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Mini-Workshop: Mathematical and Numerical Analysis of Maxwell's Equations

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Abstract. In this mini-workshop 17 leading mathematicians from Europe and United States met at the MFO to discuss and present new developments in the mathematical and numerical analysis of Maxwell's equations and related systems of partial differential equations. The report at hand offers the extended abstracts of their talks.

Mathematics Subject Classification (2010): MSC: 45xxx, 46xxx, 47xxx, 65xxx, 78xxx.

Introduction by the Organisers

The mini-workshop *Mathematical and Numerical Analysis of Maxwell's Equations*, organised by Monique Dauge (Rennes), Ulrich Langer (Linz), Peter Monk (Newark), and Dirk Pauly (Essen) was well attended with 17 participants with broad geographic representation from Europe and United States. This workshop was a nice blend of researchers with various backgrounds from Maxwell's equations.

Maxwell's equations of electro-dynamics are of huge importance in mathematical physics, engineering, and especially in mathematics, leading since their discovery to interesting mathematical problems and even to new fields of mathematical research, particularly in the analysis and numerics of partial differential equations and applied functional analysis. The deep understanding of Maxwell's equations and the possibility of their numerical solution in complex geometries and different settings have led to very efficient and robust simulation methods in Computational Electromagnetics. Moreover, efficient simulation methods pave the way for optimizing electromagnetic devices and processes. Digital communication and e-mobility are two fields where simulation and optimization techniques that are based on Maxwell's equations play a deciding role.

The workshop brought together different communities, namely people working in analysis of Maxwell's equations with those working in numerical analysis of Maxwell's equations and computational electromagnetics and acoustics.

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Mini-Workshop: Mathematical and Numerical Analysis of Maxwell's Equations

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Abstracts

High order discrete potentials

Ana Alonso Rodríguez
(joint work with Jessika Camaño, Eduardo De Los Santos, Francesca Rapetti)

In applied computations the need often arises to define, for example, a discrete field with assigned curl or to represent a div-free field in a given discrete space. In the low degree case this need can be fulfilled by involving tree and co-tree techniques (see, e.g., [2]). Our aim is to extend this techniques to high order Whitney finite elements. The key point is to identify the degrees of freedom for which the matrices representing the divergence and the gradient operators are related with the incidence matrix of a connected oriented graph. This can be done using Bernstein polynomials in the definition of the classical moments (see, e.g. [4]) usually used as degrees of freedom. A convenient visualization of these graphs can be obtained using the so-called small simplices introduced in [5].

Once it have been proved that the matrix associated with the divergence operator is the incidence matrix of a connected oriented graph, it is possible to identify an invertible square submatrix of this incidence matrix by choosing a spanning tree of the graph. This allows to easily compute the moments of a field in the space of Raviart-Thomas finite elements with assigned divergence. This approach extends to finite elements of high degree the method introduced in [3] for finite elements of degree one and can be used to construct a basis of the space of divergence-free Raviart-Thomas finite elements. The numerical tests show that the performance of the algorithm does not depend on the topology of the domain or the polynomial degree.

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Time-harmonic Maxwell equations with sign-changing coefficients

Anne-Sophie Bonnet-Ben Dhia
(joint work with Lucas Chesnel and Patrick Ciarlet)

In [1], we consider the time-harmonic Maxwell equations in a bounded domain $\Omega \subset \mathbb{R}^3$, when the dielectric permittivity ε and the magnetic permeability μ are real functions whose sign can change (ε, μ, $1/\varepsilon$ and $1/\mu$ are just supposed to be bounded functions). We assume for simplicity here that $\mu = 1$ and that Ω is topologically trivial, and we consider the variational problem:

$$\text{(1)} \quad \text{Find } E \in X \text{ such that } \forall E' \in X, \int_\Omega \operatorname{curl} E \cdot \operatorname{curl} E' - \omega^2 \varepsilon E \cdot E' = \int_\Omega J \cdot E'$$

where $\omega \in \mathbb{R}$, $J \in L^2(\Omega)^3$ is such that $\operatorname{div} J = 0$ in Ω, and X can be one of these two functional spaces:

$$H_N = \left\{ E \in L^2(\Omega)^3, \operatorname{curl} E \in L^2(\Omega)^3 \text{ and } E \times n = 0 \text{ on } \partial\Omega \right\},$$
$$\text{and } X_N(\varepsilon) = \{ E \in H_N, \operatorname{div}(\varepsilon E) = 0 \text{ in } \Omega \}.$$

The usual approach (for positive coefficients) consists in proving the following two results:

(1) Problem (1) with $X = X_N(\varepsilon)$ is equivalent to Problem (1) with $X = H_N$.
(2) The embedding of $X_N(\varepsilon)$ in $L^2(\Omega)^3$ is compact.

We prove that both assertions remain true for a sign-changing ε if

$$\text{(2)} \quad \forall f \in H^{-1}(\Omega),\ \exists! \varphi \in H^1_0(\Omega) \text{ such that } \forall \varphi' \in H^1_0(\Omega), \int_\Omega \varepsilon \nabla\varphi \cdot \nabla\varphi' = \int_\Omega f \varphi'$$

This scalar problem has been the subject of many studies in the case where the sign of ε changes across a Lipschitz boundary Σ (see for instance [2, 4, 3, 5]). Let us point out that this problem can be ill-posed, and even not Fredholm in some configurations (for instance, if a part of Σ is flat and if ε takes opposite values on each side).

To prove assertion 1, the idea is simply to notice that for every test field $E' \in H_N$, there exists $\varphi \in H^1_0(\Omega)$ such that $E' + \nabla\varphi \in X_N(\varepsilon)$.

The proof of assertion 2 relies on a T-coercivity result. More precisely, we show that, if (2) is true, there exists an operator $T \in \mathcal{L}(X_T)$ such that

$$\forall H, H' \in X_T, \quad \int_\Omega \frac{1}{\varepsilon} \operatorname{curl} H \cdot \operatorname{curl}(TH') = \int_\Omega \operatorname{curl} H \cdot \operatorname{curl} H'$$

where X_T denotes the usual space for the magnetic fields:

$$X_T = \left\{ H \in L^2(\Omega)^3, \operatorname{curl} H \in L^2(\Omega)^3, \operatorname{div} H = 0 \text{ in } \Omega \text{ and } H \cdot n = 0 \text{ on } \partial\Omega \right\}.$$

The compactness of the embedding is finally established by a proof *à la Weber* [6].

