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# Nonlinear Evolution Equations: Analysis and Numerics

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Abstract. The qualitative theory of nonlinear evolution equations is an important tool for studying the dynamical behavior of systems in science and technology. A thorough understanding of the complex behavior of such systems requires detailed analytical and numerical investigations of the underlying partial differential equations.

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## Introduction by the Organizers

The workshop *Nonlinear evolution equations*, organized by Marlis Hochbruck (Karlsruhe), Herbert Koch (Bonn), Sung-Jin Oh (Seoul) and Alexander Ostermann (Innsbruck) was well attended with around 50 participants with broad geographic representation from all continents. This workshop combined analytic and numerical aspects of nonlinear evolution equations with a focus on wave propagation.

The qualitative theory of nonlinear evolution equations is an important tool for studying the dynamical behavior of systems in science and technology. A thorough understanding of the complex behavior of such systems requires detailed analytical and numerical investigations of the underlying partial differential equations. In connection with this task one is interested in regularity and asymptotic properties of solutions as well as in efficient numerical approximations which preserve their qualitative properties on large time scales. Currently, driven by the challenging mathematical difficulties and supported by the impact of techniques from many

branches of analysis and numerics, innovative and sophisticated methods are being developed for this area of mathematics.

This workshop covered recent developments in the theory of nonlinear evolution equations, both analytical and numerical, between which there has been an increasing interaction in recent years. There is an enormous potential of a closer cooperation of the analytic and numerical communities for decisive progress in the field. The workshop focused on evolution equations describing linear and nonlinear waves such as

- wave and Schrödinger equations,
- Maxwell's equations,
- nonlinear dispersive equations,
- coupled problems (e.g. Vlasov–Maxwell equations or MHD).

In recent decades a very successful qualitative theory of the basic nonlinear dispersive equations has been developed in settings of low regularity, namely for the nonlinear wave and Schrödinger equations. This also led to a revival and sharpening of old questions like the soliton resolution conjecture, or, more generally, a global understanding of solutions to dispersive equations and the relations between asymptotic equations like the nonlinear Schrödinger equation and their role as building block in full models like water waves and wave guides.

Our understanding of complex wave propagation problems like nonlinear Maxwell equations is based on asymptotic equations, and on numerical simulations. The numerical study of evolution equations relies on efficient methods for time integration which allow for convergence results not depending on spatial discretization parameters (such as mesh size or number of basis functions). For the rigorous derivation of finite time error bounds and the study of geometric properties, one has to develop a framework for the discretization that captures the essential properties of the analytic problem. This fact makes a thorough understanding of the partial differential equation itself indispensable and it entails close thematic and methodic links between the analysis and numerics of evolution equations: splitting schemes or exponential integrators are naturally formulated within semigroup theory and functional calculi. Harmonic analysis arises in spectral methods and can be used to establish error bounds (e.g., by Strichartz estimates). Moreover, error analysis heavily relies on regularity theory and energy techniques. The numerical treatment of nonstandard boundary conditions calls for a deeper analysis of the coupling between the behavior on the boundary and in the domain. This close relationship also plays a crucial role for geometric integrators which are designed to capture the qualitative properties of solutions (such as long term behavior or energy conservation).

The rigorous error analysis of numerical methods for wave-type equations has made significant progress in recent years. However, many questions are still open and more efficient numerical methods have to be designed for particular problems. At the same time we expect that the challenges from numerics stimulate new developments in the relevant areas of analysis and numerical insights can guide analytical investigations to new frontiers.

The workshop brought together scientists from analysis and numerics to push this research forward. We gave preference to talks by young mathematicians.

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## Nonlinear Evolution Equations: Analysis and Numerics

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# Abstracts

### Blow up mechanism for quasilinear wave equations

Hajer Bahouri
(joint work with Alaa Marachli, Galina Perelman)

The main goal here is to show that the blow up mechanism initiated by Krieger, Schlag and Tataru in [1] in a semilinear framework exists as well for the quasilinear wave equation on $\mathbb{R}\times\mathbb{R}^4$:

$$\text{(NW)}\ u:=(1+u_\rho^2)\,u_{tt}-(1-u_t^2)\,u_{\rho\rho}-2u_tu_\rho u_{\rho t}+3(1+u_\rho^2-u_t^2)\Big(\frac{1}{u}-\frac{u_\rho}{\rho}\Big)=0\,,$$

which involves in Bernstein's problem. Bernstein's problem which is due to Sergei Natanovich Bernstein states as follows: if the graph of a $C^2$ function $u$ on $\mathbb{R}^n$ is a minimal surface in $\mathbb{R}^{n+1}$, does this imply that this graph is a hyperplane? Such issue amounts to ask if the solution $u$ to the following equation known as the minimal surface equation

$$\sum_{j=1}^n\partial_{x_j}\Big(\frac{u_{x_j}}{\sqrt{1+|\nabla u|^2}}\Big)=0\,,$$

is linear. This problem admits only an affirmative answer in the case of dimension $n\leq 7$, and the falsity of its extension to the case of dimension $n\geq 8$ is linked to the Simons cone which is given by

$$C_n=\big\{X=(x_1,\cdots,x_{2n})\in\mathbb{R}^{2n}\,,x_1^2+\cdots+x_n^2=x_{n+1}^2+\cdots+x_{2n}^2\big\}.$$

We refer for instance to [2, 4] and the references therein for further details on Bernstein's problem and related issues, and to [1] for detailed computations about the involvement of the quasilinear wave equation (NW) in that setting.

Our purpose in this text is to give an outline of the constructive blow up approach for (NW) by concentrating the soliton, namely the solution to the Cauchy problem:

$$\begin{cases}\ -Q_{\rho\rho}+3(1+Q_\rho^2)\Big(\frac{1}{Q}-\frac{Q_\rho}{\rho}\Big)=0\\ \ Q(0)=1\,,\,Q_\rho(0)=0\,.\end{cases}$$

First of all, let us recall that Bombieri, De Giorgi and Giusti established in [2] that the above Cauchy problem has a unique solution $Q$ in $C^\infty(\mathbb{R}^4)$ which behaves as

$$Q(\rho)=\rho+\frac{d_2}{\rho^2}\big(1+\circ(1)\big)\,,$$

when $\rho$ tends to infinity, with $d_2>0$ and which satisfies $Q(\rho)>\rho$ and $Q''(\rho)>0$, for any $\rho\geq 0$. Let us also emphasize that (NW) enjoys a scaling invariant property: defining the scaling operators, for any $\lambda>0$, any $x_0$ in $\mathbb{R}^n$ and any $t_0$ in $\mathbb{R}$,

$$\Lambda_{\lambda,x_0,t_0}u(t,x):=\lambda u\Big(\frac{t-t_0}{\lambda},\frac{x-x_0}{\lambda}\Big)\,,$$

if $u$ solves (NW) then $\Lambda_{\lambda,x_0,t_0}u$ is also a solution to (NW).

Our aim here consists in proving the existence of finite time blow up solutions to the quasilinear wave equation (NW) under the form

$$u(t,\cdot) \sim \Lambda_{\lambda(t)} Q,$$

where $\lambda(t) = t^{\nu+1}$, $\nu$ being an arbitrary large positive irrational number. More precisely, we established the following result:

For all irrational numbers $\nu > \frac{1}{2}$, there exists a positive time $T$ and a radial solution $u(t,\cdot)$ to (NW) on the interval $(0,T]$ such that

$$(u, \partial_t u) \in C((0,T], X_{L_0}) \text{ with } L_0 := 2M+1\,,\ M = \Big[\frac{3}{2}\nu + \frac{5}{4}\Big]\,,$$

and such that it blows up at time $t=0$ by concentrating the soliton profile. More precisely, writing

$$\begin{aligned} u(t,x) &= t^{\nu+1}\Big(Q\Big(\frac{x}{t^{\nu+1}}\Big) + \zeta\Big(t, \frac{x}{t^{\nu+1}}\Big)\Big)\,, \\ u_t(t,x) &= \zeta_1\Big(t, \frac{x}{t^{\nu+1}}\Big), \end{aligned}$$

we have

$$\|\nabla\zeta(t,\cdot)\|_{\dot{H}^s(R^4)} + \|\zeta_1(t,\cdot)\|_{\dot{H}^s(R^4)} \overset{t\to 0}{\longrightarrow} 0\,, \forall\, 2 < s \leq L_0 - 1\,.$$

Roughly speaking, the strategy of proof of the above blow up result proceeds in a two step process: first building an approximate solution, and then completing it to an exact solution, by controlling the remaining error via well-established arguments. The key point consists to built a good approximate solution, and to this end the analysis is done separately in the inner region corresponding to the region inside the light cone, the self-similar region which is the region around the light cone, and finally the remote region. The inner region is the region where the blow up concentrates, and where the solution is constructed as a perturbation of the profile $t^{\nu+1}Q\Big(\frac{x}{t^{\nu+1}}\Big)$. Using (NW), it turns out that in that region, we have to look for the approximate solution under the following form:

$$u_{\text{in}}^{(N)}(t,\rho) = t^{\nu+1}\sum_{k=0}^{N} t^{2\nu k} V_k\Big(\frac{\rho}{t^{\nu+1}}\Big)\,,$$

where $V_0 = Q$, and where the functions $V_k$ are obtained recursively by solving:

$$\begin{cases} \mathcal{L}V_k = F_k(V_0, \cdots, V_{k-1}) \\ V_k(0) = 0\,,\ V_k'(0) = 0\,, \end{cases}$$

with $\mathcal{L}$ the linearized operator of (NW) around $Q$. Combining the result of Bombieri, De Giorgi and Giusti together with classical ordinary differential equations arguments, one can show that the above equation admits a $C^\infty$ solution $V_k$ which grows at infinity as follows:

$$V_k(y) = \sum_{\ell=0}^{k} (\log y)^\ell \sum_{n \geq 2-2(k-\ell)} d_{n,k,\ell}\, y^{-n}\,,$$

and thus we have to restrict the construction under this form to the region $\frac{\rho}{t} \leq t^{\epsilon_1}$, with $\epsilon_1$ some fixed positive real number

To extend this approximate solution to the self-similar region, we have to take into account the matching conditions coming out from the inner region. This reduces us to look to this extension under the form:

$$u_{\rm ss}^{(N)}(t,\rho) = \rho + \lambda(t) \sum_{k=3}^{N} t^{\nu k} \sum_{\ell=0}^{\ell(k)} \big(\log t\big)^{\ell} w_{k,\ell}\Big(\frac{\rho}{\lambda(t)}\Big)\,,$$

where $w_{k,\ell}$, are determined by induction using again (NW). Here $\frac{\rho}{\lambda(t)} = \frac{1}{\sqrt{2}}$ denotes the light cone of the quasilinear wave equation that is constructed concurrently with the solution. Actually in that region, the approximate solution we construct is not $C^\infty$: this is due to the fact that the functions $w_{k,\ell}$ which are determined successively solves a system which admits a basis of solutions which are singular. The extension of the approximate solution to the remonte region can be easily determined.

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## Reducibility and growth of Sobolev norms for Schrödinger equation with time dependent unbounded perturbation


Dario Bambusi

(joint work with Benoit Grébert, Beatrice Langella, Alberto Maspero, Riccardo Montalto, Didier Robert)


In this extended abstract I will study a time dependent Schrödinger equation

$$\mathrm{i}\dot\psi = H_0\psi + \epsilon W(t)\psi \tag{1}$$

where $H_0$ is a self-adjoint, unbounded operator with pure point spectrum and $W(t)$ a time dependent family of unbounded "perturbations". The typical examples are

$$\mathrm{i}\dot\psi = (-\partial_{xx} + V(x))\psi + \epsilon W(x, -\mathrm{i}\partial_x, t)\psi\ , \quad V(x) = |x|^{2\ell}\ , \quad \ell \geq 1\ , \quad x \in \mathbb{R}\ , \tag{2}$$

and

$$\mathrm{i}\dot\psi = -\Delta\psi + \epsilon W(x,t)\psi\ , \quad x \in \mathbb{T}\ .$$

I will present some results, but the main point is that I would like to discuss a technique which seems to be quite general and consists of the idea of using first pseudodifferential calculus in order to transform the original system to a smoothing

perturbation of a suitable "normal form" and then to apply more or less standard methods in order to get the result.

In order to state precisely the assumptions on $W$, consider the classical Hamiltonian corresponding to $H_0$, namely

$$h_0 := \xi^2 + |x|^{2\ell} \ .$$

**Definition 1.** *The space $S^m$ is the space of the symbols $g \in C^\infty(\mathbb{R})$ such that $\forall k_1, k_2 \geq 0$ there exists $C_{k_1,k_2}$ with the property that*

$$\left|\partial_\xi^{k_1}\partial_x^{k_2} g(x,\xi)\right| \leq C_{k_1,k_2}\left[\left(1 + h_0(x,\xi)^{1/2\ell}\right)\right]^{m-k_1\ell-k_2} \ . \tag{3}$$

*To a symbol $g \in S^m$ we associate its Weyl quantization, namely the operator*

$$[G\psi](x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{\mathrm{i}(x-y)\cdot\xi} g\left(\frac{x+y}{2};\xi\right)\psi(y)dyd\xi \ . \tag{4}$$

An operator will be said to be of class $OPS^m$ if it is the Weyl quantization of a symbol of class $S^m$.

Furthermore we define the norms

$$\|\psi\|^2_{\mathcal{H}^s} := \langle\psi; H_0^s\psi\rangle_{L^2} \ ,$$

and the Hilbert space $\mathcal{H}^s$ as the space of the functions with finite $\mathcal{H}^s$ norm.

**Theorem 2.** *Take $\epsilon = 1$. Assume $\ell > 1$ and $W \in C^\infty(\mathbb{R}, OPS^\beta)$ with $\beta < 2\ell$. Consider a solution $\psi(t)$ of (2). For any $N \geq 0$ there exists a unitary (in $L^2$) time dependent operator $\Phi(t)$ s.t. the function $\varphi$ defined by $\Phi(t)\varphi := \psi$ satisfies the equation*

$$\mathrm{i}\dot{\varphi} = (H_0 + z_N(H_0,t) + R_N(t))\varphi \ , \tag{5}$$

*with $z_N$ a smooth symbol (and $z(H_0,t)$ spectrally defined) and $R_N(t)$ a smoothing operator. Precisely, one has*

$$\|R_N(t)\psi\|_{\mathcal{H}^{s+N}} \leq C_{N,s}\|\psi\|_{\mathcal{H}^s}$$

*Furthermore $\Phi(t)$ is bounded from $\mathcal{H}^s$ to itself.*

This theorem is a develpment of some results present in [Bam18, Bam17]; in this form it is proved in [BM18] and it exploits techniques developed in such papers and in [BGMR19].

As described in detail in the paper [Bam18], Sect. 4.2, the proof consists of two steps. First one uses the classical methods of Lie transform in order to construct an auxiliary Hamiltonian function $\chi$ whose Hamiltonian flow pushes to higher order the time dependence in the Hamiltonian system $\xi^2 + \epsilon w(x,\xi,t)$ (with $w$ the symbol of $W$). Then one quantizes $\chi$ and uses the unitary transformation it generates in order to transform the original quantum system. Actually the construction is iterative.

Using Theorem 2, by the theory of [MR17], one immediately gets that for any $\delta > 0$ and any $s > 0$ there exists a constant $C_{s,\delta}$ s.t. the solution of the time

dependent Schrödinger equation (2) fulfills the estimate

$$\|\psi(t)\|_{\mathcal{H}^s} \leq C_{s,\delta}(1+|t|)^{\delta} \tag{6}$$

One can also obtain reducibility by using the classical Kuksin's theorem [Kuk93] (or its develoments e.g. [BBM14]). For this case we have to restrict to perturbations which depend quasiperiodically on time. Thus, consider $W \in C^{\infty}(\mathbb{T}^d, OPS^{\beta})$ and consider equation (1) with $W = W(\omega t)$ and parameters $\omega$ varying in

$$\Omega := [1,2]^d \ .$$

**Theorem 3.** *Assume $\beta < 2\ell$, then $\exists \epsilon_* > 0$, $C_\lambda$ and $\forall\, |\epsilon| < \epsilon_*$ $\exists \Omega(\epsilon) \subset \Omega$ and, $\forall \omega \in \Omega(\epsilon)$ there exists a unitary (in $L^2$) time quasiperiodic operator $\Phi_\omega(\omega t)$ s.t. the function $\varphi$ defined by $\Phi_\omega(\omega t)\varphi := \psi$ satisfy the equation*

$$\mathrm{i}\dot{\varphi} = H_\infty \varphi \ , \tag{7}$$

*with $H_\infty = diag(\lambda_j^\infty)$ and*

$$\left|\lambda_j^\infty - \lambda_j\right| \leq C_\lambda \epsilon j^{\frac{\beta}{\ell+1}} \ . \tag{8}$$

*Furthermore $\forall s, r \geq 0$, $\exists \epsilon_{s,r} > 0$ and $s_r$ s.t., if $|\epsilon| < \epsilon_{s,r}$ then the map $T^n \ni \phi \mapsto U_\omega(\phi)$ is of class $C^r(T^n; B(\mathcal{H}^{s+s_r}; \mathcal{H}^s))$.*

*Finally the measure*

$$|\Omega - \Omega(\epsilon)| \overset{\epsilon \to 0}{\to} 0 \ .$$

I would like to conclude by mentioning that the theory can be considered quite complete for problems in 1 space dimensions, while in the case of higher space dimensions only a few examples are known ([BGMR18, BLM19]). At present, work is in progress in order to try to get some more general extensions to higher space dimensions and some preliminary spectral results for Sturm-Liouville type operators on tori have been obtained.

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## Small amplitude solutions of the Schrödinger-Newton-Hooke system

Piotr Bizoń

(joint work with Oleg Evnin and Filip Ficek [1])

We consider the Einstein-Klein-Gordon system (for a complex scalar field) in $d+1$ dimensions with the negative cosmological constant $\lambda = -d/\ell^2$

$$\left(\Box_g - m^2c^2\right)\phi = 0, \quad R_{\alpha\beta} - \lambda g_{\alpha\beta} = \frac{8\pi G}{c^4}\left(\partial_\alpha\phi\,\partial_\beta\bar{\phi} + \partial_\alpha\bar{\phi}\,\partial_\beta\phi + \frac{m^2c^2}{d-1}\,|\phi|^2 g_{\alpha\beta}\right).$$

Substituting $\phi = e^{-imc^2t}u(t,x)$ and taking the limits $c\to\infty$ and $\ell\to\infty$ so that $c/\ell \to \omega$, we get (setting $m = \frac{8\pi G}{d-1} = 1$)

$$i\partial_t u = -\frac{1}{2}\Delta u + \frac{1}{2}\omega^2|x|^2u + Vu, \quad \Delta V = (d-2)\,|u|^2,$$

which we call the Schrödinger-Newton-Hooke (SNH) system. Using the Green function for the Laplacian $F(x) = -\dfrac{1}{(d-2)\Omega_d}\,\dfrac{1}{|x|^{d-2}}$ (here $\Omega_d$ is the volume of $\mathbb{S}^{d-1}$), we can write the SNH system as the Hartree equation with the external harmonic potential

$$i\partial_t u = -\frac{1}{2}\Delta u + \frac{1}{2}\omega^2|x|^2u - \Omega_d^{-1}\left(|x|^{-(d-2)} * |u|^2\right)u. \tag{1}$$

We are interested in the long-time behavior of small amplitude solutions of (1). Assuming spherical symmetry, we decompose solutions into radial eigenfunctions

$$u(t,r) = \varepsilon\sum_{n=0}^{\infty}\alpha_n(t)e_n(r)e^{-iE_nt},$$

where $E_n = \left(2n+\frac{d}{2}\right)\omega$ and $e_n(r) = c_nL_n^{(\frac{d}{2}-1)}(\omega r^2)e^{-\omega r^2/2}$. This gives

$$i\dot{\alpha}_n = \varepsilon^2\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}S_{njkl}\bar{\alpha}_j\alpha_k\alpha_l e^{i(E_n+E_j-E_k-E_l)t},$$

where $S_{njkl} = \int_0^\infty r\,dr\int_0^\infty \min\{r^{d-2},s^{d-2}\}\,e_n(r)e_j(s)e_{(k}(s)e_{l)}(r)s\,ds$. Let $\tau = \varepsilon^2 t$. For small $\varepsilon$ the terms $\propto e^{i(E_n+E_j-E_k-E_l)\tau/\varepsilon^2}$ with $E_n+E_j-E_k-E_l\neq 0$ are highly oscillatory and therefore negligible. Discarding all such non-resonant terms

and keeping only the resonant quartets satisfying $E_n + E_j - E_k - E_l = 0$, we get the resonant system

$$i\,\frac{d\alpha_n}{d\tau} = \sum_{n+j=k+l} S_{njkl}\bar{\alpha}_j\alpha_k\alpha_l. \tag{2}$$

Solutions of size $\varepsilon$ of equation (1) are well approximated by the solutions of the resonant system (2) on the timescale $t \sim 1/\varepsilon^2$. Our strategy is to understand the dynamics of the resonant system and then export this knowledge to the original equation.

We note that the resonant systems of the form (2) (but with different interaction coefficients) arise for several other NLS and wave equations with cubic nonlinearities (e.g. cubic Szegő equation [2], Gross-Pitaevskii equation with a harmonic potential [3, 4], conformal wave equation on 3-sphere [5], Einstein-scalar-AdS [6]). All these systems are invariant under scaling $\alpha_n(\tau) \mapsto \varepsilon\alpha_n(\varepsilon^2\tau)$ and the global and local phase shifts: $\alpha_n(\tau) \mapsto e^{i\theta}\alpha_n(\tau)$ and $\alpha_n(\tau) \mapsto e^{in\theta}\alpha_n(\tau)$. The latter two symmetries give rise to two conserved quantities (in addition to the Hamiltonian)

$$N = \sum_{n=0}^{\infty} |\alpha_n|^2, \qquad E = \sum_{n=0}^{\infty} E_n|\alpha_n|^2.$$

The key question is: how the energy of initial data gets distributed over the modes during evolution? Does energy flow from low to high modes? Using mixed analytic and numerical methods we found that the answer depends critically on the dimension. For $\lambda \to \infty$ we have

$$S_{\lambda n,\lambda j,\lambda k,\lambda l} \sim \lambda^{\beta}, \quad \text{where} \quad \beta(d) = \begin{cases} -1/2 & \text{if } d = 3 \\ (d-6)/2 & \text{if } d \geq 5 \end{cases}$$

This suggests rapid transfer of energy to high frequencies for $d \geq 7$ which is supported by numerical simulations.

In the energy critical case $d = 4$ the resonant system (2) belongs to a class of "solvable" cubic resonant systems [7]. In this case there is an additional conserved quantity $Z = \sum_{n=0}^{\infty} \sqrt{(n+1)(n+2)}\,\bar{\alpha}_{n+1}\alpha_n$ and there exists a 3-dimensional invariant manifold given by the ansatz

$$\alpha_n = \sqrt{n+1}\,\left(b + \frac{a}{p}\,n\right)p^n,$$

where the functions $a(\tau), b(\tau), p(\tau)$ are complex-valued. The dynamics on this invariant manifold is described by the reduced Hamiltonian system that is completely integrable (thanks to three conserved quantities $N$, $E$, and $H$).

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## Asymptotic Behaviour of the Maxwell-Klein-Gordon System


Timothy Candy

(joint work with Christopher Kauffman, Hans Lindblad)


We describe recent work giving precise asymptotics for solutions to the Maxwell-Klein-Gordon system in Lorenz gauge [1]. These asymptotics have two parts, one wave-like along outgoing null cones at null infinity, and one homogeneous inside the light cone at time-like infinity. The charge plays a crucial role in imposing an oscillating factor in the asymptotics for both the phase and the gauge potential.

