$$
$$\tilde{\mathbf{A}}_K := W(\mathbb{F}_p((\overline{\pi}))^{\mathrm{perf}\wedge})$$

(where $*^{\wedge}_{(p)}$ denotes p-adic completion and $*^{\mathrm{perf}\wedge}$ denotes completed perfect closure), identifying $\pi \in \mathbf{A}_K$ with $[1+\overline{\pi}] - 1 \in \tilde{\mathbf{A}}_K$, then take $\tilde{\mathbf{A}}_K^{\dagger}$ to be the set of $x = \sum_{n=0}^{\infty} p^n [\overline{x}_n]$ for which there exist integers $a, b > 0$ (depending on x) for which

$$\overline{\pi}^{an+b} \overline{x}_n \in \widehat{\mathbb{F}_p[[\overline{\pi}]]^{\mathrm{perf}}} \qquad (n = 0, 1, \dots).$$

The action of φ and $\gamma \in \mathbb{Z}_p^{\times}$ is given by

$$\varphi(\overline{\pi}) = \overline{\pi}^p, \qquad \gamma(\overline{\pi}) = (1+\overline{\pi})^{\gamma} - 1.$$

For these three rings, the equivalence of $\mathbf{Rep}_{\mathbb{Z}_p}(G_K)$ with (φ, Γ)-modules is a mostly formal consequence of the Fontaine–Wintenberger isomorphism

$$G_{\widehat{K_\infty}} \cong G_{\widehat{K_\infty}^{\flat}}, \qquad K_\infty = K(\mu_{p^\infty}), \qquad \widehat{K_\infty}^{\flat} := \varprojlim_{x \to x^p} \widehat{K_\infty},$$

a special case of the tilting equivalence for perfectoid fields. The intersection $\mathbf{A}_K^{\dagger}$ consists of series convergent in some range $* < |\pi| < 1$; the equivalence for this ring is a deeper result of Cherbonnier–Colmez [5].

Our main result [4] is a corresponding description of $\mathbf{Rep}(G_{K,\Delta})$ where $G_{K,\Delta} := \prod_{\alpha \in \Delta} G_K$ is the product over an arbitrary finite index set Δ, using a corresponding cartesian square of ring inclusions

$$\begin{array}{ccc} \mathbf{A}_{K,\Delta}^{\dagger} & \longrightarrow & \mathbf{A}_{K,\Delta} \\ \downarrow & & \downarrow \\ \tilde{\mathbf{A}}_{K,\Delta}^{\dagger} & \longrightarrow & \tilde{\mathbf{A}}_{K,\Delta} \end{array} \tag{2}$$

These rings are certain topological products of the rings without Δ, and again can be made explicit for $K = \mathbb{Q}_p$: define

$$\mathbf{A}_{K,\Delta} := \mathbb{Z}_p[[\pi_\alpha : \alpha \in \Delta]][\pi_\alpha^{-1} : \alpha \in \Delta]^{\wedge}_{(p)}$$
$$\tilde{\mathbf{A}}_{K,\Delta} := W(\mathbb{F}_p[[\overline{\pi}_\alpha : \alpha \in \Delta]][\overline{\pi}_\alpha^{-1} : \alpha \in \Delta]^{\mathrm{perf}\wedge})$$

identifying $\pi_\alpha \in \mathbf{A}_{K,\Delta}$ with $[1+\overline{\pi}_\alpha] - 1 \in \tilde{\mathbf{A}}_{K,\Delta}$, then take $\tilde{\mathbf{A}}_{K,\Delta}^{\dagger}$ to be the set of $x = \sum_{n=0}^{\infty} p^n [\overline{x}_n]$ for which there exist integers $a, b > 0$ (depending on x) for which

$$\left(\prod_{\alpha\in\Delta}\overline{\pi}_\alpha\right)^{an+b}\overline{x}_n \in \mathbb{F}_p[[\overline{\pi}_\alpha : \alpha \in \Delta]]^{\mathrm{perf}\wedge} \qquad (n = 0, 1, \dots).$$

For each $\alpha \in \Delta$, one has actions of φ_α and $\gamma \in \Gamma_\alpha \cong \mathbb{Z}_p^\times$ given by

$$\varphi_\alpha(\overline{\pi}_\alpha) = \overline{\pi}_\alpha^p, \qquad \gamma(\overline{\pi}_\alpha) = (1 + \overline{\pi}_\alpha)^\gamma - 1,$$

where $\overline{\pi}_\beta$ remains fixed for $\beta \neq \alpha$. A $(\varphi_\Delta, \Gamma_\Delta)$-*module* over any of the rings in (2) is a finite projective (but not necessarily free) module over the corresponding ring equipped with commuting semilinear actions of $\varphi_\alpha, \Gamma_\alpha$ for each $\alpha \in \Delta$. We prove that these form a category equivalent to $\mathbf{Rep}_{\mathbb{Z}_p}(G_{K,\Delta})$.

In this result, the replacement for the Fontaine–Wintenberger isomorphism is a form of *Drinfeld's lemma* for perfectoid spaces, or more generally for *diamonds* in the sense of Scholze [15]. The original Drinfeld's lemma (as stated in [9, Lecture 4]) asserts that for two connected qcqs schemes X, Y over $\mathbb{F}_p$, the natural morphism

$$\pi_1((X \times Y)/\varphi_X) \to \pi_1(X) \times \pi_1(Y)$$

(for suitable basepoints) is an isomorphism of profinite topological groups. In this situation, the quotient by the relative Frobenius φ_X does not exist in the category of schemes, only in a suitable category of stacks. (In general $X \times Y$ need not be connected, as in the case $X = Y = \operatorname{Spec}\overline{\mathbb{F}}_p$; but $X \times Y$ does not admit any φ_X-stable disconnection, so the quotient $(X \times Y)/\varphi_X$ is connected.)

For perfectoid spaces of characteristic p, the statement of Drinfeld's lemma is the same, except that the quotient already exists in the category of perfectoid spaces. For example, if $X = Y = \operatorname{Spa}(\mathbb{C}_p^\flat)$, then $(X \times Y)/\varphi_X$ is the Fargues–Fontaine curve [6] for the perfectoid field $\mathbb{C}_p^\flat$ with coefficients in $\mathbb{C}_p^\flat$; the statement that this space is simply connected is a result of Weinstein [17] (see also [6, §8] and [16, Lecture XVI]. Crucially, this holds for more general geometric points, but this requires some additional argument with p-adic differential equations [10].

One further issue that must be addressed is that even if X and Y are quasicompact, the product $X \times Y$ need not be; for instance, in the example $X = Y = \operatorname{Spa}(\mathbb{C}_p^\flat)$, the space $X \times Y$ is neither quasicompact nor connected. In order to apply Drinfeld's lemma to multivariate (φ, Γ)-modules, we must show that étale covers of such products extend uniquely to certain compactifications thereof; this amounts to a special case of the perfectoid Riemann extension theorem, as in works of Scholze [14], André [1, 2], and Bhatt [3].

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Poitou-Tate duality in higher dimension

Alexander Schmidt
(joint work with Thomas Geisser)

The classical Poitou-Tate theorem considers the cohomology of Galois groups with restricted ramification of global fields. It states a perfect duality between Shafarevich-Tate groups, and a 9-term exact sequence relating global and local cohomology groups, cf. [NSW, (8.6.7), (8.6.10)]. In our talk we gave an overview about the proof (see [GS]) of the following generalization to higher dimension.