Finally, suppose that the scalar problem is Fredholm with a non trivial finite dimensional kernel, so that (2) is not satisfied. In that case, the first assertion is no longer true. Indeed, let φ belong to the kernel of the scalar problem. Then $E = \nabla\varphi$ is a solution of (1) with $J = 0$ for $X = X_N(\varepsilon)$ and not for $X = H_N$. However the approach can be extended to this case by setting problem (1) in an appropriate functional space $X = \tilde{X}_N(\varepsilon)$ such that $\tilde{X}_N(\varepsilon) = X_N(\varepsilon) \oplus Y_N(\varepsilon)$ where $\dim(Y_N(\varepsilon)) < +\infty$.

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On the regularity of electromagnetic fields on Lipschitz domains

Martin Costabel

The usual variational spaces X_N and X_T for time-harmonic electric or magnetic fields with PEC boundary conditions on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ are defined as

$$X_N = H(\mathrm{div}, \Omega) \cap H_0(\mathrm{curl}, \Omega) \quad \text{ and } X_T = H_0(\mathrm{div}, \Omega) \cap H(\mathrm{curl}, \Omega)\,.$$

These spaces have long been known to be contained in the Sobolev space $H^{1/2}(\Omega)$ [3, 6]. If the domain is, in addition, piecewise smooth, then there exists always an $\varepsilon > 0$ such that these spaces are contained in $H^{1/2+\varepsilon}(\Omega)$, and this additional smoothness is known to be useful in the analysis of some numerical algorithms [2, 1].

At a conference in 2017, Alberto Valli brought up the question whether such an additional regularity is also present for every Lipschitz domain. The answer turns out to be negative.

Using ideas from Nikolai Filonov's construction of a $C^{3/2}$ domain for which the usual Birman-Solomyak decomposition of X_T is not possible [5], a bounded domain Ω can be constructed that is of class C^1 and for which such an $\varepsilon > 0$ does not exist. This domain has the property that for functions in $W^{1+\frac{1}{p}+\varepsilon,p}(\Omega)$ with any $\varepsilon > 0$ and $p \geq 1$, the vanishing of the trace on $\partial\Omega$ implies the vanishing of the normal derivative. As a consequence, there exist smooth right hand sides for which the solutions of the Poisson equation with homogeneous Dirichlet or Neumann conditions (known to belong to $W^{1+\frac{1}{p},p}(\Omega)$ for $p > 1$) do not belong to $W^{1+\frac{1}{p}+\varepsilon,p}(\Omega)$ with any $\varepsilon > 0$ and $p \geq 1$. Taking gradients of these solutions and

$p = 2$ gives the non-regularity result for X_N and X_T. This construction is the subject of the article [4].

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Resonances of Optical Micro-Resonators

Monique Dauge

(joint work with Stéphane Balac and Zoïs Moitier)

Optical micro-resonators are dielectric cavities of small size (at μm scale) coupled to waveguides or fibers for light input and output [5]. The frequencies of interaction have to be close to resonance frequencies of the cavity alone (i.e. without neighboring fibers). Of particular interest are the whispering gallery modes: These special resonant modes concentrate along the boundary of the cavity and are mainly generated by curvature effects at high frequency.

As a simplified model for the three-dimensional Maxwell system (effective index method [11]), we consider two-dimensional [2D] Helmholtz equations governing transverse electric [TE] or magnetic [TM] modes. Even in this 2D framework, very few results provide asymptotic expansion of WGM at high frequency, see [6] for disk micro-resonators using expansions of Bessel functions [8].

In [3] we present a unified procedure to construct WGM in 2D cavities Ω with smooth boundaries and varying optical index n. Note that n is defined as 1 outside the cavity (in $\mathbb{R}^2 \setminus \overline{\Omega}$) and is assumed to be smooth and away from 1 in $\overline{\Omega}$. The problem under consideration is: Find non-zero u and complex number k such that—here the integer $p \in \{+1, -1\}$ distinguishes TM modes ($p = 1$) and TE modes ($p = -1$):

$$-\operatorname{div}\left(n^{p-1}\,\nabla u\right) = k^2 n^{p+1}\, u \quad \text{in} \quad \mathbb{R}^2 \tag{1}$$

complemented by suitable radiation condition (expansion at infinity in terms of Hankel functions of the first kind). Equation (1) includes the jump conditions $[u] = 0$ and $[n^{p-1}\,\partial_\nu u] = 0$ on $\partial\Omega$. Couples (k, u) are the resonant modes. In contrast with impenetrable obstacles [10], transparent obstacles or dielectric cavities may have non-real resonances super-algebraically close to the real axis [9, 7, 4].

We have found that the variations of n give rise to a new effective curvature index $\breve{\kappa} = \kappa + n'/n$ (where κ is the usual curvature of $\partial\Omega$) whose sign governs the asymptotic type: $\breve{\kappa} > 0$ causes modes to concentrate along the boundary with Airy-type profiles, $\breve{\kappa} = 0$ still causes concentration along the boundary but with Harmonic Oscillator profiles, whereas $\breve{\kappa} < 0$ may produce internal WGM types. By multiscale expansions based on a WKB approximation, we construct asymptotic quasi-resonances that have the WGM structure and satisfy suitable estimates. Our formulas are reminiscent of others obtained for eigenmodes with Dirichlet boundary conditions [2, 1]. Relying on general theory [12] we deduce that our quasi-resonances are asymptotically close to true resonances.

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The HX Preconditioner in Higher Dimensions

Jay Gopalakrishnan

(joint work with Martin Neumüller, Panayot Vassilevski)

In calculations using the n-dimensional finite element subcomplex of the de Rham complex, we often need a general purpose preconditioner for the stiffness matrix of the inner products. We report on the results obtained in [1] that generalize the construction and analysis of auxiliary space preconditioners, or HX preconditioners.

The first part of the talk focuses on developing simple computational proxies for k-forms and exterior derivatives in four dimensions (4D). For example, motivated by the fact the curl operator in two and three dimensions takes the form $\operatorname{curl} w = \varepsilon^{ij}\partial_i w_j$ and $[\operatorname{curl} w]_i = \varepsilon^{ijk}\partial_j w_k$, respectively, where we have used the Levi-Civita tensor ε and the summation convention, we define curl in 4D by $[\operatorname{Curl} w]_{ij} = \varepsilon^{ijkl}\partial_k w_l$. This produces a 4×4 skew symmetric matrix, given a 4-vector w. Similar definitions yield simple 4D proxies, implementable using scalar, vector, and matrix algebraic structures that are usually already present in current software.