We now describe these results in slightly more detail. The Maxwell-Klein-Gordon equation for a scalar field $\phi : \mathbb{R}^{1+3} \to \mathbb{C}$ and a potential $A_\alpha : \mathbb{R}^{1+3} \to \mathbb{R}$ is given by

$$\begin{aligned} D^\alpha D_\alpha \phi &= 0 \\ \partial^\beta F_{\alpha\beta} &= J_\alpha \end{aligned} \tag{1}$$

where the covariant derivative is given by $D_\alpha = \partial_\alpha + iA_\alpha$, the curvature is defined as $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, and $J_\alpha = \Im(\phi\overline{D_\alpha\phi})$ is the current. We take $x^0 = t$, and indices are raised and lowered with respect to the Minkowski metric $m = \text{diag}\,(-1, 1, 1, 1)$. The Einstein summation convention is in effect with Greek indices summed over $\alpha = 0, \dots, 3$, and Latin indices summed over the spatial variables $j = 1, 2, 3$. Thus $\partial^\alpha = m^{\alpha\beta}\partial_\beta$ and $\partial^0 = -\partial_t$. There are two key conserved quantities for the system (1), the energy

$$\mathcal{E}(\phi, A) = \int_{\mathbb{R}^3} |D\phi|^2 + |F|^2 dx$$

and the charge

$$\mathbf{q} = \int_{\mathbb{R}^3} J_0 dx.$$

The charge will be particular important, as it causes a number of corrections to the large time behaviour of solutions to (1) when compared to the evolution of solutions to the free wave equation.

The system (1) does not uniquely determine $\phi$ and $A$. In particular, for any function $\psi$, the gauge transform $\tilde{A} = A + d\psi$ gives the same field $F$, and moreover, letting $\tilde{\phi} = e^{i\psi}\phi$, if $(\phi, A)$ solve (1), then $(\tilde{\phi}, \tilde{A})$ also gives a solution to (1). Here, we fix the gauge by imposing the Lorenz gauge condition

$$\partial^\alpha A_\alpha = 0. \tag{2}$$

Given this we can rewrite the equations for the gauge potential $A_\alpha$ as the wave equation

$$\Box A_\alpha = \partial^\beta \partial_\beta A_\alpha = -J_\alpha.$$

This does not completely characterize $A$, in that we can add any one-form of the form $d\psi$ to $A$, where $\psi$ is a solution to the wave equation, and still recover the Lorenz gauge. The Lorenz gauge propagates through time, so if the solution $A_\alpha$ satisfies the Lorenz gauge condition (2) at $t = 0$, then (2) in fact holds for all times.

It is known that solutions to the Maxwell-Klein-Gordon system exist globally in time, this was first observed in similar systems by Eardly-Moncrief [2, 3] with refinement by Klainerman-Machedon [4]. Later decay estimates were shown in Lindblad-Sterbentz [5] after preliminary results in Shu [7] and Psarelli [6].

Our goal is to give a precise description of the asymptotic behaviour of the scalar (complex) field $\phi$, and the gauge $A_\mu$, evolved from data at $t = 0$. The results will be expressed in terms of *null frame*. The first two members of this are the null generators of forward and backward light cones which we define respectively as:

$$L = \partial_t + \partial_r, \qquad \underline{L} = \partial_t - \partial_r$$

where $r = |x|$. To complete the frame, we choose vector fields $\{S_B\}_{B=1,2}$ which form an orthonormal basis on each sphere $\{r = const.\}$, for each fixed time slice. The potential $A_\mu$ can be expressed in the frame $\{L, \underline{L}, S_1, S_2\}$, for instance writing $L = L^\mu \partial_\mu$, we have

$$A_L = L^\mu A_\mu = A_0 + \omega^j A_j, \qquad A_{\underline{L}} = \underline{L}^\mu A_\mu = A_0 - \omega^j A_j.$$

For later use, we note that the coefficients of the frame can also be raised and lowered using the metric $m$, thus $L_\mu = m_{\mu\nu} L^\nu$, and $L_\mu = L_\mu(\omega)$ is a function of $\omega \in \mathbb{S}^2$ only.

**Theorem 1** ([1])**.** *Assume that the data for (1) are sufficient smooth, localised, and small. Then for any $q \in \mathbb{R}$, $\omega \in \mathbb{S}^2$ the limits*

$$\Phi_0(q,\omega) = \lim_{t\to\infty} \big(re^{i\frac{1}{4\pi}\mathbf{q}\ln(1+r)}\phi\big)\big(t,(t+q)\omega\big), \qquad \frac{1}{4\pi}\mathbf{q} = \lim_{t\to\infty} (rA_L)\big(t,(q+t)\omega\big)$$

*and*

$$\mathcal{A}_{S_B}(q,\omega) = \lim_{t\to\infty} (rA_{S_B})\big(t,(q+t)\omega\big), \qquad \mathcal{A}^0_{\underline{L}}(q,\omega) = \lim_{t\to\infty} (rA^{mod}_{\underline{L}})\big(t,(t+q)\omega\big)$$

*exist, where we define*

$$A^{mod}_{\underline{L}}(t, r\omega) = A_{\underline{L}}\big(t, r\omega\big) - \frac{1}{2r}\int_{r-t}^{\infty} \mathcal{J}_{\underline{L}}(\eta, \omega) \ln\Big(\frac{\eta + t + r}{\eta + t - r}\Big) d\eta$$

*and*

$$\mathcal{J}_{\underline{L}}(q, \omega) = -2\Im\Big(\Phi_0(q, \omega)\overline{\partial_q \Phi_0(q, \omega)}\Big).$$

Note that for free solutions to the wave equation, $\Box u = 0$, on $\mathbb{R}^{1+3}$, the limit $\lim_{t\to\infty}(ru)(t, (q + t)\omega)$ exists, and the difference decays like $r^{-2}$, i.e. one order better than the decay of $u$ in the exterior $t < \frac{1}{2}|x|$. Hence the previous theorem shows that at null infinity, $A_{S_B}$ behaves like a free wave, $A_L$ converges to the charge $\mathbf{q}$, and when compared to the evolution of a free wave, the scalar field $\phi$ has phase correction at null infinity. On the other hand the bad component of gauge $A_{\underline{L}}$ only behaves like a free wave after subtracting off a term with a log growth. In particular, $A_{\underline{L}}$ has a log loss of decay when compared to the free wave equation.

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## Local time-stepping schemes for wave equations

Constantin Carle
(joint work with Marlis Hochbruck)

We consider spatial discretizations of the linear acoustic wave equation

$$\partial_{tt} u = \Delta u \qquad \text{in } (0,T)\times\Omega$$

on a bounded domain $\Omega$ equipped with Dirichlet boundary conditions and initial values. Our interest is in locally refined meshes consisting mostly of coarse mesh elements but also containing a small number of tiny elements.

On such meshes numerical time integration of the resulting semidiscrete equation

$$\mathbf{u}'' + \mathbf{L}\mathbf{u} = 0, \qquad \mathbf{u}(0) = \mathbf{u}^0, \qquad \mathbf{u}'(0) = \mathbf{v}^0$$

with an explicit scheme can be very tedious since the methods are only stable under a CFL condition: up to a constant, the timestep $\tau$ is restricted by the diameter of the smallest element in the mesh. To circumvent this issue local time-stepping schemes are popular, see, e.g., [1]. Using an explicit scheme with a large timestep $\tau$ on the coarse part of the mesh and the same scheme with a smaller timestep $\tau/p$ on the fine part of the mesh ($p \in \mathbb{N}$) one obtains schemes which hopefully are stable under a CFL condition depending only on the coarse part of the mesh.

In [1] an energy-conserving local time-stepping scheme based on the leapfrog method was derived. A stability and convergence analysis of this scheme was recently given in [3]. However, the proof of stability requires a CFL condition depending on the fine part of the mesh.

It can be shown that the local time-stepping scheme in [1] is closely related to leapfrog-Chebyshev schemes which were first proposed in [2, 6]. The following generalization motivated by [5] was recently constructed and analyzed in [4]

$$\begin{aligned} &\mathbf{u}_{n+1} - 2\mathbf{u}_n + \mathbf{u}_{n-1} = -P(\tau^2\mathbf{L})\mathbf{u}_n, \\ &\quad \mathbf{u}_0 = \mathbf{u}^0, \qquad \mathbf{u}_1 = \mathbf{u}^0 + \tau P'\big(\tau^2\mathbf{L}\big)\mathbf{v}^0 - \tfrac{1}{2}P\big(\tau^2\mathbf{L}\big)\mathbf{u}^0, \end{aligned}$$

where

$$P(z) = 2 - \tfrac{2}{T_p(\nu)} T_p\big(\nu - \tfrac{z}{\alpha_p}\big), \qquad \alpha_p = 2\tfrac{T_p'(\nu)}{T_p(\nu)}, \qquad \nu \geq 1, \tag{1}$$

and $T_p$ is the $p$th Chebyshev polynomial of first kind. Then the local time-stepping scheme in [1] can be reformulated as

$$\mathbf{u}_{n+1} - 2\mathbf{u}_n + \mathbf{u}_{n-1} = -\tau^2 \widetilde{P}(\tau^2\mathbf{L}\mathbf{R})\mathbf{L}\mathbf{u}_n,$$

where $\widetilde{P}(z) = P(z)/z$ with $P$ defined in (1) for $\nu = 1$ and $\mathbf{R}$ is a restriction matrix corresponding to the degrees of freedom belonging to the fine part of the mesh.

Using this relation and the generalized leapfrog-Chebyshev scheme we want to construct local time-stepping schemes for which one can prove stability under a CFL condition depending only on the coarse part of the mesh.

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## Discrete conservation laws for finite element discretizations of multisymplectic PDEs

Elena Celledoni
(joint work with James Jackaman)

### 1. Introduction

A multisymplectic PDE in one time and one space dimension can be written as a first order system in the form

$$K\,\mathbf{z}_t + L\,\mathbf{z}_x = \nabla \mathcal{S}\,(\mathbf{z})\,,$$

where $K$ and $L$ are skew-symmetric matrices, $\mathbf{z} \in \mathbb{R}^d$, and $\mathcal{S}\,(\mathbf{z})$ is smooth. A multisymplectic method is a space-time approximations of such a PDE, and is often based on a finite differences discretization. Such methods are designed to preserve a discrete variant of the conservation law of multisymplecticity, [2], [3]. Unfortunately, discretizations of multisymplectic PDEs (e.g. multisymplectic discretizations) are sometimes not well defined locally, and/or globally, or fail to have solutions which exist or are unique [8]. Here we consider space-time finite element discretizations focusing on the preservation of the discrete local conservation laws of momentum and energy, deriving a numerical integrator capable of simultaneously conserving nonlinear conserved quantities of different orders. These numerical integrators possess stable unique solutions, and are therefore well-posed and ideal for long time simulations.

## 2. Variational space-time formulation

Let $u = u(t,x)$ where $t \in [0,T]$ and $x \in S^1 = [0,1)$ periodic and consider the space-time variational problem

$$0 = \delta \int_{[0,T]\times S^1} L(u, u_t, u_x)\,\mathrm{d}x\,\mathrm{d}t, \tag{1}$$

where $L(u,u_t,u_x)$ is some given Lagrangian density function. Multisymplectic PDEs arise naturally from such problems through the following methodology. Introducing the auxiliary variables

$$v := \frac{\partial L}{\partial u_t}, \qquad w := \frac{\partial L}{\partial u_x},$$

and assuming that $u_t = u_t(v)$, $u_x = u_x(w)$ are invertible functions, we can define the Hamiltonian density

$$\mathcal{S}\,(u,v,w) := vu_t + wu_x - L(u, u_t(v), u_x(w)).$$

We may then express the variational principle by means of $\mathcal{S}\,(u,v,w)$ in the new variables as

$$0 = \delta \int_{[0,T]\times S^1} \left(vu_t + wu_x - \mathcal{S}\,(u,v,w)\right)\,\mathrm{d}x\,\mathrm{d}t,$$

and through taking variations $(\psi_u, \psi_v, \psi_w)^T =: \psi$ we obtain the integral form of the multisymplectic PDE

$$0 = \int_{[0,T]\times S^1} \left(K\,z_t + L\,z_x - \nabla\mathcal{S}\,(\mathbf{z})\right)\cdot\psi \quad \mathrm{d}x\,\mathrm{d}t, \qquad \forall\psi, \tag{2}$$

where

$$\mathbf{z} := \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad \psi := \begin{bmatrix} \psi_u \\ \psi_v \\ \psi_w \end{bmatrix}, \quad K = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

This underlying variational nature of multisymplectic PDEs advocates to space-time finite elements being a desirable discretization methodology.

## 3. Numerical discretization

We shall discretize the integral form of the multisymplectic PDE (2) using finite elements, however, the exact nature of the finite element formulation typically depends on the underlying PDE being considered. While multisymplectic PDEs can be hyperbolic, this is not necessarily the case, and they may be parabolic in nature. This indicates that, in practice, the optimal approach may be to discretize space and time differently. This is reflected in our finite element scheme, which is conforming and steps forward in time, but is noncomforming and solved globally in space. We note the related works of [1] in which space-time finite element methods were developed for parabolic problems, and [9] where multisymplectic finite element discretizations have been designed for a class of elliptic and hyperbolic problems.

3.1. **Space-time discretization.** We partition our spatial and temporal domain as follows:

| Time domain | discretization | temporal finite element | element length |
|---|---|---|---|
| $[0,T]$, | $0 := t_0 < t_1 < \cdots < t_N =: T$ | $\mathcal{I}_n := (t_n, t_{n+1})$ | $\tau_n := t_{n+1} - t_n$. |
| Space domain | discretization | spatial finite element | element length |
| $S^1 := [0,1)$ | $0 := x_0 < x_1 < \cdots < x_M := 1$ | $\mathcal{J}_m := (x_m, x_{m+1})$ | $h_m := x_{m+1} - x_m$. |

To facilitate our exposition we shall use $\mathcal{K}_j$ to denote either of $\mathcal{I}_n$ or $\mathcal{J}_m$ and define the finite element space as follows. Let $\mathbb{P}_r(\mathcal{K}_j)$ be the space of polynomials of degree $r$ on $\mathcal{K}_j \subset \mathbb{R}$. We define the discontinuous finite element space as

$$\mathbb{V}_r(S^1) = \{\mathbf{Z} \mid \ \mathbf{Z}|_{\mathcal{K}_j} \in (\mathbb{P}_r(\mathcal{K}_j))^d, j = 0, \ldots, J-1\},$$

and its continuous counterpart as

$$\mathbb{V}_r^C(S^1) = \mathbb{V}_r(S^1) \cap \mathcal{C}^0(S^1).$$

In addition we write $\mathbf{Z}_m^+ := \mathbf{Z}(x_m^+) = \lim_{x \to x_m^+} \mathbf{Z}(x)$, $\mathbf{Z}_m^- := \mathbf{Z}(x_m^-) = \lim_{x \to x_m^-} \mathbf{Z}(x)$ and define the following discontinuous operators:

- $[\![\mathbf{Z}_m]\!] = \mathbf{Z}_m^- - \mathbf{Z}_m^+$, the jump;
- $\{\mathbf{Z}_m\} = \frac{1}{2}(\mathbf{Z}_m^- + \mathbf{Z}_m^+)$, the average;
- $\mathcal{G}$ the discrete space derivative such that

$$\int_{S^1} \mathcal{G}(\mathbf{Z}) \cdot \phi \, \mathrm{d}x = \sum_{m=0}^{M-1} \int_{\mathcal{J}_m} \mathbf{Z}_x \cdot \phi \, \mathrm{d}x - \sum_{m=0}^{M-1} [\![\mathbf{Z}_m]\!] \cdot \{\phi_m\}, \qquad \forall \phi \in \mathbb{V}_p\left(S^1\right).$$

With this notation in mind we may define the space-time finite element method.

3.2. **Space-time finite element method.** The space-time finite element space is the tensor product of time and space finite elements:

$$\mathbb{V}_{q+1}^C([0,T]) \times \mathbb{V}_p(S^1).$$

The method is given by seeking $\mathbf{Z} \in \mathbb{V}_{q+1}^C([0,T]) \times \mathbb{V}_p(S^1)$ such that

$$\int_{[0,T]\times S^1} (K\mathbf{Z}_t + L\mathcal{G}(\mathbf{Z}) - \nabla \mathcal{S}(\mathbf{Z})) \cdot \phi \, \mathrm{d}x \, \mathrm{d}t = 0 \quad \forall \phi \in \mathbb{V}_q([0,T]) \times \mathbb{V}_p(S^1) \tag{3}$$

$$\mathbf{Z}(0,x) = \Pi \mathbf{z}(0,x),$$

where $\Pi$ is the $L_2$ projection into $\mathbb{V}_p(S^1)$. Note that since $\phi$ is allowed to be discontinuous in time we can localise our method restricting it to each finite element in time $\mathcal{I}_n$ to obtain a time-stepping method, see [4], and hence a practical implementation for both parabolic and hyperbolic problems. Localisation in space is also formally possible, however, the operator $\mathcal{G}$ acts globally.

**Theorem 1** (Discrete conserved quantities)**.** *Consider energy and momentum densities and fluxes:*

$$\begin{aligned} \mathcal{E}^D &:= \tfrac{1}{2}\mathcal{G}(\mathbf{Z}) \cdot L\mathbf{Z} + \mathcal{S}(\mathbf{Z}) & \mathcal{E}^F &:= -\tfrac{1}{2}\mathbf{Z}_t \cdot L\mathbf{Z} \\ \mathcal{M}^D &:= -\tfrac{1}{2}\mathcal{G}(\mathbf{Z}) \cdot K\mathbf{Z} & \mathcal{M}^F &:= \tfrac{1}{2}\mathbf{Z}_t \cdot K\mathbf{Z} + \mathbf{W} \cdot \mathbf{1} \end{aligned}$$

*where* $\mathbf{W} \in \mathbb{V}_q(\mathcal{I}_n) \times \mathbb{P}_p(\mathcal{J}_m)$ *depends only* $\mathbf{Z}$ *and is consistent with* $\mathcal{S}(\mathbf{Z})$.[1] *The numerical solution* $\mathbf{Z}$*, see* (3)*, conserves a discrete momentum and energy locally in time, i.e.,*

$$0 = \int_{\mathcal{I}_n \times S^1} \mathcal{M}_t^D + \mathcal{G}(\mathcal{M}^F)\,\mathrm{d}x\,\mathrm{d}t$$

*and*

$$0 = \int_{\mathcal{I}_n \times S^1} \mathcal{E}_t^D + \mathcal{G}(\mathcal{E}^F)\,\mathrm{d}x\,\mathrm{d}t,$$

*respectively. Furthermore, these conservation laws can be localised in space.*

Theorem 1 guarantees stability of our space-time finite element scheme over time, and in particular it can be seen that conservation of momentum leads to stability in a semi-norm. In addition, to the best of the author's knowledge, the simultaneous conservation of momentum and energy has not been seen before for problems of this type, and for some multisymplectic PDEs (such as the KdV equation) allows us to simultaneously conserve multiple nonlinear invariants of different orders. Further to the method being stable, it's solutions also satisfy the required existence and uniqueness properties. This can be seen for the temporal discretization following methodology outlined in [7], and the spatial discretization following [6].

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[1] $\mathbf{W}$ is such that for $m = 0, \dots, M-1$

$$\int_{\mathcal{I}_n \times \mathcal{J}_m} \mathcal{G}(\mathbf{W}) \cdot \chi - \nabla S(Z) \odot \Pi(\mathcal{G}(\mathbf{Z})) \cdot \chi \,\mathrm{d}x\,\mathrm{d}t = 0, \quad \forall \chi \in \mathbb{P}_q(\mathcal{I}_n) \times \mathbb{P}_p(\mathcal{J}_m),$$

where $\Pi$ the $L_2$ projection onto $\mathbb{P}_q(\mathcal{I}_n) \times \mathbb{P}_p(\mathcal{J}_m)$ and $\odot$ is the pointwise product.

## Uniformly accurate methods for highly-oscillatory problems with time-dependent vanishing frequency

Philippe Chartier
(joint work with M. Lemou, F. Méhats and G. Vilmart)

Highly-oscillatory evolution equations of the form

$$\dot{U}^\epsilon(t) := \frac{d}{dt}U^\epsilon(t) = \frac{1}{\epsilon}AU^\epsilon(t) + f\Big(U^\epsilon(t)\Big), \quad U^\epsilon(0) = U_0, \quad 0 \le t \le T,$$

where $T$ is a strictly positive fixed time, independent of $\epsilon$, and where the operator $A$ is supposed to be diagonalizable and to satisfy $\exp(2\pi A) = I$, have received considerable attention in the literature, from the points of view of both asymptotic analysis and numerical methods. However, allowing the parameter $\epsilon$ to take values in a whole interval of the form $]0, 1]$, prevents the use of numerical methods constructed for specific regimes. As a matter of fact, standard methods from the litterature typically have error bounds expressed as powers of the step-size $h$ of the form

$$\text{error } \le C\frac{h^p}{\epsilon^q}, \quad p > 0, \quad q > 0,$$

where $p$ is the order of the method and $q$ is equal to $p$ or $p - 1$: while suitable for the regime $\epsilon$ close to 1, they require formidable computational power for small values of $\epsilon$. At the other end of the spectrum, methods based on averaging and designed for small values of $\epsilon$ typically admit error bounds of the form

$$\text{error } \le C(h^p + \epsilon^q), \quad p > 0, \quad q > 0,$$

where $p$ is the order of the method and $q$ is the order of averaging: they thus encompass an incompressible error for larger values of $\epsilon$.

In this talk, our objective is to construct uniformly accurate methods, i.e. with an error (and at a cost) independent of the value of $\epsilon \in ]0, 1]$,

$$\text{error } \le Ch^p,$$

for equations whose frequency of oscillation *depends on time*. More precisely, we consider systems of differential equations of the form

$$\dot{U}^\epsilon(t) = \frac{\gamma(t)}{\epsilon}AU^\epsilon(t) + f\Big(U^\epsilon(t)\Big) \in R^d, \quad U^\epsilon(0) = U_0, \quad 0 \le t \le T, \tag{1}$$

where $A \in R^{d\times d}$ and where the function $f$ is assumed to be sufficiently smooth. The parameter $\epsilon$ again lies in the whole interval $(0, 1]$ and the real-valued function $\gamma$ is assumed to be continuous on $[0, +\infty)$.

Many semi-classical models for quantum dynamics also assume the form of highly oscillatory PDEs with a varying frequency , e.g. quantum models for surface hopping, graphene models, or quantum dynamics in periodic lattice. In such applications, the frequency $\gamma$ may depend on time and measures the gap between different energy bands, while the parameter $\epsilon$ is nothing but the Planck constant. *The main novel assumption* in this work is that the *function $\gamma$ vanishes at some instant $t_0$*.

Our goal is to investigate problem (1) under these new circumstances, from both the asymptotic analysis (when $\epsilon \to 0$) and the numerical approximation viewpoints. We emphasize that this situation is not covered by the standard theory of averaging and that recent numerical approaches by the same authors are ineffective. All techniques therein indeed rely fundamentally on the assumption that $\gamma(t) \geq \gamma_0$ uniformly in time, for some constant $\gamma_0 > 0$, and cannot be transposed to the context under consideration here

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## Symplectic non-squeezing and Hamiltonian PDE

Dimitrije Cicmilović
(joint work with Herbert Koch)

In [4], Gromov has formulated and proved what is now called symplectic non-sqeezing theorem in the finite-dimensional set-up. This paper has been seen as a revolution in symplectic topology and has inspired a lot of new research in this area. The result states

**Theorem 1.** *(Gromov, 1985) Let* $B_r := \{z \in \mathbb{C}^n : \|z\| \leq r\}$ *and* $\Sigma_R = \{z \in \mathbb{C}^n : |z_1| \leq R\}$. *Then there exists a symplectomorphism* $\varphi$ *such that* $\varphi(B_r) \subset \Sigma_R$ *if and only if* $r \leq R$.

