Mathematisches Forschungsinstitut Oberwolfach

Report No. 31/2018

DOI: 10.4171/OWR/2018/31

Topologie

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1 July – 7 July 2018

Abstract. The talks covered advances in algebraic K-theory and topological cyclic homology, geometric group theory, low dimensional topology relying on a mixture of combinatorial and analytic methods, classification of high-dimensional manifolds and more. Special emphasis was given to a recent breakthrough on the question of triangulability of high-dimensional manifolds.

Mathematics Subject Classification (2010): 55-xx, 57-xx, 19-xx.

Introduction by the Organisers

The workshop *Topologie* (2018) was organized by a team consisting of Mark Behrens (Notre Dame), Ruth Charney (Brandeis), Peter Teichner (Bonn) and Michael Weiss (Münster). It was unfortunate that Ruth Charney and Mark Behrens could not attend this time, but the list of invitees was managed by all four organizers, and as the meeting progressed the program for each day was decided on jointly by all four (communicating via skype and email).

The preferred calendar month for this meeting used to be September, but we moved it to July (beginning with the 2016 meeting) to make it more attractive for international participants. The list of participants at this workshop indicates that this goal was achieved. It should also be noted that many of our invitees had to decide between Oberwolfach and a topology meeting running concurrently at the Newton Institute, Cambridge. There is no indication that this lowered the standards, but it may have led to a greater-than-usual emphasis on low-dimensional topology at this meeting.

About 50 mathematicians participated in the workshop. Out of 18-19 hours total speaking time, approximately 5 hours were devoted to algebraic K- and L-theory including applications to high-dimensional manifolds, 2 hours to other homotopy theory (some of which related to manifolds), 3-4 hours to geometric group theory and related aspects of 3-manifold theory, 3 hours to overviews and applications of Seiberg-Witten theory and Heegaard-Floer homology (tools in low-dimensional topology), and 2 more hours on other aspects low dimensional topology. In addition to that, three 1-hour talks on *Triangulation and homology bordism* were delivered by Ciprian Manolescu.

We had invited Manolescu to report on his groundbreaking work ($\sim$ 2012) on the question of triangulability of high-dimensional manifolds. His first talk was accessible to all and turned on the history of the triangulation problem as well as a known reduction (going back to the late 1970s) to questions on the bordism group $\Theta^3_{\mathbb{Z}}$ of 3-dimensional homology spheres. The remaining two talks were an exposition of Heegaard-Floer homology, Seiberg-Witten theory and their uses in the investigation of $\Theta^3_{\mathbb{Z}}$. Some talks by other speakers (e.g. Stipsicz and Hom) provided additional sketches of the Heegaard-Floer and Seiberg-Witten theories for non-experts and incentives to learn more about them.

For the Wednesday morning program we selected 7 junior speakers to give talks of 20 minutes each. This was a slight deviation from the "gong shows" of previous meetings. Unresolved issues of fairness notwithstanding, these talks seemed to reach the audience very well and no doubt some of them could have been expanded into very successful one-hour talks. We mention the talks by Markus Land on K-theory, by Peter Feller on algebro-geometric aspects of knot theory and by Arunima Ray on some hitherto neglected issues in 4-dimensional topological surgery as examples.

To conclude the introduction we give a very brief chronological overview of the regular talks. More details can be found in the abstracts which form the body of this report.

Oscar Randal-Williams talked about a novel investigation of E_∞-algebras in terms of cell decompositions and related filtrations, specifically without imposing *group completeness*. Applications to algebraic K-theory were given. Akhil Mathew presented new results on the algebraic K-theory and topological cyclic homology of henselian pairs, extending an older result by Dundas and McCarthy formulated for nilpotent ideals. Cornelia Drutu talked about *median geometries* and properties of groups acting on such geometries. Andras Stipsicz gave a very accessible talk on knot concordance invariants based on knot Floer homology. Birgit Richter reported on a new "strictly commutative" model for the cochain algebra of a space; she explained this terminology and how it does not contradict the existence of the Steenrod operations (traditionally known as obstructions to commutativity). Wolfgang Lück talked about new results related to the Cannon conjecture, which concerns torsion free hyperbolic groups with boundary homeomorphic to a sphere and their realizability as fundamental groups of aspherical closed topological manifolds. Jean-Francois Lafont gave a fine overview talk on various methods to construct aspherical manifolds. Mona Merling gave a talk on the Waldhausen $S_\bullet$ construction (a key tool in the foundations of algebraic K-theory) and prospects for using variants of it to produce equivariant spectra. Francesco Lin talked on the length spectrum of hyperbolic 3-manifolds, in relation to the Hodge Laplacian. Jennifer Hom gave an overview of new results in Heegaard-Floer homology and homology sphere bordism, emphasizing the more algebraic aspects of the theory. Alessandro Sisto surprised us with a talk on statistical aspects of 3-manifolds described by a Heegaard diagram, and Matthew Hedden gave an overview on knots in relation to complex algebraic curves (not unrelated to Feller's 20 minute talk). Soren Galatius finished with a talk on the action of the absolute Galois group of $\mathbb{Q}$ on the symplectic K-theory of $\mathbb{Z}$.

The calm Oberwolfach atmosphere, good food and good weather helped to make this meeting highly successful. Our thanks go to the institute for making this possible and helping so efficiently with the organization.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows".

Workshop: Topologie

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Abstracts

On Rognes' connectivity conjecture

Oscar Randal-Williams
(joint work with Søren Galatius, Alexander Kupers)

In 1992 Rognes [2] introduced a filtration $F_\bullet\mathbb{K}(R)$ of the (free) algebraic K-theory spectrum $\mathbb{K}(R)$ of a ring R having the invariant basis number property. He identified the filtration quotients as the homotopy orbits

$$\frac{F_n\mathbb{K}(R)}{F_{n-1}\mathbb{K}(R)} \simeq \mathbb{D}(R^n)/\!\!/GL(R^n)$$

for a certain $GL(R^n)$-spectrum $\mathbb{D}(R^n)$. The kth space in this spectrum, $D^k(R^n)$, is called the k-dimensional building. When $k=1$ it is the double suspension of the Tits building of R^n, and if R is a field then it follows from the Solomon–Tits theorem that $D^1(R^n)$ is a wedge of n-spheres, so in particular is $(n-1)$-connected. Based on detailed calculations, Rognes conjectured that $\mathbb{D}(R^n)$ is $(2n-3)$-connected when R is either local or Euclidean, and he verified this for $n=2$.

If this property held, then the homotopy orbits $\mathbb{D}(R^n)/\!\!/GL(R^n)$ would also be $(2n-3)$-connected. This would mean that the spectral sequence associated to this filtration would converge rather more quickly than might be expected. While we are not able to settle Rognes' conjecture as he stated it, we are able to show that it holds for infinite fields after taking homotopy orbits: the fast convergence follows.

Theorem A. *If R is an infinite field then $\mathbb{D}(R^n)/\!\!/GL(R^n)$ is $(2n-3)$-connected.*

The majority of this talk was an outline of the proof, which is quite elementary; from now onwards suppose that R is a field. The k-dimensional buildings $D^k(R^n)$ are first compared with certain "split" k-dimensional buildings $\tilde{D}^k(R^n)$, which relate to direct-sum K-theory as the $D^k(R^n)$ relate to exact-sequence K-theory. There is a canonical $GL(R^n)$-equivariant map

$$\tilde{D}^k(R^n) \longrightarrow D^k(R^n),$$

arising from the inclusion of split exact sequences into all exact sequences. This is not an equivalence, but it induces a bijection on $GL(R^n)$-orbits of simplices in each degree, and the maps on stabiliser groups are k-fold analogues of inclusions

$$\{\text{block diagonal matrices}\} \longrightarrow \{\text{block upper-triangular matrices}\}.$$

This is of course not an isomorphism of groups, but it follows from a remarkable theorem of Nesterenko–Suslin [1] that—as long as R has "many units", which an *infinite* field does—the map induces an isomorphism on integral homology. It follows that the maps on pointed homotopy orbits

$$\tilde{D}^k(R^n)/\!\!/GL(R^n) \longrightarrow D^k(R^n)/\!\!/GL(R^n)$$

are stable homotopy equivalences, so after taking homotopy-orbits the split and non-split k-dimensional buildings may be freely interchanged.

The advantage of the $\tilde{D}^k(R^n)$ over the $D^k(R^n)$ is that the $(k+1)$-st can be produced from the k-th by a bar construction. In particular, it is easy to show that if the $\tilde{D}^k(R^n)/\!\!/GL(R^n)$ are $(2n-3+k)$-connected for all n, then the $\tilde{D}^{k+1}(R^n)/\!\!/GL(R^n)$ are $(2n-3+k+1)$-connected for all n. To see the required connectivity of $\mathbb{D}(R^n)/\!\!/GL(R^n)$ it therefore suffices to show that $D^2(R^n)$ is $(2n-1)$-connected, but this may be shown to be homeomorphic to $D^1(R^n) \wedge D^1(R^n)$, so a wedge of $2n$-spheres by the Solomon–Tits theorem.

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p-adic algebraic K-theory and topological cyclic homology

Akhil Mathew

(joint work with Dustin Clausen and Matthew Morrow)

We study the algebraic K-theory $K(R)$ of a ring R via the *cyclotomic trace*

$$K(R) \to \mathrm{TC}(R)$$

to the topological cyclic homology $\mathrm{TC}(R)$. This construction was introduced by Bökstedt-Hsiang-Madsen [1] in their results on the assembly map for the algebraic K-theory of group rings. Since then, the cyclotomic trace has been a fundamental tool for various calculations of algebraic K-theory. One reason for their power is the following result:

Theorem 1 (Dundas-Goodwillie-McCarthy [3])**.** *Let (R, I) be a pair consisting of a (not necessarily commutative) ring R and a nilpotent (two-sided) ideal I. Then the cyclotomic trace induces an equivalence on relative theories $K(R, I) \simeq \mathrm{TC}(R, I)$. Equivalently, one has the homotopy cartesian square*

$$\begin{array}{ccc} K(R) & \longrightarrow & \mathrm{TC}(R) \\ \downarrow & & \downarrow \\ K(R/I) & \longrightarrow & \mathrm{TC}(R/I) \end{array} .$$

In practice, this means that if one knows $K(R/I), \mathrm{TC}(R/I)$, then the problem of computing $K(R)$ is (more or less) reduced to that of computing $\mathrm{TC}(R)$. Calculating $\mathrm{TC}(R)$ is usually easier than $K(R)$ because the functor $\mathrm{TC}(\cdot)$ has more convenient formal properties (although it is much more difficult to define formally).

Our main result is a strengthening of the Dundas-Goodwillie-McCarthy theorem to a larger class of pairs, after profinite completion. We use the following definition from commutative algebra:

Definition 2. A pair (R, I) consisting of a commutative ring R and an ideal $I \subset R$ is called *henselian* if for every polynomial $f(x) \in R[x]$ and every $\bar{\alpha} \in R/I$ such that $f(\bar{\alpha}) = 0$ and $f'(\bar{\alpha}) \in R/I$ is a unit, then there exists $\alpha \in R$ such that $f(\alpha) = 0$ and α lifts $\bar{\alpha}$.

We have the following basic examples of henselian pairs:

(1) Any pair (R, I) such that R is I-adically complete (e.g., I nilpotent) is henselian.
(2) The subring $\mathbb{C}\{\{x\}\} \subset \mathbb{C}[[x]]$ consisting of formal power series over $\mathbb{C}$ which converge in a neighborhood of zero is henselian along its maximal ideal (x).
(3) The local rings of the Nisnevich topology of a ring R (a topology for which algebraic K-theory satisfies descent) are henselian local rings.

Next we recall the following classical result.

Theorem 3 (Gabber rigidity [4]). *Let (R, I) be a henselian pair and let p be a prime number invertible on R. Then $K(R)/p \simeq K(R/I)/p$.*

Gabber's result is preceded by work of Gillet-Thomason [8], and relying heavily on ideas introduced by Suslin [11]. Our main result generalizes both Gabber rigidity and the Dundas-Goodwillie-McCarthy theorem.

Theorem 4 (Clausen-M.-Morrow [2]). *Let (R, I) be a henselian pair and let p be a prime number. Then $K(R, I)/p \simeq \mathrm{TC}(R, I)/p$.*

Our methods are based on an imitation of several of the steps in Gabber's proof in [4], together with heavy use of the work of Geisser-Levine [7] and Geisser-Hesselholt [5] on the K-theory and topological cyclic homology of smooth $\mathbb{F}_p$-algebras (in particular calculating it explicitly for local rings). The main new ingredient is an observation about topological cyclic homology TC. The theory TC applied to a ring R is defined by constructing first the topological Hochschild homology spectrum $\mathrm{THH}(R)$, and equipping it with the structure of a cyclotomic spectrum. Then topological cyclic homology is obtained as a suitable inverse limit, which can also be identified with maps from the unit into cyclotomic spectra.

Theorem 5 ([2]). *The construction $R \mapsto \mathrm{TC}(R)/p$ commutes with filtered colimits in the ring R.*

The result is slightly surprising because of the infinitary processes involved in defining TC. Our argument for this result (and slightly more) relies on the new approach to cyclotomic spectra given by Nikolaus-Scholze [10]. In particular, they show that in the bounded-below case (of interest here), there is much redundancy in the classical definition of a cyclotomic spectrum. We show further that in forming TC, there is some additional cancellation.