The second half of the talk focuses on the generalized HX preconditioners. The construction of these preconditioners is motivated using a generalization of a decomposition of Sobolev space functions into a regular component and a potential. A discrete version of such a decomposition can now be quickly established using the modern tools of finite element exterior calculus. Extensive numerical experiments in 4D illustrate the performance of the preconditioners, practical scalability, and parameter robustness, all in accordance with the theory.

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Generalized Plane Waves & Maxwell's equations

Lise-Marie Imbert-Gérard
(joint work with J.-F. Fritsch)

Modeling for wave propagation in magnetically confined plasma motivates the development of numerical methods for smooth variable coefficient time-harmonic Maxwell's equations. The simplest of these models, the cold plasma model, reads

$$\text{curl curl}\mathbf{E} - \left(\frac{w}{c}\right)^2 \epsilon\mathbf{E} = 0$$

where the tensor ϵ is both homogeneous and anisotropic. Generalized Plane Waves (GPWs) are then introduced in the 2D variable refractive index Helmholtz framework [1, 2]. These functions are constructed to satisfy approximately the PDE, and a set of linearly independent can easily be constructed for discretization purposes. They are designed as exponential of polynomials, using Taylor expansions. The first extension of the GPW construction to the 3D vector-valued Maxwell's equation is introduced, specifying a particular ansatz for the amplitude and phase functions, and emphasizing the challenges related to the construction algorithm.

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Finite Element Methods for Maxwell's Equations

Peter Monk

In the last few decades, conforming finite element methods for approximating the time harmonic Maxwell system governing electromagnetic wave propagation have undergone profound changes. Whereas in the 1980s there was confusion about how to choose the sesquilinear form and the appropriate finite elements to obtain a convergent solver, it is now clear not only how to discretize the Maxwell system using edge elements, but also how to analyze the resulting method [11, 13]. Important spin-offs from this analysis include the Finite Element Exterior Calculus [2] and the realization that the discrete de Rham diagram is a useful tool to guarantee conservation of charge, as well as for understanding the error analysis.

Of course edge elements are not the only possibility. But alternative approaches using continuous elements require special care to control the divergence of the solution and modification at discontinuities in material properties [6]. Problems facing the use of continuous elements were first recognized by Costabel and Dauge [8] who also proposed a fix for the problem (see also [6]). Recently a different approach has been proposed by Bonito and Guermond [5].

The simplest model problem for Maxwell's equations that captures several of the difficulties associated with their solution is as follows. Suppose $\Omega \subset \mathbb{R}^3$ is a bounded polyhedral domain such that it's boundary has two connected components Γ and Σ where Σ is the boundary of the unbounded connected complement of Ω. Let $\boldsymbol{\nu}$ denote the unit outward normal to Ω and assume Ω is simply connected. Then, given a suitable tangential vector field $\mathbf{g} \in L^2(\Sigma)$ we seak $\mathbf{E}$ such that

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{E}) - \kappa^2 \varepsilon \mathbf{E} &= 0 \text{ in } \Omega \\
(\nabla \times \mathbf{E}) \times \boldsymbol{\nu} - i\kappa \mathbf{E}_T &= \mathbf{g} \text{ on } \Sigma, \\
\mathbf{E} \times \boldsymbol{\nu} &= 0 \text{ on } \Gamma.
\end{aligned}$$

Here $\mathbf{E}_T = (\boldsymbol{\nu} \times \mathbf{E}) \times \boldsymbol{\nu}$ is the tangential trace of $\mathbf{E}$ and $\kappa > 0$ is the constant wave-number of the radiation. The function ε denotes the electric permittivity of the material inside Ω and is assumed here to be a real valued piecewise smooth coefficient which is uniformly positive and bounded.

The appropriate function space for the solution is built from the standard space

$$H(\mathrm{curl};\Omega) = \left\{ \mathbf{u} \in (L^2(\Omega))^3 \mid \nabla \times \mathbf{u} \in (L^2(\Omega))^3 \right\}$$

as follows

$$X = \left\{ \mathbf{u} \in H(\mathrm{curl};\Omega) \mid \boldsymbol{\nu} \times \mathbf{u}|_\Sigma \in (L^2(\Sigma))^3,\ \boldsymbol{\nu} \times \mathbf{u} = 0 \text{ on } \Gamma \right\}$$

with norms

$$\begin{aligned}
\|\mathbf{u}\|_{H(\mathrm{curl};\Omega)} &= \sqrt{\|\mathbf{u}\|^2_{(L^2(\Omega))^3} + \|\nabla \times \mathbf{u}\|^2_{(L^2(\Omega))^3}} \\
\|\mathbf{u}\|_X &= \sqrt{\|\mathbf{u}\|^2_{H(\mathrm{curl};\Omega)} + \|\boldsymbol{\nu} \times \mathbf{u}\|^2_{(L^2(\Sigma))^3}}.
\end{aligned}$$

The associated inner products are

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u} \cdot \overline{\mathbf{v}}\, dV, \qquad \langle \mathbf{u}, \mathbf{v} \rangle_\Sigma = \int_\Sigma \mathbf{u} \cdot \overline{\mathbf{v}}\, dA.$$

Following the usual Galerkin philosophy we obtain the variational problem of finding $\mathbf{u} \in X$ such that

$$(\nabla \times \mathbf{E}, \nabla \times \boldsymbol{\phi}) - \kappa^2(\varepsilon \mathbf{E}, \boldsymbol{\phi}) - i\kappa \langle \mathbf{E}_T, \boldsymbol{\phi}_T \rangle_\Sigma = \langle \mathbf{g}, \boldsymbol{\phi}_T \rangle_\Sigma$$

for all $\boldsymbol{\phi} \in X$. Using unique continuation it is possible to prove uniqueness of any solution to this problem. Then using the Helmholtz decomposition, the Webber compactness result, and the Fredholm Alternative existence of a solution can be obtained [3, 13]. The key here is that it is essential to use a Helmholtz decomposition in the analysis. This suggests that a successful finite element space should also possess a suitable discrete Helmholtz decomposition.

Let $\mathcal{T}_h$ denote a conforming and regular tetrahedral grid covering Ω. A suitable finite element discretization is a provided by the Nédélec elements [14]. In particular the lowest order space is

$$\begin{aligned} X_h \quad = \quad & \{\mathbf{u}_h \in H(\text{curl}; \Omega) \mid \mathbf{u}_h|_K = \mathbf{a}_K + \mathbf{b}_K \times \mathbf{x} \\ & \mathbf{a}_K, \mathbf{b}_K \in \mathbb{C}^3, \quad \forall K \in \mathcal{T}_h \big\} . \end{aligned}$$

The degrees of freedom (unknowns) for this element are $\int_e \mathbf{u}_h \cdot \boldsymbol{\tau}_h\, ds$ for each edge e of each tetrahedron where $\boldsymbol{\tau}_e$ is an appropriately oriented tangent vector.