The validity of such statement for infinite dimensional Hilbert spaces remains an open problem. Motivated by finite dimensional case, we call a PDE a **Hamiltonian PDE** if it can be rewritten as

$$\dot{u}(t) = J\nabla H(u(t)),$$

for an **almost complex structure** $J$ in a Hilbert symplectic space $(\mathbb{H}, \omega)$. Hamiltonian diffeomorphisms preserve the symplectic structure, which presents us with one invariant of a Hamiltonian PDE.

First results for the infinite dimensional case are due to Bourgain ([2]) for cubic NLS on tori, followed by Kuksin ([5]) for compact perturbations of linear flows, Colliander, Keel, Staffilani, Takaoka, Tao ([3]) for KdV, Killip, Visan and Zhang ([7],[8]) for cubic NLS on the line and plane, and others. However, these results proved the non-squeezing property of aforementioned flows and were obtained by going through a finite dimensional approximation of the Hamiltonian PDE while depending on the structure of the equation, and using Gromov's original result. Natural questions arise:

1) Does a symplectic diffeomorphism in infinite dimensions which is squeezing exist?
2) Does there exist a direct proof of non-squeezing in Hilbert space?

The only previous result on 2) is a recent paper by Abbondandolo and Majer ([1]), under restrictive geometric condition that the image of the ball is convex. Our aim is to obtain a direct proof of non-squeezing under other, more natural conditions. We want to point out that our result is not implied, nor implies [1].

Moreover, another interest in non-squeezing property of a Hamiltonian PDE is that it provides some qualitative information about its flow. For example, it forbids stable attractors and uniform evacuation of fixed frequency for an open set of initial data.

Finally, we would like to address the case of wave maps in 2 dimensions, as it is still not clear how to formulate symplectic non-squeezing.

We obtain the following result:

**Theorem 2.** *(Cicmilović, Koch) Let $\varphi : (\mathbb{H}, \omega) \to (\mathbb{H}, \omega)$ be a symplectomorphism such that $\varphi$ and $\varphi^{-1}$ are continuous with respect to weak topology on $\mathbb{H}$ and such that the map*

$$D\varphi : (\mathbb{H}, \tau_{weak}) \to (B(\mathbb{H}), \|\cdot\|_{op})$$

*is continuous. Then $\varphi(\mathbb{B}_r) \subset \Sigma_R$ if and only if $r \leq R$.*

The symplectomorphism $\varphi$ induces an almost complex structure on $\mathbb{H}$ given by

$$J = \varphi_* J_{st} = d\varphi \circ J_{st} \circ d\varphi^{-1}.$$

The analogue to the holomorphic discs are maps $u : (\mathbb{D}, i) \to (\mathbb{H}, J)$ such that

$$J \circ du = du \circ i,$$

and we call them **$J$-holomorphic discs**. The proof of the theorem is based on the existence of $J$-holomorphic disc such that its boundary lies in $\partial\Sigma_R$, has area equal to $R^2\pi$ and goes through the point $\varphi(0)$.

Then the inequality is obtained from the chain of (in)equalities

$$r^2\pi \leq \int_S \omega = \int_{\varphi(S)} \omega \leq R^2\pi.$$

First one follows from the monotonicity result of Lelong for holomorphic discs, the last one from the properties of the constructed disc and from integration over a smaller set, and the middle one from the preservation of symplectic structure under $\varphi$.

For establishing existence of previously mentioned disc, we follow the approach of Sukhov and Tumanov in [6]. More precisely, rewriting the $J$-holomorphic condition in complex variables gives the equation

$$\bar\partial u(z) = A_J(u(z))\overline{\partial u}(z). \tag{1}$$

which we call Beltrami type equation. We call $A_J$ a **complex representation** of $J$ and define it as unique $\mathbb{C}$-linear operator such that $A_J\bar h = (J_{st} + J)^{-1}(J_{st} - J)h$ holds for every $h \in \mathbb{H}$. The assumptions on the symplectomorphism allow us to

generalize finite dimensional approach in [6] and prove existence of the mentioned disc.

Currently, we are working on applications to mass subcritical and critical case for NLS.

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## Around self-similar solutions to the modified Korteweg-de Vries equation


Raphaël Côte

(joint work with Simão Correia and Luis Vega)


We are interested in the dynamics near self-similar solutions for the modified Korteweg-de Vries equation:

$$\text{(mKdV)} \qquad \partial_t u + \partial_{xxx}^3 u + \epsilon \partial_x (u^3) = 0, \quad u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}.$$

The signum $\epsilon \in \{\pm 1\}$ indicates whether the equation is focusing or defocusing; in our framework, it will play no major role.

The (mKdV) equation enjoys a scaling invariance: if $u$ is a solution then

$$u_\lambda(t,x) := \lambda^{1/3} u(\lambda t, \lambda^{1/3} x)$$

is also a solution to (mKdV). As a consequence, the self-similar solutions, which preserve their shape under scaling

$$S(t,x) = t^{-1/3} V(t^{-1/3} x)$$

are of special interest. Self-similar solutions play an important role for the (mKdV) flow: they exhibit an explicit blow up behavior, and are also related with the long time description of solutions. Even for small and smooth initial data, solutions display a modified scattering where self-similar solutions naturally appear: we

refer to Hayashi and Naumkin [7, 6], which was revisited by Germain, Pusateri and Rousset [4] and Harrop-Griffiths [5].

Self-similar solutions and the (mKdV) flow are also relevant as a model for the behavior of vortex filament in fluid dynamics. More precisely, Goldstein and Petrich proposed the following geometric flow for the description of the evolution of the boundary of a vortex patch in the plane under the Euler equations:

$$\partial_t z = -\partial_{sss} z + \partial_s \bar{z} (\partial_{ss} z)^2, \quad |\partial_s z|^2 = 1,$$

where $z = z(t, s)$ is complex valued and parametrize by its arctlength $s$ a plane curve which evolves in time $t$. A direct computation shows that its curvature solves the focusing (mKdV) (with $\epsilon = 1$), and self-similar solutions with initial data

$$U(t) \rightharpoonup c\delta_0 + \alpha \, \text{v.p.} \left( \frac{1}{x} \right) \quad \text{as } t \to 0^+, \quad \alpha, c \in \mathbb{R}, \tag{1}$$

correspond to logarithmic spirals making a corner. This kind of spirals are observed in a number of fluid dynamics phenomena (we refer to [9] for more details). We were also motivated by the works by Banica and Vega (see for example [1]) on nonlinear Schrödinger type equations.

Our goal in this project is to study the (mKdV) flow around self similar solutions. Our first result is the description of self-similar solutions in Fourier space.

**Theorem 1.** *Given $c, \alpha \in \mathbb{R}$ small enough, there exists unique $a \in \mathbb{R}$, $A, B \in \mathbb{C}$ and a self-similar solution $S(t, x) = t^{-1/3} V(t^{-1/3} x)$, where $V$ satisfies*

$$\text{for } p \geq 2, \quad e^{-itp^3} \hat{V}(p) = A e^{ia \ln |p|} + B \frac{e^{3ia \ln |p| - i\frac{8}{9}p^3}}{p^3} + z(p), \tag{2}$$

$$\text{for } |p| \leq 1, \quad e^{-itp^3} \hat{V}(p) = c + \frac{3i\alpha}{2\pi} \operatorname{sgn}(p) + z(p), \tag{3}$$

*where $z \in W^{1,\infty}(\mathbb{R})$, $z(0) = 0$ and for any $k < \frac{4}{7}$, $|z(p)| + |pz'(p)| = O(|p|^{-k})$ as $|p| \to +\infty$.*

Hence self-similar solutions exhibit logarithmic oscillations for large frequencies (which are related to the critical nature of the problem), and if $\alpha \neq 0$, a jump at frequency $p = 0$ (but no oscillations). We are also able to related the constants involved $(a, A, B)$ with those appearing in the description of $V$ in physical space (as in [3] for example).

The proof consists in writing the problem as a fixed point. We compute expansions for the first three Picard iterates, for the function and is derivative; and we are able to control the remainder term in weighted $L^\infty$ based spaces indicated. The techniques involve essentially stationary phase analysis with a careful control on the error.

Our second result is concerned with local well posedness in a critical space which contains the self-similar solutions constructed above. We work with the norm (for

space time functions):

$$\|u\|_{\mathscr{E}(I)} := \sup_{t\in I} \|u(t)\|_{\mathscr{E}(t)}, \tag{4}$$

$$\|v\|_{\mathscr{E}(t)} := \|\widehat{\mathcal{G}(-t)v}\|_{L^\infty(\mathbb{R})} + t^{-1/6}\|\partial_p \widehat{\mathcal{G}(-t)v}\|_{L^2((0,+\infty))}, \tag{5}$$

where $\mathcal{G}(t)$ denote the linear KdV group and $I \subset (0,+\infty)$ is a time interval. The above norm is scaling invariant, in the sense that $\|u_\lambda(t)\|_{\mathscr{E}(t)} = \|u(\lambda t)\|_{\mathscr{E}(\lambda t)}$; also it is finite for the self-similar solutions of Theorem 1. Our result is as follows.

**Theorem 2.** *Let $u_1 \in \mathscr{E}(1)$. Then there exist $T > 1$ and a solution $u \in \mathscr{E}([1/T, T])$ to* (mKdV) *such that $u(1) = u_1$.*

*Furthermore, one has forward uniqueness. More precisely, let $0 < t_0 < t_1$ and $u$ and $v$ be two solutions to* (mKdV) *such that $\hat{u}, \hat{v} \in \mathscr{C}([t_0, t_1], L^\infty)$ and*

$$\|u\|_{\mathscr{E}([t_0,t_1])}, \|v\|_{\mathscr{E}([t_0,t_1])} < +\infty.$$

*If $u(t_0) = v(t_0)$, then for all $t \in [t_0, t_1]$, $u(t) = v(t)$.*

For small data in $\mathscr{E}(1)$, the solution is actually defined for large times, and one can describe the asymptotic behavior. This is the content of our last main result.

**Theorem 3.** *There exists $\delta > 0$ small enough such that the following holds.*

*If $\|u_1\|_{\mathscr{E}(1)} \le \delta$, the corresponding solution satisfies $u \in \mathscr{E}([1,+\infty))$. Furthermore, let $S$ be the self-similar solution such that*

$$\hat{S}(1, 0^+) = \hat{u}_1(0^+) \in \mathbb{C}.$$

*Then $\|u(t) - S(t)\|_{L^\infty} \lesssim \|u_1\|_{\mathscr{E}(1)} t^{-5/6^-}$ and there exists a profile $U_\infty \in \mathscr{C}_b(\mathbb{R} \setminus \{0\}, \mathbb{C})$, where $|U_\infty(0^+)| = \lim_{p\to+\infty} |\hat{S}(1,p)|$ is well-defined, and*

$$\left| \tilde{u}(t,p) - U_\infty(p) \exp\left( \frac{i}{4\pi} |U_\infty(p)|^2 \log t \right) \right| \lesssim \frac{\delta}{\langle p^3 t \rangle^{\frac{1}{12}}} \|u_1\|_{\mathcal{E}(1)}.$$

As a consequence, one has the asymptotics in the physical space. In the setting of the above Theorem 3, if we let

$$y = \begin{cases} \sqrt{-x/3t}, & \text{if } x < 0, \\ 0, & \text{if } x > 0. \end{cases}$$

one has, for all $t \ge 1$ and $x \in \mathbb{R}$,

$$\left| u(t,x) - \frac{1}{t^{1/3}} \operatorname{Ai}\left( \frac{x}{t^{1/3}} \right) U_\infty(y) \exp\left( \frac{i}{6} |U_\infty(y)|^2 \log t \right) \right| \lesssim \frac{\delta}{t^{1/3} \langle x/t^{1/3} \rangle^{3/10}}.$$

The main challenge we faced in proving Theorem 2 is that we cannot work with smooth data, due to the jump at frequency 0. Also, the multiplier estimates suitable for the space $\mathcal{E}(I)$ require computations on a non linear solution, and are not amenable to a fixed point scheme. Therefore, we had to rely on the resolution of an approximate problem first, followed by a compactness argument. It turns out that the approximation has to obey several constraints, which could ultimately be met by following a Friedrichs scheme with a suitably twisted cut-off in frequency. The approximate problem is solved by fixed point, in a space where

smooth function are still not dense. The compactness argument is then fairly standard. Theorem 3 follows a similar path, the expansion as $t \to +\infty$ being merely a byproduct of the analysis.

For the forward uniqueness result of Theorem 2, we consider the variation of an localized $L^2$ quantity of the difference $w$ of two solutions. Our solutions do not belong to $L^2$, but we use the improved decay of functions in $\mathscr{E}(I)$ to make sense of it. If the cut-off is furthermore chosen to be non decreasing, we can make use of a monotonicity property to control the variations of this $L^2$ quantity and conclude via a Gronwall type argument. This kind of monotonicity property was first observed and used by Kato, and is a key feature in the study of the dynamics of solitons by Martel and Merle [8]. To our knowledge, it is however the first time that is used in the context of self-similar solutions.

We are now concerned with the behavior near blow up time $t = 0$: in particular, whether the self-similar blow up is stable, and the understanding of perturbations of self-similar solutions.

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## Global almost radial solutions to supercritical dispersive equations outside the unit ball

Piero D'Ancona

I consider a defocusing semilinear wave equation

$$\Box u + |u|^{p-1}u = 0, \qquad t \geq 0, \quad |x| > 1$$

on the exterior of the unit ball of $\mathbb{R}^n$, in dimension $n = 3$ or larger, with Dirichlet boundary conditions

In a first result, we prove that for (large) radial initial data a global bounded solution exists for arbitrary power $p$, and if a suitable weighted norm of the data is bounded, the solution decays at least like $t^{-1}$ as $t \to +\infty$. Higher Sobolev regularity of the initial data propagates and Sobolev norms do not inflate.

In a second result, we prove a pointwise dispersive estimate for solutions to the exterior problem for linear wave equation with a decaying potential depending on both time and space

$$\Box u + V(t,x)u = 0.$$

The potential is assumed to decay both in time and in space, at a sufficiently fast rate. This result is based on the methods of [3] and [2].

In the main result, we prove that the radial solution previously constructed is stable for small perturbations of the initial data in a weighted high Sobolev norm of order $O(n)$ of Christodoulou type (studied in [1]), and for sufficiently high powers $p > O(n)$. This produces a fanily of global large solutions to the supercritical wave equation on the exterior of the ball, with arbitrarily high power nonlinearity.

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## Splitting methods for highly oscillatory differential equations


Benjamin Dörich

(joint work with Marlis Hochbruck)


We consider the wave equation

$$q''(t) = \Delta q(t) + G(q(t)), \quad q(0) = q_0, \quad q'(0) = q_0'$$

on a bounded domain $\Omega \subset \mathbb{R}^n$ equipped with homogeneous Dirichlet boundary conditions in the space $L^2(\Omega) \times H^{-1}(\Omega)$ and we investigate its time integration by trigonometric integrators [2, 3, 4]. If $G : L^2(\Omega) \to L^2(\Omega)$ is bounded and if the solution $q$ only satisfies the finite energy condition

$$\|q(t)\|_{H^1}^2 + \|q'(t)\|_{L^2}^2 \leq K, \qquad t \in [0,T],$$

then standard analysis based on Taylor expansion (and thus requiring at least $q''$) is not applicable. In fact, in numerical experiments one can observe order reduction and resonances at certain time steps related to the frequencies in the system. This is due to a lack of regularity of the solution.

To circumvent these problems, filters depending on the step size and the stiff linear part of the equation can be used. For appropriate even functions $\phi, \psi$ satisfying in particular

$$\phi(0) = \psi(0) = 1, \qquad \phi(k\pi) = \psi(k\pi) = 0, \quad k = 1, 2, 3, \ldots,$$

one defines filters as

$$\widehat{\Phi} = \phi\left(\tau(-\Delta)^{1/2}\right), \qquad \widehat{\Psi} = \psi\left(\tau(-\Delta)^{1/2}\right).$$

Trigonometric integrators can then be written as splitting methods applied to an averaged equation, where the nonlinearity $G$ is replaced by

$$\widetilde{G}(q) = \widehat{\Psi} G(\widehat{\Phi} q).$$

In [2, 3, 4], it has been shown that these modified schemes are convergent of second order without a CFL condition. Roughly speaking these filters smooth the nonlinearity in such a way that resonances cancel out.

Recently, in [1], the relation between trigonometric integrators and splitting methods was used to prove error bounds by techniques developed for splitting methods. As our long term goal is the analysis of full discretizations of wave equations, we first want to generalize the methods and their error analysis to an abstract Hilbert space framework which fits to splitting methods and allows for an abstract understanding of the filters. This should allow to consider much more general nonlinearities $G$ as well as different classes of averaged methods.

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## Low-rank approximation of the Boltzmann equation with applications to fluid flow

Lukas Einkemmer

Fluids play a pivotal role in virtually all fields of science and engineering. The governing partial differential equations (PDEs) are the Navier–Stokes equations. These equations are usually discretized on a grid in order to obtain a numerical solution. Historically finite difference and finite volume methods have been used extensively, while in recent years discontinuous Galerkin schemes have become more common. However, especially in the study of turbulence by direct numerical simulation (DNS), spectral methods are often preferred.

This approach is very mature and has been extensively used in practice. However, there are also a number of disadvantages. For example, numerical schemes need to satisfy the CFL condition for sound waves (which in the weakly compressible setting can be multiple orders of magnitude faster than the flow speed), the equations are relatively complicated (which puts significant constraints on the design of numerical methods), and generating an appropriate mesh can be difficult.

Here we want to propose a different approach, that at first sight is unlikely to succeed. It is well known that instead of the Navier–Stokes equation we can also solve the Boltzmann equation

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = \frac{1}{\epsilon}(f^{\mathrm{eq}}(t,x,v) - f(t,x,v)),$$

where

$$f^{\mathrm{eq}}(t,x,v) \approx \frac{\rho(t,x)}{(2\pi)^{d/2}} \exp\left(-\tfrac{1}{2}(v-u(t,x))^2\right).$$

The moments of $f$

$$\rho(t,x) = \int f(t,x,v)\,\mathrm{d}v, \qquad \rho u(t,x) = \int v f(t,x,v)\,\mathrm{d}v$$

then satisfy the Navier–Stokes equations, where $\epsilon$ is essentially the viscosity of the fluid. That is, $\epsilon$ is usually a small parameter. Mathematically, this correspondence between the kinetic approach, i.e. solving the Boltzmann equation, and the fluid approach, i.e. solving the Navier–Stokes equations, can be shown by using a Chapman–Enskog expansion (see, for example, [1]).

However, kinetic equations, such as the Boltzmann equation, are extremely challenging from a numerical point of view. This is mainly due to the fact that the problem is posed in an up to six-dimensional phase space. Three dimensions in $x$ and three dimensions in $v$. Thus, even a very coarse space discretization has an immense overhead compared to methods that directly discretize the Navier–Stokes equations.

If the parameter $\epsilon$ is small, the collision operator (the right hand side of the Boltzmann equation) is strong. This implies that as we evolve the system in time $f$ stays close to the equilibrium distribution $f^{\mathrm{eq}}$. Physically this means that the solution stays close to thermodynamic equilibrium and consequently only a small part of the phase space is accessible.

This observation gives us the opportunity to construct a numerical scheme which can be implemented efficiently. To do that we consider the weakly compressible case. In this setting we can expand the equilibrium distribution in $u$ (the flow speed $u$ is a small parameter compared to the speed of sound)

$$\begin{aligned} &f^{\mathrm{eq}}(t,x,v) \\ &= \frac{\rho(t,x)}{(2\pi)^{d/2}} \exp\left(-\frac{v^2}{2}\right)\left(1 + v\cdot u(t,x) + \frac{(v\cdot u(t,x))^2}{2} - \frac{u(t,x)^2}{2}\right) + \mathcal{O}\left(u^3\right). \end{aligned}$$

This representation is in the form of a low-rank approximation with rank $r = 10$, which we exploit in the numerical algorithm. More specifically, we write the solution of the Boltzmann equation as follows

$$f(t, x, v) = \sum_{ij} X_i(t, x) S_{ij}(t) V_j(t, v),$$

where $S_{ij}$ is an $r \times r$ matrix. This formulation is on the continuous level, but once discretized in space it can be interpreted as a singular value decomposition of $f$. To make this into a viable numerical algorithm we have to derive equations of motions for the low-rank factors $X_i(t, x)$, $S_{ij}(t)$, and $V_j(t, v)$. This has been done for the matrix case in [11] and was later extended to kinetic problems in the context of computational plasma physics [9, 8, 10]. Such methods are interesting in plasma physics as solving the full six-dimensional Vlasov–Maxwell or Vlasov–Poisson equations is very expensive and requires significant HPC resources (see, for example, [16, 2, 3, 5, 6, 7]). The main idea of the dynamic low-rank approach is to project the equations of motion onto the manifold of low-rank functions with fixed rank. Unfortunately, this results in an ill-posed problem. However, by employing the projector splitting integrator introduced in the seminal paper [12] this can be remedied (let us also remark that that these techniques have been extended to the tensor case [14, 13, 15, 9]). We then obtain the following equations

**Subproblem 1:** (only depends on $x$ but not on $v$)

$$-\partial_t K_j = \sum_l c^1_{jl} \cdot \nabla_x K_l - \frac{1}{\epsilon} \left( K_j - c^3_j(K) \rho(K) \right) \qquad \text{for} \qquad K_j = \sum_i X_i S_{ij}.$$

**Subproblem 2:** ODE

$$\partial_t S_{ij} = \sum_{lk} (d^1_{il} \cdot c^1_{jk}) S_{lk} + \frac{1}{\epsilon} \left( S_{ij} - e_{ij}(S) \right).$$

**Subproblem 3:** (only depends on $v$ but not on $x$)

$$-\partial_t L_i = -\sum_l (d^1_{il} \cdot v) L_l - \tfrac{1}{\epsilon} \left( L_i - d^3_i(L) \right) \qquad \text{for} \qquad L_i = \sum_j S_{ij} V_j.$$

We note that a QR decomposition is required to obtain $X_i$ and $S_{ij}$ from $K_j$ (in subproblem 1) and to obtain $V_j$ and $S_{ij}$ from $L_i$ in subproblem 3. For more details on the coefficients and the algorithm we refer the reader to [4].

This formulation is numerically tractable since only three-dimensional problems have to be discretized. At first sight it might not be entirely clear why discretizing the equations of motion for the low-rank factors would be advantageous compared to the Navier–Stokes equations. However, despite their appearance these equations are much simpler than the Navier–Stokes equations. For example, the transport term is a linear constant coefficient advection. This greatly facilitates the use of upwind or spectral methods. In contrast, for the Navier–Stokes equations the transport is a system of highly nonlinear conservation laws. Moreover, the nonlinearity is completely local in space and can be computed pointwise. For

numerical results and a more detailed comparison we refer the reader to [4]. A further interesting property of this low-rank approach is that it enables us to incorporate certain kinetic effects in a fluid simulation. This would be advantageous for plasmas where classical magnetohydrodynamics is insufficient, but fully kinetic simulations are too expensive.