Our results help clarify the relationship between K and TC for p-complete rings. For instance, we prove the following results:

Theorem 6 (Clausen-M.-Morrow [2]). (1) *For an arbitrary p-complete ring R, the map $\widehat{K}(R) \to \widehat{\mathrm{TC}}(R)$ (from p-adic K-theory to p-adic* TC*) exhibits the target as p-adic étale K-theory of R.*

(2) *Let R be a p-complete ring such that R/p has finite Krull dimension, and let $d \geq 1$ be such that for each $x \in \mathrm{Spec}(R/p)$, we have $[k(x) : k(x)^p] \leq p^d$. Then the map from $\hat{K}(R) \to \widehat{\mathrm{TC}(R)}$ is an equivalence in degrees $\geq d$.*

Note that in the l-adic case, the idea that the map from K-theory to its étale sheafification should be an equivalence in high enough degrees also holds for reasonably finite rings (and is expressed in the Quillen-Lichtenbaum conjecture). In both cases, the above result was previously known for smooth algebras over a perfect field k of characteristic $p > 0$ thanks to [7, 5] as well as in some other cases [6]. For algebras which are finite (as modules) over the ring of Witt vectors over a perfect field, part 2 is due to Hesselholt-Madsen [9]. The main new contribution is that one can reduce the general case to the case of fields of characteristic p via rigidity.

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Median geometry for lattices in semisimple Lie groups

Cornelia Druţu
(joint work with Indira Chatterji)

Median geometry in its various forms is relevant:

- for the recent positive answer to the virtual Haken conjecture;
- in connection with Kazhdan's property (T) and the Haagerup property (also called a-T-menability);
- as an asymptotic geometry of important topological objects such as the mapping class groups of surfaces, and the Teichmüller spaces of surfaces, endowed with the Weyl-Petersson metric;
- in optimization theory and algorithm design.

Lattices in semisimple Lie groups display various degrees of compatibility with the median geometry, ranging from the strongest kind of compatibility, the existence of a cocompact action on a CAT(0) cube complex (fundamental groups of 3-dimensional hyperbolic manifolds have such actions; this allowed I. Agol to prove the virtual Haken conjecture for such groups, building on work of F. Haglund and D. Wise) to utter incompatibility, for lattices in higher rank, whose every action on a median space must have bounded orbits, due to the property (T) that these lattices have.

In this talk, after overviewing known results, I have explained an interesting phenomenon that occurs for uniform lattices in higher dimensional rank one real simple groups and lattices in products of real rank one simple groups. These are known to be non-cubulable for products (due to work of Chatterji-Fernos-Iozzi) or presumed to be non-cubulable for some rank one cases (e.g. it is not known if arithmetic lattices of isometries of rank one real hyperbolic spaces of odd dimension are cubulable, and for certain examples in dimension 7 it is generally expected that they will not be cubulable) moreover lattices in products cannot act properly discontinuously cocompactly on median spaces of finite rank (result of Elia Fioravanti). On the other hand, all uniform lattices in real rank one simple groups or products of such groups have properly discontinuous cocompact actions on median spaces of infinite rank. This is joint work with I. Chatterji.

Homology cobordism and triangulations

Ciprian Manolescu

The study of triangulations on manifolds is closely related to understanding the three-dimensional homology cobordism group $\Theta^3_{\mathbb{Z}}$. In these three lectures, we summarized what is known about this group, with an emphasis on the local equivalence methods coming from Pin(2)-equivariant Seiberg-Witten Floer spectra and involutive Heegaard Floer homology.

The first lecture presented the classical theory of triangulations on manifolds, starting with the Hauptvermutung and the Triangulation Conjecture. To study triangulations of high-dimensional manifolds, one considers homology cobordism relations between the links of simplices. The theory involves replacing the links of simplices with PL manifold resolutions, inductively on dimension. In the process we encounter the 3-dimensional *homology cobordism group* $\Theta^3_{\mathbb{Z}}$, defined as follows:

$$\Theta^3_{\mathbb{Z}} = \{Y^3 \text{ oriented homology 3-spheres}\}/\sim$$

where the equivalence relation is given by $Y_0 \sim Y_1 \iff$ there exists a compact, oriented, PL (or, equivalently, smooth) manifold W^4 with $\partial W = (-Y_0) \cup Y_1$ and $H_*(W, Y_i; \mathbb{Z}) = 0$. If $Y_0 \sim Y_1$, we say that Y_0 and Y_1 are *homology cobordant.*

The easiest way to see that $\Theta^3_{\mathbb{Z}} \neq 0$ is to consider the *Rokhlin epimorphism*

$$\mu : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}/2, \quad \mu(Y) = \sigma(W)/8 \pmod 2, \tag{1}$$

where W is any compact, smooth, spin 4-manifold with boundary Y, and $\sigma(W)$ denotes the signature of W. Consider the exact sequence:

$$0 \longrightarrow \ker(\mu) \longrightarrow \Theta^3_{\mathbb{Z}} \xrightarrow{\mu} \mathbb{Z}/2 \longrightarrow 0. \tag{2}$$

In the 1970's, Galewski-Stern [GS80, GS79] and Matumoto [Mat78] showed

- A d-dimensional manifold M (for $d \geq 5$) is triangulable if and only if an obstruction in $H^5(M; \ker(\mu))$ vanishes.
- There exist non-triangulable manifolds in dimensions ≥ 5 if and only if the exact sequence (2) does not split.
- If they exist, triangulations on a manifold M of dimension ≥ 5 are classified (up to concordance) by elements in $H^4(M; \ker(\mu))$.

The above results provide an impetus for further studying the group $\Theta^3_{\mathbb{Z}}$, together with the Rokhlin homomorphism. Some important results, originally obtained via Yang-Mills theory, were that $\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}^\infty$ subgroup (cf. [FS90], [Fur90]) and a $\mathbb{Z}$ summand (cf. [Frø02]). It is still unknown whether $\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}^\infty$ summand, or any torsion.

The second lecture was about the Seiberg-Witten equations and their applications to homology cobordism. In many settings, the Seiberg-Witten equations can be used as a replacement for the Yang-Mills equations. For example, from the S^1-equivariant structure on Seiberg-Witten Floer homology Frøyshov extracted an epimorphism

$$\delta : \Theta^3_{\mathbb{Z}} \to \mathbb{Z},$$

and gave a new proof of the existence of a $\mathbb{Z}$ summand in $\Theta^3_{\mathbb{Z}}$.

When the Spin^c structure comes from a spin structure, the S^1 symmetry of the Seiberg-Witten equations (given by constant gauge transformations) can be expanded to a symmetry by the group Pin(2), where

$$\text{Pin}(2) = S^1 \cup jS^1 \subset \mathbb{C} \oplus j\mathbb{C} = \mathbb{H}.$$

This allows us to define a Pin(2)-equivariant Seiberg-Witten Floer homology. By imitating the construction of the Frøyshov invariant δ in this setting, we obtain three new maps

$$\alpha, \beta, \gamma : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}.$$

These are not homomorphisms, but on the other hand they are related to the Rokhlin homomorphism:

$$\alpha \equiv \beta \equiv \gamma \equiv \mu \pmod 2.$$

Under orientation reversal, the three invariants behave as follows:

$$\alpha(-Y) = -\gamma(Y), \ \beta(-Y) = -\beta(Y).$$

The properties of β suffice to prove that there are no 2-torsion elements $[Y] \in \Theta^3_{\mathbb{Z}}$ with $\mu(Y) = 1$; cf. [Man16]. Hence, the short exact sequence (2) does not split and, as a consequence of [GS80, Mat78], non-triangulable manifolds exist in every dimension ≥ 5.

Even more information can be extracted from the Seiberg-Witten equations by considering the Floer stable homotopy type from [Man03]. Stoffregen [Sto15b] constructed a *local equivalence group* $\mathcal{LE}$ out of certain equivalence classes of Pin(2)-equivariant spectra, and showed that Floer theory produces a homomorphism

$$\Theta^3_{\mathbb{Z}} \to \mathcal{LE}$$

through which the maps $\alpha, \beta, \gamma, \delta$ all factor. One application of $\mathcal{LE}$ was a new proof (due to Stoffregen) that $\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}^\infty$ subgroup.

The structure of $\mathcal{LE}$ is an open problem, but it is hoped that it could be used to produce further interesting homomorphisms from $\Theta^3_{\mathbb{Z}}$ to $\mathbb{Z}$.

The third lecture dealt with *involutive Heegaard Floer homology.* This is an invariant developed by Hendricks and the author in [HM17]. It is an analogue of $\mathbb{Z}/4$-equivariant Seiberg-Witten Floer homology, for the subgroup $\mathbb{Z}/4 = \langle j \rangle \subset$ Pin(2). Its construction is based on the Heegaard Floer theory previously built by Ozsváth and Szabó. Heegaard–Floer theory is a more computable replacement for Seiberg-Witten theory, based on counting pseudo-holomorphic curves in symmetric products.

Involutive Heegaard–Floer homology has been computed for various classes of 3-manifolds, such as Seifert fibrations and connected sums of these. This gave new proofs of some of the results about $\Theta^3_{\mathbb{Z}}$ mentioned above. Further, there is an analogue of the local equivalence group, denoted $\mathfrak{I}$, which has a simple algebraic definition; cf. [HMZ16]. There is a homomorphism $\Theta^3_{\mathbb{Z}} \to \mathfrak{I}$, which we hope to be of use in better understanding $\Theta^3_{\mathbb{Z}}$.

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Concordance invariants from knot Floer homology

András I. Stipsicz

Knot Floer homology (introduced by Ozsváth and Szabó [7] and independently by Rasmussen [8]) associates to each knot $K \subset S^3$ a finitely generated chain complex $C_K = (CFK^\infty(K), \partial)$ over the ring of Laurent polynomials $\mathbb{F}[U, U^{-1}]$ (with coefficients taken from the field $\mathbb{F}$ of two elements) admitting the following extra structures and porperties:

- C_K admits a grading M (usually called the *Maslov* grading), and the boundary operator drops M by one;
- C_K admits a filtration A (originating from a grading, the Alexander grading) respected by the boundary operator ∂;
- an algebraic filtration j (coming from a grading, which simply measures the exponent of U), also respected by ∂;
- the action of U drops the Maslov grading M by 2 and the two filtrations A and j by 1;
- the homology of C_K satisfies
$$H_*(C_K, \partial) = \mathbb{F}[U, U^{-1}].$$

The chain complex is defined through a number of additional choices: indeed, we present the knot K with a Heegaard diagram, and then apply a version of Lagrangian Floer homology in a suitable symmetric power of the Heegaard surface. According to the main result of [7], the graded, bifiltered chain homotopy type of C_K is an invariant of K.

Indeed, the above construction extends to triples (Y, K, s) where Y is a closed, oriented 3-manifold with $H_*(Y; \mathbb{Q}) \cong H_*(S^3; \mathbb{Q})$ (that is, Y is a rational homology sphere, a $\mathbb{Q}HS^3$), s is a spin^c structure on Y and $K \subset Y$ is a knot with $[K] = 0 \in H_1(Y;\mathbb{Z})$. For simplicity, $C_{(Y,K,s)}$ will be denoted by C_K.

We say that (Y_0, K_0, s_0) are *concordant* if there is a rational homology cobordism W between Y_0 and Y_1 and a spin^c structure t on W extending s_0 and s_1 and a smoothly embedded annulus $A \subset W$ between K_0 and K_1.

By an important result of J. Hom [3], if K_0 and K_1 are concordant (in the above sense) then there are graded, bifiltered chain complexes A_0 and A_1 so that $C_{K_0} \oplus A_0$ and $C_{K_1} \oplus A_1$ are (graded, bifiltered) chain homotopy equivalent and $H_*(A_0) = H_*(A_1) = 0$. (In this case, C_{K_0} and C_{K_1} are *stably equivalent.*)

Concordance invariants of knots (and of triples (Y, K, s)) from C_K can be given as follows. Notice first that C_K can be pictorially presented on the plane $\mathbb{R}^2$ by taking an $\mathbb{F}$ basis and putting a dot for each generator x of C_K to the position $(j(x), A(x)) \in \mathbb{R}^2$. The boundary operator ∂ can be symbolized by arrows, pointing from x to the components of ∂x.

Suppose that $D \subset \mathbb{R}^2$ is a closed, non-empty subset on the plane, which is not the entire plane and is of *SW type*, meaing that if $(a, b) \in D$ and $a' \leq a, b' \leq b$ then $(a', b') \in D$. To normalize matters, suppose that $(0, 0) \in \partial D$. Finally let

$$D^r = \{(a, b) \in \mathbb{R}^2 \mid (a - r, b - r) \in D\}.$$

The subcomplex $C_K(D^r)$ (which is not a sub-$\mathbb{F}[U, U^{-1}]$-module, only a sub-$\mathbb{F}[U]$-module of C_K) is generated by those $\mathbb{F}$-generators of C_K for which $(j(x), A(x)) \in D^r$. The inclusion of $C_K(D^r)$ into C_K is denoted by ι_r.

Then the following quantity is a concordance invariant of (Y, K, s):

$$\Upsilon^D_K = -2 \inf\{r \in \mathbb{R} \mid (\iota_r)_* \colon H_{d(Y,s)}(C_K(D^r)) \to H_{d(Y,s)}(C_K) \text{ is onto}\}$$

Applying this construction for the half-planes $H^t = \{y \leq \frac{t}{t-2}x\}$ for $t \in [0, 2]$ (so that H^t is of SW-type) we get a function $\Upsilon_K(t) = \Upsilon_K^{H^t}$.

The resulting function is continuous and piecewise linear, and can be effectively used to show that certain families of knots are linearly equivalent in the smooth concordance group $\mathcal{C}$. (This version of the definition was given by Livingston in [5].) Most of these results are summarized in [6], see also [1, 3, 5]; for further variants and similar invariants see [4].

As a sample of results one can show:

- There are iterated torus knots which are linearly independent of the subgroup spanned by algebraic knots in the smooth concordance group $\mathcal{C}$ (Wang [10, 9]).
- The pretzel knots $P(-2, 3, q)$ for $q \geq 7$ and odd are not concordant to positive linear combinations of algebraic knots [1].
- The subgroup of topological slice knots of $\mathcal{C}$ contains a $\mathbb{Z}^\infty$ direct summand [2, 6].