An important property of Nédélec's elements is that they contain many gradients. In the lowest order case, if

$$S_h = \{p_h \in S \mid p_h|_K \in P_1, \quad \forall K \in \mathcal{T}_h,\ p = 0 \text{ on } \Gamma,\ p \text{ is constant on } \Sigma\},$$

then $\nabla S_h \subset X_h$. Furthermore the curl satisfies a Friedrich's type inequality on the orthogonal complement of ∇S_h in X_h weighted by ϵ [13].

Using the properties of edge finite element spaces: in particular a discrete analogue of compactness [12] and an extension theorem, Gatica and Meddahi [10, 13]. prove that if h is small enough then there exists a unique finite element solution $\mathbf{E}_h \in X_h$ and

$$\|\mathbf{E} - \mathbf{E}_h\|_{H(\text{curl};\Omega)} \to 0 \text{ as } h \to 0.$$

For sufficiently smooth solutions, lowest order Nédélec elements give $O(h)$ convergence. In fact it is very beneficial to use higher order elements when κ is large [9].

For other approachs to error estimates using mixed method or the regular decomposition see [4, 11].

More recently several open source implementations of common finite element families for the Maxwell system have appeared making edge elements much more accessible. An example is the NGSolve package [15] that contains edge elements of arbitrary order, and a sophisticated interface based on Python that allows the user easy access to the underlying components of the software. This allows the easy use of edge elements in applications, for example to photonic crystals [1], in which ϵ varies markedly from place to place. This in turn raises interesting questions of approximation theory and the regularity of the solutions [7, 6].

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The time-harmonic Maxwell equations with impedance boundary conditions in domains with a polyhedral or an analytic boundary

Serge Nicaise

(joint work with Jérôme Tomezyk)

We are interested in properties of solutions of the Maxwell system with impedance boundary condition

$$\begin{cases} \operatorname{curl}\mathbf{E} - i\omega\mathbf{H} = \mathbf{0} \quad \text{and} \quad \operatorname{curl}\mathbf{H} + i\omega\mathbf{E} = \mathbf{J} \quad \text{in } \Omega, \\ \mathbf{H}\times\mathbf{n} - \lambda\,\mathbf{E_t} = \mathbf{0} \quad \text{on } \Gamma := \partial\Omega, \end{cases} \tag{1}$$

where $\mathbf{E}$ is the electric part and $\mathbf{H}$ is the magnetic part of the electromagnetic field, $\omega > 0$ corresponds to the wave number or frequency. The right hand side $\mathbf{J}$ is the current density which – in the absence of free electric charges – is divergence free, namely

$$\operatorname{div}\mathbf{J} = \mathbf{0} \quad \text{in } \Omega.$$

The impedance λ is supposed to be a smooth function satisfying

$$\lambda : \Gamma \to \mathbb{R}, \quad \text{such that} \quad \lambda(x) \neq 0, \forall x \in \partial\Omega, \tag{2}$$

see for instance [4, 3]. The case $\lambda_{\text{imp}} \equiv 1$ is also called the Silver-Müller boundary condition [1].

Note that the boundary condition in (1) is an absorbing boundary condition that is used to reduce the full space into a bounded domain.

We present two variational formulations of this problem [2, §4.4.d] and analyse more carefully the second one that consists in keeping the full electromagnetic field $(\mathbf{E}, \mathbf{H})$ as unknown, using the variational space

$$\mathbf{V} = \big\{(\mathbf{E}, \mathbf{H}) \in \big(\mathbf{H}(\text{curl}, \Omega) \cap \mathbf{H}(\text{div}, \Omega)\big)^2 : \ \mathbf{H} \times \mathbf{n} = \lambda \mathbf{E_t} \ \text{ on } \ \partial\Omega\big\}, \tag{3}$$

considering the impedance condition as an essential boundary condition.

Hence the proposed variational formulation is: Find $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$ such that

$$a(\mathbf{E}, \mathbf{H}; \mathbf{E}', \mathbf{H}') = \int_\Omega \Big(i\omega \mathbf{J} \cdot \overline{\mathbf{E}}' + \mathbf{J} \cdot \text{curl}\, \overline{\mathbf{H}}'\Big)\, dx, \forall (\mathbf{E}', \mathbf{H}') \in \mathbf{V}, \tag{4}$$

with the choice

$$a(\mathbf{E}, \mathbf{H}; \mathbf{E}', \mathbf{H}') = a_{\omega,s}(\mathbf{E}, \mathbf{E}') + a_{\omega,s}(\mathbf{H}, \mathbf{H}') - i\omega \int_{\partial\Omega} (\lambda\, \mathbf{E_t} \cdot \overline{\mathbf{E}}'_\mathbf{t} + \frac{1}{\lambda}\, \mathbf{H_t} \cdot \overline{\mathbf{H}}'_\mathbf{t})\, d\sigma,$$

with a positive real parameter $s \in [1, 2]$ appropriately fixed and

$$a_{\omega,s}(\mathbf{u}, \mathbf{v}) = \int_\Omega (\text{curl}\, \mathbf{u} \cdot \text{curl}\, \bar{\mathbf{v}} + s\, \text{div}\, \mathbf{u}\, \text{div}\, \bar{\mathbf{v}} - \omega^2 \mathbf{u} \cdot \bar{\mathbf{v}})\, dx.$$

The interest of this formulation stays on the fact that if Ω has a C^2 boundary or if it is a convex polyhedron, then $\mathbf{V}$ is embedded in $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$ and a is (weakly) coercive on $\mathbf{V}$, see [2, §4.4.d] for the smooth case and [5] for the polyhedral case. Consequently problem (4) induces a Fredholm operator of index zero from $\mathbf{V}$ into its dual and uniqueness implies existence and uniqueness.

The coerciveness of a implying that the corresponding system is an elliptic system in the Agmon-Douglis-Nirenberg sense, standard shift regularity results hold. Furthermore if the right-hand side $\mathbf{J}$ belongs to $\mathbf{H}(\text{div} = 0; \Omega)$ and $-\omega^2/s$ is not an eigenvalue of the Laplace operator Δ_{Dir}, then any solution $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$ of the variational problem (4) satisfies (1).

Another advantage of this formulation concerns its numerical approximation. Indeed as $\mathbf{V}$ is a subspace of $\mathbf{H}^1(\Omega)^2$, we can consider an *hp* C^0 finite element approximation, see [5] for polyhedral domains. But if the boundary is analytic, we cannot impose the impedance boundary condition in the finite element space. Hence we need to adopt a nonconforming approximation [6].