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## Linearized wave turbulence convergence results

Erwan Faou

We consider nonlinear three-waves dynamics of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}U_k(t) = i\omega_k U_k(t) + \varepsilon Q_k^N(U,U) + g_k(\delta, U) \tag{1}$$

where $k = (k_x, k_y) \in \mathcal{D}_N := \mathcal{D} \cap \mathbb{Z}^2/N$ where $\mathcal{D}$ is a bounded set of $\mathbb{R}^2$, and where $U_k(t)$ are typically renormalized Fourier coefficients of a sequence of $N$-periodic lattice points $U(t, j, \ell)$, $(j, \ell) \in \mathbb{Z}^2$, or of a function $U(t, x, y)$ defined on a large torus or size $N$. The nonlinearity $Q^N$ is quadratic of convolution type and symplectic in some Hamiltonian variables. The frequency vector $\omega_k$ is the trace of a smooth real function on the discrete set $\mathcal{D}_N$ and is skew symmetric with respect to $k$. The term $g_k(\delta, U)$ models a random forcing in the angles of the complex coefficients $U_k(t)$, of strength $\delta$. We moreover make the hypothesis that (1) preserve the $L^2$ norm of the $U_k$, which guarantees the existence of an invariant measure corresponding to the equirepartition of energy amongst all the modes.

Such models can be derived from nonlinear lattice equations like Kadomtsev–Petviashvili (KP) lattices, finite differences or spectral approximations of real quadratic semilinear real wave equation, or real Hamiltonian partial differential equation set of a large periodic domain of size $N$, with regularized nonlinearity truncated in frequency. To fix the ideas, we assume that the linear frequencies are given by the formula

$$\omega_k = k_x^3 + \eta k_y^2 k_x^{-1}, \tag{2}$$

corresponding to the frequency of the continuous KP equation. The parameter $\eta > 0$ is given. Note that in any case, when $N \to \infty$, (1) becomes a dispersive equation with continuous spectrum in the linear operator.

As explained in the book of Nazarenko (see [7]) such equations are universal models for two-dimensional real nonlinear waves. In the present paper, we consider the (1) in the framework of wave turbulence theory, see [8], [9], [6], [5], [3], [7], which means that we consider random initial data, and are interested in the statistical description of the solution over long times. The external forces $g_k(\delta, U)$ models the classical *Random Phase Assumption* invoked in wave turbulence theory by a stochastic forcing in the angles of the complex coefficients $U_k(t)$.

We will consider asymptotic regimes with respect to the following three parameters:

- Continuous limit in the frequency set, *i.e.* $N \to \infty$. The model thus degenerates to a dispersive equation.
- Small nonlinearity, *i.e.* $\varepsilon \to 0$, the system is thus weakly non linear.
- Small noise: The random phase forcing is driven by independent Brownian motions in the angles of the coefficients $U_k(t)$ with variance $\delta \to 0$.

Wave turbulence theory predicts that in a some asymptotic regimes with respect to these three parameters, then the expectations $\mathbb{E}\,|U_k(t)|^2$, $k = (k_x, k_y) \in \mathcal{D} \cap$

$\mathbb{Z}^2/N$ are well approximated in some time scale by $r(t,k)$, $k \in \mathcal{D} \subset \mathbb{R}^2$ the solution of a *wave kinetic* equation of the form

$$\partial_t r(t,k) = \int_{\substack{k=j+m \\ \omega_k=\omega_j+\omega_m}} \mathcal{Q}^k_{mj}\, r(t,j)\, r(t,m)\, \mathrm{d}\Sigma(j,m), \tag{3}$$

where $\mathcal{Q}^k_{mj}$ are real coefficients. The measure $\mathrm{d}\Sigma(j,m)$ is defined as the micro-canonical measure on the resonant manifold. This kinetic equation of Boltzmann type possesses stationary solutions typically of the form $|k|^{-\alpha}$ describing Kolmogorov spectra. Amongst these solutions, Rayleigh-Jeans spectra are specific stationary solutions corresponding to equirepartition of the energy according to the invariant quantities of (1). In the case of the $L^2$ norm, which is an invariant of (1), it corresponds to constant stationary solutions $r(t,k) = \sigma^2$ of (3), for some $\sigma > 0$, which is well defined as the set of frequencies considered is here bounded.

In our setting, for all given set of parameter $(N, \varepsilon, \delta)$, these particular solutions correspond to Gaussian invariant measures associated with the $L^2$ norm of the equation. If $U_k = P_k + iQ_k$ with $P_k$ and $Q_k$ real random variables, a sampling of these measures can be done by drawing all random variables $P_k$ and $Q_k$ according to the same normal law $\mathcal{N}(0, \sigma^2/2)$ for some $\sigma > 0$. For such an equirepartited initial data, the law of the solution $U_k(t)$ is stationary, and the momenta $\mathbb{E}\,|U_k(t)|^2 = \sigma^2$ are constant. We consider random initial data with small perturbations of the variance $\sigma^2$ of order $\mathcal{O}(1/N^\alpha)$, $\alpha \geq 1$. Each of the modes $P_k(0)$ and $Q_k(0)$ are drawn with respect to normal laws with variance $\frac{\sigma^2}{2} + \frac{1}{2N^\alpha} g_0(k)$ where $g_0(k)$ are functions depending smoothly on $k$. We thus have

$$\mathbb{E}|U_k(0)|^2 = \sigma^2 + \frac{g_0(k)}{N^\alpha}, \quad \alpha \geq 1. \tag{4}$$

where $\eta > 0$ is a given parameter. Under these assumptions, the main result proved in [1] can be stated as follows. We can identify the limit of the renormalized fluctuations,

$$\forall\, t \in [0,T], \qquad \lim_{\delta \to 0} \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} N^\alpha \Big( \mathbb{E}\,|U_k(\frac{t}{\varepsilon^2})|^2 - \sigma^2 \Big) = f(t,k) \quad \text{weakly}, \tag{5}$$

and prove that $f(t,k)$, $k \in \mathbb{R}^2$ satisfies the linearized kinetic equation on $[0,T]$

$$\partial_t f(t,k) = \sigma^2 \int_{\substack{k=j+m \\ \omega_k=\omega_j+\omega_m}} \mathcal{Q}^k_{mj} \big( f(t,m) + f(t,j) \big) \mathrm{d}\Sigma(j,m), \tag{6}$$

with $f(0,m) = g_0(m)$, coming from the linearization of (3) around the stationary solution $r(t,k) = \sigma^2$. Weak convergence here means that we have to consider local averages in $k$ to obtain strong convergence of *coarse-grained* quantities at a larger scale that the fine grid scale $1/N$. A precise statement, expressed in Hamiltonian variables, can be found in [1].

The second result in [1] shows that the role of the noise is crucial to obtain the kinetic equation with the same random initial data. In *generic* situations, *i.e.* for

almost all $\eta > 0$ (see (2)) we have for fixed $t \in [0, T]$,

$$\lim_{N\to\infty} \lim_{\varepsilon\to 0} \lim_{\delta\to 0} N^{\alpha} \Big( \mathbb{E}\, |U_k(\frac{t}{\varepsilon^2})|^2 - \sigma^2 \Big) = g_0(k) \tag{7}$$

contradicting (5) in this specific asymptotic regime. We can interpret this result as the fact that in the absence of random forcing ($\delta = 0$) and in the regime where $\varepsilon \to 0$ first and then $N \to \infty$, the system (1) admits generically a weak version of Birkhoff normal form reduction preserving the actions over long times. It can be explained by the fact that the discrete frequencies (2) which are given as traces of a continuous function on the discrete grid are generically non resonant for almost all $\eta$, with a control of the small denominators depending on $N$. We thus see that in this regime, the random forcing is crucial to obtain a kinetic representation of the dynamics.

As mentioned above, the noise is crucial in the emergence of the kinetic equation. Since the random phase forcing is acting only on the phase of the Fourier coefficients, the system has a degenerate noise, and can be considered as a Langevin system with small noise in large dimension. We can interpret the kinetic equation as an effect of the hypoellipticity of the system which is due to the presence of the nonlinearity.

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## In Search of a Nonlinear Convergence Proof for Waveform Relaxation

Martin J. Gander

I have shown for in my presentation a detailed nonlinear convergence proof for the parareal algorithm which has already appeared in [3], so I show here a nonlinear convergence proof for a related Waveform Relaxation (WR) method, for which various convergence results are announced in the literature, but without proof. WR methods were invented in the research laboratory of IBM in Yorktown Heights in

1982 for VLSI design by Lelarasmee, Ruehli and Sangiovanni-Vincentelli [6]. WR is a time parallel solution method for systems of ordinary differential equations of the form

$$\partial_t \mathbf{u}(t) = \mathbf{f}(t, \mathbf{u}(t)), \quad t \in (0, T], \qquad \mathbf{u}(0) = \mathbf{u}_0 \in \mathbb{R}^d, \tag{1}$$

see [4] for an overview of four different classes of time parallel time integration methods. To formulate a waveform relaxation algorithm for (1), one needs a partition function $\tilde{\mathbf{f}}(t, \mathbf{v}, \mathbf{w})$ such that

$$\tilde{\mathbf{f}}(t, \mathbf{v}, \mathbf{v}) = \mathbf{f}(t, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{R}^d,\ t \in (0, T]. \tag{2}$$

For an initial guess $\mathbf{u}^0(t)$ of the solution $\mathbf{u}(t)$ of (1), the waveform relaxation algorithm associated with the partition function (2) computes then for $k = 0, 1, 2, \ldots$

$$\partial_t \mathbf{u}^{k+1}(t) = \tilde{\mathbf{f}}(t, \mathbf{u}^{k+1}(t), \mathbf{u}^k(t)), \quad t \in (0, T], \qquad \mathbf{u}^{k+1}(0) = \mathbf{u}_0. \tag{3}$$

It is well known in the waveform relaxation community that the convergence of (3) is similar to the convergence of the Picard iteration [9],

$$\partial_t \mathbf{u}^{k+1}(t) = \mathbf{f}(t, \mathbf{u}^k(t)), \quad t \in (0, T], \qquad \mathbf{u}^{k+1}(0) = \mathbf{u}_0. \tag{4}$$

which was proved by Lindelöf [7] to be superlinear,

$$||\mathbf{u} - \mathbf{u}^k||_T \leq \frac{(LT)^k}{k!} ||\mathbf{u} - \mathbf{u}^0||_T, \tag{5}$$

where $L$ is the Lipschitz constant of $f$ and $||\mathbf{u}||_T := \max_{0 \leq t \leq T} ||\mathbf{u}(t)||$ denotes the maximum norm in $[0, T]$. It is however not easy to find a proof of this result in the literature: in [5, page 178] a convergence estimate of the form (5) is quoted under mild assumptions on the splitting function, with references to [1, 2, 8] and the original paper by Lindelöf [7]. The reference [2, equation (3.6)] contains the same result with the comment 'it is easy to prove'. A different result with an extra exponential term is shown in [1, Theorem 2.1], with the comment that 'by some standard analysis, we can easily get the convergence result'. An estimate of the form (5) is again given in [8, page 333, equation (2.13)] but for the case of linear problems only using iterated kernel estimates, and this is also indicated as the key reference for the result in the book [10, Theorem 2.5.4, see also the page just before], but again without proof. We show now how the result announced in [1, Theorem 2.1] with the extra exponential term can be proved. To do so, we need two lemmas: the first one is the well known Gronwall Lemma in the integral form:

**Lemma 1** (Gronwall Lemma (1919))**.** *Let $u(t)$, $\alpha(t)$ and $\beta(t)$ be continuous functions on $[0, T]$. If $\beta(t) \geq 0$ and*

$$u(t) \leq \alpha(t) + \int_0^t \beta(s) u(s) ds \quad \forall t \in [0, T],$$

*then*

$$u(t) \leq \alpha(t) + \int_0^t \alpha(s) \beta(s) e^{\int_s^t \beta(\tau) d\tau} ds \quad \forall t \in [0, T].$$

The next lemma is a technical lemma which will be necessary to prove the convergence estimate and illustrates how sharp one has to estimate terms in order to obtain the result announced in [1, Theorem 2.1] without proof.

**Lemma 2.** *Let* $I_0(t) := \int_0^t e^{L_1 s} ds = \frac{1}{L_1}(e^{L_1 t} - 1)$ *and*

$$I_k(t) = \frac{1}{L_1} e^{L_1 t} \frac{(L_2 t)^k}{k!} - \frac{L_2}{L_1} I_{k-1}(t) \tag{6}$$

*for* $k = 1, 2, \ldots$, *where* $L_1$ *and* $L_2$ *are two positive constants. Then*

$$L_2 I_k(t) + L_1 L_2 \int_0^t I_k(s) e^{L_1 (t-s)} ds = \frac{(L_2 t)^{k+1}}{(k+1)!} e^{L_1 t}. \tag{7}$$

*Proof.* The proof is by induction: for $k = 0$, we insert $I_0(t)$ into (7) on the left and compute

$$L_2 I_0(t) + L_1 L_2 \int_0^t I_0(s) e^{L_1 (t-s)} ds = \frac{L_2}{L_1}(e^{L_1 t} - 1) + L_2 \int_0^t (e^{L_1 s} - 1) e^{L_1 (t-s)} ds. \tag{8}$$

Now we integrate the remaining integral on the right,

$$\int_0^t (e^{L_1 s} - 1) e^{L_1 (t-s)} ds = e^{L_1 t} \int_0^t (1 - e^{-L_1 s}) ds = e^{L_1 t} \left( t + \frac{1}{L_1}(e^{-L_1 t} - 1) \right).$$

The equation (8) then gives $\frac{L_2}{L_1} e^{L_1 t} - \frac{L_2}{L_1} + L_2 e^{L_1 t} \left( t + \frac{1}{L_1}(e^{-L_1 t} - 1) \right) = L_2 t e^{L_1 t}$, which is the right hand side in (7) for $k = 0$. We thus assume that (7) holds for $k - 1$ and show that it still holds for $k$: using (6), we obtain from (7)

$$\begin{aligned}
& L_2 I_k(t) + L_1 L_2 \int_0^t I_k(s) e^{L_1 (t-s)} ds \\
=& L_2 \left( \frac{1}{L_1} e^{L_1 t} \frac{(L_2 t)^k}{k!} - \frac{L_2}{L_1} I_{k-1}(t) \right) \\
& + L_2 \int_0^t \left( e^{L_1 s} \frac{(L_2 s)^k}{k!} - L_2 I_{k-1}(s) \right) e^{L_1 (t-s)} ds \\
=& \frac{L_2}{L_1} e^{L_1 t} \frac{(L_2 t)^k}{k!} + L_2 e^{L_1 t} \int_0^t \frac{(L_2 s)^k}{k!} ds \\
& - \frac{L_2}{L_1} \left( L_2 I_{k-1}(t) + L_1 L_2 \int_0^t I_{k-1}(s) e^{L_1 (t-s)} ds \right).
\end{aligned}$$

Now the last term in parentheses is by induction hypothesis equal to $\frac{(L_2 t)^k}{k!} e^{L_1 t}$ and thus cancels with the first term, and integrating the remaining middle term, we find

$$L_2 e^{L_1 t} \int_0^t \frac{(L_2 s)^k}{k!} ds = L_2 e^{L_1 t} \frac{(L_2 t)^{k+1}}{(k+1)!} ds,$$

which concludes the proof. □

We are now ready to prove the superlinear convergence result of the waveform relaxation algorithm (3).

**Theorem 3.** *If the partition function* $\tilde{\mathbf{f}}(t,\mathbf{v},\mathbf{w})$ *in* (2) *is Lipschitz continuous in both arguments uniformly for all* $t \in [0,T]$,

$$
\begin{array}{rcl}
||\tilde{\mathbf{f}}(t,\mathbf{v}_1,\mathbf{w}) - \tilde{\mathbf{f}}(t,\mathbf{v}_2,\mathbf{w})|| & \leq & L_1||\mathbf{v}_1 - \mathbf{v}_2||, \\
||\tilde{\mathbf{f}}(t,\mathbf{v},\mathbf{w}_1) - \tilde{\mathbf{f}}(t,\mathbf{v},\mathbf{w}_2)|| & \leq & L_2||\mathbf{w}_1 - \mathbf{w}_2||,
\end{array} \tag{9}
$$

*then the waveform relaxation algorithm* (3) *satisfies the error estimate*

$$
||\mathbf{u} - \mathbf{u}^k||_T \leq e^{L_1 T}\frac{(L_2 T)^k}{k!}||\mathbf{u} - \mathbf{u}^0||_T. \tag{10}
$$

*Proof.* We start by subtracting the integral form of the waveform relaxation iteration (3) from the integral form of the problem (1),

$$
\mathbf{u}(t) - \mathbf{u}^k(t) = \int_0^t \mathbf{f}(s,\mathbf{u}(s)) - \tilde{\mathbf{f}}(s,\mathbf{u}^k(s),\mathbf{u}^{k-1}(s))ds.
$$

Now adding and subtracting the term $\tilde{\mathbf{f}}(s,\mathbf{u}^k(s),\mathbf{u}(s))$ under the integral, and using that the partition function satisfies $\mathbf{f}(s,\mathbf{u}(s)) = \tilde{\mathbf{f}}(s,\mathbf{u}(s),\mathbf{u}(s))$, we get

$$
\begin{array}{rcl}
\mathbf{u}(t) - \mathbf{u}^k(t) & = & \displaystyle\int_0^t \tilde{\mathbf{f}}(s,\mathbf{u}(s),\mathbf{u}(s)) - \tilde{\mathbf{f}}(s,\mathbf{u}^k(s),\mathbf{u}(s))ds \\
& + & \displaystyle\int_0^t \tilde{\mathbf{f}}(s,\mathbf{u}^k(s),\mathbf{u}(s)) - \tilde{\mathbf{f}}(s,\mathbf{u}^k(s),\mathbf{u}^{k-1}(s))ds.
\end{array}
$$

We take the norm on both sides and use the Lipschitz conditions (9) to obtain

$$
||\mathbf{u}(t) - \mathbf{u}^k(t)|| \leq L_1\int_0^t ||\mathbf{u}(s) - \mathbf{u}^k(s)||ds + L_2\int_0^t ||\mathbf{u}(s) - \mathbf{u}^{k-1}(s)||ds.
$$

Setting $\beta(t) := L_1$ and $\alpha(t) := L_2\int_0^t ||\mathbf{u}(s) - \mathbf{u}^{k-1}(s)||ds$, we can apply the Gronwall Lemma 1 and get

$$
||\mathbf{u}(t) - \mathbf{u}^k(t)|| \leq L_2\int_0^t ||\mathbf{u}(s) - \mathbf{u}^{k-1}(s)||ds \tag{11}
$$

$$
+ L_1 L_2\int_0^t\int_0^s ||\mathbf{u}(\tau) - \mathbf{u}^{k-1}(\tau)||d\tau e^{L_1(t-s)}ds. \tag{12}
$$

We now want to show by induction on $k$ that the bound (10) holds. For $k = 0$, the bound (10) clearly holds, so assuming that (10) holds for $k-1$, we now show that it also holds for $k$: inserting the induction hypothesis into (11), we obtain

$$
\begin{array}{rcl}
||\mathbf{u}(t) - \mathbf{u}^k(t)|| & \leq & \displaystyle L_2\int_0^t e^{L_1 s}\frac{(L_2 s)^{k-1}}{(k-1)!}||\mathbf{u} - \mathbf{u}^0||_s ds \\
& + & \displaystyle L_1 L_2\int_0^t\int_0^s e^{L_1\tau}\frac{(L_2\tau)^{k-1}}{(k-1)!}||\mathbf{u} - \mathbf{u}^0||_\tau d\tau e^{L_1(t-s)}ds.
\end{array} \tag{13}
$$

Now if we estimate the first integral term on the right by

$$
\int_0^t e^{L_1 s}\frac{(L_2 s)^{k-1}}{(k-1)!}||\mathbf{u} - \mathbf{u}^0||_s ds \leq \int_0^t e^{L_1 s}\frac{(L_2 s)^{k-1}}{(k-1)!}ds||\mathbf{u} - \mathbf{u}^0||_t,
$$

we can integrate the remaining integral by parts,

$$\int_0^t e^{L_1 s}\frac{(L_2 s)^{k-1}}{(k-1)!}ds = \frac{1}{L_1}e^{L_1 t}\frac{(L_2 t)^{k-1}}{(k-1)!} - \frac{1}{L_1}\int_0^t e^{L_1 s}\frac{(L_2 s)^{k-2}}{(k-2)!}ds,$$

and setting $I_{k-1}(t) := \int_0^t e^{L_1 s}\frac{(L_2 s)^{k-1}}{(k-1)!}ds$ we obtain the recurrence relation from the technical Lemma 2,

$$I_{k-1}(t) = \frac{1}{L_1}e^{L_1 t}\frac{(L_2 t)^{k-1}}{(k-1)!} - \frac{L_2}{L_1}I_{k-2}(t).$$

Noticing that $I_{k-1}(s)$ also appears in (13) in the second term on the right under the integral in $s$, we obtain the bound

$$||\mathbf{u}(t) - \mathbf{u}^k(t)|| \leq \left(L_2 I_{k-1}(t) + L_1 L_2 \int_0^t I_{k-1}(s)e^{L_1(t-s)}ds\right)||\mathbf{u} - \mathbf{u}^0||_t.$$

Now the value in the parentheses is precisely estimated in Lemma 2 to be $\frac{(L_2 t)^k}{k!}e^{L_1 t}$, which concludes the proof. □

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## Rational normal forms : The KillBill theory

Benoît Grébert

With Joackim Bernier (Univ Rennes) and Erwan Faou (Univ Rennes) we are interested in the long time behavior of small amplitude solutions of non-linear Hamiltonian partial differential equations on bounded domains. In this context, the competition between non-linear effects and energy conservation (typically the $H^1$ Sobolev norm) makes the problem intricate. One of the main issues is the control of higher order Sobolev norms of solutions for which typically a priori upper bounds are polynomials (see [Bou96a]).
Bambusi & Grébert have shown in [BG06] that, in a fairly general semi linear PDE framework, if an appropriate non-resonance condition is imposed on the linear part then the solution $u$ of the corresponding PDE satisfy a strong stability property:

$$(\star) \qquad \text{if } \|u(0)\|_{H^s} \leq \varepsilon \quad \text{then} \quad \|u(t)\|_{H^s} \leq 2\varepsilon \quad \text{for all} \quad t \leq \varepsilon^{-M},$$

where $\|\cdot\|_{H^s}$ denotes the Sobolev norm of order $s$, $M$ can be chosen arbitrarily large and $\varepsilon$ is supposed to be small enough, $\varepsilon < \varepsilon_0(M, s)$. The method of proof is based on the construction of Birkhoff normal forms. To verify the appropriate non-resonance condition external parameters were used –such as a mass in the case of nonlinear wave equation– and the stability result were obtained for almost every value of these parameters. Then this technic was applied to prove almost global existence results for a lot of semi linear Hamiltonian PDEs (see [Gre07, BDGS07, GIP09]). However, the case of a non-linear perturbation of a fully resonant linear PDE was not achievable by this technique. Actually for the cubic nonlinear Schrödinger equation on the two dimensional torus it is proved in [CKSTT10] that the high Sobolev norms may growth arbitrarily for some special initial data. Even in one dimension of space, it is proved in [GT12] that the quintic nonlinear Schrödinger equation on the circle does not satisfy $(\star)$.

