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Graph Theory

Organized by
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Abstract. Graph theory is a rapidly developing area of mathematics. Recent years have seen the development of deep theories, and the increasing importance of methods from other parts of mathematics. The workshop on Graph Theory brought together together a broad range of researchers to discuss some of the major new developments. There were three central themes, each of which has seen striking recent progress: the structure of graphs with forbidden subgraphs; graph minor theory; and applications of the entropy compression method. The workshop featured major talks on current work in these areas, as well as presentations of recent breakthroughs and connections to other areas. There was a particularly exciting selection of longer talks, including presentations on the structure of graphs with forbidden induced subgraphs, embedding simply connected 2-complexes in 3-space, and an announcement of the solution of the well-known Oberwolfach Problem.

Mathematics Subject Classification (2010): 05Bxx, 05Cxx, 05Dxx.

Introduction by the Organizers

Graph theory is a rapidly developing area of mathematics. Recent years have seen the development of deep theories, and the increasing importance of methods from other parts of mathematics. The Graph Theory Workshop brought together a diverse group of internationally renown experts for a stimulating week of knowledge sharing and research. The workshop showcased some of the major recent developments. There were three central themes, each of which has seen striking recent progress: the structure of graphs with forbidden subgraphs, graph minor theory, and applications of the entropy compression method. The 57 participants included many students and early career researchers, from many different geographic locations, and included a significant number of women. The organisers, Jim Geelen (Waterloo), Dan Král (Brno), and Alex Scott (Oxford), also encouraged diversity within the program: for example, two of the five longer talks were by early career researchers and one of those was a female graduate student.

The week was fully utilized with morning talks running from 9 a.m. through to lunch and afternoon talks from around 4 p.m. through to dinner, with the exception of the traditional Wednesday afternoon hike. Each day of the workshop started with a longer (50 minute) talk highlighting a recent major breakthrough in the area.

- Bojan Mohar presented surprising new results, with Zdenek Dvořák and Petr Hliněný, on obstructions to crossing number k. (The *crossing number* of a graph is the minimum number of edge crossing required to draw the graph in the plane.) For each integer k, Mohar, Dvořák, and Hliněný describe the structure of all sufficiently large minimal obstructions to crossing-number at most k.
- Johannes Carmesin, a postdoctoral fellow at Cambridge University, presented his spectacular new results on embedding simply connected 2-complexes in 3-space. This result is a 3-dimensional analogue of Kuratowski's characterization of planar graphs. Carmesin's proof is an ingenious blend of topology, graph theory, and matroid theory.
- Sophie Spirkl, a doctoral student at Princeton University, presented her exciting research with Maria Chudnovsky, Alex Scott, and Paul Seymour on induced subgraphs, which was one of the main themes of the meeting.
- In what was surely the most delightful talk of the week, Carsten Thomassen presented his new result that every graph of uncountable chromatic number contains a subgraph of infinite edge-connectivity.
- The last long talk of the week was given by Derrick Osthus who, quite fittingly, presented his proof (in joint work with Stefan Glock, Felix Joos, Jaehoon Kim and Daniella Kühn) of the Oberwolfach Problem for all sufficiently large workshops.

Continuing a well-established tradition at graph theory workshops in Oberwolfach, the first day of the meeting was mostly devoted to 5 minute mini-talks by all of the participants. In these talks the participants either posed open problems or announced recent theorems; the organizers were particularly pleased with the high quality of these short presentations. After dinner on Monday there was a lively open problem session.

During the remainder of the week there were a number of short (25 minute) talks; highlights among these talks included:

- Benny Sudakov spoke about his exciting new work, with Matija Bucić, Matthew Kwan and Alexey Pokrovsky, on Rota's Basis Conjecture. Rota's Conjecture asserts that, given n disjoint bases in an n-dimensional vector space, there exist n disjoint transversals that are bases. Sudakov et al.

proved the existence of (almost) $\frac{1}{2}n$ disjoint transversals, greatly improving on the best previous bound of $\frac{n}{\log n}$ due to Sally Dong and Jim Geelen.

- Rose McCarty, a graduate student from the University of Waterloo, presented her new result, with Jim Geelen and Paul Wollan, characterizing graphs with huge rank-width in terms of their unavoidable vertex minors. This result will, hopefully, be a first step in general structure theory for vertex-minor-closed classes of graphs.
- Maria Chudnovsky spoke about very recent work, with Alex Scott, Paul Seymour, and Sophie Spirkl, on recognizing odd-hole-free graphs. This class is important due to connections with the famous Strong Perfect Graph Theorem and because the class is χ-bounded.

The relaxed schedule format allowed a reasonable amount of time for participants to engage in research and focused discussion groups. To help start the meeting, the organizers invited three participants to run focused discussion groups; these included DP-colourings (Kostochka), vertex minors (Wollan), and tangles (Diestel). There were many other lively discussion groups held throughout the week, both in the afternoon and in the evening. The week was also a considerable success with respect to research productivity; some highlights here include:

- Maria Chudnovsky, Alex Scott and Paul Seymour extended their recognition algorithm for the class of odd-hole-free graphs to the class of graphs with no long odd hole.
- In his 5 minute talk on the Monday, Jim Geelen proposed working on a 15 year old conjecture on the Erdős-Pósa property for certain collections of paths. Jim Geelen and Sergey Norin proved this conjecture during the workshop.
- Rose McCarty completed her joint work James Davies proving that circle graphs are polynomially χ-bounded. This was a major open problem in both discrete geometry and graph theory. McCarty presented their proof in a focused discussion group on Friday afternoon.

The organizers would particularly like to thank the staff at the Institute and the Director, Prof. Dr. Gerhard Huisken, for the considerable assistance that the Institute offers in helping in the seamless organization of a thoroughly enjoyable, world-class workshop. They would also like to express their appreciation to Michelle Delcourt for her help in preparing this report.

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Workshop: Graph Theory

Table of Contents

Abstracts

Orthogonal representations of random graphs

Noga Alon
(joint work with Igor Balla, Lior Gishboliner, Adva Mond, Frank Mousset)

An orthogonal representation of a graph G is an assignment of a unit vector $x(v)$ in the d-dimensional Euclidean space R^d to every vertex v, so that for every two nonadjacent vertices u and v, the corresponding vectors $x(u), x(v)$ are orthogonal. Let $d(G)$ denote the minimum dimension d for which such a representation exists. This quantity and its analogs over other fields arise in the study of the Shannon capacity of G (see [4], [2]) and in the investigation of additional problems in Information Theory. What is the typical value of $d(G)$ for the binomial random graph $G = G(n, 0.5)$? In the (full version of) this work we show that the answer is $\Theta(n/\log n)$. This settles a question of Knuth raised in 1994 [3].

The results apply to a more general problem. The minrank of a graph G on the set of vertices $[n]$ over a field $\mathbb{F}$ is the minimum possible rank of a matrix $M \in \mathbb{F}^{n\times n}$ with nonzero diagonal entries such that $M_{i,j} = 0$ whenever i and j are distinct nonadjacent vertices of G. We obtain tight bounds for the typical minrank of the binomial random graph $G(n, p)$ over any finite or infinite field, showing that for every field $\mathbb{F} = \mathbb{F}(n)$ and every $p = p(n)$ satisfying $n^{-1} \leq p \leq 1 - n^{-0.99}$, the minrank of $G = G(n, p)$ over $\mathbb{F}$ is $\Theta(\frac{n\log(1/p)}{\log n})$ with high probability. The result for the real field implies the statement about orthogonal representations. The proof combines a recent argument of Golovnev, Regev, and Weinstein [1], who proved the above result for finite fields of size at most $n^{O(1)}$, with tools from combinatorics, probability and linear algebra, including an estimate of Rónyai, Babai, and Ganapathy [5] for the number of zero-patterns of a sequence of polynomials.

Extensions to additional geometric representations of random graphs are proved as well. The problem if for $G = G(n, 0.5)$ $d(G) = (1 + o(1))\chi(\overline{G})$ with high probability remains open.

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Coloring algorithms in the distributed setting

Marthe Bonamy

(joint work with Pierre Aboulker, Nicolas Bousquet, Louis Esperet)

This talk is concerned with efficiently coloring sparse graphs in the distributed setting with as few colors as possible. Here, we show how to extend degree-choosability results to an (efficient) algorithm in the distributed setting. In fact, our algorithm works more generally for sparse graphs (for such graphs we improve by at least one the number of colors resulting from an efficient algorithm of Barenboim and Elkin, at the expense of a slightly worst complexity). Our bounds on the number of colors turn out to be quite sharp in general. This is joint work with Pierre Aboulker, Nicolas Bousquet and Louis Esperet.

1. Coloring sparse graphs

We are concerned with the graph coloring problem in the distributed model of computation. Graph coloring plays a central role in distributed algorithms, see the recent survey book of Barenboim and Elkin [3] for more details and further references. Most of the research so far has focused on obtaining fast algorithms for coloring graphs of maximum degree Δ with $\Delta + 1$ colors, or to allow more colors in order to obtain more efficient algorithms. Our approach here is quite the opposite. Instead, we are interested in proving "best possible" results (in terms of the number of colors), in a reasonable (say polylogarithmic) round complexity. By "best possible", we mean results that match the best known existential bounds or the best known bounds following from efficient sequential algorithms. A typical example is the case of planar graphs. The famous Four Color Theorem ensures that these graphs are 4-colorable (and the proof actually yields a quadratic algorithm), but coloring them using so few colors with an efficient distributed algorithm has remained elusive. Goldberg, Plotkin, and Shannon [7] (see also [2]) obtained a deterministic distributed algorithm coloring n-vertex planar graphs with 7 colors in $O(\log n)$ rounds, but it was not known[1] whether a polylogarithmic 6-coloring algorithm exists for planar graphs.

Here we give a simple deterministic distributed 6-coloring algorithm for planar graphs, of round complexity $O(\log^3 n)$. In fact, our algorithm works in the more general list-coloring setting, where each vertex has its own list of k colors (not necessarily integers from 1 to k). The algorithm also works more generally for sparse graphs. Here, we consider the maximum average degree of a graph (see below for precise definitions) as a sparseness measure. It seems to be better suited for coloring problems than arboricity, which had been previously considered [2, 6].

[1] In [2], it is mentioned that a parallel algorithm of [7] that 5-colors plane graphs (embedded planar graphs) can be extended to the distributed setting, but this does not seem to be correct, since the algorithm relies on edge-contractions and some clusters might correspond to connected subgraphs of diameter linear in the order of the original graph. The authors of [2] acknowledged (private communication) that consequently, the problem of coloring planar graphs with 6-colors in polylogarithmic time was still open.

To state our result more precisely, we start with some definitions and classic results on graph coloring.

The *average degree* of a graph $G = (V, E)$ is defined as the average of the degrees of the vertices of G (it is equal to 0 if V is empty and to $2|E|/|V|$ otherwise). The *maximum average degree* of a graph G, denoted by $\text{mad}(G)$, is the maximum of the average degrees of the subgraphs of G. The maximum average degree is a standard measure of the sparseness of a graph. Note that if a graph G has $\text{mad}(G) < k$, for some integer k, then any subgraph of G contains a vertex of degree at most $k-1$, and in particular a simple greedy algorithm shows that G has (list)-chromatic number at most k. Therefore, for any graph G, $\chi(G) \le \text{ch}(G) \le \lfloor \text{mad}(G) \rfloor + 1$. This bound can be slightly improved when G does not contain a simple obstruction (a large clique), as will be explained below.

Most of the research on distributed coloring of sparse graphs so far [2, 6] has focused on a different sparseness parameter: The *arboricity* of a graph G, denoted by $a(G)$, is the minimum number of edge-disjoint forests into which the edges of G can be partitioned. By a classic theorem of Nash-Williams [9], we have

$$a(G) = \max\left\{ \left\lceil \tfrac{|E(H)|}{|V(H)|-1} \right\rceil \mid H \subseteq G, |V(H)| \ge 2 \right\}.$$

From this result, it is not difficult to show that for any graph G, $2a(G) - 2 \le \lceil \text{mad}(G) \rceil \le 2a(G)$ (the lower bound is attained for graphs whose maximum average degree is an even integer).

In [2], Barenboim and Elkin gave, for any $\epsilon > 0$, a deterministic distributed algorithm coloring n-vertex graphs of arboricity a with $\lfloor (2+\epsilon)a \rfloor + 1$ colors in $O(\frac{a}{\epsilon} \log n)$ rounds. In particular, their algorithm colors n-vertex graphs of arboricity a with $2a + 1$ colors in $O(a^2 \log n)$ rounds.

However it is not difficult to prove that graphs with arboricity a are $(2a-1)$-degenerate (meaning that every subgraph contains a vertex of degree at most $2a-1$), and thus $2a$-colorable, which is sharp. A natural question is whether there is a fundamental barrier for obtaining an efficient distributed algorithm coloring graphs of arboricity a with $2a$ colors. It turns out that there is such a barrier when $a = 1$, i.e. when G is a tree: It was proved by Linial [8] that coloring a path (and thus a tree) with two colors requires a linear number of rounds. Yet, our main result will easily imply that the case $a = 1$ is an exception: when $a \ge 2$, there is a fairly simple distributed algorithm running in $O(a^4 \log^3 n)$ rounds, that colors graphs of arboricity a with $2a$ colors.

2. Brooks theorem, Gallai trees, and list-coloring

A classic theorem of Brooks states that any connected graph of maximum degree Δ which is not an odd cycle or a clique has chromatic number at most Δ. This improves the simple bound of $\Delta + 1$ obtained from the greedy coloring algorithm. While most of the research in coloring in the distributed computing setting has focused on $(\Delta+1)$-coloring, Panconesi and Srinivasan [10] gave a $O(\Delta \log^3 n / \log \Delta)$

deterministic distributed algorithm that given a connected graph G of maximum degree $\Delta \geq 3$ finds a clique $K_{\Delta+1}$ or a Δ-coloring.