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Time-harmonic Scattering in exterior domains with mixed boundary conditions

Frank Osterbrink
(joint work with Dirk Pauly)

Let $\Omega \subset \mathbb{R}^3$ be an exterior weak Lipschitz domain with weak Lipschitz interface $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$ in the sense of [1, Definition 2.3, Definition 2.5]. We study the boundary value problem

$$(1) \qquad \begin{aligned} -\operatorname{rot} H + i\omega\mu E &= -F \quad \text{in} \quad \Omega, & E \times \nu &= 0 \quad \text{on} \quad \Gamma_1, \\ \operatorname{rot} E + i\omega\mu H &= G \quad \text{in} \quad \Omega, & H \times \nu &= 0 \quad \text{on} \quad \Gamma_2, \end{aligned}$$

where $\omega \in \mathbb{C} \setminus \{0\}$ and ε, μ are symmetric, uniformly positive definite L^∞-matrix fields, which are asymptotically a multiple of the identity, i.e., $\varepsilon = \varepsilon_0 \cdot \mathbb{1} + \hat{\varepsilon}$, $\mu = \mu_0 \cdot \mathbb{1} + \hat{\mu}$ with $\varepsilon_0, \mu_0 \in \mathbb{R}^+$ and

$$\hat{\varepsilon}, \hat{\mu} = \mathcal{O}(r^{-\kappa}) \quad \text{for} \quad r \longrightarrow \infty, \qquad \kappa > 1.$$

As shown in [5], working in the framework of polynomially weighted Sobolev spaces, the same methods as in [2] (see also [3]) are sufficient for solving system (1). In fact, the linear operator

$$\begin{aligned} \mathcal{M} \quad : \quad \mathsf{R}_{\Gamma_1}(\Omega) \times \mathsf{R}_{\Gamma_2}(\Omega) \subset \mathsf{L}^2_\varepsilon(\Omega) \times \mathsf{L}^2_\mu(\Omega) \quad &\longrightarrow \quad \mathsf{L}^2_\varepsilon(\Omega) \times \mathsf{L}^2_\mu(\Omega) \\ (E, H) \quad &\longmapsto \quad (-i\varepsilon^{-1} \operatorname{rot} H, i\mu^{-1} \operatorname{rot} E) \end{aligned}$$

is selfadjoint, hence $(\mathcal{M} - \omega)^{-1}$ is continuous for $\omega \in \mathbb{C} \setminus \mathbb{R}$. For $\omega \in \mathbb{R} \setminus \{0\}$, we are solving in the continuous spectrum of $\mathcal{M}$ and therefore $(\mathcal{M} - \omega)^{-1}$ exists only on a dense subset of $\mathsf{L}^2_\varepsilon(\Omega) \times \mathsf{L}^2_\mu(\Omega)$. However, using Eidus' limiting absorption principle [4] we are still able to obtain weaker so called *radiating solutions* E, H by restricting ourselves to data F, G in

$$\mathsf{L}^2_{\mathrm{s}}(\Omega) := \left\{ u \in \mathsf{L}^2_{\mathrm{loc}}(\Omega) \mid (1+r^2)^{s/2} u \in \mathsf{L}^2(\Omega) \right\}$$

for some $s > 1/2$. These solutions are then elements of

$$\mathbf{R}_{\mathrm{t}}(\Omega) := \left\{ u \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \mid \operatorname{rot} u \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \right\} \quad \forall\, t < -1/2$$

and satisfy the radiation condition

$$\left(\varepsilon_0 E - \sqrt{\varepsilon_0 \mu_0}\, \xi \times H, \mu_0 H + \sqrt{\varepsilon_0 \mu_0}\, \xi \times E \right) \in \mathsf{L}^2_s(\Omega) \times \mathsf{L}^2_s(\Omega)\,, \qquad \xi := x/r,$$

for some $s > -1/2$. In other words the resolvent $(\mathcal{M} - \omega)^{-1}$ of $\mathcal{M}$ may indeed be extended continuously to the real axis. The essential ingredients needed for the underlying limit process are the polynomial decay of eigensolutions, an a-priori-estimate for solutions corresponding to non-real frequencies, a Helmholtz-type decomposition and *Weck's local selection theorem*, i.e.,

$$\mathbf{R}_{\Gamma_1}(\Omega) \cap \mathbf{D}_{\Gamma_2}(\Omega) \hookrightarrow \mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega}) \text{ is compact.}$$

While the first two are obtained by transferring well known results for the scalar Helmholtz equation to the time-harmonic Maxwell equations using a suitable decomposition of the fields E and H the last one is an assumption on the quality of the boundary, which holds true in weak Lipschitz domains.

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On Some Compact Embeddings in Various Hilbert Complexes

Dirk Pauly

(joint work with Sebastian Bauer, Michael Schomburg, and Walter Zulehner)

We study Hilbert complexes

$$
\begin{aligned}
&\ldots \xrightarrow{\cdots} D(\mathrm{A}_0) \xrightarrow{\mathrm{A}_0} D(\mathrm{A}_1) \xrightarrow{\mathrm{A}_1} \mathsf{H}_2 \xrightarrow{\cdots} \ldots,\\
&\ldots \xleftarrow{\cdots} \mathsf{H}_0 \xleftarrow{\mathrm{A}_0^*} D(\mathrm{A}_0^*) \xleftarrow{\mathrm{A}_1^*} D(\mathrm{A}_1^*) \xleftarrow{\cdots} \ldots
\end{aligned}
\tag{1}
$$

for two densely defined and closed linear operators

$$\mathrm{A}_0 : D(\mathrm{A}_0) \subset \mathsf{H}_0 \longrightarrow \mathsf{H}_1, \qquad \mathrm{A}_1 : D(\mathrm{A}_1) \subset \mathsf{H}_1 \longrightarrow \mathsf{H}_2$$

with Hilbert space adjoints A_0^* and A_1^*. The corresponding mathematical analysis, such as closed ranges, bounded inverses, Friedrichs/Poincaré type estimates, Helmholtz type decompositions, solution theories, finite cohomology groups, benefits strongly from a certain compact embedding, i.e.,

$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1. \tag{2}$$

In the Hilbert complexes arising in mathematical physics a proper Sobolev setting with repective boundary conditions on bounded weak or strong Lipschitz domains is most important. In these cases a compact embedding (2) can be shown with the very elegant technique of regular potentials.