Let S be a nonempty set of places of a global field k, and assume that S contains the set S_∞ of archimedean places if k is a number field. Let $\mathcal{O}_S$ be the ring of S-integers in k and $\mathscr{S} = \operatorname{Spec} \mathcal{O}_S$. Let $\mathscr{X} \to \mathscr{S}$ be a regular, flat, and separated scheme of finite type of relative dimension r, $m \geq 1$ an integer invertible on $\mathscr{S}$ and $\mathscr{F}$ a locally constant, constructible sheaf of $\mathbb{Z}/m\mathbb{Z}$-modules on $\mathscr{X}$. We consider the *Shafarevich-Tate groups* defined by

$$\begin{aligned} Ш^i(\mathscr{X}, \mathscr{S}, \mathscr{F}) &= \ker\left(H^i_{\mathrm{et}}(\mathscr{X}, \mathscr{F}) \to \textstyle\prod_{v \in S} \hat{H}^i_{\mathrm{et}}(\mathscr{X} \otimes_{\mathcal{O}_S} k_v, \mathscr{F})\right) \\ Ш^i_c(\mathscr{X}, \mathscr{S}, \mathscr{F}) &= \ker\left(H^i_c(\mathscr{X}, \mathscr{F}) \to \textstyle\prod_{v \in S} \hat{H}^i_c(\mathscr{X} \otimes_{\mathcal{O}_S} k_v, \mathscr{F})\right). \end{aligned}$$

Theorem 1 (Poitou-Tate duality)**.** *Under the assumptions above, the Shafarevich-Tate groups are finite and there are perfect pairings for $i = 0, \ldots, 2r+2$:*

$$Ш^i(\mathscr{X}, \mathscr{S}, \mathscr{F}) \times Ш_c^{2r+3-i}(\mathscr{X}, \mathscr{S}, \mathscr{F}^\vee(r+1)) \longrightarrow \mathbb{Q}/\mathbb{Z}. \tag{1}$$

Recall that a homomorphism $\varphi : A \to B$ between topological groups is *strict* if it is continuous and the isomorphism $A/\ker(\varphi) \xrightarrow{\sim} \mathrm{im}(\varphi) \subset B$ is a homeomorphism. It is called *proper* if preimages of compact sets are compact.

Theorem 2 (Poitou-Tate exact sequence)**.** *For $\mathscr{X}$, $\mathscr{S}$ and $\mathscr{F}$ as above, we have an exact $6r + 9$-term sequence of abelian topological groups and strict homomorphisms*

$$0 \longrightarrow H^0_{\mathrm{et}}(\mathscr{X},\mathscr{F}) \longrightarrow P^0(\mathscr{X},\mathscr{F}) \longrightarrow H^{2r+2}_c(\mathscr{X},\mathscr{F}^\vee(r+1))^\vee \longrightarrow$$
$$\cdots$$
$$\cdots \longrightarrow H^i_{\mathrm{et}}(\mathscr{X},\mathscr{F}) \xrightarrow{\lambda_i} P^i(\mathscr{X},\mathscr{F}) \longrightarrow H^{2r+2-i}_c(\mathscr{X},\mathscr{F}^\vee(r+1))^\vee \to \tag{2}$$
$$\cdots$$
$$\cdots \to H^{2r+2}_{\mathrm{et}}(\mathscr{X},\mathscr{F}) \to P^{2r+2}(\mathscr{X},\mathscr{F}) \longrightarrow H^0_c(\mathscr{X},\mathscr{F}^\vee(r+1))^\vee \longrightarrow 0.$$

Here

$$P^i(\mathscr{X},\mathscr{F}) := \prod_{v\in S} \hat{H}^i_{\mathrm{et}}(\mathscr{X}\otimes_{\mathcal{O}_S} k_v,\mathscr{F}) \tag{3}$$

is the restricted product with respect to the subgroups $H^i_{\mathrm{nr}}(\mathscr{X}\otimes_{\mathcal{O}_S} k_v,\mathscr{F})$. The localization map λ_i is proper and has finite kernel for all i, and for $i \geq 2r+3$,

$$\lambda_i : H^i_{\mathrm{et}}(\mathscr{X},\mathscr{F}) \xrightarrow{\sim} P^i(\mathscr{X},\mathscr{F}) = \prod_{v\in S_\infty} \hat{H}^i_{\mathrm{et}}(\mathscr{X}\otimes_{\mathcal{O}_S} k_v,\mathscr{F}) \tag{4}$$

is an isomorphism. The groups in the left column of (2) are discrete, those in the middle column locally compact, and those in the right column compact.

If $\mathscr{X}$ is a smooth variety over k (i.e., S=all places), Theorem 2 was proven by S. Saito [Sa]. His proof combines classical Poitou-Tate duality with the fact that $Rf^!(\mathbb{Z}/m\mathbb{Z}) \cong \mu_m^{\otimes d}[2d]$ for any smooth, geometrically connected morphism $f : X \to Y$ of schemes with m invertible on Y [SGA4, XVIII Th. (3.2.5)]. With mild effort, Saito's argument can be extended to the case that $\mathscr{X} \to \mathscr{S}$ is *smooth*. The essential new achievement is that the assumption on $\mathscr{X}$ can be weakened from smooth to regular. We overcome the technical difficulty with $Rf^!$ by making a detour to algebraic cycle complexes, which have good base change properties by [Ge].

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Images of GL_2-type Galois representations

Jaclyn Lang
(joint work with Andrea Conti, Anna Medvedovsky)

1. Introduction

Throughout this talk, let p denote a prime and A a complete, local, noetherian, pro-p ring. Write $\mathfrak{m}_A$ for the maximal ideal of A and $\mathbb{F} = A/\mathfrak{m}_A$. Let $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(A)$ be a continuous representation unramified outside a finite set of primes. Write $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ for ρ modulo $\mathfrak{m}_A$. We will study the following question for such representations ρ under mild conditions.

Question 1. If ρ is irreducible, not induced from a character on an index 2 subgroup, and $\mathrm{Im}\,\rho$ is infinite, what is $\mathrm{Im}\,\rho$?

Let us begin with a brief history about answers to this question in certain cases. Serre showed that if ρ comes from the p-adic Tate module of a non-CM elliptic curve $E/\mathbb{Q}$, then $\mathrm{Im}\,\rho$ is open in $\mathrm{GL}_2(\mathbb{Z}_p)$ [10]. There are Galois representations attached to classical elliptic cuspidal Hecke eigenforms as well.

Theorem 2. [9], [7] *If ρ arises from a non-CM cuspidal eigenform of weight at least 2, then there is a subring $\mathcal{O} \subseteq A$ finite over $\mathbb{Z}_p$ such that either:*

(1) *$\mathrm{Im}\,\rho \cap \mathrm{SL}_2(A)$ contains (with finite index) an open subgroup of $\mathrm{SL}_2(\mathcal{O})$.*
(2) *$\mathrm{Im}\,\rho \cap \mathrm{SL}_2(A)$ contains (with finite index) an open subgroup of the norm 1 elements in a (non-split) division algebra over the fraction field of $\mathcal{O}$.*

Note that for modular forms, $\det \rho$ need not surject onto $A^\times$. This is the reason for intersecting $\mathrm{Im}\,\rho$ with $\mathrm{SL}_2(A)$ in the above (and following) results. The ring $\mathcal{O}$ in Theorem 2 is the subring of A fixed by the conjugate self-twists of ρ (see Definition 8).

