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Arithmetic of Shimura Varieties

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Abstract. Arithmetic properties of Shimura varieties are an exciting topic which has roots in classical topics of algebraic geometry and of number theory such as modular curves and modular forms. This very active research field has contributed to some of the most spectacular developments in number theory and arithmetic geometry in the last twenty years. Shimura varieties and their equal characteristic analogue, moduli spaces of shtukas, are closely related to the Langlands program (classical as well as p-adic). A particular case is given by moduli spaces of abelian varieties, a classical object of study in algebraic geometry.

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Introduction by the Organizers

The workshop *Arithmetic of Shimura varieties* was attended by 53 participants with broad geographic representation, including a number of young participants. We had 20 talks of 60 minutes each.

Arithmetic properties of Shimura varieties are an exciting topic which has roots in classical topics of algebraic geometry and of number theory such as modular curves and modular forms. This very active research field has contributed to some of the most spectacular developments in number theory and arithmetic geometry in the last twenty years. Shimura varieties and their equal characteristic analogue, moduli spaces of shtukas, are closely related to the Langlands program (classical as well as p-adic). A particular case is given by moduli spaces of abelian varieties, a classical object of study in algebraic geometry.

The topics of the talks covered the whole subject of the modern arithmetic theory of Shimura varieties, ranging from geometric and cohomological properties of Shimura varieties or moduli spaces of shtukas over the study of automorpic bundles and automorphic forms to important related theories such as Rapoport-Zink spaces or the theory of crystals.

Applications to number theory. One key part of the workshop focused on applications of the theory to number theory, and in particular to the theory of modular forms.

Fabrizio Andreatta explained in his talk joint work with Adrian Iovita. For $N \geq 5$ prime to p they give criteria determining when overconvergent p-adic modular symbols defined for the modular curve with $\Gamma_1(N) \cap \Gamma_0(p)$-level structure are classical.

The topic of Vincent Pilloni's talk on joint work with George Boxer was also p-adic interpolation of modular forms. While modular forms are classically defined via global sections of certain line bundles, they studied the first cohomology for modular curves. This is expected to be only the starting point of a theory of p-adic interpolation of cohomology classes of automorphic bundles for Shimura varieties.

Olivier Taïbi reported on recent progress with Gaëtan Chenevier in the calculation of the dimension of the spaces of cuspidal Siegel modular forms for the full Siegel group $\mathrm{Sp}_{2g}(\mathbb{Z})$. They obtain complete formulas for genus ≤ 8 and sufficiently large weights.

Benjamin Howard explained joint work with Keerthi Madapusi Pera on the Kudla program, where they show the modularity of generating series of cycles on the integral model of orthogonal Shimura varieties.

Local models and construction of Shimura varieties. Central to the theory is the availability of integral and local models for Shimura varieties. In one of the highlights of the workshop, George Pappas addressed the question of the existence and the characterization of integral models of Shimura varieties with parahoric level structure focussing on the uniqueness of such models. The central tool for controling the singularities is the local model.

For such local models, Timo Richarz explained his joint work with Thomas Haines, in which they prove the test function conjecture. It allows to identify the test function that is used in the counting points formula in order to pursue the Langlands-Kottwitz approach to the calculation of the (semi-simple) Lefschetz number.

Peter Scholze has shown that Shimura varieties of Hodge type with infinite level at p are perfectoid. This has been generalized to Shimura varieties of abelian type by Xu Shen. David Hansen sketched in his talk *Towards general perfectoid Shimura varieties* an idea towards proving this result for arbitrary Shimura varieties.

Reductions of Shimura varieties and Automorphic bundles. Several talks concerned the geometric structure of the reductions of Shimura varieties.

Eyal Goren presented joint work with Ehud De Shalit in which they construct a derivation operator Θ that is an analogue for unitary Shimura varieties to the theta derivation on modular forms. They construct a foliation over a certain union of Ekedahl-Oort strata, and Θ can be understood as the derivative in direction of this foliation.

In his talk about joint work with Yifeng Liu, Liang Xiao, Wei Zhang, and Xinwen Zhu, Yichao Tian explained the geometry of unitary Shimura varieties at places of non-quasi-split reduction. He also described as an application a level raising result for automorphic representations.

Benoît Stroh and Jean-Stefan Koskivirta both reported on positivity results for automorphic bundles. Stroh explained joint work with Yohan Brunebarbe, Wushi Goldring, and Jean-Stefan Koskivirta that gives combinatorial criteria for certain line bundles on flag spaces over Shimura varieties to be ample. This yields vanishing results on the cohomology of certain automorphic bundles. Koskivirta presented joint work with Wushi Goldring in which they study the cone of those automorphic bundles in positive characteristic that admit non-trivial global sections. These results in particular show that the cone is larger than the analogue for automorphic bundles in characteristic zero.

Kai-Wen Lan explained his joint work with Hansheng Diao, Ruochuan Liu, and Xinwen Zhu where they compare the classical Riemann-Hilbert correspondence and a p-adic Riemann-Hilbert functor and prove that the obtained local systems are isomorphic after base change to the complex numbers.

Local Shimura varieties. Rapoport-Zink spaces and their (partially still only conjecturally existing) generalizations, the local Shimura varieties, occur in the p-adic uniformization of Shimura varieties but can also be defined in situations that are not covered by the axioms of a Shimura variety.

Alexander Ivanov explained his joint work with Jared Weinstein concerning the non-special locus of connected components of Rapoport-Zink spaces at infinite level. They show that this locus is cohomologically smooth which implies that each connected component contains an open dense cohomologically smooth subspace.

Alexander Bertoloni Meli studied in his talk the cohomology of Rapoport-Zink spaces of unramified EL-type and their realization of the local Langlands correspondence. His result is dependent on a conjecture by Harris and Viehmann that now seems to within reach by the work of Scholze.

Miaofen Chen presented results on the image of the p-adic period map for local Shimura varieties. She recalled joint work with Laurent Fargues and Xu Shen in which they proved in the basic case a necessary and sufficient criterion when the image is equal to the weakly admissible locus. She then presented her work giving a criterion in the general case for quasi-split groups.

Xuhua He's talk reported on joint results with Ulrich Görtz and Michael Rapoport classifying certain extremal cases of Rapoport-Zink spaces. He explained complete criteria when affine Deligne-Lusztig varieties, which may be considered as group theoretic variants of Rapoport-Zink spaces, have dimension zero or maximal possible dimension zero.

Pascal Boyer talked in *Persistence of non-degeneracy: a local analog of Ihara's lemma* about the construction of a filtration on the cohomology of the Lubin-Tate tower induced by the Newton stratification and the description of the lattices in the graded pieces.

Crystals and Shtukas. The program was rounded off by two talks on the essential tool of crystals and Dieudonné theory and by a foundational result on the moduli space of shtukas that can be considered as an equal-characteristic analogue of a Shimura variety.

Arthur Le Bras reported on progress in joint work with Johannes Anschütz in the development of a prismatic Dieudonné theory which would significantly extend the applicability of existing Dieudonné theories.

Urs Hartl's talk presented joint work with Ambrus Pal on a Chebotarëv density theorem for convergent families of F-isocrystals parametrized by a curve over a finite field. He explained their general conjecture and also gave a proof of that conjecture for convergent F-isocrystals that are direct sums of isoclinic ones.

A complementing highlight of the workshop was the talk of Cong Xue. She explained her two fundamental finiteness results about the cuspidal cohomology of the moduli space of shtukas as a $\mathbb{Q}_\ell$-vector space and on the whole cohomology as a module over local Hecke algebras.

All participants immensely enjoyed the unique environment provided by the Mathematisches Forschungsinstitut Oberwolfach and are very grateful for its hospitality.

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Workshop: Arithmetic of Shimura Varieties

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Abstracts

Classicality of overconvergent modular symbols

Fabrizio Andreatta
(joint work with Adrian Iovita)

Fix a complete discrete valuation field K of characteristic 0, ring of integers $\mathcal{O}_K$ and residue field, a perfect field of characteristic $p > 2$. Let $N \geq 5$ be an integer not divisible by p and let $\Gamma := \Gamma_1(N) \cap \Gamma_0(p) \subset \mathrm{SL}_2(\mathbb{Z})$. Let us denote by $X := X(N,p)$ the modular curve over $\mathcal{O}_K$ which classifies generalised elliptic curves with Γ-level structure and by $\omega := \omega_{E/X}$ the invertible sheaf on X of invariant differentials of the universal generalised elliptic curve E over X. In the spirit of the classical Eichler-Shmura isomorphism, Faltings constructed in [Fa] an isomorphism, equivariant for the actions of G_K (the absolute Galois group of K) and of the Hecke operators

$$\Psi_k^{\mathrm{cl}}\colon \mathrm{H}^1\big(\Gamma, V_k\big) \otimes_K \mathbb{C}_p \cong \Big(\mathrm{H}^0\big(X, \omega^{k+2}\big) \otimes_K \mathbb{C}_p\Big) \oplus \Big(\mathrm{H}^1\big(X, \omega^{-k}\big) \otimes_K \mathbb{C}_p(k+1)\Big);$$

here $V_k := \mathrm{Symm}^k(\mathbb{Q}_p^2)$ with its natural action of Γ, $\mathbb{C}_p$ is the p-adic completion of an algebraic closure of K, $(k+1)$ refers to a Tate twist. In [AIS] we have constructed p-adic variations of the isomorphism above.

First of all, following ideas of Stevens, one can p-adically interpolate the spaces V_k using Banach spaces of analytic distributions D_k on $T_0 = \mathbb{Z}_p^* \times \mathbb{Z}_p$; here k is a K-valued p-adic weight or a family of those parametrized by a wide open $\mathcal{U}$ of the weght space. Secondly, one can p-adically interpolate the sheaves ω^k, as explained in [AIP], and get sheaves $\omega^{\dagger,k}$ at least over strict neighborhoods $X(w)$ of the ordinary locus of X (here $0 < w < p/(p+1)$ is a measure of proximity to the ordinary locus). The sections of these sheaves coincide with the overconvergent modular forms introduced and studied by Robert Coleman. The main result of [AIS] provided a G_K and Hecke equivariant morphism

$$\Psi_k\colon \mathrm{H}^1\big(\Gamma, D_k\big)\widehat{\otimes}_K \mathbb{C}_p(1) \longrightarrow \mathrm{H}^0\big(X(w), \omega_w^{\dagger,k+2}\big)\widehat{\otimes}_K \mathbb{C}_p.$$

Both spaces admit slope decomposition for the action of the U_p-operator and we proved in [AIS] that, over a wide open $\mathcal{U}$ and for every integer h, the map $\Psi_k^{\leq h}$ induced by Ψ_k on the slope $\leq h$ parts is surjective except for finitely many points of $\mathcal{U}$. This provides a direct comparison betwen the eigencurve defined in chomological terms, using modular symbols, and the eigencurve defined using overconvergent modular forms. In the present note we report on recent progress providing more information on the set of points where the map $\Psi_k^{\leq h}$ is not surjective.

Theorem. *Assume that* $\prod_{i=0}^{h-1}(k-i-1)$ *is invertible in* K *if* k *is a* K*-valued weight or in* $\mathcal{U}$ *if* k *is a family of weights parametrized by a wide open* $\mathcal{U}$*. Then the map*

$$\Psi_k^{\le h}\colon \mathrm{H}^1\big(\Gamma, D_k\big)^{\le h}\widehat{\otimes}_K\mathbb{C}_p(1) \longrightarrow \mathrm{H}^0\big(X(w), \omega_w^{\dagger, k+2}\big)^{\le h}\widehat{\otimes}_K\mathbb{C}_p$$

is surjective.

Note that this condition is necessay as for a classical integral weight $k\in\mathbb{N}$ with $k\le h-1$ the map $\Psi_k^{\le h}$ factors through Faltings' map Ψ_k^{cl} and the inclusion $\mathrm{H}^0\big(X,\omega^{k+2}\big)\subset \mathrm{H}^0\big(X(w),\omega_w^{\dagger,k+2}\big)^{\le h}$ which might be strict, providing a counterexample to surjectivity.

Sketch of the proof: First of all one may replace $\mathrm{H}^1\big(\Gamma, D_k\big)$ with the cohomology group $\mathrm{H}^1_{\mathrm{pet}}\big(X_{\mathbb{C}_p}, D_k\big)$ for Scholze's pro-étale topology on $X_{\mathbb{C}_p}$. Secondly one can replace the group $\mathrm{H}^1_{\mathrm{pet}}\big(X_{\mathbb{C}_p}, D_k\big)\widehat{\otimes}_K\mathbb{C}_p(1)$ with $\mathrm{H}^1_{\mathrm{pet}}\big(X_{\mathbb{C}_p}, D_k\widehat{\otimes}\mathcal{O}_{X_{\mathbb{C}_p}}(1)\big)$. Summarizing we have isomorphisms, equivariant for the actions of G_K and of the Hecke operators:

$$\mathrm{H}^1\big(\Gamma, D_k\big)\widehat{\otimes}_K\mathbb{C}_p(1)\cong \mathrm{H}^1_{\mathrm{pet}}\big(X_{\mathbb{C}_p}, D_k\big)\widehat{\otimes}_K\mathbb{C}_p(1)\cong \mathrm{H}^1_{\mathrm{pet}}\big(X_{\mathbb{C}_p}, D_k\widehat{\otimes}\mathcal{O}_{X_{\mathbb{C}_p}}(1)\big).$$

The main issue is to prove that the restriction map

$$\mathrm{H}^1_{\mathrm{pet}}\big(X_{\mathbb{C}_p}, D_k\widehat{\otimes}\mathcal{O}_{X_{\mathbb{C}_p}}(1)\big)\longrightarrow \mathrm{H}^1_{\mathrm{pet}}\big(X(w)_{\mathbb{C}_p}, D_k\widehat{\otimes}\mathcal{O}_{X_{\mathbb{C}_p}}(1)\big)$$

is surjective once one takes $\le h$-parts, under the hypotheses of the Theorem.

The startegy follows closely the one used by P. Kassaei in [Ka] in his proof that overconvergent forms of classical weights and small slope are classical. Denote $X(w)^\infty := X(w)$; it is the strict neighborhood of the locus of X classifying ordinary elliptic curves where the level subgroup of order p is the canonical subgroup. We also have $X(w)^0$ the analogous strict neighborhood of the locus of E classifying ordinary elliptic curves where the level subgroup of order p is *not* the canonical subgroup. We have two steps.

Step 1: Buzzard's trick. Given a class ρ in $\mathrm{H}^1_{\mathrm{pet}}\big(X(w)^\infty_{\mathbb{C}_p}, D_k\widehat{\otimes}\mathcal{O}_{X_{\mathbb{C}_p}}(1)\big)$, which is a generalized eigenclass for the operator U_p with finite slope, one can extend it to $X\backslash X(w')^0$ for any $w'>0$. This uses that U_p improves the radius of overconvergence so that, if ρ is for simplicity an eigenclass with eigenvalue α, then $\frac{U_p^n}{\alpha^n}$ extends ρ to larger and larger neighborhoods eventually covering the whole $X\backslash X(w')^0$.

Step 2: Kasseaei's series. Given a class ρ in $\mathrm{H}^1_{\mathrm{pet}}\big(X(w)^\infty_{\mathbb{C}_p}, D_k\widehat{\otimes}\mathcal{O}_{X_{\mathbb{C}_p}}(1)\big)^{\le h}$ and an extension as in Step 1, we need to extend it to further to $X(w)^0$. Over $X(w)^0$ the U_p correpondence consists of $p-1$ operators mapping $X(w)^0$ to $X(w)^\infty$, that can be used to pull-back ρ, and a "bad" operator U_p' mapping $X(w)^0$ to itself. Over $X(w)^0$ the sheaf $D_k\widehat{\otimes}\mathcal{O}_{X_{\mathbb{C}_p}}(1)$ admits an increasing filtration such that on the the cohomology $\mathrm{H}^1_{\mathrm{pet}}\big(X(w)^0_{\mathbb{C}_p}, D_k\widehat{\otimes}\mathcal{O}_{X_{\mathbb{C}_p}}(1)/\mathrm{Fil}_h D_k\widehat{\otimes}\mathcal{O}_{X_{\mathbb{C}_p}}(1)\big)$ the opertor U'_p is divisible by p^h. On the other hand the cohomology group of $\mathrm{H}^1_{\mathrm{pet}}\big(X(w)^0_{\mathbb{C}_p}, \mathrm{Fil}_h D_k \widehat{\otimes} \mathcal{O}_{X_{\mathbb{C}_p}}(1)\big)$ is expressed as extension classes of the graded pieces of the filtrations, whose cohomology groups H^1 coincide with the groups $\mathrm{H}^0\big(X(w)^0, \omega_w^{\dagger, k+2-2i}\big)\widehat{\otimes}_K \mathbb{C}_p$. It is in the computation of these extension classes that the assumptions of the theorem are used and one proves that these classes vanish. One deduces that the subspace of $\mathrm{H}^1_{\mathrm{pet}}\big(X(w)^0_{\mathbb{C}_p}, \mathrm{Fil}_h D_k \widehat{\otimes} \mathcal{O}_{X_{\mathbb{C}_p}}(1)\big)$ on which U'_p has slope $\le h$ is negligible. Passing to formal models and working modulo powers of n this alows to neglect the contribution coming from U'_p and construct the sought for extension of ρ.

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Theta operators for unitary Shimura varieties

Eyal Z. Goren

(joint work with Ehud de Shalit)

This is a report on joint work with E. De Shalit (Hebrew University). Most of the results I will be reporting on can be found in our joint papers [1, 2].

Motivation. Recall the following theorem of Edixhoven, proved in the context of his work on the weight in Serre's conjecture.

Theorem 1. *Let f be a characteristic p eigenform of type (N, k, ϵ), then $\exists 0 \le i \le p-1, k' \le p+1$ and a characteristic p eigenform g of type (N, k', ϵ) such that f and $\theta^i(g)$ have the same eigenvalues for all Hecke operators $T_\ell, \ell \ne p$.*

The proof makes use of the following: if ω is the sheaf of weight 1 modular forms, $\omega = t^*_{\mathscr{E}(univ)/X_1(N)}$ then there is an exact sequence

$$0 \to \omega^{k-p+1} \stackrel{\times h}{\to} \omega^k \to \omega^k|_{ss} \to 0,$$

where h is the Hasse invariant and ss denotes the supersingular locus. Another construction gives θ, a derivation on modular forms in characteristic p whose effect on q-expansions is

$$\theta(\sum a_n q^n) = \sum n a_n q^n.$$

Note that $\theta^p(f) = \theta(f)$, leading to the study of theta cycles.

Our goal is to develop similar structures for unitary Shimura varieties.

Unitary Shimura varieties. Let E be a quadratic imaginary field, p inert in E, $n > m > 0$, $\kappa = \mathcal{O}_E/(p) \cong \mathbb{F}_{p^2}$.

Let S be the moduli scheme over κ (of dimension nm) classifying $\underline{A} = (A, \iota, \lambda, \eta)$, where A is an abelian variety over a κ-algebra R, $\iota\colon \mathcal{O}_E \to \mathrm{End}_R(A)$ a ring homomorphism such that the induced action on the tangent space is of type (n, m), λ a principal (or prime to p) polarization inducing complex conjugation on $\mathcal{O}_E$ and η a suitable rigid prime to p level structure that would not play a role in the sequel.

$S \supseteq S_\mu$, the μ-ordinary locus (open and dense by Wedhorn).

$$x \in S_\mu(\bar{\kappa}) \Leftrightarrow A_x[p] \cong (\mu_p \otimes \mathcal{O}_E)^m \times \mathscr{G}[p]^{n-m} \times (\mathcal{O}_E/(p))^m,$$

where $\mathscr{G}$ is a fixed choice of a p-divisible group of dimension 1 and height 2. More generally, the universal abelian variety over S_μ admits a filtration by finite flat group schemes whose graded parts are gr^m, gr^o and gr^{et}, and where the geometric fibers of the graded parts are isomorphic to $(\mu_p \otimes \mathcal{O}_E)^m, \mathscr{G}[p]^{n-m}$ and $(\mathcal{O}_E/(p))^m$, respectively.

The Hodge bundle $\mathbb{E} = t^*_{\mathscr{A}^{(univ)}/S}$ has a decomposition, induced by the $\mathcal{O}_E \otimes \kappa$-action, into vector bundles of rank n and m:

$$\mathbb{E} = P \oplus Q.$$

We let $\mathcal{L} = \det(Q)$, a line bundle on S, and define $\Gamma(S, \mathcal{L}^k)$ to be the space of unitary modular forms of weight k over κ. Over S_μ there is a finer filtration by vector bundles:

$$0 \to P_0 \to P \to P_\mu \to 0,$$

of ranks $n - m, n, m$, respectively, induced by the filtration on the p-torsion $\mathscr{A}^{(univ)}[p]$. One can ask how far does this filtration extend beyond the ordinary locus. To answer that recall the Ekedahl-Oort (EO) stratification.

EO-strata. (After Ekedahl-Oort, Moonen, Wedhorn). The EO strata are parameterized in our case by the cosets $S_m \times S_n \backslash S_{m+n}$ of the symmetric groups. (The governing algebraic group is $GU(n, m)$ whose Weyl group is S_{m+n} and the signature we have fixed implies that the Weyl group of the Levi of a suitable parabolic stabilizing the Hodge filtration is $S_m \times S_n$.) Each such coset has a unique representative σ that is an (m, n)-shuffle. That is $\sigma \in S_{m+n}$ and satisfies

$$\sigma^{-1}(1) < \cdots < \sigma^{-1}(n) \;\; and \;\; \sigma^{-1}(n+1) < \cdots < \sigma^{-1}(n+m).$$

Thus, there are $\binom{n+m}{n}$ distinct strata $\{S_\sigma : \sigma \text{ an } (m, n)\text{-shuffle}\}$; $\dim(S_\sigma) = \sum_{j=1}^n (\sigma^{-1}(j) - j)$.

We let

$$S_\sharp = \cup_{\sigma \geq \mathrm{fol}} S_\sigma.$$

For example, for $GU(n, 1)$ the EO-strata have a linear structure and $S_{\mathrm{fol}} = S - S_{\mathrm{id}}$. Namely, for signature $(n, 1)$, S_{fol} is the complement of the finitely many superspecial points of S.

Extending the filtration and a new moduli problem.

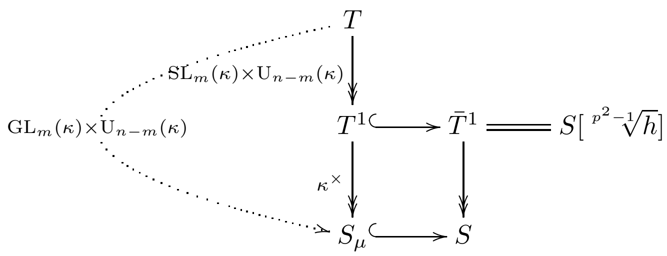
Theorem 2. *The filtration* $0 \to P_0 \to P \to P_\mu \to 0$ *extends to* $S_\sharp$ *and no further.*

We then define a new moduli problem

$$S^\sharp \to S,$$

a certain "successive blow-up" of S in $S - S_\sharp$; it classifies the same data $\underline{A}$ as S and all further extensions of P_0 into a sub-bundle of P of rank $n-m$ and killed by V. For example, for $GU(n,1)$, $S^\sharp$ is the blow-up of S at the finitely many superspecial points. The morphism $S^\sharp \to S$ is birational and an isomorphism over $S_\sharp$; $S^\sharp$ is regular.

Igusa varieties. We define a diagram



$T \to S_\mu$ is étale-Galois; $\bar{T}^1 \to S$ is Galois, purely-ramified outside S_μ; h is the Hasse invariant. Let us explain all that:

• the Hasse invariant h (constructed more generally in the unitary case by Goldring-Nicole and see also Koskivirta-Wedhorn) can be constructed as follows. Let $\mathscr{A}$ be the universal abelian variety over S. We have $V : \mathscr{A}^{(p)} \to \mathscr{A}$, $\mathcal{O}_E$-equivariant. It induces $\mathbb{E} \to \mathbb{E}^{(p)}$ and decomposing it according to $\mathcal{O}_E$-action we find: $V|_P : P \to Q^{(p)}, V|_Q : Q \to P^{(p)}$. We let h be the composition, viewed as a section of $\mathcal{L}^{p^2-1}$,

$$\det(V|_P^{(p)} \circ V|_Q) : \mathcal{L} \to \mathcal{L}^{p^2}.$$

• T classifies objects $\underline{A}$ and a pair of trivializations (locally étale liftable to the p^2-torsion), $\epsilon^m \colon (\mu_p \otimes \mathcal{O}_E)^m \to gr^m(\underline{A}[p])$, $\epsilon^o \colon \mathscr{G}[p]^{n-m} \to gr^o(\underline{A}[p])$ (the latter symplectic).