Now consider the nonlinear Schrödinger equation:

$$\text{(NLS)} \qquad iu_t = -\Delta u + \varphi(|u|^2)u, \quad x \in \mathbb{T}, \quad t \in \mathbb{R},$$

where $\varphi = \mathbb{R} \to \mathbb{R}$ is an analytic function on a neighborhood of the origin satisfying $\varphi(0) = m$ is the mass possibly 0 and $\varphi'(0) \neq 0$. Equation (NLS) is a Hamiltonian system associated with the Hamiltonian function

$$H_{\text{NLS}}(u, \bar{u}) = \frac{1}{2\pi} \int_{\mathbb{T}} \left( |\nabla u|^2 + g(|u|^2) \right) \mathrm{d}x,$$

where $g(t) = \int_0^t \varphi$, and the complex symplectic structure $i\mathrm{d}u \wedge \mathrm{d}\bar{u}$.
This is an example of fully resonant Hamiltonian PDE, as the linear frequencies are $j^2 \in \mathbb{N}$ for $j \in \mathbb{Z}$. Nevertheless in [KP96] Kuksin & Pöschel proved for such equation the persistence of finite dimensional KAM tori, a result that requires a strong non resonant property on the unperturbed Hamiltonian. Actually they considered the cubic term as part of the unperturbed Hamiltonian to modulate the resonant linear frequencies and to avoid the problem of resonances. Roughly speaking the nonlinear term generates stability. Then Bourgain in [Bou00] used

the same idea to prove that for many random small initial data the solution of (NLS) satisfies ($\star$). Although the method of proof is based on normal forms, the effective construction of the normal form depends on the initial datum in a very intricated way and actually the author does not obtain a Birkhoff normal form result for (NLS) but rather a way to break down the solution that allows him to obtain the property ($\star$).

In this work we want to construct a new type of normal form, not based on polynomial functions but on rational functions, transforming the Hamiltonian of (NLS) into an integrable one up to a small remainder, over large open sets surrounding the origin. Then stability of higher order Sobolev norms during very long time is just one of the dynamical consequences. We stress out that since our rational normal form is built on open sets, the dynamical consequences remain stable with respect to the initial datum. In particular the property ($\star$), although not verified on all a neighborhood of the origin, is locally uniform with respect to $u(0)$ in $H^s$ .

To describe our result let us introduce some notations. With a given function $u(x) \in L^2$, we associate the Fourier coefficients $(u_a)_{a\in\mathbb{Z}} \in \ell^2$ defined by

$$u_a = \frac{1}{2\pi}\int_{\mathbb{T}} u(x)e^{-iax}\mathrm{d}x.$$

In the remainder of the paper we identify the function with its sequence of Fourier coefficients $u = (u_a)_{a\in\mathbb{Z}}$ and we consider the spaces

$$\ell^1_s = \{\, u = (u_a)_{a\in\mathbb{Z}} \in \ell^2 \quad | \quad \|u\|_s := \sum_{a\in\mathbb{Z}} \langle a\rangle^s |u_a| < +\infty \,\},$$

where $\langle a\rangle = \sqrt{1+a^2}$.

Our results are divided into two parts :

• **Abstract rational normal forms** We construct a canonical transformation $\tau$ defined on an open set $\mathcal{V}_\varepsilon \subset \ell^1_s$ included in the ball of radius $\varepsilon$ centered at 0 that puts the Hamiltonian of (NLS) in normal form up to order $2r$: $H \circ \tau = Z(I) + R$ where $Z$ depends only on the actions $I = (I_a)_{a\in\mathbb{Z}}$ with $I_a = |u_a|^2$, and $R = O(\varepsilon^{2r+1})$.

Of course the open set $\mathcal{V}_\varepsilon$ is defined in a rather complex way through non-resonant relationships between actions $|u_a|^2$ and $\varepsilon$. In particular it does not contain $u \equiv 0$ which is too resonant. However, these sets are invariant by angular rotation in the sense that

$$\forall\, (\theta_a)_{a\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}, \quad u = (u_a)_{a\in\mathbb{Z}} \in \mathcal{V}_\varepsilon \Longrightarrow (e^{i\theta_a}u_a)_{a\in\mathbb{Z}} \in \mathcal{V}_\varepsilon.$$

It is then necessary to show that the flow travels within these open sets. This is achieved in a second step.

• **Generic almost preservation of the actions over very long time** For a given $\varepsilon > 0$, we set

$$u(0,x) = u^0_\varepsilon(x) = c\varepsilon \sum_{a\in\mathbb{Z}} \sqrt{I_a}e^{iax}, \tag{1}$$

where $I = (I_a)_{a\in\mathbb{Z}}$ are random variables with support included in the interval $(0, \langle a\rangle^{-2s+4})$, so that $u^0_\varepsilon$ belong to the space $\ell^1_s$ and $c = (2\pi)^{-1}\tanh\pi$ is a normalizing constant to ensure $\|u^0_\varepsilon\|_{\ell^1_s} < \varepsilon/2$ almost surely. We prove that under some assumptions on the law of $I_a$, then for essentially almost all couple $(\varepsilon, I)$ and $\varepsilon$ small enough, the initial values $u^0_\varepsilon$ of the form (1) are in the domain of definition of the normal forms, and thus have a dynamics that is essentially an integrable one over very long time. This implies the almost preservation of the actions $|u_a(t)|^2$ over times of order $\varepsilon^{-M}$ with $M$ arbitrary which in turn implies that the solution remains inside the open set $\mathcal{V}_\varepsilon$ . In particular we deduce the almost preservation of the Sobolev norm of the solution over times of order $\varepsilon^{-M}$, i.e. property $(\star)$.

As previously mentioned, the possibility of obtaining normal forms without the help of external parameters was already known in the KAM theory (see [KP96] and also [EGK16]). However these normal forms were constructed around finite dimensional tori. The originality of our analysis is that we work with truly infinite dimensional objects.

Finally let us mention two recent results that open new directions in the world of Birkhoff normal forms. In [BD] Berti-Delors have considered recently Birkhoff normal forms for a quasi linear PDE, namely the capillarity-gravity water waves equation, and thus faced unbounded nonlinearity. In this paper capillarity plays the role of the external parameter. Also in [BMP] Biasco-Masseti-Procesi, considering a suitable Diophantine condition, prove exponential stability in Sobolev norm for parameter dependent NLS on the circle.

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## Wave Maps and Bilinear Restriction

Sebastian Herr
(joint work with Timothy Candy)

In the first section, the initial value problem for the Wave Maps equation is discussed. The focus here is on the solution of the division problem, i.e. Tataru's [5] small data global well-posedness and scattering result for small initial data at the critical regularity. In the second part, the classical Fourier restriction problem is reviewed, closing with Tao's [3] endpoint estimate for the cone. In the third section, an alternative approach [1] to the division problem is described, which is based on recent extensions of bilinear (dual) Fourier restriction estimates to certain atomic function spaces [2].

### 1. Wave Maps

Wave Maps are critical points $\phi : \mathbb{R}^{1+n} \to M$ of the Lagrangian

$$\mathcal{L}(\phi) = \int_{\mathbb{R}^{1+n}} \langle \partial^\alpha, \partial_\alpha \phi \rangle_g,$$

for a Riemannian manifold $(M, g)$. Here, $\mathbb{R}^{1+n}$ is endowed with the Minkowski metric. In case the target manifold is the $2-$sphere, i.e. $M = S^2 \subset \mathbb{R}^3$, one obtains the nonlinear wave equation

$$\Box u = u(|\nabla u|^2 - |\partial_t u|^2) \tag{1}$$

for $u : \mathbb{R}^{1+n} \to \mathbb{R}^3$ and one may drop the target constraint. Now, consider the initial value problem associated with (1). The scaling critical homogeneous Sobolev space for this equation is $\dot{H}^{\frac{n}{2}}(\mathbb{R}^n)$. The global well-posedness and scattering problem has been first solved in [5] for small initial data in the (slightly smaller) scale invariant Besov space $\dot{B}^{\frac{n}{2}}_{2,1}(\mathbb{R}^n)$. This is referred to as the solution to the *division problem*, because the parametrix $\Box^{-1}$ for the wave equation is essentially the division by the symbol $|\xi|^2 - \tau^2$, which fails to be locally integrable. Then, this result [5] has been extended to small data in $\dot{H}^{\frac{n}{2}}(\mathbb{R}^n)$ in [4] by a difficult renormalization procedure. Later, the theory has been extended to other targets and initial data with energy below the threshold given by harmonic maps (in case there exist any).

Due to the null-structure

$$\partial_t u \cdot \partial_t v - \nabla u \cdot \nabla v = \tfrac{1}{2}(\Box(u \cdot v) - (\Box u) \cdot v - u \cdot (\Box v)),$$

the key difficulty in the solution of the division problem in [5] is the construction of an algebra $S$ of space-time distributions such that the estimate

$$\|\Box^{-1}(u\Box v)\|_S \lesssim \|u\|_S\|v\|_S$$

holds true and $S$ contains solutions to the homogeneous wave equation with initial data in the critical Besov space. For $u, v$ localized to the unit frequency scale, an important ingredient is to prove an estimate of the form

$$\|uv\|_{L^2(\mathbb{R}^{1+n})} \lesssim \|u\|_S\|v\|_{S^w},$$

where the $S^w$-norm can be controlled by a duality argument, and is slightly weaker than the $S$-norm. The spaces $S$ and $S^w$ constructed in [5] are difficult to explain in detail, but to some extend one can think of a typical element of $S$ as a superposition of travelling waves, i.e.

$$u(t,x) = \int_{S^{n-1}} \varphi_\omega(x+t\omega)d\sigma(\omega).$$

Now, if the directions $\omega_1$ and $\omega_2$ of two travelling waves $u$ and $v$ are transversal, one can prove an $L^2$-estimate for the product of the two rather easily. However, elements of $S^w$ are not close to being superpositions of traveling waves and a more sophisticated construction is required [5].

## 2. Fourier Restriction

The Fourier restriction operator $\mathcal{R}$ maps a Schwartz function $f : \mathbb{R}^{1+n} \to \mathbb{C}$ to the restriction $\widehat{f}|_C$ of its Fourier transform $\widehat{f}$ to some submanifold $C$ of $\mathbb{R}^{1+n}$. The Fourier restriction problem is to prove estimates of the form

$$\|\mathcal{R}f\|_{L^p(C)} \lesssim \|f\|_{L^q(\mathbb{R}^{1+n})} \tag{2}$$

for the optimal range for $p, q$, depending on the submanifold $C$. The dual operator $\mathcal{E}$ takes a function $g$ defined on $C$ and maps it to the inverse Fourier transform of this distribution. Note that if $C$ is the cone, then $u = \mathcal{E}g$ solves the wave equation. Further, (2) is equivalent to

$$\|\mathcal{E}g\|_{L^{q'}(\mathbb{R}^{1+n})} \lesssim \|g\|_{L^{p'}(C)}.$$

In the case $p = 2$ this is called the Strichartz estimate and the optimal range is $q' \geq 2\frac{n+1}{n-1}$. The proof relies on the curvature properties of the cone. While there has been a lot of progress in the past five decades, it is still an open problem to prove (2) in the optimal range. Some of the attempts rely on a bilinear approach, which allows to use transversality of wave packets similarly to what has been described above. In the case of the cone, the estimate

$$\|\mathcal{E}g_1\mathcal{E}g_2\|_{L^r(\mathbb{R}^{1+n})} \lesssim \|g_1\|_{L^2}\|g_2\|_{L^2} \tag{3}$$

holds in the extended range $r \geq \frac{n+3}{n+1}$, provided that $g_1, g_2$ have disjoint supports at the unit frequency scale [3].

### 3. Wave Maps via Bilinear Restriction

In recent joint work with Timothy Candy [1] we found an alternative approach to the division problem. Suppose that $U^2$ is defined as the atomic space with atoms

$$a(t) = \sum_I \mathbf{1}_I(t) e^{it|\nabla|} f_I, \quad \sum_I \|f_I\|^2_{L^2(\mathbb{R}^n)} = 1,$$

and $V^2$ is the space of functions $v$ such that $e^{-it|\nabla|}v$ has bounded quadratic variation. Then, one can perturb (3) to

$$\|uv\|_{L^2(\mathbb{R}^{1+n})} \lesssim \|u\|_{U^2}\|v\|_{V^2},$$

provided that $u, v$ have separated spatial Fourier supports at the unit scale. Also, there is a version for different frequency scales and more general phase functions [2]. This can be used to construct an algebra $S$ based on $U^2$ and the weak space $S^w$ based on $V^2$ which provides a new and modular solution to the division problem. For more references and all details we refer to [1]

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## Absence of positive eigenvalues of magnetic Schrödinger operators


Dirk Hundertmark

(joint work with Silvana Avramska-Lukarska, Hynek Kovarik)


For Potentials decaying to zero at infinity, a Schrödinger operator $H = -\Delta + V$ on the Hilbert space $L^2(\mathbb{R}^d)$ should not have positive eigenvalues, since these would be embedded in the essential spectrum, thus they can couple to the continuum to "dissolve." The famous Wigner von Neumann example [8], corrected in [9], shows that this picture is not always correct: There exist slowly decaying and oscillating potentials $V$ such that $-\Delta + V$ has eigenvalue 1, say.

There is by now a long history of results concerning condition on $V$ which guarantee that $-\Delta + V$ does not have positive eigenvalues, [1, 2, 3, 4, 5, 10].

We consider magnetic Schrödinger operators $H = (P-A)^2+V$, where $P = -i\nabla$ is the momentum operator, $A$ the magnetic vector potential, and $V$ the usual electric potential. The magnetic field is related to $A$ by $B = \nabla \times A$ in three dimensions, more generally by $B = dA$. See [6] for a nice discussion of gauge transformations of magnetic Schrödinger operators. Under natural conditions on

$A$ and $V$ it is shown in [11] that this yields a self-adjoint operator in $L^2(\mathbb{R}^d)$. In fact, for the magnetic vector potential one only needs $A \in L^2_{\rm loc}(\mathbb{R}^d, \mathbb{R}^d)$.

For this type of magnetic Schrödinger operators, even with $V = 0$, Miller and Simon [7] constructed examples in two space dimensions, which have dense point spectrum on $[0, \infty)$. In these examples the magnetic field is rotationally symmetric and decays slower that $|x|^{-1}$ at infinity, more precisely, $\lim_{r\to\infty} r|B(r)| = 0$. They also have some results when $\lim_{r\to\infty} r|B(r)| = c > 0$.

The known results on absence of positive eigenvalues either do not include magnetic fields, or they impose conditions directly on the magnetic vector potential, see for example [5]. This, however, is not a gauge invariant condition and thus somewhat unphysical. Moreover, in two dimensions, the most general conditions from [5] imply that the total magnetic flux is zero. Thus they do not apply in situations where one has a positive magnetic flux in two dimensions.

A typical result, which can be proven with our approach is the following: Assume that $|x||B(x)|$ is a bounded function vanishing at infinity and $V = V_1 + V_2$ with $|x|V_1(x)$ and $x \cdot \nabla V_2(x)$ bounded functions vanishing at infinity. Then the Schrödinger operator

$$(P - A)^2 + V \quad \text{on } L^2(\mathbb{R}^d)\,,$$

where $A$ is any vector potential in $L^2_{\rm loc}(\mathbb{R}^d, \mathbb{R}^d))$, with $B = dA$, has no positive eigenvalues.

Our conditions only involve the magnetic field and in light of the Miller–Simon examples, the result is sharp. Our results also extend to the Pauli operator on $L^2(\mathbb{R}^3, \mathbb{C}^2)$ and to more general magnetic fields $B$ and electric potentials $V$.

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## Well-posedness and dispersive decay of small data solutions for the Benjamin-Ono equation

Mihaela Ifrim
(joint work with Daniel Tataru)

This work is concerned with the Benjamin-Ono equation

$$(\partial_t + H\partial_x^2)\phi = \frac{1}{2}\partial_x(\phi^2), \qquad \phi(0) = \phi_0, \tag{1}$$

where $\phi$ is a real valued function $\phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$. and $H$ denotes the Hilbert transform on the real line, with the symbol convention $H(\xi) = -i \operatorname{sgn} \xi$. Thus, dispersive waves travel to the right and solitons to the left.

Our goal is two-fold. First, we revisit the $L^2$ theory for the Benjamin-Ono equation and provide a simpler, self-contained approach. Second, we consider its long time dispersive dynamics, and prove that for small localized data the solutions have (nearly) dispersive dynamics almost globally in time.

The Benjamin-Ono equation is a model for the propagation of one dimensional internal waves (see [1]). Among others, it describes the physical phenomena of wave propagation at the interface of layers of fluids with different densities (see Benjamin [1] and Ono [17]).

Equation (1) is known to be completely integrable. In particular it has an associated Lax pair, an inverse scattering transform, and an infinite hierarchy of conservation laws. The Benjamin-Ono equation is a also dispersive equation, i.e. the group velocity of waves depends on the frequency. Precisely, the dispersion relation for the linear part is given by

$$\omega(\xi) = -\xi|\xi|,$$

and the group velocity for waves of frequency $\xi$ is $v = 2|\xi|$.

There have been many developments in the $H^s$ well-posedness theory for the Benjamin-Ono equations, see: [2, 11, 14, 12, 21, 16, 18, 10, 19], leading up to the $L^2$ well-posedness results in [11] and [15]. By contrast, the critical Sobolev space for this problem is $\dot{H}^{-\frac{1}{2}}$; at this point, the range $-\frac{1}{2} < s < 0$ remains open. An important role here is played by the quasilinear character of the problem; precisely, the nonlinearity is nonperturbative and only continuous dependence on the initial data may hold, even at high regularity, see [16], [14].

Two important milestones along the way have been the $H^3$ well posedness result of Saut [19], using energy estimates, and the $H^1$ result of Tao [21], which is where the idea of renormalization is first used in the study of the Benjamin-Ono equation.

Our first goal here is to revisit the $L^2$ theory for the Benjamin-Ono equation, and (re)prove the following theorem:

**Theorem 1.** *The Benjamin-Ono equation is globally well-posed in $L^2$.*

This result was already proved in [11] and [15], but with a complicated functional setting using various modifications of $X^{s,b}$ spaces.

Given the quasilinear nature of the Benjamin-Ono equation, here it is important to specify the meaning of well-posedness:

**(i) Existence of regular solutions:** For each initial data $\phi_0 \in H^3$ there exists a unique global solution $\phi \in C(\mathbb{R}; H^3)$.

**(ii) Existence and uniqueness of rough solutions:** For each initial data $\phi_0 \in L^2$ there exists a solution $\phi \in C(\mathbb{R}; L^2)$, which is the unique limit of regular solutions.

**(iii) Continuous dependence :** The data to solution map $\phi_0 \to \phi$ is continuous from $L^2$ into $C(L^2)$, locally in time.

**(iv) Higher regularity:** The data to solution map $\phi_0 \to \phi$ is continuous from $H^s$ into $C(H^s)$, locally in time, for each $s > 0$.

**(v) Weak Lipschitz dependence:** The flow map for $L^2$ solutions is locally Lipschitz in the $H^{-\frac{1}{2}}$ topology.

The weak Lipschitz dependence part appears to be a new result, even though some estimates for differences of solutions are part of the prior proofs in [11] and [15].

Our new proof of this result is based on the idea of normal forms, introduced by Shatah [20] in the dispersive realm. In doing this, the chief difficulty we face is that the standard normal form method does not readily apply for quasilinear equations. One very robust adaptation of the normal form method to quasilinear equations, called " the quasilinear modified energy method" was introduced earlier by the authors and collaborators in [4], and then further developed in the water wave context first in [6] and later in [8, 3, 7]. There the idea is to modify the energies, rather than apply a normal form transform to the equations; this method is then successfully used in the study of long time behavior of solutions.

Here we take a slightly different tack, though the above idea is still used at a key point in the proof. Reinterpreting Tao's renormalization from a normal forms perspective, we construct a two step normal form correction, which allows us to frame the well-posedness proof simply as a bootstrap argument for linear and bilinear Strichartz estimates for the solutions.

We now arrive at the second objective of this work. The question we consider concerns the long time behavior of the Benjamin-Ono solutions with small localized data. Precisely, we are asking what is the optimal time-scale up to which the solutions have linear dispersive decay. Our main result asserts that this holds almost globally in time:

**Theorem 2.** *Assume that the initial data $\phi_0$ for* (1) *satisfies*

$$\|\phi_0\|_{L^2} + \|x\phi_0\|_{L^2} \leq \epsilon \ll 1. \tag{2}$$

*Then the solution $\phi$ satisfies the dispersive decay bounds*

$$|\phi(t,x)| + |H\phi(t,x)| \lesssim \epsilon |t|^{-\frac{1}{2}} \langle x_- t^{-\frac{1}{2}} \rangle^{-\frac{1}{2}} \tag{3}$$

*up to time*

$$|t| \lesssim T_\epsilon := e^{\frac{c}{\epsilon}}, \qquad c \ll 1.$$

The novelty in our result is that the solution exhibits dispersive decay. We also remark that better decay holds in the region $x < 0$. This is because of the dispersion relation, which sends all the propagating waves to the right.

**Remark 3.** *One does not expect this result to hold globally in time because even for small localized data this problem might still admit solitons. Then the question becomes to understand what is the earliest time the solitons could emerge. Heuristic computations, based on inverse scattering, suggest that this time is exactly the time in our theorem. Because of this our result above is likely optimal.*

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## Time integration of dispersion-managed nonlinear Schrödinger equations

Tobias Jahnke
(joint work with Marcel Mikl)

Data transfer through an optical fiber with strong dispersion management is modelled by the semilinear Schrödinger equation

$$\partial_t u = \frac{i}{\varepsilon}\gamma\left(\tfrac{t}{\varepsilon}\right)\partial_x^2 u + i|u|^2 u, \qquad x \in \mathbb{T}, \quad t > 0 \tag{1}$$

with small parameter $0 < \varepsilon \ll 1$ on the one-dimensional torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. The coefficient function $\gamma(t) := \chi(t) + \varepsilon\alpha$ is the sum of a small constant term $\varepsilon\alpha$ and a piecewise constant function

$$\chi(t) = \begin{cases} -\delta & \text{if} \quad (t \bmod 2) \in [0,1), \\ \delta & \text{if} \quad (t \bmod 2) \in [1,2) \end{cases}$$

with parameters $\alpha, \delta > 0$; cf. [3, 4]. Non-trivial solutions of (1) oscillate rapidly in time because of the factor $i/\varepsilon$ on the right-hand side. Hence, applying splitting methods, Runge-Kutta methods or other traditional time-integrators is inefficient because such methods only yield an acceptable accuracy if the oscillations are resolved by many time steps, which causes very long runtimes. The discontinuous, rapidly changing coefficient $\gamma(t/\varepsilon)$ and the nonlinear term pose additional challenges for the time-integration of (1).

In [1] we have proposed and analyzed a tailor-made method – the adiabatic midpoint rule – for the dispersion-managed nonlinear Schrödinger equation (1). This method, however, does not preserve the $L^2$ norm of the numerical solution, although it can easily be shown that

$$\|u(t,\cdot)\|_{L^2(\mathbb{T})} = \|u(0,\cdot)\|_{L^2(\mathbb{T})} \quad \text{for all} \quad t \geq 0$$

for every solution of (1). In this talk, we present an exponential version of the adiabatic midpoint rule which has the same favourable properties as its non-exponential counterpart and, in addition, provides norm conservation in $L^2$.

Both versions of the adiabatic midpoint rule are based on the unitary transformation

$$u(t,x) \mapsto \Big(y_k(t)\Big)_{k\in\mathbb{Z}}, \qquad y_k(t) := \exp\left(ik^2\phi\left(\tfrac{t}{\varepsilon}\right)\right)\hat{u}_k(t),$$

where $(\hat{u}_k)_{k\in\mathbb{Z}}$ are the Fourier coefficients of $u(t,\cdot)$ and

$$\phi(t) = \int_0^t \gamma(s)\, ds = \int_0^t \chi(s)\, ds + \varepsilon\alpha t.$$

Substituting this transform into (1) yields an infinite system of ordinary differential equations

$$\dot{y}_m(t) = i \sum_{j-k+\ell=m} y_j(t)\bar{y}_k(t)y_\ell(t) \exp\left(-i\omega_{[jk\ell m]}\phi\left(\tfrac{t}{\varepsilon}\right)\right), \qquad m \in \mathbb{Z} \tag{2}$$

with $\omega_{[jk\ell m]} := j^2 - k^2 + \ell^2 - m^2$. Eq. (2) is equivalent to

$$y'(t) = A\big(t, y(t)\big)y(t) \tag{3}$$

with operator $A$ defined by

$$\big(A(t,\mu)z\big)_m = i \sum_{j-k+\ell=m} \mu_j \bar{\mu}_k z_l \exp\left(-i\omega_{[jk\ell m]}\phi\left(\tfrac{t}{\varepsilon}\right)\right), \qquad m \in \mathbb{Z}$$

for $\mu = (\mu_m)_{m\in\mathbb{Z}}$ , $z = (z_m)_{m\in\mathbb{Z}}$ and $t \geq 0$.