Classical examples are the de Rham complex

$$
\begin{aligned}
&\{0\} \xrightarrow{\iota_{\{0\}}} \mathring{\mathsf{H}}^1 \xrightarrow{\mathring{\nabla}} \mathring{\mathsf{R}} \xrightarrow{\mathring{\mathrm{rot}}} \mathring{\mathsf{D}} \xrightarrow{\mathring{\mathrm{div}}} \mathsf{L}^2 \xrightarrow{\pi_{\mathbb{R}}} \mathbb{R},\\
&\{0\} \xleftarrow{\pi_{\{0\}}} \mathsf{L}^2 \xleftarrow{-\mathrm{div}} \mathsf{D} \xleftarrow{\mathrm{rot}} \mathsf{R} \xleftarrow{-\nabla} \mathsf{H}^1 \xleftarrow{\iota_{\mathbb{R}}} \mathbb{R},
\end{aligned}
\tag{3}
$$

which may also be generalised using alternating differential forms (even with mixed boundary conditions), the elasticity complex

$$
\begin{aligned}
&\{0\} \xrightarrow{\iota_{\{0\}}} \mathring{\mathsf{H}}^1 \xrightarrow{\mathrm{sym}\mathring{\nabla}} \mathring{\mathsf{RR}}_{\mathbb{S}}^{\top} \xrightarrow{\mathrm{Rot}\mathring{\mathrm{Rot}}_{\mathbb{S}}^{\top}} \mathring{\mathsf{D}}_{\mathbb{S}} \xrightarrow{\mathring{\mathrm{Div}}_{\mathbb{S}}} \mathsf{L}^2 \xrightarrow{\pi_{\mathsf{RM}}} \mathsf{RM},\\
&\{0\} \xleftarrow{\pi_{\{0\}}} \mathsf{L}^2 \xleftarrow{-\mathrm{Div}_{\mathbb{S}}} \mathsf{D}_{\mathbb{S}} \xleftarrow{\mathrm{RotRot}_{\mathbb{S}}^{\top}} \mathsf{RR}_{\mathbb{S}}^{\top} \xleftarrow{-\mathrm{sym}\nabla} \mathsf{H}^1 \xleftarrow{\iota_{\mathsf{RM}}} \mathsf{RM},
\end{aligned}
\tag{4}
$$

and the biharmonic (or divDiv) complex

$$
\begin{aligned}
&\{0\} \xrightarrow{\iota_{\{0\}}} \mathring{\mathsf{H}}^2 \xrightarrow{\nabla\mathring{\nabla}} \mathring{\mathsf{R}}_{\mathbb{S}} \xrightarrow{\mathring{\mathrm{Rot}}_{\mathbb{S}}} \mathring{\mathsf{D}}_{\mathbb{T}} \xrightarrow{\mathring{\mathrm{Div}}_{\mathbb{T}}} \mathsf{L}^2 \xrightarrow{\pi_{\mathsf{RT}}} \mathsf{RT},\\
&\{0\} \xleftarrow{\pi_{\{0\}}} \mathsf{L}^2 \xleftarrow{\mathrm{divDiv}_{\mathbb{S}}} \mathsf{DD}_{\mathbb{S}} \xleftarrow{\mathrm{sym}\,\mathrm{Rot}_{\mathbb{T}}} \mathsf{R}_{\mathbb{T},\mathrm{sym}} \xleftarrow{-\mathrm{dev}\,\nabla} \mathsf{H}^1 \xleftarrow{\iota_{\mathsf{RT}}} \mathsf{RT}.
\end{aligned}
\tag{5}
$$

Typical compact embeddings (2) read as follows:

$$
\begin{aligned}
&\text{in (3)}: && \mathring{\mathsf{H}}^1 \hookrightarrow \mathsf{L}^2, && \mathring{\mathsf{R}} \cap \mathsf{D} \hookrightarrow \mathsf{L}^2, && \mathring{\mathsf{D}} \cap \mathsf{R} \hookrightarrow \mathsf{L}^2, && \mathsf{H}^1 \hookrightarrow \mathsf{L}^2\\
&\text{in (4)}: && \mathring{\mathsf{H}}^1 \hookrightarrow \mathsf{L}^2, && \mathring{\mathsf{RR}}_{\mathbb{S}}^{\top} \cap \mathsf{D}_{\mathbb{S}} \hookrightarrow \mathsf{L}^2_{\mathbb{S}}, && \mathring{\mathsf{D}}_{\mathbb{S}} \cap \mathsf{RR}_{\mathbb{S}}^{\top} \hookrightarrow \mathsf{L}^2_{\mathbb{S}}, && \mathsf{H}^1 \hookrightarrow \mathsf{L}^2\\
&\text{in (5)}: && \mathring{\mathsf{H}}^2 \hookrightarrow \mathsf{L}^2, && \mathring{\mathsf{R}}_{\mathbb{S}} \cap \mathsf{DD}_{\mathbb{S}} \hookrightarrow \mathsf{L}^2_{\mathbb{S}}, && \mathring{\mathsf{D}}_{\mathbb{T}} \cap \mathsf{R}_{\mathbb{T},\mathrm{sym}} \hookrightarrow \mathsf{L}^2_{\mathbb{T}}, && \mathsf{H}^1 \hookrightarrow \mathsf{L}^2
\end{aligned}
$$

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Nonconforming Trefftz virtual elements for the Helmholtz problem

Ilaria Perugia

(joint work with Lorenzo Mascotto, Alexander Pichler)

The virtual element method (VEM) is a recent generalization of the finite element method to polytopal grids [1, 2]. The main feature of VEM is that test and trial spaces consist of functions that are not known in closed form, but that are solutions to local differential problems mimicking the target one. Despite this fact, the method is made fully computable by defining two tools, namely suitable mappings from local approximation spaces into spaces of known functions (typically polynomials), and suitable bilinear/sesquilinear stabilization forms.

I have presented a novel VEM for the two dimensional Helmholtz problem endowed with impedance boundary conditions, which was introduced in [3]. The local approximation spaces consist of Trefftz functions, namely, functions belonging to the kernel of the Helmholtz operator. The global trial and test spaces are not fully discontinuous, as in most Trefftz methods [4], but rather interelement continuity is imposed in a nonconforming fashion (à la Crouzeix-Raviart). Although their functions are only implicitly defined, as typical of the VEM framework, they contain discontinuous subspaces made of functions known in closed form and with good approximation properties (plane waves, in the considered case).

An essential ingredient in the implementation of the method is an edgewise orthogonalization-and-filtering process described in [5]. This process allows to dramatically reduce the number of basis functions without deteriorating the accuracy. It also have a positive effect on the conditioning of the overall method.