On T the vector bundles P_0, P_μ, Q are canonically trivialized. We denote by $\mathscr{E}_\rho$ any vector bundle constructed from these via a representation ρ of $\mathrm{GL}_{n-m} \times \mathrm{GL}_m \times \mathrm{GL}_m$ and refer to it as a "p-adic automorphic bundle". Note that $\mathscr{L}$ is an instance of such $\mathscr{E}_\rho$, but P is not, because the filtration does not split over S_μ.

Definition of Θ. The canonical connection $d : \mathcal{O}_T \to \Omega^1_{T/\kappa}$ induces a canonical connection $d : \mathscr{E}_\rho \to \mathscr{E}_\rho \otimes \Omega^1_{T/\kappa}$ and so we get a morphism Θ as the following composition:

$$\mathscr{E}_\rho \overset{d}{\to} \mathscr{E}_\rho \otimes \Omega^1_{T/\kappa} \overset{KS^{-1}}{\to} \mathscr{E}_\rho \otimes P \otimes Q \overset{project}{\to} \mathscr{E}_\rho \otimes P_\mu \otimes Q \cong \mathscr{E}_\rho \otimes Q^{(p)} \otimes Q,$$

where we have used the Kodaira-Spencer isomorphism, $P \otimes Q \cong \Omega^1_{X/\kappa}$ for $X = S$ or T, and the isomorphism over T or S_μ, $P_\mu \cong Q^{(p)}$.

Theorems. We have the following results concerning the operator Θ.

Theorem 3.
- (Descent) Θ *descends to an operator* $\Gamma(S_\mu, \mathscr{E}_\rho) \to \Gamma(S_\mu, \mathscr{E}_\rho \otimes Q^{(p)} \otimes Q)$.
- (Fourier-Jacobi expansions) *Let f be a section of $\mathscr{E}_\rho$ with expansion* $f = \sum_{\check{h} \in \check{H}} a(\check{h}) \cdot q^{\check{h}}$. [1] *Then,*
$$\Theta(f) = \sum_{\check{h} \in \check{H}} a(\check{h}) \otimes h \cdot q^{\check{h}}.$$
- (Analytic continuation) Θ *extends to an operator* $\Theta : \Gamma(S, \mathcal{L}^k) \to \Gamma(S, \mathcal{L}^k \otimes Q^{(p)} \otimes Q)$.[2]
- (Compatibility) *The construction of Θ is compatible with embedding of modular curves into S.*
- (Theta cycles) *Using the natural map* $\det : Q^{\otimes m} \to \det(Q) = \mathcal{L}$, *and iterating the theta operator m-times, we find an operator* $\tau : \Gamma(S, \mathcal{L}^k) \to \Gamma(S, \mathcal{L}^{k+p+1})$. *Then*
$$\tau^p(f) = h \cdot \tau(f).$$

We remark that differential operators, similar to our operator Θ, were constructed also by Eischen and Mantovan using different techniques. The intersection between our results is small.

Question. *We call $\tau(f), \ldots, \tau^p(f)$ the theta cycle of f. How does the filtration of f changes along its theta cycle? In the case of elliptic modular forms this was studied by Jochnowitz.*

Connection to foliations. We follow Jacobson, Ekedahl and Miyaoka in discussing foliations. Jacobson's theorem states that for a characteristic $p > 0$ field K there is a natural bijection,

$$\{\text{subfields } K \supseteq L \supseteq K^p\} \longleftrightarrow \{\text{sub-}p\text{-Lie algebras } A \subseteq Der(K)\},$$

under which $L \mapsto A := Der(K/L)$ and $A \mapsto L = \{x \in K : \delta(x) = 0, \forall \delta \in A\}$.

This was extended to varieties as follows. Let X be a non-singular irreducible variety over an algebraically closed field k of characteristic p. There is a natural bijection,

$$\{\text{varieties } X \to Y \to X^{(p)}\} \longleftrightarrow \{\mathscr{F} \subseteq T_X \text{ a foliation}\},$$

[1] Here $\check{H}$ is a certain lattice of rank m^2 that arises in the theory of toroidal compactifications and $a(\check{h})$ are sections of certain vector bundles on an abelian scheme of the form $A^\vee$, where A is an abelian variety with $\mathcal{O}_E$-action and signature $(n-m, 0)$. For example, for $GU(2,1)$ the expansions are $f = \sum_{n \geq 0} a(n) q^n$ and $\Theta(f) = \sum_{n \geq 0} n a(n) q^n$, where $a(n)$ are sections of line bundles on an elliptic curve with CM by $\mathcal{O}_E$.

[2] Thus, for $GU(n,1)$ we get $\Theta : \Gamma(S, \mathcal{L}^k) \to \Gamma(S, \mathcal{L}^{k+p+1})$.

where Y is normal and the morphism $X \to Y$ finite flat. By a foliation we mean a saturated subsheaf of the tangent bundle T_X of X that is closed under the Lie brackets and p-closed. Under this bijection $\mathscr{F} \mapsto Y$ where Y has structure sheaf $\mathcal{O}_Y = \{a \in \mathcal{O}_X : \xi(a) = 0, \forall \xi \in \mathscr{F}\}$, and $Y \mapsto \mathscr{F}$ where $\mathscr{F} := \{\xi \in T_X : \xi(\mathcal{O}_Y) = 0\}$. Moreover, $[k(X) : k(Y)] = p^{\mathrm{rk}\mathscr{F}}$ and Y is nonsingular if and only if $\mathscr{F}$ and $T_X/\mathscr{F}$ are vector bundles.

Theorems. Let $S_0(p)$ be the moduli scheme over S classifying $\underline{A}$ together with a subgroup $H \subset A[p]$ of rank $2m$, isotropic, $\mathcal{O}_E$-invariant and Raynaud. This scheme, for signatures $(2,1)$, was studied by Bellaiche. Two of its horizontal irreducible components (relative to the morphism $S_0(p) \to S$) are characterized by H being generically multiplicative, resp. generically étale, and we denote them $S_0(p)_m$ and $S_0(p)_{et}$. It is important to note that the morphisms $S_0(p)_m \to S, S_0(p)_{et} \to S$ are only *generically* finite.

We have the following results.

- $\mathscr{F} := (KS(P_0 \otimes Q))^{\perp}$ is a smooth rank-m^2 foliation on $S_\sharp$ that extends to $S^\sharp$.
- The foliation $\mathscr{F}$ agrees with the natural construction of an infinitesimal foliation on S_μ suggested by Moonen's theory of cascades.
- $S^\sharp \cong S_0(p)_m$ and there is a surjective finite flat morphism $S^\sharp \to S_0(p)_{et}$, which is defined by the foliation $\mathscr{F}$.

Finally, we note that in our definition of the derivation operator Θ we are essentially taking derivatives only in the direction of the foliation. This gives a geometric motivation for the seemingly arbitrary step of projecting from $P \otimes Q$ to $P_\mu \otimes Q$ appearing in the construction of Θ.

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Higher Coleman Theory

Vincent Pilloni

(joint work with George Boxer)

In the 80's, Hida introduced an ordinary projector on the space of modular forms and he constructed p-adic families of ordinary modular forms ([Hid04]). In the 90's, Coleman developed the finite slope theory ([Col97]) and Coleman and Mazur constructed the eigencurve ([CM98]). These theories have now been extended to higher dimensional Shimura varieties.

Hida and Coleman theories combine two ideas. The first one is to perform a restriction of modular forms, from the full modular curve to the ordinary locus and its neighborhoods. The additional structure on the universal p-divisible group on the ordinary locus (the canonical subgroups) allows to p-adically interpolate the sheaves of modular forms, and the cohomology. The second idea is to use the Hecke operators at p to detect when a section of the sheaf of modular forms defined on (a neighborhood of) the ordinary locus comes from a modular form.

Until recently, Hida and Coleman theories had only been used for degree 0 cohomology groups. In the recent works [Pil17], [BCGP18] we began to develop them further in order to study higher coherent cohomology of vector bundles on the Shimura varieties for the group GSp_4, and we are now convinced that Hida and Coleman theories should exist in all cohomological degrees for any Shimura variety.

The purpose of the current work is to confirm this prediction in the simple setting of modular curves and to construct Hida and Coleman theories for the degree 1 cohomology groups. We actually construct in parallel the theories for degree 0 and degree 1 cohomology, as this sheds some new light on the usual degree 0 theory. We also prove a p-adic Serre duality, which gives a perfect pairing between the theories in cohomological degree 0 and 1, but our constructions are independent of this pairing.

Let us describe the results we prove. Let $X \to \mathrm{Spec}\ \mathbb{Z}_p$ be the compactified modular curve of level $\Gamma_1(N)$, where $N \geq 3$ is an integer prime to p, and let D be the boundary divisor. Let $X_1 \to \mathrm{Spec}\ \mathbb{F}_p$ be the special fiber and X_1^{ord} be the ordinary locus.

Theorem 1 (Hida's control theorem)**.** *There is an Hecke operator T_p acting on the cohomology groups $\mathrm{R}\Gamma(X_1, \omega^k)$, $\mathrm{R}\Gamma_c(X_1^{ord}, \omega^k)$ and $\mathrm{R}\Gamma(X_1^{ord}, \omega^k)$, and an associated ordinary projector $e(T_p)$. Moreover, we have quasi-isomorphisms*

$$e(T_p)\mathrm{R}\Gamma(X_1, \omega^k) \to e(T_p)\mathrm{R}\Gamma(X_1^{ord}, \omega^k) \text{ if } k \geq 3$$

and

$$e(T_p)\mathrm{R}\Gamma_c(X_1^{ord}, \omega^k) \to e(T_p)\mathrm{R}\Gamma(X_1, \omega^k) \text{ if } k \leq -1.$$

The proof of this theorem relies on a local analysis of the cohomological correspondence T_p at supersingular points. The above theorem implies also a vanishing theorem : the ordinary cohomology of ω^k is concentrated in degree 0 if $k \geq 3$, and degree 1 if $k \leq -1$, because the ordinary locus is affine. Of course, this vanishing theorem holds true for the entire cohomology from the Riemann-Roch theorem and the Kodaira-Spencer isomorphism, but the argument above is independent and generalizes.

Let $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ be the Iwaswa algebra. Each integer $k \in \mathbb{Z}$ defines a character of $\mathbb{Z}_p^\times$, and a morphism $k : \Lambda \to \mathbb{Z}_p$.

Theorem 2. *There are two projective Λ-modules M and N carrying an action of the Hecke algebra of level prime to p, and there are canonical, Hecke-equivariant isomorphisms for all $k \geq 3$:*

1. $M \otimes_{\Lambda,k} \mathbb{Z}_p = e(T_p)\mathrm{H}^0(X, \omega^k)$,
2. $N \otimes_{\Lambda,k} \mathbb{Z}_p = e(T_p)\mathrm{H}^1(X, \omega^{2-k}(-D))$.

Moreover, there is a perfect pairing $M \times N \to \Lambda$ which interpolates the classical Serre duality pairing.

The modules M and N are obtained by considering the ordinary factor of the cohomology and cohomology with compact support of the ordinary locus with value in an interpolation sheaf of Λ-modules.

Let $X_0(p) \to \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ be the adic modular curve of level $\Gamma_1(N) \cap \Gamma_0(p)$. We have two quasi-compact opens $X_0(p)^m$ and $X_0(p)^{et}$ inside $X_0(p)$ which are respectively the locus where the universal subgroup of order p has multiplicative and étale reduction. We let $X_0(p)^{m,\dagger}$ and $X_0(p)^{et,\dagger}$ be the corresponding dagger spaces (these are the inductive limit over all strict neighborhoods of $X_0(p)^m$ and $X_0(p)^{et}$).

Theorem 3 (Coleman's classicality theorem)**.** *There is a well-defined Hecke operator U_p which is compact and has non negative slopes on $\mathrm{H}^i(X_0(p), \omega^k)$, $\mathrm{H}^i(X_0(p)^{m,\dagger}, \omega^k)$ and $\mathrm{H}^i_c(X_0(p)^{et,\dagger}, \omega^k)$. Moreover, the natural maps (where the subscript $< \star$ means slope less than $\star$ for U_p) :*

(1) $\mathrm{H}^i(X_0(p), \omega^k)^{<k-1} \to \mathrm{H}^i(X_0(p)^{m,\dagger}, \omega^k)^{<k-1}$,
(2) $\mathrm{H}^i_c(X_0(p)^{et,\dagger}, \omega^k)^{<1-k} \to \mathrm{H}^i(X_0(p), \omega^k)^{<1-k}$

are isomorphisms.

The proof of this theorem is based on some simple estimates for the operator U_p on the ordinary locus, reminicent of [Kas06]. We can again derive from this theorem a vanishing theorem for the small slope classical cohomology (without appealing to the Riemann-Roch theorem).

Coleman and Mazur constructed the eigencurve $\mathcal{C}$ of tame level $\Gamma_1(N)$. There is a weight morphism $w : \mathcal{C} \to \mathcal{W}$ where $\mathcal{W}$ is the analytic adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ associated with the Iwasawa algebra Λ.

Theorem 4. *The eigencurve carries two coherent sheaves $\mathcal{M}$ and $\mathcal{N}$ interpolating the degree 0 and 1 finite slope cohomology. For any $k \in \mathbb{Z}$, we have*

(1) $\mathcal{M}_k^{<k-1} = \mathrm{H}^0(X_0(p), \omega^k)^{<k-1}$,
(2) $\mathcal{N}_k^{<k-1} = \mathrm{H}^1(X_0(p), \omega^{2-k}(-D))^{<k-1}$,

and a there is a perfect pairing between $\mathcal{M}$ and $\mathcal{N}$, interpolating the usual Serre duality pairing.

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The test function conjecture for parahoric local models

Timo Richarz

(joint work with Thomas J. Haines)

In joint work with Thomas J. Haines [HRa, HRb], we prove the test function conjecture for local models of Shimura varieties with parahoric level constructed in [PZ13,Lev16] and their equal characteristic analogues, i.e., we express the (semi-simple) trace of Frobenius function on the sheaf of nearby cycles on the local model in terms of automorphic data as predicted by the conjecture.

Formulation of the result. Let p be a prime number. Let F be a non-archimedean local field with ring of integers $\mathcal{O}_F$ and finite residue field k_F of characteristic p and cardinality q, i.e. either $F/\mathbb{Q}_p$ is a finite extension or $F \simeq \mathbb{F}_q(\!(t)\!)$ is a local function field. Let $\bar{F}/F$ be a separable closure, and denote by Γ_F the Galois group with inertia subgroup I_F and fixed geometric Frobenius lift $\Phi_F \in \Gamma_F$.

We fix a triple $(G, \{\mu\}, \mathcal{G})$ where G is a connected reductive F-group, $\{\mu\}$ a (not necessarily minuscule) conjugacy class of geometric cocharacters defined over a finite separable extension E/F, and $\mathcal{G}$ is a parahoric $\mathcal{O}_F$-group scheme in the sense of Bruhat-Tits with generic fiber G. If $F/\mathbb{Q}_p$, we assume that $G \simeq \mathrm{Res}_{F'/F}(G')$ where F'/F is a totally (possibly wildly) ramified finite extension, and G' is a connected reductive F'-group which splits after a tamely ramified extension. Attached to the triple $(G, \{\mu\}, \mathcal{G})$ is a flat projective $\mathcal{O}_E$-scheme

$$M_{\{\mu\}} = M_{(G,\{\mu\},\mathcal{G})},$$

called the (flat) *local model*. It is defined in [PZ13, Lev16] if $F/\mathbb{Q}_p$ and in [Zhu14, Ri16a] if $F \simeq \mathbb{F}_q(\!(t)\!)$. The generic fiber $M_{\{\mu\},E}$ is naturally the Schubert variety in the affine Grassmannian of G/F associated with the class $\{\mu\}$. The special fiber $M_{\{\mu\},k_E}$ is equidimensional, but its irreducible components intersect in a complicated way in general.

Fix a prime number $\ell \neq p$, and fix $q^{\frac{1}{2}} \in \bar{\mathbb{Q}}_\ell$. Let d_μ be the dimension of the generic fiber $M_{\{\mu\},E}$, and denote the normalized intersection complex by

$$\mathrm{IC}_{\{\mu\}} \overset{\text{def}}{=} j_{!*}\bar{\mathbb{Q}}_\ell[d_\mu](\frac{d_\mu}{2}) \in D_c^b(M_{\{\mu\},E}, \bar{\mathbb{Q}}_\ell).$$

Under the geometric Satake equivalence, the complex $\mathrm{IC}_{\{\mu\}}$ corresponds to the ${}^L G_E := \hat{G} \rtimes \Gamma_E$-representation $V_{\{\mu\}}$ of highest weight $\{\mu\}$. Both $\hat{G}$ and $V_{\{\mu\}}$ are taken over $\bar{\mathbb{Q}}_\ell$.

Let E_0/F be the maximal unramified subextension of E/F, and let $\Phi_E = \Phi_{E_0} = \Phi_F^{[E_0:F]}$ and $q_E = q_{E_0} = q^{[E_0:F]}$. The semi-simple trace of Frobenius function on the sheaf of nearby cycles

$$\tau^{\mathrm{ss}}_{\{\mu\}}\colon M_{\{\mu\}}(k_E)\to\bar{\mathbb{Q}}_\ell,\quad x\mapsto(-1)^{d_\mu}\,\mathrm{tr}^{\mathrm{ss}}(\Phi_E\,|\,\Psi_{M_{\{\mu\}}}(\mathrm{IC}_{\{\mu\}})_{\bar{x}}),$$

is naturally a function in the center $\mathcal{Z}(G(E_0),\mathcal{G}(\mathcal{O}_{E_0}))$ of the parahoric Hecke algebra, cf. [Ga01, PZ13, Zhu14]. Our aim is to characterize this function uniquely in terms of "spectral information" as follows.

Theorem (The test function conjecture for parahoric local models). *Let $(G,\{\mu\},\mathcal{G})$ be a general triple as above. Let E/F be a finite separable extension over which $\{\mu\}$ is defined, and let E_0/F be the maximal unramified subextension. Then*

$$\tau^{\mathrm{ss}}_{\{\mu\}}=z^{\mathrm{ss}}_{\{\mu\}}$$

where $z^{\mathrm{ss}}_{\{\mu\}}=z^{\mathrm{ss}}_{\mathcal{G},\{\mu\}}\in\mathcal{Z}(G(E_0),\mathcal{G}(\mathcal{O}_{E_0}))$ is the unique function which acts on any $\mathcal{G}(\mathcal{O}_{E_0})$-spherical smooth irreducible $\bar{\mathbb{Q}}_\ell$-representation π by the scalar

$$\mathrm{tr}\Big(s(\pi)\;\Big|\;\mathrm{Ind}^{{}^LG_{E_0}}_{{}^LG_E}(V_{\{\mu\}})^{1\rtimes I_{E_0}}\Big),$$

where $s(\pi)\in[\hat{G}^{I_{E_0}}\rtimes\Phi_{E_0}]_{\mathrm{ss}}/\hat{G}^{I_{E_0}}$ is the Satake parameter for π [Hai15]. The function $q_{E_0}^{d_\mu/2}\tau^{\mathrm{ss}}_{\{\mu\}}$ takes values in $\mathbb{Z}$ and is independent of the choice of $\ell\neq p$ and $q^{1/2}\in\bar{\mathbb{Q}}_\ell$.

Strategy of proof. The local model $M_{\{\mu\}}$ is not semistable in general, and it is difficult to determine the value of $\tau^{\mathrm{ss}}_{\{\mu\}}$ at a given point in the special fiber directly. Instead we prove the following three statements:

a) *The Theorem for a Levi subgroup M implies the Theorem for G.*
b) *If G is anisotropic modulo center, then the Theorem for the quasi-split form G^* implies the Theorem for G.*
c) *The Theorem holds for tori.*

Let us first explain how a), b) and c) are applied to prove the Theorem: By a) it is enough to prove the Theorem for a minimal Levi subgroup, and we reduce to the case where G is anisotropic modulo center. By b), we reduce further to the case of a quasi-split group. Then the minimal Levi is a torus, and using a) again, we are reduced to the case of a torus which is c), and finishes the proof of the Theorem.

The most difficult part is the proof of a) which we sketch in the following. For simplicity of exposition, we assume that $E=E_0=F$. Let $M\subset G$ be a minimal Levi which is in good position relative to $\mathcal{G}$. As we already know that $\tau^{\mathrm{ss}}_{\{\mu\}}$ is a central function, it is uniquely determined by its image under the *injective* constant term map

$$c_M\colon\mathcal{Z}(G(F),\mathcal{G}(\mathcal{O}_F))\hookrightarrow\mathcal{Z}(M(F),M(F)\cap\mathcal{G}(\mathcal{O}_F)).$$

Our aim is to show $c_M(\tau^{\mathrm{ss}}_{\{\mu\}})=c_M(z^{\mathrm{ss}}_{\{\mu\}})$ assuming the Theorem for M. We can find a cocharacter $\chi\colon\mathbb{G}_{m,\mathcal{O}_F}\to\mathcal{G}$ whose centralizer $\mathcal{M}$ is a parahoric $\mathcal{O}_F$-group scheme with generic fiber M. Attached to χ is by the dynamic method the smooth $\mathcal{O}_F$-subgroup scheme $\mathcal{P}\subset\mathcal{G}$ whose generic fiber is a minimal parabolic subgroup

$P \subset G$ with Levi subgroup M. The natural maps $\mathcal{M} \leftarrow \mathcal{P} \rightarrow \mathcal{G}$ give rise to the diagram of Beilinson-Drinfeld Grassmannians

$$\mathrm{Gr}_{\mathcal{M}} \overset{q}{\leftarrow} \mathrm{Gr}_{\mathcal{P}} \overset{p}{\rightarrow} \mathrm{Gr}_{\mathcal{G}}, \tag{1}$$

which are $\mathcal{O}_F$-ind-schemes that degenerate the affine Grassmannian into the affine flag variety. The generic fiber of (1) is the diagram of affine Grassmannians denoted by $\mathrm{Gr}_M \overset{q_\eta}{\leftarrow} \mathrm{Gr}_P \overset{p_\eta}{\rightarrow} \mathrm{Gr}_G$, and the special fiber of (1) is the diagram on affine flag varieties denoted by $\mathcal{F}\ell_{\mathcal{M}} \overset{q_s}{\leftarrow} \mathcal{F}\ell_{\mathcal{P}} \overset{p_s}{\rightarrow} \mathcal{F}\ell_{\mathcal{G}}$. Associated with these data is the following diagram of functors:

$$\begin{array}{ccc} D_c^b(\mathrm{Gr}_G) & \xrightarrow{\Psi_{\mathcal{G}}} & D_c^b(\mathcal{F}\ell_{\mathcal{G}} \times_s \eta) \\ \mathrm{CT}_M \big\downarrow & & \big\downarrow \mathrm{CT}_{\mathcal{M}} \\ D_c^b(\mathrm{Gr}_M) & \xrightarrow{\Psi_{\mathcal{M}}} & D_c^b(\mathcal{F}\ell_{\mathcal{M}} \times_s \eta). \end{array}$$

The pair ($\Psi_{\mathcal{G}}$,$\Psi_{\mathcal{M}}$) denote the nearby cycles with target as in [SGA 7 XIII] the constructible bounded derived category of $\bar{\mathbb{Q}}_\ell$-complexes on $\mathcal{F}\ell_{\bar{s}}$ compatible with a continuous action of Γ_F. The pair $(\mathrm{CT}_M, \mathrm{CT}_{\mathcal{M}})$ are the pull-push functors given by $\mathrm{CT}_M = (q_\eta)_! \circ (p_\eta)^*\langle\chi\rangle$ (resp. $\mathrm{CT}_{\mathcal{M}} = (q_s)_! \circ (p_s)^*\langle\chi\rangle$) where $\langle\chi\rangle$ denotes a certain shift and twist associated with the cocharacter χ. The functor CT_M is well-known in Geometric Langlands [BG02,MV07] whereas the functor $\mathrm{CT}_{\mathcal{M}}$ only appears implicitly in the literature, cf. [AB09, Thm 4], [HNY13, §9].

Under the sheaf function dictionary, the nearby cycles $\Psi_{\mathcal{G}}$ are a geometrization of the Bernstein isomorphism (cf. [Ga01]), and the geometric constant term $\mathrm{CT}_{\mathcal{M}}$ is a geometrization of the map c_M in the following sense: by definition the local model $M_{\{\mu\}}$ is a closed reduced subscheme of $\mathrm{Gr}_{\mathcal{G}}$, and, up to a sign, the function $c_M(\tau^{\mathrm{ss}}_{\{\mu\}}) \in \mathcal{Z}(M(F), \mathcal{M}(\mathcal{O}_F))$ is the function associated with the complex

$$\mathrm{CT}_{\mathcal{M}} \circ \Psi_{\mathcal{G}}(\mathrm{IC}_{\{\mu\}}).$$

The following result is the geometric analogue of the compatibility of the Bernstein isomorphism with the constant term map.