There are several advantages of (2) or (3) over (1). Instead of the discontinuous function $\gamma$, the transformed equations involve the continuous function $\phi$. If the sequence $y = (y_m)_{m\in\mathbb{Z}}$ decays sufficiently fast, then the right-hand side of (2) or (3) is bounded uniformly in $\varepsilon$, which is not true for the original problem (1) due to the factor $i/\varepsilon$. Moreover, it can be shown that the solution of (2) converges to a well-defined limit (cf. [1, 2, 3, 4]) which is not the case for (1).

Let $t_n = n\tau$ and assume that approximations $y^{(n-1)} \approx y(t_{n-1})$ and $y^{(n)} \approx y(t_n)$ are available from previous steps. (The first approximation $y^{(1)}$ can be computed by performing a starting step with a one-step version of the method.) In order to compute the new approximation $y^{(n+1)} \approx y(t_{n+1})$, the evolution equation (3) is replaced on the time interval $[t_{n-1}, t_{n+1}]$ by

$$\tilde{y}'(t) = A\big(t, y^{(n)}\big)\tilde{y}(t), \qquad\qquad \tilde{y}(t_{n-1}) = y^{(n-1)}.$$

This is a *linear* equation, but with a time-dependent operator. Its solution can be approximated by

$$y(t_{n+1}) \approx \tilde{y}(t_{n+1}) \approx \exp\Big(\tau \int_{-1}^{1} A\big(t_n + \sigma\tau, y^{(n)}\big)d\sigma\Big)y^{(n-1)}.$$

In order to integrate with respect to the first variable of $A$, oscillatory integrals of the form

$$\int_a^b \exp\left(-i\omega_{[jk\ell m]}\phi\left(\tfrac{t}{\varepsilon}\right)\right)\, dt$$

have to be computed. This can be done analytically, which is one of the reasons for the efficiency of adiabatic integrators.

The convergence of the adiabatic exponential midpoint rule was analyzed in detail in [2]. We proved first-order convergence in time without any $\varepsilon$-induced step-size restriction. Moreover, we showed that under suitable regularity assumptions the accuracy improves for special step-sizes $\tau$: the error in $\ell^2$ reduces to $\mathcal{O}(\varepsilon^2 + \tau^2)$ if $\tau = \varepsilon k$ with $k \in \mathbb{N}$, and to $\mathcal{O}(\varepsilon\tau)$ if $\tau = \varepsilon/k$, respectively. The constants in the corresponding error bounds do not depend on $\varepsilon$. These results are corroborated by numerical experiments and make the adiabatic exponential midpoint rule very attractive for solving dispersion-managed nonlinear Schrödinger equations.

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## Convergence of an evolving finite element method for mean curvature flow

Balázs Kovács
(joint work with Buyang Li and Christian Lubich)

A proof of convergence was outlined for semi- and full discretizations of mean curvature flow of closed two-dimensional surfaces, based on our paper [4].

Approximating the mean curvature flow by numerical methods was first addressed by Dziuk [1] in 1990. He proposed a finite element method based on a weak formulation of the mean curvature flow as a (formally) heat-like partial differential equation, in which the moving nodes of the finite element mesh determine the approximate evolving surface. However, proving convergence of Dziuk's method or related evolving finite element methods for closed two-dimensional surfaces (or higher-dimensional hypersurfaces) has remained an open problem.

We here consider a different evolving finite element method for mean curvature flow of closed two-dimensional surfaces and prove optimal-order convergence over time intervals on which the evolving surface remains sufficiently regular. We study stability and convergence for both the finite element semi-discretization and the full discretization obtained with a linearly implicit backward difference time discretization. Our approach shares with Dziuk's method the property that the moving nodes of a finite element mesh determine the approximate evolving surface. However, the method presented here discretizes equations that are different from the equation discretized by Dziuk. In his approach, a weak formulation of the quasi-heat equation describing mean curvature flow is discretized, whereas in the present work evolution equations for the normal vector and the mean curvature are discretized, which then yield the velocity of the surface evolving under mean curvature flow. Evolution equations for geometric quantities on a surface evolving under mean curvature flow have been an important tool in the analysis of mean curvature flow ever since Huisken's 1984 paper [2], but apparently they have so far not been used in the numerical approximation of mean curvature flow.

The numerical method based on the discretization of evolution equations of geometric quantities, as presented here, is computationally more expensive than Dziuk's method (roughly by about a factor 2), but on the other hand it provides

full-order approximations to basic geometric quantities — the normal vector and mean curvature — in addition to the position and velocity of the surface.

Our numerical approach is related to our previous paper [3], where we study the convergence of finite elements on an evolving surface driven by diffusion on the surface. The convergence analysis of our method for mean curvature flow uses techniques developed in that previous paper. This numerical method admits a convergence analysis in the case of finite elements of polynomial degree at least two and backward difference formulae up to order five. The error analysis combines stability estimates and consistency estimates to yield optimal-order $H^1$-norm error bounds for the computed surface position, velocity, normal vector and mean curvature. As in [3], the stability analysis works with the matrix–vector formulation of the method and does not use geometric arguments. The geometry only enters into the analysis of the consistency error. The error analysis combines the stability estimates and consistency estimates to yield optimal-order $H^1$-norm error bounds for the computed surface position, velocity, normal vector and mean curvature.

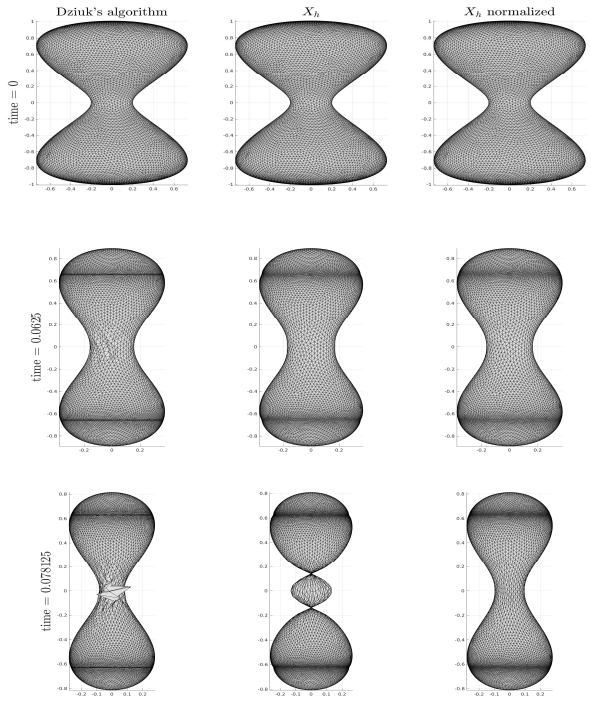


FIGURE 1. Comparison of Dziuk's algorithm, with the proposed numerical method and with its version with normalized approximate normal vector $\nu_h$ on a flow developing a pinch singularity.

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## Splitting methods for Schrödinger equations with time dependent potentials; many problems, many approaches.


Karolina Kropielnicka

(joint work with Philipp Bader, Marisa Condon, Arieh Iserles, Pranav Singh)


In this talk I will present various numerical approaches for the linear Schrödinger equations with time dependent potentials. As it often happens, the methods should be closely correlated with properties of the problem. In the first approach we suggest an application of asymptotic Zassenhauss decomposition in tandem with Munthe-Kaas–Owren basis for semiclassical scaling, see [1]. In case of high oscillations of the potential we resort to simplified commutators in Magnus expansion and integrate the potential function in the very last stage of the algorithm, like it was presented in [2]. Magnus-Lanczos methods with simplified commutators are proposed in case of Schrödinge equations with highly oscillating in time potential, see [3] for the setting of atomic scaling. Irrespectively of chosen regime, compact splittings are suggested for equations under the influence of laser matter, like it was introduces in [4]. Yet another approach will be explored, where the solution is presented as an asymptotic series in inverse powers of the oscillation frequency of the potential, [5]. I will present numerical results of those methods and compare them against other well known approaches.


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## Full-discretization of semilinear wave equations with dynamic boundary conditions

Jan Leibold
(joint work with Marlis Hochbruck)

We consider semilinear acoustic wave equations with dynamic boundary conditions. In contrast to standard boundary conditions, as of Dirichlet or Neumann type, dynamic boundary conditions do not neglect the momentum of the wave on the boundary. The mathematical modeling of such effects leads to an evolution equation in the interior domain $\Omega$ coupled to an evolution equation on the smooth boundary $\partial\Omega$. An example is the following equation with kinetic boundary conditions [4]

$$\begin{aligned} u_{tt} + (1 + \mathbf{x} \cdot \nabla) u_t - \Delta u &= |u| u \quad \text{in } (0,T) \times \Omega, \\ u_{tt} + \partial_n u + u - \Delta_{\partial\Omega} u &= |u|^2 u \quad \text{on } (0,T) \times \partial\Omega, \end{aligned} \tag{1}$$

complemented with initial conditions. Here, $\Delta_{\partial\Omega}$ denotes the Laplace-Beltrami operator.

In a compact form, (1) can be rewritten as

$$u'' + Bu' + Au = f(t,u), \qquad t \in [0,T], \tag{2}$$

where $A$ and $B$ are monotone (unbounded) linear operators and $f$ is a Lipschitz-continuous function.

We are interested in full discretizations of such equations comprising finite element discretizations in space and implicit-explicit (IMEX) time integration schemes [3]. Besides an efficient implementation of such schemes we aim at their rigorous error analysis.

IMEX schemes treat the (stiff) linear part of the differential equation implicitly and the nonlinear part explicitly. Thus they require only the solution of one linear system of equations in each time step. We designed an IMEX scheme for problems of the form (2). Numerical experiments have already shown that this scheme is more efficient than standard time integration methods like the Crank-Nicolson or the leapfrog scheme.

Boundary conditions with tangential derivatives, as, e.g., the Laplace–Beltrami operator, are intrinsically posed on domains with (piecewise) smooth, possibly curved boundaries. Therefore, numerical methods often have to make approximate the domain first which renders the approximation non-conforming. This makes the error analysis much more involved. These difficulties are addressed in [1, 2], where a unified error analysis is presented that allows to derive space discretization error bounds.

For the semi discretization in time we prove stability and a second-order error bound. Our current work is devoted to a full discretization error bound by combining the time and space discretization error analysis.

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## Recent Progress on the Half-Wave Maps Equation

Enno Lenzmann
(joint work with Patrick Gérard, Armin Schikorra)

In this talk, I discuss recent progress on the energy-critical half-wave maps equation

$$\partial_t u = u \times |\nabla| u$$

where $u : [0,T) \times \mathbb{R} \to \mathbb{S}^2$ (real line case) or $u : [0,T) \times \mathbb{T} \to \mathbb{S}^2$ (space periodic case). Here $\times$ denotes the vector product in $\mathbb{R}^3$ and $|\nabla|$ is defined by multiplication with the symbol $|\xi|$ in Fourier space.

After reviewing the classification result of traveling solitary waves obtained in [2], I report on newly found time-periodic solutions and other special solutions given by rational functions in the spatial variable $x$. The main technique behind finding these special solutions rests on a Lax pair structure recently found in [1]. More specifically, we have the Lax equation

$$\dot{L} = [L, B],$$

where $L = [H, \mathbf{U}]$ with $\mathbf{U} = \sum_{k=1}^{3} u_i \sigma_i$ (with the standard Pauli matrices $\sigma_1, \sigma_2, \sigma_3$) and $H$ denoting the Hilbert transform. For the specific form of the operator $B$, we refer to [1]. One essential analytical feature of the Lax operator $L$ is that it is finite rank if and only if the function $u$ is rational in the spatial variable $x$. In fact, we can show the following quantitive result in the real line case:

$$\text{rank of } L = \text{number of poles of } u \text{ in } \mathbb{C} \setminus \mathbb{R},$$

provided that $L$ is finite rank. By the Lax equation, the rank of $L$ stays constant in time (as long as the solutions exists). In this way, we obtain an infinite family of sets of invariant spaces under the Hamiltonian flow.

The question of stability of these rational solutions raises an interesting open problem for future research.

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## Conserved energies for the one dimensional Gross-Pitaevskii equation

Xian Liao
(joint work with Herbert Koch)

We consider the Cauchy problem for the one dimensional Gross-Pitaevskii equation

$$i\partial_t q + \partial_{xx} q = 2(|q|^2 - 1)q, \tag{1}$$

where $t, x$ denote the one dimensional time and space variables respectively and $q : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}$ is the unknown wave function. The above GP equation can be viewed as the defocusing cubic nonlinear Schrödinger equation, but with nonzero boundary condition at infinity:

$$|q(x)| \to 1, \text{ as } |x| \to \infty.$$

This nonstandard boundary condition changes dramatically the dynamics and in particular the following black (with $c = 0$) and dark (with $c \neq 0$) soliton solutions solve (1):

$$q(t, x) = \sqrt{1 - c^2} \tanh\big(\sqrt{1 - c^2}(x - 2ct)\big) + ic, \quad -1 < c < 1.$$

We would like first to search for the suitable solution spaces which consist of finite-energy solutions. We recall the conserved mass, momentum and energy for the GP equation (1):

$$\begin{aligned}
\mathcal{M}(q) &= \int_{\mathbb{R}} \big(|q|^2 - 1\big)\, dx, \\
\mathcal{P}(q) &= \operatorname{Im} \int_{\mathbb{R}} q\partial_x \bar{q}\, dx, \\
\mathcal{E}(q) &= \frac{1}{2} \int_{\mathbb{R}} \big((|q|^2 - 1)^2 + |\partial_x q|^2\big)\, dx,
\end{aligned}$$

and we observe that the energy $\mathcal{E}(q)$ is related to the $L^2$-norms of the quantities

$$|q|^2 - 1, \, \partial_x q.$$

This leads us to introduce the following solution space

$$X^s = \{q \in H^s_{\text{loc}}(\mathbb{R}) \,|\, |q|^2 - 1 \in H^{s-1}(\mathbb{R}), \quad \partial_x q \in H^{s-1}(\mathbb{R})\}/\mathbb{S}^1, \quad s \geq 0,$$

where $\mathbb{S}^1$ denotes the unit circle. For any $q \in X^s$, we define its energy norm as

$$E^s(q) = \big(\||q|^2 - 1\|^2_{H^{s-1}} + \|\partial_x q\|^2_{H^{s-1}}\big)^{\frac{1}{2}},$$

which describes well the $H^s$-regularity of $q \in X^s$ and in particular $(E^1(q))^2 = 2\mathcal{E}(q)$. Observing that $|q| \to 1$ as $|x| \to \infty$, we introduce the following distance function $d^s(u, v)$ for $u, v \in X^s$:

$$d^s(u, v) = \Big(\int_{\mathbb{R}} \inf_{|\theta|=1} \big\|\operatorname{sech}(x - y)\big(\theta u(x) - v(x)\big)\big\|^2_{H^s_x}\, dy + \big\||u|^2 - |v|^2\big\|^2_{H^{s-1}}\Big)^{\frac{1}{2}},$$

where $\operatorname{sech}(x) = \frac{2}{e^x + e^{-x}}$ and we emphasize that $\theta \in \mathbb{S}^1$ within the above integral may depend on $y$. We have the following results:

**Theorem 1.** *The space $(X^s, d^s)$, $s \geq 0$ is a complete metric space and the set $\{1+\varphi \,|\, \varphi \in C_0^\infty(\mathbb{R})\}$ is dense in it. The Gross-Pitaevskii equation* (1) *is locally-in-time well-posed in $X^s$, $s \geq 0$.*

*Let $s > \frac{1}{2}$ and $q_0 \in X^s$. Then the Gross-Pitaevskii equation is globally-in-time well-posed in $X^s$ and the energy is conserved by the Gross-Pitaevskii flow: $E^s(q(t)) \leq C(E^s(q_0))$, $\forall t \in \mathbb{R}$ where $C$ depends nonlinearly on $E^s(q_0)$.*

The local well-posedness result follows from the Strichartz's estimate: We regularize the initial data $q_0 \mapsto q_{0,\varepsilon}$ and consider the unknown function $q - q_{0,\varepsilon}$, which satisfies the Schrödinger-type equation with the initial data $q_0 - q_{0,\varepsilon} \in H^s$.

The conservation of the energy $E^s(q)$, $s > \frac{1}{2}$ may come from the classical fact that there is a Lax pair for the GP equation and hence a priori the associated transmission coefficient $T^{-1}(\lambda)$, $\lambda \in \mathbb{C}$ is invariant by time evolution. In particular in the classical framework $q \in 1 + \mathcal{S}(\mathbb{R})$, $\mathcal{S}$ being the Schwartz function space, we have the following expansion for $\ln T^{-1}(\lambda)$, with countably many conserved coefficients $(\mathcal{H}_l)_l$:

$$\ln T^{-1}(\lambda) = i \sum_{l=0}^{\infty} \mathcal{H}_l (2z)^{-l-1}, \quad |\lambda| \to \infty, \quad \mathrm{Im}\lambda > 0, \tag{2}$$

where $\lambda, z \in \mathbb{C}$ are related by $\lambda^2 = z^2 + 1$, $\mathrm{Im}z > 0$, and the first three conserved coefficients read as the mass, momentum and energy respectively:

$$\mathcal{H}_0 = \mathcal{M}, \quad \mathcal{H}_1 = \mathcal{P}, \quad \mathcal{H}_2 = 2\mathcal{E}.$$

As the mass and the momentum are not well-defined for $q \in X^s$, we have to consider the renormalised transmission coefficient $\tilde{T}^{-1}(\lambda)$ which reads, if $q \in 1+\mathcal{S}$, as $T^{-1} e^{-i\mathcal{M}(2z)^{-1} - i\mathcal{P}(2z\zeta)^{-1}}$, $\zeta = \lambda + z$, such that $\tilde{T}^{-1}$ defined for $q \in X^s$ is also invariant by the GP flow by density argument. Nevertheless the well-definedness of $\tilde{T}^{-1}$ for $q \in X^s$ and its dependence on $(|q|^2 - 1, \partial_x q)$ require more delicate analysis. In particular, when evaluated on the imaginary axis $(\lambda, z) = (i\sigma, i\tau/2)$, $\sigma^2 = \tau^2/4 - 1$, the leading term in $\big(-\tau^2 \ln \tilde{T}^{-1}(i\sigma)\big)$ reads as the following form:

$$\tilde{T}_2(i\tau) = \int_{x<y} e^{-\tau(x-y)} \big((|q|^2-1)(y)(|q|^2-1)(x) + (\partial_x q)(y)(\partial_x \bar{q})(x)\, dxdy.$$

In order to consider also the large energy case, we take the weighted energy norm $E^s_{\tau_0}(q) = \big(\int_{\mathbb{R}} (\tau_0^2 + \xi^2) \big(|\widehat{|q|^2 - 1}|^2(\xi) + |\widehat{\partial_x q}|^2(\xi)\big)\, d\xi\big)^{\frac{1}{2}}$ with the parameter $\tau_0 \geq 2$. Hence when $s \in (\frac{1}{2}, 1)$ the difference between the conserved energy defined by $\tilde{T}^{-1}$:

$$\mathcal{E}^s_{\tau_0}(q) = -\frac{2}{\pi} \sin(\pi(s-1)) \int_{\tau_0}^{\infty} (\tau^2 - \tau_0^2)(-\tau^2 \ln \tilde{T}^{-1}(i\sigma))\, d\tau$$

and the energy norm $(E^s_{\tau_0}(q))^2 = -\frac{2}{\pi} \sin(\pi(s-1)) \int_{\tau_0}^{\infty} (\tau^2 - \tau_0^2) \tilde{T}_2(i\tau)\, d\tau$ can be controlled by $C c_{\tau_0} (E^s_{\tau_0}(q))^2$ if $c_{\tau_0} := \frac{1}{\tau_0} \big( \| |q|^2 - 1 \|_{l^2_{\tau_0} DU^2} + \| \partial_x q \|_{l^2_{\tau_0} DU^2} \big) \leq C \tau_0^{-\frac{1}{2} - s} E^s_{\tau_0}$ is small enough. Thus we can choose large enough parameter $\tau_0$ (depending on $E^s(q_0)$) to obtain $|\mathcal{E}^s_{\tau_0}(q) - (E^s_{\tau_0}(q))^2| \leq \frac{1}{2} (E^s_{\tau_0}(q))^2$, such that $E^s_{\tau_0}(q) \leq 2 E^s_{\tau_0}(q_0)$ holds globally in time by virtue of the conservation of $\mathcal{E}^s_{\tau_0}(q)$. When $s \geq 1$ then

we have to do further expansions and in particular we have the following trace formular for the conserved energy $\mathcal{E}^n_{\tau_0}(q)$ when $s = n \in \mathbb{N}$:

$$\mathcal{E}^n_{\tau_0}(q) = \sum_{l=0}^{n-1} {\tau_0}^{2(n-1-l)} \binom{n-1}{l} \mathcal{H}_{2l+2},$$
$$\mathcal{H}_{2l+2} = \frac{1}{\pi} \int_{\mathbb{R}} \xi^{2l+2} \frac{1}{2} \sum_{\pm} \ln |\tilde{T}^{-1}|(\pm\sqrt{\xi^2/4+1}) d\xi - \frac{1}{2l+3} \sum_m \mathrm{Im}(2z_m)^{2l+3},$$

where $\mathcal{H}_l$ is the coefficient in (2) and $z_m = i\sqrt{1-\lambda_m^2} \in i(0,1]$ with $\{\lambda_m\}_m \subset (-1,1)$ being the possible countably many zeros of $\tilde{T}^{-1}(\lambda)$.

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## Efficient algorithms for Schrödinger equations with concentrated potentials

Maria Lopez-Fernandez
(joint work with Lehel Banjai)

We consider the efficient numerical approximation of the Schrödinger equation with potential concentrated in a discrete set of points. We find applications in the literature with linear non autonomous models of the form [8, 9]

$$\mathrm{i}\,\partial_t \psi = -\partial_{xx}\psi + \sum_{j=1}^{M} V_j(t)\delta_{x_j}\psi; \quad \psi(x,0) = \psi_0(x), \quad x \in \mathbb{R},\ t \geq 0, \tag{1}$$

for some known arbitrary time-dependent amplitudes $V_j(t)$, and also non linear models of the form [1, 5]

$$\mathrm{i}\,\partial_t \psi = -\partial_{xx}\psi + \sum_{j=1}^{M} \gamma_j |\psi|^{2\sigma}\delta_{x_j}\psi; \quad \psi(x,0) = \psi_0(x) \quad x \in \mathbb{R},\ t \geq 0, \tag{2}$$

with $\gamma_j < 0,\ \sigma \geq 0$. Extensions to higher spatial dimension are recent object of study [4, 6]. In the present report we focus on the one-dimensional case.