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A strictly commutative model for the cochain algebra of a space

Birgit Richter
(joint work with Steffen Sagave)

For a simplicial set X and an arbitrary commutative ring k, the cochain algebra on X is given by the cochain complex $C^*(K;k)$ whose cochain module in degree n is $C^n(X;k) = \mathrm{Sets}(X_n, k)$. As the coboundary map one takes the alternating sum of the coface maps. The cup-product of two cochains is induced by restricting along front and back inclusions. In particular, the cup-product is not graded-commutative on cochain level, but it induces a graded commutative k-algebra structure on homology. The cup-i-products witness the non-commutativity on cochain level, for instance

$$\delta(f \cup_1 g) = f \cup g - (-1)^{|f||g|} g \cup f,$$

so we get a homotopy that compares $f \cup g$ and $(-1)^{|f||g|} g \cup f$. The full structure is rather involved: $C^*(X;k)$ is an E_∞-algebra, i. e., its multiplication is commutative up to all higher homotopies. Mike Mandell showed that for nilpotent spaces of finite type the E_∞-algebra of the integral cochains determines the homotopy type [2].

Over the rationals the situation is drastically different. The Sullivan algebra of polynomial differential forms, $A^*_{PL}(X)$, is a differential graded commutative model of $C^*(X;\mathbb{Q})$ and rational spaces can be classified using rational commutative differential graded algebras.

Our project replaces the E_∞-algebra of cochains over an arbitrary commutative ring k by a suitable commutative monoid. We know that there exists such a model because in joint work with Brooke Shipley [4] we proved that there is a chain of

Quillen equivalence between E_∞-algebras over k and commutative monoids in the category of I-chain complexes.

Here, I is the category of finite sets and injections with objects $\underline{n} = \{1, \dots, n\}$ for $n \geq 0$ with $\underline{0} = \varnothing$. Morphisms are injective functions. Note that $\underline{0}$ is an initial object in I and that $I(\underline{n}, \underline{n})$ is the symmetric group Σ_n. In addition, I is a permutative category via $\underline{n} \oplus \underline{m} = \underline{n+m}$. An I-chain complex is a functor from the category I to the category Ch of unbounded chain complexes over k and the corresponding category Ch^I has natural transformation as morphisms. An I-chain complex X can be viewed as a coaugmented cosimplicial chain complex with additional symmetries.

For every object $\underline{m}$ of I there is an evaluation functor that sends an I-chain complex X to the chain complex $X(\underline{m})$. This functor has a left adjoint $F^I_m \colon \mathrm{Ch} \to \mathrm{Ch}^I$ with

$$F^I_m(C_*)(\underline{n}) = \bigoplus_{I(\underline{m},\underline{n})} C_*$$

for $C_* \in \mathrm{Ch}$.

The category Ch^I is symmetric monoidal via the Day convolution product, so for X, Y in Ch^I we obtain a product $X \boxtimes Y$ in Ch^I. The unit for this product is $U^I := F^I_0(S^0)$ where S^0 is the chain complex whose only non-trivial chain module is k in degree zero. We denote by $C(\mathrm{Ch}^I)$ the category of commutative monoids in Ch^I and call its object commutative I-chain complexes.

For I-chain complexes there is a Bousfield-Kan type model of the homotopy colimit: if X is an I-chain complex, then there is a chain complex, $\mathrm{hocolim}_I X$, that is the total complex associated to a simplicial chain complex built out of the nerve of I and the values of X. We show that for every X in $C(\mathrm{Ch}^I)$ the chain complex $\mathrm{hocolim}_I X$ is an E_∞-algebra over k. The proof uses an action of the Barratt-Eccles operad on the nerve of the category I – a fact that was established by Peter May in the 80's.

We construct an I-version of the polynomial forms, $A^I(X)$, for every simplicial set X as an object in $C(\mathrm{Ch}^I)$, by defining

$$A^I_\bullet = B(U^I_0, C(F^I_1(D^0)), U^I_1)$$

as a two-sided bar construction. Here D^0 is the disc complex concentrated in degrees 0 and -1 with value k and with the identity map as the only non-trivial differential and $C(F^I_1(D^0))$ denotes the free commutative I-chain complex generated by $F^I_1(D^0)$. It acts on $U^I_0 = U^I$ via the augmentation to zero and on $U^I_1 = U^I$ via the augmentation to 1. Placing D^0 in I-level 1 turns $A^I(\underline{n})_{\bullet,q}$ into a contractible simplicial k-module for all $n > 1$ and all chain degrees q [3]. Note that $A^I_\bullet$ is a simplicial object in commutative I-chain complexes.

We then define

$$A^I(X) := \mathrm{sSets}(X, A^I_\bullet).$$

For every commutative ring k and for every simplicial set X this is a commutative I-chain complex.

We show that $\mathrm{hocolim}_I A^I(X)$ is weakly equivalent to Sullivan's $A_{PL}(X)$ if k is a field of characteristic zero and we prove that for every commutative ring k $X \mapsto \mathrm{hocolim}_I A^I(X)$ is a cochain theory in the sense of Mike Mandell [1]. As a corollary we obtain that $\mathrm{hocolim}_I A^I(X)$ is weakly equivalent to $C^*(X; k)$ as an E_∞-algebra [3].

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The Stable Cannon Conjecture

Wolfgang Lück
(joint work with Steve Ferry and Shmuel Weinberger)

The following conjecture is taken from [2, Conjecture 5.1].

Cannon Conjecture.
Let G be a hyperbolic group. Suppose that its boundary is homeomorphic to S^2.
Then G acts properly cocompactly and isometrically on the 3-dimensional hyperbolic space.

If G is torsionfree, then the Cannon Conjecture reduces to

Cannon Conjecture in the torsionfree case.
Let G be a torsionfree hyperbolic group. Suppose that its boundary is homeomorphic to S^2.
Then G is the fundamental group of a hyperbolic closed 3-manifold.

We explain and present the proof of the following result taken from [3].

Theorem: The stable version of the Cannon Conjecture is true.
Let G be a hyperbolic 3-dimensional Poincaré duality group. Let N be any smooth, PL or topological manifold respectively which is closed and whose dimension is ≥ 2. Suppose that $\pi_1(N)$ is a Farrell-Jones group, i.e., satisfies both the K-theoretic and the L-theoretic Farrell-Jones Conjecture for coefficients in additive equivariant categories.

Then there is a closed smooth, PL or topological manifold M and a normal map of degree one

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^a} & \xrightarrow{\underline{f}} & \xi \times TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & BG \times N \end{array}$$

satisfying

(1) The map f is a simple homotopy equivalence;
(2) Let $\widehat{M} \to M$ be the G-covering associated to the composite of the isomorphism $\pi_1(f)\colon \pi_1(M) \xrightarrow{\cong} G \times \pi_1(N)$ with the projection $G \times \pi_1(N) \to G$. Suppose additionally that N is aspherical and $\dim(N) \geq 3$.

Then $\widehat{M}$ is homeomorphic to $\mathbb{R}^3 \times N$. Moreover, there is a compact topological manifold $\overline{\widehat{M}}$ whose interior is homeomorphic to $\widehat{M}$ and for which there is a homeomorphism of pairs $(\overline{\widehat{M}}, \partial\overline{\widehat{M}}) \to (D^3 \times N, S^2 \times N)$.

For information about the class of Farrell-Jones groups we refer for instance to [1]. It contains all hyperbolic groups, finite-dimensional CAT(0)-groups, lattices in locally compact second countable Hausdorff groups, arithmetic groups and fundamental groups of (not necessarily compact) 3-manifolds (possibly with boundary).

Its proof is based on the following input:

- Existence of normal maps with a given finite 3-dimensional Poincaré complex X as target.
- The definition of the total surgery obstruction for closed aspherical Poincaré complexes using Ranicki's algebraic surgery theory.
- Surgery for closed ANR-homology manifolds.
- Technique of pulling back the boundary.
- Quinn's obstruction and its relation to the total surgery obstruction.

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On K-theory of pullbacks

Markus Land
(joint work with Georg Tamme)

The aim of this talk was to report on a new excision result in algebraic K-theory which contains various previously known results as simple special cases. The setup is as follows. Assume that

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array} \tag{1}$$

is a pullback diagram in $\mathbb{E}_1$-ring spectra. A basic question in algebraic K-theory then asks whether the induced square obtained by applying algebraic K-theory is again a pullback, and if not, how well we understand the failure of it being a pullback. It is precisely this question that we shall address.

To state our main theorem, we recall that there is a functor

$$\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}) \longrightarrow \mathrm{Cat}_\infty^{\mathrm{ex}}$$

from the category of $\mathbb{E}_1$-ring spectra to the category of small stable ∞-categories given by sending a ring to its category of perfect modules. Furthermore, non-connective algebraic K-theory extends to a functor $K\colon \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathrm{Sp}$ and as such it is a particular instance of what is called a *localizing invariant*, i.e. it sends Verdier quotient sequences to fibre sequences of spectra and is invariant under idempotent completions. Our main result concerns general such localizing invariants, further examples of which include topological cyclic homology TC and the various forms of (topological) Hochschild homology taking the $\mathbb{T}$-action on (T)HH into account.

Theorem A. *Assume that the diagram (1) is a pullback diagram. Then there exists a natural $\mathbb{E}_1$-ring structure on the spectrum $A' \otimes_A B$ and we will denote this $\mathbb{E}_1$-ring by $A' \boxtimes_A B$. It is such that the canonical maps from A' and B, and the canonical map to B' all canonically refine to $\mathbb{E}_1$-ring maps. If $E\colon \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathrm{Sp}$ is any localizing invariant, then the induced diagram*

$$\begin{array}{ccc} E(A) & \longrightarrow & E(B) \\ \downarrow & & \downarrow \\ E(A') & \longrightarrow & E(A' \boxtimes_A B) \end{array}$$

is a pullback diagram.

Particular instances of such pullback diagrams are *Milnor squares*. These are those squares for which all rings are discrete and the map $A' \to B'$ is surjective. We obtain the following theorem.

Theorem B. *Suppose that $\mathrm{Tor}_i^A(A', B) = 0$ for $i = 1, \dots, n-1$. Then there is a long exact Mayer-Vietoris sequence in algebraic K-theory of the form*

$$K_n(A) \longrightarrow K_n(A') \oplus K_n(B) \longrightarrow K_n(B') \longrightarrow K_{n-1}(A) \longrightarrow \dots$$

Proof. The vanishing of the Tor groups together with the surjectivity assumption imply that the map $A' \boxtimes_A B \to B'$ is n-connective. Applying algebraic K-theory raises the connectivity of maps by one, so the claim follows from the long exact sequence in K-groups obtained from Theorem A. □

Given a Milnor square, let us denote the kernel of the map $A' \to B'$ by I and its unitalization by I^+, so that $\mathbb{Z}$ is a bimodule over I^+. It is a direct calculation that if $\mathrm{Tor}_i^{I^+}(\mathbb{Z},\mathbb{Z}) = 0$ for $i = 1, \ldots, n-1$, then also $\mathrm{Tor}_i^A(A', B) = 0$ for $i = 1, \ldots, n-1$. Combining this observation with Theorem B yields a result of Suslin, see [11, Theorem A].

To state the next result, we will need the notion of *truncating* invariants. Those are localizing invariants $E\colon \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathrm{Sp}$ which satisfy the following additional property: For every connective $\mathbb{E}_1$-ring A, the canonical map $E(A) \to E(\pi_0(A))$ is an equivalence.

Theorem C. *Assume that diagram (1) is a pullback diagram, that all rings are connective, and that the map $\pi_0(A') \to \pi_0(B')$ is surjective. Let E be any truncating invariant. Then there is a pullback diagram*

$$\begin{array}{ccc} E(A) & \longrightarrow & E(B) \\ \downarrow & & \downarrow \\ E(A') & \longrightarrow & E(B'). \end{array}$$

Thus, truncating invariants satisfy excision.

Proof. The conditions ensure that the map $A' \boxtimes_A B \to B'$ is an isomorphism on π_0 and that both spectra are connective. The claim then follows directly from Theorem A. □

Corollary. *The following invariants satisfy excision:*

(1) *the fibre of the cyclotomic trace $K \to \mathrm{TC}$, denoted by K^{inv}; this recovers and extends work of Geisser–Hesselholt and Dundas–Kittang, see* [6, 7, 4, 5].
(2) *the fibre of the rational Goodwillie–Jones Chern character $K_{\mathbb{Q}} \to \mathrm{HN}(-/\mathbb{Q})$, denoted by $K_{\mathbb{Q}}^{\mathrm{inf}}$; this recovers work of Cortinas, see* [1]
(3) *periodic cyclic homology over $\mathbb{Q}$, denoted by* $\mathrm{HP}(-/\mathbb{Q})$*; this recovers work of Cuntz–Quillen, see* [2].

Proof. All of the invariants are known to be localizing. By Theorem C it thus suffices to argue that they are in addition truncating. For K^{inv} this is a special case of [3, Theorem 7.0.0.2], for $K_{\mathbb{Q}}^{\mathrm{inf}}$ see [9, Main theorem], and for $\mathrm{HP}(-/\mathbb{Q})$ see [8, Theorem II.5.1]. □

In fact, from Theorem A we can deduce more properties of truncating invariants:

Theorem D. *Let E be a truncating invariant. Then*

(1) *E is nil invariant, i.e. for every unital ring A and every two-sided nilpotent ideal $I \trianglelefteq A$, the canonical map $E(A) \to E(A/I)$ is an equivalence.*
(2) *E satisfies cdh-descent.*

The proof of cdh-descent goes along the reduction steps of [10, Section 5] but becomes easier because of the following reason: Given any derived scheme $\mathcal{X}$ with underlying ordinary scheme X, then the canonical map $E(\mathcal{X}) \to E(X)$ is an equivalence if E is truncating.