When A is a ring of Krull dimension larger than 1, openness is no longer a good measure of the size of the image of ρ. Thus we introduce, for any $0 \neq \mathfrak{a}$ ideal of A,

$$\Gamma_A(\mathfrak{a}) := \ker(\mathrm{SL}_2(A) \to \mathrm{SL}_2(A/\mathfrak{a})) = \left\{ \begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix} \in \mathrm{SL}_2(A) : a, b, c, d \in \mathfrak{a} \right\}.$$

This allows us to state the following theorem, which was proved under various assumptions by a variety of people.

Theorem 3. [6], [3], [4], [5] *If $p \neq 2$, ρ arises from a non-CM Hida family, and $\bar{\rho}$ is absolutely irreducible and "regular", then there exists a ring $\mathbb{I}$ finite over $\mathbb{Z}_p[\![T]\!]$ such that $\mathrm{Im}\,\rho \supseteq \Gamma_{\mathbb{I}}(\mathfrak{a})$ for some $0 \neq \mathfrak{a}$ ideal of $\mathbb{I}$.*

We will not define the slightly technical "regular" condition in the above theorem. Instead, see Theorem 9 below for an idea of what this assumption entails. Finally, some work has been done to generalize the above theorem to the positive finite slope case.

Theorem 4. [2] *If $p \neq 2$, ρ arises from a Coleman family, and $\bar{\rho}$ is absolutely irreducible and "regular", then a certain Lie algebra attached to $\mathrm{Im}\,\rho$ is "big".*

This theorem gives a hint about how all these big image theorems are proved. Indeed, the key tool is to attach a p-adic Lie algebra to $\operatorname{Im}\rho$ and show that this Lie algebra is big. If the theory of Lie algebras is sufficiently strong, one can translate this information back into information about the group $\operatorname{Im}\rho$.

There are some problems with classical p-adic Lie algebras. For example, $\log_p$ does not behave well if A has characteristic p. Furthermore, in characteristic zero, one often needs to invert p, thus losing information at p. Most importantly for us, classical Lie algebras are not well-behaved over large dimensional rings.

2. Pink's Lie algebras

Henceforth, assume $p \neq 2$ and A is as above. Let Γ be a closed pro-p subgroup of $\mathrm{SL}_2(A)$. Pink [8] defines a function

$$\begin{aligned}\Theta : \mathrm{SL}_2(A) &\to \mathfrak{sl}_2(A)\\ x &\mapsto x - \frac{1}{2}\operatorname{tr} x.\end{aligned}$$

Following Pink, we define $L_1(\Gamma) = L_1$ to be the closed $\mathbb{Z}_p$-submodule of $\mathfrak{sl}_2(A)$ topologically generated by $\Theta(\Gamma)$. Let $L_{n+1}(\Gamma) = L_{n+1}$ be the closed $\mathbb{Z}_p$-submodule of $\mathfrak{sl}_2(A)$ generated by $[L_1, L_n]$.

For example, if $\Gamma = \Gamma_A(\mathfrak{a})$ for some A-ideal $\mathfrak{a} \neq 0$ then one can show that

$$L_n(\Gamma) = \mathfrak{sl}_2(\mathfrak{a}^n) := \left\{ \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) : a, b, c \in \mathfrak{a} \right\}.$$

Let $H_n = \Theta^{-1}(L_n) \cap \mathrm{SL}_2(A)$ and $\Gamma^{(n+1)}$ be the closed subgroup of Γ topologically generated by commutators $(\Gamma, \Gamma^{(n)})$, where $\Gamma^{(1)} := \Gamma$.

Theorem 5. [8]

- *$L_{n+1} \subseteq L_n$ for all $n \geq 1$;*
- *Γ is a normal subgroup of H_1 and H_1/Γ is abelian;*
- *$\Gamma^{(n)} = H_n$ for all $n \geq 2$.*

These Lie algebras work well in characteristic p and can be defined over large dimensional rings, but a major drawback is that L_n is only a module over $\mathbb{Z}_p$. Furthermore, it is unclear how to generalize the theory to groups other than SL_2.

3. Bellaïche's theory and our results

Let $\rho : \Pi \to \mathrm{GL}_2(A)$ be a continuous representation where:

(1) Every open subgroup $\Pi' < \Pi$ satisfies $\dim_{\mathbb{F}_p} \operatorname{Hom}(\Pi', \mathbb{F}_p) < \infty$;
(2) $\bar{\rho}$ is absolutely irreducible and projectively non-abelian;
(3) $\det\rho$ maps isomorphically to $\det\bar{\rho}$;
(4) A is generated by $\operatorname{tr}\rho(\Pi)$ as a $W(\mathbb{F})$-algebra ($\iff$ as a $W(\mathbb{F})$-module).

Theorem 6. [1] *If ρ is not induced from a character and $\operatorname{Im}\rho$ is infinite, then there exists $\mathbb{F}_p \subseteq \mathbb{F}_q \subseteq \mathbb{F}$ and closed $W(\mathbb{F}_q)$-submodules $I, B \subseteq \mathfrak{m}_A$ such that*

(1) *$W(\mathbb{F}_q)L_1 = \left(\begin{smallmatrix} I & B \\ B & I \end{smallmatrix}\right)^0 := \left\{ \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) : a \in I, b, c \in B \right\}$;*
(2) *A is generated by $W(\mathbb{F}) + I + I^2 + B$ as a $W(\mathbb{F})$-module;*

(3) *if $\bar\rho$ is not induced from a character, then $I = B$ and $I^2 \subseteq I$.*

Corollary 7. [1] *Assume further that A is a domain. If there is some $\left(\begin{smallmatrix} \lambda_0 & 0 \\ 0 & \mu_0 \end{smallmatrix}\right) \in \operatorname{Im}\bar\rho$ with $\lambda_0, \mu_0 \in \mathbb{F}_p^\times$ and $\lambda_0 \neq \pm\mu_0$, then $\operatorname{Im}\rho \supseteq \Gamma_{\mathbb{Z}_p[\![I]\!]}(\mathfrak{a})$ for some ideal $\mathfrak{a} \neq 0$ of $\mathbb{Z}_p[\![I]\!]$.*

Definition 8. A *conjugate self-twist (CST)* of ρ is a pair (σ, η_σ) such that σ is a ring automorphism of A and $\eta_\sigma : \Pi \to W(\mathbb{F})^\times$ is a homomorphism such that for all $g \in \Pi$,

$$\sigma(\operatorname{tr}\rho(g)) = \eta_\sigma(g)\operatorname{tr}\rho(g).$$

Let Σ_ρ be the subgroup of automorphisms of A consisting of all σ such that (σ, η_σ) is a conjugate self-twist of ρ for some $\eta_\sigma : \Pi \to W(\mathbb{F})^\times$. Let $\mathbb{E} = \mathbb{F}^{\Sigma_{\bar\rho}}$, the subfield of $\mathbb{F}$ fixed by every $\sigma \in \Sigma_{\bar\rho}$.

