Mathematisches Forschungsinstitut Oberwolfach

Report No. 52/2018

DOI: 10.4171/OWR/2018/52

Enveloping Algebras and Geometric Representation Theory

Organised by
Iain Gordon, Edinburgh
Bernard Leclerc, Caen
Wolfgang Soergel, Freiburg

25 November – 1 December 2018

Abstract. The workshop brought together experts investigating algebraic Lie theory from the geometric and categorical viewpoints.

Mathematics Subject Classification (2010): 17B35, 17B37, 20G15.

Introduction by the Organisers

This workshop continues a series of conferences on enveloping algebras, as the first part of the title suggests, but the organisers and also the focus of these meetings have changed over the years to reflect the newest developments in the field of algebraic Lie theory. This year the main focus was on geometric and categorical methods, with an eye to explicit and combinatorial formulas.

The meeting was attended by over 50 participants from all over the world, many of them young researchers. Typically, we had three talks in the morning and two in the afternoon, and with one exception all talks were given on the blackboard. On Tuesday we had three shorter talks by younger mathematicians. Wednesday afternoon was reserved for a walk to Sankt Roman and on Thursday evening there was a little concert in the music room.

A particular highlight seemed to us the announcement by Geordie Williamson of his joint work with Simon Riche giving a new formula for calculating characters of simple G-modules in terms of periodic p-polynomials. This is part of a big project solving the combinatorial questions of modular representation theory in equal characteristic in terms of these p-variants of the Kazhdan-Lusztig polynomials, which are themselves notoriously difficult to compute, but still give us new information in small cases and a better conceptual picture. The proof might be interpreted as a particular instance of categorification. A talk of Achar on nearby cycles in these strange world of parity sheaves might be seen as a further step in this direction. Very remarkable was also a conjectural description presented by Olivier Dudas of his joint work with Raphael Rouquier on decomposition matrices of finite general unitary groups in non describing characteristics in terms of Macdonald theory and the geometry of Hilbert schemes. Together these two breakthroughs well illustrate the fundamental divide in modular representation theory between the cases of equal and unequal characteristic as well as the fundamental impact of geometrical methods and categorification on both sides of this divide.

Many lecturers presented work that involved affine Grassmannians and Satake categories. By looking at combinatorics we do know that the dual group plays a central role in many representation theoretic questions; but Satake and the affine Grassmannian produce the only conceptual reason known currently. In this vein, Michael Finkelberg explained joint work with Vasily Krylov culminating in proof of Schieder's conjecture for a Coulomb branch realization of the enveloping algebra $U(n)$, Harold Williams described his joint with Sabin Cautis, where they construct a cluster structure for the coherent Satake theory based on factorization properties of the Beilinson-Drinfeld Grassmannians, and Peng Shan presented joint work with Eric Vasserot describing an isomorphism from the cohomology ring of affine Springer fibres to the center of the category of deformed G_1T-modules introduced by Andersen, Jantzen and Soergel.

Also, quiver varieties are still a central object of study and a successful method to advance on other problems, and were central in the talks of Kevin McGerty, David Hernandez and Peter Tingley. Related to this is also the talk of Christof Geiss. On the other hand, there is still much activity in studying the questions we are all interested in from the combinatorial side: Leonardo Patimo reported on new results on the q-coefficients of Kazhdan-Lusztig polynomials, Jacinta Torres generalized her work on Lakshmibai-Seshadri galleries,

We are happy to conclude that the representation theory and geometry born from the study of enveloping algebras is in excellent health and developed into a thriving multifaceted domain of research closely interconnected with lots of other areas of mathematics which are blooming at the moment.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Christof Geiß in the "Simons Visiting Professors" program at the MFO.

Workshop: Enveloping Algebras and Geometric Representation Theory

Table of Contents

Abstracts

The nearby cycles formalism for parity sheaves

Pramod N. Achar

This talk is based on the results of [1, 6]. Let X be a complex algebraic variety, and let $f : X \to \mathbb{C}$ be an algebraic map. Consider the diagram

$$\begin{tikzcd} X_0 \arrow[r, "\mathbf{i}"] \arrow[d, "f_0"'] & X \arrow[d, "f"'] & X_\eta \arrow[l, "\mathbf{j}"'] \arrow[d, "f_\eta"] \\ \{0\} \arrow[r] & \mathbb{C} & \mathbb{C}^\times \arrow[l] \end{tikzcd}$$

Let $D^{\mathrm{b}}_{\mathrm{c}}(X, \mathbb{C})$ be the derived category of constructible complexes of sheaves of $\mathbb{C}$-vector spaces on X. The classical (unipotent) nearby cycles functor is a functor

$$\Psi_f : D^{\mathrm{b}}_{\mathrm{c}}(X_\eta, \mathbb{C}) \to D^{\mathrm{b}}_{\mathrm{c}}(X_0, \mathbb{C})$$

Following [7, 12], this functor can be defined as

$$\Psi_f(\mathcal{F}) = \mathbf{i}^*\mathbf{j}_*(f_\eta^*\mathcal{L}_\infty \otimes^L \mathcal{F}),$$

where $\mathcal{L}_\infty$ is the (infinite-rank) pro-unipotent local system on $\mathbb{C}^\times$. This functor enjoys the following properties:

(1) The object $\Psi_f(\mathcal{F})$ is equipped with a canonical unipotent automorphism $T : \Psi_f(\mathcal{F}) \to \Psi_f(\mathcal{F})$, called the *monodromy automorphism*.
(2) The object $\Psi_f(\mathcal{F})$ is constructible. (Since $\mathcal{L}_\infty$ is not constructible, this statement is nontrivial.)
(3) The functor Ψ_f is t-exact for the perverse t-structure.

If we take the logarithm of the operator T, we can replace (1) above by:

(1′) The object $\Psi_f(\mathcal{F})$ is equipped with a canonical nilpotent endomorphism $N : \Psi_f(\mathcal{F}) \to \Psi_f(\mathcal{F})$.

Here are some ways in which one might hope to generalize this theory:

- One can replace ordinary constructible sheaves by *mixed* sheaves—either mixed Hodge modules or mixed ℓ-adic sheaves. In both settings, the "logarithm of monodromy" becomes a map of weight 2, i.e., we have a map

$$N : \Psi_f(\mathcal{F}) \to \Psi_f(\mathcal{F})(-1).$$

- One can replace constructible $\mathbb{C}$-sheaves by constructible $\Bbbk$-sheaves, where $\Bbbk$ is a field of positive characteristic. In this case, "logarithm" no longer makes sense, but properties (1), (2), and (3) still hold.

A third setting where one might want to have a nearby cycles formalism is that of the *mixed modular derived category* from [4]. This notion is based on the theory of parity sheaves [10], and it exists and behaves well only under very restrictive circumstances, such as on flag varieties. Nevertheless, it has found important applications in a number of recent advances in modular geometric representation theory (see, for instance, [5, 3]). Building on these results, it has long been expected that there should be a mixed modular analogue of Gaitsgory's construction of central sheaves on the affine flag variety [9]. This expectation is the main motivation to look for a nearby cycles formalism for parity sheaves.

The principal difficulty is that it is rather difficult to even define a sheaf functor on $D^{\mathrm{mix}}(X, \Bbbk)$ if its classical version does not send parity sheaves to parity sheaves. Even worse, the classical nearby cycles construction involves the nonconstructible, nonsemisimple local system $\mathcal{L}_\infty$. The framework of [10, 4] does not allow for nonconstructible objects or for nonsemisimple local systems.

The goal of this talk is to explain a way around these difficulties. A key technical tool in [1] is a new category $D^{\mathrm{mix}}_{\mathrm{mon}}(X, \Bbbk)$, called the *monodromic derived category*. This category (whose definition is closely modeled on that of the "free-monodromic category" in [2]) is still defined in terms of the homological algebra of parity sheaves, but it does allow both nonconstructible objects and nonsemisimple local systems. One preliminary result of [1] is the existence of a fully faithful functor

$$\mathrm{Mon} : D^{\mathrm{mix}}(X, \Bbbk) \to D^{\mathrm{mix}}_{\mathrm{mon}}(X, \Bbbk).$$

Its image is precisely the subcategory of constructible objects.

Under some assumptions on X, the paper [1] also defines a functor

$$\mathcal{J} : D^{\mathrm{mix}}_{\mathbb{G}_{\mathrm{m}}}(X, \Bbbk) \to D^{\mathrm{mix}}_{\mathrm{mon}}(X, \Bbbk)$$

that can be thought of as a parity-sheaf analogue of the operation $f_\eta^* \mathcal{L}_\infty \otimes^L (-)$. Its output is always nonconstructible, and is equipped with a endomorphism N of weight 2.

Theorem ([1])**.** For any $\mathcal{F} \in D^{\mathrm{mix}}_{\mathbb{G}_{\mathrm{m}}}(X_\eta, \Bbbk)$, the object $\mathbf{i}^*\mathbf{j}_*\mathcal{J}(\mathcal{F})$ is constructible.

The main content of [1] is a definition, rather than a theorem. Thanks to the result above, it makes sense to define a functor

$$\Psi_f : D^{\mathrm{mix}}_{\mathbb{G}_{\mathrm{m}}}(X_\eta, \Bbbk) \to D^{\mathrm{mix}}(X_0, \Bbbk) \qquad \text{by} \qquad \Psi_f(\mathcal{F}) = \mathrm{Mon}^{-1}(\mathbf{i}^*\mathbf{j}_*\mathcal{J}(\mathcal{F})(1)).$$

(The (1) is a Tate twist that is included as a normalization.) It comes with a canonical nilpotent endomorphism

$$N : \Psi_f(\mathcal{F}) \to \Psi_f(\mathcal{F})(-1).$$

This map plays the same conceptual role as the "logarithm of the monodromy" in the classical setting, but here, it is not the logarithm of anything. (When $\Bbbk$ has positive characteristic, it does not make sense to exponentiate N.) It is reasonable to expect Ψ_f to be t-exact for the perverse t-structure, but this remains conjectural at the moment.

In the companion paper [6], L. Rider and I explicitly computed $\Psi_f(\underline{\Bbbk})$ in the mixed modular derived category in two important and related settings:

(1) $X = \mathbb{C}^n$, and $f : X \to \mathbb{C}$ is the function $f(x_1, \ldots, x_n) = x_1 \cdots x_n$.
(2) X is the "global Schubert variety" associated to the first fundamental coweight of the group PGL_n. This variety comes equipped with a map

to $\mathbb{C}$. Its generic fibers are isomorphic to $\mathbb{P}^{n-1}$, and its special fiber is identified with a certain closed subset of the affine flag variety of PGL_n.

In the former case, we obtained results that closely resemble known facts about the classical nearby cycles functor for mixed ℓ-adic sheaves [11, 13].

The latter case is the first nontrivial example where one might hope to obtain instances of Gaitsgory's central sheaves. The explicit objects obtained in [6] as the nearby cycles sheaves on the affine flag variety were discovered independently by Elias [8] from a different perspective, not involving the nearby cycles formalism. Elias has shown that these objects enjoy many of the expected properties of central sheaves.

References

[1] P. Achar, *How to glue parity sheaves*, arXiv:1812.07920.

[2] P. Achar, S. Makisumi, S. Riche, and G. Williamson, *Free-monodromic mixed tilting sheaves on flag varieties*, arXiv:1703.05843.

[3] P. Achar, S. Makisumi, S. Riche, and G. Williamson, *Koszul duality for Kac–Moody groups and characters of tilting modules*, J. Amer. Math. Soc. **32** (2019), 261–310.

[4] P. Achar and S. Riche, *Modular perverse sheaves on flag varieties II: Koszul duality and formality*, Duke Math. J. **165** (2016), 161–215.

[5] P. Achar and S. Riche, *Dualité de Koszul formelle et théorie des représentations des groupes algébriques réductifs en caractéristique positive*, arXiv:1807.08690.

[6] P. Achar and L. Rider, *Nearby cycles for parity sheaves on a divisor with simple normal crossings*, arXiv:1812.07924.

[7] A. Beĭlinson, *How to glue perverse sheaves*, K-theory, arithmetic and geometry (Moscow, 1984–1986), Lecture Notes in Math., vol. 1289, Springer, 1987, pp. 42–51.

[8] B. Elias, *Gaitsgory's central sheaves via the diagrammatic Hecke category*, arXiv:1811.-06188.

[9] D. Gaitsgory, *Construction of central elements in the affine Hecke algebra via nearby cycles*, Invent. Math. **144** (2001), 253–280.

[10] D. Juteau, C. Mautner, and G. Williamson, *Parity sheaves*, J. Amer. Math. Soc. **27** (2014), 1169–1212.

[11] M. Rapoport and T. Zink, *Über die lokale Zetafunktion von Shimuravarietäten. Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik*, Invent. Math. **68** (1982), 21–101.

[12] R. Reich, *Notes on Beilinson's "How to glue perverse sheaves"*, J. Singul. **1** (2010), 94–115.

[13] T. Saito, *Weight spectral sequences and independence of ℓ*, J. Inst. Math. Jussieu **2** (2003), 583–634.

Center of G_1T-modules and cohomology of affine Springer fibres

Peng Shan

(joint work with Eric Vasserot)

In geometric representation theory, it occurs often that the center of a category of representations can be realised as cohomology of certain algebraic varieties. An important example, due to Soergel, is an isomorphism between the center of the principal block of the BGG category $\mathcal{O}$ for a complex semi-simple Lie algebra and the cohomology of its flag variety. Moreover, in the case of $\mathfrak{sl}_n$, the center of parabolic category $\mathcal{O}$ is isomorphic to the cohomology of certain Springer fibres, by results of Brundan, Stroppel. In this talk, we establish analogous results for representations of G_1T-modules.

Let G be a connected and simply-connected semi-simple algebraic group over an algebraic closed field of characteristic zero. Fix T a maximal torus in G. The kernel of the Frobenius map on G defines an infinitesimal group subscheme G_1, whose representation is equivalent to representation of the restricted enveloping algebra. We consider the category $\mathcal{C}$ of (G_1, T)-bimodules overwhich the action of T_1 as subgroups of G_1 and T coincide. Andersen-Jantzen-Soergel introduced a deformed version of this category $\mathcal{C}_R$ in [1], where R is certain commutative algebra. We choose it to be the symmetric algebra of $\mathrm{Lie}(T)$ localized at zero. The category $\mathcal{C}_R$ splits into blocks according to the usual linkage principle.

On the geometric side, we consider the affine flag variety for the Langlands dual group $G^\vee$ over $\mathbb{C}$. Pick $\gamma = z\gamma_0$, with γ_0 a semi-simple regular element in $\mathrm{Lie}(G^\vee)$, and z is the formal variable in the definition of the loop group. We consider the affine Springer fibre X_γ associated with γ. It is an equidimensional projective ind-scheme. The action of the dual maximal torus $T^\vee$ on the affine flag variety restricts to an action on X_γ. This $T^\vee$-action has isolated fixed points and satisfies GKM condition.

Our main result gives a ring isomorphism between the center of the deformed G_1T-category $\mathcal{C}_R$ and the $T^\vee$-equivariant cohomology of the neutral connected component of X_γ, base changed from $H^*_{T^\vee}(pt)$ to R. Moreover, we show that under such an isomorphism, the Springer action on the cohomology corresponds to the action of the affine Weyl group on the center via Bernstein operators associated with translation functors. We also show that a similar ring isomorphism exists between the center of any block of $\mathcal{C}_R$ and equivariant cohomology of some affine Spaltenstein varieties.

Some conjectures on the center of the non-deformed categories and the center of G_1 in type A were presented at the end of the talk.

Note that Romain Bezrukavnikov and You Qi also have some closely related results.

References

[1] H. H. Andersen, J. C. Jantzen, W. Soergel *Representations of Quantum Groups at a pth Root of Unity and of Semisimple Groups in Characteristic p: Independence of p*, Astérisque **220** (1994): 321pp.

Harish-Chandra bimodules over quantized symplectic singularities

Ivan Losev

This talk is based on [L2].

Let A be a finitely generated positively graded Poisson algebra over $\mathbb{C}$ with bracket of degree $-d$ for some $d \in \mathbb{Z}_{>0}$. We will assume throughout that $X := \mathrm{Spec}(A)$ has symplectic singularities. Examples include the algebras $A = \mathbb{C}[\mathcal{N}]$ of regular functions on the nilpotent cone $\mathcal{N}$ in a semisimple Lie algebra $\mathfrak{g}$ and the algebras $\mathbb{C}[V]^\Gamma$, where V is a finite dimensional symplectic vector space and Γ is a finite subgroup of $\mathrm{Sp}(V)$.

Let $\mathcal{A}$ be a filtered quantization of A. In the case when $A = \mathbb{C}[\mathcal{N}]$ we can take $\mathcal{A}$ to be the central reduction $\mathcal{U}_\lambda$ of the universal enveloping algebra $U(\mathfrak{g})$. Here $\lambda \in \mathfrak{h}^*/W$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and W is the Weyl group. In the case when $A := \mathbb{C}[V]^\Gamma$, we can take the spherical symplectic reflection algebras of Etingof and Ginzburg, [EG]. In fact, thanks to results of [L1], these families of algebras exhaust filtered quantizations of $A = \mathbb{C}[\mathcal{N}]$ and $A = \mathbb{C}[V]^\Gamma$, respectively.

Definition 1. Let $\mathcal{B}$ be an $\mathcal{A}$-bimodule. We say that $\mathcal{B}$ is Harish-Chandra (shortly, HC) if it admits a bimodule filtration $\mathcal{B} = \bigcup_{i\geqslant 0} \mathcal{B}_{\leqslant i}$ such that $[\mathcal{A}_{\leqslant i}, \mathcal{B}_{\leqslant j}] \subset \mathcal{B}_{\leqslant i+j-d}$ for all i, j, and $\operatorname{gr}\mathcal{B}$ is a finitely generated $\mathcal{A}$-module.

For example, the regular bimodule $\mathcal{A}$ is HC. The category of HC bimodules (where the morphisms are bimodule homomorphisms) will be denoted by $\mathrm{HC}(\mathcal{A})$.

In this talk we are interested in the structure of a certain quotient category of $\mathrm{HC}(\mathcal{A})$. Namely, to a HC bimodule $\mathcal{B}$ we can assign its support $\mathrm{Supp}(\mathcal{B})$ defined as the support in X of $\operatorname{gr}\mathcal{B}$ with respect to any filtration on $\mathcal{B}$ as in Definition 1. Let $\overline{\mathrm{HC}}(\mathcal{A})$ be the quotient of the category $\mathrm{HC}(\mathcal{A})$ by the full subcategory of all bimodules whose support is a proper subvariety of X. Note that $\mathrm{HC}(\mathcal{A})$ is a monoidal category with respect to the tensor product of bimodules and $\overline{\mathrm{HC}}(\mathcal{A})$ carries an induced monoidal structure.

It turns out that $\overline{\mathrm{HC}}(\mathcal{A})$ is equivalent to the category of representations of a certain finite group. Namikawa in [N] proved that the algebraic fundamental group $\pi_1^{alg}(X^{reg})$ is finite. For example, for $X = V/\Gamma$, this group coincides with Γ. So we write Γ for $\pi_1^{alg}(X^{reg})$ in the general case too.

It turns out, see [L2, Section 4], that there is a natural monoidal embedding $\overline{\mathrm{HC}}(\mathcal{A}) \hookrightarrow \operatorname{Rep}\Gamma$. The image depends on the choice of quantization. For example, when $X = \mathbb{C}^2/\Gamma$ is a Kleinian singularity, the parameter space for quantizations is $\mathfrak{h}^*/W$, where $\mathfrak{h}, W$ are the Cartan space and the Weyl group of the same ADE type as Γ. In [L2, Section 5] we construct a normal subgroup $\Gamma_\lambda \subset \Gamma$ from a parameter $\lambda \in \mathfrak{h}^*/W$ and prove that, for the corresponding quantization $\mathcal{A}_\lambda$, the image of $\overline{\mathrm{HC}}(\mathcal{A}_\lambda)$ in $\operatorname{Rep}\Gamma$ coincides with $\operatorname{Rep}(\Gamma/\Gamma_\lambda)$ provided Γ is not of type E_8.

Now let us proceed to the case of general X. The variety X has finitely many symplectic leaves by a result of Kaledin. Let $\mathcal{L}_1, \ldots, \mathcal{L}_k$ be the codimension two leaves. For $i =, 1 \ldots, k$, we write Σ_i for a formal slice to $\mathcal{L}_i$, it is a formal neighborhood of zero in $\mathbb{C}^2/\Gamma_i$ for a suitable subgroup $\Gamma_i \subset \mathrm{SL}_2(\mathbb{C})$. Quantizing the slice construction, we see that $\mathcal{A}$ gives rise to a filtered quantization of $\mathbb{C}[\Sigma_i]$, let us denote that quantization by $\mathcal{A}_i$. The latter, in its turn, gives rise to a normal subgroup $\Gamma_{i,\mathcal{A}} \subset \Gamma_i$ as mentioned in the previous paragraph.

Now note that there is natural homomorphism $\Gamma_i \to \Gamma$. Indeed, $\Gamma_i = \pi_1^{alg}(\Sigma_i \setminus \{0\})$ and $\Gamma = \pi_1^{alg}(X^{reg})$. The inclusion $\Sigma_i \setminus \{0\} \hookrightarrow X^{reg}$ give rise to a group homomorphism $\Gamma_i \to \Gamma$ to be denoted by φ_i.

Let $\Gamma_{\mathcal{A}}$ be the minimal normal subgroup in Γ containing $\varphi_i(\Gamma_{i,\mathcal{A}})$ for all $i = 1, \ldots, k$. The following is the main result of [L2].

Theorem 2. *Suppose that none of the subgroups $\Gamma_i \subset \mathrm{SL}_2(\mathbb{C})$ is of type E_8. Then the image of $\overline{\mathrm{HC}}(\mathcal{A})$ in $\mathrm{Rep}(\Gamma)$ coincides with $\mathrm{Rep}(\Gamma/\Gamma_{\mathcal{A}})$.*

References

[EG] P. Etingof and V. Ginzburg. *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Invent. Math. **147** (2002), 243-348.

[L1] I. Losev. *Deformations of symplectic singularities and Orbit method for semisimple Lie algebras.* arXiv:1605.00592.

[L2] I. Losev, *Harish-Chandra bimodules over quantized symplectic singularities.* arXiv:1810.07625.

[N] Y. Namikawa, *Fundamental groups of symplectic singularities.* Higher dimensional algebraic geometry–in honour of Professor Yujiro Kawamata's sixtieth birthday, 321–334, Adv. Stud. Pure Math., 74, Math. Soc. Japan, Tokyo, 2017.

Differential operators on G/U and the Gelfand-Graev action

Victor Ginzburg

(joint work with David Kazhdan)

Let G be a complex connected semisimple group and U a maximal unipotent subgroup of G. The ring $\mathscr{D}(G/U)$ of algebraic differential operators on G/U has a rich structure. In an unpublished paper written in the 1960's, I. Gelfand and M. Graev have constructed an action of the Weyl group W on $\mathscr{D}(G/U)$ by algebra automorphisms. This action is somewhat mysterious due to the fact that it does not come from a W-action on the variety G/U itself. In rank 1, the action of the nontrivial element of $W = \mathbb{Z}/2\mathbb{Z}$ is, essentially, the Fourier transform of polynomial differential operators on a 2-dimensional vector space. In the case of higher rank, the action of each individual simple reflection is defined by reducing to a rank 1 case, but it is not *a priori* clear that the resulting automorphisms of $\mathscr{D}(G/U)$ satisfy the Coxeter relations.

One of the goals of the present paper is to provide a different approach to the Gelfand-Graev action. Specifically, we will present the algebra $\mathscr{D}(G/U)$ as a quantum Hamiltonian reduction in such a way that the W-action on the algebra becomes manifest.

To explain this, recall the general setting of quantum Hamiltonian reduction. Let A be an associative ring and I a left ideal of A (there is also a counterpart of the construction for right ideals). Thus, A/I is a left A-module. The quantum Hamiltonian reduction of A with respect to I is defined to be $\big(\mathrm{End}_A\ A/I\big)^{op}$, an opposite of the associative ring of A-module endomorphisms of A/I. More explicitly, let $N(I) = \{a \in A \mid Ia \subseteq I\}$ be the *normalizer* of I in A. By construction, $N(I)$ is a subring of A such that I is a two-sided ideal of $N(I)$. For any $a \in N(I)$, the assignment $f_a : x \mapsto xa$ induces an endomorphism of A/I. Moreover, this endomorphism only depends on $a \mod I$ and we have

$$\big(\mathrm{End}_A\ A/I\big)^{op} \xleftarrow[\cong]{f_a \leftarrow\!\shortmid a} N(I)/I \;=\; \{a \in A \mid (xa - ax) \mod I = 0 \ \ \forall x \in I\}/I.$$

The following special cases of quantum Hamiltonian reduction will be especially important for us.

Example 1. (i) *Let $\mathfrak{k}$ be a Lie algebra and $\imath : \mathfrak{k} \to A$ a Lie algebra map into an associative algebra A, i.e., a linear map such that $\imath([x,y]) = \imath(x)\imath(y) - \imath(y)\imath(x)$, $\forall x, y \in \mathfrak{k}$. We let $I = A\mathfrak{k}$ be a left ideal of A generated by the image of $\imath$. In this case, we have $N(I)/I = (A/A\mathfrak{k})^{\mathfrak{k}}$, the centralizer of $\imath(\mathfrak{k})$ in $A/A\mathfrak{k}$. Similarly, one can consider a right ideal, $\mathfrak{k}A$, generated by the image of $\imath$ and the corresponding algebra $(A/\mathfrak{k}A)^{\mathfrak{k}}$.*

(ii) *Let A_1, A_2, Z, be a triple of associative rings and $\iota_i : Z \to A_i$, $i = 1, 2$, a pair of ring homomorphisms. Let I be a right ideal of $A_1^{op} \otimes A_2$ generated by the elements $\iota_1(z) \otimes 1 - 1 \otimes \iota_2(z)$, $z \in Z$. Then, we have $(A_1^{op} \otimes A_2)/I = A_1 \otimes_Z A_2$. Therefore, in this case, we obtain*

$$N(I)/I = \{a_1 \otimes a_2 \in A_1 \otimes_Z A_2 \mid \iota_1(z)a_1 \otimes a_2 = a_1 \otimes a_2\iota_2(z),\ \forall z \in Z\} = (A_1 \otimes_Z A_2)^Z,$$

where for any Z-bimodule M we put $M^Z := \{m \in M \mid zm = mz,\ \forall z \in Z\}$. Multiplication in the ring $N(I)/I$ reads as follows:

$$(a_1 \otimes a_2) \cdot (a_1' \otimes a_2') = (a_1' \cdot a_1) \otimes (a_2 \cdot a_2'), \qquad \forall a_1, a_1' \in A_1,\ a_2, a_2' \in A_2.$$

By construction, the space $A_1 \otimes_Z A_2$ comes equipped with the structure of a left $(A_1 \otimes_Z A_2)^Z$-module given by the 'inner' action $a_1 \otimes a_2 : (a_1' \otimes a_2') \mapsto (a_1' \cdot a_1) \otimes (a_2 \cdot a_2')$, and the structure of a right $A_1^{op} \otimes A_2$-module given by the 'outer' action $a_1 \otimes a_2 : (a_1' \otimes a_2') \mapsto (a_1 \cdot a_1') \otimes (a_2' \cdot a_2)$.