A *Gallai tree* is a connected graph in which each 2-connected subgraph is an odd cycle or a clique. Note that a tree is also a Gallai tree, since each block of a tree is an edge (i.e. a clique on two vertices). The degree of a vertex v in a graph G is denoted by $d_G(v)$. The proof of our main result is mainly based on the following classic theorem in graph theory proved independently by Borodin [4] and Erdős, Rubin, and Taylor [5], extending Brooks theorem (mentioned above) to the list-coloring setting.

Theorem 1 ([4, 5])**.** *If a connected graph G is not a Gallai tree, then for any list-assignment L such that for every vertex $v \in G$, $|L(v)| \geq d_G(v)$, G is L-list-colorable.*

It is not difficult to prove that Theorem 1 implies Brooks theorem. We mentioned above that for any graph G, $\chi(G) \leq \text{ch}(G) \leq \lfloor \text{mad}(G) \rfloor + 1$. Let us now see how this can be slightly improved using Theorem 1 if we exclude a simple obstruction, in the spirit of Brooks theorem.

Theorem 2 (Folklore)**.** *Let G be a graph and let $d = \lceil \text{mad}(G) \rceil$. If $d \geq 3$ and G does not contain any $(d+1)$-clique, then $\chi(G) \leq ch(G) \leq d$.*

Our main result is an efficient algorithmic counterpart of Theorem 2 in the LOCAL model of computation [8], which is standard in distributed graph algorithms. Each node of an n-vertex graph G has a unique identifier (an integer between 1 and n), and can exchange messages with its neighbors during synchronous rounds. In the LOCAL model, there is no bound on the size of the messages, and nodes have infinite computational power. Initially, each node only knows its own identifier, as well as n (the number of vertices) and sometimes some other parameters: in Theorem 3 below, each node knows its own list of d colors (in the list-coloring setting), or simply the integer d (if we are merely interested in coloring the graph with colors from 1 to d and there are no lists involved). With this information, each vertex has to output its own color in a proper coloring of the graph G. The *round complexity* of the algorithm is the number of rounds it takes for each vertex to choose a color. In the LOCAL model of computation, the output of each vertex v only depends on the labelled ball of radius r of v, where r is the round complexity of the algorithm. In particular, in this model any problem on G can be solved in a number of rounds that is linear in the diameter of G, and thus the major problem is to obtain bounds on the round complexity that are significantly better than the diameter. The reader is referred to the survey book of Barenboim and Elkin [3] for more on coloring algorithms in the LOCAL model of computation.

Theorem 3 (Main result)**.** *There is a deterministic distributed algorithm that given an n-vertex graph G, and an integer $d \geq \max(3, \text{mad}(G))$, either finds a $(d+1)$-clique in G, or finds a d-list-coloring of G in $O(d^4 \log^3 n)$ rounds. Moreover, if every vertex has degree at most d, then the algorithm runs in $O(d^2 \log^3 n)$ rounds.*

Noting that graphs of arboricity a have maximum average degree at most $2a$ and no clique on $2a+1$ vertices, we obtain the following result as an immediate consequence.

Corollary 1. *There is a deterministic distributed algorithm that given an n-vertex graph G of arboricity $a \geq 2$, finds a $2a$-list-coloring of G in $O(a^4 \log^3 n)$ rounds.*

Before we discuss other consequences of our result, let us first discuss its tightness. First, Corollary 1 improves the result of Barenboim and Elkin [2] mentioned above by at least one color in general, and Theorem 3 improves it by at least 3 colors in some cases (for instance for graphs whose maximum average degree is an even integer), and both results are best possible in general in terms of the number of colors (already from an existential point of view). On the other hand, the round complexity of our algorithm is slightly worst, but a classic result of Linial [8] shows that trees cannot be colored in $o(\log n)$ rounds with any constant number of colors, and this implies that even for fixed d or a, the round complexity in Theorem 3 and Corollary 1 cannot be replaced by $o(\log n)$. Second, another classic result of Linial [8] showing that n-vertex paths cannot be 2-colored by a distributed algorithm using $o(n)$ rounds, also shows that we cannot omit the assumption that $d \geq 3$ in the statement of Theorem 3 and the assumption that $a \geq 2$ in the statement of Corollary 1.

We also note that using network decompositions [11], we can replace the $O(d^4 \log^3 n)$ round complexity in Theorem 3 by $d^3 2^{O(\sqrt{\log n})}$, and the $O(d^2 \log^3 n)$ round complexity by $d 2^{O(\sqrt{\log n})}$ (the multiplicative factor of d is saved similarly as in [10]). These alternative bounds are not very satisfying, and in most of the applications we have in mind d is a constant anyway, so we omit the details. It remains interesting to obtain a bound on the round complexity that is sublinear in n regardless of the value of d.

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Embedding simply connected 2-complexes in 3-space

Johannes Carmesin

A classical theorem of Kuratowski characterises embeddability in the plane by two obstruction. More precisely, a graph can be embedded in the plane if and only if it does not contain the graphs K_5 and $K_{3,3}$ as a minor[2], see Figure 1.

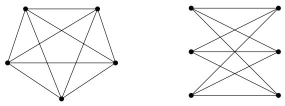

Figure 1. The graphs K_5 (on the left) and $K_{3,3}$ (on the right).

A far reaching extension is the Robertson-Seymour Theorem that says that for any minor closed class of graphs the list of minimal graphs not in that class must be finite [6]. At the heart of their proof is their structure theorem that, roughly speaking, establishes a deep connection between the minor relation and embeddings of graphs in 2-dimensional surfaces. Lovász asked how this could be extended in three dimensions. Related questions have been asked by Pardon and Wagner.

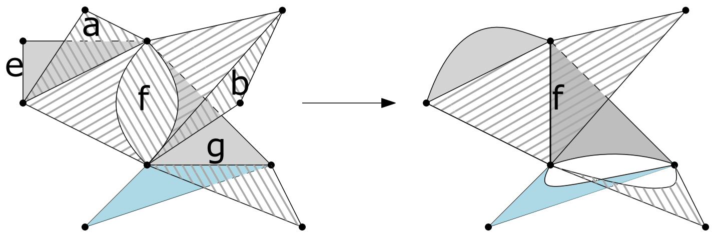

Figure 2. The 2-complex on the right is a space minor of the 2-complex on the left. More precisely, it is obtained by deleting the faces a and b, deleting the edge g, contracting the edge e, and contracting the face f.

Answering these questions, we introduce a minor relation for 2-complexes, see Figure 2. We extend Kuratowski's theorem and a related theorem of Whitney to certain simply connected 2-complexes [1, 2, 3, 4, 5].

[2]A *minor* of a graph is obtained by deleting or contracting edges.

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Detecting Odd Holes

Maria Chudnovsky
(joint work with Alex Scott, Paul Seymour, Sophie Spirkl)

A hole in a graph is an induced cycle of length at least four; and a hole is odd if it has an odd number of vertices. In 2003 a polynomial-time algorithm was found to test whether a graph or its complement contains an odd hole, thus providing a polynomial-time algorithm to test if a graph is perfect [5]. However, the complexity of testing for odd holes (without accepting the complement outcome) remained unknown. This question was made even more tantalizing by a theorem of D. Bienstock [1, 2] that states that testing for the existence of an odd hole through a given vertex is NP-complete. But in fact we can test for odd holes in polynomial time. We prove:

Theorem 1. *There is an algorithm with the following specifications:*

Input:: *A graph G.*
Output:: *Decides whether G has an odd hole.*
Running time:: $O(|G|^{12})$

The algorithm of [5] and 1 (as well as many other algorithms that test for the presence of a subdivision of a fixed induced subgraph) share the same three-step outline. First they test for a number of "easily detectable configurations". These are configurations that can be efficiently detected and whose presence guarantees that the input graph contains the required induced subgraph. The third step is a "naive" algorithm that only finds the configuration that we are looking for under very special circumstances, and would not work in general graphs. The role of the second step is to prepare the input for the naive algorithm; this step is called "cleaning".

In the cleaning step, the algorithm generates polynomially many subsets $X_1, \ldots, X_k$ of the vertex set of the input graph G, with the property that if G does in fact contain an induced subgraph of the required type, then for some

$i \in \{1, \ldots, k\}$ such a subgraph can be found in $G \setminus X_i$ using the "naive" algorithm from step three. Effectively, this means that if H is the object that we are searching for, then some X_i contains all the vertices of G that have many neighbors in H, but deleting X_i leaves H intact.

Cleaning was first used by Conforti and Rao [8] to recognize linear balanced matrices, and subsequently by Conforti, Cornuéjols, Kapoor and Vušković [7] to test for even holes, as well as in [5]. It then became a standard tool in induced subgraph detection algorithms ([3], [4], [6]). In all these papers cleaning was done (roughly) by enumerating all subsets Y_i of $V(G)$ up to a certain fixed size, and then setting X_i to be the set of vertices of $V(G)$ with certain neighbors in X_i. Trying to modify the sets Y_i from [5] seemed to be the natural approach to 1, but after fourteen years of efforts we still could not make it work. In the proof of 1 a different kind of a cleaning procedure was developed, that delves much deeper into the structure of the input graph. The potential of this "structural" cleaning is yet to be explored, but this method is likely to be of use in other questions related to induced subgraphs, both existential and algorithmic.

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Tangles outside graph theory

Reinhard Diestel

Tangles of graphs were introduced by Robertson and Seymour [13] in their series of papers on graph minors. The idea behind tangles is to capture highly cohesive parts of a graph G indirectly: not by naming their vertices and edges, but by orienting every low-order separation of G towards them. Such orientations of all these separations will be 'consistent' in that they all point towards this fixed highly cohesive region of G.

There are two major theorems on tangles in [13]. The first is the *tree-of-tangles theorem*. It finds a nested set T of separations of G, of varying order, that suffices for *distinguishing* all the (maximal) tangles of G. Note that any two maximal tangles will orient some separation of G differently: since tangles are orientations of separations, this is what it means for two tangles orienting the same set of separations to be distinct. The tree-of-tangles theorem says that such a distinguishing separation can always be found in the (nested, and hence linearly small) set T. In fact, T contains, for every pair of tangles, such a separation of minimum order among all the separations of G that distinguish those two tangles; we say that it distinguishes them *efficiently*.

The second major tangle theorem in [13] is the *tangle-tree duality theorem*. This tells us, for a given integer k, what G looks like if it has no k-tangle (one that orients precisely all the separations of G of order $< k$): it then has a nested set T of separations of order $< k$ that witnesses the nonexistence of a k-tangle in a fast-checkable way.

Tangles, and the above two theorems, can be generalized to other combinatorial structures than graphs. This was done for matroids, implicitly by Robertson and Seymour in [13], and explicity by Geelen, Gerards, Robertson and Whittle [12]. But can be done more generally still. Tangles can be used to indirectly capture clusters whenever these occur in structures that come with a natural notion of 'separation', a notion of cutting the structure in two, by orienting these separations towards the cluster. More importantly: tangles in such structures can then be thought of as 'abstract clusters' even when thay are not induced as above by known clusters. These new 'abstract clusters', however, can sometimes achieve what traditional clusters are sought for even when these latter do not exist or are difficult to find.

My talk gave outlines of this for the following example scenarios:

- Mindsets. A questionnaire S is used to poll a set V of people. Every (yes/no) question s divides V according to the answers received for s. A typical tangle will be an orientation τ of S, one way of answering all the questions, such that, for some set $X \subseteq V$ of the people polled, for each question s the answer given by τ agrees with the answer given by some 80% of X. We call τ a *mindset*: a set of views essentially shared by a sizable subset of the people polled. Importantly, there need not exist a person whose answers are precisely those of τ: these answers are typical for the people in X, but every such type as such is fictional.

Note that while the set X above is fixed, its various 80% subsets depend on s. The existence of such a set X is a claim of substance about τ: such a set will not exist for an arbitrary way of answering the questions in S. It is also possible that no mindset for S exists, that there is no such set X for any set of answers. Then our abstract version of the tangle-tree-duality theorem will provide a fast-checkable witness to this. If there are many mindsets, different ways of answering S with different corresponding sets $X \subseteq V$, our abstract version of the tree-of-tangles theorem provides a small set T of questions that suffices to distinguish all these mindsets. The questions in T may be logical combinations of the original questions from S.

If the questions in S come with an *order* signifying their significance (see below), then mindsets of low order (those choosing answers only for a few most significant questions) will evolve into more refined mindsets that choose answers also for some questions of higher order. The subset T of key questions will suffice to distinguish both basic and refined mindsets, also across different orders, by a question of the lowest possible order, the most fundamental nature. (For example, if the questions are about musical tastes, the fans of pop music will be distinguished from those of the late Beethoven string quartets by a question about pop versus classics, not just by one about chamber music of the 1820s.)

- DNA analysis. Let V be a set of DNA molecules from a sample of organisms to be classified or studied. We assume that the molecules come with aligned main strands, and thus have well-definied 'positions' at each of which they have one of four bases encoded A, C, G or T. Every position with a potential base then becomes a separation of V: of those molecules that have this base at this position versus those that do not. An orientation of S yields an assignment of one of the four bases to each position.

  Not every such assignment will be a tangle. But *typical* assignments are: assignments τ such that there exists a sizable subset $X \subseteq V$ such that, for each position, most of the molecules in X have at that position the base prescribed by τ.

  An alternative for S is to take *all* the bipartitions of V, but to assign them an order depending on how well they divide V considering the features of its elements in terms of which bases are found at which positions. A bipartition, which, for many positions, splits the sets of molecules with the same base at that position roughly in half will be considered a bad split and assigned high order.

The choice of sensible order functions is a non-trivial task that will crucially influence what tangles exist and how they are related with each other. This task can be left to the biologists, who would have to tell us which base positions are more important than others. But in the second setup suggested above it can be approached also from a mathematical point of view. This is a topic we shall discuss at a RIP workshop in Oberwolfach this November.