Theorem 9 (Conti-L.-Medvedovsky)**.**

- *L_2 is a module over $W(\mathbb{E})[\![I^2]\!]$.*
- *If there is some $\left(\begin{smallmatrix} \lambda_0 & 0 \\ 0 & \mu_0 \end{smallmatrix}\right) \in \operatorname{Im}\bar\rho$ such that $\lambda_0\mu_0^{-1} \in \mathbb{E}^\times \setminus \{\pm 1\}$ (we will call such a $\bar\rho$ **regular**), then L_3 is a module over $W(\mathbb{E})[\![I]\!]$.*
- *If $\operatorname{Im}\bar\rho \supseteq \mathrm{SL}_2(\mathbb{F}_p)$ and $p \geq 7$, then L_1 is a $W(\mathbb{E})[\![I]\!]$-module.*

Theorem 10 (Conti-L.-Medvedovsky)**.** *Let $\Pi_0 = \cap_{\sigma \in \Sigma_\rho} \ker \eta_\sigma$. Assume that the projective image of $\bar\rho$ is not abelian, $\bar\rho$ is regular, and $\bar\rho|_{\Pi_0}$ is multiplicity free. If A is a domain, then $W(\mathbb{E})[\![I]\!]$ and A^{Σ_ρ} have the same field of fractions.*

Corollary 11 (Conti-L.-Medvedovsky)**.** *Under the conditions of Theorem 10, there is some ideal $0 \neq \mathfrak{a} \subset A^{\Sigma_\rho}$ such that $\operatorname{Im}\rho \supseteq \Gamma_{A^{\Sigma_\rho}}(\mathfrak{a})$.*

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Explicit formulae for Stark Units and Hilbert's 12th problem

Samit Dasgupta
(joint work with Mahesh Kakde)

Lectures: I gave a talk entitled "Explicit formulae for Stark Units and Hilbert's 12th problem." This was a report on work in progress with Mahesh Kakde (King's College London). We are attempting to prove a conjecture of Gross and Popescu on the behavior of Stark units in towers of abelian extensions of number fields. I have proven that this conjecture implies conjectures that I have previously stated on exact p-adic formulae for Gross–Stark units. It also implies my conjectural formula for the characteristic polynomial of Gross's regulator matrix that I stated in joint work with Michael Spiess. Since the units considered (along with other easily described elements) generate the maximal abelian extension of a number field, completion of this work can be viewed as a p-adic solution to Hilbert's 12th problem.

Our method of attack is to employ the techniques in our previous work on the Gross–Stark conjecture. However we develop an integral version of the technique (as opposed to p-adic) using a construction we call "group-ring valued modular forms."

Research Collaboration: One of the great benefits of the Oberwolfach conference format is the opportunity to conduct research during the lengthy breaks and to discuss mathematics during meals. I spent most of this time working with Kakde on the problem described above. I also discussed these results and related topics with Henri Darmon.

Relations between 1-periods—Hilbert's 7th problem revisited

Annette Huber
(joint work with G. Wüstholz)

We fix an embedding $\overline{\mathbb{Q}} \to \mathbb{C}$.

1. Transcendence of periods

Metatheorem 1. *All $\overline{\mathbb{Q}}$-linear relations between 1-periods are induced from the obvious ones.*

Depending on the definition of "1-period" or "obvious relation" this meta theorem has several incarnations as actual theorems, all of them true. Note that having a $\overline{\mathbb{Q}}$-linear relation between a complex number α and 1 is equivalent to α being algebraic. Hence our work is about transcendence theory. However, we do not have anything to say about products, algebraic relations or transcendence degrees.

We first give an elementary definition of the notion of a 1-period.

Definition 2. The subset $\mathcal{P}^1 \subset \mathbb{C}$ consists of numbers of the form

$$\int_\sigma \omega$$

where

- $X/\overline{\mathbb{Q}}$ is an algebraic curve, $\omega \in \Omega^1_{\overline{\mathbb{Q}}(X)}$ (so poles are allowed);
- $\sigma = \sum_{i=1}^n a_i\gamma_i$ is a singular chain on X^{an} (i.e., $a_i \in \mathbb{Z}$ and $\gamma_i : [0,1] \to X^{\mathrm{an}}$ a continuous path avoinding the poles of ω) such that the support of the boundary divisor $\partial\sigma = \sum_{i=1}^n a_i(\gamma_i(1) - \gamma_i(0))$ is contained in $X(\overline{\mathbb{Q}})$.

This set contains many famous elements, like $2\pi i$ (transcendental by work of Lindemann), $\log(2)$ (transcendental by work of Gelfond and Schneider; solving Hilbert's 7th problem) or the periods of elliptic curves over $\overline{\mathbb{Q}}$ (transcendental by work of Siegel and Schneider). The case of closed paths (so $\partial\sigma = 0$) and general X and ω was addressed by Wüstholz. However, it is by no means true that all elements of $\mathcal{P}^1$ should be transcendental. Indeed, $\overline{\mathbb{Q}} \subset \mathcal{P}^1$ as integrals $\int_0^\alpha dt$ on the affine line. We are now able to give a complete characterisation.

Theorem 3 ([6, Theorem 5.10])**.** *An element $\alpha = \int_\sigma \omega \in \mathcal{P}^1$ is algebraic if and only if there are $f \in \overline{\mathbb{Q}}(X)^*$, $\phi \in \Omega^1_{\overline{\mathbb{Q}}(X)}$ with $\int_\sigma \phi = 0$ such that $\omega = df + \phi$.*

2. A more abstract/conceptual point of view

We get a clearer picture by passing from varieties to motives. Consider a motive M over $\overline{\mathbb{Q}}$. For our purposes we can either take the view of geometric motives as introduced by Voevodsky or Nori's abelian category of motives. In either case, there is a singular realisation $H^*_{\mathrm{sing}}(M)$ and a de Rham realisation $H^*_{\mathrm{dR}}(M)$. They are linked by Grothendieck's period isomorphism

$$H^*_{\mathrm{sing}}(M) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^*_{\mathrm{dR}}(M) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}.$$

It induces a pairing between de Rham cohomology and singular *homology.*

Definition 4. Let M be a motive over $\overline{\mathbb{Q}}$. The *period set* of M is the image of the period pairing

$$\mathcal{P}(M) = \mathrm{Im}\left(H^*_{\mathrm{dR}}(M) \times H_*^{\mathrm{sing}}(M) \to \mathbb{C}\right).$$

The *period space* of M is the $\overline{\mathbb{Q}}$-vector space generated by it:

$$\mathcal{P}\langle M\rangle = \langle \mathcal{P}(M)\rangle_{\overline{\mathbb{Q}}}.$$

Proposition 5 ([6, Remark 4.12])**.**

$$\mathcal{P}^1 = \bigcup_{M \in \mathcal{C}^1} \mathcal{P}(M)$$

where $\mathcal{C}^1$ is one of the following:

(1) *the subcategory $d_{\leq 1}\mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(\mathrm{Spec}(\overline{\mathbb{Q}}), \mathbb{Q})$ of $\mathsf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ generated by motives of smooth varieties of dimension at most 1;*

(2) *the full abelian subcategory closed under subquotients* $d_1\mathcal{MM}_{\mathrm{Nori}}$ *of the abelian category* $\mathcal{MM}_{\mathrm{Nori}}$ *generated by motives of the form* $H^1(X, Y)$ *with* X *an algebraic variety over* $\overline{\mathbb{Q}}$ *and* Y *a closed subvariety;*
(3) *Deligne's category* $1\mathrm{-Mot}_{\overline{\mathbb{Q}}}$ *of iso-1-motives over* $\overline{\mathbb{Q}}$*.*

It turns out that the translation to 1-motives is very powerful. All our results on $\mathcal{P}^1$ are deduced from the following main theorem.