We use the notation $\operatorname{Sym}\mathfrak{k}$, resp. $\mathcal{U}\mathfrak{k}$, for the symmetric, resp. enveloping, algebra of a vector space, resp. Lie algebra, $\mathfrak{k}$. For any scheme X put $\mathbb{C}[X] := \Gamma(X, \mathcal{O}_X)$. Let $\mathcal{T}^*X$, resp. $\mathscr{D}_X$ and $\mathscr{D}(X)$, denote the cotangent bundle, resp. the sheaf and ring of algebraic differential operators, on a smooth algebraic variety X.

Let G be a complex semisimple group with trivial center, and $U, \bar{U}$, a pair of opposite maximal unipotent subgroups of G. Let $\mathfrak{g}$, resp. $\mathfrak{u}, \bar{\mathfrak{u}}$, denote the Lie algebra of G, resp. $U, \bar{U}$. We fix a nondegenerate character $\psi : \bar{\mathfrak{u}} \to \mathbb{C}$, i.e., such that $\psi(f_i) \neq 0$ for every simple root vector $f_i \in \bar{\mathfrak{u}}$.

The action of U, resp. $\bar{U}$, on G by right translations gives a Lie algebra map $\mathfrak{g} \to \mathscr{D}(G)$. Let $\imath$, resp. $\bar{\imath}$, denote its restriction to the subalgebra $\mathfrak{u}$, resp. $\bar{\mathfrak{u}}$. It is well known that using the notation of Example 1(i), one has $\mathscr{D}(G/U) \cong (\mathscr{D}(G)/\mathscr{D}(G)\mathfrak{u})^{\mathfrak{u}}$. Let $\bar{\mathfrak{u}}^\psi$ be the image of the map $\bar{\mathfrak{u}} \to \mathscr{D}(G)$, $x \mapsto \bar{\imath}(x) - \psi(x)$. The algebra of *Whittaker differential operators* on $G/\bar{U}$ is defined as a quantum hamiltonian reduction $\mathscr{D}^\psi(G/\bar{U}) := (\mathscr{D}(G)/\mathscr{D}(G)\bar{\mathfrak{u}}^\psi)^{\bar{\mathfrak{u}}^\psi}$. The differential of the action of G on itself by left translations induces an algebra homomorphism $i : \mathcal{U}\mathfrak{g} \to \mathscr{D}(G/U)$, resp. $i^\psi : \mathcal{U}\mathfrak{g} \to \mathscr{D}^\psi(G/\bar{U})$.

Let T be the abstract maximal torus of G, and $\mathfrak{t} = \operatorname{Lie} T$. We have an imbedding $i_T : \mathcal{U}\mathfrak{t} \hookrightarrow \mathscr{D}(T)$ as the subalgebra of translation invariant differential operators. There is a natural T-action on G/U by right translations. The differential of this action induces an algebra homomorphism $i_r : \mathcal{U}\mathfrak{t} \to \mathscr{D}(G/U)$.

Let W be the (abstract) Weyl group, $Z\mathfrak{g}$ the center of the algebra $\mathcal{U}\mathfrak{g}$, and $hc : Z\mathfrak{g} \xrightarrow{\sim} (\mathcal{U}\mathfrak{t})^W$ the Harish-Chandra isomorphism. One has the following diagram of algebra homomorphisms:

$$\mathscr{D}^\psi(G/\bar{U}) \xleftarrow{i^\psi} \mathcal{U}\mathfrak{g} \hookleftarrow Z\mathfrak{g} \xrightarrow[\cong]{hc} (\mathcal{U}\mathfrak{t})^W \hookrightarrow \mathcal{U}\mathfrak{t} \xhookrightarrow{i_T} \mathscr{D}(T).$$

Let $\iota_1 : Z\mathfrak{g} \to \mathscr{D}(T)$, resp. $\iota_2 : Z\mathfrak{g} \to \mathscr{D}^\psi(G/\bar{U})$, be the composite homomorphism. We apply the construction of Hamiltonian reduction in the setting of Example 1(ii) for the triple $A_1 = \mathscr{D}(T)$, $A_2 = \mathscr{D}^\psi(G/\bar{U})$, $Z = Z\mathfrak{g}$, and the homomorphisms ι_1, ι_2.

With the above notation, our main result reads as follows.

Theorem. There is a natural algebra isomorphism

$$\mathscr{D}(G/U) \;\cong\; \big(\mathscr{D}(T) \bigotimes\nolimits_{Z\mathfrak{g}} \mathscr{D}^\psi(G/\bar{U})\big)^{Z\mathfrak{g}},$$

such that the map i_T, resp. i, corresponds via the isomorphism to the map $u \mapsto i_T(u) \otimes 1$, resp. $u \mapsto 1 \otimes i^\psi(u)$.

The Weyl group acts on the RHS of the isomorphism via its natural action on $\mathscr{D}(T)$, the first tensor factor. Thanks to the theorem, one can transport the W-action on the RHS to the LHS. We obtain a W-action on $\mathscr{D}(G/U)$ by algebra automorphisms. One can check, although we will not do that in the present paper, that the W-action thus obtained is the same as the Gelfand-Graev action (it is sufficient to check this for simple reflections, which reduces to a rank one computation).

A tale of two modules

Geordie Williamson

(joint work with Simon Riche)

Let G denote a reductive group over an algebraically closed field of characteristic $p > 0$. Let T denote a maximal torus, $\mathscr{X}$ its character lattice, and $\mathscr{X}_+$ the dominant weights with respect to some choice of positive roots. To any $\lambda \in \mathbb{C}hi_+$ we can associate a simple highest weight module L_λ. The L_λ are pairwise non-isomorphic and any simple algebraic representation of G is isomoprhic to some L_λ. We would like to know how big each L_λ is, and what its character is.

For SL_2 one can do everything by hand. I'm not sure who first wrote it down. The answer for SL_3 was obtained by Mark (1939) and Braden (1967). In the 70s Jantzen discovered his sum formula [6]. The sum formula gives a complete answer for Sp_4, SL_4 and G_2.

How far does one get with Jantzen's sum formula? Careful calculations of Jantzen reduce the problem to one undetermined $a \in \{1, 2\}$ for SL_5, one undetermined $d \in \{1, 2\}$ for Sp_6 and a few undetermined quantities for $Spin_7$. (This does not mean that there is only one unknown character in each case. Jantzen shows that there are a few ambiguities in each type, but that these ambiguities are all connected via the parameters above.) Groups of rank 4 and above presumably involve many more complications!

I recalled the periodic form of Lusztig's character formula [7] from 1980. It is the statement

$$[\widehat{P_A}] = \sum d_{B,\tilde{A}}(1)[\widehat{L_B}] \tag{1}$$

where we are now working in the principal block of G_1T-modules. For a p-alcove A, we let $\widehat{L_A}$ denote the simple module of highest weight λ, where λ is the unique weight in A in the orbit under the p-dilated affine Weyl group, and $\widehat{P_A}$ denotes its projective cover. The $d_{B,A}$ are Lusztig's periodic polynomials [7]. Here $A \mapsto \tilde{A}$ is the operation on p-alcoves which is uniquely determined by the following two properties: it is invariant under translations in $p\mathscr{X}$; on alcoves of the form w_0A with A in the fundamental block, it is given by $w_0A \mapsto A$.

Statement (1) implies (via Brauer-Humphreys reciprocity) character formulas for simple G_1T-modues. This in turn provides character formulas for the simple G-modules. (The key fact is that it is enough to know the character of L_λ for restricted weights, and these simple modules stay simple for G_1T.) Statement (1) is known to hold for large p above an explicit bound [2, 4]. It is also known that it is not true for many primes between h and some exponential function of h [8]. Thus it is desirable to have a feasible method of calculating these characters for small and even "medium sized" primes. For example, it would be nice if we could tell Jantzen whether $a = 1$ or 2 for SL_5!

The purpose of the lecture was to state the following new formula:

$$[\widehat{P_A}] = \sum {}^p d_{B,\tilde{A}}(1)[\widehat{L_B}]$$

This formula is valid for $p \geq 2h - 2$, where h is the Coxeter number, and has a chance to hold for all p. Here the ${}^p d_{B,A}$ are periodic p-polynomials. Lusztig observed that one may express the canonical basis in the periodic module via Kazhdan-Lusztig polynomials in the spherical module [7]. Because we know that spherical p-polynomials are, this allows us to define periodic p-polynomials via Lusztig's lemma.

Although the ${}^p d_{B,A}$ are complicated, this formula probably represents the easiest way to calculate the characters of simple G-modules beyond the cases where Jantzen's sum formula provides the answer, or where Lusztig's formula is valid. For example, Jensen and Scheinmann (work in progress) were able to verify by hand that $a = 1$ in the SL_5 case above.

The proof has three main ingredients:

(1) for A in the fundamental box, we have

$$\widehat{P_A} \cong (T_{\tilde{A}})_{|G_1T} \tag{2}$$

(proved by Jantzen [5] and Donkin [3]). This is only known to be true for $p \geq 2h - 2$, and explains why we must assume $p \geq 2h - 2$ above.

Donkin conjectures that (2) holds for all p. If his conjecture is true then our formula is valid for $p \geq h$.

(2) A formula for tilting characters recently established by Achar, Makisumi, Riche and the author [1].
(3) An embedding of the spherical category into the anti-spherical category, categorifying a well-known embedding.

Point three provides us with the two modules of our tale. I hope they live happily ever after.

References

[1] P. Achar, S. Makisumi, S. Riche, and G. Williamson, *Koszul duality for Kac-Moody groups and characters of tilting modules*, J. Amer. Math. Soc. **32** (2019), 261–310.

[2] H. H. Andersen, J. C. Jantzen, and W. Soergel. Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p. *Astérisque*, (220):321, 1994.

[3] S. Donkin. On tilting modules for algebraic groups. *Math. Z.*, 212(1):39–60, 1993.

[4] P. Fiebig. An upper bound on the exceptional characteristics for Lusztig's character formula. *J. Reine Angew. Math.*, 673:1–31, 2012.

[5] J. C. Jantzen, *Darstellungen halbeinfacher Gruppen und ihrer Frobenius-Kerne*, J. Reine Angew. Math. **317** (1980), 157–199.

[6] J. C. Jantzen. *Moduln mit einem höchsten Gewicht*, volume 750 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.

[7] G. Lusztig. *Hecke algebras and Jantzen's generic decomposition patterns*, volume 750 of *Lecture Notes in Mathematics*, Adv. Math. **37** (1980), 121–164.

[8] G. Williamson, *Schubert calculus and torsion explosion*, J. Amer. Math. Soc. **30** (2017), 1023–1046. With an appendix by A. Kontorovich, P. McNamara and G. Williamson.

Macdonald polynomials and decomposition numbers for finite unitary groups

Olivier Dudas

(joint work with Raphaël Rouquier)

We are interested in computing the decomposition numbers of finite groups, and more specifically of finite reductive groups (such as $\mathsf{GL}_n(q)$, $\mathsf{SO}_n(q)$,..., $\mathsf{E}_8(q)$). These numbers are numerical invariants of the category of representations, encoding the behaviour of irreducible representations when going from characteristic zero coefficients to positive characteristic $\ell > 0$.

There is a somewhat similar situation for highest weight categories: computing the multiplicities of the simple objects (here, the irreducible representations in characteristic ℓ) in the standard ones (here, the irreducible representations in characteristic zero).

1 - Finite general linear groups. Let $G = \mathsf{GL}_n(q)$ be the finite general linear group over a field with q elements. There is a distinguished set $\{\Delta(\lambda)\}_{\lambda \vdash n}$ of irreducible representations over $\overline{\mathbb{Q}}_\ell$, called *unipotent*, which are parametrized by partitions of n. The representation $\Delta(n)$ is the trivial representation, whereas

$\Delta(1^n)$ is the Steinberg representation (of dimension $q^{n(n-1)/2}$). When $\ell \nmid q$, the irreducible unipotent representations over $\overline{\mathbb{F}}_\ell$ can also be labeled by partitions $\{L(\lambda)\}_{\lambda \vdash n}$ in such a way that the matrix encoding the decomposition numbers — the *decomposition matrix* — has the following shape

$$\begin{array}{cc} \begin{matrix} L(n) & \cdots & L(1^n) \end{matrix} & \\ \left[\begin{array}{ccc} 1 & & (0) \\ & \ddots & \\ (*) & & 1 \\ \hline & \text{(non-unipotent)} & \end{array}\right] & \begin{matrix} \Delta(n) \\ \vdots \\ \Delta(1^n) \\ \\ \end{matrix} \end{array}$$

When ℓ is large enough, the unitriangular square matrix coincides with the decomposition matrix of a q-Schur algebra [5], which is computable by the LLT algorithm [3]. It depends only on the multiplicative order d of q in $\mathbb{F}_\ell^\times$. In addition, the category of unipotent representations for various $\mathsf{GL}_n(q)$ when n varies categorifies a level 1 Fock space for $\widehat{\mathfrak{sl}}_d$.

2 - Finite general unitary groups. Let us now consider the finite unitary group

$$\mathsf{GU}_n(q) := \{M \in \mathsf{GL}_n(q^2) \mid M^t\overline{M} = \mathsf{I}_n\}$$

where the conjugate is obtained from the involution $x \longmapsto x^q$ of $\mathbb{F}_{q^2}$. The unipotent representations (in characteristic 0 or ℓ) are also labelled by partitions of n and the decomposition matrix has the same shape as the one for $\mathsf{GL}_n(q)$. In addition, the category of unipotent representations categorifies a level 2 Fock space. However, the decomposition numbers are not computable from the corresponding canonical bases.

The representation theory of $\mathsf{GU}_n(q)$ is closely related to the representation theory of $\mathsf{GL}_n(-q)$ (which is generic in q as far as only unipotent representations are concerned). Here are some evidence for this relation:

- The order of the groups are the same (up to a sign).
- The dimension of unipotent representations in characteristic zero (which are polynomials in q) satisfy

$$\dim \Delta_{\mathsf{GU}_n}(\lambda)(q) = (-1)^{A(\lambda)} \dim \Delta_{\mathsf{GL}_n}(\lambda)(-q)$$

  where $A(\lambda) = \binom{n}{2} - \sum \binom{\lambda_i}{2}$.
- The partition of unipotent characters in the ℓ-blocks of $\mathsf{GU}_n(q)$ when d is the multiplicative order of q in $\mathbb{F}_\ell^\times$ is the same as the one for $\mathsf{GL}_n(q)$ when d is the order of $-q$.

In fact, we expect the categories of unipotent representations for $\mathsf{GU}_n(q)$ and $\mathsf{GL}_n(-q)$ to be derived equivalent. The equivalence should be perverse with respect to the dominance order and the A-function, and should send the cohomology of a Deligne–Lusztig variety $X(w)$ to the cohomology of $X(ww_0)$, where $w_0 =$

$(1, n)(2, n-1)\cdots$. Such an equivalence is predicted by the geometric version of Broué's abelian defect group conjecture.

If this derived equivalence is performed "twice" (in the sense that $X(w)$ is sent to $X(ww_0^2)$), we obtain a derived self-equivalence of $\mathsf{GL}_n(-q)$ with perversity $2A$. Such an equivalence exists at the local level, where it has a geometric interpretation in the derived category of coherent sheaves on $\mathsf{Hilb}^n(\mathbb{C}^2)$. We explain below only the numerical aspect of this equivalence, which we will need to compute the decomposition numbers.

3 - Macdonald polynomials. Let $\mathsf{Sym}_{u,v}$ be the space of symmetric functions in infinitely many variables $x_1, x_2, \ldots$ with coefficients in $\mathbb{Q}[u^{\pm 1}, v^{\pm 1}]$. The Macdonald polynomials $H_\lambda(u, v)$ form a basis of $\mathsf{Sym}_{u,v}$ satifying some triangularity property with respect to the Schur functions (transformed by a plethysm, see for example [1, §9]). One can define a linear operator ∇ on $\mathsf{Sym}_{u,v}$ by

$$\nabla H_\lambda(u, v) := u^{a(\lambda)} v^{a(\lambda^*)} H_\lambda(u, v)$$

where λ^* is the conjugate partition and $a(\lambda) = \sum (i-1)\lambda_i = \binom{n}{2} - A(\lambda^*)$.

The homogenous component of degree n of $\mathsf{Sym}_{u,v}$ is naturally isomorphic to the equivariant K-theory of $\mathsf{Hilb}^n(\mathbb{C}^2)$. Under this isomorphism, the Macdonald polynomials correspond to the skyscraper sheaves supported on the fixed points of $(\mathbb{C}^\times)^2$ on $\mathsf{Hilb}^n(\mathbb{C}^2)$, and the operator ∇ is given by tensoring with the tautological line bundle [2].

We will be interested in the matrix $M_n(u, v)$ defined as the transposed of the matrix of ∇ in the basis of the plethystic transformed Schur functions $s_\lambda[X/(1-u)]$ for $\lambda \vdash n$. It has the following properties:

- $M_n(u, v)$ is lower-triangular with respect to the antidominance order (the partition (n) labels the first row and first column), with diagonal terms equal to $u^{a(\lambda^*)} v^{a(\lambda)}$;
- $M_n(1, 1)$ is the identity matrix; and
- $M_n(u, v)^{-1} = M_n(u^{-1}, v^{-1})$.

4 - Computing the decomposition numbers. The matrix $M_n(u, v)$ encodes the local derived self-equivalence for the general linear group. The unitary group should be obtained "half-way" through this equivalence. The numerical counterpart is a matrix $D_n(u, v)$ with the following properties:

- $D_n(u, v)$ is lower-triangular with respect to the antidominance order, with diagonal terms equal to $D_{\lambda,\lambda} := \sqrt{u^{a(\lambda^*)} v^{a(\lambda)}}$;
- the terms of $D_n(u, v)$ satisfy
$$\frac{D_{\lambda,\mu}}{D_{\mu,\mu}} \in \mathbb{Z}[u^{-1}] + v\mathbb{Z}[u^{-1}, v];$$
- $M_n(u, v) = D_n(u, v) D_n(u^{-1}, v^{-1})^{-1}$.

Such a matrix exists and is unique by the properties of $M_n(u,v)$. The value at $u=v=1$ should then give the decomposition numbers of the finite unitary groups, taking into account the parity of the A-function (the perversity of the equivalence).

Conjecture. *Assume $\ell \mid q+1$. Then $D_n(1,1)$ is the square part of decomposition matrix of the unipotent blocks of $\mathsf{GU}_n(q)$, up to multiplication by the diagonal matrix $(-1)^{A(\lambda)}$ on both sides. In other words*

$$[\Delta(\lambda) : L(\mu)] = (-1)^{A(\lambda)+A(\mu)} D_n(1,1)_{\lambda,\mu}.$$

Example. For $n=3$ we have

$$M_3(u,v) = \begin{bmatrix} u^3 & \cdot & \cdot \\ (1-uv)(u+u^2) & uv & \cdot \\ (1-uv)(1-uv(u+v)) & (1-uv)(v+v^2) & v^3 \end{bmatrix}$$

and

$$D_3(u,v) = \begin{bmatrix} 1 & \cdot & \cdot \\ u^{-1}-v & 1 & \cdot \\ v^2-u^{-1}v^2-u^{-1}v & -v-v^2 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{u^3} & \cdot & \cdot \\ \cdot & \sqrt{uv} & \cdot \\ \cdot & \cdot & \sqrt{v^3} \end{bmatrix}$$

which gives

$$D_3(1,1) = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ -1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}.$$

We recover the decomposition matrix of $\mathsf{GU}_3(q)$ for $\ell \mid q+1$ computed in [4].

References

[1] F. Bergeron, *Algebraic combinatorics and coinvariant spaces*, CMS Treatises in Mathematics. Canadian Mathematical Society, Ottawa, ON; A K Peters, Ltd., Wellesley, MA, 2009.

[2] M. Haiman, *Combinatorics, symmetric functions, and Hilbert schemes*, Current developments in mathematics, 2002, 39–111, Int. Press, Somerville, MA, 2003.

[3] A. Lascoux, B. Leclerc, J.-Y. Thibon, *Hecke algebras at roots of unity and crystal bases of quantum affine algebras*, Comm. Math. Phys. **181** (1996), 205–263.

[4] T. Okuyama, K. Waki, *Decomposition numbers of* $\mathrm{SU}(3,q^2)$, J. Algebra **255** (2002), 258–270.

[5] M. Takeuchi, *The group ring of* $\mathrm{GL}_n(q)$ *and the q-Schur algebra*, J. Math. Soc. Japan **48** (1996), 259–274.

Representations of $\mathfrak{sl}(\infty)$ arising from categorical action

Vera Serganova
(joint work with Crystal Hoyt, Ivan Penkov)

Let $\mathfrak{g} = \mathfrak{sl}(\infty)$ denote the Lie algebra of traceless matrices $\{(a_{ij}) \mid i,j \in \mathbb{Z}\}$ with finitely many non-zero entries. One can consider $\mathfrak{g}$ as a Kac–Moody Lie algebra with infinite Dynkin diagram

$$\cdots - \circ - \circ - \circ - \ldots$$

We denote $e_a, f_a, a \in \mathbb{Z}$ the Chevalley–Serre generators of $\mathfrak{g}$. By V and V_* we denote the standard and costandard representation of $\mathfrak{g}$ and by $\mathbb{T}_\mathfrak{g}$ the abelian monoidal category of $\mathfrak{g}$-modules generated by V and V_*. This category was studied in [3] and [7]. For a partition λ of n and a $\mathfrak{g}$-module M denote by $S_\lambda(M)$ the Schur functor of M which is the image of a Young projector in $M^{\otimes n}$, associated with λ.

Theorem 1. [3] *The category $\mathbb{T}_\mathfrak{g}$ has enough injective objects. Every indecomposable injective object is isomorphic to $S_\lambda(V) \otimes S_\mu(V_*)$ for some partitions λ and μ. The irreducible socle $V_{\lambda,\mu}$ of $S_\lambda(V) \otimes S_\mu(V_*)$ coincides with the kernel of all contraction maps*

$$S_\lambda(V) \otimes S_\mu(V_*) \hookrightarrow V^{\otimes|\lambda|} \otimes V_*^{\otimes|\mu|} \to V^{\otimes|\lambda|-1} \otimes V_*^{\otimes|\mu|-1}.$$

Define a new category $\mathcal{C}$ as the full subcategory of $\mathfrak{g}$-modules consisting of objects M satisfying the following conditions:

(1) M has finite length and all simple subquotients of M are simple objects of $\mathbb{T}_\mathfrak{g}$.
(2) For every $m \in M$, $e_a m \neq 0$ and $f_a m \neq 0$ for finitely many $a \in \mathbb{Z}$.

Theorem 2. [5] *The category $\mathcal{C}$ has enough injective objects. The socle filtration of the indecomposable injective hull $I_{\lambda,\mu}$ of $V_{\lambda,\mu}$ has the following layers:*

$$[\operatorname{soc}_{k+1} I_{\lambda,\mu} / \operatorname{soc}_k I_{\lambda,\mu} : V_{\lambda',\mu'}] = \sum_{|\gamma|+|\delta|=k} N^{\lambda}_{\lambda',\gamma,\delta} N^{\mu}_{\mu',\gamma,\delta},$$

where $N^{\nu}_{\nu,\gamma,\delta}$ denote the Littlewood–Richardson coefficients.

The socle filtrations for $S_\lambda(V) \otimes S_\mu(V_*)$ was described in [6]. It is proven in [3] that $\mathbb{T}_\mathfrak{g}$ is Koszul and we believe that the same is true for $\mathcal{C}$.

Now let $\mathcal{O}_{m|n}$ denote the category $\mathcal{O}$ for the Lie superalgebra $\mathfrak{gl}(m|n)$ with integral weights. In [1] J. Brundan defined a categorical action of $\mathfrak{g}$ on $\mathcal{O}_{m|n}$ using translation functors. More precisely, let E and E^* denote the standard and costandard modules of $\mathfrak{gl}(m|n)$. Let $\{X_i\}$ and $\{X^i\}$ be dual bases in $\mathfrak{gl}(m|n)$ with respect to the form $\operatorname{str} XY$. The element $C := \sum_i (-1)^{p(X_i)} X_i \otimes X^i \in \mathfrak{gl}(m|n) \otimes \mathfrak{gl}(m|n)$ commutes with the action of $\mathfrak{gl}(m|n)$ on a tensor product $A \otimes B$ for any two $\mathfrak{gl}(m|n)$-modules A, B. Let $A \in \mathcal{O}$ and $E_a(A)$ (resp., $F_a(A)$) denote the generalized eigenspace of C with eigenvalue a in $A \otimes E$ (resp., $A \otimes E^*$). Then E_a and F_a are mutually adjoint exact endofunctors in $\mathcal{O}_{m|n}$.

Definition 3. Let $\mathcal{A}$ be an abelian category of modules over some superalgebra. The reduced Grothendieck group of $\mathcal{A}$ is the quotient of the usual Grothendieck group of $\mathcal{A}$ by the relation $[M] = -[M \otimes \mathbb{C}^{0|1}]$.