Key result. *The usual functorialities of nearby cycles give a natural transformation of functors*

$$\mathrm{CT}_{\mathcal{M}} \circ \Psi_{\mathcal{G}} \longrightarrow \Psi_{\mathcal{M}} \circ \mathrm{CT}_M,$$

which is an isomorphism when restricted to $\mathbb{G}_m$-equivariant complexes. Here $\mathbb{G}_m$-equivariant means with respect to the $\mathbb{G}_m$-action induced by the cocharacter χ on Gr_G.

Its proof is based on a general commutation result for nearby cycles with hyperbolic localization [Ri, Thm 3.3], and an analysis of the $\mathbb{G}_m$-action on $\mathrm{Gr}_{\mathcal{G}}$ induced by the cocharacter χ.

Now we make use of the fact that the functor in the generic fiber CT_M corresponds under the geometric Satake equivalence to the restriction of ${}^LG$-representations $V \mapsto V|_{{}^LM}$ where ${}^LM \subset {}^LG$ is the closed subgroup associated with $M \subset G$. Hence, we know that the complex $\mathrm{CT}_M(\mathrm{IC}_{\{\mu\}})$, decomposes according to the irreducible ${}^L M$-representations appearing in $V_{\{\mu\}}|_{{}^L M}$ with strictly positive multiplicities. Hence, $c_M(\tau^{\mathrm{ss}}_{\{\mu\}})$ decomposes accordingly. As $c_M(z^{\mathrm{ss}}_{\{\mu\}})$ behaves similarly, we conclude $c_M(\tau^{\mathrm{ss}}_{\{\mu\}}) = c_M(z^{\mathrm{ss}}_{\{\mu\}})$ assuming the Theorem for M.

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Ample line bundles over Shimura varieties

Benoît Stroh

(joint work with Yohan Brunebarbe, Wushi Goldring, Jean-Stefan Koskivirta)

The subject of this talk is first to introduce a fibration over a Shimura variety with flag varieties as fibers, then to prove that some line bundles are ample over such a fibration in positive characteristic for a compact Hodge-type Shimura variety, and finally to derive consequences to vanishing theorems for the coherent cohomology of the Shimura variety.

Let $g \geq 1$, $n \geq 3$ and p a prime which does not divide n. Denote by $X \to \mathrm{Spec}(\mathbb{F}_p)$ the moduli space of principally polarized abelian varieties of genus g, with a principal level n. Over X we have the universal abelian variety G with neutral section $e : X \to G$, and the vector bundle $\Omega_G = e^*\Omega^1_{G/X}$ of rank g on X. Denote by $\pi : Y \to X$ the moduli space of full flags in Ω_G ; this is a Zariski locally trivial fibration with fibers the flag variety of GL_g.

Over Y we have the graded parts of the universal flag, denoted by $\mathcal{L}_1, \cdots, \mathcal{L}_g$. They are line bundles. For any $\lambda = (k_1, \cdots, k_g) \in \mathbb{Z}^g$, denote

$$\mathcal{L}(\lambda) = \bigotimes_{i=1}^{g} \mathcal{L}_i^{\otimes k_i}$$

which is a line bundle on Y. Also denote $\mathcal{V}(\lambda) = \pi_* \mathcal{L}(\lambda)$, which is an automorphic vector bundle on X.

By Kempf theorem for the coherent cohomology of flag varieties, we have $\mathcal{V}(\lambda) \neq 0$ if and only if $k_1 \geq \cdots \geq k_g$. And in this situation, $R^i \pi_* \mathcal{L}(\lambda) = 0$ for all $i > 0$. As a consequence,

$$H^i(X, \mathcal{V}(\lambda)) = H^i(Y, \mathcal{L}(\lambda))$$

for all $i \geq 0$. Therefore (up to taking care of boundary problems, because X and Y are not proper) vanishing theorems for $H^i(X, \mathcal{V}(\lambda))$ will immediately be deduced from ampleness properties of $\mathcal{L}(\lambda)$ on Y, by using Kodaira-Raynaud vanishing theorem if $p \geq \dim(Y) = g^2$.

Our main theorem is that if $0 > k_1 > \cdots > k_g$ and moreover λ is p-orbitally close (a condition introduced in a former work by Goldring and Koskivirta), then $\mathcal{L}(\lambda)$ is ample on Y. If $p \to \infty$, the p-orbitally close condition disappears, and one recovers results of Griffiths and Schmid over the complex numbers. To prove the theorem, we use an ampleness criterion due to Kleinman and the Hasse invariants on the Ekedahl-Oort strata of Y constructed by Goldring and Koskivirta.

The smooth locus in infinite-level Rapoport-Zink spaces

Alexander B. Ivanov
(joint work with Jared Weinstein)

The results described in this report will appear in [IW]. Let p be a prime number. We are interested in smoothness properties of a Rapoport-Zink space $\mathscr{M}_{\mathcal{D},\infty}$ of EL type with infinite level structure. The space $\mathscr{M}_{\mathcal{D},\infty}$ is the limit of finite-level spaces [RZ96], each of which is a smooth rigid-analytic space, parametrizing deformations of p-divisible groups (up to isogeny) equipped with some extra structure. However, the space $\mathscr{M}_{\mathcal{D},\infty}$ itself is not of finite type over its field of scalars, hence not smooth in the usual sense. By the work of Scholze and Weinstein [SW14], $\mathscr{M}_{\mathcal{D},\infty}$ is a preperfectoid space, i.e., its base change to any perfectoid base is perfectoid. For those spaces an appropriate notion of cohomological smoothness was introduced by Scholze [Sch17].

Let C be a complete algebraically closed extension of the field of scalars of $\mathscr{M}_{\mathcal{D},\infty}$. An important consequence of cohomological smoothness of a morphism $Y \to \mathrm{Spa}(C, \mathcal{O}_C)$ is that the cohomology (with ℓ-power torsion coefficients, $\ell \neq p$) of any quasi-compact open affinoid of Y is finitely generated. In particular, $\mathscr{M}_{\mathcal{D},\infty,C}$ is *nowhere* cohomologically smooth over $\mathrm{Spa}(C, \mathcal{O}_C)$. Indeed, its set of connected components maps (via the determinant morphism) onto the locally profinite set $\mathbb{Q}_p^\times$, which is nowhere discrete. This shows that H^0 of any quasi-compact affinoid is not finitely generated. Restriction to a connected component of $\mathscr{M}_{\mathcal{D},\infty,C}$ remedies this, and our main result is the following theorem.

Theorem 1. *Let $\mathcal{D}$ be a basic EL datum. Let $\mathscr{M}^\circ_{\mathcal{D},\infty}$ be a connected component of $\mathscr{M}_{\mathcal{D},\infty,C}$. Let $\mathscr{M}^{\circ,\mathrm{non-sp}}_{\mathcal{D},\infty} \subset \mathscr{M}^\circ_{\mathcal{D},\infty}$ be the non-special locus, corresponding to p-divisible groups without extra endomorphisms. Then $\mathscr{M}^{\circ,\mathrm{non-sp}}_{\mathcal{D},\infty}$ is cohomologically smooth over C.*

In particular, $\mathscr{M}^\circ_{\mathcal{D},\infty}$ contains an open dense subset which is cohomologically smooth over C. We mention the following application of Theorem 1 to Shimura varieties, which are closely related to Rapoport-Zink spaces. Consider the tower of classical modular curves $X(p^m)$. We regard them as rigid spaces over C. There is a perfectoid space $X(p^\infty)$ over C for which $X(p^\infty) \sim \varprojlim_n X(p^n)$, and a Hodge-Tate period map $\pi_{HT}\colon X(p^\infty) \to \mathbb{P}^1_C$ [Sch15], which is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant. Let $X(p^\infty)^\circ \subset X(p^\infty)$ be a connected component.

Corollary 2. *The following are equivalent for a C-point x of $X(p^\infty)^\circ$.*

(1) *The point x corresponds to an elliptic curve E, such that the p-divisible group $E[p^\infty]$ has* $\mathrm{End}\, E[p^\infty] = \mathbb{Z}_p$.
(2) *The stabilizer of $\pi_{HT}(x)$ in* $\mathrm{PGL}_2(\mathbb{Q}_p)$ *is trivial.*
(3) *There is a neighborhood of x in $X(p^\infty)^\circ$ which is cohomologically smooth over C.*

Here, the equivalence of (1) and (2) is not hard to see, the direction (2) ⇒ (3) follows from Theorem 1, and (3) ⇒ (2) follows from the explicit description of the special affinoids in the Lubin-Tate curve studied by Weinstein [Wei16].

We now discuss Theorem 1 in slightly more detail. For simplicity we only consider Rapoport-Zink spaces of special type: let H be a p-divisible group of height n and dimension d over a perfect field k of characteristic p. According to [SW14] the corresponding Rapoport-Zink space $\mathscr{M}_{H,\infty}$ parametrizes triples (G, ρ, α) consisting of a p-divisible group, a quasi-isogeny with the framing object and a trivialization of the Tate module. (Note also that $\mathscr{M}_{H,\infty}$ is basic if and only if H is isoclinic). Moreover, it admits a description in terms of modifications of vector bundles on the Fargues-Fontaine curve. For perfectoid S over $W(k)[1/p]$, let $X_S = X_{S^\flat}$ denote the corresponding (schematic, relative) Fargues-Fontaine curve, equipped with a divisor $i\colon \infty \hookrightarrow X_S$ corresponding to the untilt S of $S^\flat$. Let $M(H)$ be the covariant isocrystal over k attached to H, and let $\mathcal{E}_S(M(H))$ denote the corresponding vector bundle on X_S. Put $\mathcal{E}_S(H) = \mathcal{E}_S(M(H)) \otimes_{\mathcal{O}_{X_S}} \mathcal{O}_{X_S}(1)$ (thus, pointwise on S, slopes of H coincide with HN-slopes of $\mathcal{E}_S(H)$).

Theorem 3 ([SW14])**.** *The functor $\mathscr{M}_{H,\infty}$ (restricted to perfectoids over $W(k)[1/p]$) is isomorphic to the functor sending a perfectoid S over $W(k)[1/p]$ to the set of exact sequences*

$$0 \to \mathcal{O}^n_{X_S} \to \mathcal{E}_S(H) \to i_*W \to 0,$$

where W is a projective $\mathcal{O}_S$-module quotient of $M(H) \otimes_{W(k)[1/p]} \mathcal{O}_S$.

Using this description one can reinterpret $\mathscr{M}^{\circ}_{H,\infty}$ as a moduli problem of global sections of a smooth scheme Z over X_C, i.e., for an affinoid perfectoid T/C, $\mathscr{M}^{\circ}_{H,\infty}(T) = \mathrm{Mor}_{X_C}(X_T, Z)$. The scheme Z is constructed by some dilatation process and for a flat morphism $y\colon Y \to X_C$, the set $Z(Y)$ consists of all morphisms $f\colon \mathcal{O}^n_Y \to y^*\mathcal{E}$, such that the rank of f is equal to $n-d$ everywhere on Y_∞, and the determinant of f is the modification $\mathcal{O}_Y \to \det(y^*\mathcal{E})$, corresponding to the chosen connected component. This interpretation allows to use the following perfectoid version of the Jacobian criterion to prove Theorem 1:

Theorem 4 (Fargues–Scholze, [FS]). *Let S be some perfectoid space in characteristic p and let $Z \to X_S$ be a smooth morphism. Let $\mathscr{M}_Z$ be the subfunctor of the functor of functor global sections of Z/X_S,*

$$T \mapsto \mathscr{M}_Z(T) = \left\{ X_S\text{-morphisms } X_T \xrightarrow{t} Z \text{such that all slopes of } t^*\mathrm{Tan}_{Z/X_S} \text{ are } > 0 \right\}$$

Then $\mathscr{M}_Z$ is cohomologically smooth.

Thus Theorem 1 is reduced to a question about the HN-slopes of the pull-back of the tangent bundle Tan_{Z/X_C} along a section $f\colon Z \to X_C$. There is a natural action of $\mathrm{GL}_n(\mathbb{Q}_p) \times D^\times$ (where $D = \mathrm{End}H$ in the isogeny category) on $\mathscr{M}_{H,\infty}$, given in the above terms by $(\alpha,\beta)\colon f \mapsto \beta \circ f \circ \alpha$. For an S-point x of $\mathscr{M}_{H,\infty}$ corresponding to a modification f define the $\mathbb{Q}_p$-algebra $A_x := \{(\alpha,\beta) \in M_n(\mathbb{Q}_p) \times D\colon f \circ \alpha = \beta \circ f\}$. The special locus $\mathscr{M}^{\mathrm{sp}}_{H,\infty,C}$ in $\mathscr{M}_{H,\infty,C}$ is cut out by the closed condition $A_x \neq \mathbb{Q}_p$, and the non-special locus $\mathscr{M}^{\mathrm{non-sp}}_{H,\infty,C}$ is its open complement in $\mathscr{M}_{H,\infty,C}$. From Theorem 4 and the interpretation of $\mathscr{M}^{\circ}_{H,\infty}$ as moduli of global sections, it is clear that Theorem 1 now follows from

Theorem 5. *Assume that H is isoclinic. Let $x \in \mathscr{M}^{\circ}_{H,\infty}(C)$ correspond to a global section f of Z over X_C. The following are equivalent:*

(i) $x \in \mathscr{M}^{\circ,\mathrm{sp}}_{H,\infty,C}$

(ii) $f^*\mathrm{Tan}_{Z/X_C}$ *has a subbundle of slope* ≤ 0.

The proof of this theorem is the main technical content of the present work. It requires an interpretation of the vector bundle $f^*\mathrm{Tan}_{Z/X_C}$ in terms of more accessible vector bundles on X_C, and reduces at the end to linear algebra.

Finally, we mention an observation due to Scholze, which indicates a potential application of Theorem 1. Consider the case of the Lubin-Tate curve, i.e., let H be of dimension $d = 1$ and height $n = 2$. The resulting perfectoid space $\mathscr{M}_{H,\infty,C}$ was studied by Weinstein [Wei16]. Every finite level space $\mathscr{M}_{H,m,C}$ admits a semistable model $\hat{\mathscr{M}}_m$, but these models cannot be put into a *compatible* system of formal schemes $\cdots \to \hat{\mathscr{M}}_1 \to \hat{\mathscr{M}}_0$. The reason is that the number of irreducible components of the special fiber grows unbounded around CM-points (= points in the special locus). This problem can be resolved by removing the CM-points:

Theorem 6 (Weinstein [Wei16] Theorem 1.0.5). *The tower $\{\mathscr{M}^{\mathrm{non-sp}}_{H,m,C}\}_{m\geq 0}$ of non-special loci in the Lubin-Tate curve admits a sequence of compatible semistable* *models, with finite transition maps. The irreducible components of the special fibers of these models belong to a finite list of curves. More precisely, if* $(C_m)_{m\geq 0}$ *are such components, such that* $C_{m+1} \to C_m$ *and the genus of some* C_m *is positive, then for* m *big enough, the transition maps* $C_{m+1} \to C_m$ *are purely inseparable.*

The original proof of Theorem 6 in [Wei16] used the already known local Langlands correspondence in these cases to prove that the genus of the curves cannot grow infinitely. Using cohomological smoothness it should be possible to reprove this result without referring to the local Langlands correspondence.

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Prismatic Dieudonné theory

Arthur-César Le Bras

(joint work with Johannes Anschütz)

The goal of the ongoing project [1] on which I reported is to establish classification theorems for p-divisible groups, using the new prismatic formalism of Bhatt-Scholze [4]. This is a work in progress, and the results are not yet definitive.

In all the text, p is a fixed prime number.

Definition 1. *A ring* R *is quasi-syntomic if* R *is* p*-complete with bounded* p^∞*-torsion and if the cotangent complex* $L_{R/\mathbb{Z}_p}$ *has* p*-complete Tor-amplitude in* $[-1,0]$*. The category of all quasi-syntomic rings is denoted by* QSyn.

Similarly, a map $R \to R'$ *of* p*-complete rings with bounded* p^∞*-torsion is a quasi-syntomic cover if* R' *is* p*-completely faithfully flat over* R *and* $L_{R'/R} \in D(R')$ *has* p*-complete Tor-amplitude in* $[-1,0]$.

We endow $\mathrm{QSyn}^{\mathrm{op}}$ *with the structure of a site using quasi-syntomic covers.*

Remark 2. *This definition, due to Bhatt-Morrow-Scholze [3], extends (in the p-complete world) the usual notion of a local complete intersection ring to the non-Noetherian, non finite-type setting. The interest of this formulation, apart from being more general, is that it more clearly shows why this category of rings is relevant : the key property of quasi-syntomic rings is that they have a well-behaved (p-completed) cotangent complex.*

The objective of [1] is to make progress towards a complete classification of p-divisible groups over quasi-syntomic rings. Let us give some examples of such rings.

Examples 3. *(i) Any p-complete l.c.i. Noetherian ring is in* QSyn*. For example, $\mathcal{O}_K \in$* QSyn*, for any finite extension K of $\mathbb{Q}_p$.*

(ii) There are also has big rings in QSyn*. For example, any (integral) perfectoid ring is in* QSyn*.*

(iii) As a consequence of (ii), the p-completion of a smooth algebra over a perfectoid ring is also quasi-syntomic, as well as the quotient of an integral perfectoid ring by a finite regular sequence. Examples of such : $\mathcal{O}_{\mathbb{C}_p}/p$ or $\mathbb{F}_p[T^{1/p^\infty}]/(T-1)$.

The authors of [3] isolated an interesting class of quasi-syntomic rings.

Definition 4. *A ring R is quasi-regular semi-perfectoid if $R \in$* QSyn *and there exists a perfectoid ring S mapping surjectively to R.*

Example 5. *Any perfectoid ring, or any quotient of a perfectoid ring by a finite regular sequence, is quasi-regular semi-perfectoid.*

A nice property of quasi-regular semi-perfectoid rings is that they form a basis of the quasi-syntomic topology on QSyn. Using *quasi-syntomic descent*, one is thus reduced to establish a classification theorem for p-divisible groups over quasi-regular semi-perfectoid rings[1]. In the rest of the text, I will therefore only discuss the case of quasi-regular semi-perfectoid rings.

From now on, *we fix* a quasi-regular semi-perfectoid ring R, and a perfectoid ring S mapping surjectively to R. Let $\tilde{\xi} \in A_{\inf}(S)$ be a generator of the kernel of the map $A_{\inf}(S) \to S$.

The following definition comes from [4].

Definition 6. *The prismatic site R_Δ of R is the opposite category of bounded prisms (A, I) together with a map $R \to A/I$, with the obvious notion of morphisms, where covers are faithfully flat morphisms of prisms (i.e. maps $(A, I) \to (B, IB)$ such that B is (p, I)-completely flat over A).*

The functor sending $(A, I) \in R_\Delta$ to A (resp. A/I) defines a sheaf $\mathcal{O}_\Delta$ (resp $\overline{\mathcal{O}}_\Delta$) on R_Δ.

As $(A_{\inf}(S), (\tilde{\xi}))$ is a final object of S_Δ, the site R_Δ has a final object $(\Delta_R, (\tilde{\xi}))$: the prismatic envelope of $\ker(A_{\inf}(S) \to R)$ in $A_{\inf}(S)$, which also coincides with the derived prismatic cohomology of R over $A_{\inf}(S)$.

Examples 7. *(i) If R is perfectoid, $\Delta_R = A_{\inf}(R)$.*

(ii) If $pR = 0$, one has a natural isomorphism

$$\alpha : \Delta_R \to A_{\mathrm{crys}}(R),$$

given by Frobenius in the perfect case.

[1] Here, one sees that it is useful to be able to work with pretty big rings !

Set :

$$\mathcal{N}^{\geq 1}\Delta_R = \{x \in \Delta_R, \varphi(x) \in \tilde{\xi}\Delta_R\}$$

(the first piece of the Nygaard filtration). One has a natural isomorphism, induced by the Frobenius morphism :

$$\Delta_R/\mathcal{N}^{\geq 1}\Delta_R \simeq R.$$

One can define a divided Frobenius $\varphi_1 : \mathcal{N}^{\geq 1}\Delta_R \to \Delta_R$, by the formula $\varphi_1 = \varphi/\tilde{\xi}$ (recall that Δ_R is $\tilde{\xi}$-torsion free).

Definition 8. *The category* $\mathrm{DD}(R)$ *of divided prismatic Dieudonné modules over* R *is the category of collections*

$$\underline{M} = (M, \mathrm{Fil}\ M, \Phi, \Phi_1),$$

consisting of a projective Δ_R*-module* M *of finite type, a* Δ_R*-submodule* $\mathrm{Fil}\ M \subset M$*, and* φ*-linear maps* $\Phi : M \to M$ *and* $\Phi_1 : \mathrm{Fil}\ M \to M$*, such that :*

- $\mathcal{N}^{\geq 1}\Delta_R \cdot M \subset \mathrm{Fil}\ M$ *and* $M/\mathrm{Fil}\ M$ *is projective as an* R*-module.*
- *If* $x \in \mathcal{N}^{\geq 1}\Delta_R$*,* $m \in M$*,* $\Phi_1(xm) = \varphi_1(x)\Phi(m)$*.*
- $\Phi_1(\mathrm{Fil}\ M)$ *generates* M *as a* Δ_R*-module.*

The morphisms are Δ_R*-linear morphisms preserving the filtrations and commuting with* Φ *and* Φ_1*.*

One recovers familiar objects for perfectoid rings.

Proposition 9. *Let* R *be a perfectoid ring. The functor :*

$$(M, \mathrm{Fil}\ M, \Phi, \Phi_1) \mapsto (\mathrm{Fil}\ M, \tilde{\xi}\Phi_1)$$

is an equivalence between $\mathrm{DD}(R)$ *and the category of minuscule Breuil-Kisin-Fargues modules over* Δ_R *(with respect to* $\tilde{\xi}$*).*

Let $\mathrm{BT}(R)$ be the category of p-divisible groups over R.

Hope 10. *There is a natural functor :*

$$\mathrm{BT}(R) \to \mathrm{DD}(R),\ G \mapsto \underline{M}_R(G) = (M_R(G), \mathrm{Fil}\ M_R(G), \Phi, \Phi_1).$$

This functor is an equivalence of categories.

This hope is inspired by the work of Zink and Lau on *windows*[2]. Via Proposition 9, it is compatible with known results in the perfectoid case (Lau, Scholze-Weisntein).

We can define a functor $\underline{M}_R$ by adapting Berthelot-Breen-Messing's definition ([2]) to the prismatic setting. The module $M_R(G)$ is defined by evaluating on the initial prism $(\Delta_R, (\tilde{\xi}))$ the *prismatic crystal* :

$$\mathcal{E}xt^1_{R_\Delta}(i_*G, \mathcal{O}_\Delta)$$

[2] Note though that we expect the previous statement to be true for all p-divisible groups and all p.

(i_*G is the abelian sheaf sending $(A, I) \in R_\Delta$ to $G(A/I)$) and is equipped with a Frobenius, coming from the Frobenius of the prismatic structure sheaf. Its submodule Fil $M_R(G)$ is defined using the Nygaard filtration.

When $pR = 0$ and $G \in \mathrm{BT}(R)$, one has, as a corollary of the *crystalline comparison theorem* for prismatic cohomology, a canonical isomorphism of Dieudonné modules :

$$\alpha^* M_R(G) = M_R^{\mathrm{crys}}(G),$$

where $\alpha : \Delta_R \simeq A_{\mathrm{crys}}(R)$ is the isomorphism introduced in Remark 7 (ii) and $M_R^{\mathrm{crys}}(G)$ is the object defined and studied in [2].

We know how to prove that our functor $\underline{M}_R$ is an equivalence at least when $pR = 0$ or when R is p-torsion free. The general case is still open.

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Cohomologies of stacks of shtukas

Cong Xue

Let X be a smooth projective geometrically irreducible curve over a finite field $\mathbb{F}_q$. We denote by F its function field, $\mathbb{A}$ the ring of adèles of F and $\mathbb{O}$ the ring of integral adèles.

Let G be a connected split reductive group over $\mathbb{F}_q$.

1. Automorphic forms

Let Ξ be a cocompact subgroup in $Z_G(F)\backslash Z_G(\mathbb{A})$, where Z_G is the center of G. Then the quotient $Z_G(F)\backslash Z_G(\mathbb{A})/Z_G(\mathbb{O})\Xi$ is finite.

Let $N \subset X$ be a finite subscheme. We denote by $\mathcal{O}_N$ the ring of functions on N and $K_N := \mathrm{Ker}(G(\mathbb{O}) \to G(\mathcal{O}_N))$.

Let ℓ be a prime number not dividing q. We are interested in the vector space of automophic forms with compact support $C_c(G(F)\backslash G(\mathbb{A})/K_N\Xi, \mathbb{Q}_\ell)$.

We have the notion of cuspidal automorphic form. An automorphic form is said to be cuspidal if its image under the constant term morphism along any proper parabolic subgroup of G is zero. We denote by C_c^{cusp} the space of cuspidal automorphic forms. A theorem of Harder ([Har74], Theorem 1.2.1) says that C_c^{cusp} is of finite dimension.