As with mindsets, neither of the variants for S discussed need have a tangle. Indeed, for a contaminated DNA sample it should not. But if it does not, then we can prove this by our tangle-tree duality theorem.

On the other hand if there are many tangles, then our tree-of-tangles theorem displays a tree structure in which they are organized, and shows how high-order tangles evolve from low-order ones. If V contains DNA from many species, their phylogenetic tree can be obtained in this way. If V is more homogeneous, consisting, say, of DNA from related bacteria or viruses, its tangles will identify DNA segments that are typical for this sample. There may be several types, one for each tangle, and one may hope to develop drugs targeting each of these types rather than a plethora of bacteria or viruses each with their individual DNA.

- Image segmentation. Here, V is the set of pixels of an image. Every pixel has a number of parameters, such as darkness, colour and so on. Separations will be bipartitions of V obtained by cutting the picture along some line. They will have an order depending on how well they do that: lines cutting through many pairs of similar adjacent pixels will be considered bad and given high order.

The tangles in a picture will correspond roughly to regions of similar pixels. Unlike in classical image sementation or clustering, however, we do not have to assign a tangle to each pixel. The image of a letter L with two serifs, for example, may have only two tangles of some fixed k, one for each serif, but most of the pixels in the shape of this L will be far from either serif.

The tree-of-tangles in image segmentation comes in the form of a set of non-crossing lines (nested bipartitions of V) which roughly delineate the main regions of the picture. Thus, a picture is captured not by storing masses of information about all its pixels, but rather more succinctly as a set of a few lines dividing it up, much as a caricaturist would draw a rough version of it. See [9] for details.

The formal framework for abstract tangles, not just of the example types just discussed, is described in [2]. The general tree-of-tangles theorem is proved in [5], the general tangle-tree duality theorem in [8]. Mathematical applications of these can be found in [3, 7]. Further theoretical developments can be found in [1, 4, 6, 10, 11].

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Dual Circumference and Collinear Sets

Vida Dujmović
(joint work with Pat Morin)

For a planar graph G, we say that a set $S \subseteq V(G)$ is a *collinear set* if G has a non-crossing straight-line drawing in which the vertices of S are all collinear. A *plane graph* is a planar graph G along with a particular non-crossing drawing of G. The *dual* $G^\star$ of a plane graph G is the graph whose vertex set $V(G^\star)$ is the set of faces in G and in which $fg \in E(G^\star)$ if and only if the faces f and g of G have at least one edge in common. The *circumference*, $c(G)$, of a graph G is the length of its longest cycle. We prove the following theorem:

Theorem 1. *Let T be a triangulation of maximum degree Δ whose dual $T^\star$ has circumference ℓ. Then T has a collinear set of size $\Omega(\ell/\Delta^4)$.*

It is known that any planar graph of maximum degree Δ can be triangulated so that the resulting triangulation has maximum degree $\lceil 3\Delta/2\rceil + 11$ [8]. This fact, together with Theorem 1 and the current best lower bounds on the length of longest cycles in 3-regular 3-connected graphs [9], implies the following corollary:

Corollary 1. *Every n-vertex triangulation of maximum degree Δ contains a collinear set of size $\Omega(n^{0.8}/\Delta^4)$.*

It is known that every planar graph G has a collinear set of size $\Omega(\sqrt{n})$ [1, 5]. Corollary 1 therefore improves on this bound for bounded-degree planar graphs and, indeed for the family of n-vertex planar graphs of maximum degree $\Delta \in O(n^\delta)$, with $\delta < 0.075$.

Very recently, Dujmović et al. [6] showed that, if S is a collinear set in a triangulation T then, for any point set $X \subset \mathbb{R}^2$ with $|X| = |S|$, T has a non-crossing straight-line drawing in which the vertices of S are drawn on the points in X. Because of this, collinear sets have numerous applications in graph drawing and related areas. For example, it is known that every n-vertex planar geometric graph can be untangled while keeping $\Omega(n^{0.25})$ vertices fixed [1] and that there are n-vertex planar geometric graphs that cannot be untangled while keeping $O(n^{0.4948})$ vertices fixed [2]. Although asymptotically tight bounds are known for paths [3], trees [7], outerplanar graphs [7], planar graphs of treewidth two [10], and planar graphs of treewidth three [4], progress on the general case has been stuck for 10 years due to the fact that the exponent 0.25 comes from two applications of Dilwerth's Theorem. Thus, some substantially new idea appears to be needed. By relating collinear/free sets to dual circumference, the results presents an effective new idea. Indeed, Corollary 1 implies that every bounded-degree n-vertex planar geometric graph can be untangled while keeping $\Omega(n^{0.4})$ vertices fixed. Note that, even for bounded-degree planar graphs, $\Omega(n^{0.25})$ was the best previously-known lower bound.

Our work opens two avenues for further progress:

(1) Lower bounds on the circumference of 3-regular 3-connected graphs is an active area of research. Indeed, the $\Omega(n^{0.8})$ lower bound of Liu, Yu, and

Zhang [9] is less than a year old. Any further progress on these lower bounds will translate immediately to an improved bound in Corollary 1 and all its applications.

(2) It is possible that the dependence on Δ can be removed from Theorem 1 and Corollary 1, thus making these results applicable to all planar graphs, regardless of maximum degree.

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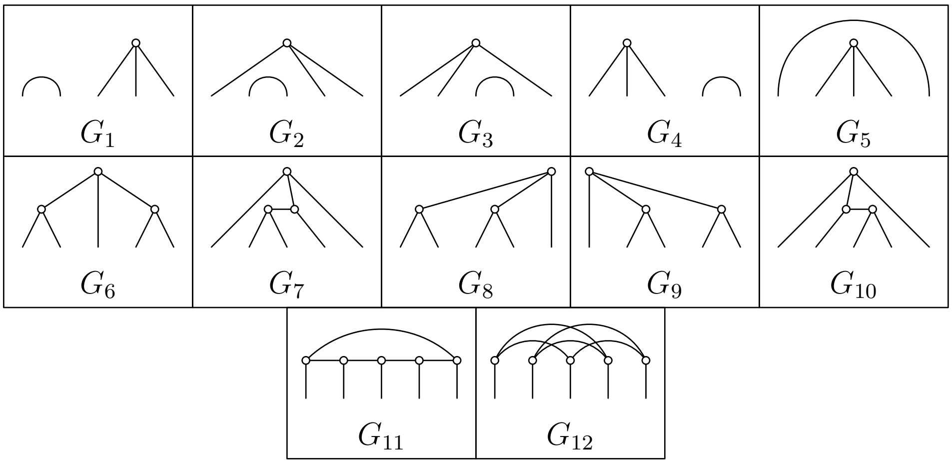
Coloring count cones of planar graphs

Zdeněk Dvořák

(joint work with Bernard Lidický)

For a plane near-triangulation G with the outer face bounded by a cycle C, let $n^\star_G$ denote the function that to each 4-coloring ψ of C assigns the number of ways ψ extends to a 4-coloring of G. The block-count reducibility argument [1, 2] (which has been developed in connection with attempted proofs of the Four Color Theorem) is equivalent to the statement that $n^\star_G$ belongs to a certain cone (depending only on the length of C).

For $|C| = 5$, the cone has 12 rays, corresponding to the following graphs (in the dual setting of numbers of 3-edge-colorings of cubic graphs with 5 half-edges; for such a graph H, we denote by n_H the function giving the numbers of extensions of precolorings of these half-edges).



The Four Color Theorem is equivalent to the fact that there exists no plane near-triangulation G with the outer face bounded by a 5-cycle such that $n^\star_G$ is a multiple of $n_{G_{12}}$. We conjecture that actually more is true: the last ray can be omitted from the list entirely.

Conjecture 1. *For every plane near-triangulation G with the outer face bounded by a 5-cycle, $n^\star_G$ is a non-negative linear combination of $n_{G_1}, \ldots, n_{G_{11}}$.*

As a supporting evidence for this conjecture, we have performed computational experiments, in particular showing that it holds for all near-triangulations with at most 26 faces.

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On Ramsey numbers

Jacob Fox

The talk would be about recent progress on problems motivated by the classical problem of improving the bounds on Ramsey numbers. Randomness, quasirandomness, books, and games play an important role in the talk.

Degree condition forcing oriented cycles of a given length

Andrzej Grzesik
(joint work with Roman Glebov, Jan Volec)

The longstanding Caccetta-Häggkvist Conjecture [1] says that every n-vertex oriented graph G with minimum out-degree $\delta^+(G) \geq \frac{n}{\ell}$ contains an oriented cycle of length at most ℓ. Despite quite intensive work on this problem (see for example a summary of results and open problems related to the conjecture [2]), even the case of $\ell = 3$ still remains open. The weaker conjecture with assumption on the minimal semidegree (minimum of out-degrees and in-degrees over all vertices) is also open.

Motivated by this conjecture, Kelly, Kühn and Osthus [3] made a conjecture on minimal semidegree forcing appearance of a directed cycle of a given length and proved it for cycles of length not divisible by 3. The asymptotic version of the conjecture was proven by Kühn, Osthus and Piguet [4] when the cycle length is large enough.

In the talk we present constructions showing that in general the conjectured threshold is not correct. Then we prove the optimal threshold for each length of the cycle greater than 3.

Theorem 1. *For any $\ell \geq 4$ every big enough n-vertex oriented graph G with semidegree $\delta^\pm(G) \geq \frac{n}{k} + \frac{k-1}{2k}$ contains a directed cycle of length ℓ, where k is the smallest integer greater than 2 that does not divide ℓ.*

Moreover, if $\ell \not\equiv 3 \pmod{12}$, then this is the best possible threshold. If $\ell \equiv 3 \pmod{12}$, then $\delta^\pm(G) \geq \frac{n}{4} + \frac{1}{4}$ is already forcing an ℓ-cycle and this is the best possible threshold.

The proof for the case $k = 4$ is different than the proof for larger values of k. In particular, for $k \geq 5$ one can prove the following stability version, which is not true for $k = 4$.

Theorem 2. *For $\ell \geq 4$ let k being the smallest integer greater than 2 that does not divide ℓ. If $k \geq 5$, then any oriented graph H with $\delta^\pm(H) \geq \frac{n}{k}\left(1 - \frac{1}{30k}\right)$ that does not contain a closed walk of length ℓ, is a subgraph of a blow-up of C_k.*

In order to prove this, we firstly obtain that such a graph has bounded directed diameter by providing bounds on the sizes of neighborhoods. Then we show using Frobenius coin problem that it cannot contain short cycles with one edge reversed. This is enough to define the wanted blobs and prove the theorem.

To prove the main theorem for $k \geq 5$, we use the regularity lemma for oriented graphs and the above theorem, to obtain a structure of the graph G with some small fraction of additional edges and vertices. Then, by application of some results on additive combinatorics, we prove that the assumed semidegree threshold gives enough many additional edges and vertices to construct the wanted cycle of length ℓ.

The case of $k = 4$ requires a different approach. In particular, notice that for $\ell = 6$ one blob of a blow-up of C_4 can contain arbitrary one-way oriented bipartite graph. Also, for any odd ℓ, one can reverse edges of any C_4 contained in a blow-up of C_4 keeping the semidegree assumption and avoiding cycles of odd lengths.

The proof of the main theorem for $k = 4$, in particular for small cycle lengths, needs directed diameter bound, that cannot be obtained in the same way as for $k \geq 5$. To achieve this, we use the method of flag algebras created by Razborov. The proof, especially the cases of $\ell = 6$ and $\ell = 9$, need also more detailed analysis, because the small length of the wanted cycle can cause complications in combining some partial structures to obtain the cycle of length ℓ.

It would be interesting to prove similar statement, but with only minimal out-degree assumption. The crucial step in the proof, where our methods use also the minimal in-degree assumption, is to obtain the bound on the directed diameter. Thus, it would be very interesting to prove some bound on the directed diameter of such graphs.

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Large independent sets in triangle-free subcubic graphs

Gwenaël Joret

(joint work with Wouter Cames van Batenburg and Jan Goedgebeur)

Heckman and Thomas [4] showed that every n-vertex planar triangle-free graph with maximum degree 3 has an independent set of size at least $3n/8$, confirming a conjecture of Albertson, Bollobas and Tucker [1]. In this talk I will present a strengthening of this result: There exists a set S of six nonplanar graphs (each of order at most 22), such that every n-vertex triangle-free graph with maximum degree 3 and having no member of S as a subgraph has an independent set of size at least $3n/8$. This proves a conjecture of Fraughnaugh and Locke [3]. A corollary of this result is that every 2-connected n-vertex triangle-free graph with maximum degree 3 has an independent set of size at least $3n/8$, except for the six graphs in S.

Whether these results can be extended to fractional coloring remains an intriguing open problem. Recall that $n/\alpha \leq \chi_f$ for every n-vertex graph G with independence number α and fractional chromatic number χ_f. Dvořák, Sereni, and Volec [2] proved that $\chi_f \leq 14/5$ when G is triangle free with maximum degree 3, generalizing the old result of Staton [5] that $n/\alpha \leq 14/5$. Heckman and Thomas [4] conjectured that $\chi_f \leq 8/3$ if G is moreover planar, which would generalize their theorem. For all we know, this might even be true in the setting we considered, that is, when G has none of the six graphs in S as subgraph.

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Critical subgraphs of Schrijver graphs

Tomáš Kaiser

(joint work with Matěj Stehlík)

Recall that the vertices of the *Kneser graph* $KG(n,k)$ (where $n \geq 2k$ and $k \geq 1$) are all k-subsets of $\{1,\dots,n\}$, with two subsets joined by an edge if they are disjoint. In 1979, Lovász [4] proved the conjecture of Kneser [3] from 1955 that the chromatic number of $KG(n,k)$ is $n-2k+2$. Shortly afterwards, Schrijver [5] constructed spanning subgraphs of the Kneser graphs with the same chromatic number and the property that they are *vertex-critical* — that is, the removal of any vertex decreases the chromatic number. The *Schrijver graph* $SG(n,k)$ is the induced subgraph of $KG(n,k)$ on the set of all k-sets containing no pair of consecutive elements $\{i,i+1\}$ nor the pair $\{1,n\}$.