Theorem 4. [1] *Let* $\mathbf{K}_{m|n}$ *be the complexified reduced Grothendieck group of* $\mathcal{O}_{m|n}$. *Then* E_i, F_i *induce linear operators* $e_i, f_i : \mathbf{K}_{m|n} \to \mathbf{K}_{m|n}$. *Furthermore* $e_i, f_i, i \in \mathbb{Z}$ *satisfy Serre's relation for the infinite-dimensional Lie algebra* $\mathfrak{sl}(\infty)$ *with the Dynkin diagram* A_∞. *Thus,* $\mathbf{K}_{m|n}$ *is an integrable* $\mathfrak{sl}(\infty)$*-module,* $\mathfrak{sl}(\infty)$*-weight spaces correspond to the blocks in* $\mathcal{O}_{m|n}$.

Theorem 5. [5],[2]

(1) $\mathbf{K}_{m|n}$ *is an injective object of* $\mathcal{C}$.
(2) *The submodule* $V^{\otimes m} \otimes V_*^{\otimes n} \hookrightarrow \mathbf{K}_{m|n}$ *is isomorphic to the subgroup generated by the classes of all Verma modules.*
(3) *The socle of* $\mathbf{K}_{m|n}$ *is isomorphic to the subgroup generated by the classes of all projective objects in* $\mathcal{O}_{m|n}$.

Let $\mathcal{F}_{m|n}$ be the category of finite-dimensional $\mathfrak{gl}(m|n)$-modules semisimple over the Cartan subalgebra $\mathfrak{h}$ and $\mathbf{J}_{m|n}$ denote its complexified reduced Grothendieck group.

For $M \in \mathcal{O}_{m|n}$ denote by ΓM the subset of all $\mathfrak{gl}(m|n)_0$-finite vectors. Then Γ defines a left exact functor $\mathcal{O}_{m|n} \to \mathcal{F}_{m|n}$. Its derived functor Γ^i is called the Zuckerman functor.

Theorem 6. [5]

(1) $\mathbf{J}_{m,n}$ *is the injective hull of* $V_{1^m,1^n}$.
(2) *The map* $[M] \to \sum(-1)^i[\Gamma^i M]$ *defines an* $\mathfrak{sl}(\infty)$*-equivariant map* $\gamma : \mathbf{K}_{m|n} \to \mathbf{J}_{m|n}$.
(3) *The restriction of* γ *to* $V^{\otimes m} \otimes V_*^{\otimes n} \hookrightarrow \mathbf{K}_{m|n}$ *coinsides with the natural projector to* $\Lambda^m V \otimes \Lambda^n V_*$.

In order to understand the categorical meaning of the socle filtration of $\mathbf{K}_{m|n}$ recall the definition of DS functor, [4]. Let

$$X := \{x \in \mathfrak{gl}(m|n)_1 \mid [x,x] = 0\},$$

$$X_k := \{x \in X \mid \mathrm{rk}(x) = k\},\ k \leq \min(m,n).$$

For a $\mathfrak{gl}(m|n)$-module M define $DS_x M := \ker x_M / \operatorname{im} x_M$.

Theorem 7. [5] *Let* $x \in X_k$.

(1) $DS_x : \mathcal{O}_{m|n} \to \mathcal{O}_{m-k|n-k}$ *is a symmetric monoidal functor which commutes with translation functors* E_i, F_i.
(2) *Passage to the Grothendieck groups induces a homomorphism of* $\mathfrak{sl}(\infty)$*-modules* $ds_x : \mathbf{K}_{m|n} \to \mathbf{K}_{m-k|n-k}$.

Remark 8. *Although* DS_x *is not exact, for an exact sequence* $0 \to N \to M \to L \to 0$ *we have a canonical exact sequence*

$$0 \to R \to DS_x N \to DS_x M \to DS_x L \to R \otimes \mathbb{C}^{0|1} \to 0.$$

This ensures the existence of the corresponding map ds_x for the reduced Grothendieck groups.

Let $k \leq \min(m,n)$, and $\mathcal{O}^k_{m|n}$ be the subcategory of $\mathcal{O}_{m|n}$ consisting of all modules M such that $DS_x M = 0$ for all $x \in X_k$. This category is not abelian but Karoubian and monoidal. Let $\mathbf{K}^k_{m|n}$ be the subgroup in $\mathbf{K}_{m|n}$ generated by the classes of all objects in $\mathcal{O}^k_{m|n}$.

Conjecture 9. $\mathrm{soc}_k \mathbf{K}_{m|n} = \mathbf{K}^k_{m|n}$.

References

[1] J. Brundan, *Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra* $\mathfrak{gl}(m|n)$, J. Amer. Math. Soc. 16 (2003), no. 1, 185–231.

[2] K. Coulembier, V. Serganova, *Homological invariants in category O for the general linear superalgebra*, Trans. Amer. Math. Soc. 369 (2017), no. 11, 7961–7997.

[3] E. Dan-Cohen, I. Penkov, V. Serganova, A Koszul category of representations of finitary Lie algebras, Advances in Mathematics 289 (2016), 250–278.

[4] M. Duflo, V. Serganova, *On associated variety for Lie superalgebras*, arXiv:Math/0507198.

[5] C. Hoyt, I. Penkov, V.Serganova, *Integrable $sl(\infty)$-modules and category $\mathcal{O}$ for $\mathfrak{gl}(m|n)$*, Journal of LMS, doi.org/10.1112/jlms.12176.

[6] I. Penkov, K. Styrkas, *Tensor representations of classical locally finite Lie algebras*, in Developments and Trends in Infinite-Dimensional Lie Theory, Progress in Mathematics 288, Birkhäuser, 2011, 127–150.

[7] S. Sam, A. Snowden, *Stability patterns in representation theory*, Forum of Mathematics, Sigma, vol. 3, e11, 2015.

Semiinfinite highest weight and ϵ-stratified categories

Catharina Stroppel

(joint work with Jon Brundan)

Highest weight categories naturally appear in representation theory and were studied in quite some detail in an impressive bulk of works by many people including in particular Cline, Parshall and Scott, Donkin and Ringel. Slightly more general notions of (properly) stratified categories were pioneered via different approaches in particular by Cline Parshall and Scott [CPS1], by Dlab or in the work of Dlab, Agoston and Lucasz [ADL]. A unifying approach seems however so far missing. Recent developments around categorifications suggest to generalize these constructions further to a *semi-infinite setting*. Our goal of our work is to provide such a construction, which at the same time unifies the existing approaches to stratified categories in the classical setting.

Let $\mathbb{F}$ be any field. Consider a *finite Abelian category*, that is, a category $\mathcal{R}$ equivalent to the category A-mod$_{\mathrm{fd}}$ of finite-dimensional left A-modules for some finite-dimensional $\mathbb{F}$-algebra A. Let $\mathbf{B}$ be a finite set indexing a full set of pairwise inequivalent irreducible objects $\{L(b)|b \in \mathbf{B}\}$. Let $P(b)$ (resp. $I(b)$) be a projective cover (resp. injective hull) of $L(b)$.

A *stratification* of $\mathcal{R}$ is the data of a function $\rho : \mathbf{B} \to \Lambda$ for some poset Λ. For $\lambda \in \Lambda$, let $\mathcal{R}_{\leq\lambda}$ (resp. $\mathcal{R}_{<\lambda}$) be the Serre subcategory of $\mathcal{R}$ generated by the irreducibles $L(b)$ for $b \in \mathbf{B}$ with $\rho(b) \leq \lambda$ (resp. $\rho(b) < \lambda$). Then define the *stratum* $\mathcal{R}_\lambda$ to be the Serre quotient $\mathcal{R}_{\leq\lambda}/\mathcal{R}_{<\lambda}$ with quotient functor $j^\lambda : \mathcal{R}_{\leq\lambda} \to \mathcal{R}_\lambda$. For $b \in \mathbf{B}_\lambda := \rho^{-1}(\lambda)$, let $L_\lambda(b) := j^\lambda L(b)$. These give a full set of pairwise inequivalent irreducible objects in $\mathcal{R}_\lambda$. Let $P_\lambda(b)$ (resp. $I_\lambda(b)$) be a projective cover (resp. an injective hull) of $L_\lambda(b)$ in $\mathcal{R}_\lambda$. Now j^λ has a left adjoint $j_!^\lambda$ and a right adjoint j_*^λ, which we refer to as the *standardization* and *costandardization functors*, respectively, following [LW]. Then we introduce the *standard, proper standard, costandard* and *proper costandard* objects of $\mathcal{R}$ for $\lambda \in \Lambda$ and $b \in \mathbf{B}_\lambda$:

$$(1)\quad \Delta(b) := j_!^\lambda P_\lambda(b), \quad \bar{\Delta}(b) := j_!^\lambda L_\lambda(b), \quad \nabla(b) := j_*^\lambda I_\lambda(b), \quad \bar{\nabla}(b) := j_*^\lambda L_\lambda(b).$$

Fix a sign function $\epsilon : \Lambda \to \{\pm\}$ and define the ϵ-*(co)standard*

$$(2)\quad \Delta_\epsilon(b) := \begin{cases} \Delta(b) & \text{if } \epsilon(\rho(b)) = + \\ \bar{\Delta}(b) & \text{if } \epsilon(\rho(b)) = - \end{cases}, \quad \nabla_\epsilon(b) := \begin{cases} \bar{\nabla}(b) & \text{if } \epsilon(\rho(b)) = + \\ \nabla(b) & \text{if } \epsilon(\rho(b)) = - \end{cases}.$$

Then $\mathcal{R}$ is a *finite ϵ-stratified category* if for every $b \in \mathbf{B}$, the projective object $P(b)$ has a Δ_ϵ-flag with sections $\Delta_\epsilon(c)$ for $c \in \mathbf{B}$ with $\rho(c) \geq \rho(b)$.

This definition includes the classical highest weight categories (the case when all strata are simple) and the finite (co)standardly stratified categories from the existing literature (as the cases when ϵ is constant $-$ respectively $+$). It also includes the cases of *fully stratified categories* [ADL, Definition 1.3], that is categories which are both standardly and costandardly stratified.

Many fully stratified categories arise in the context of categorification. This includes the pioneering examples of categorified tensor products of finite dimensional irreducible representations for the quantum group attached to $\mathfrak{sl}_k$, see [FKS, in particular Remark 2.5], but more importantly the axiomatix definition of *tensor product categorification* by Losev and Webster [LW]. On such categories, there is a fully stratified structure which gives a categorical interpretation of Lusztig's construction of tensor product of based modules for a quantum group.

We introduce the notion of a *lower finite ϵ-stratified category* where we allow the poset Λ to be infinite, but require the intervals $(\infty, \lambda]$ for $\lambda \in \Lambda$ to be finite and adapt the definition from above to this case. The main special example for this is the category $\mathrm{Rep}(G)$ of finite-dimensional rational representations of a connected reductive algebraic group. We also introduce *upper finite ϵ-stratified categories* where we allow the poset Λ to be infinite, but upper finite. One of the main examples are the representation categories of the socalled Deligne categories.

Theorem 1. *In either case ϵ-tilting objects exist (that is objects possessing both an* ascending Δ_ϵ-flag *and a* descending ∇_ϵ-flag*). Isomorphism classes of indecomposables ϵ-tilting objects are in canonical bijection with* $\mathbf{B}$.

In the lower finite setting this follows easily from a truncation to the finite case, whereas in the upper finite setting this result is rather surprising.

We introduce Ringel dual categories and Ringel duality and show the following:

Theorem 2. *Ringel duality connects lower and upper finite stratified categories:*

$$\left\{\begin{array}{c}\textit{lower finite}\\ \epsilon\textit{-stratified categories}\end{array}\right\} \overset{\textit{Ringel duality}}{\longleftrightarrow} \left\{\begin{array}{c}\textit{upper finite}\\ (-\epsilon)\textit{-stratified categories}\end{array}\right\}.$$

In particular the Ringel dual of a lower respectively an upper highest weight category is an upper respectively a lower highest weight category. In concrete terms the duality pairs certain categories of comodules with certain categories of locally finite dimensional modules over locally finite dimensional algebras. Since modules over a finite dimensional $\mathbb{F}$-algebra are nothing else than comodules over its vector space dual coalgebra, this distinction is not visible in the finite setting. Theorem 2 in particular can be applied to pair blocks of category $\mathcal{O}$ for affine Kac-Moody Lie algebras of integral weights at negative level with a corresponding integral block at positive level via the above Ringel duality.

References

[ADL] I. Ágoston, V. Dlab and E. Lukács, Stratified algebras, *Math. Rep. Acad. Sci. Canada* **20** (1998), 22–28.

[CPS1] E. Cline, B. Parshall and L. Scott, Finite dimensional algebras and highest weight categories, *J. Reine Angew. Math.* **391** (1988), 85–99.

[FKS] I. Frenkel, M. Khovanov and C. Stroppel, A categorification of finite-dimensional irreducible representations of quantum $\mathfrak{sl}_2$ and their tensor products, *Selecta Math.* (N.S.) **12** (2006), 379–431.

[LW] I. Losev and B. Webster, On uniqueness of tensor products of irreducible categorifications, *Selecta Math.* **21** (2015), 345–377.

Minimal reflections and the coefficient of q in Kazhdan-Lusztig polynomials

Leonardo Patimo

Let (W, S) be a Coxeter system. For any pair of elements $x, y \in W$ with $x \leq y$, the Kazhdan-Lusztig polynomial $P_{x,y}(q)$ is defined in [1] via an elementary recursive formula. KL polynomial are nowadays ubiquitous in representation theory as they appear in many character formulas, e.g. for simple modules in category $\mathcal{O}$. The original definition, however, does not explain precisely what kind of information one needs to compute the Kazhdan-Lusztig polynomial $P_{x,y}(q)$. To address this question, a powerful approach is provided by the theory of moment graphs.

The Bruhat graph $\mathcal{G}$ associated to W is the labeled directed graph whose vertices are the elements of W and where for any reflection t such that $\ell(xt) > \ell(x)$ there is an arrow $x \to xt$, labeled by the positive root corresponding to t. By work of Braden-MacPherson [2] and Fiebig [4], we can in fact compute the KL polynomials $P_{x,y}(q)$ as the Poincaré polynomial of the stalks of a certain sheaves on the moment graph $\mathcal{G}$. One of the advantages of this description is that we can directly see that the polynomial $P_{x,y}(q)$ depends only on the restriction of $\mathcal{G}$ to the Bruhat interval $[x, y]$.

In the 1980s, M. Dyer and G. Lusztig independently conjectured a much stronger statement: they claimed that the polynomial $P_{x,y}(q)$ should only depend on the poset structure of the Bruhat interval $[x, y]$. This is today known as the "Combinatorial Invariance Conjecture". It is equivalent to saying that knowledge of the labels in the Bruhat graph is superfluous when computing the KL polynomials, or in other words, this conjecture says that there should be a completely combinatorial formula to compute the coefficients of $P_{x,y}(q)$. The goal of my work is to describe a formula for the linear coefficient of $P_{x,y}(q)$. This is the main result in [3].

Consider the Hasse diagram $H_{x,y}$ of the Bruhat interval $[x, y]$. For a set F of edges in $H_{x,y}$, we denote by $F^\diamond$ the smallest set of edges which contains F and such that whenever we have a 4-cycle $\mathcal{D}$ in $H_{x,y}$, if two adjacent edges of $\mathcal{D}$ are in $F^\diamond$ then also the other two edges must be in $F^\diamond$. We call $F^\diamond$ the *diamond closure* of F, and we say that F is *diamond generating* if $F^\diamond$ contains all the edges of $H_{x,y}$. We define $g_{x,y}$ to be the minimal cardinality of a diamond generating set of $H_{x,y}$.

Let $q_{x,y}$ denote the coefficient of q in $P_{x,y}(q)$. Let $c_{x,y}$ denote the number of coatoms in $[x, y]$, i.e. the number of elements z in the interval $[x, y]$ with $\ell(z) = \ell(y) - 1$. Then we have $q_{x,y} \geq c_{x,y} - g_{x,y}$. If W is of type ADE, we can also prove the opposite inequality, that is, we have

$$q_{x,y} = c_{x,y} - g_{x,y}. \tag{1}$$

To show this equality, we have to relate the coefficient $g_{x,y}$ to that of smaller Bruhat intervals. This is actually an easy task if there exists a simple reflection s such that $xs > x$ and $ys < y$. Unfortunately, such a simple reflection does not always exist.

In type ADE, we use recent work of Tsukermann-Williams [5] and Caselli-Sentinelli [6] to obtain a workaround. For any Bruhat interval $[x, y]$ in such groups, these authors define a distinguished reflection t, called *minimal reflection*, which retains some of the behaviour of simple reflections. They show for example that $y \geq xt \gtrdot x$, $y \gtrdot yt \geq x$ and

$$R_{x,y}(q) = (q-1)R_{x,yt}(q) + qR_{xt,yt}(q).$$

In [3], we show that a further property of simple reflections generalizes to minimal reflections: if t is a minimal reflection for (x, y), maximal chains between x and yt exist in the set $W^t := \{x \in W \mid xt \gtrdot x\}$. This is the crucial and final ingredient of our proof of (1).

References

[1] D. Kazhdan and G. Lusztig. *Representations of Coxeter groups and Hecke algebras.* Invent. Math., **53**(2):165–184, 1979.

[2] T. Braden and R- MacPherson. *From moment graphs to intersection cohomology.* Math. Ann., **321**(3):533–551, 2001.

[3] L. Patimo, *A combinatorial formula for the coefficient of q in Kazhdan-Lusztig polynomials.* ArXiv 1811.08184.

[4] P. Fiebig. The combinatorics of Coxeter categories. *Trans. Amer. Math. Soc.*, **360**(8):4211–4233, 2008.
[5] E. Tsukerman and L. Williams. *Bruhat interval polytopes.* Adv. Math., **285**, 766–810, 2015.
[6] F. Caselli and P. Sentinelli. *The generalized lifting property of Bruhat intervals.* J. Algebraic Combin., **45**(3):687–700, 2017.

Cuspidal D-modules and generalized quantum Hamiltonian reduction

Sam Gunningham

Let G be a connected complex reductive group acting on $\mathfrak{g} = \mathrm{Lie}(G)$ by the adjoint action. We consider the abelian category $\mathrm{Dmod}(\mathfrak{g})^G$ of strongly G-equivariant D-modules on $\mathfrak{g}$. In [Gun18] we proved the following generalized Springer theorem for $\mathrm{Dmod}(\mathfrak{g})^G$.

Theorem 1. *There is an equivalence of abelian categories*

$$\mathrm{Dmod}(\mathfrak{g})^G \simeq \bigoplus_{(L,\mathcal{E})}^{\perp} \mathrm{Dmod}\left(\mathfrak{z}(\mathfrak{l})\right)^{W_{G,L}},$$

where the (finite) sum is indexed by cuspidal data $(L, \mathcal{E})$ *in the sense of Lusztig* [Lus84].[1] *Here, L is a Levi subgroup of G, $\mathfrak{z}(\mathfrak{l})$ denotes the center of $\mathfrak{l} = Lie(L)$ and $W_{G,L} = N_G(L)/L$ is the relative Weyl group.*

Example 2. *The block* $\mathrm{Dmod}(\mathfrak{g})^G_{Spr}$ *corresponding to the unique cuspidal datum $(T, \mathcal{E})$ for a maximal torus T is known as the Springer block. The theorem in particular gives an equivalence*

$$\mathrm{Dmod}(\mathfrak{g})^G_{Spr} \simeq \mathrm{Dmod}(\mathfrak{t})^W = \mathfrak{D}_{\mathfrak{t}}\#W - \mathrm{mod}$$

where W is the Weyl group of G (with respect to T). In the case $G = GL_n$ (or PGL_n) the Springer block is the entire category. Thus there is an equivalence

$$\mathrm{Dmod}(\mathfrak{gl}_n)^{GL_n} \simeq \mathrm{Dmod}(\mathbb{A}^n)^{S_n} \simeq \mathfrak{D}_{\mathbb{A}^n}\#S_n - \mathrm{mod}$$

Example 3. *In the case $G = SL_2$, there is a unique simple cuspidal D-module $\mathfrak{E}$ supported on the nilpotent cone of G. It may be characterized as the intersection cohomology (IC) extension of the unique non-trivial local system on the regular nilpotent orbit.*[2] *In this case the generalized Springer decomposition has two blocks - the Springer block and the cuspidal block:*

$$\mathrm{Dmod}(\mathfrak{sl}_2)^{SL_2} \simeq \mathrm{Dmod}(\mathbb{A}^1)^{\mathbb{Z}/2\mathbb{Z}} \oplus \langle \mathfrak{E} \rangle$$

The cuspidal block $\langle \mathfrak{E} \rangle$ is a semisimple category with a single simple object $\mathfrak{E}$, thus it is equivalent to the category of vector spaces.

[1] A cuspidal datum consists of a pair $(L, \mathcal{E})$ of a Levi subgroup of G and a simple D-module supported on the nilpotent cone for L which is cuspidal: it vanishes upon parabolic restriction to every proper Levi subgroup of L. Considering the simple objects of the subcategory of equivariant D-modules supported on the nilpotent cone of G recovers Lusztig's generalized Springer correspondence.

[2] In fact the extension is clean, so the IC extension agrees with the star and shriek extensions.

The goal of this project is to understand the interaction of the generalized Springer decomposition with the underlying G-representation on the global sections of the D-module.

To give a sense of of what such an understanding might look like, let us first examine the Springer block. Consider the equivariant D-module $\mathfrak{P} := \mathfrak{D}_\mathfrak{g}/\mathfrak{D}_\mathfrak{g}\mathrm{ad}(\mathfrak{g})$, which represents the functor of quantum Hamiltonian reduction (i.e. G-invariants of the global sections of a strongly equivariant module). Combining the results of Hotta-Kashiwara [HK84] with Theorem 1 one can show the following:

Theorem 4. *The Hamiltonian reduction D-module $\mathfrak{P}$ is a projective generator of the Springer block; under the generalized Springer correspondence*

$$\mathrm{Dmod}(\mathfrak{g})^G_{Spr} \simeq \mathrm{Dmod}(\mathfrak{t})^W$$

it corresponds to the object $\mathfrak{D}_\mathfrak{t}$ (with its canonical W-equivariant structure). In particular the Springer block is characterized by the property that any non-zero object has a G-invariant vector.

It is natural to ask: is the a similar characterization of the other generalized Springer blocks?

Let us first describe a natural generalization of the quantum Hamiltonian reduction D-module $\mathfrak{P}$. For each (finite dimensional) representation V of G, there is a canonical strongly equivariant D-module

$$\mathfrak{P}(V) := \mathfrak{D}_\mathfrak{g} \otimes_{\mathbb{U}\mathfrak{g}} V$$

where the homomorphism $\mathrm{ad} : \mathbb{U}\mathfrak{g} \to \mathfrak{D}_\mathfrak{g}$ is induced from the infinitesimal adjoint action. The modules $\mathfrak{P}(V)$ are compact and projective; they represent the functor which assigns to each equivariant D-module its V-multiplicity space. The collection of all $\mathfrak{P}(V)$ as V ranges over finite dimensional representations of G forms a set of compact projective generators of $\mathrm{Dmod}(\mathfrak{g})^G$.

Now our question can be made more precise: how do the modules $\mathfrak{P}(V)$ under the generalized Springer decomposition?

This question has a nice partial answer in the case $G = SL_n$ (though we shall see that the situation is necessarily more complicated for other types). In general, the component group $\pi_0 Z(G) = Z(G)/Z^\circ(G)$ acts on the category $\mathrm{Dmod}(\mathfrak{g})^G$, giving an orthogonal decomposition indexed by the characters $\widehat{\pi_0 Z(G)}$. In the case $G = SL_n$, we may show that this decomposition agrees with the generalized Springer decomposition, in the sense that for each character $\overline{k} \in \widehat{\pi_0 Z(G)} \simeq \mathbb{Z}/n\mathbb{Z}$ there is a corresponding cuspidal datum $(L_{\overline{k}}, \mathcal{E}_{\overline{k}})$ such that the corresponding generalized Springer block is characterized by the fact that $\pi_0(Z(G))$ acts via the character $\overline{k}$. In particular, for an irreducible representation V, $Z(G)$ acts on V via a character $\overline{k}$ and the D-module $\mathfrak{P}(V)$ lies entirely in the corresponding generalized Springer block. If the central character $\overline{k}$ of V is relatively prime to n then the D-module $\mathfrak{P}(V)$ is a direct sum of a number of copies of the unique simple cuspidal object with the same central character.

Example 5. *If $G = SL_2$ and $V_n = \mathrm{Sym}^n(\mathbb{C}^2)$ the irreducible representation of dimension $n+1$, $\mathfrak{P}(V_n)$ is cuspidal exactly when n is odd. In that case $\mathfrak{P}(V_n) \simeq \mathfrak{E}^{\oplus m_n}$ for some non-negative integers m_n. By the universal property of the objects $\mathfrak{P}(V_n)$, the multiplicities m_n may also be interpreted as the multiplicity of the representation V_n in the global sections of $\mathfrak{E}$.*[3] *On the other hand, for even values of n $\mathfrak{P}(V_n)$ lies in the Springer block which is equivalent to $\mathrm{Dmod}(\mathfrak{t})^W$. Under this equivalence, $\mathfrak{P}(V_n)$ corresponds to $\mathfrak{D}_\mathfrak{t}$ (respectively $\mathfrak{D}_\mathfrak{t} \otimes sgn$) if $n \equiv 0 \mod 4$ (respectively $n \equiv 2 \mod 4$).*

One can see that the situation for $G = SL_n$ cannot hold in general: there are groups G of adjoint type (so $Z(G) = 1$) which admit non-trivial cuspidal data. In those cases one can show that the D-module $\mathfrak{P}(V)$ always have a summand which is contained in the Springer block (controlled by the zero weight space of V together with its Weyl group representation). But for certain representations there may be other summands which live in other generalized Springer blocks.

This naturally leads to the following set of problems, which the author hopes to investigate in forthcoming work.

- Let us call a representation *quasi-cuspidal* if $\mathfrak{P}(V)$ admits a cuspidal summand (equivalently, submodule or quotient). How to characterize the quasi-cuspidal representations?
- Find a minimal set of irreducible representations $V_1, \ldots V_n$ such that $\mathfrak{P}(V_1), \ldots, \mathfrak{P}(V_n)$ generate the category $\mathrm{Dmod}(\mathfrak{g})^G$.
- For each cuspidal datum $(L, \mathcal{E})$, characterize the corresponding block of $\mathrm{Dmod}(\mathfrak{g})^G$ in terms of the irreducible representations of G appearing as a summand of the global sections.