Theorem 6 ([6, Theorem 3.8])**.** *Let* M *be an iso-1-motive over* $\overline{\mathbb{Q}}$*,* $\sigma \in V_{\mathrm{sing}}(M)$ *and* $\omega \in V_{\mathrm{dR}}^\vee(M)$ *such that* $\omega(\sigma) = 0$*. Then there is a short exact sequence of iso-1-motives*

$$0 \to M_1 \xrightarrow{\nu} M \xrightarrow{p} M_2 \to 0$$

such that $\sigma = \nu_*(\sigma_1)$*,* $\omega = p^*\omega_2$*.*

Note that the converse is a direct consequence of functoriality of the period pairing. This is one of the obvious relations alluded to in Meta Theorem 1.

Idea of proof. Deligne attaches to M a certain algebraic group $M^\natural$ such that $V_{\mathrm{dR}}(M) = \mathrm{Lie}(M^\natural)$. By construction, $\sigma \in \mathrm{Lie}(M^\natural)^{\mathrm{an}}$ such that $\exp(\sigma) \in M^\natural(\overline{\mathbb{Q}})$ and $\omega \in \mathrm{Ann}(\sigma) \subset \mathrm{coLie}(M^\natural)$. We apply the analytic subgroup theorem of Wüstholz [7] to $M^\natural, \sigma, \omega$. □

Besides the results mentioned before, Theorem 6 also allows us to give an explicit formula for the $\overline{\mathbb{Q}}$-dimension of $\mathcal{P}\langle M\rangle$ for all $M \in 1\mathrm{-Mot}_{\overline{\mathbb{Q}}}$, see [6, Theorem 6.2]. This should be seen as a generalisation of Baker's theorem on the $\overline{\mathbb{Q}}$-dimension of a space spanned by $\log(\alpha)$ for α running through a finite set of algebraic numbers.

3. The period conjecture

We now explain how these results relate to the period conjecture. It goes back to Grothendieck [3] with details spelled out e.g. by André in his monograph [1]. An alternative view point is due to Kontsevich [5], making use of results of Nori. This section is based on joint work of the author and Müller-Stach, see [4].

The category $\mathcal{MM}_{\mathrm{Nori}}$ is Tannakian with fibre functor H^*_{sing}. By Tannaka duality, it is equivalent to the finite dimensional representations of a pro-algebraic group G_{Nori}, Nori's motivic Galois group of $\overline{\mathbb{Q}}$. Also by Tannaka-duality, there is a G_{Nori}-torsor $X = \mathrm{Spec}(\tilde{\mathcal{P}})$ of isomorphisms between the fibre functors H^*_{sing} and H^*_{dR}. The period isomorphism defines a $\mathbb{C}$-valued point of X, or equivalently, a $\overline{\mathbb{Q}}$-algebra homomorphism $\tilde{\mathcal{P}} \to \mathbb{C}$. There is an explicit description of $\tilde{\mathcal{P}}$ in terms of generators and relations, see [4, Definition 13.1.1]. This description makes it clear that $\mathcal{P}$ is the image of $\tilde{\mathcal{P}}$ under the period map. We call $\tilde{\mathcal{P}}$ the algebra of *formal periods over* $\overline{\mathbb{Q}}$.

Conjecture 7 (Kontsevich [5])**.** *The evalution map* $\tilde{\mathcal{P}} \to \mathcal{P}$ *is injective.*

We prefer to break this up into smaller problems. Let M be a motive. We put $\langle M\rangle \subset \mathcal{MM}_{\mathrm{Nori}}$ the full abelian subcategory closed under subquotients generated by M and $\langle M\rangle^{\otimes} \subset \mathcal{MM}_{\mathrm{Nori}}$ the full Tannakian subcategory closed under subquotients. Nori's construction attaches formal period spaces $\tilde{\mathcal{P}}\langle M\rangle$ and $\tilde{\mathcal{P}}\langle M\rangle^{\otimes}$ to these. In the second case this is just Tannaka theory again. The image of $\tilde{\mathcal{P}}\langle M\rangle$ under the evaluation map is $\mathcal{P}\langle M\rangle$.

Conjecture 8 (Kontsevich's period conjecture for M)**.** *The evaluation map* $\tilde{\mathcal{P}}\langle M\rangle \to \mathcal{P}\langle M\rangle$ *is injective.*

Conjecture 9 (Grothendieck's period conjecture for M [4, Conjecture 13.2.5])**.** *The scheme* $X(M) = \mathrm{Spec}(\tilde{\mathcal{P}}\langle M\rangle^{\otimes})$ *is connected and its dimension equals the transcendence degree* $\mathrm{trdeg}\overline{\mathbb{Q}}(\mathcal{P}(M))$ *of the field generated by* $\mathcal{P}(M)$*.*

Proposition 10 ([4, Proposition 13.2.6, Proposition 7.5.9])**.** *The following are equivalent:*

(1) *Conjecture 7,*
(2) *Conjecture 8 for all* $M \in \mathcal{MM}_{\mathrm{Nori}}$*,*
(3) *Conjecture 9 for all* $M \in \mathcal{MM}_{\mathrm{Nori}}$*.*

Going back to the very beginnning, we can now make Meta Theorem 1 more precise.

Theorem 11 (Kontsevich's period conjecture for 1-motives, [6, Theorem 5.3])**.** *Conjecture 8 holds for all* $M \in d_1\mathcal{MM}_{\mathrm{Nori}}$*.*

Remark 12. Another consequence of the main theorem is the fully faithfulness of the functor $1\!-\!\mathrm{Mot}_{\overline{\mathbb{Q}}} \to (\overline{\mathbb{Q}}, \mathbb{Q})\!-\!\mathrm{Vect}$ attaching to every 1-motive the triple $(V_{\mathrm{dR}}(M), V_{\mathrm{sing}}(M), \phi)$ where ϕ is the period isomorphism, see [6, Theorem 5.4]. This was independently shown by Andreatta, Barbieri-Viale and Bertapelle, see [2].

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Recent progress on the global Gan-Gross-Prasad conjecture for unitary groups

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Let E/F be a quadratic extension of number fields and $\mathbf{A}$ be the adèle ring of F. Let $W \subset V$ be hermitian spaces over E of respective dimensions n and $n+1$. We set $H := U(W)$ and $G := U(W) \times U(V)$. These are connected algebraic reductive groups defined over F and we consider H as a subgroup of G through the diagonal embedding $H \hookrightarrow G$. To such a subgroup we can associate an 'automorphic period' that is to say the linear form

$$P_H : \mathcal{A}_{cusp}(G) \to \mathbf{C}$$

given by

$$P_H(\varphi) := \int_{[H]} \varphi(h)dh$$

where $[H] := H(F)\backslash H(\mathbf{A})$, $\mathcal{A}_{cusp}(G)$ denotes the space of cuspidal automorphic forms for G (a certain space of functions on $[G] = G(F)\backslash G(\mathbf{A})$) and the integral is taken with respect to the Tamagawa measure on $[H]$. A conjecture of Gan-Gross-Prasad [4] relates this period to the non-vanishing of certain automorphic L-functions at their center of symmetry.