Schrijver graphs in general do not have the stronger property of being *critical* (or *edge-critical*), namely that the removal of any edge yields a graph of smaller chromatic number. This is only true in the extreme cases $k=1$ or $n \leq 2k+1$. To our knowledge, there are no known constructions of critical subgraphs of Schrijver graphs with the same chromatic number.

In our talk, we provide such a construction in the case $k=2$: for each $n \geq 5$, we define a spanning subgraph $XG(n,2)$ of $SG(n,2)$ as follows. Let C_n be a cycle with vertices $1,\dots,n$ in order. A *chord* of C_n is an edge of the complement of C_n. The vertices of $XG(n,2)$ are the chords of C_n. Two chords are joined by an edge if they cross (their vertices are interlaced on C_n), or if the vertex 1 is separated by the endvertices of one of the chords from the other chord.

By constructing a homomorphism from the Mycielski graph over $XG(n-1,2)$ to $XG(n,2)$, for each $n \geq 6$, we show that the chromatic number of $XG(n,2)$ is $n-2$ (that is, the same as that of $SG(n,2)$). In addition, we prove that $XG(n,2)$ is critical and that asymptotically, $XG(n,2)$ contains 2/3 of the edges of $SG(n,2)$.

The graphs $XG(n,2)$ were discovered in the course of our work on quadrangulations of projective spaces [1, 2], and indeed $XG(n,2)$ quadrangulates the projective space of dimension $n-4$. In the talk, we will also discuss the attempts (currently work in progress) to extend the construction to arbitrary k.

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Euler tours in hypergraphs

Daniela Kühn

(joint work with Stefan Glock, Felix Joos, Deryk Osthus)

Finding an *Euler tour* in a graph is a problem as old as graph theory itself: Euler's negative resolution of the Seven Bridges of Königsberg problem in 1736 is widely considered the first theorem in graph theory. Euler observed that if a (multi-)graph contains a closed walk which traverses every edge exactly once, then all the vertex degrees are even. He also stated that every connected graph with only even vertex degrees contains such a walk, which was later proved by Hierholzer and Wiener.

There are several ways of generalising the concept of paths/cycles, and similarly Euler trails/tours, to hypergraphs. Not least due to its connection to universal cycles, we focus on the so-called 'tight' regime. Given a k-graph G (i.e. a k-uniform hypergraph G), a cyclic sequence of vertices $x_1x_2\dots x_\ell x_1$ is an *Euler tour of* G if $\{x_i, x_{i+1}, \dots, x_{i+k-1}\} \in E(G)$ for all $i \in [\ell]$, and every edge of G appears exactly once in this way.

The problem of deciding whether a given 3-graph has an Euler tour has been shown to be NP-complete [8]. Thus, when $k > 2$, there is probably no simple characterisation of k-graphs having an Euler tour.

It is easy to see that if an Euler tour of the complete k-graph $K_n^{(k)}$ on n vertices exists, then k divides $\binom{n-1}{k-1}$, i.e. k divides the degree of every vertex of $K_n^{(k)}$. In 1989, Chung, Diaconis and Graham conjectured that the converse should also be true, at least if n is sufficiently large, and offered \$100 for the resolution of this problem. (In their paper, they actually phrased this conjecture in terms of universal cycles for $\binom{[n]}{k}$.)

Conjecture 1 (Chung, Diaconis, Graham [1, 2]). *For every $k \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $K_n^{(k)}$ has an Euler tour whenever k divides $\binom{n-1}{k-1}$.*

It is easy to see that Conjecture 1 is true for $k = 2$. Numerous partial results have been obtained. In particular, Jackson [7] proved the conjecture for $k = 3$, and Hurlbert [6] confirmed the cases $k \in \{3, 4, 6\}$ if n and k are coprime. Approximate versions of Conjecture 1 have been obtained in [3, 4]. We prove Conjecture 1 in a strong form by showing the existence of tight Euler tours in quasi-random k-graphs G (provided that G satisfies the necessary divisibility condition that k divides all vertex degrees of G).

Our proof goes as follows. In the first step, we find a 'spanning' walk $\mathcal{W}$ in G, where spanning means that every ordered $(k-1)$-set of vertices appears at least once consecutively in the vertex sequence of $\mathcal{W}$. For this, we show that a self-avoiding random walk yields such a walk $\mathcal{W}$ (after an appropriate number of steps) with high probability. This step will use only a small fraction of the edges of G. We then extend $\mathcal{W}$ to a closed walk $\mathcal{W}'$. Subsequently, we remove $E(\mathcal{W}')$ from G and decompose the remaining k-graph into tight cycles using results on the existence of F-designs from [5]. Each such cycle can be incorporated into $\mathcal{W}'$, which finally yields a tight Euler tour.

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Erdős-Pósa property of H-induced subdivisions

O-joung Kwon

(joint work with Jean-Florent Raymond)

Kim and Kwon [1] proved that holes (induced cycles of length at least 4) have the Erdos-Posa property; that is, given a graph G and an integer k, either G contains $k+1$ vertex-disjoint holes or it contains a vertex set of size $O(k^2 \log k)$ hitting all holes. On the other hand, we show that this property does not hold for holes of length at least 5.

We say that subdivisions of a graph H have the induced Erdős-Pósa property if there is a function f satisfying that for every graph G and every integer k, either G contains $k+1$ vertex-disjoint induced subdivisions of H, or a vertex set of size at most $f(k)$ hitting all such induced subdivisions. Motivated from the previous work, we naturally ask for which graph H, whether H-subdivisions have the induced Erdős-Pósa property. In this work, we identify necessary conditions on H for the class of H-subdivisions to have the induced Erdős-Pósa property.

We completely settle the case when H is a forest; we show that H-subdivisions have the induced Erdős-Pósa property if and only if H contains no component containing two vertices of degree at least 3. So, it is sufficient to look at graphs H containing a cycle. We show that H-subdivisions have no induced Erdős-Pósa property if one of the following holds: (1) H contains an induced cycle of length at least 5, (2) H contains a cycle and two adjacent vertices having no neighbors in the cycle, (3) H contains a cycle and three vertices having no neighbors in the cycle, (4) H is non-planar, (5) $H = K_{2,n}$ for $n \geq 3$.

Among remaining graphs, we prove that if H is either the diamond, the 1-pan (C_3 with one leaf), or the 2-pan (the graph obtained from the disjoint union of C_3 and K_2 by adding an edge), then the class of H-subdivisions has the induced Erdős-Pósa property. All other cases remains open. An interesting open problem is to determine when $H = K_4$ or H is a fan with path of length 3.

Generally, determining whether some induced substructures have the Erdős-Pósa property or not. We posed one specific problem; do even cycles have the induced Erdős-Pósa property?

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The Graham-Pollak Theorem for Hypergraphs

Imre Leader

(joint work with Luka Milicevic, Ta Sheng Tan)

Let $f_r(n)$ be the minimum number of complete r-partite r-graphs needed to partition the edge-set of the complete r-uniform hypergraph on n vertices. Graham and Pollak showed that $f_2(n) = n - 1$. An easy construction shows that $f_r(n) \leq (1 - o(1))\binom{n}{\lfloor r/2 \rfloor}$, and it has been unknown if this upper bound is asymptotically sharp.

We show that this is not the case: in fact $f_4(n) \leq \frac{14}{15}(1 - o(1))\binom{n}{2}$. Based on this, if we write the limiting value of the constant as c_r (in the sense that we have $c_4 \leq \frac{14}{15}$), it follows that $c_r \leq \frac{14}{15}$ for all even $r \geq 4$.

It turns out that, to obtain $c_r \to 0$, one would need to know that $c_r < 1$ for some *odd* value of r. We give a separate argument to show that $c_{295} < 1$, and hence we have that $c_r \to 0$.

On minimal Ramsey graphs in multiple colours

Anita Liebenau
(joint work with Dennis Clemens, Damian Reding)

For an integer $q \geq 2$, a graph G is called q-Ramsey for a graph H if every q-colouring of the edges of G contains a monochromatic copy of H. If G is q-Ramsey for H, yet no proper subgraph of G has this property then G is called q-Ramsey-minimal for H. Generalising a statement by Burr, Faudree and Schelp [2] from 1977 we prove in [4] that, for $q \geq 3$, if G is a graph that is not q-Ramsey for some graph H then G is contained as an induced subgraph in an infinite number of q-Ramsey-minimal graphs for H, as long as H is 3-connected or isomorphic to the triangle. For such H, the following are some consequences.

- For $2 \leq r < q$, every r-Ramsey-minimal graph for H is contained as an induced subgraph in an infinite number of q-Ramsey-minimal graphs for H.
- For every $q \geq 3$, there are q-Ramsey-minimal graphs for H of arbitrarily large maximum degree, genus, and chromatic number.
- The collection $\{\mathcal{M}_q(H) : H \text{ is 3-connected or } K_3\}$ forms an antichain, where $\mathcal{M}_q(H)$ denotes the set of all graphs that are q-Ramsey-minimal for H.

We also address the question which pairs of graphs satisfy $\mathcal{M}_q(H_1) = \mathcal{M}_q(H_2)$, in which case H_1 and H_2 are called q-equivalent. This notion was introduced by Szabó, Zumstein and Zürcher [7] in 2010. We show that two graphs H_1 and H_2 are q-equivalent for even q if they are 2-equivalent, and that in general q-equivalence for some $q \geq 3$ does not necessarily imply 2-equivalence. Finally we indicate that for connected graphs this implication may hold: Results by Nešetřil and Rödl [6] and by Fox et al. [5] imply that the complete graph is not 2-equivalent to any other connected graph. We prove that this is the case for an arbitrary number of colours.

There are numerous open problems remaining. Here are a few. Axenovich, Rollin and Ueckerdt [1] show that if two graphs H and H' are 2-equivalent and $H \subseteq H'$ then H and H' are q-equivalent for every $q \geq 3$. It would be desirable to remove the condition $H \subseteq H'$. We prove that 2-equivalence (without the subgraph requirement) implies q-equivalence for *even* q. More generally, if two graphs H and H' are q- and r-equivalent (for some $q, r \geq 2$) then they are $(aq + br)$-equivalent for all integers $a, b \geq 1$. The following is open:

Problem. Prove that if two graphs are 2-equivalent then they are also 3-equivalent. Or does there exist a pair of graphs that are 100-equivalent, but not 101-equivalent?

There are pairs of graphs H and H' that are 3-equivalent but not 2-equivalent. All those examples have at least one of H or H' being disconnected and we wonder whether this is a coincidence.

Problem. Let H and H' be both connected graphs that are 3-equivalent. Is it true that they are 2-equivalent as well?

This question may have an affirmative answer for the trivial reason that there are no two non-isomorphic graphs H and H' that are q-equivalent for any $q \geq 2$. This question was first posed in 2014 by Fox et al. [5] for two colours, and we extend it here to any number of colours.

Problem. For given $q \geq 2$, are there two non-isomorphic connected graphs H and H' that are q-equivalent?

It is known that the complete graph K_k is Ramsey equivalent to K_k+H where H is a collection of vertex-disjoint cliques, see, e.g., [7, 3]. What other graphs H have that property? We concentrate on the 2-colour case to highlight how little is known. We know that K_k and $K_k + K_k$ are not Ramsey equivalent (since the clique on $R_2(k)$ vertices is a distinguisher) and that K_k and K_k+K_{k-1} are Ramsey equivalent [3].

Problem.

- What is the largest value of $t = t(k)$ such that there is a connected graph H on t vertices so that K_k and $K_k + H$ are Ramsey equivalent?
- What is the largest value of $t = t(k)$ such that K_k and K_k+S_t are Ramsey equivalent, where by S_t we denote the star with t vertices (in alignment with the previous question)?
- What is the largest value of $t = t(k)$ such that K_k and K_k+P_t are Ramsey equivalent, where by P_t we denote the path with t vertices?

The second question is due to Fox et al. [5]. Note that the equivalence of K_k and $K_k + K_{k-1}$ implies that the answer to these questions is at least $k - 1$. Moreover, it is easy to obtain an upper bound of roughly $R(k)$, i.e. exponential in k. To the best of our knowledge nothing better is known.

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Clustered coloring on old graph coloring conjectures

Chun-Hung Liu

Hadwiger conjectured in 1943 that every K_{t+1}-minor free graph is properly t-colorable. This conjecture is known to be equivalent with the Four Color Theorem when $t \leq 5$, but it is open for every $t \geq 6$. Even though Hadwiger's conjecture is considered to be very difficult, two strengthenings were proposed. Hajós in 1940's conjectured that every K_{t+1}-topological minor free graph is properly t-colorable; Gerards and Seymour conjectured that every odd K_{t+1}-minor free graph is properly t-colorable. Hajós' conjecture is known to be false in general but the Gerards-Seymour conjecture remains open. One direction to approach or rectify those conjectures is to consider clustered coloring which is a relaxation of the notion of proper coloring.

We say that a class of graphs is *clustered k-colorable* if there exists a constant N such that for every graph in this class, one can color its vertices such that every monochromatic component contains at most N vertices. Joint with Wood, we prove the following results.