From Theorem A we can furthermore deduce the following strengthening of a result of Geisser–Hesselholt, see [7, Theorem 3.1], about pro-excision for algebraic K-theory.

Theorem E. *Let $f\colon A \to B$ a map of simplicial rings and assume that f sends a 2-sided ideal $I \trianglelefteq A$ to a 2-sided ideal $J \trianglelefteq B$. Assume that*

(1) *f induces a weak equivalence of pro-spectra $\{I^n\} \xrightarrow{\simeq} \{J^n\}$, i.e. a pro-isomorphism on all homotopy groups, and*
(2) *f induces a weak equivalence of pro-spectra $\{A/I^n \otimes_A B\} \xrightarrow{\simeq} \{B/J^n\}$.*

Then the diagram of pro-spectra

$$\begin{array}{ccc} K(A) & \longrightarrow & K(B) \\ \downarrow & & \downarrow \\ \{K(A/I^n)\} & \longrightarrow & \{K(B/J^n)\} \end{array}$$

is a weak pullback diagram, i.e. it induces a long exact Mayer-Vietoris sequence of pro-homotopy groups.

We remark that in the case where both A and B are ordinary commutative and noetherian, condition (2) of the theorem is always fulfilled.

We end with an example that shows that the ring structure on $A' \boxtimes_A B$ is in general non-commutative, even if all rings in the diagram (1) are commutative.

Theorem F. *Consider the diagram*

$$\begin{array}{ccc} k & \longrightarrow & k[x] \\ \downarrow & & \downarrow \\ k[x^{-1}] & \longrightarrow & k[x^{\pm 1}] \end{array}$$

Then the ring $k[x] \boxtimes_k k[x^{-1}]$ provided by Theorem A is isomorphic to

$$k\langle x, y\rangle/(yx - 1),$$

where $k\langle x, y\rangle$ denotes the free k-algebra on two (non-commuting) variables.

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Four-manifolds up to connected sum with complex projective planes

Daniel Kasprowski
(joint work with Mark Powell and Peter Teichner)

We study the diffeomorphism classification of orientable, closed, connected, smooth 4-manifolds up to connected sum with complex projective planes. More precisely, two such manifolds M_1, M_2 are $\mathbb{C}P^2$-stably diffeomorphic if there exist k_1, k_2, n_1, n_2 such that $M_1 \# k_1 \mathbb{C}P^2 \# n_1 \overline{\mathbb{C}P}^2$ and $M_2 \# k_2 \mathbb{C}P^2 \# n_2 \overline{\mathbb{C}P}^2$ are diffeomorphic. Note that we do not fix orientations on M_1 and M_2 and hence do not assume the diffeomorphism to respect any orientations.

Stabilizing a manifold with $\mathbb{C}P^2$ does not change the fundamental group. Hence two manifold can only be $\mathbb{C}P^2$-stably diffeomorphic if they have isomorphic fundamental groups. We restrict our study to manifolds with fundamental group isomorphic to a given group π. We will denote the classifying map by $c\colon M \to B\pi$.

Theorem 1 (Kreck [Kre99])**.** Two orientable, closed, connected, smooth 4-manifolds M_1 and M_2 are $\mathbb{C}P^2$-stably diffeomorphic if and only if they have the same fundamental group and the images of the fundamental classes $(c_1)_*[M_1]$ and $(c_2)_*[M_2]$ coincide in $H_4(\pi;\mathbb{Z})/\pm\operatorname{Aut}(\pi)$.

Here the quotient by $\operatorname{Aut}(\pi)$ takes care of the different choices of identifications c_* of the fundamental groups with π, and the sign $\pm$ removes dependency on the choice of fundamental class.

The 2-*type* of a connected manifold M consists of its fundamental group $\pi_1(M)$ together with the second homotopy group $\pi_2(M)$ as a $\mathbb{Z}[\pi]$-module and the k-invariant $k(M) \in H^3(\pi_1(M);\pi_2(M))$ that classifies the fibration

$$K(\pi_2(M),2) \to P_2(M) \to K(\pi_1(M),1)$$

corresponding to the second stage $P_2(M)$ of the Postnikov tower for M.

Our main theorem says that this 2-type $(\pi_1 \pi_2, k)$, considered up to stable isomorphism as discussed below, classifies 4-manifolds up to $\mathbb{C}P^2$-stable diffeomorphism in many cases.

Theorem 2 ([KPT, Theorem A])**.** Let π be a group that is

(1) torsion-free; or
(2) infinite with one end; or
(3) finite with $H_4(\pi;\mathbb{Z})$ annihilated by 4 or 6.

Then two closed, connected, smooth 4-manifolds with fundamental group π are $\mathbb{C}P^2$-stably diffeomorphic if and only if their 2-types (π, π_2, k) are stably isomorphic.

Remark 3. The statement of Theorem 2 also holds for non-orientable manifolds if the orientation character is added to the 2-type. It also holds in the topological category if either the Kirby-Siebenmann invariant is added or if one also allows connected sum with the Chern manifold $*\mathbb{C}P^2$ which has non-trivial Kirby-Siebenmann invariant.

A connected sum with $\mathbb{C}P^2$ changes the second homotopy group by adding a free summand $\pi_2(M\#\mathbb{C}P^2) \cong \pi_2(M) \oplus \mathbb{Z}[\pi]$. The k-invariant $k(M) \in H^3(\pi;\pi_2(M))$ maps via $(k(M), 0)$ to $k(M\#\mathbb{C}P^2)$ under the composition

$$H^3(\pi;\pi_2(M)) \to H^3(\pi;\pi_2(M)) \oplus H^3(\pi;\mathbb{Z}[\pi]) \xrightarrow{\cong} H^3(\pi;\pi_2(M\#\mathbb{C}P^2)).$$

This leads to the notion of *stable isomorphism* of 2-types: a pair (φ_1, φ_2) consisting of an isomorphism $\varphi_1\colon \pi_1(M_1) \to \pi_1(M_2)$, together with an isomorphism, for some $r, s \in \mathbb{N}_0$,

$$\varphi_2\colon \pi_2(M_1) \oplus Z[\pi]^r \xrightarrow{\cong} \pi_2(M_2) \oplus \mathbb{Z}[\pi]^s \text{ satisfying } \varphi_2(g \cdot x) = \varphi_1(g) \cdot \varphi_2(x)$$

for all $g \in \pi_1(M_1)$ and for all $x \in \pi_2(M_1) \oplus \mathbb{Z}[\pi]^r$. We also require that (φ_1, φ_2) preserves k-invariants in the sense that

$$\begin{array}{rcl} (\varphi_1^{-1}, \varphi_2)\colon H^3(\pi_1(M_1);\pi_2(M_1) \oplus \mathbb{Z}[\pi]^r) & \to & H^3(\pi_1(M_2);\pi_2(M_2) \oplus \mathbb{Z}[\pi]^s) \\ (k(M_1), 0) & \mapsto & (k(M_2), 0). \end{array}$$

A consequence of Theorem 2 is that, under the above assumptions, the stable 2-type of a 4-manifold determines the isometry class of its equivariant intersection form up to stabilisation by standard forms (± 1) on $\mathbb{Z}[\pi]$. In the simply connected case, this follows from the classification of odd indefinite forms by their rank and signature, since for two given simply connected 4-manifolds, the rank and signatures of their intersection forms can be equalised by $\mathbb{C}P^2$-stabilisation. For general fundamental groups, the underlying module does not algebraically determine the intersection form up to stabilisation, but Theorem 2 says that equivariant intersection forms of 4-manifolds with the appropriate fundamental group are controlled in this way.

The following example demonstrates that the hypotheses of Theorem 2 are necessary. We consider a class of infinite groups with two ends, namely $\pi =$

$\mathbb{Z}\times\mathbb{Z}/p$. In this case the 2-type does not determine the $\mathbb{C}P^2$-stable diffeomorphism classification, as the following example shows.

Example 4. Let L_{p_1,q_1} and L_{p_2,q_2} be two 3-dimensional lens spaces, which are closed, oriented 3-manifolds with cyclic fundamental group $\mathbb{Z}/p_i$ and universal covering S^3. Assume that $p_i \geq 2$ and $1 \leq q_i < p_i$. The 4-manifolds $M_i := S^1 \times L_{p_i,q_i}$, $i = 1, 2$ have $\pi_2(M_i) = \{0\}$. Whence their 2-types are stably isomorphic if and only if $\pi_1(L_{p_1,q_1}) \cong \pi_1(L_{p_2,q_2})$, that is if and only if $p_1 = p_2$. However, the 4-manifolds M_1 and M_2 are $\mathbb{C}P^2$-stably diffeomorphic if and only if L_{p_1,q_1} and L_{p_2,q_2} are homotopy equivalent.

It is a classical result that there are homotopically inequivalent lens spaces with the same fundamental group. In fact it was shown by J.H.C. Whitehead that L_{p,q_1} and L_{p,q_2} are homotopy equivalent if and only if their $\mathbb{Q}/\mathbb{Z}$-valued linking forms are isometric.

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Uniqueness of embeddings of the affine line into affine spaces

Peter Feller

(joint work with Immanuel van Santen)

Intrigue. *A topological perspective on the embedding problem in affine geometry.*

In [6], Kraft described challenging problems on affine space such as the following.

Embedding Problem. *Fix integers $n > d > 0$. Given embeddings f and g of $\mathbb{C}^d$ into $\mathbb{C}^n$, does there exist an automorphism $\Psi\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $\Psi \circ f = g$?*

Here an *automorphism* is a polynomial self-map that has a polynomial inverse and an *embedding* is a polynomial map that has a polynomial left-inverse. In full generality, the embedding problem is open. The only general result concerns the case of large codimension: If $n \geq 2d + 2$, then for all pairs of embeddings $f, g\colon \mathbb{C}^d \to \mathbb{C}^n$, there exists an automorphism $\Psi\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $\Psi \circ f = g$; see [2, 4, 5, 10]. This result (and its proof) can be understood as an analogue of the weak Whitney isotopy theorem: For a d-dimensional smooth closed manifold M, if $n \geq 2d+2$, then all smooth embeddings of M into $\mathbb{R}^n$ are ambient isotopic.

From now on we specialize to $d = 1$. The story starts with a result by Abhyankar and Moh [1] and Suzuki [11]: if a polynomial map $\mathbb{C} \to \mathbb{C}^2, t \mapsto (p(t), q(t))$ is an embedding, then $\deg(p(t))|\deg(q(t))$ or $\deg(q(t))|\deg(p(t))$. A short argument shows that the Abhyankar-Moh-Suzuki result resolves the embedding problem for $d = 1$ and $n = 2$ in the positive. Thus (together with the above mentioned result for large codimension), the embedding problem for $d = 1$ and $n \neq 3$ is solved.[1] The case $n = 3$ remains open. We consider three examples of embeddings $\mathbb{C} \to \mathbb{C}^3$:

$$f_0 \colon t \mapsto (t, 0, 0),\ f_1 \colon t \mapsto (t^3, t^4, t^5 + t),\ \text{and } f_2 \colon t \mapsto (t^3 - 3t, t^4 - 4t^2, t^5 - 10t).$$

The embedding f_1 stands in contrast to the strong restrictions on degrees for embeddings $\mathbb{C} \to \mathbb{C}^2$; however, Craighero explicitly described an automorphism Ψ such that $\Psi \circ f_0 = f_1$ [2]. Shastri discovered f_2 through knot theory considerations (the restriction of f_2 to $\mathbb{R} \to \mathbb{R}^3$ is knotted) and asked (the still open) question, whether there exists an automorphism Ψ such that $\Psi \circ f_0 = f_2$; see [9].

Question. *Do classical knot invariants, such as the knot group or knot polynomials, have analogues in the setting of embeddings $\mathbb{C} \to \mathbb{C}^3$ which establishes that there does not exists an automorphism Ψ such that $\Psi \circ f_0 = f_2$?*

In a different direction, we generalized the resolution of the embedding problem for $d = 1$ and $n \neq 3$ to embeddings of $\mathbb{C}$ into more general varieties.

Theorem ([3]). *Let X be the underlying variety of a connected affine algebraic group. Then two embeddings of the affine line $\mathbb{C}$ into X are the same up to an automorphism of X provided that X is not isomorphic to a product of a torus $(\mathbb{C}^*)^k$ and one of the three varieties $\mathbb{C}^3$, $\mathrm{Sl}_2(\mathbb{C})$, and $\mathbb{P}\mathrm{Sl}_2(\mathbb{C})$.*

For the proof, the main technic available to treat embeddings into $\mathbb{C}^n$—generic projections to linear subspaces—is replaced with the study of projections to quotients by unipotent subgroups.

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[1] **Connection to topology.** Since the Abhyankar-Moh-Suzuki result was first proven, many different proofs have emerged. In [8], Rudolph provides a surprising proof based on two smooth topology ingredients. Firstly, the Jordan-Schoenflies theorem for smooth curves in $\mathbb{R}^2$. Secondly, the fact that non-trivial torus knots (and certain iterates thereof) $K \subset S^3$ are not *slice*; i.e., $K \subset S^3 = \partial B^4$ does not arise as the boundary of a smooth closed 2-disc in the 4-ball B^4 with boundary. This proof has led to fruitful generalizations; see e.g. [7].

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Formal Moduli Problems and Partition Lie Algebras

Lukas Brantner
(joint work with Akhil Mathew)

Overview. If k is a field of characteristic zero, a theorem of Lurie–Pridham asserts an equivalence between formal moduli problems and d.g. Lie algebras over k. We generalise this equivalence to arbitrary fields by using "partition Lie algebras". These new gadgets are intimately related to the equivariant topology of the partition complex, which allows us to access the operations acting on their homotopy groups.