To be more specific, let $\pi \hookrightarrow \mathcal{A}_{cusp}(G)$ be a cuspidal automorphic representation of $G(\mathbf{A})$ and let π_E denote its quadratic base-change with respect to the extension E/F. This base-change is known to exist in full generality by recent work of Mok [10] and Kaletha-Minguez-Shin-White [9] on the endoscopic classification of automorphic representations of unitary groups; it is an automorphic representation of $G_E \simeq GL_{n,E} \times GL_{n+1,E}$ and as such decomposes as an exterior tensor product $\pi_E = \pi_{W,E} \otimes \pi_{V,E}$ of automorphic representations of $GL_{n,E}$ and $GL_{n+1,E}$ respectively. Then the L-function appearing in the Gan-Gross-Prasad conjecture is the Rankin-Selberg L-function of pair $L(s, \pi_{W,E} \times \pi_{V,E})$ constructed by Jacquet-Piatetskii-Shapiro-Shalika [7] and that we shall denote by $L(s, \pi_E)$ in what follows.

To state the Gan-Gross-Prasad conjecture precisely it is actually more convenient to consider several groups at the same time. More precisely, for W' an hermitian space of the same dimension n as W, we define similarly a pair of groups

$$H_{W'} := U(W') \hookrightarrow G_{W'} := U(W') \times U(V')$$

and an automorphic period

$$P_{H_{W'}} : \mathcal{A}_{cusp}(G_{W'}) \to \mathbf{C}$$

where V' is the unique hermitian space (up to isomorphism) containing W' and such that $V'/W' \simeq V/W$. In such a situation, for almost all places v of F there is a natural isomorphism $G_{W',v} \simeq G_v$ well-defined up to inner conjugation (here, $G_{W',v}$ and G_v denote the groups of F_v-points of $G_{W'}$ and G respectively) and we say that two cuspidal automorphic representations π, π' of $G(\mathbf{A})$ and $G_{W'}(\mathbf{A})$ respectively are *nearly equivalent* if $\pi_v \simeq \pi'_v$ for almost all places v of F.

Finally, recall that following Arthur a cuspidal automorphic representation π is said to be of *Ramanujan type* if for almost every place v the local component π_v is generic (these are precisely the cuspidal automorphic representations for which we expect a generalization of the Ramanujan conjecture i.e. π_v tempered for all v). We can now state the global Gan-Gross-Prasad conjecture for unitary groups (similar conjectures exist for all classical groups see [4] Conjecture 26.1).

Conjecture 1 (Gan-Gross-Prasad)**.** Let π be a cuspidal automorphic representation of $G(\mathbf{A})$ which is of Ramanujan type. Then, the following are equivalent:

(1) $L(\frac{1}{2}, \pi_E) \neq 0$;
(2) there exists a hermitian space W' of dimension n and a cuspidal automorphic representation π' of $G_{W'}(\mathbf{A})$ nearly equivalent to π such that $P_{H_{W'}} \mid_{\pi'} \neq 0$.

This conjecture has been further refined by Ichino-Ikeda [6] (the unitary case being actually due to R. N. Harris [5]) in the form of a precise formula relating $L(\frac{1}{2}, \pi_E)$ to the automorphic period $P_H \mid_\pi$. The statement is as follows. In the $n = 1$ case this essentially recovers a celebrated result of Waldspurger [11] on toric periods for $GL(2)$.

Conjecture 2 (Ichino-Ikeda, R.N. Harris)**.** Assume that the cuspidal automorphic representation π is everywhere tempered (i.e. π_v is tempered for all places v). Then, for every factorizable vector $\varphi = \otimes_v \varphi_v \in \pi$ and any sufficiently large finite set S of places of F (depending on φ), we have

$$\frac{|P_H(\varphi)|^2}{\langle \varphi, \varphi \rangle_{Pet}} = 2^{-\beta} \Delta^S \frac{L^S(\frac{1}{2}, \pi_E)}{L^S(1, \pi, Ad)} \prod_{v \in S} P_{H_v}(\varphi_v, \varphi_v)$$

where

- $\langle ., . \rangle_{Pet}$ denotes the Petersson inner product on $\mathcal{A}_{cusp}(G)$ (i.e. the L^2 inner product for the Tamagawa measure on $[G]$);
- the P_{H_v} are the so-called *local periods* defined by
$$P_{H_v}(\varphi_v, \varphi_v) := \frac{\int_{H_v} \langle \pi_v(h_v)\varphi_v, \varphi_v \rangle_v dh_v}{\langle \varphi_v, \varphi_v \rangle_v}, \quad 0 \neq \varphi_v \in \pi_v$$
where $\langle ., . \rangle_v$ is any invariant inner product on (the space of) π_v and the local Haar measures are supposed to factorize the global (Tamagawa) Haar measure on $H(\mathbf{A})$: $d_{Tam}h = \prod_v dh_v$;
- Δ^S is a certain product of special values of (partial) Hecke L-functions (for a precise description we refer to [5]);
- 2^β is the size of the 'centralizer of the Langlands parameter of π' (although such a Langlands parameter is still in general a purely conjectural object, we can describe what ought to be its centralizer purely in terms of the base-change π_E, see [9] Section 1.3.4).

A lot of progress have been made on these two conjectures in recent years. Starting with the seminal work of Wei Zhang [14], [15] whom, following an approach due originally to Jacquet-Rallis [8] of comparing two relative trace formulas and the proof by Z.Yun and J. Gordon [13] of the relevant fundamental lemma, has established both Conjecture 1 and Conjecture 2 under some local assumptions. More precisely, Zhang has proved Conjecture 1 under the following hypothesis (see Theorem 1.1 of [14]):

- All Archimedean places of F split in E;
- There exist two non-Archimedean places v_0, v_1 of F which split in E such that π_{v_0} is supercuspidal and π_{v_1} is tempered.

In the subsequent paper [15], Zhang was able to obtain Conjecture 2 under the following, more stringent, additional assumption (see Theorem 1.2 of *loc. cit.*):

- for every non-split place v of F, π_v is supercuspidal or π_v is unramified and the residual characteristic of v is larger than a constant $c(n)$ which depends only on $n = \dim(W)$.

As we said, in his work Zhang follows a strategy laid down by Jacquet-Rallis of comparing two relative trace formulas (RTF). These close cousins of the Arthur-Selberg trace formula allow, very roughly, to express periods of automorphic forms in terms of *relative orbital integrals* i.e. integrals of test functions on adelic double cosets (typically, in the case at hand, double cosets of $H(\mathbf{A})$ in $G(\mathbf{A})$). One of the RTF to be compared involves the Gan-Gross-Prasad period P_H whereas the other is living on the group $G' := GL_{n,E} \times GL_{n+1,E}$ (note that the base-change π_E of π is an automorphic representation of this group) and involves two periods: one with respect to the diagonally embedded subgroup $H_1 := GL_{n,E}$ is related to central L-values of the form $L(\frac{1}{2}, \pi_E)$ through Rankin-Selberg theory and the other one with respect to the subgroup $H_2 = GL_{n,F} \times GL_{n+1,F}$ (eventually twisted by a certain quadratic character) was studied by Rallis and Flicker and allows to detect representations coming by base-change from G. The comparison of these two RTF follows the, now classical, paradigm of endoscopy: there is a correspondence between orbits (or rather double cosets) on both sides and we look for test functions with matching orbital integrals. Using the Eulerian nature of orbital integrals this boils down to two local problems that are: a fundamental lemma (i.e. the characteristic functions of suitable hyperspecial compact subgroups match at almost every places) and the existence of smooth transfer (i.e. existence of sufficiently many matching test functions).