References

[CEE09] Damien Calaque, Benjamin Enriquez, and Pavel Etingof, *Universal KZB equations: the elliptic case*, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I, Progr. Math., vol. 269, Birkhäuser Boston, Inc., Boston, MA, 2009, pp. 165–266.

[Gun18] Sam Gunningham, *Generalized Springer theory for D-modules on a reductive Lie algebra*, Selecta Mathematica, New Series (2018).

[HK84] Ryoshi Hotta and Masaki Kashiwara, *The invariant holonomic system on a semisimple Lie algebra*, Inventiones Mathematicae **75** (1984), no. 2, 327–358.

[Lus84] G. Lusztig, *Intersection cohomology complexes on a reductive group*, Inventiones Mathematicae **75** (1984), no. 2, 205–272.

[3]Since giving the talk, the author has learned that these multiplicities (and corresponding ones for $G = SL_n$) have been computed by Calaques-Enriques-Etingof [CEE09] using a certain Schur-Weyl duality functor with values in a rational Cherednik algebra: we have $m_n = n$.

Generalized Lakshmibai-Seshadri galleries

Jacinta Torres

Let $\mathfrak{g}$ be a complex, simple Lie algebra, and $L(\lambda)$ a simple module over the complex numbers of highest weight $\lambda \in X^+$. Consider the set

$$\Gamma = \{\gamma : [0,1] \longrightarrow X_{\mathbb{R}} : \gamma(1) \in X,\ \gamma(0) = 0\}$$

of piecewise linear paths starting at the origin and ending at an integral weight. In [2], Littelmann defined a finite set of paths $\mathcal{P}(\gamma_\lambda) \subset \Gamma$ depending on a single path $\gamma_\lambda \in \Gamma$ with endpoint $\gamma_\lambda(1) = \lambda$ and contained in the dominant Weyl chamber, for which the character of $L(\lambda)$ is a generating function in the following sense:

$$\mathrm{ch}(L(\lambda)) = \sum_{\gamma \in P(\gamma_\lambda)} x^{\gamma(1)}. \tag{1}$$

The set $\mathcal{P}(\gamma_\lambda)$ is constructed by applying certain root operators f_α, one for every simple root α, to γ_λ. Together with their partial inverses e_α, they endow the set $\mathcal{P}(\gamma_\lambda)$ with the structure of a crystal isomorphic to $B(\lambda)$.

Question. Given a path $\gamma \in \Gamma$, how can one decide if it belongs to $P(\gamma_\lambda)$ for some γ_λ contained in the dominant Weyl chamber?

In this note, we give necessary conditions for *galleries of alcoves*, that is, certain sequences of alcoves in the corresponding affine Coxeter complex (which has the same underlying space $X_{\mathbb{R}}$), with the first alcove containing the origin and a choice of integral weight contained in the last one. Our ongoing work generalizes work of Gaussent-Littelmann in [1], where they introduce so-called Lakshmibai-Seshadri galleries, for which the formula (1) makes sense and still holds after replacing $\eta(1)$ for a path η by the endpoint $e(\gamma)$ of a gallery γ.

1. Galleries

A **gallery of alcoves**, or just a **gallery**, is a sequence

$$\gamma := (v_0, a_0, p_1, \cdots, p_r, a_r, v_r) \tag{2}$$

of alcoves and panels (i.e. codimension one faces of alcoves) $a_{i-1} \supset p_i \subset a_i$ for $i \in [1, r]$. In addition we require a choice of a final vertex $v_r \in X$, an integral weight contained in the last alcove, and ask that the first alcove a_0 contains the origin $v_0 = 0$. The integral weight v_r is the endpoint $e(\gamma)$ of the gallery γ. By labelling the codimension one faces of the fundamental alcove with colours $I = \{i_0, ..., i_n\}$, where n is the rank of the Lie algebra $\mathfrak{g}$, we obtain a labelling of every panel. The label of a panel p will be denoted by $t(p)$ and referred to as the **type** of the panel. The type of a vertex v is the set $t(v) = \{t(p) : v \in p\}$ of types of the panels containing v. The type of γ is the sequence of types

$$t(\gamma) := (t(p_1), ..., t(p_r))$$

of its panels and endpoint. Given a dominant integral weight $\lambda \in X^+$, we will denote by

$$\gamma_\lambda := (v_0', a_0', p_1', \cdots, p_r', a_r', v_r') \tag{3}$$

a gallery with $v_r' = \lambda$ and such that each one of its alcoves and panels is contained in the dominant Weyl chamber. We will call such a gallery **dominant**.

1.1. **Dimension and properties of galleries.** Let γ be a gallery, with notation as in (2). A hyperplane H is a **load-bearing wall** for γ at $i \in [0, r]$ if it contains the i-th panel $p_i \subset H$ (or the origin $v_0 \in H$ if $i = 0$) and if $a_i \subset H^-$. The **dimension** of the gallery γ is the number of load bearing walls, counting repetitions:

$$\dim(\gamma) = \sharp \{j | \text{ there exists a load-bearing wall } H \text{ for } \gamma \text{ at } j\}.$$

We say that there is a **fold** at a panel p_i if $a_i = a_{i-1}$, and a **crossing** if $a_{i-1} \neq a_i$. The fold or crossing is positive (respectively negative) if $a_i \subset H^+$ (respectively $a_i \subset H^-$), where H is the hyperplane on which the panel p_i is contained.

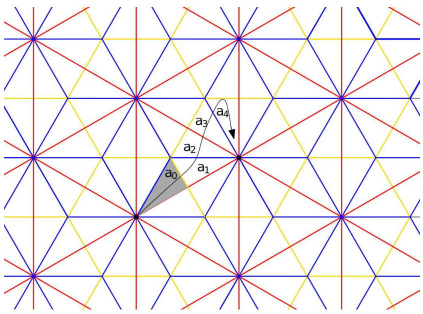


FIGURE 1. A dominant gallery of dimension 10 for the simple exceptional Lie algebra of type G_2. It consists of six alcoves: the five distinct ones a_0, the fundamental alcove, a_1, a_2, a_3, a_4, which are noted in the picture, and $a_5 = a_4$. The corresponding fold at the panel p_5 is negative. The panels are indicated by the black path, and the origin and endpoint are indicated by thick black dots.

Let γ_λ be a dominant gallery, with notation as in (3). Let $D \subset [1, r]$ be the set of indices i such that there is a negative crossing or fold in γ_λ at p_i'. Consider the set of galleries of the same type as γ_λ. Within this set, we look at those which at the panels with indices in D have either a negative crossing or a negative fold. Out of these, we restrict ourselves to those which, at every other panel, have either a crossing or a positive fold. We will denote this set by $\Gamma^+(\gamma_\lambda)$. Now, for every

$\mu \leq \lambda$ consider the subset of $\Gamma^+_{max}(\gamma_\lambda, \mu) \subset \Gamma^+(\gamma_\lambda)$ of galleries γ with maximal dimension having endpoint $e(\gamma) = \mu$, and

$$\Gamma^+_{max}(\gamma_\lambda) = \bigcup_{\mu \leq \lambda} \Gamma^+_{max}(\gamma_\lambda, \mu).$$

Remark 1. *If γ_λ is minimal as defined in* [1]*, then the set $\Gamma^+_{max}(\gamma_\lambda)$ is the set of Lakshmibai-Seshadri galleries (LS galleries for short) introduced by Gaussent-Littelmann in a geometric context. If this is the case, an LS gallery $\gamma \in \Gamma^+_{max}(\gamma_\lambda)$ has dimension $\langle \lambda + e(\gamma), \rho^\vee \rangle$.*

We are now ready to state our result which generalizes work of Gaussent-Littelmann and partially answers Question 1.

Theorem 2. *The set $\Gamma^+_{max}(\gamma_\lambda)$ is stable under the action of the root operators f_α. Moreover, it only contains one dominant gallery, that is γ_λ.*

References

[1] S. Gaussent and P. Littelmann. LS galleries, the path model, and MV cycles. *Duke Math. J.*, 127(1):35–88, 2005.

[2] P. Littelmann. Paths and root operators in representation theory. *Ann. of Math. (2)*, 142(3):499–525, 1995.

Semisimplification of representation categories

Thorsten Heidersdorf

(joint work with Rainer Weissauer)

0.1. **Semisimplification.** Let $\mathcal{C}$ denote a k-linear (k a field) braided rigid monoidal category with unit object $\mathbf{1}$ and $End(\mathbf{1}) \cong k$. For such a category one can define the trace $Tr(f) \in End(\mathbf{1}) \cong k$ of an endomorphism $f \in End(X)$ for any $X \in \mathcal{C}$ and the dimension of X via $dim(X) = Tr(id_X)$. The negligible morphisms

$$\mathcal{N}(X,Y) = \{f : X \to Y \mid Tr(g \circ f) = 0 \ \forall g : Y \to X\}$$

can be seen as an obstruction to the semisimplicity of $\mathcal{C}$. The negligible morphisms form a tensor ideal of $\mathcal{C}$ and the quotient category $\mathcal{C}/\mathcal{N}$ is again a k-linear braided rigid monoidal category. Under some mild assumptions on $\mathcal{C}$ [AK02] the quotient is semisimple. We call this the *semisimplification* of $\mathcal{C}$.

0.2. **Representation categories.** Examples of such categories are often coming from representation theory.

(1) $\mathcal{C} = Rep(G,k)$, the category of finite dimensional representations of an algebraic group over a field k; or $\mathcal{C} = Tilt(G,\mathbb{F}_q) \subset Rep(G,\mathbb{F}_q)$, the category of tilting modules for a semisimple, simply connected algebraic group G.
(2) $\mathcal{C} = Rep(U_q(\mathfrak{g}))$, finite dimensional modules of type 1 for Lusztig's quantum group $U_q(\mathfrak{g})$ for a complex semisimple Lie algebra $\mathfrak{g}$; or $\mathcal{C} = Tilt(U_q(\mathfrak{g}))$, the subcategory of tilting modules.
(3) $\mathcal{C} = Del_t$, one of the Deligne categories associated to $GL(n)$, $O(n)$ or S_n for $t \in \mathbb{C}$, or its abelian envelope.

Of particular importance in this list is $Tilt(U_q(\mathfrak{g}))$ (studied e.g. in [AP95]) since the semisimple quotient is a modular tensor category. For other examples see [EO18]. André and Kahn [AK02] studied the case where $\mathcal{C} = Rep(G)$, the category of representations of an algebraic group over a field k of characteristic 0. In this case $\mathcal{C}/\mathcal{N}$ is of the form $Rep(G^{red})$ where G^{red} is a pro-reductive group, the *reductive envelope* of G (this is false in $char(k) > 0$).

0.3. **Representations of supergroups.** The results of [AK02] generalize partially to algebraic supergroups if k is algebraically closed. Using a characterization of super tannakian categories by Deligne [Del02], the quotient $Rep(G)$ of representations of an algebraic supergroup on finite dimensional super vector spaces by the negligible morphisms is of the form $Rep(G^{red},\varepsilon)$ where G^{red} is an affine supergroup scheme and $\varepsilon : \mathbb{Z}/2\mathbb{Z} \to G^{red}$ such that the operation of $\mathbb{Z}/2\mathbb{Z}$ gives the $\mathbb{Z}_2$-graduation of the representations [He15]. A determination of G^{red} is typically out of reach. More amenable is the full monoidal subcategory $Rep(G)^I$ of direct summands in iterated tensor products of irreducible representations of $Rep(G)$. The irreducible representations of the quotient category $Rep(G)^I/\mathcal{N} \cong Rep(H,\varepsilon')$ correspond to indecomposable direct summands of non-vanishing superdimension in such iterated tensor products. The aim is then to determine H. For an irreducible representation $L(\lambda)$ consider its image in $Rep(H)$ and take the tensor category generated by it. This category is of the form $Rep(H_\lambda,\varepsilon')$ for a reductive group H_λ and $L(\lambda)$ corresponds to an irreducible faithful representation V_λ of H_λ.

0.4. **The category $Rep(GL(m|n))$.** Let $\mathcal{T}_{m|n}$ be the category of finite dimensional representations of $GL(m|n)$. The categories $\mathcal{T}_{m|n}$ are not semisimple for $m,n \geq 1$. As above we consider only objects that are retracts of iterated tensor products of irreducible representations $L(\lambda)$. This subcategory is called $\mathcal{T}^I_{m|n}$ and we denote the pro-reductive group of its semisimple quotient by $H_{m|n}$. The crucial tool to determine $H_{m|n}$ is the Duflo-Serganova functor [DS05] [HW14] $DS : \mathcal{T}_{m|n} \to \mathcal{T}_{m-1|n-1}$. It allows us to reduce the determination of $H_{m|n}$ to lower rank.

Theorem. [HW18, Theorem 5.15] a) $H_{m|n}$ is a pro-reductive group. b) DS restricts to a tensor functor $DS : \mathcal{T}^I_{m|n} \to \mathcal{T}^I_{m-1|n-1}$ and gives rise to a functor $DS : \mathcal{T}^I_{m|n}/\mathcal{N} \to \mathcal{T}^I_{m-1|n-1}/\mathcal{N}$. c) There is an embedding $H_{m-1|n-1} \to H_{m|n}$ and DS can be identified with the restriction functor.

We specialize now to $GL(n|n)$ and use the notation $G_n = (H_{n|n})^0_{der}$ and $G_\lambda = (H_\lambda)^0_{der}$. We also suppose that $sdim(L(\lambda)) > 0$ since we can replace $L(\lambda)$ by its parity shift. We say a representation is weakly selfdual (SD) if it is selfdual after restriction to $SL(n|n)$.

Theorem. [HW18, Theorem 6.2] $G_\lambda = SL(V_\lambda)$ if $L(\lambda)$ is not (SD). If $L(\lambda)$ is (SD) and $V_\lambda|_{G_{\lambda'}}$ is irreducible, $G_\lambda = SO(V_\lambda)$ respectively $G_\lambda = Sp(V_\lambda)$ according to whether $L(\lambda)$ is orthogonal or symplectic selfdual. If $L(\lambda)$ is (SD) and $V_\lambda|_{G_{\lambda'}}$ decomposes into at least two irreducibe representations, then $G_\lambda \cong SL(W)$ for $V_\lambda|_{G_{\lambda'}} \cong W \oplus W^\vee$.

We conjecture that the last case in the theorem doesn't happen. The ambiguity in the determination of G_λ is only due to the fact that we cannot exclude special elements with 2-torsion in $\pi_0(H_{n|n})$.

Theorem. [HW18, Theorem 6.8] Let $\lambda \sim \mu$ if $L(\lambda) \cong L(\mu)$ or $L(\lambda) \cong L(\mu)^\vee$ after restriction to $SL(n|n)$. Then

$$G_n \cong \prod_{\lambda \in X^+/\sim} G_\lambda.$$

In down to earth terms, these theorems give

- the decomposition law of tensor products of indecomposable modules in $\mathcal{T}^I_{m|n}$ up to indecomposable summands of superdimension 0; and
- a classification (in terms of the highest weights of H_λ and H_μ) of the indecomposable modules of non-vanishing superdimension in iterated tensor products of $L(\lambda)$ and $L(\mu)$.

We remark that the statement about $G_{n|n}$ implies a strange disjointness property of iterated tensor products of irreducible representations of non vanishing superdimension. For the general $\mathcal{T}_{m|n}$-case ((where $m \geq n$) recall that every maximal atypical block in $\mathcal{T}_{m|n}$ is equivalent to the principal block of $\mathcal{T}_{n|n}$. We denote the image of an irreducible representation $L(\lambda)$ under this equivalence by $L(\lambda^0)$.

Conjecture. *(work in progress)* Suppose that $\mathrm{sdim}(L(\lambda)) > 0$. Then $H_\lambda \cong GL(m-n) \times H_{\lambda^0}$ and $L(\lambda)$ corresponds to the representation $L_\Gamma \otimes V_{\lambda^0}$ of H_λ. Here L_Γ is an irreducible representation of $GL(m-n)$ which only depends on the block Γ (the *core* of Γ).

References

[AP95] Henning Haahr Andersen and Jan Paradowski, *Fusion categories arising from semisimple Lie algebras.*, Commun. Math. Phys., 169, 1995.

[AK02] Yves André and Bruno Kahn, *Nilpotence, radicaux et structures monoïdales.* Rend. Semin. Mat. Univ. Padova, 108, 2002.

[Del02] Deligne, P., *Catégories tensorielles. (Tensor categories).*, Mosc. Math. J. , 2, 2, 2002.

[Del90] Deligne, P., *Catégories tannakiennes. (Tannaka categories).*, The Grothendieck Festschrift, Collect. Artic. in Honor of the 60th Birthday of A. Grothendieck. Vol. II, Prog. Math. 87, 111-195 (1990)..

[DS05] Duflo, M. and Serganova. V., *On associated variety for Lie superalgebras.*, arXiv:math/0507198v1, 2005.

[EO18] Etingof, P. and Ostrik, V., On semisimplification of tensor categories, ArXiv e-prints, 1801.04409, 2018.

[He15] Heidersdorf, T., On supergroups and their semisimplified representation categories, to appear in: Algebr. Represent. Theory ArXiv e-prints, 1512.03420, (2015).

[HW14] Heidersdorf, T. and Weissauer, R., *Cohomological tensor functors on representations of the General Linear Supergroup* to appear in Mem. Am. Math. Soc., ArXiv e-prints, 1406.0321, 2014.

[HW18] Heidersdorf, T. and Weissauer, R., On classical tensor categories attached to the irreducible representations of the General Linear Supergroups $GL(n|n)$, ArXiv e-prints, 1805.00384, (2018).

Braidings, cacti and crystals

Jacob Greenstein

(joint work with Arkady Berenstein and Jian-Rong Li)

Let $\mathfrak{g}$ be a reductive Lie algebra and let $U_q(\mathfrak{g})$ be the corresponding quantized universal enveloping algebra (in fact our results can be formulated in a more general framework of symmetrizable Kac-Moody algebras with extended weight lattices). Consider the category $\mathcal{O}_q^{int}(\mathfrak{g})$ whose objects are direct sums (possibly infinite) of simple finite dimensional $U_q(\mathfrak{g})$-modules. Let W be the Weyl group of $\mathfrak{g}$. After Lusztig ([Lus]), the braid group Br_W corresponding to W acts on the category $\mathcal{O}_q^{int}(\mathfrak{g})$. This means that for every object V in $\mathcal{O}_q^{int}(\mathfrak{g})$ we have a group homomorphism $\mathrm{Br}_W \to GL(V)$, and these homomorphisms are natural with respect to morphisms in $\mathcal{O}_q^{int}(\mathfrak{g})$.

On the other hand, let Cact_W be the *cactus group* associated with W (cf [Los, B, DJS]). For $W = S_n$ the corresponding group appeared in the study of Deligne-Mumford compactification of the moduli space of stable rational curves with $n+1$ marked points and their applications in mathematical physics, as well as in the context of coboundary categories introduced by Drinfeld [D]. One example of such categories is provided by crystals, and the corresponding action of Cact_{S_n}, was studied in [HK, S].

In this talk we discuss an action of Cact_W on the category $\mathcal{O}_q^{int}(\mathfrak{g})$ ([BGL1]) which is connected to Lusztig's symmetries. Our action on modules preserves their lower and upper crystal lattices of Kashiwara and factors through to an action on the corresponding crystal bases. We also study the action of the "largest cactus" (the standard generator of Cact_W which is mapped to the longest element of W under the natural surjection $\mathrm{Cact}_W \to W$) on Gelfand-Kirillov model for the category $\mathcal{O}_q^{int}(\mathfrak{g})$. We prove that it is an algebra anti-involution, and show that it preserves the upper global crystal basis and hence the upper global crystal basis in each simple highest weight module. In fact, these involutions are closely related to remarkable quantum twists studied by Kimura and Oya ([KO]).

Our motivation stems from the study of monomial braidings ([BGL2]). Namely, starting from a vector space V and a braiding on $V \otimes V$, we obtain families of braidings on $V^{\otimes n} \otimes V^{\otimes n}$ parametrized by bi-transitive (also known as bipartite) relations on $\{1, \dots, n\}$. This can be further generalized and yields the notion, as well as infinite families of examples, of multibraidings. The first example was constructed in [BG] in connection with quantum folding. Using the results of [BGL1] and quantum Howe duality ([TL]) we connect the study of spectral properties of such (multi)braidings to the purely combinatorial question of determining the multiplicities of ± 1 eigenvalues of certain involutions in the cactus group on tensor products of highest weight modules.

References

[BG] A. Berenstein and J. Greenstein, *Quantum folding*, Int. Math. Res. Not. **2011** (2011), no. 21, 4821–4883

[BGL1] A. Berenstein, J. Greenstein, J.-R. Li, *On cacti and crystals*, preprint arXiv:1803.11330 (to appear)

[BGL2] A. Berenstein, J. Greenstein, J.-R. Li, *Monomial braidings* (in preparation)

[B] C. Bonnafé, *Cells and cacti*, Int. Math. Res. Not. **2016** (2016) no. 19, 5775–5800

[D] V. G. Drinfel'd, *Quasi-Hopf algebras*, Algebra i Analiz **1** (1989), no. 6, 114–148

[HK] A. Henriques and J. Kamnitzer, Crystals and coboundary categories, Duke Math. J. **132** (2006), no. 2, 191–216

[KO] Y. Kimura and H. Oya, *Twist automorphisms on quantum unipotent cells and dual canonical bases*, preprint arXiv:1701.02268.

[Los] I. Losev, *Cacti and cells*, preprint, arXiv:1506.04400

[Lus] G. Lusztig, Introduction to quantum groups, Progress in Mathematics, vol. 110, Birkhäuser Boston, Inc., Boston, MA, 1993.

[DJS] M. Davis, T. Januszkiewicz, and R. Scott, R., *Fundamental groups of blow-ups*, Adv. Math., **177** (2003), no. 1, 115–179

[S] A. Savage, Crystals, quiver varieties, and coboundary categories for Kac-Moody algebras, Adv. Math. **221** (2009), no. 1, 22–53

[TL] V. Toledano Laredo, *A Kohno-Drinfeld theorem for quantum Weyl groups*, Duke Math. J. **112** (2002), no. 3, 421–451.

On the pure cohomology of multiplicative quiver varieties.

Kevin McGerty

(joint work with Tom Nevins)

The study of moduli spaces of classes of representations of quivers has a long and important history: A celebrated example of the structures such study has revealed is Lusztig's geometric construction of the canonical basis. The perverse sheaves which occur in Lusztig's canonical basis can be characterised microlocally in terms of the moduli spaces of representations of the preprojective algebra of Dlab and Ringel, and framed versions of these moduli spaces also arose in the work of Kronheimer and Nakajima on moduli spaces of instantons. Nakajima's later generalization of this work, and the rich theory that revealed have made quiver varieties central objects of geometric representation theory.

More recently, the study of moduli spaces of connections, and in particular the Deligne-Simpson problem, led Crawley-Boevey and Shaw [2] to introduce certain "multiplicative" analogues of quiver varieties: these are symplectic varieties which are moduli spaces for a multiplicative version of the preprojective algebra, and they include as special cases character varieties of Riemann surfaces with punctures.

Work of Boalch [1] has also show how such varieties arise as moduli spaces of irregular connections. In the present talk we describe recent joint work with T. Nevins on the pure cohomology of multiplicative quiver varieties, showing that it is generated by the Chern classes of tautological bundles. (Unlike the setting of Nakajima's quiver varieties, the full cohomology ring of a multiplicative quiver variety need not be pure, so the pure part is the largest subalgebra one can hope to generate by tautological classes.) Our main tool in establishing this result is the construction of a compactification of a multiplicative quiver variety to which the tautological bundles naturally extend. To find such a compactification, we study the moduli space of graded representations of a kind of "graded version" of the Crawley-Boevey-Shaw multiplicative preprojective algebra.

References

[1] P. Boalch, *Global Weyl groups and a new theory of multiplicative quiver varieties*, Geom. Topol. **19** (2016) no. 6, 3467–3536.

[2] W. Crawley-Boevey, P. Shaw, *Multiplicative preprojective algebras, middle convolution and the Deligne-Simpson problem*, Adv. Math. **201** (2006), 180–208.

The Coherent Satake Category

Harold Williams

(joint work with Sabin Cautis)

Let G be a complex reductive group. The Satake category $\mathcal{P}^{G(\mathcal{O})}(\mathrm{Gr}_G)$ of $G(\mathcal{O})$-equivariant perverse sheaves on the affine Grassmannian Gr_G plays a fundamental role in geometric representation theory and, in particular, the geometric Langlands program. Its structure is well understood via the geometric Satake equivalence of [11, 5, 13], which states that $\mathcal{P}^{G(\mathcal{O})}(\mathrm{Gr}_G)$ is monoidally equivalent to the representation category of the Langlands dual group $G^\vee$.

Like its constructible counterpart, the derived category of $G(\mathcal{O})$-equivariant coherent sheaves on Gr_G has a perverse t-structure [2] which is finite length and stable under convolution. This gives us the *coherent* Satake category $\mathcal{P}_{coh}^{G(\mathcal{O})}(\mathrm{Gr}_G)$. In contrast with $\mathcal{P}^{G(\mathcal{O})}(\mathrm{Gr}_G)$ this monoidal category is not semi-simple and is poorly understood.

In the joint work [3] we pursue the structure theory of the coherent Satake category. Our main results are that

(1) $\mathcal{P}_{coh}^{G(\mathcal{O})}(\mathrm{Gr}_G)$ admits renormalized r-matrices,
(2) $\mathcal{P}_{coh}^{G(\mathcal{O})}(\mathrm{Gr}_G)$ is rigid as a monoidal category (i.e. every object has a left and right dual), and
(3) $\mathcal{P}_{coh}^{GL_n(\mathcal{O})}(\mathrm{Gr}_{GL_n})$ is a monoidal cluster categorification.