An introduction to formal moduli problems. In order to study a given kind of algebro–geometric object over a ground field k in families (e.g. elliptic curves or GL_n-bundles), it is desirable to construct a representing geometric object X satisfying the following informal identity for all k-algebras R:

$$\mathrm{Map}(\mathrm{Spec}(R), X) \simeq \{\ \mathrm{Spec}(R)\text{-families of objects of the given kind }\}.$$

Usually, we cannot find a variety or scheme with this property due to the presence of automorphisms. This obstacle can be circumvented by passing to *stacks*, i.e. functors $X : \{\text{Commutative } k\text{-algebras}\} \to \{\text{Groupoids}\}$ satisfying suitable geometricity conditions. By definition, we have $X(R) = \mathrm{Map}(\mathrm{Spec}(R), X)$.

Recent advances in distinct branches of mathematics (e.g. [3],[6],[8]) have highlighted the importance of "homotopical enhancements" of algebraic geometry. Following Toën–Vezzosi [10] and Lurie [5][7], one can proceed in two ways: derived algebraic geometry replaces commutative k-algebras with *simplicial* commutative k-algebras, whereas spectral algebraic geometry is based on connective $\mathbb{E}_\infty$-k-algebras. The former theory seems more suitable for algebro–geometric applications, whereas the latter applies in homotopical contexts. If $\mathrm{char}(k) = 0$, the two theories agree. We shall focus on the derived case, but will comment on how our results can be modified to apply in the spectral setting. Families of derived algebro–geometric objects of a given kind can often be represented by *derived stacks*, i.e. functors $X : \mathrm{SCR}_k \to \mathcal{S}$, from the ∞-category of simplicial commutative k-algebras to the ∞-category $\mathcal{S}$ of spaces, satisfying suitable geometricity conditions.

The formal neighbourhood of a k-valued point $x \in X(k)$ in a derived stack X is then described by the functor $\mathrm{SCR}_k^{\mathrm{art}} \to \mathcal{S}$ given by $R \mapsto X(R) \times^h_{X(k)} \{x\}$. Here, $\mathrm{SCR}_k^{\mathrm{art}}$ denotes the ∞-category of all $A \in \mathrm{SCR}_k$ such that $\pi_0(A)$ is local Artinian with residue field k and $\dim_k(\pi_*(A)) < \infty$. If X represents some family of derived algebro–geometric objects, then a point $x \in X(k)$ corresponds to a specific object

defined over $\mathrm{Spec}(k)$, and $X_x^\wedge$ is the space of its infinitesimal deformations.
In sufficiently geometric situations, the functor $X_x^\wedge$ satisfie the following conditions:

Definition 1. A *formal moduli problem* is a functor $X : \mathrm{SCR}_k^{\mathrm{art}} \to \mathcal{S}$ such that $X(k) \simeq *$ and whenever $A \simeq B \times_D^h C$ is a pullback in $\mathrm{SCR}_k^{\mathrm{art}}$ with $\pi_0(B) \twoheadrightarrow \pi_0(D)$, $\pi_0(C) \twoheadrightarrow \pi_0(D)$ surjective, applying X gives a pullback $X(A) \simeq X(B) \times_{X(D)}^h X(C)$. Write $\mathrm{Moduli}_k \subset \mathrm{Fun}(\mathrm{SCR}_k^{\mathrm{art}}, \mathcal{S})$ for the ∞-category of formal moduli problems.

If $\mathrm{char}(k) = 0$, then formal moduli problems are controlled by d.g. Lie algebras. More precisely, let $\mathfrak{D} : (\mathrm{SCR}_k^{\mathrm{aug}})^{op} \to \mathrm{DGLA}_k$ be the right adjoint to the Chevalley-Eilenberg cochains functor from the ∞-category of d.g. Lie algebras to the ∞-category of augmented simplicial commutative k-algebras. The underlying chain complex of $\mathfrak{D}(R)$ is the linear dual of the cotangent fibre $\cot(R)$, which can be computed explicitly as $\cot(R) = |\operatorname{Bar}_\bullet(1, \mathrm{Sym}^*, \mathfrak{m}_R)|$ for $\mathfrak{m}_R$ the augmentation ideal of R and Sym^* the monad parametrising nonunital simplicial commutative k-algebras. The following theorem of Lurie [7] and Pridham [9] clarifies previous seminal work of Deligne, Drinfel'd, Feigin, Hinich, Kontsevich-Soibelman, Manetti, and others:

Theorem 2. *(Lurie , Pridham) If k is a field of characteristic zero, the functor* $\mathrm{DGLA}_k \to \mathrm{Moduli}_k$ *given by* $\mathfrak{g} \mapsto \big(R \mapsto \mathrm{Map}_{\mathrm{DGLA}_k}(\mathfrak{D}(R), \mathfrak{g})\big)$ *is an equivalence.*

Partition Lie Algebras. We generalise Theorem 2 to arbitrary fields k and thus give a Lie–algebraic description of the infinitesimal structure of moduli stacks.

To construct our equivalence, we want to define a functor $\mathfrak{D} : (\mathrm{SCR}_k^{\mathrm{aug}})^{op} \to \Lambda$ to some ∞-category Λ of generalised Lie algebras in a way that makes the induced functor $\Lambda \to \mathrm{Moduli}_k$ given by $\mathfrak{g} \mapsto (R \mapsto \mathrm{Map}_\Lambda(\mathfrak{D}(R), \mathfrak{g}))$ an equivalence.

In a first attempt to define $\mathfrak{D}$ and Λ, we observe that the tangent fibre functor $\cot^\vee : (\mathrm{SCR}_k^{\mathrm{aug}})^{op} \to \mathrm{Mod}_k$ admits a left adjoint. Writing L for the monad associated with this adjunction, we obtain a functor $(\mathrm{SCR}_k^{\mathrm{aug}})^{op} \to \mathrm{Alg}_{\mathrm{L}}(\mathrm{Mod}_k)$. Unfortunately, this very natural functor *does not* allow us to establish an equivalence between $\mathrm{Alg}_{\mathrm{L}}(\mathrm{Mod}_k)$ and Moduli_k. Roughly speaking, the monad L fails to preserve sifted colimits because it involves a double dualisation, which in turn prohibits us from using Lurie's ∞-categorical version of the Barr-Beck theorem. Even though $\mathrm{Alg}_{\mathrm{L}}(\mathrm{Mod}_k)$ is therefore the wrong target category, the assignment $(A \mapsto \cot(A)^\vee)$ is still the correct functor whenever $A \in \mathrm{SCR}_k^{\mathrm{art}}$ is Artinian. We therefore want to replace L with a sifted-colimit-preserving monad L^π that agrees with L on some full subcategory of Mod_k containing $\cot(A)^\vee$ for all Artinian A.

Indeed, let $\mathrm{Coh}_k^{\leq 0}$ be the full subcategory of Mod_k spanned by all coconnective k-module spectra with finite-dimensional homotopy groups in all degrees. Any Artinian $A \in \mathrm{SCR}_k^{\mathrm{art}}$ has $\cot(A)^\vee \in \mathrm{Coh}_k^{\leq 0}$. In fact, the monad L from above preserves $\mathrm{Coh}_k^{\leq 0}$ (cf. [5, Proposition 3.2.14.]) and is well-behaved on this subcategory: if $X_\bullet$ is a simplicial diagram in $\mathrm{Coh}_k^{\leq 0}$ with $|X_\bullet| \in \mathrm{Coh}_k^{\leq 0}$, then $|\mathrm{L}(X_\bullet)| \simeq \mathrm{L}(|X_\bullet|)$. From this, we can show that $\mathrm{L}|_{\mathrm{Coh}_k^{\leq 0}}$ lies in the image of the fully faithful monoidal restriction functor $\mathrm{End}_\Sigma^{\mathrm{Coh}_k^{\leq 0}}(\mathrm{Mod}_k) \to \mathrm{End}(\mathrm{Coh}_k^{\leq 0})$. Here, $\mathrm{End}_\Sigma^{\mathrm{Coh}_k^{\leq 0}}(\mathrm{Mod}_k)$ is the ∞-category of sifted-colimit-preserving endofunctors of Mod_k which preserve $\mathrm{Coh}_k^{\leq 0}$. Let L^π be the unique monad lifting $\mathrm{L}\,|_{\mathrm{Coh}_k^{\leq 0}}$ under the above restriction functor.

Definition 3. A *partition Lie algebra* is an algebra over the monad L^π on Mod_k.

If $M \in \mathrm{Mod}_k^{\leq 0}$ is represented by a cosimplicial k-module, $\mathrm{L}^\pi(M)$ is given by

$$\mathrm{L}^\pi(M) = \bigoplus_{n\geq 1} (\tilde{C}^\bullet(\Sigma|\Pi_n|^\diamond, k) \otimes M^{\otimes n})^{\Sigma_n}.$$

Here, $\Sigma|\Pi_n|^\diamond$ denotes the reduced-unreduced suspension of the n^{th} partition complex $|\Pi_n|$, i.e. the realisation of the poset of proper nontrivial partitions of $\{1,\ldots,n\}$. The functor $\tilde{C}^\bullet(-,k)$ sends a space X to the cosimplicial set of reduced k-valued singular cochains on X, and the functor $(-)^{\Sigma_n}$ takes strict Σ_n-fixed points.

Since L^π and L agree on $\mathrm{Coh}_k^{\leq 0}$, we have a functor $\cot(-)^\vee$ from finitely presented simplicial commutative k-algebras to partition Lie algebras. Extending in a filtered-limit-preserving way, we obtain a functor $\mathfrak{D} : (\mathrm{SCR}_k^{\mathrm{aug}})^{op} \to \mathrm{Alg}_{\mathrm{L}^\pi}(\mathrm{Mod}_k)$. This assignment is not fully faithful; for example, $k[x] \to k[[x]]$ gives an equivalence after applying $\mathfrak{D}$. However, it becomes fully faithful on a suitable subcategory:

Definition 4. An augmented simplicial commutative k-algebra $A \in \mathrm{SCR}_k^{\mathrm{aug}}$ is *complete Noetherian* if $\pi_0(A)$ is Noetherian and complete with respect to $\mathfrak{m} = \ker(\pi_0(A) \to k)$ and $\pi_i(A)$ is finitely generated over $\pi_0(A)$ for all i.

Theorem 5. *The functor $\mathfrak{D}$ restricts to an equivalence between the ∞-category $\mathrm{SCR}_k^{\mathrm{cNoet}}$ of complete Noetherian $A \in \mathrm{SCR}_k^{\mathrm{aug}}$ and the ∞-category $\mathrm{Alg}_{\mathrm{L}^\pi}(\mathrm{Coh}_k^{\leq 0})$ of partition Lie algebras whose underlying module lies in $\mathrm{Coh}_k^{\leq 0}$.*

Using this equivalence and an analysis of the functor $\mathfrak{D}$ on "π_0-surjective pullbacks", we can prove that $\mathfrak{D}$ defines a deformation theory in the sense of Lurie (cf. [7, Definition 12.3.3.2.]), which in turn implies our generalisation of Theorem 2:

Theorem 6. *For any field k, the functor $\mathrm{Alg}_{\mathrm{L}^\pi}(\mathrm{Mod}_k) \to \mathrm{Moduli}_k$ given by $\mathfrak{g} \mapsto (R \mapsto \mathrm{Map}_{\mathrm{Alg}_{\mathrm{L}^\pi}(\mathrm{Mod}_k)}(\mathfrak{D}(R), \mathfrak{g}))$ establishes an equivalence between the ∞-category of partition Lie algebras and the ∞-category of formal moduli problems.*

Remark 7. There is a parallel equivalence between formal moduli problems based on connective $\mathbb{E}_\infty$-k-algebras and an ∞-category of *spectral partition Lie algebras.*

Remark 8. Relying on the connection between partition complexes and partition Lie algebras (or spectral partition Lie algebras), we compute the homotopy groups of free objects (thus extending computations in [1],[2],[4]). These groups parametrise operations acting on the homotopy groups of partition Lie algebras.

Remark 9. Our equivalence generalises to mixed characteristic contexts, where we can describe infinitesimal deformations of $\mathrm{Spec}(A)$-valued families for A Noetherian with a map to a field k such that $\pi_0(A)$ is local with residue field k and complete with respect to the augmentation ideal.

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Thickening CW complexes to manifolds

Grigori Avramidi

A *thickening* of a finite complex X is a manifold homotopy equivalent to it.

Question. What is the minimal dimension of a thickening of X?

In this talk I explained the relation between thickening obstructions and classical embedding obstructions [2], as well as the relation between thickening obstructions and the following basic conjecture about L^2-Betti numbers:

Conjecture 1 (Singer)**.** *The L^2-Betti numbers of a closed aspherical n-manifold are concentrated in dimension $n/2$.*

The relation to thickening obstructions is via a recent result of Okun and Schreve [4] who showed that the Singer conjecture is equivalent to the following

Conjecture 2 (Davis-Okun[3])**.** *If a finite aspherical complex X has non-vanishing k-th L^2-Betti number, then X does not have a thickening of dimension $< 2k$.*

In other words, the Singer conjecture predicts that L^2-Betti numbers give aspherical thickening obstructions. Moreover, this latter version of the conjecture puts the focus squarely on the fundamental group $\Gamma := \pi_1 X$. It has been verified for many, but not all, of the groups studied by geometric group theorists.

At the end of the talk I outlined a construction showing that thickening obstructions often vanish rationally. More precisely, say that a manifold M is a *rational thickening* of X if there is a π_1-isomorphism $X \to M$ that induces a rational homology isomorphism of universal covers $H_*(\widetilde{X};\mathbb{Q}) \cong H_*(\widetilde{M};\mathbb{Q})$.