Let v be a place in $X \smallsetminus N$. Let $\mathcal{O}_v$ be the complete local ring at v and let F_v be its field of fractions. Let $\mathscr{H}_{G,v} := C_c(G(\mathcal{O}_v)\backslash G(F_v)/G(\mathcal{O}_v), \mathbb{Q}_\ell)$ be the Hecke algebra of G at the place v. The space of automophic forms is equipped with an action of $\mathscr{H}_{G,v}$ by convolution on the right. The following proposition is well known by some experts:

Proposition 1. *For any place v of $X \smallsetminus N$, $C_c(G(F)\backslash G(\mathbb{A})/K_N\Xi, \mathbb{Q}_\ell)$ is a $\mathscr{H}_{G,v}$-module of finite type.*

2. Stacks of shtukas

Let $\widehat{G}$ be the Langlands dual group of G over $\mathbb{Q}_\ell$ (its roots and weights are the coroots and coweights of G, and vice-versa). Let I be a finite set and W be a finite dimensional $\mathbb{Q}_\ell$-linear representation of $\widehat{G}^I$. Associated to these data, in [Var04] Varshavsky has defined the stacks classifying G-shtukas (denoted by $\mathrm{Cht}_{G,N,I,W}$) over $(X \smallsetminus N)^I$. (The case of Drinfeld's shtukas is for $G = GL_n$, $I = \{1, 2\}$ and $W = St \boxtimes St^*$).

He has also defined the ℓ-adic cohomology groups with compact support in degree j of $\mathrm{Cht}_{G,N,I,W}$ (denoted by $H^j_{G,N,I,W}$). The cohomology group $H^j_{G,N,I,W}$ is defined as an inductive limit of Harder-Narasimhan truncations $H^{j,\,\leq\mu}_{G,N,I,W}$, which are finite dimensional $\mathbb{Q}_\ell$-vector spaces. However, $H^j_{G,N,I,W}$ may be infinite dimensional.

In particular, when $I = \emptyset$ and $W = \mathbf{1}$ (the one-dimensional trivial representation of the trivial group $\widehat{G}^\emptyset$), we have $H^0_{G,N,\emptyset,\mathbf{1}} = C_c(G(F)\backslash G(\mathbb{A})/K_N\Xi, \mathbb{Q}_\ell)$.

The cohomology group $H^j_{G,N,I,W}$ is equipped with an action of Hecke algebra $\mathscr{H}_{G,v}$ by Hecke correspondence for the stacks of shtukas (which is analogous to the Hecke correspondence for the Shimura varieties).

3. Results

Let P be a parabolic subgroup of G and let M be its Levi quotient. As in [Var04], we can define the stack of P-shtukas $\mathrm{Cht}_{P,N,I,W}$ and the stack of M-shtukas $\mathrm{Cht}_{M,N,I,W}$. The morphisms $G \hookleftarrow P \twoheadrightarrow M$ induce a correspondence

$$\mathrm{Cht}_{G,N,I,W} \leftarrow \mathrm{Cht}_{P,N,I,W} \rightarrow \mathrm{Cht}_{M,N,I,W}\,.$$

From this, in [Xue18a], we construct a constant term morphism

$$C_G^{P,\,j} : H^j_{G,N,I,W} \to H^j_{M,N,I,W}.$$

Then we define the cuspidal cohomology $H^{j,\,\mathrm{cusp}}_{G,N,I,W} \subset H^j_{G,N,I,W}$ as the intersection of the kernels of the constant term morphisms for all proper parabolic subgroups. In particular, when $I = \emptyset$ and $W = \mathbf{1}$, the constant term morphism $C_G^{P,\,0}$ coincides with the usual constant term morphism for automorphic forms and we have $H^{0,\,\mathrm{cusp}}_{G,N,\emptyset,\mathbf{1}} = C_c^{\mathrm{cusp}}$.

Using constant term morphisms $C_G^{P,\,j}$ and the contractibility of deep enough Harder-Narasimhan strata established in [Xue18a], we show that

(a) ([Xue18a]) there exists μ_0 large enough, such that

$$H^{j,\,\mathrm{cusp}}_{G,N,I,W} \subset \mathrm{Im}(H^{j,\,\leq\mu_0}_{G,N,I,W} \to H^j_{G,N,I,W}).$$

It implies

Theorem 2. *(loc.cit. Theorem 0.0.1) The $\mathbb{Q}_\ell$-vector space $H^{j,\,\mathrm{cusp}}_{G,N,I,W}$ is of finite dimension.*

This theorem is a generalization of Theorem 1.2.1 in [Har74].

We denote by $H^{j,\,\mathrm{Hf}}_{G,N,I,W}$ the "(rationally) Hecke-finite cohomology" defined in [Laf18] (a class is said to be (rationally) Hecke-finite if it belongs to a finite dimensional subspace stable by all Hecke algebras).

Proposition 3. *(loc.cit. Proposition 0.0.1) The two $\mathbb{Q}_\ell$-vector subspaces $H^{j,\,\mathrm{cusp}}_{G,N,I,W}$ and $H^{j,\,\mathrm{Hf}}_{G,N,I,W}$ of $H^{j}_{G,N,I,W}$ are equal.*

This proposition is a generalization of Proposition 8.23 in [Laf18].

(b) ([Xue18b]) for any place v of $X \smallsetminus N$, there exists μ_1 large enough, such that

$$H^{j}_{G,N,I,W} = \mathscr{H}_{G,v} \cdot H^{j,\,\leq\mu_1}_{G,N,I,W}.$$

It implies

Theorem 4. *(loc.cit. Theorem 0.0.2) For any place v of $X \smallsetminus N$, $H^{j}_{G,N,I,W}$ is a $\mathscr{H}_{G,v}$-module of finite type.*

This theorem is a generalization of Proposition 1.

As an application, we extend the action of $\mathrm{Weil}(\overline{F}/F)^I$ (where $\overline{F}$ is an algebraic closure of F) from $H^{j,\,\mathrm{Hf}}_{G,N,I,W}$ to $H^{j}_{G,N,I,W}$ (by lemma of Drinfeld), then generalize the excursion operators in [Laf18] from C_c^{cusp} to $C_c(G(F)\backslash G(\mathbb{A})/K_N\Xi, \mathbb{Q}_\ell)$.

Let $\mathscr{I}$ be an ideal of $\mathscr{H}_{G,v}$ of finite codimension (for example the kernel of a character). Similarly to [Laf18], we obtain a canonical decomposition of the quotient vector space

$$C_c(G(F)\backslash G(\mathbb{A})/K_N\Xi, \overline{\mathbb{Q}}_\ell)/\mathscr{I} \cdot C_c(G(F)\backslash G(\mathbb{A})/K_N\Xi, \overline{\mathbb{Q}}_\ell)$$

indexed by $\widehat{G}(\overline{\mathbb{Q}}_\ell)$-conjugacy classes of morphisms $\sigma : \mathrm{Weil}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$, which is compatible with the Satake isomorphism at every place of $X \smallsetminus N$ and is compatible with parabolic induction.

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Local systems over Shimura varieties: a comparison of two constructions

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(joint work with Hansheng Diao, Ruochuan Liu, and Xinwen Zhu)

Let (G, X) be any Shimura datum, where G is a reductive algebraic group over $\mathbb{Q}$ and X is a $\mathrm{G}(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{C}^\times \to \mathrm{G}(\mathbb{R})$ satisfying certain axioms. Given any neat open compact subgroup K of $\mathrm{G}(\mathbb{A}^\infty)$, by results of Baily–Borel and Borel, the double coset space $\mathrm{Sh}^{\mathrm{an}}_{K,\mathbb{C}} := \mathrm{G}(\mathbb{Q})\backslash\big(\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)/K\big)$ can be identified with the complex analytification of a canonical quasi-projective variety $\mathrm{Sh}_{K,\mathbb{C}}$ over $\mathbb{C}$. More precisely, the whole tower $\{\mathrm{Sh}_{K,\mathbb{C}}\}_K$ with its right action by $\mathrm{G}(\mathbb{A}^\infty)$ has a canonical algebraic structure. Furthermore, by results of Shimura, Deligne, Borovoi, and Milne, among others, the whole tower $\{\mathrm{Sh}_{K,\mathbb{C}}\}_K$ with its canonical right action by $\mathrm{G}(\mathbb{A}^\infty)$ has a *canonical model* $\{\mathrm{Sh}_K\}_K$ over the *reflex field* E, which is a number field E depending only on (G, X) but not on K. We shall call any of these varieties the *Shimura varieties* associated with (G, X). For simplicity of exposition, we shall assume that $E = \mathbb{Q}$ in what follows.

Let G^c denote quotient of G by the maximal $\mathbb{Q}$-anisotropic $\mathbb{R}$-split subtorus of the center of G. For any coefficient field F, we shall denote by $\mathrm{Rep}_F(\mathrm{G}^c)$ the category of algebraic representations of G^c over F.

Suppose $V \in \mathrm{Rep}_{\mathbb{Q}}(\mathrm{G}^c)$, with $V_{\mathbb{C}} := V \otimes_{\mathbb{Q}} \mathbb{C}$. Then the local sections of $\mathrm{G}(\mathbb{Q})\backslash\big((\mathsf{X} \times V_{\mathbb{C}}) \times \mathrm{G}(\mathbb{A}^\infty)/K\big) \to \mathrm{G}(\mathbb{Q})\backslash\big(\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)/K\big)$ defines a canonical Betti *local system* ${}_{\mathrm{B}}\underline{V}_{\mathbb{C}}$ over $\mathrm{Sh}^{\mathrm{an}}_{K,\mathbb{C}}$. There is also a canonical (algebraic) filtered regular connection $({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$ over $\mathrm{Sh}_{K,\mathbb{C}}$ (satisfying Griffiths transversality) such that $({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla)$ corresponds to ${}_{\mathrm{B}}\underline{V}_{\mathbb{C}}$ under Deligne's classical Riemann–Hilbert correspondence [5], and such that $\mathrm{Fil}^\bullet$ is induced by the Hodge cocharacters μ_h given by $h \in \mathsf{X}$. Such local systems and filtered connections are well-known complex analytically constructed objects over Shimura varieties.

On the other hand, for each prime number $p > 0$, consider $V_{\mathbb{Q}_p} := V \otimes_{\mathbb{Q}} \mathbb{Q}_p$, together with the canonical *p-adic étale local system* (i.e., lisse p-adic étale sheaf) ${}_{\text{ét}}\underline{V}_{\mathbb{Q}_p}$ over Sh_K defined using the tower of canonical models $\{\mathrm{Sh}_{K'}\}_{K' \subset K}$. By [11], this p-adic étale local system is *de Rham* in the sense that its geometric stalks over all classical points (defined by finite extensions of $\mathbb{Q}_p$) are de Rham as p-adic Galois representations. As in the case above over $\mathbb{C}$, but by using instead the *algebraic p-adic Riemann–Hilbert functor* (over $\mathbb{Q}_p$) we constructed (in [7, §6]), we also obtain a canonical (algebraic) filtered regular connection $({}_{p\text{-dR}}\underline{V}_{\mathbb{Q}_p}, \nabla, \mathrm{Fil}^\bullet)$ over $\mathrm{Sh}_{K,\mathbb{Q}_p}$. By base change under any field homomorphism from $\mathbb{Q}_p$ to $\mathbb{C}$, we obtain a filtered regular connection $({}_{p\text{-dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$ over $\mathrm{Sh}_{K,\mathbb{C}}$.

Note that the above base change from $\mathbb{Q}_p$ to $\mathbb{C}$ makes sense because we are working with *algebraic* filtered connections! The constructions over the analytification of $\mathrm{Sh}_{K,\mathbb{Q}_p}$ as in [16] and [11] are insufficient because canonical extensions and algebraizations generally do not exist in the rigid analytic world, unlike in the complex analytic world. Rather, we constructed (in [7, §5]) an analytic *logarithmic Riemann–Hilbert functor*, by working with pro-Kummer étale sites and log de Rham period sheaves over suitable smooth compactifications, which provides the desired canonical extensions to which GAGA applies. Crucially, we showed that all the *eigenvalues of residues* are in $\mathbb{Q} \cap [0, 1)$, and we made essential uses of the finiteness of $[k : \mathbb{Q}_p]$ and the theory of decompletions.

By the classical Riemann–Hilbert correspondence again (in the easier direction), $({}_{p\text{-dR}}\underline{V}_{\mathbb{C}}, \nabla)$ defines a Betti local system ${}_{p\text{-B}}\underline{V}_{\mathbb{C}}$ over $\mathrm{Sh}^{\mathrm{an}}_{K,\mathbb{C}}$. Such ${}_{p\text{-B}}\underline{V}_{\mathbb{C}}$ and $({}_{p\text{-dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$ are our new *p-adic analytically constructed* objects (with coefficient field $\mathbb{C}$!) over Shimura varieties. It is natural to ask how these objects compare with their complex analytically constructed counterparts.

Our main result is that ${}_{p\text{-B}}\underline{V}_{\mathbb{C}}$ and $({}_{p\text{-dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$ can be canonically identified with ${}_{\mathrm{B}}\underline{V}_{\mathbb{C}}$ and $({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$, respectively, in a way compatible with the Hecke action of $\mathrm{G}(\mathbb{A}^\infty)$, with morphisms of Shimura varieties induced by morphisms of Shimura data, and with descent to canonical models of filtered connections (as in Harris's and Milne's works; see [10] and [14]). (See [7, §7], where we treated more general $V \in \mathrm{Rep}_{\overline{\mathbb{Q}}}(\mathrm{G}^c)$.)

Our proof uses several of the most general results and techniques available for Shimura varieties and their canonical models, from the (known) abelian case of Fontaine–Mazur conjecture [9] to Deligne's and Blasius's results [1, 6] that Hodge cycles on abelian varieties over number fields are *absolute Hodge* and *de Rham*, and then from Margulis's *superrigidity theorem* [12] and Borel's *density theorem* [2, 3] to a construction credited to Piatetski-Shapiro by Borovoi [4] and Milne [13].

Consequently, by the p-adic de Rham comparison results (for general smooth varieties over $\mathbb{Q}_p$) in [7, §6], we know that $H^i_{\text{ét}}(\mathrm{Sh}_{K,\overline{\mathbb{Q}}_p}, {}_{\text{ét}}\underline{V}_{\mathbb{Q}_p})$ is *de Rham* as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, and that the Hodge–Tate weights of this representation is determined by the dimensions of certain coherent cohomology (and hence relative Lie algebra cohomology) given by Faltings's dual BGG complexes (see [8] and [10]). We also obtain a new proof of the degeneracy of the Hodge–de Rham spectral sequence for $H^i_{\mathrm{dR}}(\mathrm{Sh}_{K,\mathbb{C}}, {}_{\mathrm{dR}}\underline{V}_{\mathbb{C}})$ on the E_1 page, based on p-adic Hodge theory instead of complex Hodge theory. (In particular, we have not used Saito's theory of mixed Hodge modules [15].) We will extend these results and treat the compactly supported cohomology and interior cohomology in a forthcoming work.

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The Cohomology of EL-Type Rapoport-Zink Spaces and the Local Langlands Correspondence

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Our goal in this talk is study the relationship between the cohomology of Rapoport-Zink spaces and the local Langlands correspondence. We study Rapoport-Zink spaces of unramified EL-type which we denote $\mathbb{M}_{b,\mu}$ (see [3], [8]). These are moduli spaces of p-divisible groups coming from an unramified EL-datum consisting of

(1) a finite unramified extension F of $\mathbb{Q}_p$,
(2) a finite dimensional F vector space V which defines the group $G = \mathrm{Res}_{F/\mathbb{Q}_p}\mathrm{GL}(V)$,
(3) a conjugacy class of cocharacters $[\mu]$ with $\mu : \mathbb{G}_m \to G_{\overline{\mathbb{Q}_p}}$,

(4) an element b of a finite set $\mathbf{B}(G,\mu)$ which defines a group J_b which is an inner twist of a Levi subgroup M_b of G.

Roughly one can think of b,μ as specifying the Newton and Hodge polygons of a p-divisible group and J_b as the automorphism group of the isocrystal b.

The spaces $\mathbb{M}_{b,\mu}$ are formal schemes over $\widehat{\mathbb{Q}_p^{ur}}$. One constructs a tower of rigid spaces $\mathbb{M}^{rig}_{U,b,\mu}$ over the generic fiber $\mathbb{M}^{rig}_{b,\mu}$ of $\mathbb{M}_{b,\mu}$, where the index U runs over compact open subgroups of $G(\mathbb{Q}_p)$. Associated to such a tower we have a cohomology space $[H^\bullet(G,b,\mu)]$ which is an element of the Grothendieck group $\mathrm{Groth}(G(\mathbb{Q}_p)\times J_b(\mathbb{Q}_p)\times W_E)$ of admissible representations of $G(\mathbb{Q}_p), J_b(\mathbb{Q}_p)$ and W_E, where the latter group is the Weil group of the reflex field, E, of $[\mu]$. This construction can be thought of as an alternating sum of a direct limit over $U\subset G$ of l-adic cohomology groups with the actions of $G(\mathbb{Q}_p)$ and $J_b(\mathbb{Q}_p)$ arising from Hecke correspondences and isogenies of p-divisible groups, respectively.

The cohomology object $[H^\bullet(G,b,\mu)]$ gives rise to a map of Grothendieck groups

$$\mathrm{Mant}_{G,b,\mu}:\mathrm{Groth}(J_b(\mathbb{Q}_p))\to\mathrm{Groth}(G(\mathbb{Q}_p)\times W_E)$$

which maps a representation ρ to the alternating sum of the $J_b(\mathbb{Q}_p)$-linear Ext groups of $[H^\bullet(G,b,\mu)]$ and ρ.

In [9], Shin proved an averaging formula for $\mathrm{Mant}_{G,b,\mu}$ which is key to our work. He defined a map

$$\mathrm{Red}_b:\mathrm{Groth}(G(\mathbb{Q}_p))\to\mathrm{Groth}(J_b(\mathbb{Q}_p))$$

which up to a character twist is given by composing the un-normalized Jacquet module

$$\mathrm{Jac}^G_{P_b}:\mathrm{Groth}(G(\mathbb{Q}_p))\to\mathrm{Groth}(M_b(\mathbb{Q}_p))$$

with the Jacquet-Langlands map of Badulescu [2]

$$\mathrm{LJ}:\mathrm{Groth}(M_b(\mathbb{Q}_p))\to\mathrm{Groth}(J_b(\mathbb{Q}_p)).$$

Shin uses global methods and so necessarily works with a large but inexplicit class of representations which he denotes *accessible*. This set loosely consists of those representations appearing as the p-component of an automorphic representation of a certain global group. In particular, the essentially square integrable representations in $\mathrm{Groth}(G(\mathbb{Q}_p))$ are accessible.

In what follows $r_{-\mu}$ is a finite dimensional representation of $\widehat{G}\rtimes W_E$ which restricts to the representation of highest weight $-\mu$ on $\widehat{G}$, and LL is the semisimplifed local Langlands correspondence. Shin shows the following result.

Theorem 1 (Shin). *Assume π is an accessible representation of $G(\mathbb{Q}_p)$. Then*

$$\sum_{b\in\mathbf{B}(G,\mu)}\mathrm{Mant}_{G,b,\mu}(\mathrm{Red}_b(\pi))=[\pi][r_{-\mu}\circ\mathrm{LL}(\pi)|_{W_E}],$$

where the above formula is correct up to a Tate twist which we omit for clarity.

Additionally we have the conjecture of Harris and Viehmann which allows us to write $\mathrm{Mant}_{G,b,\mu}$ for non-basic b (b is basic when it corresponds to an isocrystal with a single slope) in terms of $\mathrm{Mant}_{G',b',\mu'}$ such that G' is a general linear group of smaller rank than G. This conjecture was formulated in [4], [7] and is expected to be proven in forthcoming work of Scholze. In what follows, Ind is the unnormalized parabolic induction functor.

Conjecture 2 (Harris-Viehmann)**.**

$$\mathrm{Mant}_{G,b,\mu} = \sum_{(M_b,\mu')\in\mathcal{I}^{G,\mu}_{M_b,b'}} \mathrm{Ind}^{G}_{P_b}(\otimes^{k}_{i=1}\mathrm{Mant}_{M_{b'_i},b'_i,\mu'_i}),$$

where we omit a Tate twist discussed in [1].

Shin's averaging formula and the Harris Viehmann conjecture allow one to compute $\mathrm{Mant}_{G,b,\mu} \circ \mathrm{Red}_b$ recursively. The latter lets us compute $\mathrm{Mant}_{G,b,\mu}$ for non-basic b given that we know $\mathrm{Mant}_{G',b',\mu'}$ for G' of lower rank and the former lets us compute $\mathrm{Mant}_{G,b,\mu}$ for the unique basic $b \in \mathbf{B}(G,\mu)$ if we know it for all non-basic $b \in \mathbf{B}(G,\mu)$. One of the main results of [1] and the focus of our talk, is to give a non-recursive description of $\mathrm{Mant}_{G,b,\mu} \circ \mathrm{Red}_b$ which we now describe.

Let $G = \mathrm{Res}_{F/\mathbb{Q}_p}\mathrm{GL}(V)$ as before and choose a rational Borel subgroup B of G, and a rational maximal torus $T \subset B \subset G$. Then we consider pairs (M_S, μ_S) where $M_S \subset T$ is a Levi subgroup of a parabolic subgroup P_S containing B, and $\mu_S \in X_*(T)$ is dominant as a cocharacter of M_S. We call a pair of the above form a *cocharacter pair* for G.

We associate to a cocharacter pair (M_S, μ_S) the map of representations $[M_S, \mu_S] : \mathrm{Groth}(G(\mathbb{Q}_p)) \to \mathrm{Groth}(G(\mathbb{Q}_p) \times W_{E_{\{\mu_S\}_{M_S}}})$, which up to a Tate twist is given by

$$\pi \mapsto [(\mathrm{Ind}^{G}_{P_S} \circ [\mu_S] \circ \mathrm{Jac}^{G}_{P_S})(\pi)]$$

and

$$[\mu_S] : \mathrm{Groth}(M_S(\mathbb{Q}_p)) \to \mathrm{Groth}(M_S(\mathbb{Q}_p) \times W_{E_{\{\mu_S\}_{M_S}}})$$

given by

$$\pi \mapsto [\pi][r_{-\mu_S} \circ LL(\pi)]$$

Then our main result is

Theorem 3. *Suppose* $\mathrm{Mant}_{\mathrm{G,b},\mu}$ *corresponds to a tower of unramified Rapoport-Zink spaces of EL-type. We assume that the Harris-Viehmann conjecture is true. Then if* $\rho \in \mathrm{Groth}(G(\mathbb{Q}_p))$ *is essentially square-integrable, we have*

$$\mathrm{Mant}_{G,b,\mu}(\mathrm{Red}_b(\rho)) = \sum_{(M_S,\mu_S)\in\mathcal{R}_{G,b,\mu}} (-1)^{L_{M_S,M_b}}[M_S,\mu_S](\rho),$$

where $\mathcal{R}_{G,b,\mu}$ *is a collection of cocharacter pairs with a combinatorial definition and* $(-1)^{L_{M_S,M_b}}$ *is an easily determined sign.*

A crucial part of the proof of the above theorem is the following unconditional result, which is perhaps interesting in its own right.

Theorem 4 (Imprecise version, see [1]). *For general quasisplit G and a cochcaracter μ (not necessarily minuscule), combinatorial analogues of Shin's formula and the Harris-Viehmann conjecture hold true.*

This result suggests that perhaps the combinatorics of cocharacter pairs is related to $\mathrm{Mant}_{G,b,\mu}$ in cases more general than Rapoport-Zink spaces of unramified EL-type.

We conclude by discussing how our combinatorial formula can be used to prove the EL-type cases of a conjecture of Harris ([4, Conj 5.4]) describing $\mathrm{Mant}_{G,b,\mu}(I^G_M(\rho))$ for ρ a supercuspidal representation of $M(\mathbb{Q}_p)$ for M a Levi subgoup of G. In this case, I^G_M denotes normalized parabolic induction. In particular, we show the following result.

Theorem 5 (Harris conjecture). *We assume that Shin's averaging formula holds for all admissible representations of $G(\mathbb{Q}_p)$ and that the Harris-Viehmann conjecture is true. Let ρ be a supercuspidal representation of $M(\mathbb{Q}_p)$. Then up to a precise Tate twist,*

$$\mathrm{Mant}_{G,b,\mu}(JL^{-1}\delta^{\frac{1}{2}}_{G,P_b} I^{M_b}_M(\rho)) = [I^G_M(\rho)] \left[\bigoplus_{(M,\mu')\in \mathrm{Rel}^{G,\mu}_{M,b}} r_{-\mu'} \circ LL(\rho) \right]$$

for an explicit set of cocharacter pairs $\mathrm{Rel}^{G,\mu}_{M,b}$.