Theorem 1. *Let p, q be positive integers. Let $\mathcal{F}$ be a class of graphs where every graph in $\mathcal{F}$ does not contain $K_{p,q}$ as a subgraph.*

(1) If there exists a planar graph H such that every graph in $\mathcal{F}$ is H-minor free, then $\mathcal{F}$ is clustered $(p+1)$-choosable.
(2) If there exists a graph H such that every graph in $\mathcal{F}$ is H-minor free, then $\mathcal{F}$ is clustered $(p+2)$-colorable.
(3) If there exists a graph H such that every graph in $\mathcal{F}$ is odd H-minor free, then $\mathcal{F}$ is clustered $(2p+1)$-colorable.
(4) If there exists a graph H of maximum degree d, where $p+3d \geq 7$, such that every graph in $\mathcal{F}$ is H-topological minor free, then $\mathcal{F}$ is clustered $(p+3d-4)$-colorable.

These results not only generalize results about graphs of bounded maximum degree proved by Alon, Ding, Oporowski and Vertigan [1] and by Liu and Oum [3] but also can be applied to the clustered coloring version of Hajós' conjecture and Gerards-Seymour conjecture. In particular, we prove that the clustered coloring version of Hajós' conjecture is true for graphs of bounded treewidth and has a linear upper bound for the required number of colors, and we improve the currently best known upper bound of the required number of colors for the clustered coloring version of Gerards-Seymour conjecture provided by Kang and Oum [2].

We conjecture that the clustered coloring version of Hajós' conjecture and Gerards-Seymour conjecture are true.

Conjecture 1. *For every positive integer t, the class of K_{t+1}-topological minor free graphs and the class of odd K_{t+1}-minor free graphs are clustered t colorable.*

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A Fast New Algorithm for Weak Graph Regularity

László Miklós Lovász
(joint work with Jacob Fox, Yufei Zhao)

Regularity lemmas are very useful tools in graph theory and theoretical computer science. A version, known as the Frieze-Kannan or "weak" regularity lemma [4], says the following. Given a partition $\mathcal{P}$ of the vertices of a graph G, let $G_{\mathcal{P}}$ be the graph obtained by taking, between any pair of parts, weighted edges, with each edge having weight equal to the density between the two parts. We say that a partition is Frieze-Kannan-ϵ-regular, or weakly ϵ-regular, if for any two sets of vertices S and T, then number of edges between S and T in G, and the number of edges between S and T in $G_{\mathcal{P}}$, differs by at most ϵn^2, where n is the number of vertices of G. The lemma then says that for any $\epsilon > 0$, any graph G has a weakly ϵ-regular partition $\mathcal{P}$ of the vertices into at most $2^{2/\epsilon^2}$ parts.

In fact, Frieze and Kannan prove something slightly stronger. They show that the graph can be approximated by a weighted sum of at most $1/\epsilon^2$ complete bipartite graphs between a pair of subsets of vertices, so that for any sets of vertices S and T, the number of edges between S and T in the approximation and in the original graph differs by at most ϵn^2. This is, essentially, a stronger version of the result, because given such an approximation, it can be shown that if $\mathcal{P}$ is the joint refinement of all the sets that arise as one side of a bipartite graph in the sum, then $\mathcal{P}$ is a weakly 2ϵ-regular partition.

Prior to our work, there were two deterministic algorithms for weak regularity, due to Dellamonica, Kalyanasundaram, Martin, Rödl, and Shapira [1, 2], running in time $\epsilon^{-O(1)}n^{\omega+o(1)}$ or $2^{2^{\epsilon^{-O(1)}}}n^2$, where $\omega < 2.373$ is the matrix multiplication constant. These algorithms give weakly regular partitions, but it is not obvious how to use them to give an approximation as the sum of weighted bipartite graphs. In this talk, I'll discuss a recent deterministic algorithm that finds, in $\epsilon^{-O(1)}n^2$ time, an ϵ-regular Frieze–Kannan partition of a graph on n vertices [3]. Furthermore, the algorithm outputs an approximation of a given graph as a weighted sum of $\epsilon^{-O(1)}$ many complete bipartite graphs, thus giving an algorithm for the stronger form of the Frieze-Kannan regularity lemma.

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A grid theorem for vertex minors

Rose McCarty

(joint work with Jim Geelen, O-joung Kwon, Paul Wollan)

Locally complementing at a vertex v of a graph G replaces the induced subgraph on the set of neighbours of v by its complement. A graph H is a *vertex minor* of G if H can be obtained from G by a sequence of vertex deletions and local complementations. Thus, as an example, every induced subgraph of G is also a vertex minor of G. Vertex minors were introduced by Bouchet [2] to study isotropic systems. We will discuss two main examples of graph classes that are closed under taking vertex minors: circle graphs and graphs of bounded rank-width.

A graph G is a *circle graph* if there is a set of chords $\mathcal{C}$ of a circle so that $V(G) = \mathcal{C}$ and distinct vertices $u, v \in V(G)$ are adjacent if and only if the chords u and v intersect. The class of circle graphs is closed under taking vertex minors, and Bouchet [1] gave a characterization of circle graphs by three forbidden vertex minors.

Now, let A denote the adjacency matrix of a graph G. The *rank* of a set $X \subseteq V(G)$, denoted $\rho(X)$, is the rank over the binary field of the submatrix of A with rows X and columns $V(G) \setminus X$. As a function with domain $2^{V(G)}$, ρ is symmetric and submodular. Furthermore, ρ is invariant under local complementation. So ρ gives rise to a nice width parameter that does not increase when taking vertex minors. We define this parameter next.

Firstly, a *rank-decomposition* of G is a tree T so that the set of leaves of T is equal to $V(G)$, and so that all vertices of T have degree either one or three. The *width* of an edge e of T is the rank of the set of all leaves of one of the components of $T - e$. Finally, the *rank-width* of a graph G is the minimum over all rank-decompositions T of G, of the maximum over all edges e of T of the width of e. Rank-width was introduced by Oum and Seymour [6], and Oum [5] proved a number of results on rank-width and vertex minors which we use in our work. We also use a theorem of Kwon and Oum [4] as a base case. We prove the following.

Theorem 1. *For every circle graph H, there is an integer r_H so that if G is any graph with rank-width at least r_H, then G has a graph isomorphic to H as a vertex minor.*

This is analogous to the Grid Theorem of Robertson and Seymour [7], which states that for every planar graph H, there is an integer t_H so that if G is any graph with tree-width at least t_H, then G has a minor isomorphic to H. It is important to point out that the class of all circle graphs has unbounded rank-width. Like the Grid Theorem, we reduce Theorem 1 to the case where H is a certain type of circle graph which we call comparability grids. For a positive integer n, the $n \times n$ *comparability grid* is the graph with vertex set $\{(i,j) : i,j \in \{1,2,\dots,n\}\}$ where there is an edge between distinct vertices (i,j) and (i',j') if either $i \le i'$ and $j \le j'$, or $i \ge i'$ and $j \ge j'$. While there is a similarity between the Grid Theorem and Theorem 1, as far as we know neither implies the other.

Let $\mathcal{F}$ be any proper class of graphs that is closed under isomorphism and taking vertex minors. We conclude with some conjectures. The first is due to Geelen; see [3, Conjecture 1].

Conjecture 1. *(Chi-boundedness Conjecture) There is a function f so that for every graph $G \in \mathcal{F}$ with maximum clique of size ω, the chromatic number of G is no more than $f(\omega)$.*

Conjecture 2. *(Max-Clique Testing Conjecture) There is a polynomial-time algorithm that, given a graph $G \in \mathcal{F}$, computes the size of a maximum clique of G.*

Conjecture 3. *(Well-Quasi-Ordering Conjecture) There are, up to isomorphism, finitely many excluded vertex minors for $\mathcal{F}$.*

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Weak coloring numbers of planar graphs

Piotr Micek
(joint work with Gwenaël Joret)

Given a graph G, an integer $r \geq 0$, and a linear order L of the vertices of G, we say that v *r-reaches* x, if there exists a path P from v to x in G of length at most r (we count the edges) and with x being the least vertex of P in L. Let

$$W_r^L(v) = \{x \in V(G) \mid x \text{ is } r\text{-reachable from } v\},$$
$$\mathrm{wcol}_r(G) = \min_L \max_v |W_r^L(v)|.$$

The value $\mathrm{wcol}_r(G)$ is the *r-th weak coloring number* of G. It was introduced by Kierstead and Yang in 2003 in [3] where they write *"While our main motivation is the study of game chromatic number, there have been other applications of these ideas and we expect there will be more"*.

We have discussed two recent applications of weak coloring numbers. One is a bound on the chromatic number of exact-distance graphs. Given a graph G, by $G^{[\#d]}$ we denote a graph on the same set of vertices as G and two vertices are adjacent in $G^[\#d]$ if they are at distance d in G. Van den Heuvel, Kierstead and Quiroz [4] proved that for every positive odd integer d and every graph G we have

$$\chi(G^{[\#d]}) \leq \mathrm{wcol}_{2d-1}(G).$$

The second application is a bound for poset dimension. Given a poset P of height h and with a cover graph G, let $c = \mathrm{wcol}_{3h}(G)$. Then

$$\dim(P) \leq 4^c.$$

This was proved by Joret, Micek, Ossona de Mendez and Wiechert [2]. Both results improve on previous bounds by a large margin and both have very simple proofs.

Since there are more and more interesting bounds in terms of the weak coloring numbers it is important to have good bounds for weak coloring numbers themselves. A significant progress was made recently by van den Heuvel, Ossona de Mendez, Quiroz, Rabinovich, and Siebertz [1]. They proved (among other results) that the r-th weak coloring number of planar graphs is $O(r^3)$, while the best known lower bound is $\Omega(r^2)$. In fact the cubic bound works also for graphs with bounded genus and a more general bound for graphs excluding K_t as a minor is $O(r^{t-1})$.

We show the following new bounds.

Theorem 1. *There are planar graphs G with* $\mathrm{wcol}_r(G) = \Omega(r^2 \log r)$.

Theorem 2. *Planar graphs G of treewidth at most* 3 *have* $\mathrm{wcol}_r(G) = O(r^2 \log r)$. *Moreover, this bound is best possible.*

Theorem 3. *Outerplanar graphs G have* $\mathrm{wcol}_r(G) = O(r \log r)$. *Moreover, this bound is best possible.*

We leave the following questions for further research.

Conjecture 1. *Planar graphs G have* $\mathrm{wcol}_r(G) = O(r^2 \log r)$.

Question 1. *What is the bound on* wcol_r *for* $K_{3,t}$*-minor free graphs?*

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Structure and rough characterization of crossing-critical graphs

Bojan Mohar

(joint work with Zdeněk Dvořák, Petr Hliněný)

We study c-crossing-critical graphs, which are the minimal graphs that require at least c edge-crossings when drawn in the plane. For any fixed $c > 1$ there exist infinitely many c-crossing-critical graphs. It has been previously shown that c-crossing-critical graphs have bounded path-width and contain only a bounded number of internally disjoint paths between any two vertices. We expand on these results, providing a more detailed description of the structure of crossing-critical graphs. On the way towards this description, we prove a new structural characterisation of plane graphs of bounded path-width. Then we show that every c-crossing-critical graph can be obtained from a c-crossing-critical graph of bounded size by replicating bounded-size parts that already appear in narrow "bands" or "fans" in the graph. This also gives an algorithm to generate all the c-crossing-critical graphs of at most given order n in polynomial time per each generated graph.

1. Introduction

The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of crossings of edges in a drawing of G in the plane. If c is a positive integer, a graph G is *c-crossing-critical* if $cr(G) \geq c$, but every proper subgraph G' of G has $cr(G') < c$.

Minimizing the number of edge-crossings in a graph drawing in the plane is considered one of the most important attributes of a "nice drawing" of a graph, and this question has found numerous other applications (see, for example, [6], [9], or the monograph [8]). Consequently, a great deal of research work has been invested into understanding what forces the graph crossing number to be high. There exist strong quantitative lower bounds, such as the famous Crossing Lemma [1, 6]. However, the quantitative bounds show their strength typically in dense graphs, and hence they do not shed much light on the structural properties of graphs with bounded crossing number.

There are only two 1-crossing-critical graphs without degree-2 vertices, the Kuratowski graphs K_5 and $K_{3,3}$, but it has been known already since Širáň's [10] construction that the structure of c-crossing-critical graphs is quite rich and nontrivial for any $c \geq 2$. Already the first nontrivial case of $c = 2$ shows a dramatic increase in complexity of the problem. Yet, Bokal, Oporowski, Richter, and Salazar recently succeeded in obtaining a full description [2] of all the 2-crossing-critical graphs up to finitely many "small" exceptions.

To our current knowledge, there is no hope of extending the explicit description from [2] to any value $c > 2$. We, instead, give for any fixed positive integer c an asymptotic structural description of all sufficiently large c-crossing-critical graphs. Our contribution can be summarized as follows:

(1) There exist three kinds of local arrangements—a crossed band of uniform width, a twisted band, or a twisted fan—such that any optimal drawing of a sufficiently large c-crossing-critical graph contains at least one of them.
(2) There are well-defined local operations (replacements) performed on such bands or fans that can reduce any sufficiently large c-crossing-critical graph to one of finitely many basic c-crossing-critical graphs.
(3) A converse—a well-defined bounded-size expansion operation—can be used to iteratively construct each c-crossing-critical graph from a c-crossing-critical graph of bounded size. This yields a way to enumerate all the c-crossing-critical graphs of at most given order n in polynomial time per each generated graph. More precisely, the total runtime is $O(n)$ times the output size.

Structural properties of crossing-critical graphs have been studied for more than two decades, and we now briefly review two of the basic results. First, we remark that a c-crossing-critical graph may have no drawing with only c crossings (examples exist already for $c = 2$). Richter and Thomassen [7] proved the following upper bound:

Theorem 1 ([7])**.** *Every c-crossing-critical graph has a drawing with at most $\lceil 5c/2 + 16 \rceil$ crossings.*

Hliněný [4] proved that c-crossing-critical graphs have bounded path-width, and he and Salazar [5] showed that c-crossing-critical graphs can contain only a bounded number of internally disjoint paths between any two vertices.