Theorem 3 ([1])**.** *Suppose X is a finite aspherical complex whose fundamental group is a d-dimensional duality group, $d \geq 3$. Then X has a $(d+3)$-dimensional rational thickening.*

Since L^2-Betti numbers cannot tell the difference between rational thickenings and genuine thickenings, it follows from this theorem that there can be no rational analogue of the Davis-Okun conjecture. Using the method of [4], one gets from this that there is no rational analogue of the Singer conjecture. Say that a manifold is *rationally aspherical* if its universal cover has the rational homology of a point.

Theorem 4 ([1])**.** *There is a closed, rationally asphrical n-manifold M whose L^2-Betti numbers are not concentrated in dimension $n/2$.*

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Smooth embeddings of a triangulated manifold

Pedro Boavida de Brito

(joint work with Pascal Lambrechts, Paul Arnaud Songhafouo and Dan Pryor)

I reported on work in progress whose goal is to describe the space of smooth embeddings between smooth manifolds in terms of a triangulation of the source manifold. Our main result is the following:

Let M^m and N^n be smooth manifolds, and let X be a simplicial set with a homeomorphism $|X| \cong M$. Then there is a cosimplicial space $Z^\bullet$ such that

(i) the homotopy limit of $Z^\bullet$ computes the space of smooth embeddings of M in N when $n - m \geq 3$,
(ii) for each integer $p \geq 0$, the space Z^p has the weak homotopy type of $\mathrm{emb}(X_p \times \mathbb{R}^m, N)$, the space of framed configurations of X_p points in N.

When M is the interval, a relative (with boundary) version of this theorem recovers Sinha's cosimplicial models for knot spaces [2]. There is also a dual statement, describing the factorization homology over M as the homotopy colimit of a simplicial space with p-simplices given by the factorization homology over $X_p \times \mathbb{R}^d$.

The proof uses manifold calculus [1]. It proceeds by carefully gluing Weiss covers of the simplices of the triangulation in order to obtain a Weiss cover of $|X|$, along with a judicious use of the isotopy invariance properties of the functors involved.

A clarification is due (thanks to Oscar Randal-Williams for bringing this to my attention after the talk): while Z^p does not depend on the smooth structure of M for each p, the smooth structure is implicitly used to define the cosimplicial maps.

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The 4-dimensional sphere embedding theorem

Arunima Ray
(joint work with Mark Powell and Peter Teichner)

In 1982, Mike Freedman proved the 4-dimensional topological disc embedding theorem, which he used to prove the topological h-cobordism theorem and the topological Poincaré conjecture [2], both in dimension four. He also outlined a proof for 4-dimensional topological surgery, which implies the classification of simply connected 4-manifolds. Briefly, in order for the surgery sequence to be exact in high dimensions, one needs to realize half of a hyperbolic basis for the kernel on H_2 of an $(n-1)$-connected normal map $M \to X$ for a $2n$-manifold M and a $2n$-dimensional Poincaré complex X, with the goal of then performing surgery on these embedded spheres to kill the kernel. Freedman's outline succeeds in constructing such embedded spheres for a 4-manifold. However, in this dimension, we additionally need to arrange for geometrically transverse spheres for these embedded spheres, since otherwise performing surgery might result in a non-simply connected outcome. Such geometrically transverse spheres can be produced using an improved version of the disc embedding theorem, which we give below. The symbols λ and μ refer to the homological intersection pairing and the self-intersection number respectively (see [1, 3] for detailed definitions).

Theorem 1 (Disc embedding theorem with transverse spheres)**.** *Let W be a compact 4-manifold with $\pi_1(W)$ good. Let $f_1, \ldots, f_k$ be a collection of immersed discs in W, that is,*

$$f_i\colon (D^2, S^1) \looparrowright (W, \partial W),$$

for all i, such that the collection of $f_i\colon S^1 \to \partial W$ is pairwise disjoint. Suppose there are framed immersions

$$g_1, \ldots, g_k \colon S^2 \looparrowright W$$

such that $\lambda(f_i, g_j) = \delta_{ij}$, $\lambda(g_i, g_j) = \mu(g_i) = 0$ for all $i, j = 1, \ldots, k$.

Then there exist disjointly embedded locally flat discs $f_1', \ldots, f_k'\colon D^2 \hookrightarrow W$, with geometrically transverse spheres $g_1', \ldots, g_k'$, such that f_i' and f_i have the same framed boundary.

Recall that a *good* group is one for which the π_1-null disc lemma holds [1, 3]. In particular, the trivial group is good. Since W is merely claimed to be a topological manifold, we are using the notion of topological transversality [1], whose known proofs rely on the original disc embedding theorem (without transverse spheres).

The above statement appears in [1], but no construction for the claimed geometrically transverse spheres is given. We give the construction of these geometrically transverse spheres, thereby completing the proof of the disc embedding theorem with transverse spheres. We follow the proof outlined in [1], and given in greater detail in the upcoming book [3]. The first step consists of constructing a 4-dimensional object called a 1-*storey capped tower*. Then one shows that certain 1-storey capped towers contain an infinite iterated 4-dimensional object, which we call a *skyscraper*. Both 1-storey capped towers and skyscrapers are built using layers of surfaces and discs. Lastly, techniques from decomposition space theory are used to show that any skyscraper is homeomorphic to a 2-handle, relative to the attaching region. The following proposition is the key tool in our construction, where we utilise the power of having a transverse capped surface.

Proposition 2. *Let W be a compact 4-manifold with $\pi_1(W)$ good. Let $f_1, \dots, f_k$ be a collection of immersed discs in W, that is,*

$$f_i \colon (D^2, S^1) \looparrowright (W, \partial W),$$

for all i, such that the collection of $f_i \colon S^1 \to \partial W$ is pairwise disjoint. Suppose that $\{\Sigma_i^c\}$ is a collection of transverse capped surfaces for $\{f_i\}$ such that

$$\lambda(C_\ell, C_m) = \mu(C_\ell) = 0$$

for every pair of caps C_ℓ and C_m of $\{\Sigma_i^c\}$. Assume in addition that each Σ_i, namely the body of Σ_i^c, is contained in a neighbourhood of ∂W.

Then there exists a collection of 1-storey capped towers $\{\mathcal{T}_i^c\}$ with arbitrarily many surface stages, such that $\mathcal{T}_i^c$ and f_i has the same framed boundary, with a collection of transverse spheres $\{R_i\}$. Moreover, each R_i is obtained from Σ_i^c by contraction.

With our methods, we prove the following theorem for (topologically) embedding spheres in 4-manifolds.

Theorem 3 (Sphere embedding theorem with transverse spheres)**.** *Let W be a compact 4-manifold with $\pi_1(W)$ good. Suppose there exist immersions*

$$f_1, \dots, f_k \colon S^2 \looparrowright W,$$

with $\lambda(f_i, f_j) = \mu(f_i) = 0$ for all $i, j = 1, \dots, k$. Suppose moreover that there exist algebraically transverse spheres for the f_i, that is, there are framed immersions

$$g_1, \dots, g_k \colon S^2 \looparrowright W$$

with $\lambda(f_i, g_j) = \delta_{ij}$. Then there exist topologically embedded disjoint framed spheres

$$f'_1, \dots, f'_k \colon S^2 \hookrightarrow W,$$

with each f'_i regularly homotopic to f_i, with geometrically transverse spheres,

$$g'_1, \dots, g'_k \colon S^2 \looparrowright W.$$

In other words, for each i, f'_i and g'_i intersect at a single point transversally, and f'_i and g'_j are disjoint for $i \neq j$.

The above theorem can be directly applied to show that the surgery sequence is exact in dimension four for manifolds with good fundamental group.

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Closed aspherical manifolds

Jean-François Lafont

This talk was focused on constructions of closed aspherical manifolds, with an emphasis on geometric methods. A manifold M is aspherical if its universal cover is contractible, or equivalently, if the only non-trivial homotopy group is $\pi_1(M)$. It is closed provided it is compact with no boundary. There are three basic methods for producing closed aspherical manifolds: (i) take a quotient of a contractible Lie group by a discrete cocompact subgroup, (ii) use some version of non-positive curvature, or (iii) assemble it by gluing together pieces that are aspherical. We will focus on methods (ii), and to a lesser extent (iii).

For geometric constructions of aspherical manifolds, the key result is the Cartan-Hadamard theorem, which states that a complete, simply-connected Riemannian manifold of non-positive sectional curvature has to be contractible.

Within the class of closed aspherical manifolds, there are various flavors of non-positive curvature, giving rise to the following subclasses:

(1) closed locally CAT(0) manifolds,
(2) closed Riemannian manifolds of non-positive sectional curvature, and
(3) closed locally symmetric manifolds of non-compact type.

Each class is successively more specialized, with strict containments $(3) \subset (2) \subset (1)$. I will outline the known constructions of manifolds in each of these three subclasses – and specialize a bit further to focus on the *negatively* curved examples. We will be interested in manifolds up to finite covers, or equivalently, to groups up to commensurability (two groups are commensurable if they have finite index subgroups that are isomorphic to each other).

1. Locally symmetric manifolds

Going back to Cartan, we have a complete classification of non-positively curved symmetric manifolds. The negatively curved spaces fall into three families (real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$, complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$, and quaternionic hyperbolic space $\mathbb{H}^n_{\mathbb{O}}$) and one exceptional space (the Cayley hyperbolic plane $\mathbb{H}^2_{Cay}$ of real dimension 16). In addition to these, there are a number of *higher rank* examples, which contain totally geodesic isometric embeddings of flat planes, and hence have some zero curvature.

The basic construction of lattices is number theoretic. Let us illustrate this for the special case of (real) hyperbolic manifolds. In this case, we have $Isom(\mathbb{H}^n_{\mathbb{R}}) \cong SO(n,1;\mathbb{R})$, so we want to produce a lattice inside this Lie group. Choose a totally real number field K of degree k over $\mathbb{Q}$, and a $(n+1)$-dimensional K-vector space V equipped with a quadratic form Q of signature $(n,1)$. With some hypotheses on σ, the group $SO(\sigma;\mathcal{O}_K)$ of linear maps of V, preserving σ with entries in the ring $\mathcal{O}_K \subset K$ of algebraic integers, defines a lattice in $Isom(\mathbb{H}^n_{\mathbb{R}})$. When n is odd, there is another possible arithmetic construction that relies on division algebras. Any lattice that is commensurable to one of these is called arithmetic. In these types of arithmetic hyperbolic manifolds, any $(k+1)$-dimensional subspace $W \leq V$ with the property that $\sigma|_W$ has signature $(k,1)$ will give rise to a closed, immersed, k-dimensional hyperbolic submanifold. Thus such totally geodesic submanifolds are very abundant.

In the higher rank case, Margulis showed that *all* lattices are arithmetic. Corlette showed this is also true for lattices in $Isom(\mathbb{H}^n_{\mathbb{O}}) \cong Sp(n,1)$ as well as in $Isom(\mathbb{H}^2_{Cay}) \cong F_{4,-20}$. In contrast, Gromov–Piatetski-Shapiro gave constructions of non-arithmetic lattices in $Isom(\mathbb{H}^n_{\mathbb{R}}) \cong SO(n,1)$. They take two arithmetic lattices constructed as above, which contain embedded totally geodesic separating codimension one submanifolds. Cutting the two arithmetic hyperbolic manifolds along the submanifolds, they then glue together half of one manifold with half of the other manifold. When the original arithmetic lattices are non-commensurable, the resulting hyperbolic manifold is non-arithmetic. Non-arithmetic lattices are also known in $SU(2,1)$ and $SU(3,1)$. Whether or not non-arithmetic lattices exist in $Isom(\mathbb{H}^n_{\mathbb{C}}) \cong SU(n,1)$, $n \geq 3$ is a well-known open problem.

2. Riemannian manifolds

More general than the locally symmetric manifolds, we have the Riemannian manifolds of negative (or non-positive) curvature. Other than the locally symmetric examples, there are only two other known constructions of negatively curved Riemannian manifolds: the Gromov–Thurston branched coverings, and the Ontaneda smooth hyperbolization (discussed in the next section).

To describe the Gromov–Thurston manifolds, consider a closed hyperbolic n-manifold M, and assume that it contains a totally geodesic codimension two submanifold N with the property that $[N] \in H_{n-2}(M;\mathbb{Z})$ is an s-divisible class, i.e. the equation $s \cdot x = [N]$ has a solution $x \in H_{n-2}(M;\mathbb{Z})$. Then one can form $\overline{M}$, an s-fold branched cover of M, ramified over N. Pulling back the Riemannian metric from M to $\overline{M}$, we obtain a metric on $\overline{M}$ which is hyperbolic almost everywhere. The only exception is along the branching locus $N \hookrightarrow \overline{M}$ where the metric has a singularity – orthogonal to the branching locus, one has total angle $2s\pi > 2\pi$. Gromov–Thurston showed that this singularity can be smoothed out, while maintaining strictly negative curvature.

To produce these examples, one needs to find totally geodesic codimension two submanifolds, satisfying the s-divisibility condition. Generalizing what we saw in the last section, these are easy to find inside arithmetic hyperbolic manifolds: one first finds a totally geodesic codimension one submanifold $Y^{n-1} \subset M^n$ that separates, and then a further codimension two submanifold $N \subset Y^{n-1}$ that separates Y. Since $[N] = 0$, it is s-divisible for all s.