One of the main achievements of Zhang was to prove the existence of smooth transfer for non-Archimedean places. This was subsequently extended (in a weaker form) to Archimedean places by Hang Xue [12] therefore allowing to circumvent the first assumption above for applications to the Gan-Gross-Prasad conjecture (Conjecture 1). On the other hand, to get the precise formula conjectured by R. N. Harris and Ichino-Ikeda (Conjecture 2), Zhang faced an additional difficulty: once the comparison of the two RTF is effective, we naturally end up with a factorization of the (square of the) period P_H in terms of different local periods than in Conjecture 2 and which are moreover living on the local groups G'_v. Thus to get back Conjecture 2, we need to compare these local periods; or rather the natural distributions derived from them that are called *relative characters* (as they extend the usual notion of trace of a representation) when applied to matching test functions. Since the notion of matching test functions is defined using relative orbital integrals, this is in some sense asking for a spectral characterization of it. Zhang made a very precise conjecture in this direction ([15] Conjecture 4.4) and was further able to prove it for supercuspidal as well as unramified representations in sufficiently large residual characteristic (this last case is, essentially, a consequence of the fundamental lemma). This explains the last local condition above. In [1], I was able to extend Zhang method to obtain the required identity for all tempered representations at every non-Archimedean place. Finally, this result is reproved in the forthcoming paper [2] by a different method allowing to treat uniformly both the Archimedean and non-Archimedean places. Together with recent results of Chaudouard-Zydor [3] on comparison of the fine geometric expansions of Jacquet-Rallis RTFs this allows to state the following theorem:

Theorem 1 (Zhang, Xue, B.-P., Chaudouard-Zydor)**.** Both Conjecture 1 and Conjecture 2 hold under the following unique local condition: there exists a place v of F such that $\pi_{E,v}$ is supercuspidal.

Finally, a word on this last local condition: it actually originates from the use of simple versions of Jacquet-Rallis RTFs (allowing to ignore all analytical difficulties). To get rid of it, one would need the fine spectral expansions of these RTFs which seems a very hard problem of its own. See however [16] for the most up-to-date progress on this problem.

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Cohomology of p-adic Stein spaces

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(joint work with Pierre Colmez, Gabriel Dospinescu)

Let p be a prime. Let $\mathscr{O}_K$ be a complete discrete valuation ring of mixed characteristic $(0,p)$ with perfect residue field k and fraction field K. Let F be the fraction field of the ring of Witt vectors $\mathscr{O}_F = W(k)$ of k. Let $\overline{K}$ be an algebraic closure of K and let $C = \widehat{\overline{K}}$ be its p-adic completion; let $\mathscr{G}_K = \mathrm{Gal}(\overline{K}/K)$.

0.1. **The p-adic étale cohomology of Drinfeld half-space.** We reported on some results of our research project that aims at understanding the p-adic (pro-)étale cohomology of p-adic symmetric spaces [2]. The main question of interest being: does this cohomology realize the hoped for p-adic local Langlands correspondence in analogy with the ℓ-adic situation ? When we started this project we did not know what to expect and local computations were rather discouraging: geometric p-adic étale cohomology groups of affinoids and their interiors are huge and not invariant by base change to a bigger complete algebraically closed field. However there was one computation done long ago by Drinfeld that stood out. Let us recall it.

Assume that $[K:\mathbb{Q}_p] < \infty$ and let $\mathbb{H}_K = \mathbb{P}^1_K \setminus \mathbb{P}^1(K)$ be the Drinfeld half-plane, thought of as a rigid analytic space. It admits a natural action of $G := \mathrm{GL}_2(K)$.

Fact 1. (Drinfeld) If ℓ is a prime number (including $\ell = p$!), there exists a natural isomorphism of $G \times \mathscr{G}_K$-representations

$$H^1_{\text{ét}}(\mathbb{H}_C, \mathbb{Q}_\ell(1)) \simeq (\mathrm{Sp}^{\mathrm{cont}}(\mathbb{Q}_\ell))^*,$$

where $\mathrm{Sp}^{\mathrm{cont}}(\mathbb{Q}_\ell) := \mathscr{C}(\mathbb{P}^1(K), \mathbb{Q}_\ell)/\mathbb{Q}_\ell$ is the continuous Steinberg representation of G with coefficients in $\mathbb{Q}_\ell$ equipped with a trivial action of $\mathscr{G}_K$ and $(-)^*$ denotes the weak topological dual.

The proof is very simple: it uses Kummer theory and vanishing of the Picard groups (of the standard Stein covering of $\mathbb{H}_K$) [1, 1.4]. This result was encouraging because it showed that the p-adic étale cohomology was maybe not as pathological as one could fear.

Drinfeld's result was generalized, for $\ell \neq p$, to higher dimensions by Schneider-Stuhler [4]. Let $d \geq 1$ and let $\mathbb{H}^d_K$ be the Drinfeld half-space of dimension d, i.e.,

$$\mathbb{H}^d_K := \mathbb{P}^d_K \setminus \bigcup_{H \in \mathscr{H}} H,$$

where $\mathscr{H}$ denotes the set of K-rational hyperplanes. We set $G := \mathrm{GL}_{d+1}(K)$. If $1 \le r \le d$, and if ℓ is a prime number, denote by $\mathrm{Sp}_r(\mathbb{Q}_\ell)$ and $\mathrm{Sp}_r^{\mathrm{cont}}(\mathbb{Q}_\ell)$ the generalized locally constant and continuous Steinberg $\mathbb{Q}_\ell$-representations of G, respectively, equipped with a trivial action of $\mathscr{G}_K$.

Theorem 1. (Schneider-Stuhler) *Let $r \ge 0$ and let $\ell \ne p$. There are natural $G \times \mathscr{G}_K$-equivariant isomorphisms*

$$H^r_{\text{ét}}(\mathbb{H}^d_C, \mathbb{Q}_\ell(r)) \simeq \mathrm{Sp}_r^{\mathrm{cont}}(\mathbb{Q}_\ell)^*, \quad H^r_{\text{proét}}(\mathbb{H}^d_C, \mathbb{Q}_\ell(r)) \simeq \mathrm{Sp}_r(\mathbb{Q}_\ell)^*.$$

The computations of Schneider-Stuhler work for any cohomology theory that satisfies certain axioms, the most important being the homotopy property with respect to the open unit ball, which fails rather dramatically for the p-adic (pro-)étale cohomology since the p-adic étale cohomology of the unit ball is huge. Nevertheless, we prove the following result.

Theorem 2. *Let $r \ge 0$.*

(1) *There is a natural isomorphism of $G \times \mathscr{G}_K$-locally convex topological vector spaces (over $\mathbb{Q}_p$).*

$$H^r_{\text{ét}}(\mathbb{H}^d_C, \mathbb{Q}_p(r)) \simeq \mathrm{Sp}_r^{\mathrm{cont}}(\mathbb{Q}_p)^*.$$

These spaces are weak duals of Banach spaces.