By saying $\mathcal{P}_{coh}^{G(\mathcal{O})}(\mathrm{Gr}_G)$ admits renormalized r-matrices, we mean that for any nonzero objects $\mathcal{F}, \mathcal{G} \in \mathcal{P}_{coh}^{G(\mathcal{O})}(\mathrm{Gr}_G)$ there is a canonical nonzero map

$$\mathbf{r}_{\mathcal{F},\mathcal{G}} : \mathcal{F} * \mathcal{G} \to \mathcal{G} * \mathcal{F},$$

and that collectively these maps satisfy a list of several axioms. Roughly, these axioms say that although the maps $\mathbf{r}_{\mathcal{F},\mathcal{G}}$ do not constitute a braiding ($\mathcal{F} * \mathcal{G}$ and $\mathcal{G} * \mathcal{F}$ are not isomorphic in general), their failure to do so is controlled in a precise sense by certain inequalities. The construction and characteristic properties of the maps $\mathbf{r}_{\mathcal{F},\mathcal{G}}$ are similar to those of the renormalized r-matrices which appear in the representation theory of quantum loop algebras [7] and KLR (or quiver Hecke) algebras [8], hence our terminology.

The origin of renormalized r-matrices in $\mathcal{P}^{G(\mathcal{O})}_{coh}(\mathrm{Gr}_G)$ is the Beilinson-Drinfeld (BD) Grassmannian [1]. The BD Grassmannians of $\mathbb{A}^1$ and its powers are ind-schemes $\mathrm{Gr}_{G,\mathbb{A}^n}$ over $\mathbb{A}^n$ for $n > 0$. Collectively they form the prototypical example of a factorization space, meaning they satisfy certain compatibility conditions on their restrictions to diagonals and disjoint loci. Their role in the construction of renormalized r-matrices is closely related to their role in establishing the commutativity constraint used to prove the geometric Satake equivalence.

As in the setting of quantum loop algebras and KLR algebras [9], renormalized r-matrices together with certain properties implied by rigidity strongly constrain the behavior of real simple objects – that is, objects whose convolution square is again simple. On the other hand, the coherent Satake category has an abundance of real simple objects: for any integer ℓ and dominant coweight $\lambda^\vee$ an example is given by the restriction of the line bundle $\mathcal{O}(\ell)$ on Gr_G to the $G(\mathcal{O})$-orbit closure $\overline{\mathrm{Gr}}^{\lambda^\vee}_G$, shifted to lie in cohomological degree $-\frac{1}{2}\dim \overline{\mathrm{Gr}}^{\lambda^\vee}_G$.

Since $\mathcal{P}^{G(\mathcal{O})}_{coh}(\mathrm{Gr}_G)$ is finite length, the classes of simple objects form a basis in its Grothendieck ring $K^{G(\mathcal{O})}(\mathrm{Gr}_G)$. Following the framework developed in [10] to study the upper global or dual canonical bases of quantum groups [12, 6], results (i) and (ii) above provide a mechanism by which this basis – in particular, the subset formed by classes of real objects – may come to include the structure of a cluster algebra.

A cluster algebra is a ring with a partial basis of a certain form: it contains special elements (cluster variables) grouped into overlapping subsets (clusters) such that the monomials in any cluster are again basis elements [4]. Any two clusters are connected by a sequence of mutations, an operation which creates a new cluster by exchanging a single cluster variable for a new one.

For any particular G, to deduce that the classes of simple perverse coherent sheaves endow $K^{G(\mathcal{O})}(\mathrm{Gr}_G)$ with the structure of a cluster algebra (i.e. to dedue result (iii)) from results (i) and (ii) using the strategy of [10] one must show, by hand, that a finite subset of would-be cluster variables and mutations in $K^{G(\mathcal{O})}(\mathrm{Gr}_G)$ lift to suitable real simple objects and exact sequences. In [3] this is carried out fully in the case $G = GL_n$, however a similar result is anticipated in the case of a general reductive group.

References

[1] A. Beilinson and V. Drinfeld, *Quantization of Hitchin's fibration and Langlands' program*, Algebraic and geometric methods in mathematical physics **19**, 1–332.

[2] R. Bezrukavnikov, M. Finkelberg and I. Mirković, *Equivariant homology and K-theory of affine Grassmannians and Toda lattices*, Compositio Mathematica **141** (2005), 746–768.

[3] S. Cautis and H. Williams, *Cluster theory of the coherent Satake category*, arXiv:1801.08111 (2018).

[4] S. Fomin and A. Zelevinsky, *Cluster algebras. I. Foundations*, Journal of the American Mathematical Society **15** (2002), 497–529.

[5] V. Ginzburg, *Perverse sheaves on a loop group and Langlands duality*, arXiv:alg-geom/9511007 (1995).

[6] M. Kashiwara, *On crystal bases of the Q-analogue of universal enveloping algebras*, Duke Mathematical Journal **63** (1991), 465–516.

[7] M. Kashiwara, *On level-zero representations of quantized affine algebras*, Duke Mathematical Journal **112** (2002), 117–175.

[8] S.-J. Kang, M. Kashiwara, and M. Kim, *Symmetric quiver Hecke algebras and R-matrices of quantum affine algebras*, arXiv:1304.0323 (2013).

[9] S.-J. Kang, M. Kashiwara, M. Kim, and S.-j. Oh, *Simplicity of heads and socles of tensor products*, Compositio Mathematica **151** (2015), 377–396.

[10] S.-J. Kang, M. Kashiwara, M. Kim, and S.-j. Oh, *Monoidal categorification of cluster algebras*, Journal of the American Mathematical Society **31** (2018), 349–426.

[11] G. Lusztig, *Singularities, character formulas, and a q-analog of weight multiplicities*, Astérisque **101** (1983), 208–229.

[12] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, Journal of the American Mathematical Society **3** (1990), 447–498.

[13] I. Mirković and K. Vilonen, *Perverse sheaves on affine Grassmannians and Langlands duality*, Mathematical Research Letters **17** (2000), 13–24.

Grothendieck ring isomorphisms, cluster algebras and Kazhdan-Lusztig polynomials

David Hernandez

(joint work with Hironori Oya)

1. Main question

Let $\mathfrak{g}$ be a simple complex finite-dimensional Lie algebra and $\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$ its loop algebra. Drinfeld and Jimbo associated to each complex number $q \in \mathbb{C}^*$ a Hopf algebra $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ called a quantum group. Its representation theory is important, in particular from the point of view of quantum integrable systems. Though it has been intensively studied from several geometric, algebraic, combinatorial perspectives, some basic questions are still open, such as the dimension and character of simple finite-dimensional modules.

Inspired by the general framework of monoidal categorification of cluster algebras [HL1], we get new results in this direction.

2. Quantum Grothendieck ring

Let $\mathcal{C}$ be the monoidal category of finite-dimensional modules of $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$. We assume $q \in \mathbb{C}^*$ is not a root of unity. The category $\mathcal{C}$ is non semi-simple, and non braided. It has a very intricate structure. The simple objects in $\mathcal{C}$ have been classified by Chari-Pressley in terms of Drinfeld polynomials. The fundamental modules in $\mathcal{C}$ are distinguished simple modules whose classes generate the Grothendieck ring $K(\mathcal{C})$.

For simply-laced types, Nakajima [N] established a remarkable Kazhdan-Lusztig algorithm to compute the multiplicity $P_{m,m'}$ of a simple module $L(m')$ in a standard module $M(m)$, that is a tensor product of fundamental modules. Here m, m' belong to an ordered set of monomials $(\mathcal{M}, \leq)$ which parametrizes both simple and standard objects. This allows to calculate the classes $[L(m)]$ in terms of the classes of standard modules whose dimensions and characters are known. This gives an answer to the initial problem : the multiplicities $P_{m,m'}$ are proved to be the evaluation at $t = 1$ of analogues $P_{m,m'}(t)$ of Kazhdan-Lusztig polynomials. These polynomials are constructed from the structure of the quantum Grothendieck ring $K_t(\mathcal{C})$ which is a t-deformation of $K(\mathcal{C})$ in a certain quantum torus [N, VV]. The polynomials $P_{m,m'}(t)$ are defined as the transition matrix from a basis $[M(m)]_t$ obtained as a t-deformation in $K_t(\mathcal{C})$ of $[M(m)]$, to a basis $[L(m)]_t$ which is characterized as being canonical. Besides the coefficients of the polynomials $P_{m,m'}(t)$ are positive [N]. These results are based on the geometric realization of standard modules in terms of quiver varieties known only for simply-laced types (note however that geometric characters formulas for standard modules have been obtained in [HL3] for all types).

For general types, the question is still open. A conjectural answer was proposed by the speaker in [H] by giving a different construction of the quantum Grothendieck $K_t(\mathcal{C})$ and the corresponding polynomials $P_{m,m'}(t)$ and canonical basis $[L(m)]_t$ which can be extended to general types. The ambient quantum torus is not obtained from a convolution product for quiver varieties, but by considering properties of vertex operators appearing in the theory of q-characters of Frenkel-Reshetikhin [FR].

This leads to a general precise conjectural formula for the multiplicity of simple modules in standard modules :

Conjecture 1 (Hernandez, 2004)**.**

$$[M(m)] = [L(m)] + \sum_{m'<m} P_{m,m'}(1)[L(m')], \tag{1}$$

and the polynomials $P_{m,m'}(t)$ *are positive.*

The first point of the conjecture above means that $[L(m)]_t$ is $[L(m)]$ at $t = 1$.

In the ADE-cases, a submonoidal category $\mathcal{C}'$ of $\mathcal{C}$ is introduced in [HL2] so that $K_t(\mathcal{C}') \simeq \mathcal{U}_t(\mathfrak{n})$, the canonical bases being identified with the Lusztig dual canonical bases. The corresponding polynomials $P_{m,m'}(t)$ are the actual Kazhdan-Lusztig polynomials expressing dual PBW-bases in terms of the dual canonical bases.

3. Isomorphisms and cluster algebras

We denote by $\mathcal{C}_X$ the category $\mathcal{C}$ for $\mathfrak{g}$ of type X.

Theorem 2 (Hernandez-Oya, 2018). *There is a ring isomorphism*

$$K_t(\mathcal{C}_{B_n}) \simeq K_t(\mathcal{C}_{A_{2n-1}})$$

preserving the canonical bases. Moreover the polynomials $P_{m,m'}$ are positive in type B_n.

A crucial point in the proof is the input from cluster algebra theory. Indeed the quantum Grothendieck rings are subrings of quantum tori, respectively $\mathcal{Y}_{t,A_{2n-1}}, \mathcal{Y}_{t,B_n}$, which a priori are different. However, using cluster algebra structures introduced in [HL3], these quantum tori can be mutated in the sense of the theory of quantum cluster algebras. After a distinguished sequence of mutations that we introduce, the ambient quantum tori can be identified and the quantum Grothendieck rings are proved to be isomorphic :

$$\begin{tikzcd}
\mathcal{Y}_{t,A_{2n-1}} \arrow[r, squiggly, "mutation"] & \mathcal{Y}'_{t,A_{2n-1}} \arrow[r] & \mathcal{Y}_{t,B_n} \\
& K_t(\mathcal{C}_{A_{2n-1}}) \arrow[ul, hook] \arrow[u, hook] \arrow[r] & K(\mathcal{C}_{B_n}) \arrow[u, hook]
\end{tikzcd}$$

4. Application to the Kazhdan-Lusztig algorithm

Simultaneously, Kashiwara-Kim-Oh [KKO] introduced functors

$$\mathcal{C}_{B_n} \longleftarrow \text{KLR-algebra modules} \longrightarrow \mathcal{C}_{A_{2n-1}}$$

from a category of module of a quiver-Hecke (KLR) algebras of type A_∞ and the categories of finite-dimensional modules under study. The functors are obtained as Schur-Weyl dualities generalizing quantum affine Schur-Weyl dualities [CP]. It implies the existence of an isomorphism between *classical* Grothendieck rings

$$K(\mathcal{C}_{B_n}) \simeq K(\mathcal{C}_{A_{2n-1}})$$

preserving the basis of simple modules.

We prove that our isomorphism of quantum Grothendieck rings specializes at $t = 1$ to the isomorphism of [KKO] (note that, as far as the speaker knows, the isomorphism of quantum Grothendieck rings can not be deduced directly from the result of [KKO]). The following diagram is commutative :

$$\begin{tikzcd}
K_t(\mathcal{C}_{B_n}) \arrow[r, "{[HO]}"] \arrow[d, "t=1"'] & K_t(\mathcal{C}_{A_{2n-1}}) \arrow[d, "t=1"] \\
K(\mathcal{C}_{B_n}) \arrow[r, "{[KKO]}"] & K(\mathcal{C}_{A_{2n-1}})
\end{tikzcd}$$

Hence, combining all these results, from geometric representation theory, cluster algebras isomorphism and quiver Hecke functors, we obtain that for the category $\mathcal{C}_{B_n}$ the classes $[L(m)]_t$ are specialized to the classes $[L(m)]$:

Theorem 3 (Hernandez-Oya, 2018)**.** *The conjecture of* [H] *is true in type B : a Kazhdan-Lusztig algorithm gives the dimensions and characters of simple finite-dimensional modules.*

References

[CP] V. Chari and A. Pressley, *Quantum affine algebras and affine Hecke algebras*, Pacific J. Math. **174** (1996), no. 2, 295–326.

[FR] E. Frenkel and N. Reshetikhin, *The q-characters of representations of quantum affine algebras and deformations of W-Algebras*, Contemp. Math. **248** (1999), 163–205.

[H] D. Hernandez, *Algebraic approach to q,t-characters*, Adv. Math. **187**, 1–52 (2004).

[HL1] D. Hernandez and B. Leclerc *Cluster algebras and quantum affine algebras*, Duke Math. J. **154** (2010), no. 2, 265–341.

[HL2] D. Hernandez and B. Leclerc, *Quantum Grothendieck rings and derived Hall algebras*, J. Reine Angew. Math. **701**, 77–126 (2015).

[HL3] D. Hernandez and B. Leclerc, *A cluster algebra approach to q-characters of Kirillov-Reshetikhin modules*, J. Eur. Math. Soc. **18**, no. 5, 1113–1159 (2016).

[HO] D. Hernandez and H. Oya, *Quantum Grothendieck ring isomorphisms, cluster algebras and Kazhdan-Lusztig algorithm*, Preprint arXiv:1803.06754

[KKO] M. Kashiwara, M. Kim and S.-j. Oh, *Monoidal categories of modules over quantum affine algebras of type A and B*, Preprint arXiv:1710.06627

[N] H. Nakajima, *Quiver varieties and t-analogs of q-characters of quantum affine algebra*, Ann. of Math. (2) **160**, no. 3, 1057–1097 (2004).

[VV] M. Varagnolo and E. Vasserot, *Perverse sheaves and quantum Grothendieck rings*, Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000), Progr. Math. **210**, Birkhäuser Boston, Boston, MA, 345–365 (2003).

Supported by ERC with Grant Agreement number 647353 Qaffine.

The classification of Gelfand-Tsetlin modules and the Braverman-Finkelberg-Nakajima construction

Ben Webster

(joint work with Oded Yacobi, Alex Weekes)

One very challenging problem in the representation theory of the Lie algebra $\mathfrak{gl}_n$ is the classification of Gelfand-Tsetlin modules, that is, the finitely generated modules where the Gelfand-Tsetlin subalgebra Γ generated by the centers of the universal enveloping algebras $U(\mathfrak{gl}_1) \subset U(\mathfrak{gl}_2) \subset \cdots \subset U(\mathfrak{gl}_n)$ acts locally finitely. See [FGR, H] for a more general discussion of this problem.

The heart of our approach is the use of the generalized weight functors

$$W_{\mathfrak{m}}(M) = \{m \in M \mid \mathfrak{m}^N m = 0 \ \forall N \gg 0\}$$

for the different maximal ideals $\mathfrak{m} \in \mathrm{MaxSpec}(\Gamma)$. These functors are exact, and for any Gelfand-Tsetlin module $M \cong \bigoplus_{\mathfrak{m} \in \mathrm{MaxSpec}(\Gamma)} W_{\mathfrak{m}}(M)$. On very general grounds, the category of Gelfand-Tsetlin modules is thus controlled by the category whose objects are these functors, with morphisms given by natural transformations.

This category becomes much easier to analyze when we realize $U(\mathfrak{gl}_n)$ as a quantum Coulomb branch, in the sense of [BFN]. This allows us to identify the space of natural transformations between two weight functors as the homology of a Steinberg type space; in fact, when we consider the endomorphisms of an appropriate sum of weight spaces, it is precisely a completed weighted KLR algebra as defined in [W2], following the approach of [W1, KTWWY]. This fact allows us to complete the desired classification, and answer many questions about the structure of simple Gelfand-Tsetlin modules, by giving a finite dimensional algebra whose simple representations are in bijection with Gelfand-Tsetlin modules of a fixed weight. In particular, we identify the set of simple integrable Gelfand-Tsetlin modules with fixed central character with the dual canonical basis of the zero weight space of a tensor product of $U(\mathfrak{n}) \otimes (\mathbb{C}^n)^{\otimes n}$ of the inversal enveloping algebra of lower triangular matrices $\mathfrak{n}$ with n copies of the standard representation of $\mathfrak{sl}_n$. The other weight spaces of this tensor product correspond to similar module categories for W-algebras or orthogonal Gelfand-Tsetlin algebras [M].

This same approach can be applied to other quantized Coulomb branches, such as rational Cherednik algebras, as well as other principal Galois orders (as introduced in [H]). However, such algebras which are not Coulomb branches will require new calculations.

References

[BFN] Alexander Braverman, Michael Finkelberg, and Hiraku Nakajima, *Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II*, Preprint arXiv:1601.03586.

[FGR] Vyacheslav Futorny, Dimitar Grantcharov, and Luis Enrique Ramirez, *Singular Gelfand-Tsetlin modules of $\mathfrak{gl}(n)$*. Adv. in Math. **290** (2016), 453–482.

[FO] Vyacheslav Futorny and Serge Ovsienko, *Fibers of characters in Gelfand-Tsetlin categories*, Trans. Amer. Math. Soc. **366** (2014), no. 8, 4173–4208. MR 3206456

[H] Jonas T. Hartwig, *Principal Galois orders and Gelfand-Zeitlin modules*, Preprint arXiv:1710.04186.

[KTWWY] Joel Kamnitzer, Peter Tingley, Ben Webster, Alex Weekes, and Oded Yacobi, *On category $\mathcal{O}$ for affine Grassmannian slices and categorified tensor products*, Preprint arXiv:1806.07519.

[M] Volodymyr Mazorchuk, *Orthogonal Gelfand-Zetlin Algebras I*, Contributions to Algebra and Geometry **40** (1999), no. 2, 399–415.

[MV] Ian M. Musson and Michel Van den Bergh, *Invariants under tori of rings of differential operators and related topics*, Mem. Amer. Math. Soc. **136** (1998), no. 650, viii+85.

[W1] Ben Webster, *Koszul duality between Higgs and Coulomb categories $\mathcal{O}$*, Preprint arXiv:1611.06541.

[W2] ———, *Weighted Khovanov-Lauda-Rouquier algebras*, Preprint arXiv:1209.2463.

Nilpotent Slodowy slices and collapsing levels for W-algebras

Anne Moreau
(joint work with Tomoyuki Arakawa)

Let G be a complex connected simple algebraic group of adjoint type, with Lie algebra $\mathfrak{g}$. The *nilpotent Slodowy slice* associated with a nilpotent orbit $\mathbb{O}$ in $\mathfrak{g}$ and an $\mathfrak{sl}_2$-triple (e,h,f) of $\mathfrak{g}$ is the intersection

$$\mathscr{S}_{\mathbb{O},f} := \overline{\mathbb{O}} \cap \mathscr{S}_f,$$

where $\mathscr{S}_f \cong f+\mathfrak{g}^e$ is the Slodowy slice of the $\mathfrak{sl}_2$-triple (e,h,f). Nilpotent Slodowy slices associated with the principal nilpotent orbit $\mathbb{O}_{prin}$ and a subregular nilpotent element f_{subreg} for the types A, D, E have a simple singularity of the same type as G. More generally, the singularities of an arbitrary nilpotent Slodowy slices are understood best for Gf a minimal degeneration of $\mathbb{O}$ [13, 14, 10].

In another direction, it is known that nilpotent Slodowy slices appear as associated varieties of W-algebras [2, 5], as we explain below. With every vertex algebra V one associates a Poisson algebra R_V, called the *Zhu C_2-algebra*, as follows. Let $C_2(V)$ be the subspace of V spanned by the elements $a_{(-2)}b$, where $a, b \in V$, and set $R_V = V/C_2(V)$. Here $a_{(n)}$, $n \in \mathbb{Z}$, stands for the Fourier mode of the field $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}$ corresponding to $a \in V$. Then R_V has naturally a Poisson algebra structure and the *associated variety* of V is the affine Poisson variety $X_V := \operatorname{Specm} R_V$.

The universal W-algebra associated with $(\mathfrak{g}, f)$ at level $k \in \mathbb{C}$ is

$$\mathcal{W}^k(\mathfrak{g}, f) := H^0_{DS,f}(V^k(\mathfrak{g})),$$

where $H^{\bullet}_{DS,f}(?)$ is the BRST cohomology functor of the quantized Drinfeld-Sokolov reduction associated with $(\mathfrak{g}, f)$, [8, 11]. The associated variety $X_{\mathcal{W}^k(\mathfrak{g},f)}$ is isomorphic to the Slodowy slice $\mathscr{S}_f$, [9]. For any quotient V of the universal affine vertex algebra $V^k(\mathfrak{g})$ associated with $\mathfrak{g}$ at level k, $H^0_{DS,f}(V)$ is a quotient of $\mathcal{W}^k(\mathfrak{g}, f)$ provided that it is nonzero, and we have [2]: $X_{H^0_{DS,f}(V)} = X_V \cap \mathscr{S}_f$, which is a $\mathbb{C}^*$-invariant subvariety of $\mathscr{S}_f$.

In particular, if $X_{L_k(\mathfrak{g})}$ is the Zariski closure $\overline{\mathbb{O}}$ of some nilpotent orbit $\mathbb{O}$ in $\mathfrak{g}$, where $L_k(\mathfrak{g})$ is simple quotient vertex algebra of $V^k(\mathfrak{g})$, and if $f \in \overline{\mathbb{O}}$, then $X_{\mathcal{W}_k(\mathfrak{g},f)} \subset \mathscr{S}_{\mathbb{O},f}$ (conjecturally, the equality holds), where $\mathcal{W}_k(\mathfrak{g}, f)$ is the simple quotient vertex algebra of $\mathcal{W}_k(\mathfrak{g}, f)$. It is known that this occurs if k is an admissible level[1] by [2], or else if $\mathfrak{g}$ belongs to the Deligne exceptional series and if $k = -h^\vee/6 - 1 + n$, where $n \in \mathbb{Z}_{\geq 0}$ is such that $k \notin \mathbb{Z}_{\geq 0}$ by [4].

Definition 1. Let $\mathfrak{g}^\natural$ be the centralizer in $\mathfrak{g}$ of the $\mathfrak{sl}_2$-triple (e,h,f). We say that the level k is *collapsing* if $\mathcal{W}_k(\mathfrak{g}, f) \cong L_{k^\natural}(\mathfrak{g}^\natural)$, where $k^\natural$ is a complex number entirely determined by the level k and the reductive Lie algebra $\mathfrak{g}^\natural$. For example, if $\mathcal{W}_k(\mathfrak{g}, f) \cong \mathbb{C}$, then k is collapsing.

[1] A level k is *admissible* if $L_k(\mathfrak{g})$ is an admissible $\widehat{\mathfrak{g}}$-module which happens if and only if $k = -h^\vee + p/q$, with $p, q \in \mathbb{Z}_{>0}$, $(p, q) = 1$ and either $(q, r^\vee) = 1$ and $p \geq h^\vee$, or $(q, r^\vee) = r^\vee$ and $p \geq h$. Here $r^\vee$ is the lacety of $\mathfrak{g}$, and h (resp. $h^\vee$) is its (resp. dual) Coxeter number [12].

The notion of collapsing levels goes back to Adamović et al. [1]. There is a full classification of collapsing levels for the case where $f = f_{min}$ is a minimal nilpotent element[2] [1], as well as of pairs $(\mathfrak{g}, k)$ such that $\mathcal{W}_k(\mathfrak{g}, f_{min}) \cong \mathbb{C}$, [4, 1].

Although the minimal nilpotent case is well-understood, little or almost nothing is known for collapsing levels for non-minimal nilpotent elements. In this context, associated varieties and singularities of nilpotent Slodowy slices are proving to be very useful tools to find new collapsing levels. Let us outline the main idea.

It may happen that a nilpotent Slodowy slice $\mathscr{S}_{\mathbb{O},f}$ is isomorphic to a nilpotent orbit Zariski closure $\overline{\mathbb{O}^\natural}$ in the reductive Lie algebra $\mathfrak{g}^\natural$. Many examples can be exhibited from [13, 14, 10]. If so, and if $\overline{\mathbb{O}}$ and $\overline{\mathbb{O}^\natural}$ are the associated varieties of some affine vertex algebras $L_k(\mathfrak{g})$ and $L_{k^\natural}(\mathfrak{g}^\natural)$, respectively, one may ask whether k is a collapsing level. Naturally, the knowledge of the associated varieties is far from sufficient to ensure that given vertex algebras are isomorphic, but in some favourable cases we are able to conclude.

If k is an admissible level, and if V is either the simple affine vertex algebra $L_k(\mathfrak{g})$ or its Drinfeld-Sokolov reduction $H^0_{DS,f}(L_k(\mathfrak{g}))$, then its *normalized character* χ_V admits a "nice" asymptotic behavior [12]. Then we prove the following.

Theorem 2 (Arakawa-Moreau). *Assume that k and $k^\natural$ are admissible levels for $\mathfrak{g}$ and $\mathfrak{g}^\natural$, respectively, that $f \in X_{L_k(\mathfrak{g})}$ and that $\chi_{H^0_{DS,f}(L_k(\mathfrak{g}))}(\tau) \sim \chi_{L_{k^\natural}(\mathfrak{g}^\natural)}(\tau)$, as $\tau \downarrow 0$. Then $H^0_{DS,f}(L_k(\mathfrak{g})) \cong \mathcal{W}_k(\mathfrak{g}, f)$ is simple and k is a collapsing level.*

In this way we discovered a large number of collapsing levels for $f \neq f_{min}$.