3. CAT(0) and CAT(-1) examples

The CAT(0) (or CAT(-1)) condition is a metric analogues of non-positive (or negative) curvature. It requires that every geodesic triangle inside the space is thinner than a comparison triangle inside Euclidean (or hyperbolic) space. Generalizing the Cartan–Hadamard theorem, one knows that simply connected geodesic spaces that are CAT(0) are contractible. There is also a local-to-global principal: a space is locally CAT(0) if and only if its universal cover is CAT(0). Thus any closed manifold with a locally CAT(0) (or CAT(-1)) metric is automatically aspherical.

A space is piecewise Euclidean or piecewise hyperbolic if it is assembled by gluing together tiles, where each tile is either a Euclidean or hyperbolic manifold with piecewise geodesic boundary. For such spaces, it is easy to determine whether or not the glued up space is locally CAT(0) or CAT(-1). At each point in the space, one can look at the space of unit tangent vectors. This naturally inherits a piecewise spherical metric, and one merely needs to check Gromov's *large link condition* – that the links do not contain any geodesic loop of length $< 2\pi$. With this criterion in hand, constructing locally CAT(0) or CAT(-1) manifolds turns into a problem of identifying the "building blocks" to glue together, and specifying the gluing instructions in order to obtain large links (method (iii) from our introduction).

One basic method for producing CAT(0) examples is the Davis complex for certain Coxeter groups. This starts with a triangulation of the $(n-1)$-sphere, and produces a closed n-manifold with a locally CAT(0) cubulation (each cube isometric to $[0,1]^n$). The triangulation of the sphere specifies the instructions for how the cubes should be glued together, and all vertex links are combinatorially a copy of the triangulated sphere.

Another basic method is the Charney–Davis strict hyperbolization. This builds on a non-strict hyperbolization process by Gromov, which inputs a simplicial complex, and outputs a locally CAT(0) cube complex. Gromov's process preserves the topology of the links, so hyperbolizing a manifold produces a locally CAT(0) manifold. Charney–Davis explain how to change the building blocks from Euclidean cubes to hyperbolic manifolds with boundary. Their pieces are obtained from the arithmetic hyperbolic manifolds described earlier, by cutting them along a collection of totally geodesic codimension one submanifolds which pairwise intersect orthogonally, and have the same symmetries as that of a cube. Replacing the cubical building blocks in Gromov's hyperbolization by these hyperbolic building blocks does not change the vertex links, so maintains Gromov's large link condition, and hence produces a locally CAT(-1) space. Notice that this metric is a priori non-Riemannian, as near the gluing locus you again have "too much angle". A remarkable recent result of Ontaneda shows that the building blocks can be chosen so that the singularities can be smoothed away, while still preserving negative curvature (producing new examples of negatively curved Riemannian manifolds).

4. Conclusion

As we saw, all the methods above rely on finding totally geodesic submanifolds within arithmetic hyperbolic manifolds. In trying to extend these various constructions, one is naturally led to studying totally geodesic submanifolds in other locally symmetric manifolds. A key problem is to identify the possible homology classes represented by such submanifolds.

G-manifolds and algebraic *K*-theory

Mona Merling
(joint work with Cary Malkiewich)

Waldhausen's celebrated construction of $\mathbf{A}(X)$, the algebraic K-theory spectrum of a space X, introduced in [10], provides a critical link in the chain of homotopy theoretic constructions that show up in the classification of manifolds and their diffeomorphisms. The "stable parametrized h-cobordism theorem" [9] [12] gives a decomposition

$$\mathbf{A}(X) \simeq \Sigma^\infty X_+ \times \mathbf{Wh}(X), \tag{1}$$

where $\mathbf{Wh}(X)$ is a spectrum with the property that for a smooth compact manifold M, the underlying infinite loop space of $\Omega\mathbf{Wh}(M)$ is equivalent to the stable h-cobordism space

$$\mathcal{H}^\infty(M) = \operatorname{colim}\, \mathcal{H}(M \times [0,1]^k),$$

and Weiss and Williams show that $\mathcal{H}^\infty(M)$ provides the information that accesses the diffeomorphism group of M in a stable range [13].

In this talk, taking G to be a finite group, I described a joint project with Malkiewich, which is motivated by Goodwillie's vision that for a compact smooth G-manifold M, there should exist a G-equivariant extension $\mathbf{A}_G(M)$ of $\mathbf{A}(M)$ that satisfies a splitting

$$\mathbf{A}_G(M) \simeq \Sigma^\infty_G M \times \mathbf{Wh}_G(M) \tag{2}$$

analogous to the nonequivariant one from equation (1), and where the factor $\mathbf{Wh}_G$ is expected to encode information about equivariant h-cobordisms of M. It is worthwhile to notice that in a program aimed at establishing a chain of homotopy-theoretic constructions that relate the behavior of compact G-manifolds to that of their underlying equivariant homotopy types, "genuine" stable equivariant homotopy theory, i.e., the need to stabilize with respect to G-representations, comes in at the very first step. The first obstruction to a CW-complex being homotopy equivalent to a smooth manifold is Poincaré duality, and in order to talk about

Poincaré duality for G-manifolds one needs to consider $RO(G)$-graded homology and cohomology. The homotopy theoretic obstructions in the equivariant classification story should live in the genuine G-equivariant stable homotopy category; in particular the terms in (2) are supposed to be genuine G-equivariant spectra.

In order to illustrate why one should expect that the stabilization of equivariant h-cobordisms that shows up in the definition of $\mathbf{Wh}_G(M)$ should also be with respect to representations, we go on to describe an equivariant version of the classical h-cobordism theorem due to Araki and Kawakubo [AK88]. The classical h-cobodism and s-cobordism theorems due to Smale, and Barden, Mazur and Stallings, respectively, say that the Whitehead torsion of an h-cobordism $M \hookrightarrow W$ is the trivial element of the Whitehead group $Wh(M)$ if and only if the h-cobordism is trivial. Now suppose $(W; M, N)$ is an equivariant h-cobordism, namely W is a compact G-manifold which is a cobordism between G-manifolds M and N, and the inclusions $M \hookrightarrow W$ and $N \hookrightarrow W$ are G-homotopy equivalences. An equivariant h-cobordism W is trivial if W is G-homotopy equivalent to $M \times [0,1]$. Araki and Kawakubo show that the equivariant Whitehead torsion of an equivariant h-cobordism $M \hookrightarrow W$ is the trivial element of the equivariant Whitehead group $Wh_G(M)$ if and only if there exists a G-representation V such that the equivariant h-cobordism $(W \times D(V); M \times D(V), N \times D(V))$ is trivial, where $D(V)$ is the unit disk in the representation V.

A thorough treatment of equivariant Whitehead groups and equivariant Whitehead torsion is given in [1]. In particular, Lück proves that the equivariant Whitehead group of a G-space X splits as

$$Wh_G(X) \cong \bigoplus_{(H) \leq G} Wh(X^H_{hWH})$$

where WK is the Weyl group $N_H K/K$, and $(H) \leq G$ denotes conjugacy classes of subgroups. This splitting is reminiscent of the tom Dieck splitting for genuine G-suspension spectra

$$(\Sigma^\infty_G X_+)^G \cong \bigvee_{(H) \leq G} \Sigma^\infty_+ X^H_{hWH}$$

and suggests that the variant of A-theory that will fit in equation (2) should in fact be a genuine G-spectrum, whose fixed points have a similar splitting. In [4], using the newly developed technology of spectral Mackey functors as a model of G-spectra [6, 8, 7, 5, 3], we give such a construction.

Theorem 1. *For G a finite group, there exists a functor $\boldsymbol{A}_G$ from G-spaces to genuine G-spectra whose fixed points exhibit a "tom Dieck style" splitting*

$$\boldsymbol{A}_G(X)^G \simeq \prod_{(H) \leq G} \boldsymbol{A}(X^H_{hWH}),$$

and a similar formula for the fixed points of each subgroup H.

Badzioch and Dorabiala study our fixed points in [2], and they show that the inclusion $(\Sigma_G^\infty X_+)^G \to \mathbf{A}_G(X)^G$ respects the splittings. Also, in forthcoming work, Goodwillie and Igusa define a space $\mathcal{H}^\infty(M)^G$ of equivariant h-cobordisms on a compact smooth G-manifold M, stabilized not only with respect to intervals, but with respect to representation disks, which exhibits a similar tom Dieck style splitting. Our work in progress is to combine these results to show that there is a fiber sequence of G-spectra

$$\boldsymbol{\mathcal{H}}_G(M) \to \Sigma_G^\infty M_+ \to \mathbf{A}_G(M)$$

where the underlying infinite loop space of the fixed point spectrum $\boldsymbol{\mathcal{H}}_G(M)^G$ is the stable space of equivariant h-cobordisms $\mathcal{H}^\infty(M)^G$, stabilized with respect to representation disks. The next step will be to show that the map $\Sigma_G^\infty M_+ \to \mathbf{A}_G(M)$ is split injective, so that we get the desired splitting displayed as (2).

We end with a remark on a different definition of equivariant A-theory. The Waldhausen category whose K-theory has the tom Dieck style splitting exhibited by $\mathbf{A}_G(X)^H$ is the category of finite retractive H-spaces and H-maps with H-weak equivalences (i.e., H-maps that induce equivalences on all fixed points). When X is a G-space, the category $R(X)$ of all retractive spaces over X inherits a G-action by conjugation (precomposing the inclusion map with g^{-1} and postcomposing the retraction map with g). In [4] we identify the homotopy fixed point category $R(X)^{hH}$ with the category of retractive H-spaces with equivariant inclusion and retraction maps. However, the weak equivalences in the H-fixed point subcategories are *coarse*: they are nonequivariant equivalences that are H-equivariant maps. This motivates us to give the name $\mathbf{A}_G^{\text{coarse}}$ to the version of equivariant A-theory whose H-fixed points are the K-theory of $R(X)^{hH}$. Although $\mathbf{A}_G^{\text{coarse}}(X)$ does not match our expected input for the h-cobordism theorem, it does have a surprising connection to the bivariant A-theory of Williams [11], which is studied in [4].

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The Seiberg-Witten equations and the length spectrum of hyperbolic three-manifolds

Francesco Lin

(joint work with Michael Lipnowski)

While in the last three decades both Floer homology and geometrization have been extremely successful when addressing problems in three-dimensional topology, their interplay (if any) is still extremely mysterious. Geometrization implies that Seifert and hyperbolic three-manifolds are the protagonists of topology in three-dimensions; and while gauge-theoretic equations are very well-understood on the former class (see for example [4], [8] for a complete description of the set of solutions), essentially nothing is known about the latter class. By Mostow rigidity the geometric invariants of a hyperbolic three-manifold Y are also topological invariants, so the following is a natural question.

Question. For a hyperbolic three-manifold Y, is there any relationship between the topological invariants arising from the hyperbolic geometry of Y (e.g. the volume, injectivity radius, lengths of geodesics, etc.) and the invariants arising from Floer homology?

In this direction, in work in preparation joint with M. Lipnowski [7], we discuss for a hyperbolic-three manifold Y with $b_1(Y) = 0$ a relationship between the following:

- on the hyperbolic geometry side, the volume $\mathrm{vol}(Y)$ and the complex length spectrum $\mathcal{L}(Y)$, i.e. the set of lengths and holonomies of closed geodesics in Y (with multiplicities) [9];
- on the Floer-theoretic side, the existence of irreducible solutions to the Seiberg-Witten equations on Y [5].

Our main result is the following.

Theorem 1. For several hyperbolic three-manifolds in the Hodgson-Weeks census (including for examples the manifolds labeled $0, 2, 3, 15, 25, 31, 39$ and 44) the Seiberg-Witten equations do not admit any irreducible solutions.

These are the first examples of hyperbolic three-manifolds on which the set of solutions to the Seiberg-Witten equations is determined explicitly. As a direct consequence, their reduced monopole Floer homology *HM* is trivial (this was previously shown by Dunfield [3] using surgery exact triangles and the Geometrization theorem).

Our approach to the main theorem exploits in an essential way the underlying hyperbolic metric, and uses as stepping stone the spectral geometry of the Hodge Laplacian $\Delta = (d + d^*)^2$ acting on coexact 1-forms. Here, the key role is played the least eigenvalue λ_1^* (which is also a topological invariant by Mostow rigidity). Its relation with Floer homology comes from the fact that, as a special case of the main result of [6], for a hyperbolic rational homology sphere the estimate $\lambda_1^* \leq 4$ holds when there are irreducible solutions to the Seiberg-Witten equations. On the other hand, we relate λ_1^* to the volume and geodesic spectrum of Y via representation-theoretic techniques by using a specialization of the Selberg trace formula to the case hyperbolic three-manifolds and coclosed 1-forms. To prove our main result, we perform numerical computations that take as input geometric data from built-in functions in SnapPy [2]. Our method is inspired by the work [1] on the Selberg eigenvalue conjecture.

As another application of this interaction between Seiberg-Witten theory, hyperbolic geometry and spectral theory we have the following.

Theorem 2. For several hyperbolic three-manifolds in the Hodgson-Weeks census (including for examples the manifolds labeled 1, 4, 6 and 7) we can provide precise numerically certified bounds for λ_1^*. For example, for the manifold 1 (also known as the Meyerhoff manifold), we have $0.337 < \lambda_1^* < 0.339$.

The key observation here is that for the manifolds in the list the reduced monopole Floer homology group *HM* is non-trivial, which implies the existence of an irreducible solution to the Seiberg-Witten equations.

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Heegaard Floer and homology cobordism

Jennifer Hom
(joint work with Kristen Hendricks and Tye Lidman)

We use Heegaard Floer homology to define an invariant of homology cobordism. This invariant is isomorphic to a summand of the reduced Heegaard Floer homology and is analogous to Stoffregen's connected Seiberg-Witten Floer homology. The definition relies on the involutive Heegaard Floer package of Hendricks and Manolescu. We discuss applications to the homology cobordism group and the relationship with the involutive correction terms $\underline{d}$ and $\overline{d}$.

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What does a generic 3-manifold look like?