After carrying out an h-version error analysis of the method, I have presented numerical tests reported in [5], which *i)* demonstrate the h-version theoretical convergence rates; *ii)* show that, in case of smooth solutions, the p-version achieves exponential convergence; *iii)* show that, in case of singular solutions, the *hp*-version on graded meshes achieves exponential convergence; *iv)* show that the new method is competitive with other plane wave-based methods in terms of number of degrees of freedom needed in order to achieve a given accuracy, and is able to reach a higher accuracy, before onset of instabilities.

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Memoryless Evolutionary Problems as Dynamic Abstract Friedrichs Systems

Rainer Picard

Following the setting presented in [2] many models of mathematical physics share a common form

$$\partial_t V + AU = F,$$

where ∂_t denotes time-differentiation and A is a maximal accretive linear operator. Indeed, in standard cases A is simply skew-selfadjoint in an underlying real Hilbert space H. The unknowns U, V are here linked by a so-called material law

$$V = \mathcal{M}U.$$

In a number of studies, see e.g. [1, 4, 5], it has been illustrated, that this simple framework is indeed suitable for a large number of complex applications including even time-delay problems and fractional time derivatives. A typical and simple case, which we shall focus on here is

$$\mathcal{M} = M_0 + \partial_t^{-1} M_1, \tag{1}$$

where M_0, M_1 are continuous linear operators in H, in particular, M_0 is selfadjoint. The operator ∂_t^{-1} appearing here is forward causal time-integration, which can be properly realized in a weighted Hilbert space

$$H_{\rho,0}(\mathbb{R}, H) := \left\{ f \in L^{2,\mathrm{loc}}(\mathbb{R}, X) \mid \int_{\mathbb{R}} \langle f(t) | f(t) \rangle_H \ \exp(-2\rho t)\ dt < \infty \right\}$$

with the natural weighted inner product

$$(f, g) \mapsto \langle f|g \rangle_{\rho,0,0} := \int_{\mathbb{R}} \langle f(t) | g(t) \rangle_H \ \exp(-2\rho t)\ dt.$$

In this setting the time-derivative ∂_0 is a normal, strictly positive operator, [2], [3, Chapter 6], indeed we have

$$\partial_t \geq \rho \tag{2}$$

in the sense that we have for the numerical range of ∂_t that

$$\langle U|\partial_t U\rangle_{\rho,0,0} \geq \rho \langle U|U\rangle_{\rho,0,0}$$

for all U in the domain $\operatorname{dom}(\partial_t)$ of ∂_t. This strict positivity carries over to a basic well-posedness constraint for material laws of the form (1) such that

$$\rho M_0 + M_1 \geq c_0 > 0 \tag{3}$$

holds for some real number c_0 and all sufficiently large positive $\rho \in \mathbb{R}$. Since in this case $\partial_t M(\partial_t) = \partial_t M_0 + M_1$ is indeed a local operator in time, we speak of a memoryless material law.

For A skew-selfadjoint these systems can be transformed into operator equations of the form

$$1 + \mathcal{A},$$

where now $\mathcal{A}$ is skew-selfadjoint. Since the classical Friedrichs systems can also be brought into this formal form, we speak of abstract Friedrichs systems. With the skew-selfadjointness of A playing a central role, we present a number of tools, useful for modeling concrete problems, to construct such operators. These tools as well as the utility of the general setting are illustrated with various applications from mathematical physics.

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Solving Elasticity in H(curl)

Joachim Schöberl
(joint work with Astrid Pechstein, Michael Neunteufel, Philip Lederer)

Vector valued finite elements with tangential or normal continuity are well established in the electromagnetics community. We show how to solve the elasticity equation with tangentially continuous H(curl) finite elements, and why this make sense. The TDNNS mixed method proposed by A. Pechstein and J. Schöberl uses the H(curl) function space for the displacement, and H(div div) for the stresses. The canonical finite elements for the stress space are normal-normal continuous symmetric matrix valued elements. It is shown that these elements are robust for anisotropic meshes, and hand over to plate and shell models. A recent proposal by Gopalakrishnan, Lederer and Schöberl switches the continuity: The vector vector variable u is normal-continuous, while the matrix variable sigma is now normal-tangential continuous, and non-symmetric. This method provides an exactly divergence vector field u, and is thus useful for incompressible flows. We show how all these spaces fit into a complex similar to the de Rham complex.

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On the helicity of a bounded domain and the Biot–Savart operator

Alberto Valli

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, with Lipschitz boundary $\partial\Omega$ and unit outward normal vector $\mathbf{n}$ on $\partial\Omega$. Consider the Hilbert space

$$\boldsymbol{\mathcal{V}} = \{\mathbf{v} \in (L^2(\Omega))^3 \,|\, \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}\,.$$

The helicity of a vector field $\mathbf{v} \in \boldsymbol{\mathcal{V}}$, a concept introduced by Woltjer (1958) and named by Moffatt (1969), is given by

$$H(\mathbf{v}) = \frac{1}{4\pi} \int_\Omega \int_\Omega \mathbf{v}(\mathbf{x}) \times \mathbf{v}(\mathbf{y}) \cdot \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \, d\mathbf{x} \, d\mathbf{y}\,. \tag{1}$$

It is a "measure of the extent to which the field lines wrap and coil around one another" (Cantarella et al. (2001)).

The Biot–Savart operator BS is defined in $\boldsymbol{\mathcal{V}}$ as

$$BS(\mathbf{v})(\mathbf{x}) = \frac{1}{4\pi} \int_\Omega \mathbf{v}(\mathbf{y}) \times \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \, d\mathbf{y}\,. \tag{2}$$

Hence we can rewrite $H(\mathbf{v}) = \int_\Omega \mathbf{v} \cdot BS(\mathbf{v})$.

It is well-known that $BS(\mathbf{v})$ satisfies in Ω the relations $\operatorname{curl} BS(\mathbf{v}) = \mathbf{v}$ and $\operatorname{div} BS(\mathbf{v}) = 0$. Let us define $\widehat{BS}(\mathbf{v})$ the $(L^2(\Omega))^3$-orthogonal projection of $BS(\mathbf{v})$ over $\boldsymbol{\mathcal{V}}$; since it differs from $BS(\mathbf{v})$ by a gradient of a scalar function, it satisfies

$$\left\{ \begin{array}{ll} \operatorname{curl} \widehat{BS}(\mathbf{v}) = \mathbf{v} & \text{in } \Omega \\ \operatorname{div} \widehat{BS}(\mathbf{v}) = 0 & \text{in } \Omega \\ \widehat{BS}(\mathbf{v}) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \,, \end{array} \right. \tag{3}$$

and one also easily sees that $H(\mathbf{v}) = \int_\Omega \mathbf{v} \cdot \widehat{BS}(\mathbf{v})$.