Theorem 3 (Arakawa-Moreau). *Assume that $\mathfrak{g} = \mathfrak{sl}_n$ (we have similar results for $\mathfrak{so}_n$ and $\mathfrak{sp}_n$). Write $n = mq + s$, with $m, q > 0$ and $s \geq 0$. Assume that $(q, s) = 1$ and that the partition associated with the nilpotent orbit Gf is $(q^m, 1^s)$. Then $\mathcal{W}_{-n+n/q}(\mathfrak{sl}_n, f) \cong L_{-s+s/q}(\mathfrak{sl}_s)$ so that $k = -n + n/q$ is a collapsing level.*

Example 4. *Here are a few examples in the exceptional types (we do not list all the examples obtained), where we write for f the label in the Bala-Carter classification of its nilpotent orbit:*

$$\mathcal{W}_{-12+12/5}(E_6, A_4) \cong L_{-2+2/5}(A_1), \quad \mathcal{W}_{-18+18/3}(E_7, E_6) \cong L_{-2+2/13}(A_1),$$
$$\mathcal{W}_{-18+18/7}(E_7, (A_5)'') \cong L_{-4+4/7}(G_2), \quad \mathcal{W}_{-9+9/7}(F_4, B_3) \cong L_{-2+2/7}(A_1), \quad \textit{etc.}$$

All these examples are obtained by exploiting [2] *and* [10] *to detect the levels, and then applying Theorem 2 to prove that the isomorphisms indeed hold.*

It was observed by physicists that nilpotent Slodowy slices appear as the *Higgs branches* of Argyres-Douglas theories in four-dimensional $N = 2$ superconformal field theories (see e.g. [15]). The Higgs branch of a four-dimensional $N = 2$ superconformal field theory $\mathcal{T}$ is conjecturally [7] isomorphic to the associated variety of the vertex algebra corresponding to $\mathcal{T}$ via the 4d/2d-duality discovered in [6]. The reader is referred to [3] for a recent survey on this conjecture. Now, typical examples of vertex algebras corresponding to the Argyres-Douglas theories are the vertex algebras $L_{-h^\vee+h^\vee/q}(\mathfrak{g})$, $\mathcal{W}_{-h^\vee+h^\vee/q}(\mathfrak{g}, f)$, for $\mathfrak{g}$ of type

[2]The work of Adamović et al. [1] includes the case where $\mathfrak{g}$ is a simple affine Lie superalgebra.

A, D, E. Such exemples have occured in Theorem 3 and Example 4. As noticed in [15], a given Argyres-Douglas theory can be realized in several ways. Whenever this happens, it means that we have an isomorphisms between W-algebras. We believe that such a phenomenon essentially reflects that the level is collapsing, provided that one of the W-algebras in an affine vertex algebra. Actually, from the geometry of nilpotent Slodowy slices, it is sometimes possible to predict isomorphisms between non-trivial W-algebras. For example, we conjecture that $\mathcal{W}_{-7+7/3}(\mathfrak{sl}_7, f) \cong \mathcal{W}_{-4+4/3}(\mathfrak{sl}_4, f')$, where f belongs to the nilpotent orbit of $\mathfrak{sl}_7$ associated with the partition $(3, 2^2)$ and f' to that of $\mathfrak{sl}_4$ associated with (2^2).

Acknowledgments. Part of this work was done during author's stay at the RIMS of Kyoto as a visiting professor. The author wishes to express her thanks to this institute for its hospitality. She is supported by the ANR Project GeoLie Grant number ANR-15-CE40-0012 and the Labex CEMPI (ANR-11-LABX-0007-01).

References

[1] D. Adamović, V. Kac, F. Möseneder, P. Papi and O. Perše, *An application of collapsing levels to the representation theory of affine vertex algebras*, to appear in IMRN.

[2] T. Arakawa, *Associated varieties of modules over Kac-Moody algebras and C_2-cofiniteness of W-algebras*, IMRN **2015** (2015), n°22, 11605–11666.

[3] T. Arakawa, *Associated varieties and Higgs branches (a survey)*, Contemp. Math. **711** (2018), 37–44.

[4] T. Arakawa and A. Moreau, *Joseph ideals and lisse minimal W-algebras*, J. Inst. Math. Jussieu, **17** (2018), n°2, 397–417.

[5] T. Arakawa and A. Moreau, *On the irreducibility of associated varieties of W-algebras*, J. of Algebra, **500** (2018), 542–568.

[6] C. Beem, M. Lemos, P. Liendo, W. Peelaers, L. Rastelli, and B. C. van Rees, *Infinite chiral symmetry in four dimensions*, Comm. Math. Phys. **336** (2015), n°3, 1359–1433.

[7] C. Beem and Leonardo Rastelli, *Vertex operator algebras, Higgs branches, and modular differential equations*, preprint, arXiv:1707.07679[hep-th].

[8] B. Feigin and E. Frenkel, *Quantization of the Drinfel′d-Sokolov reduction*, Phys. Lett. B **246** (1990), n°1-2, 75–81.

[9] A. De Sole and V. Kac, *Finite vs affine W-algebras*, Jpn. J. Math. **1** (2006), n°1, 137–261.

[10] B. Fu, D. Juteau, P. Levy, and E. Sommers, *Generic singularities of nilpotent orbit closures*, Adv. Math. **305** (2017), 1–77.

[11] V. Kac, S. Roan, and M. Wakimoto, *Quantum reduction for affine superalgebras*, Comm. Math. Phys. **241** (2003), n°2-3, 307–342.

[12] V. Kac and M. Wakimoto, *Classification of modular invariant representations of affine algebras*, Adv. Ser. Math. Phys. **7**, 138–177. World Sci. Publ., Teaneck, NJ, 1989.

[13] H. Kraft and C. Procesi, *Minimal singularities in GL_n*, Invent. Math. **62** (1981), n°3, 503–515.

[14] H. Kraft and C. Procesi, *On the geometry of conjugacy classes in classical groups*, Comment. Math. Helv. **57** (1982), n°4, 539–602.

[15] J. Song, D. Xie, and W. Yan, *Vertex operator algebras of Argyres-Douglas theories from M5-branes*, J. High Energy Phys. **12** (2017), n°123, front matter+35.

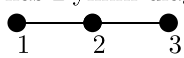
Quiver varieties and crystals in symmetrizable type via modulated graphs

PETER TINGLEY
(joint work with Vinoth Nandakumar)

Kashiwara and Saito [2] gave a geometric construction of the infinity crystal for any symmetric Kac-Moody algebra. The underlying set consists of the irreducible components of the nilpotent representation varieties of Lusztig's preprojective algebra from [3]. We generalize this to symmetrizable types by replacing Lusztig's preprojective algebra with a more general one due to Dlab and Ringel [1]. We also discuss extensions and relationships to other recent constructions. Much of this can be found in more detail in [5].

1.1. **Crystals.** The infinity crystal $B(\infty)$ is a combinatorial object which, among other things, parameterizes a basis for $U_q^-(\mathfrak{g})$, the lower triangular part of a quantized universal enveloping algebra. It consists of a set along with some partial permutations f_i, and is usually defined algebraically using Kashiwara's modification $\widetilde{F}_i$ of the Chevalley generators. One can also realize the same combinatorics in other ways. For example, for $\mathfrak{sl}_n$, the infinity crystal is realized using multisegments, and the related finite crystals are realized using Young-tableaux. These realizations can shed light on the underlying representation theory. For instance, the connection with Young tableaux gives a way to see how Littlewood-Richardson coefficients are given by counting Littlewood-Richardson tableaux (which was known before crystals were around, but it is still a nice example!)

1.2. **Kashiwara and Saito's construction.** We will mostly review this using the example of A_3 (so $\mathfrak{sl}_4$). This has Dynkin diagram



The *path algebra* has a basis consisting of paths, and multiplication is concatenation if possible, and 0 otherwise. The *preprojective algebra* Λ is the quotient by a generic relation between 2-step paths starting and ending at each vertex.

For $\mathbf{v} \in \mathbb{N}_I$ (I indexes nodes of the diagram), let $\Lambda(\mathbf{v})$ be the variety of nilpotent representations of Λ on $\oplus_I \mathbb{C}^{v_i}$, where each lazy path e_i projects onto the corresponding subspace. Kashiwara and Saito showed that $\sqcup_{\mathbf{v}} \mathrm{Irr}\Lambda(\mathbf{v})$ realizes $B(\infty)$ with the crystal operators f_i acting as follows: Fix $X \in \mathrm{Irr}\Lambda(\mathbf{v})$. Let S_i be the simple one-dimensional module over i. For generic $x \in X$ and a generic extension

$$0 \to S_i \to (V', x') \to (V, x) \to 0,$$

(V', x') is in a single irreducible component $Y \in \mathrm{Irr}\Lambda(V')$. Then $Y = f_i X$.

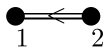
Proving this is not particularly hard! There is a second set of operators, f_i^*, defined in the same way, but adding S_i to the head instead of the socle. $B(\infty)$ also has a second set of operators, those twisted by Kashiwara's involution. In fact $B(\infty)$ can be characterized as a set with operators f_i and f_i^* satisfying some straightforward conditions, including that f_i, f_j^* commute except occasionally when $i = j$. Now it is just a matter of checking the conditions match exactly, which they do.

1.3. **Punch line.** About 8 years ago, two things bothered me about this:

- Every proof using this realization was only valid in symmetric type, even though most of the combinatorial results could be extended to symmetric type by “folding.” It was annoying to have this extra step!
- The name preprojective algebra. It seemed like it should mean something, but the people I asked didn’t know where it came from (other than that it is the term Lusztig used).

Then, at the 2010 Auslander Lectures, several talks mentioned “preprojective modules,” and even a preprojective algebra! A mystery was about to be resolved! Next, Hugh Thomas pointed me to a paper of Dlab and Ringel [1] studying preprojective algebras in symmetrizable types...which actually dates to before Lusztig’s work! Now for my punchline: Dlab and Ringel’s preprojective algebra (over infinite fields) can be used to realize $B(\infty)$ for any symmetrizable Kac-Moody algebra, with everything analogous to Kashiwara-Saito’s work. Let me explain a bit more.

1.4. **Symmetrizable quiver varieties.** Consider e.g. B_2, with Dynkin diagram



We need a field extension of degree 2 to handle the different length roots, so we use $\mathbb{R} \subset \mathbb{C}$. We also need a pair of bimodules for each edge, one for each orientation. We use ${}_{\mathbb{C}}M_{\mathbb{R}}$ and ${}_{\mathbb{R}}\overline{M}_{\mathbb{C}}$, where both M and $\overline{M}$ are copies of $\mathbb{C}$, and all actions are just multiplication. Here M corresponds to orienting the edge left to right.

The path algebra is replaced by the tensor algebra: a copy of $\mathbb{R}$ at vertex 1, a copy of $\mathbb{C}$ at 2, M, $\overline{M}$, $\overline{M} \otimes_{\mathbb{C}} M$, $M \otimes_{\mathbb{R}} \overline{M}$, $\overline{M} \otimes_{\mathbb{C}} M \otimes_{\mathbb{R}} \overline{M}$, etc. Multiplication is tensor product, or 0 if the ends don’t match up. The preprojective algebra is the quotient by a degree 2 element starting and ending at each vertex. Here we use

$$\tau_1 = 1 \otimes 1 \in \overline{M} \otimes_{\mathbb{C}} M \quad \text{and} \quad \tau_2 = 1 \otimes i + i \otimes 1 \in M \otimes_{\mathbb{R}} \overline{M}.$$

We need these to satisfy the condition that $z\tau_2 = \tau_2 z$ for any $z \in \mathbb{C}$, which is clearly true above. Dlab and Ringel use the canonical elements for non-degenerate bilinear forms, which implies the necessary condition in general.

The variety $\Lambda(\mathbf{v})$ (over $\mathbb{R}$) consists of representations of this algebra on $\mathbb{R}^{v_1} \oplus \mathbb{C}^{v_2}$. $\mathrm{Irr}\Lambda(V)$ is the set of irreducible components of $\Lambda(V)$, considered as a subset of

$$\mathrm{Hom}_{\mathbb{C}}(M \otimes_{\mathbb{R}} \mathbb{R}^{v_1}, \mathbb{C}^{v_2}) \oplus \mathrm{Hom}_{\mathbb{R}}(\overline{M} \otimes_{\mathbb{C}} \mathbb{C}^{v_2}, \mathbb{R}^{v_1}).$$

We mean this in a naive sense, not considering components of the abstract variety that do not have sufficient $\mathbb{R}$ points to appear. Now the definition of the crystal operators is like in the symmetric case, and so is the proof that this realizes $B(\infty)$!

1.5. **Affine $\mathfrak{sl}_2$.** This case is symmetric, but we have more freedom to choose the relations in the preprojective algebra than usual, and this gives non-isomorphic algebras that can still realize $B(\infty)$. Studying those may be interesting.

1.6. **Relation with folding.** Our construction is related to the method of folding by a diagram automorphism to realize symmetrizable types: Taking the fixed points of the composition of a diagram automorphism and a Galois automorphism of the same order acting on the symmetric quiver variety gives an instance of our construction. Well, conjecture I guess, this certainly hasn't been written up.

1.7. **Relation to work of Geiss, Leclerc and Schroer.** In [4] they consider a similar construction, but using nilpotent polynomial rings in place of our fields. They can then work over an algebraically closed field, a clear advantage. But there are disadvantages as well. For instance, they must restrict to only consider representations that admit a filtration of a fixed form. They also find irreducible components of lower dimension which they must ignore.

Anyway, they use $\mathbb{C}$ and $\mathbb{C}[\epsilon]/\epsilon^2$ where we use $\mathbb{R}$ and $\mathbb{C}$ above. It seems there should be a direct relationship with our construction, roughly as follows:

- Our construction works fine using the field extension $\mathbb{C}((\epsilon^2)) \subset \mathbb{C}((\epsilon))$.
- One would look for an "integral" part, defined over $\mathbb{C}[[\epsilon^2]] \subset \mathbb{C}[[\epsilon]]$.
- One would then set $\epsilon^2 = 0$, and hope nothing important collapses...

Much needs to be checked, and it is probably more complicated than this. Christof Geiss will discuss something kind of like this in his talk.

References

[1] V. Dlab, C. M. Ringel, The preprojective algebra of a modulated graph, Representation theory II, Lecture Notes in Math 832, Springer, Berlin 1980, 216-231.

[2] M. Kashiwara and Y. Saito, Geometric construction of crystal bases. *Duke Math. J.* **89** (1997), 9-36.

[3] G. Lusztig. Quivers, perverse sheaves, and quantized enveloping algebras. *J. Amer. Math. Soc.*, **4** (2):365–421, 1991.

[4] C. Geiss, B. Leclerc, J. Schroer. Quivers with relations for symmetrizable Cartan matrices IV: Crystal graphs and semicanonical functions. *Selecta Math.* (N.S.) **24** (2018) no. 4, 3283–3348.

[5] V. Nandakumar and P. Tingley. Quiver varieties and crystals in symmetrizable type via modulated graphs. *Math. Res. Lett.* **25** No. 1 (2018) pp 159–180.

Drinfeld-Gaitsgory-Vinberg interpolation Grassmannian and geometric Satake equivalence

Michael Finkelberg

(joint work with Vasily Krylov)

0.1. Let $\Lambda = \bigoplus_n \Lambda^n$ be the ring of symmetric functions equipped with the base of Schur functions s_λ. It also carries a natural coproduct. The classical Schubert calculus is the based isomorphism of the bialgebra Λ with $H^\bullet(\mathrm{Gr}, \mathbb{Z})$ (doubling the degrees) taking s_λ to the fundamental class of the corresponding Schubert variety σ_λ. Here Gr is the infinite Grassmannian $\mathrm{Gr} = \varinjlim \mathrm{Gr}(k, m) \simeq BU(\infty)$, and the coproduct on $H^\bullet(\mathrm{Gr}, \mathbb{Z})$ comes from the H-space structure on the classifying space $BU(\infty)$.

Here is a more algebraic geometric construction of the coproduct on $H^\bullet(\mathrm{Gr},\mathbb{Z})$. We have $H^{2n}(\mathrm{Gr},\mathbb{Z}) = H^{2n}(\overline{\mathrm{Sch}}_n,\mathbb{Z}) = H^{2n}_c(\overline{\mathrm{Sch}}_n,\mathbb{Z}) = H^{2n}_c(\mathrm{Sch}_n,\mathbb{Z})$ where $\mathrm{Sch}_n \subset \mathrm{Gr}$ (resp. $\overline{\mathrm{Sch}}_n \subset \mathrm{Gr}$) stands for the union of all n-dimensional (resp. $\leq n$-dimensional) Schubert cells (with respect to a fixed flag).

Recall the Calogero-Moser phase space $\mathcal{C}_n$: the space of pairs of $n\times n$-matrices (X,Y) such that $[X,Y]+\mathrm{Id}$ has rank 1, modulo the simultaneous conjugation of X,Y. The integrable system $\pi_n\colon \mathcal{C}_n \to \mathbb{A}^{(n)}$ takes (X,Y) to the spectrum of X. Wilson [10] has discovered the following two key properties of the Calogero-Moser integrable system:

(a) for $n_1+n_2=n$, a *factorization* isomorphism

$$\mathcal{C}_n \times_{\mathbb{A}^{(n)}} (\mathbb{A}^{(n_1)}\times\mathbb{A}^{(n_2)})_{\mathrm{disj}} \xrightarrow{\sim} (\mathcal{C}_{n_1}\times\mathcal{C}_{n_2}) \times_{(\mathbb{A}^{(n_1)}\times\mathbb{A}^{(n_2)})} (\mathbb{A}^{(n_1)}\times\mathbb{A}^{(n_2)})_{\mathrm{disj}}.$$

(b) For $x\in\mathbb{A}^1$, an isomorphism $\pi_n^{-1}(n\cdot x) \xrightarrow{\sim} \mathrm{Sch}_n$.

Now the desired coproduct

$$\Delta = \bigoplus_{n_1+n_2=n} \Delta_{n_1,n_2}\colon H^{2n}_c(\mathrm{Sch}_n,\mathbb{Z}) \to \bigoplus_{n_1+n_2=n} H^{2n_1}_c(\mathrm{Sch}_{n_1},\mathbb{Z})\otimes H^{2n_2}_c(\mathrm{Sch}_{n_2},\mathbb{Z})$$

is nothing but the cospecialization[1] morphism for the compactly supported cohomology of the fibers of π_n restricted to the subfamily $\pi_n^{-1}(n_1\cdot x + n_2\cdot y) \subset \mathcal{C}_n$ (from the fibers over the diagonal $x=y$ to the off-diagonal fibers $x\neq y$), cf. [5].

0.2. Given a reductive complex algebraic group G, Schieder [9] constructed a bialgebra $\mathcal{A}$ playing the role of $\bigoplus_n H^{2n}_c(\mathrm{Sch}_n,\mathbb{C})$ for the affine Grassmannian Gr_G in place of Gr. In order to explain his construction, we set up the basic notations for G and Gr_G.

We fix a Borel and a Cartan subgroup $G\supset B\supset T$, and denote by W the Weyl group of (G,T). Let N denote the unipotent radical of the Borel B, and let N_- stand for the unipotent radical of the opposite Borel B_-. Let Λ (resp. $\Lambda^\vee$) be the coweight (resp. weight) lattice, and let $\Lambda^+\subset\Lambda$ (resp. $\Lambda^{\vee+}\subset\Lambda^\vee$) be the cone of dominant coweights (resp. weights). Let also $\Lambda_+\subset\Lambda$ (resp. $\Lambda^\vee_+\subset\Lambda^\vee$) be the submonoid spanned by the simple coroots (resp. roots) $\alpha_i,\ i\in I$ (resp. $\alpha^\vee_i,\ i\in I$). We denote by $G^\vee\supset T^\vee$ the Langlands dual group, so that Λ (resp. $\Lambda^\vee$) is the weight (resp. coweight) lattice of $G^\vee$.

Let $\mathcal{O}$ denote the formal power series ring $\mathbb{C}[[z]]$, and let $\mathcal{K}$ denote its fraction field $\mathbb{C}((z))$. The affine Grassmannian $\mathrm{Gr}_G = G_\mathcal{K}/G_\mathcal{O}$ is an ind-projective scheme, the union $\bigsqcup_{\lambda\in\Lambda^+}\mathrm{Gr}^\lambda_G$ of $G_\mathcal{O}$-orbits. The closure of Gr^λ_G is a projective variety $\overline{\mathrm{Gr}}^\lambda_G = \bigsqcup_{\mu\leq\lambda}\mathrm{Gr}^\mu_G$. The fixed point set Gr^T_G is naturally identified with the coweight lattice Λ; and $\mu\in\Lambda$ lies in Gr^λ_G iff $\mu\in W\lambda$.

For a coweight $\nu\in\Lambda=\mathrm{Gr}^T_G$, we denote by $S_\nu\subset\mathrm{Gr}_G$ (resp. $T_\nu\subset\mathrm{Gr}_G$) the orbit of $N(\mathcal{K})$ (resp. of $N_-(\mathcal{K})$) through ν. The intersections $S_\nu\cap\overline{\mathrm{Gr}}^\lambda_G$ (resp. $T_\nu\cap\overline{\mathrm{Gr}}^\lambda_G$) are the *attractors* (resp. *repellents*) of $\mathbb{C}^\times$ acting via its homomorphism 2ρ to the Cartan torus T acting on $\overline{\mathrm{Gr}}^\lambda_G$: $S_\nu\cap\overline{\mathrm{Gr}}^\lambda_G = \{x\in\overline{\mathrm{Gr}}^\lambda_G : \lim_{c\to 0}2\rho(c)\cdot x = \nu\}$ and

[1] terminology of [9, 6.2.7].

$T_\nu \cap \overline{\mathrm{Gr}}^\lambda_G = \{x \in \overline{\mathrm{Gr}}^\lambda_G : \lim_{c\to\infty} 2\rho(c)\cdot x = \nu\}$. Going to the limit $\mathrm{Gr}_G = \lim_{\lambda\in\Lambda^+} \overline{\mathrm{Gr}}^\lambda_G$, S_ν (resp. T_ν) is the attractor (resp. repellent) of ν in Gr_G. The closure $\overline{S}_\nu$ is the union $\bigsqcup_{\mu\le\nu} S_\mu$, while $\overline{T}_\nu = \bigsqcup_{\mu\ge\nu} T_\mu$.

Definition 1. (a) For $\theta \in \Lambda_+$ we denote by Sch_θ (resp. $\overline{\mathrm{Sch}}_\theta$) the intersection $S_\theta \cap T_0$ (resp. $\overline{S}_\theta \cap \overline{T}_0$).[2] It is equidimensional of dimension $\langle\rho^\vee,\theta\rangle$.

(b) We set $\mathcal{A}_\theta := H_c^{\langle 2\rho^\vee,\theta\rangle}(\mathrm{Sch}_\theta,\mathbb{C}) = H_c^{\langle 2\rho^\vee,\theta\rangle}(\overline{\mathrm{Sch}}_\theta,\mathbb{C})$, and $\mathcal{A} := \bigoplus_{\theta\in\Lambda_+}\mathcal{A}_\theta$.

Given a smooth curve X and $\theta \in \Lambda_+$, the open zastava space $\overset{\circ}{Z}{}^\theta$ (see e.g. [2]) is equipped with the projection $\pi_\theta\colon \overset{\circ}{Z}{}^\theta \to X^\theta$ to the degree θ configuration space of X. It enjoys the factorization property, and for any $x \in X$, we have a canonical isomorphism $\pi_\theta^{-1}(\theta\cdot x) \xrightarrow{\sim} \mathrm{Sch}_\theta$. Given $\theta_1,\theta_2 \in \Lambda_+$ such that $\theta_1+\theta_2=\theta$, the coproduct $\Delta_{\theta_1,\theta_2}\colon \mathcal{A}_\theta \to \mathcal{A}_{\theta_1}\otimes\mathcal{A}_{\theta_2}$ is defined just like in 0.1 via the cospecialization morphism for the subfamily $\pi_\theta^{-1}(\theta_1\cdot x+\theta_2\cdot y)$.

To construct the product $\mathsf{m}\colon \bigoplus_{\theta_1+\theta_2=\theta}\mathcal{A}_{\theta_1}\otimes\mathcal{A}_{\theta_2}\to\mathcal{A}_\theta$ we need the Drinfeld-Gaitsgory interpolation $\widetilde{\mathrm{Sch}}_\theta \to \mathbb{A}^1$ [4] constructed with respect to the $\mathbb{C}^\times$-action on $\overline{\mathrm{Sch}}_\theta$ arising from the cocharacter 2ρ of T. The key property of $\widetilde{\mathrm{Sch}}_\theta \to \mathbb{A}^1$ is that the fibers over $a \neq 0$ are all isomorphic to $\overline{\mathrm{Sch}}_\theta$, while the zero fiber $(\widetilde{\mathrm{Sch}}_\theta)_0$ is isomorphic to the disjoint union $\bigsqcup_\lambda \overline{\mathrm{Sch}}{}^{+,\lambda}_\theta \times \overline{\mathrm{Sch}}{}^{-,\lambda}_\theta$. Here λ (a coweight in Λ_+ such that $\lambda\le\theta$) runs through the set of $\mathbb{C}^\times$-fixed points of $\overline{\mathrm{Sch}}_\theta$, and $\overline{\mathrm{Sch}}{}^{+,\lambda}_\theta$ (resp. $\overline{\mathrm{Sch}}{}^{-,\lambda}_\theta$) stands for the corresponding attractor (resp. repellent). It is easy to see that $H_c^{\langle 2\rho^\vee,\theta\rangle}((\widetilde{\mathrm{Sch}}_\theta)_0,\mathbb{C}) = \bigoplus_{\theta_1+\theta_2=\theta} H_c^{\langle 2\rho^\vee,\theta_1\rangle}(\mathrm{Sch}_{\theta_1},\mathbb{C})\otimes H_c^{\langle 2\rho^\vee,\theta_2\rangle}(\mathrm{Sch}_{\theta_2},\mathbb{C})$, and the desired product m is nothing but the cospecialization morphism for the compactly supported cohomology of the fibers of the Drinfeld-Gaitsgory family.

Schieder conjectured that the bialgebra $\mathcal{A}$ is isomorphic to the universal enveloping algebra $U(\mathfrak{n}^\vee)$ of $\mathrm{Lie}(N^\vee)$ where $N^\vee \subset B^\vee \subset G^\vee$ is the unipotent radical of Borel subgroup of $G^\vee$. The goal of the present work is a proof of Schieder's conjecture.