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Automorphic weights

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(joint work with Wushi Goldring)

Let $(\mathbf{G}, \mathbf{X})$ be a Shimura datum of Hodge type, where $\mathbf{G}$ is a connected reductive $\mathbf{Q}$-group. For each compact open subgroup $K \subset \mathbf{G}(\mathbf{A}_f)$, denote by $Sh_K(\mathbf{G}, \mathbf{X})$ the attached Shimura variety. It admits a canonical model over its reflex field E. If p is a prime number of good reduction, then for each place $\mathfrak{p}|p$ in E, Kisin and Vasiu have constructed a canonical model $\mathcal{S}_K$ over the ring $\mathcal{O}_{E_\mathfrak{p}}$.

One can attach to the Shimura datum a cocharacter $\mu : \mathbf{G}_{m,\mathbf{C}} \to \mathbf{G}_\mathbf{C}$ and a parabolic subgroup $\mathbf{P} \subset \mathbf{G}_\mathbf{C}$. Fix a Borel pair $(\mathbf{B}, \mathbf{T})$ such that $\mathbf{B} \subset \mathbf{P}$. Then for each character $\lambda \in X^*(\mathbf{T})$, there is a vector bundle $\mathscr{V}(\lambda)$ on $Sh_K(G, X)$ (defined possibly after a finite extension of E). This vector bundle is modeled on the $\mathbf{P}$-representation

$$\mathbf{V}(\lambda) = H^0(\mathbf{P}/\mathbf{B}, \mathcal{L}_\lambda) \tag{1}$$

where $\mathcal{L}_\lambda$ is the line bundle naturally attached to λ on the flag variety $\mathbf{P}/\mathbf{B}$. Since all the parabolic subgroups $\mathbf{P}, \mathbf{B}$ extend (uniquely) to $\mathcal{O}_{E_\mathfrak{p}}$, the family of vector bundles $\mathscr{V}(\lambda)$ extends naturally to $\mathcal{S}_K$ as well.

Let F be an algebraically closed field endowed with a map $\mathcal{O}_{E_\mathfrak{p}} \to F$. In this talk, we study the following set:

$$C_K(F) := \{\lambda \in X^*(\mathbf{T}) |\ H^0(\mathcal{S}_K \otimes_{\mathcal{O}_{E_\mathfrak{p}}} F, \mathscr{V}(\lambda)) \neq 0\}.$$

It is easy to see that the set $C_K(F)$ is a cone inside $X^*(\mathbf{T})$. Since it depends highly on the level K, it is not reasonable to expect a simple description of this set. However, for a cone $C \subset X^*(\mathbf{T})$, denote by $\langle C \rangle$ the saturated cone of C:

$$\langle C \rangle := \{\lambda \in X^*(\mathbf{T}) |\ \exists N \geq 1,\ N\lambda \in C\}.$$

Then one can show that $\langle C_K(F) \rangle$ is independent of K. The philosophy of Shimura varieties implies that this set should admit a group-theoretic description. This is the case for $F = \mathbf{C}$ (at least when $Sh_K(\mathbf{G}, \mathbf{X})$ is compact). Denote by $\mathbf{L}$ the Levi subgroup of $\mathbf{P}$ containing $\mathbf{T}$ and write $\Phi^+, \Phi^+_\mathbf{L}, \Delta, \Delta_\mathbf{L}$ for the positive roots, the positive roots of $\mathbf{L}$, the simple roots, the simple roots of $\mathbf{L}$, respectively. One has by [2]:

$$\langle C_K(\mathbf{C}) \rangle = \left\{ \lambda \in X^*(\mathbf{T}) \,\middle|\, \begin{array}{lll} \langle \lambda, \alpha^\vee \rangle > 0 & \text{for all} & \alpha \in \Phi^+_\mathbf{L} \\ \langle \lambda, \alpha^\vee \rangle < 0 & \text{for all} & \alpha \in \Phi^+ \setminus \Phi^+_\mathbf{L} \end{array} \right\}. \tag{2}$$

The positivity conditions that appear in (2) were first observed by Griffiths and Schmid, so we write C_{GS} for the above cone and call it the Griffiths-Schmid cone.

In this talk, we try to obtain a similar, group-theoretic description of $C_K(\overline{\mathbf{F}}_p)$. One sees easily that $C_K(\mathbf{C}) \subset C_K(\overline{\mathbf{F}}_p)$, hence by the above result one has an inclusion $C_{GS} \subset \langle C_K(\overline{\mathbf{F}}_p) \rangle$, but in general this inclusion is strict. To determine $C_K(\overline{\mathbf{F}}_p)$, it is useful to consider the smooth and surjective map of stacks constructed by Zhang ([7])

$$\zeta : \mathcal{S}_K \otimes \overline{\mathbf{F}}_p \to G\text{-}\mathtt{Zip}^\mu$$

where $G\text{-}\mathtt{Zip}^{\mu}$ is the stack of G-zips of Moonen-Wedhorn and Pink-Wedhorn-Ziegler ([4], [5], [6]). Here, G is the special fiber of a reductive $\mathbf{Z}_p$-model of $\mathbf{G}_{\mathbf{Q}_p}$ and μ is a cocharacter of $G_{\overline{\mathbf{F}}_p}$ obtained by reducing the one of $\mathbf{G}_{\mathbf{C}}$. The vector bundles $\mathscr{V}(\lambda)$ exist on the stack $G\text{-}\mathtt{Zip}^{\mu}$, so we may define another cone:

$$C_{\mathrm{zip}} := \{\lambda \in X^*(\mathbf{T}) |\ H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathscr{V}(\lambda)) \neq 0\}.$$

Since ζ is surjective, we have an inclusion

$$H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathscr{V}(\lambda)) \subset H^0(S_K, \mathscr{V}(\lambda))$$

where $S_K = \mathcal{S}_K \otimes_{\mathcal{O}_{E_{\mathfrak{p}}}} \overline{\mathbf{F}}_p$. Hence we also have inclusions $C_{\mathrm{zip}} \subset C_K(\overline{\mathbf{F}}_p)$ and $\langle C_{\mathrm{zip}} \rangle \subset \langle C_K(\overline{\mathbf{F}}_p) \rangle$. We conjectured in [1] that one has an equality $\langle C_{\mathrm{zip}} \rangle = \langle C_K(\overline{\mathbf{F}}_p) \rangle$ and showed this in several cases (Hilbert modular varieties, Siegel surfaces, Picard surfaces at split primes). In any case, one can try to determine C_{zip} as an approximation of $C_K(\overline{\mathbf{F}}_p)$. Denote by G, P, B, L, T the special fibers of the unique integral models of $\mathbf{G}, \mathbf{P}, \mathbf{B}, \mathbf{L}, \mathbf{T}$. We have again a P-representation

$$V(\lambda) = H^0(P/B, \mathcal{L}_\lambda)$$

similarly to (1). As for $\mathbf{V}(\lambda)$, its highest weight is λ but it may no longer be irreducible. For each character $\chi \in X^*(T)$, denote by $V(\lambda)_\chi$ the χ-weight space of $V(\lambda)$. We define a subspace $V_{\leq 0} \subset V(\lambda)$ by

$$V_{\leq 0} := \bigoplus_{\substack{\langle \chi, \alpha^\vee \rangle \leq 0 \\ \forall \alpha \in \Delta \setminus \Delta_L}} V(\lambda)_\chi$$

From now on, we assume for simplicity that the cocharacter μ is defined over $\mathbf{F}_p$. Denote by $G\text{-}\mathtt{Zip}^{\mu}_{\mathrm{ord}} \subset G\text{-}\mathtt{Zip}^{\mu}$ the ordinary locus of $G\text{-}\mathtt{Zip}^{\mu}$ (it is an open substack whose inverse image by ζ is the usual ordinary locus of S_K).

Theorem 1 ([3]). *There is a commutative diagram where the horizontal maps are isomorphisms*

$$\begin{array}{ccc} H^0(G\text{-}\mathtt{Zip}^{\mu}_{\mathrm{ord}}, \mathscr{V}(\lambda)) & \xrightarrow{\sim} & V(\lambda)^{L(\mathbf{F}_p)} \\ \uparrow & & \uparrow \\ H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathscr{V}(\lambda)) & \xrightarrow{\sim} & V(\lambda)^{L(\mathbf{F}_p)} \cap V_{\leq 0} \end{array}$$

Hence we deduce that the cone C_{zip} consists of those $\lambda \in X^*(T)$ for which the intersection $V(\lambda)^{L(\mathbf{F}_p)} \cap V_{\leq 0}$ is nonzero. This is a representation-theoretic description, but it is still not explicit. For any $f \in V(\lambda)$, we can form the product

$$N(f) := \prod_{s \in L(\mathbf{F}_p)} s \cdot f$$

It is an element of $V(l\lambda)$, where l denotes the cardinality of $L(\mathbf{F}_p)$, and it is $L(\mathbf{F}_p)$-invariant by construction. Hence one can try to find an element $f \in V(\lambda)$ such that $N(f)$ lies in $V_{\leq 0}$. Let $f_\lambda \in V(\lambda)$ be the unique (up to scalar) B-equivariant vector. Then we have:

Theorem 2. *The element $N(f_\lambda)$ is in $V_{\leq 0}$ if and only if for all $\alpha \in \Delta \setminus \Delta_L$, one has*

$$\sum_{w\in W_L(\mathbf{F}_p)} \sum_{i=0}^{r_\alpha -1} p^{i+\ell(w)} \langle w\lambda, \sigma^i \alpha^\vee \rangle \leq 0$$

where r_α denotes the smallest integer $r \geq 1$ such that α is defined over $\mathbf{F}_{p^r}$, $\ell(w)$ is the length of the element w, and σ denotes the action of the Frobenius element.

For example, in the case when $G = GSp(2n)$, the Shimura variety parametrizes principally polarized abelian varieties of dimension n with level structure. The weight of an automorphic vector bundle is in this case an n-tuple $\lambda = (k_1, ..., k_n) \in \mathbf{Z}^n$ such that $k_1 \leq k_2 \leq ... \leq k_n$. The theorem implies that if λ satisfies the condition

$$p^{n-1}k_1 + p^{n-2}k_2 + ... + pk_{n-1} + k_0 \leq 0$$

then there exists a global section over G-$\mathtt{Zip}^\mu$ (and hence also S_K) of weight $l\lambda$ where $l = |GL_n(\mathbf{F}_p)|$. This condition is much weaker than in the characteristic zero situation, where $\langle C_K(\mathbf{C})\rangle$ consists of those $(k_1, ..., k_n)$ such that $k_1 \leq k_2 \leq ... \leq k_n \leq 0$.

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Level one algebraic cusp forms: non-existence and counting

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(joint work with Gaëtan Chenevier)

We report on progress on two related problems: the classification of level one algebraic cusp forms in small weight, and giving explicit formulas for discrete multiplicities for classical groups over $\mathbb{Z}$, also in level one. We first formulate the most classical application of our results, to Siegel modular forms.

We denote by $\mathrm{S}_k(\Gamma_g)$ and $\mathrm{S}_{\underline{k}}(\Gamma_g)$ respectively the space of cuspidal Siegel modular forms for the full Siegel modular group $\Gamma_g = \mathrm{S}_{2g}(\mathbb{Z})$, which are either scalar-valued of weight $k \in \mathbb{Z}$, or more generally vector-valued of weight $\underline{k} = (k_1, k_2, \ldots, k_g)$ in $\mathbb{Z}^g$ with $k_1 \geq k_2 \geq \cdots \geq k_g$. The question of determining the dimension of $\mathrm{S}_{\underline{k}}(\Gamma_g)$, very classical for $g = 1$, has a long and rich history for $g > 1$. It has first been attacked for $g = 2$ using geometric methods, in which case concrete formulas were obtained by Igusa (1962) in the scalar-valued case, and by Tsushima (1983) for the weights $k_1 \geq k_2 \geq 3$. An analogue of Igusa's result for $g = 3$ was proved by Tsuyumine in 1986, but only quite recently (2011) a conjectural formula was proposed by Bergström, Faber and van der Geer, in the vector valued case $k_1 \geq k_2 \geq k_3 \geq 4$, based on counting genus three curves over finite fields. Their formula, and more generally a formula for $\dim \mathrm{S}_{\underline{k}}(\Gamma_g)$ for arbitrary $g \leq 7$ and $k_g > g$ was proved by the second author in [9]. Actually, the general formulas given loc. cit. apply to any genus g and any weights with $k_g > g$. However, they involve certain rational numbers, that we shall refer to later as *masses*, that are rather difficult to compute; a number of algorithms were provided loc. cit. to determine them, allowing one to numerically compute those masses for $g \leq 7$. Our first main result is a completely different and comparatively much easier method to compute the aforementioned masses. This method allows us to recover, in a uniform and rather "effortless" way, all the computations of masses done in [9] for $g \leq 7$, and even to go further:

Theorem 1. *There is an explicit and implemented formula computing* $\dim \mathrm{S}_{\underline{k}}(\Gamma_g)$ *for any $g \leq 8$, and any $\underline{k}$ with $k_g > g$.*

We also computed $\dim \mathrm{S}_{\underline{k}}(\Gamma_g)$ for $13 \geq k_1 \geq \cdots \geq k_g > g$. The condition $k_g > g$ corresponds to holomorphic discrete series representations of $\mathrm{Sp}_{2g}(\mathbb{R})$. Using recent work of Moeglin-Renard [8], we also obtain the following theorem for Siegel modular forms of low scalar weight.

Theorem 2. *Up to multiplication by a scalar, there are exactly three Siegel cusp forms in level one, scalar weight $k \leq 13$ in genus $g \geq k$, occurring for $(k, g) \in \{(12, 12), (13, 16), (13, 24)\}$.*

Our proofs of these theorems rely on automorphic methods, building on a strategy developed in the recent works [5], [4], [9]. Important ingredients are Arthur's L^2-Lefschetz trace formula [2] and Arthur's endoscopic classification of the discrete automorphic spectrum of classical groups in terms of general linear groups [3], including the so-called multiplicity formula. A special feature of this approach is that even if we were only interested in $\dim \mathrm{S}_{\underline{k}}(\Gamma_g)$, we would be forced to compute first the dimension of various spaces of automorphic forms for several split classical groups over $\mathbb{Z}$ of smaller dimension, not limited to symplectic groups but also special orthogonal groups. An important gain, however, is that in the end we do not only compute $\dim \mathrm{S}_{\underline{k}}(\Gamma_g)$, but also the dimension of its subspace of cuspforms of any possible endoscopic type, a quantity which is arguably more interesting than the whole dimension itself.

Let us briefly recall how Arthur's classification links spaces of Siegel cusp forms to self-dual algebraic regular cuspidal automorphic representations for general linear groups. Let $m \geq 1$ be an integer and π a cuspidal automorphic representation of PGL_m over $\mathbb{Q}$. We say that π has level one if π_p is unramified for each prime p. We say that π is algebraic if the infinitesimal character of π_∞, seen as a semisimple conjugacy class in $\mathrm{M}_m(\mathbb{C})$, has its eigenvalues in $\frac{1}{2}\mathbb{Z}$, say $w_1 \geq w_2 \geq \cdots \geq w_m$, and with $w_i - w_j \in \mathbb{Z}$. Those eigenvalues w_i are called the weights of π, and the important integer $w(\pi) := 2w_1$ is called the motivic weight of π. We have $w_{m+1-i} = -w_i$ for all i, and in particular, $w(\pi) \geq 0$. For an integer $m \geq 1$, we denote by W_m the set of $\underline{w} = (w_i)_i$ in $\frac{1}{2}\mathbb{Z}^m$ with $w_1 \geq w_2 \geq \cdots \geq w_m$, $w_i - w_j \in \mathbb{Z}$ and $w_i = -w_{m+1-i}$ for all $1 \leq i, j \leq m$. We denote by $\mathrm{N}(\underline{w})$ the number of level 1 cuspidal algebraic automorphic representations of PGL_m whose weights are the w_i, and by $\mathrm{N}^{\perp}(\underline{w})$ the number of those π satisfying furthermore $\pi^\vee \simeq \pi$. Let us say that an element $\underline{w} \in \mathrm{W}_m$ is regular if we have $w_i \neq w_j$ for all $i \neq j$, unless perhaps we have $m \equiv 0 \bmod 4$ is even, $w_1 \in \mathbb{Z}$ and $i = j - 1 = m/2$ (hence $w_i = w_j = 0$). As was observed in [5], [4], [9], the level 1 self-dual π of regular weights are the exact building blocks for Arthur's endoscopic classification of the discrete automorphic representations of split classical groups over $\mathbb{Z}$ which are unramified at all finite places, and discrete series at the Archimedean place, with a very concrete form of Arthur's multiplicity formula (relying on [1]).

Key fact. *Fix $g \geq 1$ and $\underline{k} = (k_1, \ldots, k_g) \in \mathbb{Z}^g$ with $k_1 \geq k_2 \geq \cdots \geq k_g > g$. Then the dimension of $\mathrm{S}_{\underline{k}}(\Gamma_g)$ is an explicit function of the (finitely many) quantities $\mathrm{N}^{\perp}(\underline{w})$ with $\underline{w} = (w_i) \in \mathrm{W}_m$ regular, $m \leq 2g + 1$ and $w_1 \leq k_1 - 1$.*

Before we explain our strategy for proving Theorems 1 and 2, let us state the second main result of our work, which is a partial classification of the level 1 cuspidal algebraic π of PGL_m over $\mathbb{Q}$ having motivic weight ≤ 24, without any assumption on m. The first statement of this type, proved in [4, Thm. F], asserts that there are exactly 11 such π of motivic weight ≤ 22. In our new work we significantly simplify the proof of this result, and also prove the following theorem for motivic weights 23 (we also prove a classification theorem for motivic weight 24, but only for regular weights).

Theorem 3. *There are exactly 13 level 1 cuspidal algebraic automorphic representations of GL_m over $\mathbb{Q}$, with m varying, with motivic weight 23, and having the weight 23 with multiplicity 1:*

(i) 2 *representations of* PGL_2 *generated by the eigenforms in* $\mathrm{S}_{24}(\mathrm{SL}_2(\mathbb{Z}))$*,*
(ii) 3 *representations of* PGL_4 *of weights* $\pm 23/2, \pm v/2$ *with* $v = 7, 9$ *or* 13*,*
(iii) 7 *representations of* PGL_6 *of weights* $\pm 23/2, \pm v/2, \pm u/2$ *with*

$$(v, u) = (13, 5), (15, 3), (15, 7), (17, 5), (17, 9), (19, 3) \text{ and } (19, 11),$$

(iv) 1 *representation of* PGL_{10} *of weights* $\pm 23/2$*,* $\pm 21/2$*,* $\pm 17/2$*,* $\pm 11/2$*,* $\pm 3/2$*.*

There are all self-dual (of symplectic type) and uniquely determined by their weights.

Despite our efforts, we have not been able to classify the π of motivic weight 23 such that multiplicity of the weight 23 is > 1. We could only prove that there is an explicit list $\mathcal{L}$ of 192 weights $\underline{w} = (w_i)$ with $w_1 = w_2 = 23$ such that (a) the weight of any such π belongs to $\mathcal{L}$, (b) for any $\underline{w}$ in $\mathcal{L}$ there is at most one π with this weight (necessarily self-dual symplectic), except for the single weight $\underline{w} = (v_i/2)$

in $W_{14} \cap \mathcal{L}$ with $(v_1, v_2, \ldots, v_7) = (23, 23, 21, 17, 13, 7, 1)$, for which there might also be two such π which are the dual of each other.

Our proof of Theorem 3 is in the same spirit of the one of [4, Thm. F]. All the representations appearing in the theorem were already known to exist by the works [5], [9], so the main problem is to show that there are no others. The basic idea that we use for doing so is to consider an hypothetical π, consider an associate L-function of π, and show that this function cannot exist by applying to it the so-called explicit formula for suitable test functions. This is a classical method by now, that was developed by Mestre in [6] and applied to the standard L-function of π, and later by Miller [7] to the Rankin-Selberg L-function of π.

Two importants novelties were discovered in [4] in order to obtain the classification result in motivic weight ≤ 22. The first one is a finiteness result which implies that there are only finitely many level 1 cuspidal algebraic automorphic representations π of PGL_m, with m varying, of motivic weight ≤ 23. This result is effective and produces a finite but large list of possible weights for those π. The hardest part is then to eliminate most of those remaining weights. Improving on [4], we discovered a criterion that is (in practice!) much more efficient than the aforementioned ones. The basic idea is to apply the explicit formula to the Rankin-Selberg L-function of to all linear combinations $\lambda\pi \oplus \lambda_1\pi_1 \oplus \cdots \lambda_s\pi_s$ where π is unknown of given weight, the π_i are known (in the sense that they exist and we know their weight), and λ and the λ_i are arbitrary nonnegative *real* numbers.

For the strategy above to be successful in disproving the existence of π, it is important to know the existence of π_i's having small motivic weight. These are obtained using dimension formulas such as Theorem 1. Our proof of Theorem 1 goes in the other direction. That is, it uses the knowledge that for some regular $\underline{w} \in W_m$ we have $N^{\perp}(\underline{w}) = 0$ to indirectly compute masses, simply by solving linear equations. More precisely, Arthur's L^2-Lefschetz trace formula [2] is applied to a symplectic or special orthogonal group G over $\mathbb{Z}$ such that $G(\mathbb{R})$ has discrete series. For a dominant weight λ for $G(\mathbb{C})$ and for the unit of the unramified Hecke algebra, the geometric side of this trace formula is

$$\mathrm{EP}(G; \lambda) = \sum_{i \geq 0} (-1)^i \dim \mathrm{H}^i(\mathfrak{g}, K; \mathcal{A}^2(G) \otimes V_\lambda^\vee) \in \mathbb{Z}$$

where $\mathfrak{g}$ is the complexified Lie algebra of $G(\mathbb{R})$, K is maximal compact subgroup of $G(\mathbb{R})$, V_λ the irreducible algebraic representation of $G(\mathbb{C})$ with highest weight λ, and $\mathcal{A}^2(G)$ the space of square-integrable automorphic forms for G. Similarly to the Key fact above, Arthur's classification allows us to express this integer in terms of $N^{\perp}(w(\lambda))$, for a certain regular $w(\lambda) \in W_m$ associated to λ and m the dimension of the standard representation of the dual group of G, and similar integers in smaller dimensions. The trace formula reads

$$\mathrm{EP}(G; \lambda) \;=\; \mathrm{T}_{\mathrm{geom}}(G; \lambda)$$

where the geometric side $\mathrm{T}_{\mathrm{geom}}(G; \lambda)$ is a finite sum of terms indexed by conjugacy classes of Levi subgroups of G. The main term, corresponding to G itself, can be expressed as

$$\mathrm{T}_{\mathrm{ell}}(G;\lambda) = \sum_{c\in\mathcal{C}(G)} \mathrm{m}_c\,\mathrm{trace}(c\,|\,V_\lambda)$$

where $\mathcal{C}(G)$ denotes the set of $G(\overline{\mathbb{Q}})$-conjugacy classes of finite order elements in $G(\mathbb{Q})$ and the rational number m_c is called the mass of c. The other terms can be expressed in terms of elliptic terms for smaller groups. Thus our inductive strategy to compute all masses m_c is to exhibit a set Λ of dominant weights for $G(\mathbb{C})$ satisfying the two following properties.

- For all $\lambda \in \Lambda$ we have $\mathrm{N}^\perp(w(\lambda)) = 0$.
- The $\Lambda \times \mathcal{C}(G)$ matrix $(\mathrm{trace}(c\,|\,V_\lambda))_{(\lambda,c)}$ has rank $\mathcal{C}(G)$.

To prove the first property, we use the aforementioned method using the explicit formula applied to Rankin-Selberg L-functions. Once masses are computed by solving this linear system, one can evaluate the formula for arbitrary λ, and inductively compute all $\mathrm{N}^\perp(\underline{w})$ for all regular $\underline{w}$.

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Fargues-Rapoport conjecture in the non-basic case

Miaofen Chen

Let F be a finite extension of $\mathbb{Q}_p$, and let $\breve{F}$ be the p-adic completion of the maximal unramified extension of F, with the Frobenius action σ. Let $(G, b, \{\mu\})$ be a local Shimura datum over F, where G is a reductive group over F, $b \in G(\breve{F})$ and $\{\mu\}$ is a geometric conjugacy class of a minuscule cocharacter μ of G with the relation $[b] \in B(G, \mu)$, where $B(G, \mu)$ is the Kottwitz subset [Ko] inside the set $B(G)$ of σ-conjugacy classes in $G(\breve{F})$. Let E be the field of definition of $\{\mu\}$, and let $\breve{E}$ be the p-adic completion of the maximal unramified extension of E. Associated to this local Shimura datum, Scholze defined a tower of rigid analytic spaces $\mathcal{M}(G,b,\mu)_{K\subset G(F)}$ over $\breve{E}$ which are called local Shimura varieties [SW], where $K \subset G(F)$ runs through the open compact subgroups of $G(F)$. They are generalization of Rapoport-Zink spaces defined in [RZ]. On this tower of local Shimura varieties, there is a p-adic period mapping

$$\pi : \mathcal{M}(G,b,\mu)_K \to \mathcal{F} := \mathcal{F}(G,\mu)$$

which is an étale morphism between rigid analytic spaces over $\breve{E}$ with $\mathcal{F}$ the flag variety associated to (G,μ). Its image

$$\mathcal{F}^a := \mathcal{F}(G,b,\mu)^a := \mathrm{Im}(\pi)$$

is open in the flag variety $\mathcal{F}$ which is called the p-adic period domain or admissible locus in $\mathcal{F}$.