Theorem 2 ([4])**.** *Every c-crossing-critical graph has path-width at most $\lceil 2^{6(72 \log_2 c + 248)c^3 + 1} \rceil$.*

2. Structure of large crossing-critical graphs

The proof of our structural characterisation of crossing-critical graphs can be roughly divided into two main parts. The first one establishes the existence of specific plane bands (resp. fans) and their tiles in crossing-critical graphs. In the second part we closely analyse these bands and tiles.

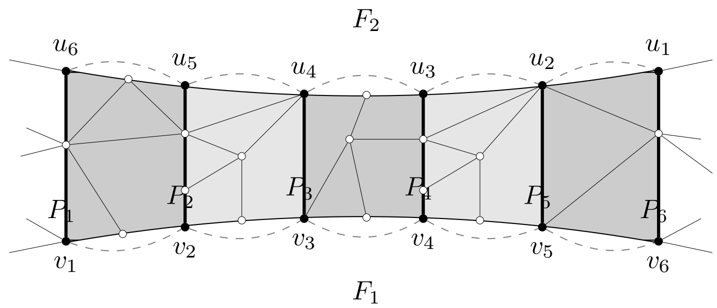

Figure 1. An example of paths $P_1, \dots, P_6$ (bold lines) forming an (F_1, F_2)-band of length 6, cf. Definition 1. The five tiles of this band, as in Definition 2, are shaded in grey and the dashed arcs represent α_i and α'_i from that definition.

Definition 1. *Let G be a 2-connected plane graph. Let F_1 and F_2 be distinct faces of G and let $v_1, v_2, \dots, v_m$, and $u_1, u_2, \dots, u_m$ be some of the vertices incident with F_1 and F_2, respectively, listed in the cyclic order along the faces. If $P_1, \dots, P_m$ are pairwise vertex-disjoint paths in G such that P_i joins v_i with u_{m+1-i}, for $1 \le i \le m$, then we say that $(P_1, \dots, P_m)$ forms an (F_1, F_2)-*band of length m*. Note that P_i may consist of only one vertex $v_i = u_{m+1-i}$.*

*Let F_1 and $v_1, v_2, \dots, v_m$ be as above. If u is a vertex of G and $P_1, \dots, P_m$ are paths in G such that P_i joins v_i with u, for $1 \le i \le m$, and the paths are pairwise vertex-disjoint except for their common end u, then we say that $(P_1, \dots, P_m)$ forms an (F_1, u)-*fan of length m*. The (F_1, u)-fan is* proper *if u is not incident with F_1.*

Definition 2. *Let $(P_1, \dots, P_m)$ be either an (F_1, F_2)-band or an (F_1, u)-fan of length $m \ge 3$. For $1 \le i \le m-1$, let α_i be an arc between v_i and v_{i+1} drawn inside F_1, and let α'_i be an arc drawn between u_i and u_{i+1} in F_2 in the case of the band; α'_i are null when we are considering a fan. Furthermore, choose the arcs to be internally disjoint. Let θ_i be the closed curve consisting of P_i, α_i, P_{i+1}, and α'_{m-i}. Let λ_i be the connected part of the plane minus θ_i that contains none of the paths P_j ($1 \le j \le m$) in its interior. The subgraphs of G drawn in the closures of $\lambda_1, \dots, \lambda_{m-1}$ are called* tiles *of the band or fan (and the tile of λ_i includes $P_i \cup P_{i+1}$ by this definition). The union of these tiles is the* support *of the band or fan.*

Our key results are summarized below in Theorems 3 and 4. Unfortunately, the space limitation does not allow us to state all necessary definitions. We refer to the full version of the paper [3] and to the monograph [8] for the terms that we do not define here.

Theorem 3. *Let c be a positive integer, and let $g : \mathbf{N} \to \mathbf{N}$ be an arbitrary non-decreasing function. There exist integers w_0 and n_0 such that the following holds. Let G be a 2-connected c-crossing-critical graph, and let G' be the plane graph associated with a drawing of G with the minimum number of crossings. Let*

Y denote the set of crossing vertices of G'. If $|V(G)| \geq n_0$, then for some $w' \leq w_0$, G' contains an (F_1, F_2)-band or a proper (F_1, u)-fan (where F_1 and F_2 are distinct faces and u is a vertex) of length at least $g(w')$ and with support disjoint from Y, such that each of its tiles has size at most w'.

Theorem 4. *For every integer $c \geq 1$, there exists a positive integer n_0 such that the following holds. If G is a 2-connected c-crossing-critical graph, then there exists a sequence $G_0, G_1, \ldots, G_m$ of 2-connected c-crossing-critical graphs such that $|V(G_0)| \leq n_0$, $G_m = G$, and for $i = 1, \ldots, m$, G_i is an n_0-bounded expansion of G_{i-1}.*

Moreover, the generating sequences claimed by Theorem 4 can be turned into an efficient enumeration procedure to generate all c-crossing-critical graphs of order at most n, for each fixed c.

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Claw-free matroids

Peter Nelson
(joint work with Kazuhiro Nomoto)

A *matroid* is a pair $M = (E, G)$, where G is a finite binary projective geometry $PG(n-1, 2)$, and E is any subset of the points of G. A matroid N is an *induced submatroid*) of M if $N = (E \cap F, F)$ for some subgeometry F of G; write $M|F$ for this matroid. If M has no induced restriction isomorphic to N, then M is *N-free.*

If G has dimension 3 and E is a basis of G, then (E, G) is a *claw*. A matroid with no such induced restriction is *claw-free*. Claws (and larger matroids whose ground set is a basis) play a role in the induced submatroid order similar to that played by trees in the order on induced subgraphs; one can thus formulate extremal and structural statements from induced subgraphs in the setting of matroids.

Our main result is an exact structure theorem for claw-free matroids, analogous to the structure theorem for claw-free graphs proved by Chudnovsky and Seymour. The theorem states that claw-free matroids can all be constructed from matroids in one of three 'basic classes' of claw-free matroids via a single 'join' operation that preserves the property of being claw-free.

We also present a conjecture along the lines of the Gyarfas-Sumner conjecture for graphs, concerning the boundedness of a parameter analogous to chromatic number in certain classes that are closed under taking induced submatroids. While this conjecture fails, we present an alternative conjecture that relates matroids to algebraic varieties in $\mathbb{F}_2^n$.

Resolution of the Oberwolfach problem

Deryk Osthus

(joint work with Stefan Glock, Felix Joos, Jaehoon Kim, Daniela Kühn)

A central theme in Combinatorics and related areas is the decomposition of large discrete objects into simpler or smaller objects. In graph theory, this can be traced back to the 18th century, when Euler asked for which orders orthogonal Latin squares exist (which was finally answered by Bose, Shrikhande, and Parker [1]). This question can be reformulated as the existence question for resolvable triangle decompositions in the balanced complete tripartite graph. (Here a resolvable triangle decomposition is a decomposition into edge-disjoint triangle factors.) In the 19th century, Walecki proved the existence of decompositions of the complete graph K_n (with n odd) into edge-disjoint Hamilton cycles and Kirkman formulated the school girl problem. The latter triggered the question for which n the complete graph on n vertices admits a resolvable triangle decomposition, which was finally resolved in the 1970s by Ray-Chaudhuri and Wilson [8] and independently by Jiaxi. This topic has developed into a vast area with connections e.g. to statistical design and scheduling, Latin squares and arrays, graph labellings as well as combinatorial probability.

A far reaching generalisation of Walecki's theorem and Kirkman's school girl problem is the following problem posed by Ringel in Oberwolfach in 1967.

Problem 1 (Oberwolfach problem)**.** *Let $n \in \mathbb{N}$ and let F be a 2-regular graph on n vertices. For which (odd) n and F does K_n decompose into edge-disjoint copies of F?*

Addressing conference participants in Oberwolfach, Ringel fittingly formulated his problem as a scheduling assignment for diners: assume n people are to be seated around round tables for $\frac{n-1}{2}$ meals, where the total number of seats is equal to n, but the tables may have different sizes. Is it possible to find a seating chart such that every person sits next to any other person exactly once?

We answer this affirmatively for all sufficiently large n. Note that the Oberwolfach problem does not have a positive solution for every odd n and F (indeed, there are four known exceptions).

A further generalisation is the Hamilton-Waterloo problem; here, two cycle factors are given and it is prescribed how often each of them is to be used in the decomposition. We also resolve this problem in the affirmative (for large n) via the following even more general result. We allow an arbitrary collection of types of cycle factors, as long as one type appears linearly many times. This immediately implies that the Hamilton-Waterloo problem has a solution for large n for any bounded number of given cycle factors. (In [5], we actually state and prove an even more general result.)

Theorem 1 ([5]). *For every $\alpha > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all odd $n \geq n_0$ the following holds. Let $F_1, \dots, F_k$ be 2-regular graphs on n vertices and let $m_1, \dots, m_k \in \mathbb{N}$ be such that $\sum_{i\in[k]} m_i = (n-1)/2$ and $m_1 \geq \alpha n$. Then K_n admits a decomposition into graphs $H_1, \dots, H_{(n-1)/2}$ such that for exactly m_i integers j, the graph H_j is isomorphic to F_i.*

The Oberwolfach problem and its variants have attracted the attention of many researchers, resulting in more than 100 research papers covering a large number of partial results. Most notably, Bryant and Scharaschkin [2] proved it for infinitely many n. Most classical results in the area are based on algebraic approaches, often by exploiting symmetries. More recently, major progress for decomposition problems has been achieved via absorbing techniques in combination with approximate decomposition results (often also in conjunction with probabilistic ideas). In [5], at a very high level, we also pursue such an approach. As approximate decomposition results, we exploit a hypergraph matching argument due to Alon and Yuster (which in turn is based on the Rödl nibble via the Pippenger-Spencer theorem) and a bandwidth theorem for approximate decompositions due to Condon, Kim, Kühn, and Osthus [3]. Our absorption procedure utilizes as a key element a very special case of a recent result of Keevash on resolvable designs [6].

Earlier, Kim, Kühn, Osthus, and Tyomkyn [7] considered approximate decompositions into graphs of bounded degree in host graphs G satisfying weaker quasirandom properties (namely, ε-superregularity). Their resulting blow-up lemma for approximate decompositions was a key ingredient for [3] (and thus for Theorem 1). It already implies that an approximate solution to the Oberwolfach problem can always be found (the latter was obtained independently by Ferber, Lee, and Mousset [4]).

While considerably more general than the Oberwolfach problem, Theorem 1 may be just the tip of the iceberg, and it seems possible that the following is true.

Conjecture 1. *For all $\Delta \in \mathbb{N}$, there exists an $n_0 \in \mathbb{N}$ so that the following holds for all $n \geq n_0$. Let $F_1, \dots, F_t$ be n-vertex graphs such that F_i is r_i-regular for* *some* $r_i \leq \Delta$ *and* $\sum_{i\in[t]} r_i = n-1$. *Then there is a decomposition of* K_n *into* $F_1, \dots, F_t$.

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Counting matroids of a fixed rank

Rudi Pendavingh

(joint work with Remco van der Hofstad, Jorn van der Pol)

Let $m(n,r)$ denote the number of matroids of rank r on a fixed ground set E of cardinality n. We present the following result.

Theorem 1. *For each* $r > 3$*:*

$$\ln m(n,r) = \binom{n}{r}\frac{\ln(n)+1-r+o(1)}{n} \qquad \text{as } n\to\infty$$

A matroid M is *paving* if each dependent set of M has cardinality at least $r(M)$. Let $p(n,r)$ be the number of paving matroids. I a previous paper [3], we demonstrated that

$$\ln p(n,r) \leq \ln m(n,r) \leq \left(1+\frac{r+o(1)}{n-r+1}\right)\ln p(n,r) \qquad \text{as } n\to\infty.$$

for each fixed rank r. It follows that the count paving matroids will dominate the upper bound on the number of matroids, at least on this logarithmic scale. To obtain the sharp upper bound of Theorem 1, we now show:

Theorem 2. *For each* $r > 3$*:*

$$\ln p(n,r) \leq \binom{n}{r}\frac{\ln(n)+1-r+o(1)}{n} \qquad \text{as } n\to\infty$$

To show this, we show that any paving matroid is uniquely determined by a *sparse paving* matroid together with a modest amount of further information. We show a tradeoff between the amount of 'further information' and the amount of information needed to describe the sparse paving matroid. This tradeoff is such that a small increase in the extra information implies a sharp decrease in the complexity of the sparse paving matroid, at least for paving matroids of rank 4 or greater. As a result, the number of paving matroids is bounded closely to the number of sparse paving matroids. Our proof of Theorem 1 adapts an entropy counting method of Linial and Luria [2], and further relies on a sharp lower bound on the number of sparse paving matroids due to Bennett and Bohman [1].

The information tradeoff observed in matroids of rank > 3 does not extend to matroids of rank 3, and so the above argument then fails. Using a completely different set of techniques, we obtained:

Theorem 3.

$$\binom{n}{3}\frac{\ln(n) - 2 + o(1)}{n} \leq \ln p(n,3) \leq \binom{n}{3}\frac{\ln(n) + 0.35 + o(1)}{n} \qquad \textit{as } n \to \infty$$

We are not entirely convinced that the constant $c = 0.35$ in this upper bound is best possible, but we do think that in rank 3 the gap between $p(n, r)$ and $s(n, r)$ is more pronounced than in higher rank. Specifically, let $p'(n, 3)$ denote the number of paving matroids of rank three without hyperplanes of cardinality > 4.

Conjecture 1. *There is a constant $c > -2$ such that*

$$\ln p(n,3) \approx \ln p'(n,3) = \frac{1}{n-r+1}\binom{n}{r}(\ln(n-r+1)+c) \quad \textit{as } n \to \infty.$$

In other words, we expect that in rank 3 the type of matroid which dominates the logarithmic bound is a paving matroid with hyperplanes of cardinality $2, 3$ and 4.

Our results are described in a recent preprint on arxiv [5], and they were a part of the PhD thesis of Jorn van der Pol [4].