0.3. In order to produce an isomorphism $\mathcal{A} \xrightarrow{\sim} U(\mathfrak{n}^\vee)$, we construct an action of $\mathcal{A}$ on the geometric Satake fiber functor. More precisely, we denote by $r_{\nu,+}$ (resp. $r_{\nu,-}$) the locally closed embedding $S_\nu \hookrightarrow \mathrm{Gr}_G$ (resp. $T_\nu \hookrightarrow \mathrm{Gr}_G$). We also denote by $\iota_{\nu,+}$ (resp. $\iota_{\nu,-}$) the closed embedding of the point ν into S_ν (resp. into T_ν).

According to [1, 4], there is a canonical isomorphism of functors $\iota^*_{\nu,-}r^!_{\nu,-} \simeq \iota^!_{\nu,+}r^*_{\nu,+}\colon D^b_{G_\mathcal{O}}(\mathrm{Gr}_G) \to D^b(\mathrm{Vect})$. For a sheaf $\mathcal{P} \in D^b_{G_\mathcal{O}}(\mathrm{Gr}_G)$, its hyperbolic stalk at ν is defined as $\Phi_\nu(\mathcal{P}) := \iota^*_{\nu,-}r^!_{\nu,-}\mathcal{P} \simeq \iota^!_{\nu,+}r^*_{\nu,+}\mathcal{P}$. According to [8], for $\mathcal{P} \in \mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G)$, the hyperbolic stalk $\Phi_\nu(\mathcal{P})$ is concentrated in degree $\langle 2\rho^\vee,\nu\rangle$, and there is a canonical direct sum decomposition $H^\bullet(\mathrm{Gr}_G,\mathcal{P}) = \bigoplus_{\nu\in\Lambda}\Phi_\nu(\mathcal{P})$. Moreover, the abelian category $\mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G)$ is monoidal with respect to the convolution operation $\star$, and the functor $H^\bullet(\mathrm{Gr}_G,-)\colon (\mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G),\star) \to (\mathrm{Vect},\otimes)$

[2] Here Sch stands for Schieder.

is a fiber functor identifying $(\mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G), \star)$ with the tensor category $\mathrm{Rep}(G^\vee)$ (geometric Satake equivalence).

We define a morphism of functors $\mathcal{A}_\theta \otimes \Phi_\nu \to \Phi_{\nu+\theta}$. To this end (and also in order to check various tensor compatibilities) we consider the *Drinfeld-Gaitsgory-Vinberg interpolation Grassmannian*: a relative compactification $\mathrm{VinGr}_G^{\mathrm{princ}}$ of the Drinfeld-Gaitsgory interpolation $\widetilde{\mathrm{Gr}}_G \to \mathbb{A}^1$. We also consider an extended version $\mathrm{VinGr}_G \to T_{\mathrm{ad}}^+ := \mathrm{Spec}\,\mathbb{C}[\Lambda_+^\vee]$ and its version VinGr_{G,X^n} for the Beilinson-Drinfeld Grassmannian. It was implicit already in Schieder's work, and it was made explicit by D. Gaitsgory (private communication, cf. an earlier work [6]). We believe it is a very interesting object in its own right. For example, let $\omega \in \Lambda^+$ be a minuscule dominant coweight. Then the Schubert variety Gr_G^ω is isomorphic to a parabolic flag variety G/P_ω, and the corresponding subvariety VinGr_G^ω of VinGr_G is isomorphic to Brion's degeneration of Δ_{G/P_ω} in $\mathrm{Hilb}(G/P_\omega \times G/P_\omega) \times T_{\mathrm{ad}}^+$ [3, §3].

Remark. (a) By the geometric Satake equivalence, for $\mathcal{P} \in \mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G)$, the cohomology $H^\bullet(\mathrm{Gr}_G, \mathcal{P})$ is equipped with an action of $U(\mathfrak{g}^\vee)$. For example, the action of the Cartan subalgebra $U(\mathfrak{t}^\vee) \subset U(\mathfrak{g}^\vee)$ comes from the grading $H^\bullet(\mathrm{Gr}_G, \mathcal{P}) = \bigoplus_{\nu \in \Lambda} \Phi_\nu(\mathcal{P})$. The action of $U(\mathfrak{n}^\vee)$ comes from the geometric action of the Schieder bialgebra $\mathcal{A}$ on the geometric Satake fiber functor, and the isomorphism $\mathcal{A} \xrightarrow{\sim} U(\mathfrak{n}^\vee)$. Finally, the action of $U(\mathfrak{n}_-^\vee)$ is conjugate to the action of $U(\mathfrak{n}^\vee)$ with respect to the Lefschetz bilinear form on $H^\bullet(\mathrm{Gr}_G, \mathcal{P})$.

(b) By construction, the Schieder algebra $\mathcal{A}$ comes equipped with a basis (fundamental classes of irreducible components of Sch_θ). On the other hand, $U(\mathfrak{n}^\vee)$ is equipped with the *semicanonical basis* [7].[3] In the simplest example when $G = SL(2)$, $\mathcal{A}$ is $\mathbb{N}$-graded, and each graded component $\mathcal{A}_n$ is one-dimensional with the basis vector e_n; one can check $e_n e_m = \binom{n+m}{n} e_{n+m}$. Hence the two bases match under the isomorphism $\mathcal{A} \xrightarrow{\sim} U(\mathfrak{n}^\vee)$. However, J. Kamnitzer has checked recently that for $G = SL(6)$ the two bases do not match. Thus the Higgs branch realization [7] of $U(\mathfrak{n}^\vee)$ is different from the Coulomb branch realization [9] of $U(\mathfrak{n}^\vee)$.

References

[1] T. Braden, *Hyperbolic localization of intersection cohomology*, Transform. Groups **8** (2003), 209–216.

[2] A. Braverman, M. Finkelberg, D. Gaitsgory and I. Mirković, *Intersection cohomology of Drinfeld's compactifications*, Selecta Math. **8** (2002), no. 3, 381–418. *Erratum*: Selecta Math. **10** (2004), no. 3, 429–430.

[3] M. Brion, *Group completions via Hilbert schemes*, J. Algebraic Geom. **12** (2003), no. 4, 605–626.

[4] V. Drinfeld, D. Gaitsgory, *On a theorem of Braden*, Transform. Groups **19** (2014), 313–358.

[5] M. Finkelberg, V. Ginzburg, *Calogero-Moser space and Kostka polynomials*, Adv. Math. **172** (2002), no. 1, 137–150.

[6] D. Gaitsgory, D. Nadler, *Spherical varieties and Langlands duality*, Mosc. Math. J. **10** (2010), no. 1, 65–137.

[3] constructed under the assumption that G is simply laced.

[7] G. Lusztig, *Semicanonical bases arising from enveloping algebras*, Adv. Math. **151** (2000), no. 2, 129–139.
[8] I. Mirković, K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. (2) **166** (2007), no. 1, 95–143.
[9] S. Schieder, *Monodromy and Vinberg fusion for the principal degeneration of the space of G-bundles*, arXiv:1701.01898.
[10] G. Wilson, *Collisions of Calogero-Moser particles and an adelic Grassmannian*, Invent. Math. **133** (1998), 1–41.

Supports for linear degenerations of flag varieties

Markus Reineke
(joint work with Xin Fang)

1. Supports

Suppose given a proper algebraic map $f : X \to Y$ between irreducible complex algebraic varieties, with X being smooth. Then the variation of cohomology of the fibres $X_y = f^{-1}(y)$ is encoded in the complex of constructible sheaves $Rf_*\mathbb{Q}_X$, in particular the cohomology of the fibres of f is given by the stalks of its cohomology sheaves,

$$H^*(X_y) = \mathcal{H}^*_y(Rf_*\mathbb{Q}_X).$$

By the Decomposition Theorem, $Rf_*\mathbb{Q}_X$ is isomorphic to a finite direct sum of shifts of intersection cohomology complexes,

$$Rf_*\mathbb{Q}_X \simeq \bigoplus_{i=1}^{n} \mathbf{IC}(\overline{S_i}, L_i)[d_i],$$

where the S_i are certain locally closed subvarieties of Y, the L_i are local systems on S_i, and $d_i \in \mathbb{Z}$. We call $\{\overline{S_1}, \ldots, \overline{S_n}\}$ the *set of supports* of f. This is thus an invariant of the map f, which encodes the variation of cohomology along its fibres in terms of the singularities (or, more precisely, their local intersection cohomology) of the subvarieties $\overline{S_i}$.

For an introduction to the concept of supports and an overview of known results, see [4]. For example, if f is a semismall map, the supports are precisely the so-called relevant strata, and if f is flat with irreducible d-dimensional fibres, all supports have codimension less than d (Goresky-Macpherson). For certain abelian fibrations, Ngo's support theorem describes the supports, and a result of Migliorini and Shende localizes the supports inside so-called higher discriminant loci.

It is an interesting problem to describe the supports for proper maps featuring in contexts of Geometric Representation Theory, which we will do in the following for the flat family of so-called linear degenerations of flag varieties.

2. Linear degenerations

Let $n \geq 1$, and let V be an $(n+1)$-dimensional complex vector space. Then the flag variety $\mathrm{SL}(V)/B$ is realized as the closed subset

$$\mathrm{Fl}(V) = \{U_1 \subset U_2 \subset \ldots \subset U_n \subset V \mid \dim U_i = i\}$$

of the product $\mathrm{Gr} = \prod_{i=1}^{b} \mathrm{Gr}_i(V)$ of Grassmannians of V.

We would like to degenerate $\mathrm{Fl}(V)$ inside Gr by relaxing the containment relation between the subspaces constituting the flag. For a tuple $f_* = (f_1, \ldots, f_{n-1}) \in \mathrm{Hom}(V,V)^{n-1}$, we define

$$\mathrm{Fl}^{f_*}(V) = \{(U_1, \ldots, U_n) \in \mathrm{Gr} \,|\, f_i(U_i) \subset U_{i+1}\}$$

as the *f_*-degenerate flag variety.* For example, if all f_i are isomorphisms, we recover Fl, and in the most degenerate case $f_i = 0$, we find Gr. We call $\mathbf{r}(f_*) = (\mathrm{rank}(f_{j-1} \circ \ldots \circ f_i))_{i<j}$ the *rank tuple* of f_* and note that $\mathrm{Fl}^{f_*}(V)$ only depends on $\mathbf{r} = \mathbf{r}(f_*)$; thus we can denote it by $\mathrm{Fl}^{\mathbf{r}}(V)$. The following is proved in [1]:

- The degenerate flag variety $\mathrm{Fl}^{f_*}(V)$ is irreducible of dimension $n(n+1)/2$ if and only if $\mathbf{r}(f_*) \geq \mathbf{r}^1 = (n+1+i-j)_{i<j}$.
- In this case, $\mathrm{Fl}^{f_*}(V)$ is a normal locally complete intersection variety, admitting an affine paving and an algebraic group action with a dense orbit.
- $\mathrm{Fl}^{\mathbf{r}^1}$ is isomorphic to E. Feigin's degenerate flag variety.
- The degenerate flag variety $\mathrm{Fl}^{f_*}(V)$ is of dimension $n(n+1)/2$ if and only if $\mathbf{r}(f_*) \geq \mathbf{r}^2 = (n+i-j)_{i<j}$, in which case it is a locally complete intersection variety admitting an affine paving.
- The number of irreducible components of $\mathrm{Fl}^{\mathbf{r}^2}(V)$ equals the n-th Catalan number.

Define $\mathcal{U} \subset \mathrm{Hom}(V,V)^{n-1}$ as the open subset of all tuples f_* such that $\mathbf{r}(f_*) \geq \mathbf{r}^1$, on which the group $G = \mathrm{GL}(V)^n$ acts naturally via base change; the orbits $\mathcal{O}(\mathbf{r})$ for this action can be indexed by rank tuples $\mathbf{r} \geq \mathbf{r}^1$. We define the linear degeneration family

$$\mathcal{F}l = \{(f_*, U_*) \in \mathcal{U} \times \mathrm{Gr} \,|\, f_i(U_i) \subset U_{i+1}\} \subset \mathcal{U} \times \mathrm{Gr},$$

which admits G-equivariant maps

$$\mathrm{Gr} \xleftarrow{p} \mathcal{F}l \xrightarrow{\pi} \mathcal{U}$$

such that p is open in a homogeneous bundle (and thus identifying $\mathcal{F}l$ as smooth and irreducible), and π is projective and flat with irreducible fibres $\pi^{-1}(f_*) = \mathrm{Fl}^{f_*}(V)$ by the above theorem. We want to determine the supports of π.

3. Main result

To a *Motzkin path* of length n, given by a tuple $x_* = (0 = x_0, x_1, \ldots, x_n = 0)$ such that $x_i \geq 0$ and $x_i - x_{i-1} \in \{-1, 0, 1\}$, we associate a rank tuple $\mathbf{r}(x_*)$, called a *Motzkin rank tuple*, by

$$\mathbf{r}(x_*)_{im} = n + 1 - \max_{i \leq j \leq k \leq l \leq m} (x_{k-1} + x_k - x_{j-1} - x_l).$$

The main result is:

The set of supports of the family $\pi : \mathcal{F}l \to \mathcal{U}$ is the set of Motzkin rank tuples $\overline{\mathcal{O}(\mathbf{r}(x_*))}$. In other words, we have

$$R\pi_* \mathbb{Q}_{\mathcal{F}l} \simeq \bigoplus_{x_*} \mathbf{IC}(\overline{\mathcal{O}(\mathbf{r}(x_*))}) \otimes V(x_*)^\bullet$$

for certain (unknown) graded $\mathbb{Q}$-vector spaces $V(x_*)^\bullet$ encoding multiplicities and shifts.

4. Techniques

To prove the main result, we use G. Lusztig's geometric realization of quantized enveloping algebras and their canonical bases [3]. Denoting by Q the linearly oriented type A_n quiver, for any dimension type $\mathbf{d} \in \mathbb{N}^n$ we consider the variety $R_{\mathbf{d}}(Q)$ parametrizing $\mathbf{d}$-dimensional representations of Q, on which an algebraic group $G_{\mathbf{d}}$ acts naturally. The space

$$\bigoplus_{\mathbf{d}} K_0(\mathrm{Perv}^{G_{\mathbf{d}}}(R_{\mathbf{d}}(Q))) \otimes_{\mathbb{Z}} \mathbb{Q}(v)$$

of suitably extended Grothendieck groups of the categories of $G_{\mathbf{d}}$-equivariant perverse sheaves on $R_{\mathbf{d}}(Q)$ carries a natural $\mathbb{N}^n$-graded associative convolution product $*$, and a natural basis $\mathcal{B}$ given by intersection cohomology complexes along the (finitely many) orbits of each $R_{\mathbf{d}}(Q)$. The resulting based algebra is isomorphic to $\mathcal{U}_v^+(\mathfrak{sl}_{n+1})$, with Chevalley generators E_i subject to quantized Serre relations, together with Lusztig's canonical basis.

The relation to linear degenerations of flag varieties is provided by the observation that $\mathcal{U} \subset R_{\mathbf{d}}(Q)$ for $\mathbf{d} = (n+1, \ldots, n+1)$, that the canonical basis elements $b(\mathbf{r})$ corresponding to intersection cohomology complexes on this variety are naturally indexed by rank tuples $\mathbf{r}$, and that $\overline{\mathcal{O}(\mathbf{r})}$ is a support of $\pi : \mathcal{F}l \to \mathcal{U}$ if and only if $b(\mathbf{r})$ appears with non-zero coefficient in the expansion of the monomial

$$E_1^{(n)} \ldots E_n^{(1)} E_1^{(1)} \ldots E_n^{(n)} = \sum_{\mathbf{r} \geq \mathbf{r}^1} \gamma(\mathbf{r}) b(\mathbf{r})$$

in the canonical basis. Using the Knight-Zelevinsky multisegment duality [2] and explicit degree estimates for the coefficients $\gamma(\mathbf{r})$, the above description of supports is achieved.

References

[1] G. Cerulli Irelli, X. Fang, E. Feigin, G. Fourier, M. Reineke, *Linear degenerations of flag varieties* , Math. Z. **287** (2017), no. 1–2, 615–654.
[2] H. Knight, A. Zelevinsky, *Representation of quivers of type A and the multisegment duality*, Adv. Math **117** (1996) , 273–293.
[3] G. Lusztig, *Quivers, perverse sheaves, and quantized enveloping algebras* , J. Amer. Math. Soc. **4** (1991), no. 2, 365–421.
[4] L. Migliorini, *Support theorems for algebraic maps* , Milan J. Math. Vol. **83** (2015), 21–45.

Elliptic quantum groups

Valerio Toledano Laredo
(joint work with Sachin Gautam)

The dynamical Yang–Baxter equations (DYBE) were introduced by Felder in 1994 to circumvent the fact that the Yang–Baxter equations do not admit elliptic solutions outside of type A_n [3]. Felder showed that the classical DYBE possess elliptic solutions for all Lie types, proposed that elliptic quantum groups be the quantum groups associated to elliptic solutions of the quantum YBE via the RTT construction of Faddeev–Reshetikhin–Takhtajan, and gave an elliptic solution of the quantum DYBE in type A.

Jimbo–Konno–Odake–Shiraishi [11] and Etingof–Schiffmann [1] later showed that elliptic solutions of the quantum DYBE exist for any complex semisimple Lie algebra $\mathfrak{g}$ and finite–dimensional representation V of the quantum loop algebra $U_q(L\mathfrak{g})$ of $\mathfrak{g}$. This led to a tentative definition of the elliptic quantum group $E_{\hbar,\tau}(\mathfrak{g})$ as the algebra $U_q(L\mathfrak{g})$, where $q = \exp(\pi\iota\hbar)$, but with a different, dynamical coproduct obtained by twisting the standard coproduct of $U_q(L\mathfrak{g})$ by the corresponding fusion operator.

A more intrinsic presentation was obtained by Kojima–Konno [4] for $\mathfrak{g} = \mathfrak{sl}_n$ and, more recently, by Farghly–Konno–Oshima [2] for any complex semisimple $\mathfrak{g}$, under the assumption that the elliptic parameter $p = \exp(2\pi\iota\tau)$ is formal, so that the odd theta function $\vartheta(z) = z^{1/2}\prod_{n>0}(1 - zp^n)(1 - z^{-1}p^{n-1})(1 - p^n)^{-2}/2\pi\iota$ is regarded as a formal power series in p with coefficients in $\mathbb{C}[z^{\pm 1}]$. The presentation is given in terms of Drinfeld full currents, or equivalently integer moded generators, which satisfy commutation relations involving ϑ.

In joint work with Sachin Gautam [10], we propose a definition of the category of finite–dimensional representations of $E_{\hbar,\tau}(\mathfrak{g})$ which is intrinsic, uniform for all Lie types, and valid for numerical values of p. Our presentation of $E_{\hbar,\tau}(\mathfrak{g})$ is given in terms of Drinfeld half rather than full currents, which are elliptic rather than distributional functions of a complex parameter, and recovers Farghly–Konno–Oshima's presentation if p is regarded as formal. Interestingly, however, our presentation cannot be given by canonical generators and relations for p numerical since $\vartheta(z)$ has essential singularities at $z = 0, \infty$ in that case.

We classify simple objects in $\mathrm{Rep}_{\mathrm{fd}}(E_{\hbar,\tau}(\mathfrak{g}))$ in terms of elliptic Drinfeld polynomials. Our classification is new even for $\mathfrak{g} = \mathfrak{sl}_2$, and is analogous to Drinfeld's and Chari–Pressley's highest weight classification of irreducible finite–dimensional representations of Yangians and quantum loop algebras [6, 5]. This analogy does not extend to proofs, however, since $E_{\hbar,\tau}(\mathfrak{g})$ does not seem to admit Verma modules. We circumvent this issue by constructing a functor Θ from finite–dimensional representations of the quantum loop algebra $U_q(L\mathfrak{g})$ to those of $E_{\hbar,\tau}(\mathfrak{g})$, which is such that $\Theta(\mathcal{V})$ is irreducible if and only if $\mathcal{V}$ is, and relying on the classification of simples in $\mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$.

The functor Θ is governed by the monodromy of the p–difference equations $f_i(pz) = \Psi_i(z)f_i(z)$ determined by the commuting (half–)currents of $U_q(L\mathfrak{g})$ on a finite–dimensional representation $\mathcal{V}$. It is a trigonometric version of the functor from finite–dimensional representations of the Yangian $Y_\hbar\mathfrak{g}$ to those of $U_q(L\mathfrak{g})$ we constructed in our previous work [7, 8, 9].

Our presentation of $E_{\hbar,\tau}(\mathfrak{g})$ is valid for an arbitrary symmetrisable Kac–Moody algebra $\mathfrak{g}$, as is our classification, provided finite–dimensionality is replaced by an integrability and category $\mathcal{O}$ conditions.

References

[1] P. Etingof, O. Schiffmann, *On highest weight modules over elliptic quantum groups*, Lett. Math. Phys. **47** (1999), 179–188.

[2] R. Farghly, H. Konno, K. Oshima, *Elliptic algebra $U_{q,p}(\widehat{\mathfrak{g}})$ and quantum Z–algebras*, Algebr. Represent. Theory, **18** (2015), 103–135.

[3] G. Felder, *Conformal field theory and integrable systems associated to elliptic curves*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 1247–1255, Birkhäuser, Basel, 1995.

[4] T. Kojima, H. Konno, *Elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}_N})$ and the Drinfeld realization of the elliptic quantum group $B_{q,\lambda}(\widehat{\mathfrak{sl}_N})$*, Comm. Math. Phys. **239** (2003), 405-447.

[5] V. Chari, A. Pressley, *Quantum affine algebras and their representations*, Representations of groups (Banff, AB, 1994), 59–78, CMS Conf. Proc., 16, Amer. Math. Soc., 1995.

[6] V. Drinfeld, *A new realization of Yangians and quantum affine algebras*, Soviet Math. Dokl. **36** (1988), 212–216.

[7] S. Gautam, V. Toledano Laredo, *Yangians and quantum loop algebras*, Selecta Math. (N.S.) **19** (2013), 271–336.

[8] S. Gautam, V. Toledano Laredo, *Yangians, quantum loop algebras, and abelian difference equations*, J. Amer. Math. Soc., **29** (2016), 775–824.

[9] S. Gautam, V. Toledano Laredo, *Meromorphic tensor equivalence for Yangians and quantum loop algebras*, Publ. Math. Inst. Hautes Études Sci. **125** (2017), 267–337.

[10] S. Gautam, V. Toledano Laredo, *Elliptic quantum groups and their finite–dimensional representations*, arXiv:1707.06469, 52 pages.

[11] M. Jimbo, H. Konno, S. Odake, J. Shiraishi, *Quasi-Hopf twistors for elliptic quantum groups*, Transform. Groups, **4** (1999), 303–327.

Rigid indecomposable modules and real Schur roots

Christof Geiss

(joint work with Bernard Leclerc, Jan Schröer)

Let A be a finite dimensional hereditary K-algebra. Its Grothendieck group $K_0(A) = \mathbb{Z}^I$ for $I = \{1, 2, \dots, n\}$ comes with the canonical basis $(\alpha_i)_{i\in I}$ given by the classes of the simple A-modules. The homological bilinear form

$$(M, N) \mapsto \dim_K \operatorname{Hom}_A(M, N) - \dim_K \operatorname{Ext}^1_A(M, N),$$

on the category of finite dimensional A-modules descends to an integral bilinear form on $K_0(A)$ determined by

$$\langle \alpha_i, \alpha_j \rangle_{(C,D,\Omega)} = \begin{cases} c_i c_{ij} & \text{if } (i, j) \in \Omega, \\ c_i & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

Here, $C \in \mathbb{Z}^{I\times I}$ is a symmetrizable generalized Cartan matrix, $D = \operatorname{diag}((c_i)_{i\in I})$ is a (left) symmetrizer of C and $\Omega \subset I \times I$ is an orientation for C. We may suppose that $(i, j) \in \Omega$ implies $i < j$. In other words, $K_0(A)$ has the structure of a generalized Cartan lattice with an orthogonal exceptional sequence in the sense of Hubery and Krause [8, Sec. 3]. We can consider (C, D, Ω) as a basic combinatorial invariant of A. Note that C will be symmetric if K is algebraically closed. From Kac's theorem and its relatives it follows that for many finite dimensional hereditary algebras the classes of the finite dimensional indecomposable A-modules correspond precisely to the positive roots of the corresponding Kac-Moody Lie algebra $\mathfrak{g}(C)$. This holds in particular if K is algebraically closed or finite, or if C is of finite or of affine type. The Weyl group $W \subset \operatorname{Aut}(\mathbb{Z}^I)$ is generated by the simple reflections $(s_i)_{i\in I}$, where

$$s_i(\alpha) := \alpha - \frac{2(\alpha, \alpha_i)}{(\alpha_i, \alpha)}\alpha_i,$$

and $(-, -)$ denotes the symmetrized bilinear form. The real roots $\Phi_{\mathrm{re}}(C) := \cup_{i\in I} W\alpha_i$ can be considered as a subset of the roots of the Kac-Moody Lie algebra $\mathfrak{g}(C)$. The *real Schur roots* are a subset of the positive real roots $\Phi^+_{\mathrm{re}}(C)$ which can be described combinatorially, following [8, Cor. 4.8], in terms of non-crossing partitions, namely

$$\Phi_{\mathrm{rS}}(C, \Omega) = \{\alpha \in \Phi^+(C) \mid 1 < s_\alpha < s_1 s_2 \cdots s_n\},$$

where $<$ denotes the absolute order on W.

By work of Crawley-Boevey [2] and Ringel [9] the real Schur roots can be identified with the classes of the rigid indecomposable A-modules for *any* finite dimensional hereditary K-algebra A with combinatorial invariant (C, D, Ω). Here, *rigid* means that the module has no self-extensions. From the work of Crawley-Boevey and Ringel follows also that the endomorphism ring of each rigid indecomposable A-module is isomorphic to the endomorphism ring of a simple A-module, and thus by Schur's Lemma it is isomorphic to a division algebra.