Associated to the local Shimura datum $(G,b,\{\mu\})$, Rapoport and Zink also constructed in [RZ] an open subspace

$$\mathcal{F}^{wa} := \mathcal{F}(G,b,\mu)^{wa}$$

containing the admissible locus $\mathcal{F}^a$ inside $\mathcal{F}(G,\mu)$, which is called the weakly admissible locus. For any finite extension $K_1|\breve{E}$, the points in $\mathcal{F}(G,b,\mu)^{wa}(K_1)$ correspond to weakly admissible filtered isocrystals with G-structure. Colmez and Fontaine showed in [CF] that the weakly admissible locus is an approximation of admissible locus in the sense that these two rigid analytic spaces have the same classical points. On the other hand, Hartl showed in [Ha] that these two spaces in general are NOT the same. Therefore it's natural to ask in which cases these two spaces coincide. The Fargues-Rapoport conjecture answers this question when b is basic.

Theorem 1 (Fargues-Rapoport conjecture, [CFS] Theorem 6.1)**.** *Suppose b is basic. Then $\mathcal{F}(G,b,\mu)^a = \mathcal{F}(G,b,\mu)^{wa}$ if and only if $(G,\{\mu\})$ is fully Hodge-Newton decomposable.*

Recall that the pair $(G,\{\mu\})$ is called fully Hodge-Newton decomposable, if for any non basic $[b'] \in B(G,\mu)$, the triple $(G,b',\{\mu\})$ is Hodge-Newton decomposable, roughly speaking, which means the Newton polygon of b' has a break point on the Hodge polygon of μ. In [GHN] there is a complete classification of all the fully Hodge-Newton decomposable pairs $(G,\{\mu\})$, and also equivalent conditions for $(G,\{\mu\})$ being fully Hodge-Newton decomposable.

Without the b basic assumption, Hartl gives in [Ha] Theorem 8.3 a criterion for $\mathcal{F}^a = \mathcal{F}^{wa}$ when $G = \mathrm{GL}_h$. More precisely, he shows that for $G = \mathrm{GL}_h$, we have $\mathcal{F}(G,b,\mu)^a = \mathcal{F}(G,b,\mu)^{wa}$ if and only if the Newton slopes of b are one of the following forms

- $(1^{(n_1)}, \frac{1}{n}^{(n)}, 0^{(n_0)})$ for $n_0, n, n_1 \in \mathbb{N}$ and $n_0 + n + n_1 = h$;
- $(1^{(n_1)}, \frac{n-1}{n}^{(n)}, 0^{(n_0)})$ for $n_0, n, n_1 \in \mathbb{N}$ and $n_0 + n + n_1 = h$;
- $(1^{(n_1)}, \frac{1}{2}^{(4)}, 0^{(n_0)})$ for $n_0, n_1 \in \mathbb{N}$ and $n_0 + 4 + n_1 = h$;

Inspired of Hartl's result, we can also give a similar group theoretic criterion for $\mathcal{F}^a = \mathcal{F}^{wa}$ for general reductive group G. For the simplicity of the statement, in the following theorem, we will assume that G is quasi-split. For general reductive group G, by passing to the quasi-split inner form of the adjoint group G^{ad}, we also have a similar description.

Theorem 2. *Suppose G is quasi-split. Let M be the standard Levi subgroup of G, such that $[b] \in B(M, \mu)$, Newton vector ν_b of b is G-dominant and (M, b, μ) is Hodge-Newton indecomposable. Then the equality $\mathcal{F}(G, b, \mu)^a = \mathcal{F}(G, b, \mu)^{wa}$ holds if and only if $(M, \{\mu\})$ is fully Hodge-Newton indecomposable and $[b]$ is basic in $B(M)$.*

The strategy of proof consists of two steps. In the first step, we relate the admissible locus and weakly admissible locus for different groups G and M in order to reduce to the Hodge-Newton indecomposable case. In the second step, we show that in the Hodge-Newton indecomposable case with b non-basic, the admissible locus is stable under a group action while the weakly admissible locus is not, which implies that these two subspaces don't coincide. This allows us to reduce to b basic case, and can therefore apply the Fargues-Rapoport conjecture to conclude.

In the cases $\mathcal{F}^a \neq \mathcal{F}^{wa}$, we can ask further questions where live the weakly admissible points outside the admissible locus. To formulate a more precise question, we need to introduce a different point of view of the admissible locus via the Fargues-Fontaine curve. Let C be a complete algebraically closed extension of $\breve{E}$. Associated to the tilt $C^\flat$ of C, we can define the Fargues-Fontaine curve X over F equipped with a closed point ∞ with residue field $k(\infty) = C$ and $\widehat{\mathcal{O}}_{X,\infty} = B_{dR}^+(C)$ ([FF]). We have a bijection between $B(G)$ and isomorphism classes of G-bundles over X ([Fa1]):

$$B(G) \xrightarrow{\sim} H^1_{\text{ét}}(X, G)$$
$$[b] \mapsto [\mathcal{E}_b].$$

For each $x \in \mathcal{F}(C)$ one can construct a modification $\mathcal{E}_{b,x}$ of $\mathcal{E}_b$ à la Beauville-Laszlo [CS, Fa3, Fa2]. This is given by gluing $\mathcal{E}_{b|X\setminus\{\infty\}}$ and the trivial G-bundle on $\mathrm{Spec}(\mathrm{B}^+_{\mathrm{dR}}(C))$ via the gluing datum given by x.

The isomorphism class of $\mathcal{E}_{b,x}$ defines a stratification of $\mathcal{F}(G, \mu)$ [CS, CFS]

$$\mathcal{F}(G, \mu) = \coprod_{[b'] \in B(G)} \mathcal{F}(G, b, \mu)^{[b']},$$

with open stratum $\mathcal{F}^{[1]} = \mathcal{F}^a$ the admissible locus. Rapoport showed in [Ra] that when b is basic, $\mathcal{F}(G, b, \mu)^{[b']}$ is non empty if and only if $[b']$ lies in the generalized Kottwitz subset $B(G, 0, \nu_b \mu^{-1})$ (cf. [CFS]). We can ask the following natural question:

Question 3. *For which $[b'] \in B(G)$, $\mathcal{F}(G, b, \mu)^{[b']} \cap \mathcal{F}(G, b, \mu)^{wa} \neq \emptyset$?*

When b is basic, we showed in [CFS] that for $[b'] \in B(G, 0, \nu_b \mu^{-1})$, if $(G, b', \nu_b \mu^{-1})$ is Hodge-Newton decomposable, then $\mathcal{F}(G, b, \mu)^{[b']}$ does not contain any weakly admissible point; if $(G, b', \nu_b\mu^{-1})$ is Hodge-Newton indecomposable, and $[b']$ is very close to $[1]$, then $\mathcal{F}(G, b, \mu)^{[b']}$ contains weakly admissible points. But we don't know for other strata. In fact, we guess that all the other strata contain weakly admissible points.

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Persistence of non-degeneracy: a local analog of Ihara's lemma

Pascal Boyer

Fix prime numbers $l \neq p$ and a finite extension K of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$. For $d \geq 1$, denote $\widehat{\mathcal{M}}_{LT,d,n}$ the pro-formal scheme representing the functor of isomorphism classes of deformations by quasi-isogenies of the formal $\mathcal{O}_K$-module over $\overline{\mathbb{F}}_p$ with height d and with level structure n. We denote $\mathcal{M}_{LT,d,n}$ its generic fiber over $\widehat{K}^{un}$. For $\Lambda = \overline{\mathbb{Q}}_l, \overline{\mathbb{Z}}_l$ or $\overline{\mathbb{F}}_l$, consider both

$$\mathcal{U}^{d-1}_{LT,d,\Lambda} := \varinjlim_{n} H^{d-1}(\mathcal{M}_{LT,d,n} \widehat{\otimes}_{\widehat{K}^{un}} \widehat{\overline{K}}, \Lambda)$$

and

$$\mathcal{V}^{d-1}_{LT,d,\Lambda} := \varinjlim_{n} H_c^{d-1}(\mathcal{M}_{LT,d,n} \widehat{\otimes}_{\widehat{K}^{un}} \widehat{\overline{K}}, \Lambda).$$

There is a natural action of $GL_d(K) \times D^\times_{K,d} \times W_k$ on $\mathcal{U}^{d-1}_{LT,d,\Lambda}$ and $\mathcal{V}^{d-1}_{LT,d,\Lambda}$, where $D_{K,d}$ (resp. W_k) is the central division algebra over K with invariant $1/d$ (resp. the Weil group of K). For $\Lambda = \overline{\mathbb{Q}}_l$, both $\mathcal{U}^{d-1}_{LT,d,\overline{\mathbb{Q}}_l}$ and $\mathcal{V}^{d-1}_{LT,d,\overline{\mathbb{Q}}_l}$ are described in [2], in terms of the local Langlands and Jacquet-Langlands correspondences. For $\Lambda = \overline{\mathbb{Z}}_l$, we moreover proved, in [3], that these $\overline{\mathbb{Z}}_l$-modules are free so that we know the semi-simplification of $\mathcal{U}^{d-1}_{LT,d,\overline{\mathbb{F}}_l}$ and $\mathcal{V}^{d-1}_{LT,d,\overline{\mathbb{F}}_l}$. In this talk, we propose to construct a particular filtration of $\mathcal{U}^{d-1}_{LT,d,\overline{\mathbb{Q}}_l}$ and $\mathcal{V}^{d-1}_{LT,d,\overline{\mathbb{Q}}_l}$ and to describe the lattices of the graduate.

To do so, we classically propose to deduce this construction from a global situation through the Berkovich's comparison theorem, by studying the perverse sheaf of nearby cycles Ψ_v over some tower of Kottwitz-Harris-Taylor Shimura varieties $(X_I)_I$, cf. [6], at a fixed place v of its reflex field F such that $F_v \simeq K$, and where I described small enough open compact subgroups: recall that the corresponding reductive $G/\mathbb{Q}$ group is some similitude group with signature $(1, d-1), (0, d) \cdots (0, d)$ at infinity. With the help of the Newton stratification and the usual adjunction properties $j_! j^* \longrightarrow \mathrm{id}$ and $\mathrm{id} \longrightarrow j_* j^*$, we then construct a filtration of Ψ_v given rise, by considering its fiber at some geometric supersingular point, to the desired filtration of the local objects. We then obtain the following interesting property, very similar to the global Ihara's lemma as stated by Clozel-Harris-Taylor in [5].

Theorem. *The persistence of non-degeneracy property holds for* $\mathcal{V}^{d-1}_{LT,d,\overline{\mathbb{Z}}_l} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ *and* $\mathcal{U}^{d-1}_{LT,d,\overline{\mathbb{Z}}_l,free} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$*, i.e. any irreducible* $GL_d(K)$*-equivariant subspace is non-degenerate.*

To obtain such a property, we introduce the mirabolic subgroup $M_d(K)$ of matrices such that the last row is $(0, \cdots, 0, 1)$ which is classically used in the theory of admissible representations of $GL_d(K)$ through the notion of level of degeneracy, both for $\overline{\mathbb{Q}}_l$ and $\overline{\mathbb{F}}_l$-representations. In particular, up to isomorphism, there is only one irreducible non-degenerate representation of $M_d(K)$ denoted τ_{nd} (resp. $r_l(\tau_{nd})$) over $\overline{\mathbb{Q}}_l$ (resp. $\overline{\mathbb{F}}_l$): an irreducible representation of $GL_d(K)$, over $\overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$, is then called non-degenerate if its restriction to $M_d(K)$ contains the non-degenerate representation. In particular if $\bar{\pi}$ is an irreducible degenerate $\overline{\mathbb{F}}_l$-representation of $GL_d(K)$, then $r_l(\tau_{nd})$ is not a subquotient of $\bar{\pi}_{|M_d(K)}$.

The idea is then to prove a $M_d(K)$-version of the previous theorem through a $M_d(F_v)$-equivariant filtration of Ψ_v. For this we use the Newton stratification

$$X^{\geq d}_{I,\bar{s}_v} \subset X^{\geq d-1}_{I,\bar{s}_v} \subset \cdots \subset X^{\geq 1}_{I,\bar{s}_v} = X_{I,\bar{s}_v}$$

of the geometric special fiber of X_I at v, where we recall that each $X^{=h}_{I,\bar{s}_v} := X^{\geq h}_{I,\bar{s}_v} \setminus X^{\geq h+1}_{I,\bar{s}_v}$ is of pure dimension $d-h$ and parabolically induced in the sense that there exists a closed sub-scheme $X^{=h}_{I,\bar{s}_v,1}$ stabilized by the standard parabolic subgroup $P_{h,d-h}(F_v)$ with Levi $GL_h(F_v) \times GL_{d-h}(F_v)$, such that

$$X^{=h}_{I,\bar{s}_v} = X^{=h}_{I,\bar{s}_v,1} \times_{P_{h,d-h}(F_v)} GL_d(F_v).$$

Consider now $j_{\neq 1} : X^{\geq 1}_{I,\bar{s}_v} \setminus X^{\geq 1}_{I,\bar{s}_v,1} \hookrightarrow X^{\geq 1}_{I,\bar{s}_v}$ which is an affine morphism, and denote $i^1_1 : X^{\geq 1}_{I,\bar{s}_v,1} \hookrightarrow X^{\geq 1}_{I,\bar{s}_v}$.

Lemma. *The perverse sheaf* $\Psi_{v,1} := i^1_{1,*}{}^p\mathcal{H}^0 i_1^{1,*}\big(\Psi_v\big)$ *is free, and in the following short exact sequence which is* $M_d(F_v)$*-equivariant*

$$0 \to j_{\neq 1,!}j^*_{\neq 1}\Psi_v \longrightarrow \Psi_v \longrightarrow \Psi_{v,1} \to 0,$$

the three terms are free perverse sheaves in the sense of [4].

The strategy is then to use the other Newton strata, to construct a filtration of $\Psi_{v,1}$ and take its fiber at some geometric point of $X^{=d}_{I,\bar{s}_v}$ so that to obtain a filtration of $\mathcal{U}^{d-1}_{LT,d,\overline{\mathbb{Z}}_l}$ whose every graduates verify the persistence of non-degeneracy property for $M_d(K)$. During the process we have to deal with the following issue relatively to free perverse $\overline{\mathbb{Z}}_l$-sheaves A, B simple over $\overline{\mathbb{Q}}_l$ such that the following short exact sequence splits over $\mathbb{Q}_l$

$$0 \to A \longrightarrow P \longrightarrow B \to 0.$$

We can then write $0 \to \tilde{B} \longrightarrow P \longrightarrow \tilde{A} \to 0$ with $A \hookrightarrow \tilde{A} \twoheadrightarrow T$ and $\tilde{B} \hookrightarrow B \twoheadrightarrow T$ where T is torsion. The question is then to give criterion for $T = 0$ so that P is split. For this we recall the following tools of the theory of representations of $M_d(K)$, cf [1] §4. Denote $V_d(K)$ the unipotent radical of M_d and for a fixed non-trivial character ψ of K, let θ be the character of $V_d(K)$ defined by $\theta((m_{i,j})) = \psi(m_{d-1,d})$. For $H = GL_r(K)$ or $M_r(K)$, we denote $\mathrm{Alg}(H)$ the abelian category of algebraic representations of H and consider

$$\Psi^- : \mathrm{Alg}(M_d(K)) \longrightarrow \mathrm{Alg}(GL_{d-1}(K)), \qquad \Phi^- : \mathrm{Alg}(M_d) \longrightarrow \mathrm{Alg}(M_{d-1}(K))$$

defined by $\Psi^- = r_{V_d,1}$ (resp. $\Phi^- = r_{V_d,\theta}$) the functor of V_{d-1} coinvariants (resp. (V_{d-1},θ)-coinvariants), cf. [1] 1.8 or [7] over $\overline{\mathbb{F}}_l$.

Consider first the following embedding $GL_r(K) \times M_s(K) \hookrightarrow M_{r+s}(K)$ sending

$$A \times M \mapsto \begin{pmatrix} A & U \\ 0 & M \end{pmatrix}.$$

By inducing we then define a functor

$$\rho \times \tau \in \mathrm{Alg}(GL_r(K)) \times \mathrm{Alg}(M_s(K)) \mapsto \rho \times \tau \in \mathrm{Alg}(M_{r+s}(K).$$

Secondly we consider $M_r(K) \times GL_s(K) \hookrightarrow M_{r+s}(K)$ sending

$$\begin{pmatrix} A & V \\ 0 & 1 \end{pmatrix} \times B \mapsto \begin{pmatrix} A & U & V \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and we define

$$\tau \times \rho \in \mathrm{Alg}(M_r(K)) \times \mathrm{Alg}(GL_s(K)) \mapsto \tau \times \rho\nu^{-1/2} \in \mathrm{Alg}(M_{r+s}(K)),$$

to be the compact induced representation.

Proposition. *(cf. [1] 4.13) Let* $\rho \in \mathrm{Alg}(GL_r(K))$*,* $\sigma \in \mathrm{Alg}(GL_t(K))$ *and* $\tau \in \mathrm{Alg}(M_s(K))$*.*

(a) *In* $\mathrm{Alg}(M_{r+t}(K)$*, we have*

$$0 \to (\rho_{|M_r(K)}) \times \sigma \longrightarrow (\rho \times \sigma)_{|M_{r+t}(K)} \longrightarrow \rho \times (\sigma_{|M_t(K)}) \to 0.$$

(b) *If* Ω *is one of the functors* Ψ^-, Φ^-*, then* $\rho \times \Omega(\tau) \simeq \Omega(\rho \times \tau)$*.*

(c) $\Psi^-(\tau \times \rho) \simeq \Psi^-(\tau) \times \rho$ *and*

$$0 \to \Phi^-(\tau) \times \rho \longrightarrow \Phi^-(\tau \times \rho) \longrightarrow \Psi^-(\tau) \times (\rho_{|M_r(K)}) \to 0.$$

(d) *Suppose* $r > 0$. *Then for any non-zero* $M_{s+t}(K)$*-submodule* $\omega \subset \tau \times \rho$, *we have* $\Phi^-(\omega) \neq (0)$.

We can then remark that $(\rho_{|M_r(K)}) \times \sigma$ tends to have high derivative subspaces while $\rho \times (\sigma_{|M_t(K)})$ has small derivative. Using this informal observation, we are able to move all the zero dimensional perverse sheaves of $\Psi_{v,1}$ so that they, all together, form a quotient of it. The non-degeneracy property is then proved inductively using point (d) of the last proposition.

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Geometry of non-quasi-split unitary Shimura varieties and arithmetic applications

Yichao Tian

(joint work with Yifeng Liu, Liang Xiao, Wei Zhang and Xinwen Zhu)

In this report, I will explain some intermediate results in my on-going project joint with Yifeng Liu, Liang Xiao, Wei Zhang and Xinwen Zhu [LTXZZ]. These geometric results will be essential for later applications to the Beilinson-Bloch-Kato conjecture for Rankin–Selberg motives.

Let F be an imaginary quadratic extension of $\mathbf{Q}$. [1] We fix an embedding $\iota : F \hookrightarrow \mathbb{C}$. Let $(\mathrm{V}, (\_,\_))$ be an n-dimensional hermitian space over F of signature $(n-1, 1)$ with $n \geq 2$, and $\mathrm{G} = \mathrm{U}(\mathrm{V})$. Let h be the Deligne homomorphism

$$h : \mathrm{Res}_{\mathbf{C}/\mathbf{R}}(\mathbb{G}_m) \to \mathrm{G}(\mathbf{R}) \cong \mathrm{U}(n-1, 1)$$

[1] Actually, our main results in [LTXZZ] are also valid for general CM extensions. To simply the presentation, we will focus on imaginary quadratic number field.

sending $z \in \mathbf{C}^\times$ to the diagonal matrix $\mathrm{diag}(1, \cdots, 1, \bar{z}/z)$. For a open compact neat subgroup $K \subset \mathrm{G}(\mathbb{A}_f)$, we have the Shimura variety $\mathrm{Sh}(G, K)$ defined over F attached to the Shimura datum (G, h). Consider now a prime p inert in F such that the hermitian space V is non-split at p, i.e. $V_p := V \otimes \mathbf{Q}_p$ admits no self-dual lattice. This implies that V_p admits a lattice Λ_p such that $\Lambda_p \subseteq \Lambda_p^\vee$ with $\dim_{\mathbf{F}_{p^2}}(\Lambda_p^\vee/\Lambda_p) = 1$, where $\Lambda_p^\vee$ denotes the dual lattice of Λ_p with respect to the hermitian pairing on V_p. Note that when n is even, the unitary group G is non-quasi-split at p. Let $K_p \subset \mathrm{G}(\mathbf{Q}_p)$ be the stabilizer of Λ_p. In this report, we will assume that the level K has the form $K = K^p K_p$, with K^p a neat open compact subgroup of $\mathrm{G}(\mathbb{A}_f^p)$.

To study the modulo p geometry of $\mathrm{Sh}(\mathrm{G}, K)$, one has to first define an integral model of $\mathrm{Sh}(G, h)_K$ over $\mathcal{O}_{F,(p)}$. However, $\mathrm{Sh}(\mathrm{G}, K)$ is a Shimura variety of abelian type, and we do not have yet a general theory for integral models of Shimura varieties of abelian type at non-quasi-split primes. In our case, a nice integral model for $\mathrm{Sh}(G, h)_K$ was introduced by Bruinier, Howard, Kudla, Rapoport and Yang in [BHKRY], and generalized by Rapoport-Smithling-Zhang [RSZ] to general CM extensions. Their idea is to consider an extra torus $\mathrm{T}_0 := \mathrm{Res}_{F/\mathbf{Q}}(\mathbb{G}_m)$ together with the Deligne homomorphism

$$h_{\mathrm{T}_0} : \mathrm{Res}_{\mathbf{C}/\mathbf{R}}(\mathbb{G}_m) \to \mathrm{T}_0(\mathbf{R}) = \mathbf{C}^\times$$

given by identity. Fix a neat open compact subgroup $K_{\mathrm{T}_0} \subset \mathrm{T}_0(\mathbb{A}_f)$. Put $\widetilde{\mathrm{G}} = \mathrm{G} \times \mathrm{T}_0$, $\widetilde{h} = h \times h_{\mathrm{T}_0}$, and $\widetilde{K} = K \times K_{\mathrm{T}_0}$. Then the Shimura variety $\mathrm{Sh}(\widetilde{\mathrm{G}}, \widetilde{K})$ attached to the Shimura datum $(\widetilde{\mathrm{G}}, \widetilde{h})$ admits a nice moduli interpretation. Using this moduli interpretation, one can define a quasi-projective scheme $\mathcal{M}(\widetilde{G}, \widetilde{K})$ over $\mathrm{Spec}(\mathcal{O}_{F,(p)})$ such that its generic fibre is isomorphic to

$$\mathrm{Sh}(\widetilde{G}, \widetilde{K}) = \mathrm{Sh}(G, K) \times_{\mathrm{Spec}(F)} \mathrm{Sh}(\mathrm{T}_0, K_{\mathrm{T}_0}).$$

To describe the special fiber of $\mathcal{M}(\widetilde{G}, \widetilde{K})$, we need to introduce some notation. Let V' be the unique hermitian space over F such that

- $\mathrm{V}' \otimes \mathbb{A}_f^p$ is isomorphic to the hermitian space $\mathrm{V} \otimes \mathbb{A}_f^p$,
- $\mathrm{V}'_p := \mathrm{V}' \otimes \mathbf{Q}_p$ is split, i.e. it admits a self-dual lattice,
- $\mathrm{V}' \otimes \mathbf{R}$ has signature $(n, 0)$ (i.e. it is positive definite).