Alessandro Sisto
(joint work with Peter Feller, Pierre Mathieu and Samuel Taylor)

By "generic" 3-manifold we mean a manifold obtained randomly choosing the gluing map of a Heegaard splitting. This construction will be discussed and made precise below, and it was first considered by Dunfield and Thurston [3]. One can similarly consider generic fibred 3-manifolds, obtained, in this case, as mapping tori of randomly chosen mapping classes. In both cases one obtains hyperbolic 3-manifolds (with high probability), and describing features of their hyperbolic metrics is the main goal of this report.

1. Basic notations and definition

Let use denote by Σ_g the (closed connected orientable) surface of genus g, and we will always assume $g \geq 2$ in what follows.

Given a handlebody H and a self-homeomorphism $\phi : \partial H \to \partial H$, one can form a closed 3–manifold, which we denote by H_ϕ, by gluing two copies of H along their boundaries using ϕ as gluing map. Heegaard proved that all (closed connected orientable) 3–manifolds can be obtained in this way. Such decompositions of 3–manifolds into handlebodies are called *Heegaard splittings*.

Since isotopic gluing maps give rise to homeomorphic manifolds, we can define H_ϕ for ϕ a mapping class, i.e. an isotopy class of an orientation preserving homeomorphism of Σ_g.

Another important construction of 3–manifolds is that of *mapping tori* of surface homemorphisms. Given ϕ a self-homeomorphism of Σ_g, the mapping torus M_ϕ is obtained from $\Sigma_g \times [0, 1]$ by gluing $\Sigma_g \times \{0\}$ to $\Sigma_g \times \{1\}$ using ϕ. 3-manifolds obtained this way are exactly the 3-manifolds that fibre over S^1.

Once again, it turns out that isotopic gluing maps give rise to homeomorphic manifolds, so that the gluing map is best thought of as a mapping class.

2. What does a generic 3–manifold look like?

Given a class of mathematical objects, it is natural to ask what a "generic" one looks like. In order to make this question precise in the case of 3–manifolds, we will use random walks on mapping class groups.

A *(simple) random walk* (w_n) on a finitely generated group G is obtained as follows. Given a fixed finite symmetric generating set S of G, w_n is the group element corresponding to a word in S of length n chosen uniformly at random. Our G will be the mapping class group of Σ_g, and the results we discuss hold for any choice of finite symmetric generating set S.

We will call "random 3-manifold" the 3–manifold H_{w_n} obtained from a Heegaard splitting where the gluing map is chosen using a random walk on the mapping class group of Σ_g. Similarly, we will call M_{w_n} "random fibred 3–manifold".

In what follows, we discuss a few of the known results about random 3–manifolds and random fibred 3–manifolds. Mapping tori are much better understood than Heegaard splittings, and hence much more is known about random fibred 3–manifolds than about random 3–manifolds. Hence, we start by discussing the former.

Random fibred 3–manifolds. We will say that a statement about w_n holds asymptotically almost surely (a.a.s) if the probability that it holds tends to 1 as n tends to infinity.

First of all, M_{w_n} a.a.s. admits a hyperbolic metric, that is a metric of constant sectional curvature -1. Just like the other results below, this follows combining a 3-manifold/hyperbolic geometry theorem and a random walk theorem, namely in this case:

(1) a theorem of Thurston that says that M_ϕ is hyperbolic if and only if ϕ is a so-called "pseudo-Anosov" (meaning that no power of it preserves the isotopy class of a non-trivial simple closed curve), and
(2) the fact that w_n is a.a.s. pseudo-Anosov.

The latter result was proven independently by Maher and Rivin [7, 9].

There can only be one hyperbolic metric on a given 3–manifold by Mostow's rigidity theorem, so the next natural question is how to describe the hyperbolic metric on M_{w_n}. A few geometric properties of the metric are known; we will briefly discuss 2 of them.

The first one is that a.a.s. we have $vol(M_{w_n}) \asymp n$, where vol denotes the volume (more precisely, there exists a constant C so that a.a.s. we have $vol(M_{w_n}) \in [n/C, Cn]$). The 3-manifold part of this theorem is a result of Brock [1], which gives an estimate for the volume of M_ϕ in terms of ϕ. Namely, the theorem says that the volume is roughly proportional to the translation distance $\tau_{WP}(\phi)$ of ϕ with respect to the Weil-Petersson metric on Teichmüller space. In a sense, the knowledge about random walks is more refined than the knowledge about hyperbolic volume, meaning that there is a central limit theorem for $\tau_{WP}(w_n)$ (as well as several other related quantities), which I proved with Mathieu [8]. So not only we have a.a.s. $vol(M_{w_n}) \asymp n$, but one might hope for a central limit theorem for $vol(M_{w_n})$. However, it is not clear whether this holds.

The second result, conjectured by Rivin and proved by Taylor and myself [10], is that a.a.s. we have $inj(M_{w_n}) \asymp 1/\log^2 n$, where inj denotes the injectivity radius. This relies on deep work of Brock-Canary-Minsky, culminating in [2], that, in particular, gives a concrete biLipschitz model of M_ϕ in terms of ϕ. The biLipschitz model can also be used to describe several other features of M_{w_n}, but in the interest of conciseness we will stop here.

Random 3–manifolds. Once again, H_{w_n} a.a.s. admits a hyperbolic metric, that is to say "random 3–manifolds are hyperbolic". This was shown by Maher [6], and the proof relies on a criterion due to Hempel [5] and the geometrisation theorem for 3-manifolds, proven by Perelman.

Unfortunately, there is no analogue of the biLipschitz model for Heegaard splittings. However, in work in progress with Feller we use topological and hyperbolic-geometry techniques, including the work of Brock-Canary-Minsky, to give a partial description of the hyperbolic metric on 3-manifolds admitting Heegaard splittings under certain rather natural conditions. Roughly, the first condition is admitting a simple closed curve on the Heegaard surface at sufficiently large curve-graph distance from the disk sets, and the second condition (that allows for more precise results) is that the subsurface projection of the disk sets on the complement of the curve are far apart. I would like to remark that we do not rely on the geometrisation theorem, so that part of our work is a geometrisation theorem for random 3-manifolds.

The main point is that due to (essentially) known random-walk results, these conditions hold a.a.s., and from the partial description of the hyperbolic metric we conclude that the volume of H_{w_n} grows at least linearly, while the injectivity radius goes to 0.

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An overview of knot theory and algebraic curves

Matthew Hedden

Low-dimensional topology has often borne witness to the beautiful phenomenon whereby a rigid analytic or geometric object can be characterized topologically. Mostow rigidity stands out in three dimensions, while dimension four calls to mind the work of Donaldson [5] and Gompf [7] equating Lefschetz pencils with symplectic structures. My talk explored the interaction between a fundamentally topological concept: the knot theory of embedded curves in 3-manifolds, and an analytic concept: the geometry of algebraic curves in Stein surfaces. I gave a survey of this area, guided by the following question

Question: Which link types arise as the intersection of an algebraic curve $V \subset \mathbb{C}^2$ with the unit sphere $S^3 = \{(z,w) \in \mathbb{C}^2 |\ |z|^2 + |w|^2 = 1\}$?

I'll call a link as above a $\mathbb{C}$-*link*. I'm particularly interested in how to detect if a given link type can be realized as a $\mathbb{C}$-link. Are there topological obstructions or characterizations of these links?

The history of this question has roots in Newton, who proposed an algorithm to solve a polynomial equation $p(z,w) = 0$ [1]. In the case that the polynomial has a non-trivial partial derivative, this is possible by the implicit function theorem. When the gradient vanishes, Newton's insight was that he could algorithmically solve for one variable as a fractional power series in the other (in today's language). The pertinence of this algorithm is that it allows one to parametrize an algebraic curve near a singular point, a procedure which would connect algebraic curves to knot theory some 250 years later.

Indeed, in the early twentieth century algebraic geometers were attempting to employ branched covering techniques, so successful in the study of algebraic curves, to the classification of algebraic surfaces. Here, they were met with the problem of understanding the topology of the complement of an algebraic curve (the branch locus) near a singular point. They realized that this was equivalent to understanding the topology of the complement of the intersection of the curve with a small radius sphere. This intersection is the *link of the singularity*, and the knots arising are often called *algebraic knots*. Newton's parametrization allowed a complete topological classification of these by Brauner in 1928 terms of *iterated torus knots* [4] c.f. [6].

Forty years later, Milnor's treatise on singular points of complex hypersurfaces [11] elucidated the fundamental fibration structure present on the complement of an algebraic knot, and in that book he asked the question - later referred to as the "Milnor Conjecture" - as to whether the genus of the fiber of the aforementioned fibration coincides with the "Überschneidungszahl" or *unknotting number* of the algebraic knot. This question had a profound impact on low-dimensional topology, and continues to offer guidance to the area despite its resolution by Kronheimer and Mrowka in the early 90's using gauge theory [10]

Rudolph was the first to treat the general question above, and in the early eighties he examined what link types arise when one looks at an algebraic curve from afar [12]. Indeed, even in the case of smooth algebraic curves, he showed that one can get very interesting links by intersecting with a fixed radius sphere, thereby enclosing more of the global topology of an affine curve. To state his result, let B_n denote the braid group on n strands, generated by elements σ_i, $i = 1, .., n-1$ subject to the braid relations. Call an element $\beta \in B_n$ *quasipositive* if it can be written as a product of conjugates of the positive generators:

$$\beta = \Pi_{k=1}^{m} w_k \sigma_{i_k} w_k^{-1}.$$

Rudolph showed that the closure of a quasipositive braid is a $\mathbb{C}$-link. In my talk I sketched a more constructive proof of his result which I recently used to extend his theorem to subcritical Stein domains (see below).

Kronheimer and Mrowka's resolution of the Milnor conjecture proved the stronger result that a smooth complex curve bounded by a knot realizes the smooth 4-genus i.e. is of minimal genus amongst all smooth surfaces in the 4-ball bounded by the knot. This drew attention to Rudolph's work as it allowed him to produce many examples of knot types whose smooth 4-genus can be computed. For instance, he quickly produced many examples of knots which are topologically slice yet smoothly non-slice [13]. Moreover, he was able to connect quasipositivity to contact geometry and proved a general bound on the smooth 4-genus of an arbitrary knot in these terms, nowadays referred to as the *slice-Bennequin inequality* [14].

The fundamental importance of quasipositivity was driven home in the work of Boileau and Orevkov, who used the deep work of Gromov on the genericity of foliations of $S^2 \times S^2$ by J-holomorphic spheres to prove the converse of Rudolph's theorem, arriving at the following characterization of $\mathbb{C}$-links.

Theorem (Rudolph 1983 [12], Boileau and Orevkov 2001 [3]) *A link $L \subset S^3$ is a $\mathbb{C}$-link if and only if L can be represented as the closure of a quasipositive braid.*

Inspired by the beauty of their characterization, it is natural to wonder if such descriptions extend to the study of $\mathbb{C}$-links in other 3-manifolds. I recently gave a similar characterization of $\mathbb{C}$-links in the next simplest Stein domains: the 4-ball with a collection of Stein 1-handles attached (called *subcritical Stein domains*). The boundary of the ball with k Stein 1-handles is $\#^k S^1 \times S^2$, and the characterization of $\mathbb{C}$-links is in terms of the braid group of a k-punctured disk, denoted B_n^k.

Theorem [9] *Let $L \subset \#^k S^1 \times S^2$ be a link. Then L bounds a complex curve in a subcritical Stein filling if and only if L can be represented as the closure of a quasipositive braid in the kernel of $\phi^{tw} : B_n^k \to \mathbb{Z}^k$.*

Here, quasipositive is exactly as above - a conjugate of positive generators - and the homomorphism ϕ^{tw} counts the signed number of times the braid goes around the punctures.

Even more recently, work of Hayden [8] proved an analogue of Boileau and Orevkov's result which holds in complete generality:

Theorem [8] Every $\mathbb{C}$-link in the boundary of a Stein domain is Stein quasipositive.

In his theorem, Stein quasipositivity means that the link can be represented by an element in the braid group of a punctured surface which (identifying the braid group with the appropriate mapping class group) can be expressed by a product of *right-handed* half-twists and Dehn twists.

With Baykur, Etnyre, Hayden, Kawamuro, and Van Horn-Morris, we proved:

Theorem [2] Every Stein quasipositive braid bounds a complex curve in some Stein filling of the open book.

Together with Hayden's result, this gives a topological characterization of $\mathbb{C}$-knots in any 3-manifold.

Acknowledgement: Matthew Hedden was partially supported by NSF grant DMS-RC107281.

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Galois action on the symplectic K-theory of $\mathbb{Z}$

Søren Galatius
(joint work with Tony Feng and Akshay Venkatesh)

Symplectic K-theory $\mathrm{KSp}(\mathbb{Z})$ is defined from the symplectic groups $\mathrm{Sp}_{2g}(\mathbb{Z})$ in the same way as $\mathrm{K}(\mathbb{Z})$ is defined from $\mathrm{GL}_n(\mathbb{Z})$. There is an expected value of the homotopy groups $\mathrm{KSp}_*(\mathbb{Z})$ in terms of ordinary K-theory as well as L-theory, at least away from 2. In degree $4i+2$ it is the direct sum of an infinite cyclic group and a finite group of known order. Taking this expected value for granted, we study the action of the absolute Galois group of $\mathbb{Q}$ on the p-completion of $\mathrm{KSp}_*(\mathbb{Z})$, arising from $\mathcal{A}_g$, the moduli space of genus g principally polarized abelian varieties. We find that the action is a highly non-trivial extension of a cyclotomic character by a trivial representation; in fact the universal such extension in a sense that I will explain. Time permitting, I will speculate on a possible upgrade from homotopy groups to a spectrum or space level statement.

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