In previous work [5] we introduced for any field $\mathbb{F}$ an 1-Iwanaga Gorenstein $\mathbb{F}$-algebra $H := H_{\mathbb{F}}(C, D, \Omega)$, defined in terms of a quiver with relations. Its modules of finite projective dimension form a hereditary exact category. They behave in many aspects like the modules over an hereditary algebra with the same combinatorial invariants. For example the Grothendieck group of those modules can be identified with $(\mathbb{Z}^I, \langle -, -\rangle_{(C,D,\Omega)})$. In this spirit we can now show the following result:

Theorem. Let $H = H_{\mathbb{F}}(C, D, \Omega)$ be as above. Then the following holds:

(a) Taking classes in the Grothendieck group of modules of finite projective dimension, induces a bijection between the isoclasses of indecomposable rigid H-modules of finite projective dimension, and the real Schur roots $\Phi_{\mathrm{rS}}(C, \Omega)$.
(b) If M is a rigid indecomposable H-module of finite projective dimension, we have $\mathrm{End}_H(M) \cong \mathbb{F}[\epsilon]/(\epsilon^{c_i})$ for some $i \in I$, and M is free as an $\mathrm{End}_H(M)$-module.
(c) Taking classes in the Grothendieck group of all finite dimensional H-modules induces a bijection between the isoclasses of left finite H-bricks and dual real Schur roots $\Phi_{\mathrm{rS}}(C^T, \Omega)$.

Here, a module is called a *brick* if its endomorphism ring is a division algebra, and it is called *left finite* if the smallest torsion class which contains it is functorially finite.

In order to approach this result we note that the algebras $(H_{\mathbb{F}}(C, kD, \Omega))_{k\in\mathbb{N}_+}$ form a directed system with elements $\epsilon_k \in Z(H_{\mathbb{F}}(C, kD, \Omega))$ and canonical isomorphisms

$$H_{\mathbb{F}}(C, kD, \Omega)/(\epsilon_k^l H_{\mathbb{F}}(C, kD, \Omega)) \cong H(C, lD, \Omega)$$

for $l < k$, such that ϵ_k is projected onto ϵ_l, see [6, Sec. 2.2]. It follows that

$$\widehat{H} := \varprojlim_k H_{\mathbb{F}}(C, kD, \Omega)$$

is naturally a noetherian $\mathbb{F}[\![\epsilon]\!]$-algebra. Moreover we consider the localization $\widetilde{H} = \widetilde{H}_{\mathbb{F}(\!(\varepsilon)\!)}(C, D, \Omega) = \widehat{H}_{\varepsilon}$. It is easy to see that $\widehat{H}$ is free of finite rank as an $\mathbb{F}[\![\varepsilon]\!]$-module and (except for trivial cases) of global dimension 2. On the other hand $\widetilde{H}$ is a finite-dimensional hereditary $\mathbb{F}(\!(\varepsilon)\!)$-algebra of type (C, D, Ω). The diagram

$$\begin{tikzcd} \widehat{H} \arrow[r, "\mathrm{loc}"] \arrow[d, "\mathrm{proj}"'] & \widetilde{H} \\ H & \end{tikzcd}$$

allows us to relate the $\mathbb{F}[\![\varepsilon]\!]$-lattices of $\widehat{H}$, via the reduction functor $H \otimes_{\widehat{H}} -$, with the finite dimensional H-modules of finite projective dimension, and similarly, via the localization functor $\widetilde{H} \otimes_{\widehat{H}} -$, with the finite dimensional modules of the hereditary $\mathbb{F}(\!(\epsilon)\!)$-algebra $\widetilde{H}$. This is quite similar to a p-modular system.

Although some of our basic constructions are closely related to [3], we have to work with the exchange graph of support tilting modules rather than with exceptional sequences. The following facts are crucial for the proof of our theorem:

- τ-rigid H-modules have in fact projective dimension at most 1 (L. Demonet), thus the exchange graph of support tilting H-modules is $|I|$-regular [1].
- The exchange graph of support tilting modules for $\widetilde{H}$ is connected and $|I|$-regular [7].
- Statement (c) is an almost formal consequence of (a) and (b) by [4, Thm. 4.1].

Finally it is worth to mention that the algebra $\widehat{H} = \widehat{H}_{\mathbb{F}[\![\varepsilon]\!]}(C, D, \Omega)$ can be described as

$$\mathbb{F}\langle\langle Q(C, \Omega)\rangle\rangle / I,$$

where $\mathbb{F}\langle\langle Q(C, \Omega)\rangle\rangle$ is the completed path algebra of the quiver $Q(C, \Omega)$, and I is the ideal generated by the relations (H2), which were both introduced in [5, Sec. 1.4]. In this description, $\widehat{H}$ carries a $\mathbb{F}[\![\varepsilon]\!]$-algebra structure via the map which sends ε to $\sum_{i \in I} \varepsilon_i^{c_i} \in \widehat{H}$. For example, with

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \Omega = \{(1, 2), (2, 3)\}$$

we get

$$Q(C, \Omega) = \begin{tikzcd} 1 \arrow[loop above, "\varepsilon_1"] & 2 \arrow[l, "\alpha"'] \arrow[loop above, "\varepsilon_2"] & 3 \arrow[l, "\beta"'] \arrow[loop above, "\varepsilon_3"] \end{tikzcd} \quad \text{and} \quad (H2) = (\varepsilon_1^2 \alpha - \alpha \varepsilon_2, \varepsilon_2 \beta - \beta \varepsilon_3^2).$$

References

[1] T. Adachi, O. Iyama, I. Reiten: *τ-tilting theory.* Compos. Math. **150** (2014), no. 3, 415–452.

[2] W. Crawley-Boevey: *Exceptional sequences of representations of quivers.* Representations of Algebras (Ottawa, ON, 1992), 117–124, CMS Conf. Proc. **14**, Amer. Math. Soc., 1993.

[3] W. Crawley-Boevey: *Rigid integral representations of quivers.* Representation theory of algebras (Cocoyoc, 1994), 155–163, CMS Conf. Proc., **18**, Amer. Math. Soc., 1996.

[4] L. Demonet, O. Iyama, G. Jasso: *τ-tilting finite algebras, bricks and g-vectors.* Int. Math. Res. Not. IMRN (2017): 1–41.

[5] Ch. Geiß, B. Leclerc, J. Schröer: *Quivers with relations for symmetrizable Cartan matrices I: Foundations.* Invent. Math. **209** (2017), no. 1, 61–158.

[6] Ch. Geiß, B. Leclerc, J. Schröer: *Quivers with relations for symmetrizable Cartan matrices II: Change of symmetrizers.* IMRN Vol. 2018 no. 9, 2866–2898.

[7] A. Hubery: *The cluster complex of an hereditary artin algebra.* Algebr. Represent. Theory **14** (2011), no. 6, 1163–1185.

[8] A. Hubery, H. Krause: *A categorification of non-crossing partitions.* J. Eur. Math. Soc. **18** (2016), no. 10, 2273–2313.

[9] C.M. Ringel: *The braid group action on the set of exceptional sequences of a hereditary Artin algebra.* Abelian Group Theory and Related Topics (Oberwolfach, 1993), 339–352, Contemp. Math. **171**, Amer. Math. Soc., 1994

Conformal blocks for Galois covers of algebraic groups

Shrawan Kumar
(joint work with Jiuzu Hong)

1. Introduction

Wess-Zumino-Witten model is a type of two dimensional conformal field theory, which associates to an algebraic curve with marked points and integrable highest weight modules of an affine Kac-Moody Lie algebra associated to the points, a finite dimensional vector space consisting of conformal blocks. The space of conformal blocks has many important properties including Propogation of Vacua and Factorization. Deforming the pointed algebraic curves in a family, we get a sheaf of conformal blocks. This sheaf admits a projectively flat connection when the family of pointed curves is a smooth family. The mathematical theory of conformal blocks was first established by Tsuchiya-Ueno-Yamada. All the above properties are important ingredients in the proof of the celebrated Verlinde formula for the dimension of the space of conformal blocks.This theory has a geometric counterpart in the theory of moduli spaces of principal bundles over algebraic curves and also the moduli of curves and its stable compactification.

In this paper we study a twisted theory of conformal blocks on Galois covers of algebraic curves. More precisely, we consider an algebraic curve Σ with an action of a finite group Γ. Moreover, we take a group homomorphism $\phi : \Gamma \to \mathrm{Aut}(\mathfrak{g})$ of Γ acting on a simple Lie algebra $\mathfrak{g}$. Given any smooth point $q \in \Sigma$, we attach an affine Lie algebra $\hat{L}(\mathfrak{g}, \Gamma_q)$ (in general a twisted affine Lie algebra), where Γ_q is the stabilizer group of Γ at q. The integrable highest weight representations of $\hat{L}(\mathfrak{g}, \Gamma_q)$ of level c (where c is a positive integer) are parametrized by certain finite set $D_{c,q}$ of dominant weights of the reductive Lie algebra $\mathfrak{g}^{\Gamma_q}$, i.e., for any $\lambda \in D_{c,q}$ we attach an integrable highest weight representation $\mathscr{H}(\lambda)$ of $\hat{L}(\mathfrak{g}, \Gamma_q)$ of level c and conversely. Given a collection $\vec{q} := (q_1, \cdots, q_s)$ of smooth points in Σ such that their Γ-orbits are disjoint and a collection of weights $\vec{\lambda} = (\lambda_1, \dots, \lambda_s)$ with $\lambda_i \in D_{c,q_i}$, we consider the representation $\mathscr{H}(\vec{\lambda}) := \mathscr{H}(\lambda_1) \otimes \cdots \otimes \mathscr{H}(\lambda_s)$. Now, define the associated space of *twisted covacua* (or *twisted dual conformal blocks*) as follows:

$$\mathscr{V}_{\Sigma,\Gamma,\phi}(\vec{q}, \vec{\lambda}) := \frac{\mathscr{H}(\vec{\lambda})}{\mathfrak{g}[\Sigma \backslash \Gamma \cdot \vec{q}]^{\Gamma} \cdot \mathscr{H}(\vec{\lambda})},$$

where $\mathfrak{g}[\Sigma \backslash \Gamma \cdot \vec{q}]^{\Gamma}$ is the Lie algebra of Γ-equivariant regular functions from $\Sigma \backslash \Gamma \cdot \vec{q}$ to $\mathfrak{g}$ acting on the i-th factor $\mathscr{H}(\lambda_i)$ of $\mathscr{H}(\vec{\lambda})$ via its Laurent series expansion at q_i.

The following *Propogation of Vacua* is the first main result of our work.

Theorem 1. *Assume that Γ stabilizes a Borel subalgebra of $\mathfrak{g}$. Let q be a smooth point of Σ such that q is not Γ-conjugate to any point $\vec{q}$. Then, we have the* *following isomorphism of spaces of twisted covacua:*

$$\mathscr{V}_{\Sigma,\Gamma,\phi}(\vec{q},\vec{\lambda}) \simeq \mathscr{V}_{\Sigma,\Gamma,\phi}\Big((\vec{q},q),(\vec{\lambda},0)\Big).$$

In fact, a stronger version of Propogation Theorem is proved. In our equivariant setting we need to generalize some important ingredients. Further, the fact that

“ The endormorphism $X_{-\theta}\otimes f$ of $\mathscr{H}$ is locally nilpotent for all $f\in\mathscr{O}(U)$ ”

can not easily be generalized to the twisted case. To prove an analogous result, we need to assume that Γ stabilizes a Borel subalgebra of $\mathfrak{g}$. It will be interesting to see if this assumption can be removed.

Let q be a nodal point in Σ. Assume that the action of Γ at q is stable and the stabilizer group Γ_q does not exchange the two formal branches around q. Let Σ' be the normalization of Σ at the points $\Gamma\cdot q$, and let q',q'' be the two smooth points in Σ' over q. The following *Factorization Theorem* is our second main result.

Theorem 2. *Assume that* Γ *stabilizes a Borel subalgebra of* $\mathfrak{g}$*. Then, there exists a natural isomorphism:*

$$\mathscr{V}_{\Sigma,\Gamma,\phi}(\vec{q},\vec{\lambda}) \simeq \bigoplus_{\mu\in D_{c,q''}} \mathscr{V}_{\Sigma',\Gamma,\phi}\Big((\vec{q},q',q''),(\vec{\lambda},\mu^*,\mu)\Big),$$

where μ^* *is the dominant weight of* $\mathfrak{g}^{\Gamma_{q'}}$ *such that* $V(\mu^*)$ *is the dual representation* $V(\mu)^*$ *of* $\mathfrak{g}^{\Gamma_q}=\mathfrak{g}^{\Gamma_{q'}}=\mathfrak{g}^{\Gamma_{q''}}$*.*

The formulation of the Factorization Theorem in the twisted case is a bit more delicate, since the parameter sets $D_{c,q'}$ and $D_{c,q''}$ attached to the points q',q'' are different in general; nevertheless they are related by the dual of representations under the assumption that the action of Γ at the node q is stable and the stabilizer group Γ_q does not exchange the branches. Its proof requires additional care (from that of the untwisted case) at several places. The assumption that Γ stabilizes a Borel subalgebra of $\mathfrak{g}$ also appears in this theorem as we use the Propogation Theorem in its proof.

Given a family $(\Sigma_T,\vec{q})$ of s-pointed Γ-curves over a connected scheme T and weights $\vec{\lambda}=(\lambda_1,\dots,\lambda_s)$ with $\lambda_i\in D_{c,q_i}$ as above, one can attach a functorial coherent sheaf $\mathscr{V}_{\Sigma_T,\Gamma,\phi}(\vec{q},\vec{\lambda})$ of twisted covacua over the base T. We prove the following theorem.

Theorem 3. *Assume that the family* $\Sigma_T\to T$ *is a smooth family over a smooth base* T*. Then, the sheaf* $\mathscr{V}_{\Sigma_T,\Gamma,\phi}(\vec{q},\vec{\lambda})$ *of twisted covacua is locally free of finite rank over* T*. In fact, there exists a projectively flat connection on* $\mathscr{V}_{\Sigma_T,\Gamma,\phi}(\vec{q},\vec{\lambda})$*.*

This theorem relies mainly on the Sugawara construction for the twisted affine Kac-Moody algebras. In the untwisted case, this construction is quite well-known. In the twisted case, the formulae are written in terms of the abstract Kac-Moody presentation of $\hat{L}(\mathfrak{g},\sigma)$, where σ is a finite order automorphism of $\mathfrak{g}$. For our application, we require the formulae in terms of the affine realization of $\hat{L}(\mathfrak{g},\sigma)$ as a central extension of the twisted loop algebra $\mathfrak{g}((t))^\sigma$. We present such a formula in our work, which might be new (to our knowledge).

Let $\overline{\mathscr{H}M}_{g,\Gamma,\eta}$ be the Hurwitz stack of Γ-stable s-pointed Γ-curves of genus g with marking data η at the marked points such that the set of Γ-orbits of the marked points contains the full ramification divisor. It was proved by Bertin-Romagny that $\overline{\mathscr{H}M}_{g,\Gamma,\eta}$ is a smooth and proper Deligne-Mumford stack. We can attach a collection $\vec{\lambda}$ of dominant weights to the marking data η, and associate a coherent sheaf $\mathscr{V}_{g,\Gamma,\phi}(\eta,\vec{\lambda})$ of twisted covacua over the Hurwitz stack $\overline{\mathscr{H}M}_{g,\Gamma,\eta}$. We prove the following theorem.

Theorem 4. *Assume that Γ stabilizes a Borel subalgebra of $\mathfrak{g}$. Then, the sheaf $\mathscr{V}_{g,\Gamma,\phi}(\eta,\vec{\lambda})$ is locally free over the stack $\overline{\mathscr{H}M}_{g,\Gamma,\eta}$.*

Our proof of this theorem follows closely the work of Looijenga in the non-equivariant setting; in particular, we use the canonical smoothing deformation of nodal curves and gluing tensor elements. The Factorization Theorem also plays a crucial role in the proof. In the case Γ is cyclic, this theorem together with the Factorization Theorem allows us to reduce the computation of the dimension of the space of twisted covacua to the case of Galois covers of projective line with three marked points.

In the usual (untwisted) theory of conformal blocks, the space of covacua has a beautiful geometric interpretation in that it can be identified with the space of generalized theta functions on the moduli space of parabolic G-bundles over the algebraic curve, where G is the simply-connected simple algebraic group associated to $\mathfrak{g}$. This identification was proved by Beauville-Laszlo, Faltings, Kumar-Narasimhan-Ramanathan, Laszlo-Sorger and Pauly. In the setting of Γ-curves Σ as well, it is expected that the space of twisted covacua can be identified with the generalized theta functions on the moduli space of parabolic $\mathcal{G}$-bundles on $\bar{\Sigma}$, where $\bar{\Sigma}$ is the quotient of Σ by Γ, and $\mathcal{G}$ is the parahoric Bruhat-Tits group scheme over $\bar{\Sigma}$ obtained via the construction of Γ-invariants of the Weil restriction from Σ to $\bar{\Sigma}$. In fact, this natural question has been formulated earlier by Pappas-Rapoport in a more general setting. Along this direction, there has been some results recently by Hacen when Γ is of order 2 acting on $\mathfrak{g} = sl_n$ by certain involutions.

There were some earlier works related to the twisted theory of conformal blocks. For example Frenkel-Szczesny studied the twisted modules over Vertex algebras on algebraic curves, and Kuroki-Takebe studied a twisted Wess-Zumino-Witten model on elliptic curves. When Γ is of prime order and the marked points are unramified, the space $\mathscr{V}_{\Sigma,\Gamma,\phi}(\vec{q},\vec{\lambda})$ has been studied recently by Damiolini, where she proved similar results as ours. Our work is a vast generalization of her work, since we do not need to put any restrictions on the Γ-orbits, and the only restriction on Γ is that Γ stabilizes a Borel subalgebra of $\mathfrak{g}$ (when Γ is a cyclic group it automatically holds). In particular, when Γ has nontrivial stabilizers at the marked points $\vec{q}$, twisted affine Kac-Moody Lie algebras and their representations occur naturally in this ramification theory of conformal blocks. We also learnt from S. Mukhopadhyay that he has obtained some results (unpublished) in this direction.

Our work is motivated by a conjectural connection predicted by Fuchs-Schweigert between the trace of diagram automorphism on the space of conformal blocks and certain conformal field theory related to twisted affine Lie algebras. A Verlinde type formula for the trace of diagram automorphism on the space of conformal block has been proved recently by J. Hong, where the formula involves the twisted affine Kac-Moody algebras mysteriously. We hope that it can shed some light on the dimension of twisted conformal blocks; in particular, when Γ acts on $\mathfrak{g}$ by diagram automorphisms.

Reporter: Leonardo Patimo

Participants

Prof. Dr. Pramod N. Achar
Department of Mathematics
Louisiana State University
Baton Rouge LA 70803-4918
UNITED STATES

Prof. Dr. Henning Haahr Andersen
TGMRC
China Three Gorges University
No. 8, Daxue Road, Xiling
Yichang City, Hubei Province 443 002
CHINA

Prof. Dr. Alexander Braverman
Perimeter Institute for Theoretical Physics
31 Caroline St. N.
Waterloo ON N2L 2Y5
CANADA

Dr. Adrien Brochier
Department Mathematik
Universität Hamburg
Bundesstrasse 55
20146 Hamburg
GERMANY

Prof. Dr. Vyjayanthi Chari
Department of Mathematics
University of California, Riverside
900, West Big Springs Road
Riverside CA 92521-0135
UNITED STATES

Dr. Stephanie Cupit-Foutou
Fakultät für Mathematik
Ruhr-Universität Bochum
Gebäude NA 4/67
44780 Bochum
GERMANY

Dr. Olivier Dudas
UFR de Mathématiques
Université Paris Diderot
5, rue Thomas Mann
75205 Paris Cedex 13
FRANCE

Dr. Claude Eicher
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge, MA 02139-4307
UNITED STATES

Prof. Dr. Mikhail Finkelberg
Department of Mathematics
National Research University
Higher School of Economics
6 Usacheva St.
Moscow 119 048
RUSSIAN FEDERATION

Dr. Iordan Ganev
Institute of Science and
Technology Austria (IST Austria)
Am Campus 1
3400 Klosterneuburg
AUSTRIA

Prof. Dr. Sachin Gautam
Department of Mathematics
The Ohio State University
100 Mathematics Building
231 West 18th Avenue
Columbus, OH 43210-1174
UNITED STATES

Prof. Dr. Christof Geiss
Instituto de Matemáticas
UNAM
Ciudad Universitaria
04510 Ciudad de México Ciudad de México
MEXICO

Prof. Dr. Victor Ginzburg
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
UNITED STATES

Dr. Simon Goodwin
School of Mathematics
The University of Birmingham
Edgbaston
Birmingham B15 2TT
UNITED KINGDOM

Prof. Dr. Iain Gordon
School of Mathematics
University of Edinburgh
James Clerk Maxwell Building
King's Buildings
Mayfield Road
Edinburgh EH9 3FD
UNITED KINGDOM

Prof. Dr. Maria Gorelik
Department of Mathematics
The Weizmann Institute of Science
P.O. Box 26
Rehovot 76100
ISRAEL

Prof. Dr. Jacob Greenstein
Department of Mathematics
University of California, Riverside
900, West Big Springs Road
Riverside, CA 92521-0135
UNITED STATES

Dr. Sam Gunningham
School of Mathematics
The University of Edinburgh
James Clerk Maxwell Building
The King's Buildings
Peter Guthrie Tait Road
Edinburgh EH9 3FD
UNITED KINGDOM

Dr. Thorsten Heidersdorf
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Anthony Henderson
School of Mathematics and Statistics
The University of Sydney
Sydney NSW 2006
AUSTRALIA

Prof. Dr. David Hernandez
IMJ - PRG
Université Paris VII - Paris Diderot
Bat. S. Germain
CP 7012
75205 Paris Cedex 13
FRANCE

Prof. Dr. Jens Carsten Jantzen
Matematisk Institut
Aarhus Universitet
Ny Munkegade 118
8000 Aarhus C
DENMARK

Dr. Thorge Jensen
Département de Mathématiques
LMBP UMR 6620
Université Clermont Auvergne
24, Avenue des Landais
63177 Aubière Cedex
FRANCE

Dr. David Jordan
School of Mathematics
University of Edinburgh
James Clerk Maxwell Bldg.
King's Buildings, Mayfield Road
Edinburgh EH9 3JZ
UNITED KINGDOM

Dr. Daniel Juteau
IMJ - PRG
Université Paris Diderot (Paris VII)
CNRS UMR 7586
Bâtiment Sophie Germain
Case Postale 7012
75205 Paris Cedex 13
FRANCE

Prof. Dr. Shrawan Kumar
Department of Mathematics
University of North Carolina at Chapel Hill
Phillips Hall
Chapel Hill, NC 27599-3250
UNITED STATES

Dr. Martina Lanini
Dipartimento di Matematica
Universita degli Studi di Roma II
Tor Vergata
Via della Ricerca Scientifica
00133 Roma
ITALY

Prof. Dr. Bernard Leclerc
Département de Mathématiques
Université de Caen
14032 Caen Cedex
FRANCE

Prof. Dr. Ivan Loseu
Department of Mathematics
University of Toronto
40 St. George Street
Toronto ON M5S 2E4
CANADA

Dr. Carl Mautner
Department of Mathematics
University of California, Riverside
900 University Avenue
Riverside, CA 92521
UNITED STATES

Prof. Dr. Kevin R. McGerty
Mathematical Institute
Oxford University
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Peter J. McNamara
School of Mathematics and Statistics
The University of Melbourne
Parkville, VIC 3010
AUSTRALIA

Prof. Dr. Anne Moreau
Laboratoire Paul Painleve, UMR 8524
Université des Sciences et
Technologie de Lille 1
Cite Scientifique, Bat. M2
59655 Villeneuve d'Ascq Cedex
FRANCE

Dr. Emily Norton
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Dr. Leonardo Patimo
Mathematisches Institut
Universität Freiburg
Ernst-Zermelo-Strasse 1
79104 Freiburg i. Br.
GERMANY

Tomasz Przezdziecki
School of Mathematics and Statistics
University of Glasgow
University Gardens
Glasgow G12 8QW
UNITED KINGDOM

Dr. Fan Qin
School of Mathematical Sciences
Shanghai Jiao Tong University
800 Dongchuan Road
Shanghai Shi 200 240
CHINA

Prof. Dr. Markus Reineke
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstrasse 150
44801 Bochum
GERMANY

Prof. Dr. Jan Schröer
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Beatrix Schumann
Mathematisches Institut
Universität zu Köln
Weyertal 86 - 90
50931 Köln
GERMANY

Prof. Dr. Vera V. Serganova
Department of Mathematics
University of California
Berkeley CA 94720-3840
UNITED STATES

Dr. Peng Shan
Yau Mathematical Sciences Center
Tsinghua University
Beijing 100 084
CHINA

Prof. Dr. Wolfgang Soergel
Mathematisches Institut
Universität Freiburg
Ernst-Zermelo-Strasse 1
79104 Freiburg i. Br.
GERMANY

Prof. Dr. Catharina Stroppel
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Peter Tingley
Department of Mathematics and
Statistics
Loyola University of Chicago
Chicago, IL 60626-5385
UNITED STATES

Prof. Dr. Valerio Toledano Laredo
Department of Mathematics
Northeastern University
567 Lake Hall
360 Huntington Avenue
Boston MA 02115-5000
UNITED STATES

Dr. Jacinta Torres
Fakultät für Mathematik
Karlsruher Institut für Technologie
(KIT)
Englerstrasse 2
76131 Karlsruhe
GERMANY

Prof. Dr. Michela Varagnolo
Département de Mathématiques
Université de Cergy-Pontoise
Site Saint-Martin, BP 222
2, Avenue Adolphe Chauvin
95302 Cergy-Pontoise Cedex
FRANCE

Prof. Dr. Eric Vasserot
Equipe de Theorie de Groupes
Université Paris VII
175, rue de Chevaleret
75013 Paris Cedex
FRANCE

Prof. Dr. Ben Webster
Perimeter Institute for Theoretical Physics
31 Caroline St. N.
Waterloo ON N2L 2Y5
CANADA

Dr. Harold Williams
Department of Mathematics
University of Texas
2515 Speedway Stop
Austin, TX 78712-1202
UNITED STATES

Prof. Dr. Geordie Williamson
School of Mathematics and Statistics
The University of Sydney
Sydney NSW 2006
AUSTRALIA

Prof. Dr. Oksana Yakimova
Mathematisches Institut
Friedrich-Schiller-Universität Jena
07737 Jena
GERMANY