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Local Versions of Coloring Theorems

Luke Postle

(joint work with Marthe Bonamy, Michelle Delcourt, Thomas Kelly, Peter Nelson, Thomas Perrett)

In the 1970s, Vizing and independently Erdos, Rubin and Taylor introduced list coloring. Erdos, Rubin and Taylor and independently Borodin then proved the list coloring version of Brooks' theorem. More than that, they proved a 'local list version of Brooks' theorem', namely they showed that if $|L(v)| \geq d(v) \geq \omega(v) := \omega(G[N[v]])$ for every vertex v in a graph G, then G has an a L-coloring. Surprisingly, few other local list versions were known for other coloring theorem. Local list versions are nice however since they generalize and simultaneously interpolate list coloring theorems.

Here we discuss local list versions of other prominent graph coloring theorems that we have recently proved. In particular, Bonamy, Kelly, Nelson and I proved a list local version of Johannson's theorem that $\chi(G) \leq O(\Delta/\ln\Delta)$, subject to the somewhat necessary condition that the range of degrees is at most exponential (i.e. $\delta(G) \geq \text{polylog}\Delta(G)$. Kelly and I proved a list local version of the epsilon version of Reed's conjecture. Namely, Delcourt and I had proved that $\chi(G) \leq (1-\varepsilon)(\Delta(G)+1) + \epsilon\omega(G)$ for large enough $\Delta(G)$ with $\varepsilon = 1/13$, improving on earlier works by Reed ($\varepsilon = 10^{-8}$), King and Reed ($\varepsilon = \frac{1}{320e^6}$) and Bonamy, Perrett and I ($\varepsilon = \frac{1}{26}$). Kelly and I managed to prove the list local version of this subject again to an at most exponential range of degrees with ε roughly $\frac{1}{100}$ and subject to the condition $\omega(v) < d(v) - \text{polylog}d(v)$ for every v.

There remain many open questions about local list versions such as: what happens when ω is close to d? Is an at most exponential range of degrees necessary? Can these versions be 'simultaneously localized'?

Concatenating bipartite graphs

Paul Seymour

(joint work with Maria Chudnovsky, Alex Scott, Sophie Spirkl)

Let $x, y \in (0,1]$; and let A, B, C be disjoint nonempty subsets of a graph G, where every vertex in A has at least $x|B|$ neighbours in B, and every vertex in B has at least $y|C|$ neighbours in C. We denote by $\phi(x,y)$ the maximum z such that, in all such graphs G, there is a vertex $v \in C$ that is joined to at least $z|A|$ vertices in A by two-edge paths. The function ϕ is interesting, and we have been investigating some of its properties, in [1]. For instance:

- $\phi(x,y) = \phi(y,x)$ for all x, y; and
- for each integer $k > 1$, there is a discontinuity in $\phi(x,x)$ when $x = 1/k$: $\phi(x,x) \leq 1/k$ when $x \leq 1/k$, and $\phi(x,x) \geq \frac{2k-1}{2k(k-1)}$ when $x > 1/k$.

There are several open questions: for instance, if $x > 1/3$ does it follows that $\phi(x,x) \geq 1/2$? We are able to prove that $\phi(x,x) \geq 3/7$. In general, is it true that for all integers $k \geq 1$, if $x, y \in (0,1]$ with $x + ky > 1$ and $kx + y \geq 1$, then $\phi(x, y) \geq 1/k$?

We examined in detail which pairs (x, y) satisfy $\phi(x, y) \geq z$, for $z = 1/2, 2/3$ and $1/3$. and plotted the answer in the (x, y)-plane. This results in two curves, for each value of z: below the first the answer is no, above the second the answer is yes, and in between we are not sure. The curves are very irregular, different sections corresponding to different theorems and counterexamples; but they are gratifyingly close in some areas, particularly the curves for $z = 2/3$.

What if we require in addition that every vertex in B has at least $x|A|$ neighbours in A, and every vertex in C has at least $y|B|$ neighbours in B? Call the corresponding function $\psi(x, y)$. Then we can prove that for all integers $k \geq 1$, if $x, y \in (0, 1]$ with $x + ky > 1$ and $kx + y \geq 1$, then $\psi(x, y) \geq 1/k$, and a number of other results. Is it true that $\psi(x, y) = \psi(y, x)$ for all x, y?

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Sparse graphs with forbidden induced subgraphs

Sophie Spirkl

(joint work with Maria Chudnovsky, Jacob Fox, Anita Liebenau, Marcin Pilipczuk, Alex Scott, Paul Seymour)

The celebrated Erdős-Hajnal conjecture [9, 10] asserts that for every graph H, there is a $\delta > 0$ such that every n-vertex graph G that does not contain H as an induced subgraph has a clique or a stable set of size at least n^{δ}. This has been proved for a few graphs H (e.g. [6], see also [3]), but remains open in general. The conjecture is known to be true when n^{δ} is replaced by $2^{\delta\sqrt{\log n}}$ [10].

On general properties of H-free graphs, Rödl [13] proved the following:

Theorem 1. *For all graphs H and $\varepsilon > 0$, there is a $\delta > 0$ such that for every H-free graph G, there is an induced subgraph J of G such that $|V(J)| \geq \delta|V(G)|$ and either J or J^c is ε-*sparse*, i.e. every vertex has degree at most $\varepsilon|V(J)|$.*

This motivates the study of sparse H-free graphs. We consider the following question: For which graphs H does there exist $\varepsilon > 0$ such that every n-vertex ε-sparse H-free graph G with $n > 1$ contains two sets A, B of vertices such that

(a) $|A|, |B| \geq \varepsilon n$;
(b) $|A| \geq \varepsilon n$ and $|B| \geq n^{\varepsilon}$; resp.
(c) $|A|, |B| \geq n^{\varepsilon}$.

and either all edges between A and B are present in G (A is *complete* to B), or no edges between A and B are present in G (A is *anticomplete* to B)?

Question (c) was solved in the affirmative by Erdös, Hajnal, and Pach [11]. Question (b) was conjectured to be true for all H by Conlon, Fox, and Sudakov

[8], and was proved for triangle-free graphs H whose vertex set can be partitioned into a stable set and a matching [4]. If Question (b) is true for all graphs, a similar method could likely be used to show a bound of $2^{\delta\sqrt{\log n \log\log n}}$ in the Erdős-Hajnal conjecture, which was done for $H = C_5$ in [5].

Question (a) has received considerable attention and is known as the *strong Erdős-Hajnal property*; in particular, if it holds for graphs H and J, then the conclusion of the Erdős-Hajnal conjecture holds for graphs G that are H-free and J^c-free. While Question (a) is known to be false when H is not a forest [2], it has been proved successively for paths [1], hooks (paths with an additional leaf at the third vertex) [2], subdivided caterpillars [12], and finally for all forests [7].

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Degree conditions for embedding trees

Maya Stein
(joint work with Guido Besomi, Matías Pavez-Signé)

This talk was based on joint work [1, 2] with Guido Besomi and Matías Pavez-Signé. The question we are interested in is the following: which degree bounds ensure that a graph G has to contain each tree on k edges as a subgraph? There are several conjectures in this respect, famous examples are the Erdős-Sós conjecture and the Loebl-Komlós-Sós conjecture, which use conditions on the average and on the median degree of G, respectively.

Recently, new conjectures conditioning on a combination of the minimum and the maximum degree of the host graph have sbeen suggested, the first of these being the 2/3 conjecture due to Havet, Reed, Stein and Wood [3], another one is the 2k-conjecture, which was proposed in [1]. We show approximate versions for large dense host graphs and bounded degree trees of both conjectures, and of the Erdős-Sós conjecture [1]. We then propose a unification of the two new conjectures, and prove a version for bounded trees and large dense host graphs [2].

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Rota's Basis Conjecture

Benny Sudakov
(joint work with Matija Bucić, Matthew Kwan, Alexey Pokrovskiy)

Given bases $B_1, \dots, B_n$ in an n-dimensional vector space V, a *transversal basis* is a basis of V containing a single distinguished vector from each of $B_1, \dots, B_n$. Two transversal bases are said to be *disjoint* if their distinguished vectors from B_i are distinct, for each i. In 1989, Rota conjectured (see [10, Conjecture 4]) that for any vector space V over a characteristic-zero field, and any choice of $B_1, \dots, B_n$, one can always find n pairwise disjoint transversal bases.

Despite the apparent simplicity of this conjecture, it remains wide open, and has surprising connections to apparently unrelated subjects. Specifically, it was discovered by Huang and Rota [10] that there are implications between Rota's basis conjecture, the Alon–Tarsi conjecture [2] concerning enumeration of even and odd Latin squares, and a certain conjecture concerning the supersymmetric bracket algebra.

Rota also observed that an analogous conjecture could be made in the much more general setting of *matroids*, which are objects that abstract the combinatorial properties of linear independence in vector spaces. Specifically, a finite matroid $M = (E, \mathcal{I})$ consists of a finite ground set E (whose elements may be thought of as vectors in a vector space), and a collection $\mathcal{I}$ of subsets of E, called independent sets. The defining properties of a matroid are that:

- the empty set is independent (that is, $\emptyset \in \mathcal{I}$);
- subsets of independent sets are independent (that is, if $A' \subseteq A \subseteq E$ and $A \in \mathcal{I}$, then $A' \in \mathcal{I}'$);

- if A and B are independent sets, and $|A| > |B|$, then an independent set can be constructed by adding an element of A to B (that is, there is $a \in A \backslash B$ such that $B \cup \{a\} \in \mathcal{I}$). This final property is called the *augmentation property.*

Observe that any finite set of elements in a vector space (over any field) naturally gives rise to a matroid, though not all matroids arise this way. A *basis* in a matroid M is a maximal independent set. By the augmentation property, all bases have the same size, and this common size is called the *rank* of M. The definition of a transversal basis generalises in the obvious way to matroids, and the natural matroid generalisation of Rota's basis conjecture is that for any rank-n matroid and any bases $B_1, \ldots, B_n$, there are n disjoint transversal bases.

Although Rota's basis conjecture remains open, various special cases have been proved. Several of these have come from the connection between Rota's basis conjecture and the Alon–Tarsi conjecture. Specifically, due to work by Drisko [6] and Glynn [9] on the Alon–Tarsi conjecture, Rota's original conjecture for vector spaces over a characteristic-zero field is now known to be true whenever the dimension n is of the form $p \pm 1$, for p a prime. Wild [12] proved Rota's basis conjecture for so-called "strongly base-orderable" matroids, and used this to prove the conjecture for certain classes of matroids arising from graphs. Geelen and Humphries proved the conjecture for "paving" matroids [7], and Cheung [4] computationally proved that the conjecture holds for matroids of rank at most 4.

Various authors have also proposed variations and weakenings of Rota's basis conjecture. For example, Aharoni and Berger [1] showed that in any matroid one can cover the set of all the elements in $B_1, \ldots, B_n$ by at most $2n$ "partial" transversals, and Bollen and Draisma [3] considered an "online" version of Rota's basis conjecture, where the bases B_i are revealed one-by-one. In 2017, Rota's basis conjecture received renewed interest when it was chosen as the twelfth "Polymath" project, in which amateur and professional mathematicians from around the world collaborated on the problem. Some of the fruits of the project were a small improvement to Aharoni and Berger's theorem, and improved understanding of the online version of Rota's basis conjecture [11].

One particularly natural direction to attack Rota's problem is to try to find lower bounds on the number of disjoint transversal bases. Rota's basis conjecture asks for n disjoint transversal bases, but it is not completely obvious that even two disjoint transversal bases must exist! Wild [12] proved some lower bounds for certain matroids arising from graphs, but the first nontrivial bound for general matroids was by Geelen and Webb [8], who prove that there must be $\Omega\left(\sqrt{n}\right)$ disjoint transversal bases. Recently, this was improved by Dong and Geelen [5], who used a beautiful probabilistic argument to prove the existence of $\Omega\left(n/\log n\right)$ disjoint transversal bases. In this paper we improve this substantially and obtain the first linear bound.

Theorem 1. *For any $\varepsilon > 0$, the following holds for sufficiently large n. Given bases $B_1, \ldots, B_n$ of a rank-n matroid, there are at least $(1/2 - \varepsilon)\, n$ disjoint transversal bases.*

Of course, since matroids generalise vector spaces, this also implies the same result for bases in an n-dimensional vector space.

We also mention the following strengthening of Rota's basis conjecture due to Kahn (see [10]). This is simultaneously a strengthening of the Dinitz conjecture on list-colouring of $K_{n,n}$, solved by Galvin.

Conjecture 1. *Given a rank-n matroid and bases $B_{i,j}$ for each $1 \leq i, j \leq n$, there exist representatives $b_{i,j} \in B_{i,j}$ such that each of the sets $\{b_{1,j}, \ldots, b_{n,j}\}$ and $\{b_{i,1}, \ldots, b_{i,n}\}$ are bases.*

The methods developed in this paper also are also suitable for studying 1. In particular, the argument used to prove 1 can readily be modified to show the following natural partial result towards Kahn's conjecture.

Theorem 2. *For any $\varepsilon > 0$ the following holds for sufficiently large n. Given a rank-n matroid and bases $B_{i,j}$ for each $1 \leq i \leq n$ and $1 \leq j \leq f = (1-\varepsilon)n/2$, there exist representatives $b_{i,j} \in B_{i,j}$ and $L \subseteq \{1, \ldots, f\}$ such that each $\{b_{i,j} : i \in L\}$ is independent, and such that $\{b_{i,1}, \ldots, b_{i,n}\}$ is a basis for any $i \in L$ and $|L| \geq (1/2 - \varepsilon)n$.*

Note also that if, for each fixed j, the bases $B_{1,j}, \ldots, B_{n,j}$ are all equal, then Kahn's conjecture reduces to Rota's basis conjecture. This observation also shows that 2 implies 1.