The solution $\psi(x,t)$ to equations (1) and (2) can be uniquely determined by the values of $\psi$ at the points where the potential is concentrated, this is, the scalar functions

$$q_j(t) := \psi(x_j, t), \qquad t \geq 0. \tag{3}$$

In the case of (2), the functions $q_j$ satisfy the following system of Volterra equations

$$q_j(t) + \sum_{j=1}^{M} \gamma_j \int_0^t k(t-s, x_j - x_k)|q_j(s)|^{2\sigma} q_j(s)\, ds = (\mathcal{U}(t)\psi_0)(x_j), \tag{4}$$

for $j = 1, \dots, M$, with $k$ the Green function for the (free) Schrödinger equation in the whole space [7]

$$k(t,x) = \frac{\mathrm{e}^{\mathrm{i}\,\pi/4}}{2\sqrt{\pi t}}\,\mathrm{e}^{\mathrm{i}\,|x|^2/4t}, \quad x \in \mathbb{R},\ t > 0, \tag{5}$$

and $\mathcal{U}(t)\psi_0$ the solution to the free Schrödinger equation with initial data $\psi_0$. The solution $\psi(x,t)$ at any $x \in \mathbb{R}$ can then be expressed as

$$\psi(x,t) + \gamma \sum_{k=1}^{M} \int_0^t k(t-s, x-x_k)|q_k(s)|^{2\sigma} q_k(s)\,ds = (\mathcal{U}(t)\psi_0)(x). \tag{6}$$

A linear Volterra system, analogous to (4), is satisfied by the $q_j$ corresponding to (1) [8, 9].

A careful numerical study of the so-called *charge equations* (4) can be found in [5]. The method proposed in [5] is reliable but turns out to be rather expensive both from the computational and the storage point of view. Here we proposed a generalization of the ideas in [2] in order to reduce both the number of operations and the memory requirements, while keeping the target accuracy in the final approximation. Our proposal is a special algorithm for the implementation of the Runge–Kutta based Convolution Quadrature (CQ) method. The Runge–Kutta CQ is first presented in [10] and then analyzed in [3] for convolution kernels like those in our application to (4). The CQ is designed to approximate abstract-time convolutions of the form

$$f(t) := \int_0^t \widetilde{k}(t-s)\,g(s), \tag{7}$$

at equispaced points $t_n = nh$, for some fixed $h > 0$, $n = 1, \dots, N$. In the case of (4) we actually have several different kernels $\widetilde{k}(t) = k(t, x_j - x_k)$, $j,k = 1, \dots, M$, depending on all possible distances between the points $x_j$ and $x_k$. The CQ based on a Runge–Kutta method with abscissae $c_1, \dots, c_s$, approximates (7) by a discrete convolution of the form

$$f(t_n) \approx f_n = \sum_{j=0}^{\infty} \boldsymbol{\omega}_j (g(t_n - t_j + c_l h))_{l=1}^{s}\,. \tag{8}$$

The (row vector of) convolution weights $\boldsymbol{\omega}_j$ depend on the Laplace transform $\widetilde{K}$ of the kernel $\widetilde{k}$ and the particular Runge–Kutta method of choice, which must be implicit and satisfy certain hypotheses [3]. In this way $\widetilde{k}$ is never evaluated by the numerical method but only $\widetilde{K}$. For the application to (4) it is

$$\widetilde{K}(z) = K(z,x) = \frac{1}{2\sqrt{z/\,\mathrm{i}}} \exp\left(-|x|\sqrt{z/\,\mathrm{i}}\right), \quad x \in \mathbb{R}. \tag{9}$$

We propose an implementation of the CQ method in (8) which requires:

- $O(|\log(\varepsilon)|N \log N)$ multiplications.
- $O(|\log(\varepsilon)|\log(N))$ evaluations of $K(z,x)$, for every $x \in \mathbb{R}$.
- $O(|\log(\varepsilon)|\log(N))$ storage.

In the above, $\varepsilon$ denotes the target accuracy in the computation of the weights $\boldsymbol{\omega}_j$, $j = 0, \ldots, N$. Our algorithm is based on the following representation of the $\boldsymbol{\omega}_j$ associated to (9):

**Theorem 1.** *Let us consider an L-stable Runge–Kutta method with abscissae* $\mathbf{c} = (c_i)_{i=1}^s$, *weights* $\mathbf{b} = (b_j)_{j=1}^s$ *and coefficient matrix* $\mathbf{A} = (a_{ij})_{i,j=1}^s$. *Let us assume that the Runge–Kutta method satisfies the assumptions in* [3]. *Then, for fixed* $x \in \mathbb{R}$, *the CQ weights associated to the given Runge–Kutta method and* (9) *satisfy*

$$\boldsymbol{\omega}_n(x) = \frac{h}{2\pi\,\mathrm{i}} \left( \int_{\Gamma_+ \cup \Gamma_-} K(z,x)\mathbf{e}_n(hz)\,dz + \int_0^\xi G(\lambda,x)\mathbf{e}_n(-h\lambda)\,d\lambda, \right), \tag{10}$$

*where, for* $r$ *the stability function of the Runge–Kutta method,* $\mathbf{e}_n(z) := r(z)^n \mathbf{q}(z)$ *with* $\mathbf{q}(z) = \mathbf{b}^T(I - z\mathbf{A})^{-1}$, $\Gamma_\pm = \{-\xi \pm \mathrm{i}\, y : y > 0\}$ *and*

$$G(\lambda, x) = \frac{\mathrm{i}\, e^{\mathrm{i}\,\pi/4}}{\sqrt{\lambda}} \cosh(x e^{i\pi/4}\sqrt{\lambda}).$$

We show that for a given target accuracy $\varepsilon > 0$, a convenient choice of $\xi > 0$ can be found in such a way that the integrals along $\Gamma_\pm$ in (10) can be neglected for the computation of the $\boldsymbol{\omega}_j$. For the integral along the real interval $[0, \xi]$ we derive a quadrature rule which provides a uniform approximation of accuracy $\varepsilon$, for every $n = n_0, \ldots, N$, with $n_0$ depending on $|x|$ and $h$. Setting $\mathbf{g}_j = (g(t_j + c_\ell h))_{\ell=1}^s$, such an approximation of the CQ weights leads to the following splitting of the summation in (8):

$$\sum_{j=0}^{n} \boldsymbol{\omega}_j \mathbf{g}_{n-j} = \sum_{j=0}^{n_0} \boldsymbol{\omega}_j \mathbf{g}_{n-j} + \sum_{j=n_0+1}^{n} \boldsymbol{\omega}_j \mathbf{g}_{n-j} = L_n + H_n.$$

The *history term* $H_n$ is then approximated by

$$H_n \approx \sum_{\ell=1}^{N_Q} w_\ell \sum_{j=n_0+1}^{n} (r(-hx_\ell))^j \mathbf{q}(-hx_\ell)\mathbf{g}_{n-j} = h \sum_{k=1}^{N_Q} w_k (r(-hx_k))^{n_0+1} Q_{n,k},$$

with

$$Q_{n,k} = \sum_{j=0}^{n-n_0-1} (r(-hx_k))^j \mathbf{q}(-hx_k)\mathbf{g}_{n-n_0-1-j},$$

which satisfy the recursion:

$$Q_{n,k} = r(-hx_k)Q_{n-1,k} + \mathbf{q}(-hx_k)\mathbf{g}_{n-n_0-1}, \qquad Q_{n_0,k} = 0. \tag{11}$$

We prove that $N_Q$ can be taken proportional to $|\log(\varepsilon)|\log(N)$. This together with the recursive procedure (11) leads to the claimed complexity and storage requirements. A semi-implicit scheme based on this algorithm is applied to solve (4), with results in good agreement with those reported in [5].

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## Stability and convergence of time discretizations of quasi-linear evolution equations of Kato type

Christian Lubich
(joint work with Balázs Kovács)

In a very insightful paper published in 1975, Kato [11] presents a concise framework for quasi-linear evolution equations in a Banach space, proves local well-posedness of the initial value problem within this framework and shows that the framework and results apply to a variety of quasi-linear partial differential equations. He lists symmetric hyperbolic systems of the first order, wave equations, Korteweg–de Vries equation, Navier–Stokes and Euler equations, equations for compressible fluids, magnetohydrodynamic equations, coupled Maxwell and Dirac equations — and adds "etc.". Particularly noteworthy appears the application to symmetric hyperbolic systems in the sense of Friedrichs (in arbitrary space dimension), which is a large and fundamental class of problems.

While Kato's paper has been influential in the analysis of nonlinear hyperbolic and dispersive partial differential equations, it has apparently gone unnoticed in the numerical literature for such equations. Kato's framework has been modified and generalized to further classes of partial differential equations, by himself and coauthors in [9, 10] shortly after [11], and by other researchers until recently, e.g., in [5, 13]. To our knowledge, the only numerical paper related to Kato's framework before our paper [12] is the work of Hochbruck & Pažur [7] who study the implicit Euler time discretization in a modified Kato framework that was developed by

Müller [13] for dealing with a class of quasi-linear Maxwell equations; see also [8] for subsequent work in this direction.

Here we show that Kato's original framework from [11], when restricted to Hilbert spaces (which are typically used in the applications), combines remarkably well with the technique of "energy estimates" for time discretizations, that is, with the use of positive definite and semi-definite bilinear forms for proving stability and error bounds. We show this for the implicit Runge–Kutta methods such as the Gauss and Radau IIA methods of arbitrary orders, which have the properties of algebraic stability and coercivity, notions that are due to Burrage & Butcher [1] and Crouzeix [2] (for algebraic stability) and to Crouzeix and Raviart [3] (for coercivity); see also [4, 6]. Although these notions were developed and recognized as important properties in the context of stiff ordinary differential equations in the same decade in which Kato's paper appeared, it seems that no link between these analytical and numerical theories was made. With a delay of some decades, this is now done in the present talk based on our paper [12] — in view of both, the perfectly fitting connection of the analytical framework and the numerical methods, and the undiminished significance of the considered evolution equations in applications.

The distinguishing feature in Kato's theory is a commutator condition, which enables the transfer between stronger and weaker norms and is here combined with energy estimates for the analysis of numerical methods. The talk presented stability estimates and optimal-order convergence results for time discretizations of classes of quasi-linear hyperbolic and dispersive partial differential equations on $\mathbb{R}^d$ or on a $d$-dimensional torus.

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## Space discretization of quasilinear wave-type equations

Bernhard Maier
(joint work with Marlis Hochbruck)

We consider the numerical approximation of solutions to quasilinear wave-type equations of the form

$$\Lambda\big(y(t)\big)y'(t) = Ay(t) + Q\big(y(t)\big)y(t), \tag{1}$$

subject to initial values and Dirichlet boundary conditions on a bounded domain $\Omega$ with smooth boundary. Here, $A$ is a linear skew-adjoint operator and the nonlinearities $\Lambda$ and $Q$ model for example nonlinear material laws such as the Kerr nonlinearity.

As shown in [4] by Kato and for a refined framework in [7], the evolution equation (1) is locally well-posed on a time intervall $(0, T)$.

Recently, time integration for the evolution equation of (1) was investigated. In [5] the authors show rigorous error bounds for algebraically stable Runge-Kutta methods in Kato's original framework [4]. The time integration within the refined framework [7] was considered in [2] and [3].

Here, we present an error analysis for space discretizations of (1) with finite elements. For discretized versions $\Lambda_h, A_h$ and $Q_h$ of the operators $\Lambda, A$ and $Q$ we consider the semi-discrete equation

$$\Lambda_h\big(y_h(t)\big)y_h'(t) = A_h y_h(t) + Q_h\big(y_h(t)\big)y_h(t), \tag{2}$$

again subject to initial values. We want to emphasize that we consider also nonconforming discretizations, because, in general, domain approximations due to the smoothness of the boundary of $\Omega$ (which is required for the well-posedness of (1)) are necessary. Hence, we use a lift operator $L_h$ to compute differences between functions from the discrete and continuous function spaces.

Based on the Picard-Lindelöf theorem we first prove local well-posedness with a mesh-dependent minimal time of existence $T_h$. Next, we prove the following error bound between the exact solution $y$ of (1) and the discrete approximation $y_h$ of (2) in the energy norm.

$$\|y(t) - L_h y_h(t)\| \leq C_y(1+t)e^{C_y t}E,$$

with a constant $C_y$ depending on the solution $y$ of (1) and its time derivative, but which is independent of the mesh width $h$. The error term $E$ consists of interpolation errors, the error of the discretized initial values and differences between discrete and continuous operators.

The proof of this result is based on [1] and [6], where the authors present a unified error analysis for linear and semilinear wave-type equations, respectively. To conclude, we use an inverse inequality in order to extend the minimal time of existence $T_h$ to $T$.

Our aim is to provide a rigorous error analysis for the full discretization by generalizing the techniques from [3]. Hence, we consider as a next step the full discretization of (1), combining the techniques from space and time discretization analysis.

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## Randomized final data problem for nonlinear Schrödinger equations


Kenji Nakanishi

(joint work with Takuto Yamamoto)


This is a report on our recent work [7]. We study asymptotic behavior of solutions for large time of the nonlinear Schrödinger equation

$$i\dot{u} - \Delta u = |u|^p u, \quad u(t,x) : R^{1+d} \to C.$$

More precisely, we are interested in the scattering problem in $L^2(R^d)$, that is whether the solution $u$ exists globally in time ($t \in R$) and

$$\|u(t) - v(t)\|_{L^2(R^d)} \to 0 \quad (t \to \infty)$$

for some free solution $v(t) = e^{-it\Delta}\phi$ with $\phi \in L^2(R^d)$.

The existence of such a global solution $u$ for every $\phi \in L^2(R^d)$ was proven in [4] for $d \geq 3$ and $2/d < p < 4/d$, but the uniqueness is an open problem in the $L^2$ setting, unless one assumes that the final data $\phi$ is more localized (such as

$|x|^s\phi \in L^2(R^d)$ with $s \geq 2/p - d/2$, cf. [2, 5]). The range of power is optimal since for $p \leq 2/d$ the asymptotic profile needs to be modified nonlinearly (cf. [8]), and for $p \geq 4/d$ there are plenty of solutions blowing up at large time, even though starting from $t = \infty$ with the asymptotically free profiles (cf. [6]).

The uniqueness is difficult, since it is a super critical problem in view of the scaling invariance:

$$u(t,x) \mapsto u_a(t,x) := a^{2/p}u(a^2t, ax)$$

of the equation, where $\|u_a(0)\|_{L^2(R^d)} = a^{2/p-d/2}\|u(0)\|_{L^2(R^d)} \to 0$ as $a \to +0$ for $p < 4/d$, while the time scale of $u_a$ becomes larger. Since scattering of $u$ is equivalent to that of $u_a$, it implies that the scattering problem is no easier by restricting to small $L^2$ norm of $\phi$ and $u$, nor by restricting to large time $t > 0$. Therefore perturbative argument, dominating the nonlinearity by the linear part, can not possibly work. As we have little technique to compare solutions of nonlinear dispersive equations except the perturbative arguments, it leaves the uniqueness question widely open.

Nevertheless, by introducing randomness into the final data $\phi$, Murphy [3] proved that if the power $p$ is bigger than the Strauss power $p_0(d)$

$$p > p_0(d) := \frac{\sqrt{d^2+12d+4} - d + 2}{2d},$$

one can find a unique global solution $u$ (in some function space) asymptotic to $e^{-it\Delta}\phi^\omega$ almost surely, for every $\phi \in L^2(R^d)$, where $\phi^\omega$ is a randomized final data defined by

$$\phi^\omega(x) = \sum_{k\in Z^d} g_k(\omega)\chi(x-k)\phi(x),$$

with $\chi \in C_0^\infty(R^d)$ smoothly decomposing $R^d$ into cubes

$$1 = \sum_{k\in Z^d} \chi(x-k),$$

while $\{g_k\}_{k\in Z^d}$ is a set of i. i. d. random variables with zero mean and appropriate distribution, e.g., the Gaussian. The key estimate is an improvement by the randomization of the Strichartz estimate:

$$\|e^{-it\Delta}\phi^\omega\|_{L^\alpha_\omega L^q_t L^r_x(R^{1+d})} \leq C\sqrt{\alpha}\|\phi\|_{L^2(R^d)},$$

which holds for all $(q,r,\alpha) \in [2,\infty)$ satisfying $1/q > d/2 - d/r$ and $\alpha \geq \max(q,r)$. The range of exponents (integrability) is much wider than the deterministic case

$$\|e^{-it\Delta}\phi\|_{L^q_t L^r_x(R^{1+d})} \leq C\|\phi\|_{L^2(R^d)},$$

which requires $2/q = d/2 - d/r$. Roughly speaking, the randomized version is as good as if $\phi \in L^1 \cap L^2$, except the $L^\infty$ cases.

However, the condition $p > p_0(d)$ excludes the physically important case $p = 1$ on $R^3$, namely the quadratic nonlinear Schrödinger equations in three dimensions,

since $p_0(3) = 1$. For example, there is a model system [1] for the laser-plasma interactions:

$$\begin{cases} i\dot{u}_1 - \Delta u_1 = \bar{u_2} u_3, \\ i\dot{u}_2 - \Delta u_2 = \bar{u_1} u_3, \\ i\dot{u}_3 - \Delta u_3 = u_1 u_2. \end{cases}$$

So we slightly extend the lower bound of $p$ to include such cases, precisely to

$$p > p_1(d) := \frac{\sqrt{d^2 + 24d + 16} - d + 4}{4d} \ (< p_0(d)),$$

though we need to replace the function space for $u$ to ensure its uniqueness. The numerical values of the two powers are

$$\begin{array}{llll} p_1(1) = 2.35... & p_1(2) = 1.28... & p_1(3) = 0.90... & p_1(4) = 0.70... \\ p_0(1) = 2.56... & p_0(2) = 1.41... & p_0(3) = 1 & p_1(4) = 0.78... \end{array}$$

The meaning of those critical exponents can be understood in view of the scaling invariance and the dispersive decay estimate:

$$\|e^{-it\Delta}\phi\|_{L^{r^*}(R^d)} \le C|t|^{-d/2+d/r}\|\phi\|_{L^r(R^d)},$$

where $1 \le r \le 2$ and $r^* := \frac{r}{r-1}$ denotes the Hölder conjugate. Let $L^q(R^d)$ be the invariant initial space for the invariant scaling $u_a$, namely

$$\|u_a(0)\|_{L^q(R^d)} = \|u(0)\|_{L^q(R^d)} \iff q = \frac{dp}{2}.$$

Then the four critical exponents are respectively characterized by the invariant exponent $q$ as follows. Let $v(t) = e^{-it\Delta}\phi$ and $t > 0$.

(1) Fujita exponent $p = 2/d \iff q = 1$. The lower bound to apply the decay estimate. The situation is the same for the semilinear heat equation.
(2) Mass-critical exponent $p = 4/d \iff q = 2$. The upper bound to apply the decay estimate. It is not a restriction for the heat equation.
(3) Strauss exponent $p = p_0(d) \iff q = (p+2)^*$.

$$\phi \in L_x^{(p+2)^*} \implies v(t) \in L_x^{p+2} \implies |v(t)|^p v(t) \in L_x^{(p+2)^*},$$

so that we can use the iteration argument with the decay estimate.
(4) Our exponent $p = p_1(d) \iff q = (2p+2)^*$. Then

$$\phi \in L_x^{(2p+2)^*} \implies v(t) \in L_x^{2p+2} \implies |v(t)|^p v(t) \in L_x^2,$$

which is the lower bound to apply the decay estimate to the nonlinearity.

Let me conclude this report with an open question: What happens if we consider the same data at $t = 0$ instead of $t = \infty$, namely the randomized Cauchy problem:

$$i\dot{u} - \Delta u = |u|^p u, \quad u(0) = \phi^\omega, \quad 2/d < p < 4/d.$$

Since $\phi^\omega \in L^2(R^d)$ almost surely, we have a unique global solution $u \in C(R; L^2(R^d))$, cf. [9]. The free profiles cannot possibly complete the asymptotic description, since the equation has solitons, with arbitrarily small size of $L^2(R^d)$ (as well as higher Sobolev norms). Then it seems natural to expect

Randomized soliton resolution conjecture. Let $2/d < p < 4/d$ and $\phi \in L^2(R^d)$. Then for almost every $\omega$, there exists a sequence of solitons in the form $u_j(t,x) = e^{ita_j + ixk_j}\phi_j(x - c_j t)$ and $\phi \in L^2(R^d)$ such that the solution $u$ of the randomized Cauchy problem satisfies

$$\|u(t) - \sum_j u_j(t) - e^{-it\Delta}\phi\|_{L^2(R^d)} \to 0 \quad (t \to \infty).$$

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## Fully discrete approximation of general nonlinear Sobolev equations


Serge Nicaise

(joint work with Fatiha Bekkouche, Wided Chikouche)


We consider Sobolev's type equation (see [6])

$$\begin{cases} A_1(t,u)u_t + A_2(t,u)u = f(t,u) & \text{in } V', \quad 0 < t \le T, \\ u(0) = u_0 & \text{in } V, \end{cases}$$

where $V$ is a Hilbert space with norm $\|\cdot\|$, $V'$ its dual with respect to a pivot space $H$, $A_1(t,u) : V \to V'$ and $A_2(t,u) : V \to V'$ are linear continuous operators from $V$ into $V'$ that may depend on $t$ and $u$, but $A_1(t,u)$ is not continuous from $V$ into $H$, see [3] for the case when $A_1$ is independent of $t$ and $u$. Typical examples are the heat conduction involving two temperatures for isotropic materials [4] or the effects of pulsed electric fields on heterogeneous media [1].

Under the assumptions that $a_1(t;u;\cdot,\cdot)$ and $a_2(t;u;\cdot,\cdot)$ are uniformly continuous with respect to $t$ and $u$, and that $a_1(t;u;\cdot,\cdot)$ is uniformly coercive with respect

to $t$ and $u$, the previous problem is equivalent to the (vector-valued) ordinary differential equation

$$\left\{ \begin{array}{ll} u_t = F(t,u) & \text{in } V, \quad 0 < t \leq T, \\ u(0) = u_0 & \text{in } V, \end{array} \right. \tag{1}$$

where $F(t,u) = g(t,u) - \mathcal{A}(t,u)$, with $g(t,u) = A_1(t,u)^{-1} f(t,u)$ and $\mathcal{A}(t,u) = A_1(t,u)^{-1} A_2(t,u)$. As already noticed in [3], since $\mathcal{A}(t,u)$ is bounded from $V$ into $V$, this problem is nonstiff and therefore explicit (in time) schemes can be used to approximate such a problem. This last property added to some Lipschitz-type assumptions on $a_i$ and $f$ allows to apply Theorem 6 of [5] to get a local existence and uniqueness result, see [2].

We further perform a general analysis for a fully discrete scheme of problem (1) by combining some error estimates of explicit semi-discrete schemes in time of ordinary differential equations (adapted to Hilbert valued equations) with new error estimates of the corresponding fully discrete schemes based on some "regularity" assumptions and interpolation error estimates. Alltogether, if $U_{n,h}$ is the fully discrete approximation of the solution $u$ at time $t_n$ obtained by explicit Euler's scheme or explicit two-step Runge-Kutta's scheme (Heun's scheme), we prove the error estimate

$$\|u(t_n) - U_{n,h}\| \leq C((\Delta t)^p + h^{q(s)}), \tag{2}$$

for all $n = 1, \ldots, N$, where $p = 1$ (resp. $p = 2$) for Euler's scheme (resp. Runge-Kutta's scheme) and $q(s)$ is related to our abstract assumptions (but in practice it depends on the regularity of the initial datum and the chosen finite element space), and $C$ is a positive constant independent of $h$ and $\Delta t$. Different numerical examples from [2] are confirming the optimality of this error estimate.