In Valli (2019) it is shown that the couple $(\widehat{BS}(\mathbf{v}), \mathbf{0})$ is the solution $(\mathbf{u}, \mathbf{q}) \in \boldsymbol{\mathcal{Z}} \times \boldsymbol{\mathcal{H}}$ of the following constrained least-square problem:

$$\begin{array}{llll} \int_\Omega \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{w} + \int_\Omega \mathbf{q} \cdot \mathbf{w} & = & \int_\Omega \mathbf{v} \cdot \operatorname{curl} \mathbf{w} & \forall\, \mathbf{w} \in \boldsymbol{\mathcal{Z}} \\ \int_\Omega \mathbf{u} \cdot \mathbf{p} & = & 0 & \forall\, \mathbf{p} \in \boldsymbol{\mathcal{H}} \,, \end{array} \tag{4}$$

where

$$\begin{aligned} \boldsymbol{\mathcal{X}} &= \{\mathbf{w} \in H(\operatorname{curl};\Omega) \,|\, \operatorname{curl} \mathbf{w} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\} \,, \\ \boldsymbol{\mathcal{Z}} &= \{\mathbf{w} \in \boldsymbol{\mathcal{X}} \mid \oint_{\gamma_j} \mathbf{w} \cdot \mathbf{t}_j = 0 \text{ for } j = 1, \ldots, g\} \,, \\ \boldsymbol{\mathcal{H}} &= \operatorname{grad} H^1(\Omega) \,, \end{aligned}$$

and the closed curves $\gamma_j \subset \partial\Omega$ are a basis of the first homology group of $\overline{\Omega}$.

It is known that $\widehat{BS}$ is a self-adjoint and compact operator in $\boldsymbol{\mathcal{V}}$ (see, e.g., Cantarella et al. (2001)), thus its spectrum is discrete. Defining the helicity of a domain Ω by

$$H_\Omega = \sup_{\mathbf{v} \in \boldsymbol{\mathcal{V}} \setminus \mathbf{0}} \frac{|H(\mathbf{v})|}{\|\mathbf{v}\|^2_{L^2(\Omega)}} \,, \tag{5}$$

we easily obtain the spectral representation

$$H_\Omega = |\lambda^\Omega_{\max}| \,,$$

where $\lambda^\Omega_{\max}$ is the eigenvalue of $\widehat{BS}$ in Ω of maximum absolute value.

A finite element scheme for calculating the eigenvalues of $\widehat{BS}$ has been proposed in Alonso Rodríguez et al. (2018), and permits to compute the helicity of Ω.

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On Nonlocal *H*-convergence

Marcus Waurick

In classical homogenisation problems for elliptic-type equations, one studies coefficients

$$M(\alpha, \beta, \Omega) := \{a \in L^\infty(\Omega; \mathbb{R}^{d\times d}; \Re a(x) \geq \alpha, \Re a(x)^{-1} \geq 1/\beta \,(\text{a.e. } x \in \Omega)\}$$

for some fixed $0 < \alpha < \beta$ and $\Omega \subseteq \mathbb{R}^d$ open and bounded. The inequalities are understood in the sense of positive definiteness. Then a sequence $(a_n)_n$ in $M(\alpha, \beta.\Omega)$ is said to *locally H-converge* to $a \in M(\alpha, \beta, \Omega)$, if for all $f \in H^{-1}(\Omega)$ and corresponding solutions $(u_n)_n$ in $H_0^1(\Omega)$ of the Dirichlet problem

$$\langle a_n \operatorname{grad} u_n, \operatorname{grad} \phi\rangle = f(\phi) \quad (\phi \in H_0^1(\Omega))$$

we have that $u_n \rightharpoonup u \in H_0^1(\Omega)$ and $a_n \operatorname{grad} u_n \rightharpoonup a \operatorname{grad} u \in L^2(\Omega)^d$, where u is the unique solution of

$$\langle a \operatorname{grad} u, \operatorname{grad} \phi\rangle = f(\phi) \quad (\phi \in H_0^1(\Omega)).$$

In this talk, we address the situation, where the condition on $(a_n)_n$ to be L^∞-matrices is relaxed to the extend that $(a_n)_n$ is now only assumed to live in

$$\mathcal{M}(\alpha, \beta, \Omega) := \{a \in \mathcal{B}(L^2(\Omega)^d); \Re a \geq \alpha, \Re a^{-1} \geq 1/\beta\}.$$

It is relatively easy to see that the above definition has to be amended to be fitted also for this potentially nonlocal situation. In fact, for the definition of nonlocal H-convergence, one needs to introduce a second variational problem with the curl-operator (without boundary condition) replacing the above grad.

Having introduced the new notion of nonlocal H-convergence, we show that this convergence induces a compact, metrisable Hausdorff topology on (subsets of) $\mathcal{M}(\alpha, \beta, \Omega)$. Moreover, we show that nonlocal H-convergence coincides with local H-convergence if restricted to $M(\alpha, \beta, \Omega)$. It turns out that different choices for the boundary conditions for grad and curl lead to different topologies. Thus, nonlocal H-convergence 'sees' the boundary, a property not shared by local H-convergence. Furthermore, we provide a div-curl-type characterisation for nonlocal H-convergence, which facilitates the computation of nonlocal H-limits in practice. The results have applications to Maxwell's equations with nonlocal, that is, convolution-type material law relevant in nonlocal response theory for electromagnetics; see also [1, Chapter 10] and for the application part see [2] and the survey paper [4].

The talk is based on results in [3].

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Geometric decompositions in finite element exterior calculus

Ragnar Winther

(joint work with Richard S. Falk)

The theory for approximation of Hilbert complexes, presented in [2], explains the importance of bounded cochain projections for the construction of stable numerical methods for the Hodge Laplace problems. The purpose of this talk is to give a review of various constructions of such operators, and to discuss their properties.

The classical cochain projections are the canonical projections constructed from the degrees of freedom. However, these operators are not bounded in the desired Sobolev norms. Motivated by this observation we reviewed the construction of the so-called "smoothed projections," as it was presented in [1]. This gives L^2 bounded projections, but in contrast to the canonical projections, they are nonlocal. We then followed up by presented the construction of local cochain projections given in [3]. In the final part of the talk we briefly reviewed some results for the bubble transform, see [4], which potentially can lead to cochain projections with bounds independent of the polynomial degree.

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