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Every graph of uncountable chromatic number has a subgraph of infinite edge-connectivity.

Carsten Thomassen

Erdős and Hajnal [1] conjectured in 1966 that every graph of uncountable chromatic number contains a subgraph of infinite connectivity. This problem is also discussed in [3] and [7]. Komjáth [2] proved that every graph of uncountable chromatic number contains a subgraph of uncountable chromatic number and of any finite connectivity. He has also proved that every graph of uncountable chromatic number contains a subgraph with infinite vertex degrees and of any finite connectivity, see [3]. He proved in [4] that it is consistent that there is an uncountable chromatic graph with no infinitely connected uncountable chromatic subgraph. More recently, a ZFC example has been given by Soukup [6].

In this talk we prove that the edge-connectivity version of the conjecture is true. The same holds if "chromatic number" is replaced by "coloring number" in both the assumption and conclusion of the result. It is consistent that it also holds for "list-chromatic number" since Komjáth [5] proved that it is consistent that the list-chromatic number equals the coloring number (when these are infinite). We also prove that, if each orientation of a graph G has a vertex of infinite outdegree, then G contains an uncountable subgraph of infinite edge-connectivity. All these results generalize to arbitrary infinite regular cardinals.

The proofs are all based on the general result that if the graph H can be obtained from the graph G by a generalized sequence of finite-cut-deletions, then G can be obtained from H by a generalized sequence of finite-cut-additions. The precise meaning is explained below.

If D is a cut in G, then $G - D$ is obtained from G by a *cut-deletion*. We also say that G is obtained from $G - D$ by a *cut-addition*.

As usual, a *sequence* of elements in a set S can be described as a collection a_n of elements in S where n is a natural number. If the indices n are chosen from a set of ordinals (smaller than some fixed large ordinal), then we speak of a *generalized sequence*.

If G is a graph, then a subgraph H of G is obtained from G by a *generalized sequence of cut-deletions* if there exists a generalized sequence of subgraphs G_α of G such that the following hold:

(i) $G = G_1$,

(ii) $H = G_{\alpha_0}$ for some ordinal α_0, and

(iii) If α is a limit ordinal, $\alpha \leq \alpha_0$, then G_α is the intersection of all G_β with $\beta < \alpha$, and

(iv) If α is an ordinal, $\alpha < \alpha_0$, then $G_{\alpha+1}$ is obtained from G_α by a cut-deletion.

If G is a graph, and H is a subgraph of G, then G is obtained from H by a *generalized sequence of cut-additions* if there exists a generalized sequence of subgraphs H_α of G such that the following hold:

(v) $H = H_1$,

(vi) $G = H_{\alpha_0}$ for some ordinal α_0, and

(vii) If α is a limit ordinal, $\alpha \leq \alpha_0$, then H_α is the union of all H_β with $\beta < \alpha$, and

(viii) If α is an ordinal, $\alpha < \alpha_0$, then $H_{\alpha+1}$ is obtained from H_α by a cut-addition.

Note that, if H is obtained from the graph G by a generalized sequence of finite-cut-deletions, and G is obtained from H by a generalized sequence of finite-cut-additions, then some of the added cuts may have to be distinct from all the deleted cuts. It would be interesting to investigate to which extent the same (or almost same) deleted cuts can be used in the generalized sequence of finite-cut-additions.

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Coloring rings

Kristina Vušković

(joint work with Frédéric Maffray, Irena Penev)

A *hole* is a chordless cycle of length at least 4. A *ring* R (of *length* k) is a graph whose vertex set can be partitioned into $k \geq 4$ nonempty sets $X_1, \dots, X_k$ such that for all $i \in \{1, \dots, k\}$ the set X_i can be ordered as $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ so that $X_i \subseteq N[u_i^{|X_i|}] \subseteq \dots \subseteq N[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$. In particular, each X_i is a clique and every hole of R is of length k. A *hyperhole* is a ring with further property that X_i is complete to X_{i+1} for every i.

Hyperholes are a subclass of proper circular arc graphs, for which it is known that coloring, stable set and clique problems can be solved in polynomial time. Rings are a subclass of circular arc graphs, but they are not necessarily proper circular arc (as they may contain claws). It is known that for circular arc graphs clique and stable set problems can be solved in polynomial time, but that coloring is NP-hard. For proper circular arc graphs coloring is also known to be solvable in polynomial time.

We observe that rings have unbounded clique-width, but seemingly simple structure. For example, it is easy to see that for some vertex v of a ring R, $R[N[v]]$ and $R \setminus N[v]$ are both chordal. These properties are well suited for clique and stable set problems, but do not immediately help with coloring. In this talk we present the following result from [3].

Theorem 1. [3] *If R is a ring, then $\chi(R) = max\{\chi(H) : H$ is a hyperhole in $R\}$. Furthermore, rings can be colored in polynomial time.*

Rings whose length is even are perfect and quite easy to color. Coloring of rings whose length is odd turned out to be much more complicated. Odd rings are even-hole-free, and their study is also motivated by this class, for which complexity of coloring and stable set problems are still open.

Rings also appear as a basic class in the decomposition theorem for the following class of graphs. A Truemper configuration is any theta, pyramid, prism or wheel. A *wheel* (H, x) is induced by a hole H and a vertex x that has at least three neighbors in H. If x has exactly three neighbors in H, that are furthermore consecutive on H, then (H, x) is a *twin wheel*. If x is adjacent to all vertices of H, then (H, x) is a *universal wheel*. Let $\mathcal{G}_{UT}$ be the class of graphs that out of all Truemper configurations may contain only twin and universal wheels. The following decomposition theorem is proved in [1]. A hole or a ring is *long* if it is of length at least 5.

Theorem 2. [1] *If $G \in \mathcal{G}_{UT}$ then either G has a clique cutset or G belongs to one of the following classes.*

(1) *G has exactly one nontrivial anticomponent and this anticomponent is a long ring.*
(2) *$\alpha(G) \leq 2$ and every anticomponent of G is either a 5-hyperhole or it is (long hole, $K_{2,3}$, $\overline{C_6}$)-free.*
(3) *$\alpha(G) \geq 3$ and G is (long hole, $K_{2,3}$, $\overline{C_6}$)-free.*

It is known that the clique problem is NP-hard on $\mathcal{G}_{UT}$, in fact on (long hole, $K_{2,3}$, $\overline{C_6}$)-free graphs. Recently it was shown that for (long hole, $\overline{C_6}$)-free graphs the stable set problem can be solved in polynomial time [2], which by Theorem 2 implies that the stable set problem can be solved in polynomial time for $\mathcal{G}_{UT}$. The coloring problem on this class remains open.

Theorem 1 implies that Hadwiger's Conjecture holds for rings and hence for graphs that satisfy (1). From known results it can be derived that Hadwiger's Conjecture holds for graphs that satisfy (2). So proving Hadwiger's Conjecture for $\mathcal{G}_{UT}$ reduces to proving it for graphs that satisfy (3).

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Triples in matroid circuits

Geoff Whittle
(joint work with Jim Geelen, Bert Gerards)

Paul Seymour described the structure that arises when three elements of a binary matroid are not contained in a circuit. In this talk I consider the analogous problem for representable matroids. In general, three elements of a vertically 5-connected matroid are not contained in a circuit if and only if they are joints of a frame matroid. For binary matroids, or indeed matroids representable over a field of characteristic two, we can replace 5 by 4 in the above statement. For other representable matroids specific obstacles arise that prevent the result from generalising to matroids representable over fields of odd characteristic.

Some natural open questions arise. It is likely that our results extend to non-representable matroids. This seems achievable but would require more difficult proof techniques. Also, the obstacles that arise that prevent the result holding for vertically 4-connected matroids are structure, but their structure is not fully understood. It would also be helpful to have a better understanding of the structure of the obstacles for vertically 4-connected matroids.

Around Ryser-Stein-Brualdi conjecture

Liana Yepremyan

A *Latin square* of order n is an $n \times n$ array filled with n different symbols, each occurring exactly once in each row and each column. A *partial transversal* of order k in a Latin square is a collection of k cells which do not share the same row, column or symbol. A *transversal* in a Latin square of order n is a partial tranvsersal of order n.

In 1967 Ryser conjectured [16] that the number of transversals in a Latin square of order n has the same parity as n. So a weaker form of this conjecture says that any Latin square of odd order has a transversal. Note that for even n this is not true; for such n the addition table of a cyclic group of order n is a Latin square with no transversal. In fact, in 1985 Maillet [10] gave many examples of group tables having no transversals, one particular case being $\mathbb{Z}_n$ (recently, this is extended further by Cavenagh and Wanless [3]).

Ryser's original conjecture is also false for odd n, because there are known Latin squares of odd order such that they have even number of transversals. However, the weaker conjecture still remains open. To this end, Brualdi [2] conjectured that every Latin square of order n has a partial transversal of size $n-1$ and moreover, if n is odd, it has a transversal. Stein [17] conjectured that even stronger statement holds, and the same outcome should hold even in an $n \times n$ array filled with the numbers $1, 2, \ldots, n$ such that every number occurs exactly n times in total. Very recently this was disproved by Pokrovskiy and Sudakov [14]; they constructed such arrays with no partial transversal of order $n - \frac{1}{42} \ln n$. The following conjecture is often referred in the literature as Ryser-Stein-Brualdi conjecture.

Conjecture 1. *Every Latin square of order n has a partial transversal of size $n-1$ and moreover, if n is odd, it has a transversal.*

This problem received a lot of attention and led to the development of important tools in extremal combinatorics, such as the first application of the Lopsided Lovász Local Lemma by Erdős and Spencer [4]. The current best lower bound on the size of the partial transversal is $n - O(\log^2 n)$ proved by Hatami and Shor [7]. In this report we discuss two variations of the Ryser-Stein-Brualdi conjecture.

First let us see what is the corresponding graph theory problem. To every Latin square one can assign an edge-colouring of the complete bipartite graph $K_{n,n}$ by colouring the edge ij by the symbol in the cell (i, j). This is a proper colouring, i.e., one in which any edges which share a vertex have distinct colours. Identifying the cell (i, j) with the edge ij, a partial transversal corresponds to a rainbow matching of the same size. So the Ryser-Stein-Brualdi conjecture says that any proper edge-colouring of $K_{n,n}$ contains a rainbow matching of size $n-1$. Aharoni and Berger conjectured the following.

Conjecture 2. *Let G be a bipartite multigraph that is properly edge-coloured with n colours and has at least $n + 1$ edges of each colour. Then G has a rainbow matching using every colour.*

To see that this conjecture is a generalization of Ryser-Stein-Brualdi conjecture, consider any proper edge-colouring of $K_{n,n}$ with n colours and let G be that $K_{n,n}$ plus a new edge of multiplicity n using all n colours. It is easy to see that G has a rainbow matching using every colour if and only if $K_{n,n}$ has a rainbow matching of size $n-1$. Pokrovskiy [13] showed that Conjecture 2 is asymptotically true, in that the conclusion holds if there are at least $n + o(n)$ edges of each colour. In a joint work with Peter Keevash [9], we considered the same question without the bipartiteness assumption and obtained a result somewhat analogous to Pokrovskiy's. We showed that any multigraph with edge multiplicities $o(n)$ that is properly edge-coloured by n colours with at least $n + o(n)$ edges of each colour contains a rainbow matching of size $n - c$, for some large absolute constant $c > 0$. Our algorithm is deterministic; it uses some switching method ideas for extending a matching (we call it reachibility). A similar result to ours was also obtained independently by Gao, Ramadurai, Wanless and Wormald [5]. Their result is closer than ours to the spirit of Conjecture 2, as they obtain a perfect rainbow matching (whereas we allow a constant number of unused colors). However, they also require a stronger bound on edge multiplicities. Their method is randomized, in particular they use the differential equation method. I think it is worth to pursue both deterministic (as in [9, 13]) and probabilistic (as in [5]) ideas in making further progress on Conjecture 2.

A second variation of Ryser-Stein-Brualdi conjecture follows. It is generally believed that if extra symbols are allowed then finding a transversal in Latin squares might become easier. To this end, Akbari and Alipour conjectured that every generalized Latin square with at least $n^2/2$ symbols has a transversal, where a *generalized Latin square* of order n is an $n \times n$ array filled with an arbitrary number of symbols such that no symbol appears twice in the same row or column. Very recently we confirmed this conjecture [8] in a stronger form, showing that for sufficiently large n, already $n^{2-\varepsilon}$ symbols suffice, for $\varepsilon = 1/200$. Our method employs regularity-type arguments, the probabilistic method and a greedy algorithm based on some switching method (this is in the spirit of the previously mentioned reachability algorithm). We did not try to optimize ε, but due to the auxiliary regularity-type arguments, our result cannot give something of form $n^{1+o(1)}$. However, there is no known barrier to believe that this number would not suffice. Note that when n is odd, if we believe Ryser-Stein-Brualdi conjecture, then we believe that $o(1)$ term can be even taken to be zero! As a first step can we answer the following question?

Question 2. *Does every generalized Latin square with at least $n^{1.5}$ symbols have a transversal?*

Note that independently Akbari-Alipour conjecture was also confirmed by Montgomery, Pokrovskiy and Sudakov [11], but their ε is implicit. In fact, they developed a very general and powerful toolkit for embedding spanning rainbow structures in graphs. As an auxillary result, they showed that every properly edge-coloured, quasi-random-like balanced bipartite graph on $2n$ vertices where every colour appears at most $(1-o(1))n$ times has $(1-o(1))n$ pairwise disjoint rainbow perfect matchings. This is done via a careful application of a variant of the famous Rödl nibble, where as an iterative step they find an approximate rainbow matching which behaves like a uniformly randomly chosen one.

One approach for Question 2 could be to combine their method together with ours. The same goes for attacking Ryser's conjecture. Can we improve the term $O(\log^2 n)$ in Hatami and Shor's result to some absolute constant?

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