Let $\mathrm{G}' := \mathrm{U}(\mathrm{V}')$ and fix an isomorphism $\mathrm{G}(\mathbb{A}_f^p) \cong \mathrm{G}'(\mathbb{A}_f^p)$ so that one regards K^p as an open compact subgroup of $\mathrm{G}'(\mathbb{A}_f^p)$. Fix a chain of lattices in V'_p:

$$\Lambda_p^\circ \subset \Lambda_p^\bullet \subset p^{-1}\Lambda_p^\circ$$

such that $\Lambda_p^{\circ,\vee} = \Lambda_p^\circ$ and $p\Lambda_p^\bullet \subseteq \Lambda_p^{\bullet,\vee}$ with

$$\dim_{\mathbf{F}_{p^2}}(\Lambda_p^{\bullet,\vee}/p\Lambda_p^\bullet) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Let $K_p^\circ \subset \mathrm{G}'(\mathbf{Q}_p)$ (resp. $K_p^\bullet \subset \mathrm{G}'(\mathbf{Q}_p)$) be the stabilizer of Λ_p° (resp. $\Lambda_p^\bullet$). Put $K^\circ = K^p K_p^\circ$ and $K^\bullet = K^p K_p^\bullet$. For $? = \circ, \bullet$, one has the discrete Shimura set

$$\mathrm{Sh}(\mathrm{G}', K^?) \coloneqq \mathrm{G}'(\mathbf{Q})\backslash \mathrm{G}'(\mathbb{A}_f)/K^?.$$

Similarly, for $\widetilde{K}^? = K^? \times K_{\mathrm{T}_0}$ with $? = \circ, \bullet$, one can define the moduli scheme $\mathcal{M}(\widetilde{\mathrm{G}}', \widetilde{K}^?)$ which is finite and étale over $\mathcal{O}_{F,(p)}$.

Our main geometric result is the following

Theorem 1 (Liu–T.–Xiao–Zhang–Zhu)**.** *The following statements hold:*

(1) *The scheme $\mathcal{M}(\widetilde{\mathrm{G}}, \widetilde{K})$ is strictly semi-stable over $\mathcal{O}_{F,(p)}$ such that its special fiber $\mathcal{M}(\widetilde{\mathrm{G}}, \widetilde{K})_{\mathbf{F}_{p^2}}$ is the union of two strata X° and $X^\bullet$, called the balloon stratum and ground stratum respectively, which are both smooth varieties over $\mathbf{F}_{p^2}$ of dimension $n-1$.*

(2) *There exists a natural projection $\mathrm{pr}^\circ \colon X^\circ \to \mathcal{M}(\widetilde{\mathrm{G}}', \widetilde{K}^\circ)_{\mathbf{F}_{p^2}}$ such that each geometric fibre of pr° is isomorphic to $\mathbb{P}^{n-1}$.*

(3) *Let $X^\dagger \coloneqq X^\circ \cap X^\bullet$, and $\mathrm{pr}^\dagger \colon X^\dagger \to \mathcal{M}(\widetilde{\mathrm{G}}', \widetilde{K}^\circ)_{\mathbf{F}_{p^2}}$ be the restriction of pr° to $X^\dagger$. Then each geometric fiber of $\mathrm{pr}^\dagger$ is isomorphic to the Fermat hypersurface in $\mathbb{P}^{n-1}$ defined by the homogeneous equation*

$$x_0^{p+1} + \cdots + x_{n-1}^{p+1} = 0.$$

(4) *There exists a correspondence*

$$\mathcal{M}(\widetilde{\mathrm{G}'}, \widetilde{K}^\bullet)_{\mathbf{F}_{p^2}} \xleftarrow{\ \mathrm{pr}^\bullet\ } Y^\bullet \xrightarrow{\ \iota^\bullet\ } X^\bullet$$

such that

- *each geometric fiber of $\mathrm{pr}^\bullet$ is a certain Deligne–Luzstig variety that is connected and smooth of dimension $\lfloor n/2 \rfloor$,*
- *$\iota^\bullet$ is a closed immersion if $K^\bullet$ is sufficiently small,*
- *and the supersingular locus of $\mathcal{M}(\widetilde{\mathrm{G}}, \widetilde{K})_{\mathbf{F}_{p^2}}$ is exactly the union of X° and the image of $\iota^\bullet$.*

Example 2. *When $n = 2$, then $\iota^\bullet : Y^\bullet \xrightarrow{\sim} X^\bullet$ is an isomorphism, and each geometric fiber of $\mathrm{pr}^\bullet$ is a $\mathbb{P}^1$. So the whole $\mathcal{M}(\widetilde{\mathrm{G}}, \widetilde{K})_{\overline{\mathbf{F}}_{p^2}}$ consists of two collections of $\mathbb{P}^1$'s indexed respectively by $\mathcal{M}(\widetilde{\mathrm{G}'}, \widetilde{K}^\circ)(\overline{\mathbf{F}}_p) = \mathrm{G}'(\mathbf{Q})\backslash \mathrm{G}'(\mathbb{A}_f)/K^\circ \times \mathbb{A}_{F,f}^\times/F^\times K_{\mathrm{T}_0}$ and $\mathcal{M}(\widetilde{\mathrm{G}'}, \widetilde{K}^\bullet)(\overline{\mathbf{F}}_p) = \mathrm{G}'(\mathbf{Q})\backslash \mathrm{G}'(\mathbb{A}_f)/K^\bullet \times \mathbb{A}_{F,f}^\times/F^\times K_{\mathrm{T}_0}$.*

We describe now some arithmetic applications of our geometric results. From now on, we suppose that $n = 2r \geq 2$ is even. Consider an irreducible cuspidal automorphic representation $\pi' = \bigotimes_{v \leq \infty} \pi'_v$ of $\mathrm{G}'(\mathbb{A})$, where $\mathbb{A}$ denotes the adele ring of $\mathbf{Q}$, such that

- π'_∞ is trivial,
- π'_p is hyperspecial, i.e. $(\pi'_p)^{K_p^\circ} \neq 0$,
- π' admits a base change to an irreducible cuspidal automorphic representation Π' of $\mathrm{GL}_n(\mathbb{A}_F)$.

Since p is inert in F, the Satake parameter of Π'_p consists of r unordered pairs of algebraic numbers $\{\alpha_1, \alpha_1^{-1}\}, \cdots, \{\alpha_r, \alpha_r^{-1}\}$. Let $E \subset \mathbf{C}$ be a number field such that the finite part of Π' is defined over E, and that all α_i's are in E. We fix a finite place λ of E of residue characteristic ℓ, and an isomorphism $\overline{\mathbf{Q}}_\ell \cong \mathbf{C}$. Then thanks to the work of Harris-Taylor, Shin, Clozel, Chenevier-Harris, one can attach to Π' a ℓ-adic Galois representation

$$\rho_{\Pi'}:\ \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathcal{O}_{E_\lambda})$$

such that $\rho^c_{\Pi'} \cong \rho^\vee_{\Pi'}(1-n)$, where $\rho_{\Pi'}$ denotes the complex conjugate of $\rho_{\Pi'}$ and $\rho^\vee_{\Pi'}$ is the contragredient representation. Let $\bar{\rho}_{\Pi'}$ denote the residue Galois representation of $\rho_{\Pi'}$, which is well-defined up to semi-simplification.

Theorem 3 (Liu–T.–Xiao–Zhang–Zhu)**.** *Under the above notation, we make the following assumptions:*

(1) $\ell \nmid p(p^2-1)$.
(2) *No* α_i *is congruent to* -1 *modulo* λ [2]*,*
(3) *Exactly one pair of* $\{\alpha_i, \alpha_i^{-1}\}$*'s is congruent to* $\{p, p^{-1}\}$ *modulo* λ*.*
(4) $\bar{\rho}_{\Pi'}$ *is absolutely irreducible.*
(5) $\bar{\rho}_{\Pi'}$ *satisfies Caraiani-Scholze's generic decomposed condition [CS].*

Then there exists a cuspidal automorphic representation π *of* $\mathrm{G}(\mathbb{A})$ *such that*

- π_p *has level* K_p*, i.e.* $\pi_p^{K_p} \neq 0$*,*
- *and* π *admits a base change to a cuspidal automorphic representation* Π *of* $\mathrm{GL}_n(\mathbb{A}_F)$ *with* $\bar{\rho}_\Pi = \bar{\rho}_{\Pi'}$*.*

Recall that G is not quasi-split at p so that the base change Π is not hyperspecial at p. Our result thus predicts the existence of a cuspidal automorphic representation Π of deeper level at p such that its Hecke eigensystem is congruent to that of Π'. This is an analogue for even unitary groups of Ribet's classical level-raising result for holomorphic modular forms [Ri, Theorem 7.3]. Note that, for such a level-raised automorphic representation Π to exist, it is necessary that at least one of the pairs $\{\alpha_i, \alpha_i^{-1}\}$ is congruent to $\{p, p^{-1}\}$ modulo λ. Here, we can only treat the case when there exists exactly one such pair. In this sense, this is the minimal level-raising phenomenon for even unitary groups.

To conclude this note, let us just mention that a key geometric ingredient in the proof of Theorem 3 is an explicit computation of the intersection matrix of algebraic cycles $X^\dagger$ and $Y^\bullet$ on the ground stratum $X^\bullet$ in Theorem 1, and such a computation uses geometric Satake equivalence.

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Modularity of generating series of cycles

Benjamin Howard
(joint work with Keerthi Madapusi Pera)

Let V be quadratic space over $\mathbb{Q}$ of signature $(n,2)$. Assume, for simplicity, that there exists an even unimodular lattice $L \subset V$. From this data one can construct a Shimura datum reflex field $\mathbb{Q}$, and an n-dimensional orthogonal Shimura variety $M \to \mathrm{Spec}(\mathbb{Q})$. The underlying reductive group of the Shimura datum is the group $\mathrm{GSpin}(V)$ of spinor similitudes of V.

Fix an integer $1 \le d \le n$. There is a family of special cycles $Z(T) \to M$, indexed by symmetric positive semi-definite matrices $T \in \mathrm{Sym}_d(\mathbb{Q})$. Loosely speaking, these cycles are images of smaller orthogonal Shimura varieties associated to subspaces of V. The cycle $Z(T)$ has codimension $\mathrm{rank}(T)$, but we can correct it to a cycle class

$$C(T) = Z(T) \cdot \underbrace{\omega^{-1} \cdots \omega^{-1}}_{d-\mathrm{rank}(T)} \in \mathrm{CH}^d(M) \tag{1}$$

in codimension d by repeatedly intersecting with a distinguished line bundle ω.

It was conjectured by Kudla, and then proved by Bruinier and Westerholt-Raum [2] (building on earlier work of Zhang [5] and Borcherds [1]) that the formal generating series

$$\phi(\tau) = \sum_{T \in \mathrm{Sym}_d(\mathbb{Q})} C(T) \cdot q^T \in \mathrm{CH}^d(M)[[q]] \tag{2}$$

is a Siegel modular form of weight $1 + \frac{n}{2}$. Here τ is the variable in the genus g Siegel space, and $q^T = e^{2\pi i \mathrm{Trace}(\tau T)}$.

By work of Kisin [4] (extended by Kim and Madapusi Pera [3]) it is known that the Shimura variety M admits a canonical extension to a smooth scheme $\mathcal{M} \to \mathrm{Spec}(\mathbb{Z})$. The purpose of the talk was to explain how to extend the above modularity result to the integral model $\mathcal{M}$, by first extending the construction of cycle classes to

$$\mathcal{C}(T) \in \mathrm{CH}^d(\mathcal{M}), \tag{3}$$

and then proving the modularity of the corresponding generating series

$$\phi(\tau) = \sum_{T \in \mathrm{Sym}_d(\mathbb{Q})} \mathcal{C}(T) \cdot q^T \in \mathrm{CH}^d(\mathcal{M})[[q]]. \tag{4}$$

As $\mathcal{M}$ has dimension $n+1$, we now allow $1 \le d \le n+1$.

One of the main difficulties is that, while there is a naive construction of cycles $\mathcal{Z}(T) \to \mathcal{M}$, these are typically not equidimensional. This prevents one from defining (3) simply by imitating the construction (1). The way around this is to use a result of Gillet-Soule, which identifies the codimension d Chow group of $\mathcal{M}$ with a graded piece of the coniveau filtration on the Grothendieck of coherent sheaves on $\mathcal{M}$. This allows us to define (3) by constructing certain coherent sheaves on $\mathcal{M}$, as opposed to constructing physical cycles.

Once the cycle classes (3) have been defined, the next step is to prove the modularity of (4) under the assumption $d < n/3$. This assumption guarantees that the naive cycles $\mathcal{Z}(T)$ are equidimensional, normal, flat, and every irreducible component of every mod p-reduction meets the boundary of a toroidal compactification of $\mathcal{M}$. Using these properties, one can show that the proof of modularity of (2) applies also to (4) largely unchanged.

It then remains to remove the assumption $d < n/3$. This is done by first choosing an embedding of $\mathcal{M}$ into a larger Shimura variety $\mathcal{M}'$ associated to a quadratic space of signature $(n', 2)$. This larger Shimura variety has its own generating series of cycles

$$\phi'(\tau) = \sum_T \mathcal{C}'(T) \cdot q^T \in \mathrm{CH}^d(\mathcal{M}')[[q]],$$

which is modular if we choose n' large enough that $d < n'/2$. The pullback of this generating series to $\mathcal{M}$) is then also modular, and one can deduce the modularity of (4) from this.

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Local models and canonical integral modules of Shimura varieties

George Pappas

The talk addressed the question of characterizing and showing uniqueness of integral models of Shimura varieties at primes where the level subgroup is parahoric. In general, these models are not smooth, and the approach of Milne, Faltings-Chai and Vasiu-Zink, which uses a Neron type extension property for characterizing them cannot be applied.

1.1. Let G be a connected reductive algebraic group over $\mathbb{Q}_p$ and $\{\mu\}$ the $G(\bar{\mathbb{Q}}_p)$-conjugacy class of a minuscule cocharacter $\mu : \mathbb{G}_{\mathrm{m}\bar{\mathbb{Q}}_p} \to G_{\bar{\mathbb{Q}}_p}$. Let $E \subset \bar{\mathbb{Q}}_p$ be the reflex field and X_μ the G-homogeneous variety of parabolic subgroups of G of type μ. Denote by $\breve{E}$ the completion of the maximal unramified extension of E in $\bar{\mathbb{Q}}_p$ and by k its (algebraicaly closed) residue field. We write $W = W(k)$ for the ring of Witt vectors. We use F to denote a finite extension of E or $\breve{E}$.

Let $\mathcal{G}$ be a smooth affine connected group scheme over $\mathbb{Z}_p$ with generic fiber G. We assume the following purity condition:

(P) Every $\mathcal{G}$-torsor over $\mathrm{Spec}(W[[u]]) - \{(p, u)\}$ is trivial.

Assume that there is a closed group embedding

$$\iota : \mathcal{G} \hookrightarrow \mathrm{GL}_n = \mathrm{GL}(\Lambda)$$

such that $\{\iota(\mu)\}$ is minuscule for GL_n and $\iota(\mathcal{G})$ contains the scalars. We can realize $\mathcal{G}$ as the common stabilizer of a finite list of tensors $(s_a)_a$ in the tensor algebra $\Lambda^\otimes$ of $\Lambda = \mathbb{Z}_p^n$. There is an equivariant closed embedding $X_\mu \subset \mathrm{Gr}(d, n)_E$, where d corresponds to $\iota(\mu)$. Here $\mathrm{Gr}(d, n)$ is the Grassmannian which has a natural model $\mathrm{Gr}(d, \Lambda)$ over $\mathbb{Z}_p$. Denote by $\mathcal{M}$ the closure of X_μ in $\mathrm{Gr}(d, \Lambda)_{\mathcal{O}_E}$. It admits an action of $\mathcal{G}$ and there is an equivariant closed embedding

$$\iota_* : \mathcal{M} \hookrightarrow \mathrm{Gr}(d, \Lambda)_{\mathcal{O}_E}.$$

1.2. Now suppose that R is a p-adically complete *normal* $\mathcal{O}_E$-flat algebra which is, either topologically of finite type, or complete local with residue field k. Set $A = W(R)$ for the ring of (p-typical) Witt vectors with coefficients in R. Denote by $\phi : W(R) \to W(R)$ the Frobenius and by $I_R \subset W(R)$ the kernel of the 0-th Witt coordinate. Since $\phi(I_R) \subset pW(R)$, the Frobenius gives

$$\bar{\phi} : R = W(R)/I_R \to A/pA.$$

1.3. Let $\mathcal{F} \subset \Lambda \otimes_{\mathbb{Z}_p} R$ correspond to the image under ι_* of an R-valued point of $\mathcal{M}$. Compose this with $\bar{\phi} : R \to A/pA$ to obtain

$$\bar{\phi}^* \mathcal{F} \subset \Lambda \otimes_{\mathbb{Z}_p} A/pA.$$

Let $U \subset \Lambda \otimes_{\mathbb{Z}_p} A$ be the inverse image of $\bar{\phi}^* \mathcal{F}$ under reduction modulo p. We have $U^\otimes \subset U^\otimes[1/p] = \Lambda_A^\otimes[1/p]$. As in [2] §3.2, we can show that the tensors $s_a \otimes 1 \in \Lambda_A^\otimes \subset \Lambda_A^\otimes[1/p]$ belong to $U^\otimes$ and that the scheme of isomorphisms that respect the tensors

$$\mathcal{Q} := \underline{\mathrm{Isom}}_{s_a \otimes 1}(U, \Lambda_A)$$

is a $\mathcal{G}$-torsor over $\mathrm{Spec}(A)$. More generally, suppose that $\mathcal{P}$ is a $\mathcal{G}$-torsor over A given together with a $\mathcal{G}$-equivariant morphism $q : \mathcal{P} \otimes_A R \to \mathcal{M}$. Then, by the previous construction and descent, we obtain a $\mathcal{G}$-torsor $\mathcal{Q}$ together with an isomorphism of G-torsors

$$\alpha : \mathcal{Q}[1/p] \xrightarrow{\sim} \phi^* \mathcal{P}[1/p]$$

over $A[1/p]$. We may think of $\mathcal{Q}$ as a "modification of $\phi^* \mathcal{P}$ along the divisor $p = 0$, bounded by $\mathcal{M}$". This construction allows us to give:

Definition: A $(\mathcal{G},\mathcal{M})$-display $\mathcal{D}$ over R is a triple $(\mathcal{P},q,\Psi)$ of:

- A $\mathcal{G}$-torsor $\mathcal{P}$ over $W(R)$,
- a $\mathcal{G}$-equivariant morphism $q:\mathcal{P}\otimes_{W(R)}R\to\mathcal{M}$ over $\mathcal{O}_E$,
- a $\mathcal{G}$-isomorphism $\Psi:\mathcal{Q}\xrightarrow{\sim}\mathcal{P}$, where $\mathcal{Q}$ is the $\mathcal{G}$-torsor above.

The Frobenius of the display $\mathcal{D}$ is the G-isomorphism $F:\phi^*\mathcal{P}[1/p]\xrightarrow{\sim}\mathcal{P}[1/p]$ given by $p\cdot\Psi\cdot\alpha^{-1}$.

If R is complete local with residue field k and $p\geq 3$, there is a similar definition of a Dieudonné $(\mathcal{G},\mathcal{M})$-display in which, instead of $W(R)$, we use Zink's variant $\hat{W}(R)$ of the Witt vectors.

1.4. Suppose that $\rho:\mathrm{Gal}(\bar{F}/F)\to\mathcal{G}(\mathbb{Z}_p)\subset\mathrm{GL}(\Lambda[1/p])$ is a crystalline Galois representation such that the Hodge filtration on $D_{\mathrm{dR}}(\Lambda[1/p])$ is given by a G-cocharacter conjugate to μ. The theory of Breuil-Kisin modules combined with arguments from [2] §3.3, allows us to construct a corresponding Dieudonné $(\mathcal{G},\mathcal{M})$-display $\hat{\mathcal{D}}(\rho)$ over $\mathcal{O}_F$.

1.5. Let (G,X) be a Shimura datum with reflex field E. Fix a prime $p\neq 2$, a place $v|(p)$ of E and set $E=\mathsf{E}_v$ and $G=\mathsf{G}\otimes_{\mathbb{Q}}\mathbb{Q}_p$. Let $\{\mu\}$ be the conjugacy class of minuscule cocharacters given by X. Let $\mathsf{K}=\mathsf{K}_p\mathsf{K}^p\subset\mathsf{G}(\mathbb{A}_f)$ where $\mathsf{K}_p\subset\mathsf{G}(\mathbb{Q}_p)$, $\mathsf{K}^p\subset\mathsf{G}(\mathbb{A}_f^p)$ are compact open subgroups. Assume K^p is sufficiently small and consider the Shimura variety

$$\mathrm{Sh}_K:=\mathrm{Sh}_{\mathsf{K}}(\mathsf{G},X)=\mathsf{G}(\mathbb{Q})\backslash X\times\mathsf{G}(\mathbb{A}_f)/\mathsf{K}.$$

The map $\varprojlim_{\mathsf{K}_p}\mathrm{Sh}_{\mathsf{K}_p\mathsf{K}^p}\to\mathrm{Sh}_{\mathsf{K}_p\mathsf{K}^p}$ gives a pro-étale K_p-cover. Restriction over each $\xi\in\mathrm{Sh}_{\mathsf{K}_p\mathsf{K}^p}(F)$ gives a K_p-representation ρ_ξ of $\mathrm{Gal}(\bar{F}/F)$. We also have a corresponding p-adic étale G-local system $\mathcal{L}$ over $\mathrm{Sh}_{\mathsf{K}_p\mathsf{K}^p}$.

Suppose now that G splits over a tame extension of $\mathbb{Q}_p$ and that p does not divide $|\pi_1(G_{\mathrm{der}})|$. Fix K_p *parahoric*. Then $\mathsf{K}_p=\mathcal{G}(\mathbb{Z}_p)$, where $\mathcal{G}$ is the corresponding connected Bruhat-Tits group scheme. By [2] §1.4, the purity (P) is satisfied. Assume in addition, partly for simplicity, that the Shimura datum (G,X) is of Hodge type. Then ι as above exists and the closure $\mathcal{M}$ agrees with the *local model* $\mathrm{M}^{\mathrm{loc}}=\mathrm{M}^{\mathrm{loc}}(\mathcal{G},\{\mu\})$ of [3]; this is independent of choices, see [4] and [1].

We ask for models $\mathcal{S}_{\mathsf{K}}$ (of finite type and flat over $\mathcal{O}_E$) of the Shimura variety Sh_{K} which are normal and also support the following additional structure:

1) A convergent filtered Frobenius G-isocrystal T over the formal scheme $\hat{\mathcal{S}}_{\mathsf{K}}$ which is "associated"[1] to the p-adic étale G-local system $\mathcal{L}$.
2) A $(\mathcal{G},\mathrm{M}^{\mathrm{loc}})$-display $\mathcal{D}=(\mathcal{P},q,\Psi)$ over $\hat{\mathcal{S}}_{\mathsf{K}}$.

We ask that for $\mathsf{K}'^p\subset\mathsf{K}^p$, there are finite étale $\mathcal{S}_{\mathsf{K}_p\mathsf{K}'^p}\to\mathcal{S}_{\mathsf{K}_p\mathsf{K}'^p}$ which extend $\mathrm{Sh}_{\mathsf{K}_p\mathsf{K}'^p}\to\mathrm{Sh}_{\mathsf{K}_p\mathsf{K}^p}$ and are compatible with T and $\mathcal{D}$. In addition, we require:

a) Set $\mathcal{S}_{\mathsf{K}_p}=\varprojlim_{\mathsf{K}^p}\mathcal{S}_{\mathsf{K}_p\mathsf{K}^p}$. For any mixed characteristic dvr R over $\mathcal{O}$,

$$\mathcal{S}_{\mathsf{K}_p}(R[1/p])=\mathcal{S}_{\mathsf{K}_p}(R).$$

[1] after pulling back to a suitable semi-stable model via an alteration.

b) The Frobenius G-torsor $\mathcal{P}[1/p]$ is given by evaluating T at $W(\hat{\mathcal{S}}_{\mathsf{K}})$ and the morphism $q[1/p]$ is given by the filtered G-structure on T.

c) For all $\xi : \mathrm{Spec}(\mathcal{O}_F) \to \mathcal{S}_{\mathsf{K}}$, the Galois representation ρ_ξ is crystalline and the pull-back $\xi^*\mathcal{D}$ naturally descends to the Dieudonné $(\mathcal{G}, \mathrm{M}^{\mathrm{loc}})$-display $\hat{\mathcal{D}}(\rho_\xi)$ over $\mathcal{O}_F$.

d) For all $\bar{x} \in \mathcal{S}_{\mathsf{K}}(k)$, the pull-back of $\mathcal{D}$ to the completion $\hat{\mathcal{O}}_{\mathcal{S}_{\mathsf{K}},\bar{x}}$ descends to a Dieudonné $(\mathcal{G}, \mathrm{M}^{\mathrm{loc}})$-display $\hat{\mathcal{D}}_{\bar{x}}$ over $\hat{\mathcal{O}}_{\mathcal{S}_{\mathsf{K}},\bar{x}}$.

e) For all $\bar{x} \in \mathcal{S}_{\mathsf{K}}(k)$, if s is a section of $\hat{\mathcal{P}}_{\bar{x}}$ over $\hat{W}(\widehat{\mathcal{O}}_{\mathcal{S}_{\mathsf{K}},\bar{x}})$ which is "rigid in the first order at $\bar{x}$"[2], then $q \cdot s : \mathrm{Spec}(\widehat{\mathcal{O}}_{\mathcal{S}_{\mathsf{K}},\bar{x}}) \to \mathrm{M}^{\mathrm{loc}}$ gives an isomorphism

$$\widehat{\mathcal{O}}_{\mathcal{S}_{\mathsf{K}},\bar{x}} \simeq \widehat{\mathcal{O}}_{\mathrm{M}^{\mathrm{loc}},(q\cdot s)(\bar{x})}.$$

The main result is the following

Theorem: *Under the above assumptions, the models $\mathcal{S}_{\mathsf{K}}$ of the Shimiura varieties* $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X)$ *constructed in [2], support T and $\mathcal{D}$ as above. Any two systems $(\mathcal{S}_{\mathsf{K}}, T, \mathcal{D})$ and $(\mathcal{S}'_{\mathsf{K}}, T', \mathcal{D}')$ obtained by two different choices in the construction of* loc. cit.*, are isomorphic. In particular, the models $\mathcal{S}_{\mathsf{K}}$ constructed in [2], are independent of the choices in their construction.*

In fact, we expect that any two systems $(\mathcal{S}_{\mathsf{K}}, T, \mathcal{D})$ and $(\mathcal{S}'_{\mathsf{K}}, T', \mathcal{D}')$ that satisfy the above are isomorphic to each other (i.e. regardless of how they are obtained). This stronger statement would provide a characterization of the integral models.