(2) *There is a strictly exact sequence of $G \times \mathscr{G}_K$-Fréchet spaces*

$$0 \longrightarrow (\Omega^{r-1}(\mathbb{H}^d_K)/\ker d)\widehat{\otimes}_K C \longrightarrow H^r_{\text{proét}}(\mathbb{H}^d_C, \mathbb{Q}_p(r)) \longrightarrow \mathrm{Sp}_r(\mathbb{Q}_p)^* \longrightarrow 0.$$

(3) *The natural map $H^r_{\text{ét}}(\mathbb{H}^d_C, \mathbb{Q}_p(r)) \to H^r_{\text{proét}}(\mathbb{H}^d_C, \mathbb{Q}_p(r))$ identifies étale cohomology with the space of G-bounded vectors*[1] *in the pro-étale cohomology.*

Hence, the p-adic étale cohomology is given by the same dual of a Steinberg representation as its ℓ-adic counterpart and is invariant by scalar extension to bigger C's. However, the p-adic pro-étale cohomology is a nontrivial extension of the same dual of a Steinberg representation that describes its ℓ-adic counterpart by a huge space that depends very much on C.

Remark 3. In [1] we have generalized the above computation of Drinfeld in a different direction, namely, to the Drinfeld tower in dimension 1. We have shown that, if $K = \mathbb{Q}_p$, the p-adic local Langlands correspondence for de Rham Galois representations of dimension 2 (of Hodge-Tate weights 0 and 1 and not trianguline) can be realized inside the p-adic étale cohomology of the Drinfeld tower (see [1, Theorem 0.2] for a precise statement). The two important cohomological inputs were

[1] Recall that a subset X of a locally convex vector space over $\mathbb{Q}_p$ is called *bounded* if $p^n x_n \mapsto 0$ for all sequences $\{x_n\}, n \in \mathscr{N}$, of elements of X. In the above, x is called a *G-bounded vector* if its G-orbit is a bounded set.

(1) a p-adic comparison theorem that allows us to recover the p-adic pro-étale cohomology from the de Rham complex and the Hyodo-Kato cohomology; the latter being compared to the ℓ-adic étale cohomology computed, in turn, by non-abelian Lubin-Tate theory,
(2) the fact that the p-adic étale cohomology is equal to the space of G-bounded vectors in the p-adic pro-étale cohomology.

In contrast, here we obtain the third part of Theorem 2 only after proving the two previous parts. In fact, for a general rigid analytic variety, the natural map from p-adic étale cohomology to p-adic pro-étale cohomology is not an injection.

0.2. **A comparison theorem for p-adic pro-étale cohomology.** The proof of Theorem 2 uses the result below, which is our main theorem. It generalizes the classical p-adic comparison theorem to rigid analytic Stein spaces[2] over K with a semistable reduction. Let the field K be as at the beginning of the introduction.

Theorem 4. *Let $r \geq 0$. Let X be a semistable Stein weak formal scheme over $\mathcal{O}_K$. There exists a commutative $\mathcal{G}_K$-equivariant diagram of Fréchet spaces*[3]

$$\begin{array}{ccc} H^r_{\text{proét}}(X_C, \mathbb{Q}_p(r)) & \longrightarrow & (H^r_{\text{HK}}(X_k)\widehat{\otimes}_F \mathcal{B}^+_{\text{st}})^{N=0,\varphi=p^r} \\ \downarrow{\scriptstyle \widetilde{\beta}} & & \downarrow{\scriptstyle \iota_{\text{HK}}\otimes\theta} \\ \Omega^r(X_K)_{d=0}\widehat{\otimes}_K C & \xrightarrow{\text{can}} & H^r_{\text{dR}}(X_C) \end{array}$$

The horizontal maps are strictly surjective and their kernels are isomorphic to $(\Omega^{r-1}(X_K)/\ker d)\widehat{\otimes}_K C$. The maps $\widetilde{\beta}$ and $\iota_{\text{HK}} \otimes \theta$ are strict (and have closed images). Moreover,

$$\ker(\widetilde{\beta}) \simeq \ker(\iota_{\text{HK}} \otimes \theta) \simeq (H^r_{\text{HK}}(X_k)\widehat{\otimes}_F \mathcal{B}^+_{\text{st}})^{N=0,\varphi=p^{r-1}}.$$

Here $H^r_{\text{HK}}(X_k)$ is the overconvergent Hyodo-Kato cohomology of Grosse-Klönne [3], $\iota_{\text{HK}} : H^r_{\text{HK}}(X_k) \otimes_F K \xrightarrow{\sim} H^r_{\text{dR}}(X_C)$ is the Hyodo-Kato isomorphism, $\mathcal{B}^+_{\text{st}}$ is the semistable ring of periods defined by Fontaine, and $\theta : \mathcal{B}^+_{\text{st}} \to C$ is Fontaine's projection.

Example 2. In the case the Hyodo-Kato cohomology vanish we obtain a particularly simple formula. Take, for example, the rigid affine space $\mathbb{A}^d_K$. For $r \geq 1$, we have $H^r_{\text{dR}}(\mathbb{A}^d_K) = 0$ and, by the Hyodo-Kato isomorphism, also $H^r_{\text{HK}}(\mathbb{A}^d_K) = 0$. Hence the above theorem yields an isomorphism

$$H^r_{\text{proét}}(\mathbb{A}^d_C, \mathbb{Q}_p(r)) \xleftarrow{\sim} (\Omega^{r-1}(\mathbb{A}^d_K)/\ker d)\widehat{\otimes}_K C.$$

Remark 5. (i) We think of the above theorem as a one-way comparison theorem, i.e., the pro-étale cohomology $H^r_{\text{proét}}(X_C, \mathbb{Q}_p(r))$ is the pullback of the diagram

$$(H^r_{\text{HK}}(X_k)\widehat{\otimes}_F \mathcal{B}^+_{\text{st}})^{N=0,\varphi=p^r} \xrightarrow{\iota_{\text{HK}}\otimes\theta} H^r_{\text{dR}}(X_K)\widehat{\otimes}_K C \xleftarrow{\text{can}} \Omega^r(X_K)_{d=0}\widehat{\otimes}_K C$$

[2]Recall that a rigid analytic space Y is Stein if it has an admissible affinoid covering $Y = \cup_{i\in\mathcal{N}} U_i$ such that $U_i \Subset U_{i+1}$. The key property we need is the acyclicity of cohomology of coherent sheaves.

[3]The completed tensor product is taken with respect to a Stein covering of X_K.

built from the Hyodo-Kato cohomology and a piece of the de Rham complex.

(ii) When we started doing computations of pro-étale cohomology groups (for the affine line), we could not understand why the p-adic pro-étale cohomology seemed to be so big while the Hyodo-Kato cohomology was so small (actually 0 in that case): this was against what the proper case was teaching us. If X is proper, $\Omega^{r-1}(X_K)/\ker d = 0$ and the upper line of the above diagram becomes

$$0 \to H^r_{\text{proét}}(X_C, \mathbb{Q}_p(r)) \to (H^r_{\text{HK}}(X_k)\widehat{\otimes}_F \mathscr{B}^+_{\text{st}})^{N=0,\varphi=p^r} \\ \to (H^r_{\text{dR}}(X_K)\widehat{\otimes}\mathscr{B}^+_{\text{dR}})/\text{Fil}^r \to 0.$$

Hence the huge term on the left disappears, and an extra term on the right shows up. This seemed to indicate that there was no real hope of computing p-adic étale and pro-étale cohomologies of big spaces. It was learning about Drinfeld's result that convinced us to look further.

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