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Extremal cases of Rapoport-Zink spaces

Xuhua He

(joint work with Ulrich Görtz and Michael Rapoport)

This talk is based on the joint work with U. Görtz and M. Rapoport [4].

The main object of this talk is the (union of) affine Deligne-Lusztig variety

$$X(\{\mu\}, b)_K := \{g\breve{K} \in \breve{G}/\breve{K}; g^{-1}b\sigma(g) \in \breve{K}\mathrm{Adm}(\{\mu\})\breve{K}\}.$$

Here F is a nonarchimedean local field, $\breve{F}$ is the completion of the maximal unramified extension of F, σ is the Frobenius morphism of $\breve{F}$ over F, $\mathbb{G}$ is a connected

[2]with respect to a connection on $\mathcal{P}$, which is given by the display structure.

reductive group over F, $\{\mu\}$ is a conjugacy class of cocharacters of $\mathbb{G}$, $\mathrm{Adm}(\{\mu\})$ is the $\{\mu\}$-admissible set, $b \in \breve{G} := \mathbb{G}(\breve{F})$, and $\breve{K}$ is a parahoric subgroup of $\breve{G}$.

If F is a function field, $X(\{\mu\}, b)_K$ is a closed scheme of locally finite type inside the partial affine flag variety $\breve{G}/\breve{K}$. If F is a p-adic field, then $X(\{\mu\}, b)_K$ is a closed perfect scheme of locally finite type inside the p-adic partial affine flag variety $\breve{G}/\breve{K}$ in the sense of Bhatt-Scholze and Zhu.

The motivation to study $X(\{\mu\}, b)_K$ arises from arithmetic geometry. If F is a p-adic field and $\{\mu\}$ is minuscule, then $X(\{\mu\}, b)_K$ arises as the set of geometric points of the *underlying reduced set* of a Rapoport-Zink formal moduli space of p-divisible groups, cf. [8]. Something analogous holds in the function field case for formal moduli spaces of *Shtukas*, cf. [9].

In this talk, we focus on the case where $b = \tau$ is a basic element in $\breve{G}$ that is associated to $\{\mu\}$. There are two important families of these formal schemes: the *Lubin-Tate case* and the *Drinfeld case*. In the first case, the underlying reduced scheme has dimension 0. In the second case, the underlying reduced scheme reaches the maximal possible dimension (i.e. of dimension $\langle \mu, 2\rho \rangle$).

We address the question of classifying the cases when $X(\{\mu\}, b)_K$ has minimal dimension zero (as in the Lubin-Tate case) or maximal dimension $\langle \mu, 2\rho \rangle$ (as in the Drinfeld case). We give the classification in terms of *enhanced Coxeter data* [7].

To state the main result, we assume that $\mathbb{G}$ is adjoint and simple over F and that $\{\mu\}$ is non-central in every simple factor of $\mathbb{G}$.

Theorem 1. *The scheme $X(\{\mu\}, \tau)_K$ is of dimension 0 if and only the enhanced Coxeter datum is $(\tilde{A}_{n-1}, \omega_1^\vee, id, \emptyset)$ (i.e. the Lubin-Tate case).*

Theorem 2. *The scheme $X(\{\mu\}, \tau)_K$ is equi-dimensional with dimension equals to $\langle \mu, 2\rho \rangle$ if and only if the enhanced Coxeter datum is one of the following:*

(1) $(\tilde{A}_{n-1}, \omega_1^\vee, Ad(\tau)^{-1}, \emptyset)$ *(i.e. the Drinfeld case);*
(2) $(\tilde{A}_3, \omega_2^\vee, Ad(\tau)^{-1}, \emptyset)$*;*
(3) $(\tilde{A}_{n-1} \times \tilde{A}_{n-1}, (\omega_1^\vee, \omega_{n-1}^\vee), {}^1\varsigma_0, \emptyset)$ *(i.e. the Hilbert-Blumenthal case).*

It is remarkable that in all three cases the parahoric subgroup K is the Iwahori subgroup. This implies the following characterization of the Drinfeld case.

Corollary 3. *The scheme $X(\{\mu\}, \tau)_K$ is equi-dimensional with dimension equal to $\langle \mu, 2\rho \rangle$ for every parahoric subgroup K if and only if the enhanced Coxeter datum is equal to $(\tilde{A}_{n-1}, \omega_1^\vee, Ad(\tau)^{-1}, \emptyset)$.*

This answers a question of Scholze of characterizing the Drinfeld case through the dimension of its underlying reduced scheme.

Now let us mention a key ingredient of the proof.

It is conjectured by Kottwitz and Rapoport and proved in [6] that $X(\{\mu\}, b)_K \neq \emptyset$ if and only if $[b] \in B(G, \{\mu\})$, the set of neutrally acceptable σ-conjugacy classes of $\breve{G}$.

In [3], we introduced the fully Hodge-Newton decomposable pairs. By definition, the pair $(G, \{\mu\})$ is fully Hodge Newton decomposable if for any nonbasic $[b] \in$

$B(G, \{\mu\})$, the pair $((\{\mu\}, b)$ is Hodge-Newton decomposable, i.e., the Newton polygon of b has a break point on the Newton polygon of $\{\mu\}$. One main result of [3] is to show that the pair $(G, \{\mu\})$ is fully Hodge-Newton decomposable if and only if for some K, the (union of) affine Deligne-Lusztig variety $X(\{\mu\}, b)_K$ in the partial affine flag variety $\breve{G}/\breve{K}$ is naturally a union of classical Deligne-Lusztig varieties in a finite partial flag variety. Note that a priori, the latter condition depends on the parahoric subgroup. However, it is shown that it is in fact independent of the parahoric level structure (as the parahoric subgroups do not appear in the definition of fully Hodge-Newton decomposable pairs).

It is interesting that if the reduced scheme $X(\{\mu\}, \tau)_K$ is of dimension 0 or is equi-dimensional with dimension equals to $\langle \mu, 2\rho \rangle$, then the pair $(G, \{\mu\})$ is a fully Hodge-Newton decomposable pair. However, not all the fully Hodge-Newton decomposable pairs occur. Moreover, the parahoric subgroup $\breve{K}$ also plays an important role here. To achieve the classification, we combine the results on the fully Hodge-Newton decomposable pairs with the method of partial conjugation action [5] and the notion of critical index introduced by Drinfeld [2].

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Crystalline Chebotarëv Density Theorems

Urs Hartl
(joint work with Ambrus Pál)

We formulate a conjectural analogue of Chebotarëv's density theorem for convergent F-isocrystals over a smooth geometrically irreducible curve defined over a finite field using the Tannakian formalism. We prove this analogue for several large classes, including direct sums of isoclinic convergent F-isocrystals and semisimple convergent F-isocrystals which have an overconvergent extension and such that their pull-back to a sufficient small non-empty open sub-curve has connected monodromy.

1. A Conjecture

Let U be a smooth, geometrically irreducible curve over a finite field $\mathbb{F}_q$ having q elements and characteristic p, and denote by $|U|$ the set of closed points of U. Let $F\colon U \to U$ be the (absolute) q-Frobenius which is the identity on $|U|$ and the q-power map on the structure sheaf. Let $K/\mathbb{Q}_q$ be a totally ramified finite field extension. We take $F = \mathrm{id}_K$ as a lift of the q-Frobenius $F = \mathrm{id}_{\mathbb{F}_q}$. For every $x \in |U|$ let $\mathbb{F}_x$, $\deg(x)$ and q_x denote the residue field of x, its degree over $\mathbb{F}_q$ and its cardinality, respectively. For $e \in \mathbb{N}$ let K_e be the unramified field extension of K of degree e. Then $\mathrm{Gal}(K_e/K) = \langle \mathrm{Frob}_q \rangle$.

Let $F\text{-}\mathrm{Isoc}_K(U)$ denote the K-linear rigid abelian tensor category of *K-linear convergent F-isocrystals on U*; see [Cre92, Chapter 1] for details. It is a Tannakian category with fiber functors ω_x for every $x \in |U|$ with $e := \deg(x)$ given by

$$\omega_x\colon\ F\text{-}\mathrm{Isoc}_K(U) \longrightarrow (K_e\text{-vector spaces})\,, \quad \mathcal{F} \mapsto x^*\mathcal{F}\,.$$

This fiber functor is non-neutral if $e > 1$. Actually, $x^*\mathcal{F}$ is an F-isocrystal over $\mathbb{F}_x$, that is an object of

$$F\text{-}\mathrm{Isoc}_K(\mathbb{F}_x) \;:=\; \big\{\, (W, F_W)\colon\ W \text{ a } K_e\text{-vector space}, \\ F_W : W \to W \text{ a } \mathrm{Frob}_q\text{-semilinear automorphisms} \,\big\}\,.$$

So $F_W^e : W \xrightarrow{\sim} W$ is a K_e-linear automorphism of W.

Now fix a base point $u \in U(\mathbb{F}_q)$. (We assume $\deg(u) = 1$ only for simplicity of the exposition.) For $\mathcal{F} \in F\text{-}\mathrm{Isoc}_K(U)$ let $\langle\!\langle \mathcal{F} \rangle\!\rangle \subset F\text{-}\mathrm{Isoc}_K(U)$ be the Tannakian subcategory generated by $\mathcal{F}$. Its *monodromy group* is defined as

$$\mathrm{Gr}(\mathcal{F}/U, u) \;:=\; \mathrm{Aut}^{\otimes}(\omega_u|\langle\!\langle \mathcal{F} \rangle\!\rangle)\,.$$

It is a linear algebraic group over K, *not necessarily connected.*

For every $x \in U(\mathbb{F}_{q^e})$ there is a non-canonical isomorphism of fiber functors $\omega_x \otimes_{K_e} \overline{K} \cong \omega_u \otimes_K \overline{K}$, where $\overline{K}$ is an algebraic closure of K. This induces a non-canonical isomorphism

$$\mathrm{Aut}^{\otimes}(\omega_x|\langle\!\langle \mathcal{F} \rangle\!\rangle) \times_{K_e} \overline{K} \;\cong\; \mathrm{Gr}(\mathcal{F}/U, u) \times_K \overline{K}\,. \tag{1}$$

We define $\mathrm{Frob}_x(\mathcal{F}) \subset \mathrm{Gr}(\mathcal{F}/U,u)(\overline{K})$ as the conjugacy class of the image under the isomorphism (1) of $F_w^e \in \mathrm{Aut}^{\otimes}(\omega_x|\langle\langle\mathcal{F}\rangle\rangle)$. This conjugacy class is independent of the choice of the isomorphism (1), and hence it is K-rational.

Conjecture A. *For every subset $S \subset |U|$ of Dirichlet density one the set*

$$\bigcup_{x\in S} \mathrm{Frob}_x(\mathcal{F}) \;\subset\; \mathrm{Gr}(\mathcal{F}/U,u)$$

is Zariski-dense.

2. Applications

Corollary. *Let $\mathcal{F},\mathcal{G} \in F\text{-}\mathrm{Isoc}_K(U)$ be convergent F-isocrystals on U of the same rank with $\mathrm{Tr}(\mathrm{Frob}_x(\mathcal{F})) = \mathrm{Tr}(\mathrm{Frob}_x(\mathcal{G}))$ for all points x in a subset $S \subset |U|$ of Dirichlet density one. If Conjecture A holds for the direct sum $\mathcal{F}^{ss} \oplus \mathcal{G}^{ss}$ of the semi-simplifications then $\mathcal{F}^{ss} \cong \mathcal{G}^{ss}$.*

Proof. Since $\mathcal{F}^{ss}$ lies in $\langle\langle\mathcal{F}\rangle\rangle$ there is an epimorphism of linear algebraic groups $\mathrm{Gr}(\mathcal{F}/U,u) \twoheadrightarrow \mathrm{Gr}(\mathcal{F}^{ss}/U,u)$ under which $\mathrm{Frob}_x(\mathcal{F})$ maps onto $\mathrm{Frob}_x(\mathcal{F}^{ss})$. The two spaces $\omega_u(\mathcal{F}^{ss})$ and $\omega_u(\mathcal{G}^{ss})$ are semi-simple representations of the group $\mathrm{Gr}\big((\mathcal{F}^{ss} \oplus \mathcal{G}^{ss})/U,u\big)$. By our hypothesis their trace functions coincide on the subset $\mathrm{Frob}_x(\mathcal{F}^{ss} \oplus \mathcal{G}^{ss})$. By Conjecture A the two trace functions coincide on all of $\mathrm{Gr}\big((\mathcal{F}^{ss} \oplus \mathcal{G}^{ss})/U,u\big)$. This implies that the two representations are isomorphic; see [Ser98, Lemma in § I.2.3 on p. I-11]. And therefore the convergent F-isocrystals $\mathcal{F}^{ss}$ and $\mathcal{G}^{ss}$ are isomorphic. □

Example. *If A and B are abelian varieties over U with Dieudonné isocrystals $\mathcal{F} = D(A)$ and $\mathcal{G} = D(B)$, this gives an isogeny criterion for A and B.*

3. Cases we can prove

Definition. *Let $\mathcal{F} \in F\text{-}\mathrm{Isoc}_K(U)$. We define*

- *the* slopes of $\mathcal{F}$ at $x \in |U|$ *as*

$$\tfrac{1}{\deg(x)} \cdot \mathrm{ord}_p\big(\textit{eigenvalues of } \mathrm{Frob}_x(\mathcal{F}) \textit{ on } \omega_u(\mathcal{F})\big)\,.$$

- *$\mathcal{F}$ to be* isoclinic *if for all $x \in |U|$, $\mathcal{F}$ has a single slope at x (which is then the same for every x).*
- *$\mathcal{F}$ to be* unit-root *if it is isoclinic of slope zero.*

Proposition. *Conjecture A holds for unit-root convergent F-isocrystals $\mathcal{F}$ on U.*

Proof. Choose a geometric base point $\bar{u}$ above u and let $\pi_1^{\text{ét}}(U,\bar{u})$ be the étale fundamental group of U. By a result of R. Crew [Cre87, Theorem 2.1 and Remark 2.2.4] the full subcategory of $F\text{-}\mathrm{Isoc}_K(U)$ consisting of unit-root F-isocrystals is tensor equivalent to the category $\mathrm{Rep}_K\, \pi_1^{\text{ét}}(U,\bar{u})$ of continuous representations of $\pi_1^{\text{ét}}(U,\bar{u})$ on finite dimensional K-vector spaces. Moreover, under this equivalence the fiber functor $\omega_u\colon F\text{-}\mathrm{Isoc}_K(U) \to (K\text{-vector spaces})$ and the forgetful fiber functor $\omega_{\mathrm{forget}}\colon \mathrm{Rep}_K\, \pi_1^{\text{ét}}(U,\bar{u}) \to (K\text{-vector spaces})$ become isomorphic over $\overline{K}$.

Let $\rho_{\mathcal{F}} \colon \pi_1^{\text{ét}}(U, \bar{u}) \to \mathrm{GL}_r(K)$ be the representation corresponding to a unit-root F-isocrystal $\mathcal{F}$. Then $\mathrm{Gr}(\mathcal{F}/U, u)$ is a closed subgroup of $\mathrm{GL}_{r,K}$ such that $\mathrm{Gr}(\mathcal{F}/U, u) \times_K \overline{K}$ equals the Zariski-closure of the image of $\rho_{\mathcal{F}}$. Moreover, for all points $x \in |U|$ the $\mathrm{Gr}(\mathcal{F}/U, u)(\overline{K})$-conjugacy classes of $\rho(x_* \,\mathrm{Frob}_x^{-1})$ and $\mathrm{Frob}_x(\mathcal{F})$ coincide, where $\mathrm{Frob}_x^{-1} \in \mathrm{Gal}(\overline{\mathbb{F}}_x/\mathbb{F}_x)$ is the geometric Frobenius at x which maps $a \in \overline{\mathbb{F}}_x$ to a^{1/q_x} for $q_x = \#\mathbb{F}_x$.

If $S \subset |U|$ is a subset of Dirichlet density one, then by the classical Chebotarëv density theorem [Ser63, Theorem 7] the union of the Frobenius conjugacy classes Frob_x^{-1} for the points $x \in S$ are dense in $\pi_1^{\text{ét}}(U, \bar{u})$ with respect to the pro-finite topology. Since this topology is finer than the restriction of the Zariski topology from $\mathrm{Gr}(\mathcal{F}/U, u)$, the set $\bigcup_{x \in S} \mathrm{Frob}_x(\mathcal{F})$ is Zariski-dense in $\mathrm{Gr}(\mathcal{F}/U, u)$. □

Example. *Let $\mathcal{C}$ be the pullback to U of the constant F-isocrystal on $\mathbb{F}_q$ of rank one given by $(K, F = \pi^s)$ with $s \in \mathbb{Z}$, where $\pi \in K$ is a uniformizing parameter. If $s \neq 0$ then $\mathrm{Gr}(\mathcal{F}/U, u) = \mathbb{G}_{m,K}$. Indeed, $\mathrm{Gr}(\mathcal{F}/U, u)$ is a closed subgroup of $\mathrm{Aut}_K(u^*\mathcal{C}) = \mathbb{G}_{m,K}$ which contains $\mathrm{Frob}_x(\mathcal{C}) = \{\pi^{s \deg(x)}\}$. Since this set is infinite, the only such group is $\mathbb{G}_{m,K}$. The set $\bigcup_{x \in U} \mathrm{Frob}_x(\mathcal{F}) \subset \pi^{\mathbb{Z}s} \subset \mathbb{G}_{m,K}$ is Zariski-dense. However, this set is discrete in $\mathbb{G}_m(K)$ for the p-adic topology. For that reason we can only expect density for the Zariski-topology.*

Theorem 1. *Conjecture A holds for the direct sum $\mathcal{F} = \bigoplus_i \mathcal{F}_i$ of isoclinic convergent F-isocrystals $\mathcal{F}_i$ on U.*

Idea of the proof. We twist away the slope of $\mathcal{F}_i$ by a constant rank one F-isocrystal $\mathcal{C}_i$ (after enlarging K). Then $\mathcal{G}_i := \mathcal{F}_i \otimes \mathcal{C}_i$ is unit-root. We set $\mathcal{G} := \bigoplus_i \mathcal{G}_i$ and $\mathcal{C} := \bigoplus_i \mathcal{C}_i$. Since $\mathcal{F} \in \langle\langle \mathcal{G} \oplus \mathcal{C} \rangle\rangle$ it is enough to prove Conjecture A for $\mathcal{G} \oplus \mathcal{C}$. In the diagram

$$
\begin{array}{ccc}
\mathrm{Gr}(\mathcal{G} \oplus \mathcal{C}) = \mathrm{Gr}(\mathcal{G}) & \underset{\mathrm{Gr}(\langle\langle \mathcal{G} \rangle\rangle \cap \langle\langle \mathcal{C} \rangle\rangle)}{\times} & \mathrm{Gr}(\mathcal{C}) \\
\uparrow & & \uparrow \\
C := \rho_{\mathcal{G}}\big(\pi_1^{\text{ét}}(U, \bar{u})\big) & & F_{\mathcal{C}}^{\mathbb{Z}}
\end{array}
$$

the subgroup C is compact, and hence a p-adic Lie group by [Ser92, Part II, § V.9, Corollary to Theorem 1 on page 155]. We now count the cardinality of

$$\left\{\, c \in C \colon \exists\, x \in S \text{ with } (c, F_{\mathcal{C}}^{\deg(x)}) \in \mathrm{Frob}_x(\mathcal{G} \oplus \mathcal{C}) \,\right\}.$$

A lower bound is provided by the Chebotarëv density for $\mathcal{G}$. If $\bigcup_{x \in S} \mathrm{Frob}_x(\mathcal{G} \oplus \mathcal{C})$ was contained in a hyperplane we would obtain a contradicting upper bound by a result of Oesterlé [Oes82]. □

For the next result we let $\mathcal{F} \in F\text{-}\mathrm{Isoc}_K(U)$ be semi-simple. By the slope filtration theorem of Grothendieck and Katz [Kat79, Corollary 2.3.2] there is a non-empty open subcurve $f \colon V \hookrightarrow U$ such that $f^*\mathcal{F}$ has a slope filtration with isoclinic subquotients. It is always true that

$$\mathrm{Gr}(f^*\mathcal{F}/V, u) \hookrightarrow \mathrm{Gr}(\mathcal{F}/U, u)$$

is a closed immersion. Note, that in contrast to ℓ-adic and p-adic Galois representations this closed immersion can be strict for F-isocrystals.

Conjecture B. $\mathrm{Gr}(f^*\mathcal{F}/V,u) \hookrightarrow \mathrm{Gr}(\mathcal{F}/U,u)$ *is a parabolic subgroup.*

Theorem 2. *Conjecture B for $\mathcal{F}$ implies Conjecture A for $\mathcal{F}$.*

Theorem 3. *If $\mathrm{Gr}(f^*\mathcal{F}/V,u)$ is connected and $\mathcal{F}$ extends to an overconvergent F-isocrystal on U, then Conjecture A holds for $\mathcal{F}$.*

Idea of the proofs for both theorems. We use Theorem 1 for $(f^*\mathcal{F})^{ss}$ which is a direct sum of isoclinic convergent F-isocrystals on V. In the diagram

$$\begin{array}{ccc} \mathrm{Gr}(f^*\mathcal{F}/V,u) & \overset{}{\underset{\beta}{\hookrightarrow}} & \mathrm{Gr}(\mathcal{F}/U,u) \\ \downarrow{\scriptstyle \alpha} & & \\ \mathrm{Gr}\big((f^*\mathcal{F})^{ss}/V,u\big) & & \end{array} \qquad \begin{array}{ccc} \mathrm{Frob}_x(f^*\mathcal{F}) & \hookrightarrow & \mathrm{Frob}_x(\mathcal{F}) \\ \downarrow & & \\ \mathrm{Frob}_x\big((f^*\mathcal{F})^{ss}\big) & & \end{array}$$

the vertical morphism α identifies $\mathrm{Gr}\big((f^*\mathcal{F})^{ss}/V,u\big)$ with the maximal reductive quotient of $\mathrm{Gr}(f^*\mathcal{F}/V,u)$. We then (develop and) use the theory of maximal quasi-tori as in the following

Definition. *Let G be a linear algebraic group over an algebraically closed field L of characteristic zero, which is not necessarily connected. A closed subgroup $T \subset G$ is a* maximal quasi-torus *if the morphism $\alpha\colon G \twoheadrightarrow G/R_uG =: \widetilde{G}$ onto the maximal reductive quotient $\widetilde{G}$ of G induces an isomorphism $\alpha\colon T \xrightarrow{\sim} \alpha(T) \subset \widetilde{G}$ and there is a maximal torus and a Borel subgroup $\widetilde{T}^\circ \subset \widetilde{B} \subset \widetilde{G}^\circ$ such that $\alpha(T)$ equals the intersection $N_{\widetilde{G}}(\widetilde{B}) \cap N_{\widetilde{G}}(\widetilde{T}^\circ)$ of the normalizers. (Then $\alpha(T)^\circ = \widetilde{T}^\circ$).*

Now Conjecture B (and likewise the hypotheses of Theorem 3) implies that β maps any maximal quasi-torus of $\mathrm{Gr}(f^*\mathcal{F}/V,u)$ onto a maximal quasi-torus T of $\mathrm{Gr}(\mathcal{F}/U,u)$. Then we (prove and) use that in the reductive group $\mathrm{Gr}(\mathcal{F}/U,u)$ the Zariski-density of the union $\bigcup_{x\in S}\mathrm{Frob}_x(\mathcal{F})$ is equivalent to the Zariski-density in T of $\bigcup_{x\in S} T\cap\overline{\mathrm{Frob}_x(\mathcal{F})}$. □

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