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## Asymptotically stable selfsimilar blow-up of the supercritical gKdV

Lisa Onkes
(joint work with Herbert Koch)

The **(gKdV)** reads for $p \geq 2$

$$\text{(gKdV)} \qquad \begin{cases} \partial_t u + \partial_{xxx} u + \partial_x \left( |u|^{p-1} u \right) = 0 \\ u(0,x) = u_0 \end{cases} .$$

This equation is invariant under translation in time and space; and under the scaling

$$u_\lambda(x,t) = \lambda^{\frac{2}{p-1}} u(\lambda^3 t, \lambda x).$$

Therefore the homogeneous Sobolev spaces $\dot{H}^{s_c}$ with the *critical exponent* $s_c = \frac{1}{2} - \frac{2}{p-1}$ are scaling invariant:

$$\|u\|_{\dot{H}^{s_c}} = \|u_\lambda\|_{\dot{H}^{s_c}}.$$

If $s_c = 0$ we call the problem *critical*, for $s_c < 0$ *subcritical* and for $s_c > 0$ *supercritical*.

In the slightly supercritical case Koch [2] proved the existence of a selfsimilar solution to the (gKdV) and Lan [4] showed the presence of an open set of initial data in $H^1$ leading to selfsimilar blow-up solutions. We are able to prove the asymptotic stability of the selfsimilar blow-up solution found by Koch.

**Theorem 1.** *Let $0 < s_c < s < 1/2$ and assume a spectral condition. Then there exists a selfsimilar blow-up solution in $\dot{H}^s$ to the (gKdV) and a neighborhood of this solution, such that every solution in this set converges exponentially fast (in selfsimilar coordinates) to the selfsimilar solution.*

### The Spectral Assumption.

The (gKdV) has time-periodic solutions of the form

$$u(t,x) = Q(x-t),$$

where $Q \in H^1(\mathbb{R})$ solves

$$\partial_{xx} Q - Q + Q|Q|^{p-1} = 0.$$

There exists a unique real nonnegative radially symmetric exponentially decreasing $C^2$ solution $Q$ to this equation – the *ground state* – and every other solution is a translate of this solution (see [1], [3]).

The to $Q$ corresponding solution $u(t,x) = Q(x-t)$ of the (gKdV) is called the *soliton solution* and represents the solution where the dispersive behavior of the Laplacian is compensated by the focusing nonlinearity.

Let $0 < s_c < s < \frac{1}{2}$ and $u$ be a selfsimilar solution to the (gKdV) of the form

$$u(t,x) = \lambda^{\frac{2}{p-1}} V(\lambda x).$$

Therefore the *selfsimilar profile* $V$ has to satisfy

$$\frac{2}{p-1}V + yV_y - V_{yyy} - (|V|^{p-1}V)_y = 0,$$

if

$$\frac{\lambda_t}{\lambda} = -\lambda^3;$$

i.e. $\lambda(t) = (3t)^{-1/3}$.

We change the coordinates so that the bifurcation from the soliton equation becomes visible : Let $b > 0$,

$$\frac{\lambda_t}{\lambda} = -b\lambda^3.$$

Then the *selfsimilar profile* $V$ fulfills

$$\underbrace{b\left(\frac{2}{p-1}V + yV_y\right)}_{\Lambda V} - \left(V_{yy} + |V|^{p-1}V\right)_y = 0.$$

We will require

$$\int VQ_x \mathrm{d}x = 0,$$

such that this equation has a unique solution. We linearize the (gKdV) in selfsimilar coordinates around $V \in \dot{H}^s(\mathbb{R})$ and define the *linearized operator* $\mathcal{L}[b,p]$ given by

$$\mathcal{L}A := b\left(\frac{2}{p-1}A + yA_y\right) - A_{yyy} - \left(p|V|^{p-1}A\right)_y$$

which satisfies

- $\mathcal{L}(V_y) = -bV_y$ by the scaling invariance and
- $\mathcal{L}(\Lambda V) = -3b\Lambda V$ by the translation invariance.

We can now formulate our Assumption

**Assumption 2.** *There exists a $\delta > 0$ such that*

$$\|e^{\theta\mathcal{L}}\phi\|_{\dot{H}^s} \leq Ce^{-\delta\theta}\|\phi\|_{\dot{H}^s}$$

*for all $\theta \geq 0$ and $\phi \in (\mathrm{Ker}(\mathcal{L}^* + 3b) \oplus \mathrm{Ker}(\mathcal{L}^* + b))^{\perp}$.*

This Assumption states, that the only discrete eigenvalues of $e^{\theta\mathcal{L}}$ are the once induced by the invariances of the equation. We are able to rigorously prove this close to the critical case. Further away from the critical case we need to take this as a condition which might be verified numerically.

Under our Assumption we are able to construct a contraction

$$\varphi : B_\delta(V) \to V + (\mathrm{Ker}(\mathcal{L}^* + 3b) \oplus \mathrm{Ker}(\mathcal{L}^* + b))^{\perp},$$

and therefore prove Theorem 1.

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## A dynamical low-rank integrator for the Vlasov–Maxwell equations

Chiara Piazzola

(joint work with Lukas Einkemmer and Alexander Ostermann)

The Vlasov–Maxwell system models collisionless magnetized plasmas. The simulation of this type of plasma is crucial in many applications, for example, for magnetically confined nuclear fusion. A kinetic description of the charged particles in a plasma under the influence of an electromagnetic field is required. The particle distribution $f(t, x, v)$ is a function of position $x$ and velocity $v$, and it is given by the Vlasov equation

$$\partial_t f(t, x, v) + v \cdot \nabla_x f(t, x, v) + F(f) \cdot \nabla_v f(t, x, v) = 0.$$

We consider here just one particle species, the electrons, in an uniform ion background. They are subject to the Lorentz force $F = -(E + v \times B)$, where $E$ and $B$ denote the electric and magnetic field, respectively. Then the Vlasov equation has to be coupled with the Maxwell equations for the evolution of the electromagnetic field.

Kinetic models are very challenging from a computational point of view, due to the high dimensionality. The system is posed in an up to three-dimensional physical space $\Omega_x$ and a three-dimensional velocity space $\Omega_v$. A full discretization in dimension $d$ with $n$ points in each direction would lead to storage requirements $\mathcal{O}(n^{2d})$ with $d = 1, 2, 3$.

Recently a *dynamical low-rank approach* has been proposed for the Vlasov–Poisson system [1]. The key idea is to constrain the dynamics of the system to the low-rank manifold, as proposed in the seminal papers [4, 5]. The distribution function $f$ is approximated as a combination of functions in $x$ and in $v$

$$f(t, x, v) \approx \sum_{i,j=1}^{r} X_i(t, x) S_{ij}(t) V_j(t, v).$$

The number $r$ gives the rank of the approximation. The computational complexity of the problem is drastically reduced when $r \ll n$. This approximation is justified also for the Vlasov–Maxwell equations. By employing a Fourier analysis, one can show that in the linear regime the distribution function is of low-rank. The full analysis is reported in [3].

A rank-$r$ approximation of $f$ is obtained by means of an orthogonal projection onto the tangent space of the low-rank manifold. This approach, called *projector-splitting integrator* in [5], consists of projecting the right-hand side of the Vlasov equation onto the tangent space. The resulting differential equation is then solved by splitting methods. Three subflows have to be considered. For the first subflow the following partial differential equation is obtained

$$\partial_t K_j(t,x) = -\sum_l c^1_{jl} \cdot \nabla_x K_l(t,x) + \sum_l c^2_{jl} \cdot E(t,x) K_l(t,x) + \sum_l c^3_{jl} \cdot B(t,x) K_l(t,x),$$

where the unknown is $K_j(t,x) = \sum_i X_i(t,x) S_{ij}(t)$. The factor $S_{ij}$ is then updated by solving

$$\frac{\mathrm{d}}{\mathrm{d}t} S_{ij}(t) = \sum_{k,l} \left( c^1_{jl} \cdot d^2_{ik} - c^2_{jl} \cdot d^1_{ik} - c^3_{jl} \cdot d^3_{kl} \right) S_{kl}(t).$$

The third partial differential equation for $L_i(t,x) = \sum_j S_{ij}(t) V_j(t,v)$ is then given by

$$\partial_t L_i(t,v) = -\sum_k d^2_{ik} \cdot v L_k(t,v) + \sum_k d^1_{ik} \cdot \nabla_v L_k(t,v) + \sum_k d^3_{ik} \cdot (\nabla_v L_k(t,v) \times v).$$

We refer to [3] for the details on the coefficients $c^{1,2,3}_{jl}$ and $d^{1,2,3}_{ik}$. The differential equations can be solved by any suitable numerical method and employing any suitable phase space discretization. Note that a QR decomposition is required after the first and the third step to compute $X$ and $V$ from the resulting matrices $K$ and $L$. The advantage of this approach is that the physical and the velocity space decouple. The equation for $K_j$ is posed in position space, whereas the equation for $L_i$ depends only on velocity. Thus, two differential equations in an up to three-dimensional space have to be solved.

Moreover, the employed low-rank approximation combines very well with Maxwell's equations, which are posed in the position space only. The time evolution of the electromagnetic field is given by

$$\begin{aligned} \partial_t E(t,x) &= \nabla \times B(t,x) - j(t,x) \\ \partial_t B(t,x) &= -\nabla \times E(t,x). \end{aligned}$$

The source terms, the charge density and current density, are given by

$$\rho(t,x) = \int_{\Omega_v} f(t,x,v)\,\mathrm{d}v \quad \text{and} \quad j(t,x) = \int_{\Omega_v} v f(t,x,v)\,\mathrm{d}v,$$

respectively. At the continuous level, the electromagnetic field is self-consistent with the charge function obtained from Maxwell's equations. The two divergence constraints, the so-called Gauss laws, are automatically satisfied

$$\begin{aligned} \nabla \cdot E(t,x) &= \frac{1}{|\Omega_x|} \int_{\Omega_x} \rho(t,x)\,\mathrm{d}x - \rho(t,x) \\ \nabla \cdot B(t,x) &= 0. \end{aligned}$$

The divergence constraint for the electric field is a direct consequence of the conservation of charge. At the discrete level, however, the divergence of the electric

field is not consistent with the charge density obtained from the Vlasov equation. A solution violating Gauss' law is wrong from the physical point of view. To overcome this problem we propose a correction procedure to explicitly enforce the divergence constraint at each time step. This approach was used, e.g., in [6]. Some related ideas have been used in [2] for the conservation of mass and momentum for the Vlasov–Poisson equation. Let us denote the incorrect value of the electric field after one time step by $\bar{E}^{n+1}$. This value does not satisfy Gauss' law. We adjust it by including a correction potential $\phi$, such that the final value of the electric field is

$$E^{n+1} = \bar{E}^{n+1} - \tau \nabla \phi.$$

The correction potential is computed from the following Poisson equation

$$-\Delta \phi = \frac{\frac{1}{|\Omega_x|} \int_{\Omega_x} \rho^{n+1} \, \mathrm{d}x - \rho^{n+1} - \nabla \cdot \bar{E}^{n+1}}{\tau}.$$

Note that the correction is constructed by taking into account the specific numerical integrator employed to get $\bar{E}^{n+1}$. This strategy enforces the divergence constraint up to machine precision.

The proposed low-rank integrator succeeds in capturing the main features of the Vlasov–Maxwell system. In the linear regime the physical behaviour is already very well reproduced even with a very small rank. Highly nonlinear problems require a higher rank. This is, for example, the case of the Weibel instability after the saturation of the magnetic field. For more details on the numerical results we refer the reader to [3].

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## A pseudo-spectral approach for the linearized Korteweg–de Vries equation

Mirko Residori
(joint work with Lukas Einkemmer and Alexander Ostermann)

We propose a new method for solving the linearized Korteweg–de Vries (KdV) equation

$$u_t + g(x)u_x + u_{xxx} = 0, \quad (t,x) \in [0,T] \times [a,b] \tag{1}$$

with initial data $u(0,x) = u_0(x)$, subject to transparent boundary conditions. We follow a splitting strategy in order to solve separately the advective and the dispersive part of the equation. For this purpose, we employ a Lie–Trotter splitting.

Let $0 = t_0 < t_1 < \cdots < t_M = T$ be an equidistant time discretization with time step $\tau$ ($t_m = m\tau$) and let $\Phi^\tau$ and $\Psi^\tau$ be the partial flows of

$$\begin{cases} v_t + g(x)v_x = 0, \\ v(0,x) = v_0(x), \end{cases} \tag{2}$$

$$\begin{cases} w_t + w_{xxx} = 0, \\ w(0,x) = w_0(x), \end{cases} \tag{3}$$

respectively. The numerical approximation to (1) at time $t = t_{m+1}$ is given by

$$u^{m+1}(x) = \Psi^\tau \circ \Phi^\tau(u^m)(x), \tag{4}$$

where $u^m$ is the numerical solution to (1) at time $t = t_m$.

We solve (2) by the explicit Euler scheme, since the CFL condition determined by the advection $g(x)v_x$ is generally not prohibitive. For solving (3) we choose the implicit Euler method. The numerical scheme (4) thus takes the form

$$u^{m+1} + \tau\partial_x^3 u^{m+1} = u^m - \tau g(x)u_x^m, \quad u^0(x) = u(0,x). \tag{5}$$

The transparent boundary conditions are then designed for the numerical scheme (5) in the same spirit as in [2]. We emphasise that such conditions are non-local in time. To summarize, we obtain the following numerical scheme

$$\begin{cases} u^{m+1} + \tau\partial_x^3 u^{m+1} = u^m - \tau g(x)u_x^m, \quad (t,x) \in [0,T] \times (a,b), \\ u(0,x) = u^0(x), \\ u^{m+1}(a) = f_1(t_{m+1}, u^{m+1}), \\ u^{m+1}(b) = f_2(t_{m+1}, u^{m+1}), \\ u_x^{m+1}(b) = f_3(t_{m+1}, u^{m+1}). \end{cases} \tag{6}$$

The functions $f_1$, $f_2$ and $f_3$ depend on the time history of the numerical solution at the boundaries. Further, we highlight their dependence on $u^{m+1}$.

We now take advantage of the employed Lie–Trotter splitting by carrying out the spatial discretization by a pseudo-spectral approach, as it has been done for the pure dispersive equation in [3]. This approach gives us a very accurate spatial numerical solution using a modest number of grid points. To proceed, let $\mathcal{P}_N$ be

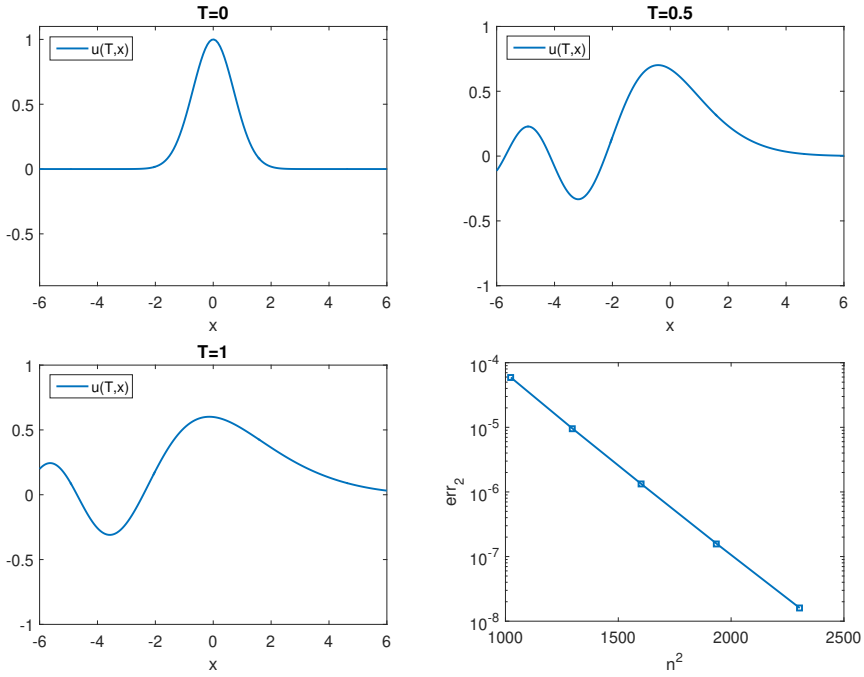


FIGURE 1. Top left: initial data $u^0(x)$. Top right: numerical solution at time $T = 0.5$. Bottom left: numerical solution at time $T = 1$. Bottom right: semilog plot of the spatial error.

the polynomial space of degree $N$ and let us define

$$\begin{aligned} V_N = \{u \in \mathcal{P}_N \,|\, u(a) &= f_1(t_{m+1}, u^{m+1}), \\ u(b) &= f_2(t_{m+1}, u^{m+1}), \\ u_x(b) &= f_3(t_{m+1}, u^{m+1})\} \end{aligned}$$

and let $V_N^*$ be the *dual* space of $V_N$. The variational formulation of (6) reads: find $u \in V_N$ such that for every $v \in V_N^*$

$$(u^{m+1}, v) + \tau(\partial_x^3 u^{m+1}, v) = (u^m - \tau g(x) u_x^m, v). \tag{7}$$

By choosing the basis functions of $V_N, V_N^*$ properly (similarly to [1]) we obtain banded matrices that discretize in space equation (7). Therefore, we can solve the linear system with $O(N)$ operations.

In Fig. 1 we present a numerical simulation that shows the effect of the transparent boundary conditions as well as the fast convergence of the implemented dual-Petrov–Galerking method. In particular, we consider

$$[a, b] = [-6, 6], \quad u^0(x) = \mathrm{e}^{-x^2}, \quad g(x) = g_4 x^4 + g_3 x^3 + g_2 x^2 + g_1 x + g_0$$

and a final time $T = 1$. The polynomial $g(x)$ is assumed to be constant for $x \in \mathbb{R} \setminus [-6, 6]$ and chosen everywhere differentiable. Therefore, we set $g_0 = 2 + 4g_1 - 18g_2$, $g_1 = 1/5$, $g_2 = 1/10$, $g_3 = -g_1/108$, $g_4 = -g_2/72$.

We employ an equidistant time discretization with $m = 2^{13}$ time steps and a space discretization with $N = 2^8$ grid points. We plot the numerical solution at times $T = 0$, 0.5 and 1. We notice no visible reflections at the boundaries as the solution "leaves" the domain smoothly thanks to the transparent boundary conditions. In a semi-logarithmic plot we show the super-polynomial spatial convergence of the employed pseudo-spectral scheme.

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## Error analysis of Runge-Kutta schemes for quasilinear hyperbolic problems

Roland Schnaubelt

(joint work with Marlis Hochbruck, Tomislav Pažur)

Quasilinear hyperbolic evolution equations describe a wide range of phenomena in physics, including in particular the Maxwell system with nonlinear instantaneous constitutive laws. There is a well established analytical theory for such problems. On the other hand, despite their importance, for quasilinear hyperbolic problems there are only very few rigorous convergence results concerning time integration methods. For instance, in [1] it is shown that the approximations of the semi-implicit Euler method for (1) are well-posed and converge with order 1/2.

Kato's approach from [4] provides a unified framework for a large class of quasilinear hyperbolic evolution equations. In this work a local well-posedness theory is established in spaces like $H^3$ in the case of first order systems. However, the setting of [4] cannot directly be applied to problems like quasilinear Maxwell equations, as noted on p. 53 of [4]. One has to invoke state dependent (energy-type) norms in addition. This approach was elaborated by Kato in [5] for a large class of evolution equations. We work within a refinement of Kato's theory due to Müller [8] whose framework is closer to the applications we have in mind than that of [5].

In this framework two of the present authors recently showed well-posedness and first order convergence of the semi-implicit and implicit Euler approximations to quasilinear hyperbolic evolution equations, see [2]. Very recently, in [6] implicit Runge–Kutta schemes have been analyzed in Kato's original framework of [4].

In the present paper we study implicit Runge–Kutta methods for the quasilinear hyperbolic evolution equation

$$\Lambda(u(t))u'(t) = Au(t) + Q(u(t))u(t), \qquad u(0) = u_0. \tag{1}$$

Let $Z \hookrightarrow Y \hookrightarrow X$ be densely embedded Hilbert spaces and $R > 0$ a fixed radius. We assume that $A$ is a linear skew-adjoint operator in $X$ which induces a map $A \in \mathcal{B}(Z, Y)$, that $\Lambda(v) \in \mathcal{B}(X)$ is a symmetric positive definite operator, and that $Q(v) \in \mathcal{B}(X)$ is uniformly bounded for $v$ in the ball $B_Y(R)$. Moreover, the maps $\Lambda : B_Y(R) \to \mathcal{B}(X)$, $\Lambda^{-1} : B_Y(R) \to \mathcal{B}(Y)$, and $Q : B_Y(R) \to \mathbb{B}(Z, Y)$ are Lipschitz. Let $v \in B_Y(R) \cap B_Z(r)$ for some $r \geq 1$. Set $A(v) = \Lambda(v)^{-1}(A + Q(v))$. The core condition for (1) is the existence of an isomorphism $S : Z \to X$ and operators $B(v) \in \mathcal{B}(X)$ bounded by $\beta(r)$ such that $SA(v)S^{-1} = A(v) + B(v)$. (We omit further minor conditions which can be found in our paper [3].)

These assumptions are valid for quasilinear Maxwell and other wave-type equations on the full space or with Dirichlet boundary conditions, see [2] or [8] for a precise statement. Here $X$, $Y$ and $Z$ are suitable subspaces of $L^2$, $H^2$ and $H^3$, respectively. The Maxwell system with the usual boundary conditions of a perfect conductor can be treated in our framework only for special nonlinearities, cf. Proposition 4.8 in [8].

Under these hypotheses, the problem (1) is locally wellposed in $Z$ due to the main results in [8] or [5]. In particular, for $u_0 \in B_Y(R/c_0) \cap B_Z(r)$ there is a time $T_0 = T_0(r) = c_0'/(r + \beta(r)) > 0$ and a unique solution $u \in C([0, T_0], Z) \cap C^1([0, T_0], Y)$ of (1), cf. Theorem 2.3 in [3]. These constants and those below only depend on the given operators (and the Runge–Kutta scheme), see [3] for details.

The general $m$-stage Runge–Kutta method with $m$ distinct nodes $0 \leq c_i \leq 1$ and weights $a = (a_{ij})_{i,j=1}^m$ and $b = (b_i)_{i=1}^m$ is given by

$$\begin{aligned} U_{ni} &= u_n + \tau \sum_{j=1}^{s} a_{ij} A(U_{ni}) U_{nj}, \\ u_{n+1} &= u_n + \tau \sum_{i=1}^{s} b_i A(U_{ni}) U_{ni} \end{aligned} \tag{2}$$

for $n \in \{0, \dots, N-1\}$, $i \in \{1, \dots, m\}$, a step size $\tau > 0$, and $N\tau \geq T$. Here, $T$ is given below, $u_n \approx u(t_n)$ approximates the solution $u$ to (1) at time $t_n = n\tau$, and $U_{ni} \approx u(t_n + c_i\tau)$ are the internal stages.

We assume that $a$ is invertible, that the scheme has order of at least $m+1$, and that it is *algebraically stable* and *coercive*; i.e,

- $(b_i a_{ij} + b_j a_{ji} - b_i b_j)$ is positive semidefinite, $b_i \geq 0$ for all $i$,
- there is a number $\alpha > 0$ and a positive diagonal matrix $D \in \mathbb{R}^{m \times m}$ with $\xi^\top D a^{-1} \xi \geq \alpha \xi^\top D \xi$ for $\xi \in \mathbb{R}^m$.

Typical examples are Gauß or Radau IA and IIA collocations methods. To obtain full classical convergence order, one would need rather strong additional regularity assumptions, cf. Section 4.5 of [6], which we want to avoid here.

In Proposition 4.3 of our paper [3] we showed that there are solutions of (2) for $u_0 \in B_Y(R/c_1) \cap B_Z(r)$ and $n \le T_1/\tau$, where $c_1 \ge c_0$, $T_1 = T_1(r) = c_1'/(r + \beta(r))$, $\tau \in (0, \tau_0(r)]$, and $\tau_0(r) \in (0, 1]$ is maximal time step size. Moreover, the solutions $u_n$ are bounded by

$$\|u_n\|_Z \le c\mathrm{e}^{\kappa(r+\beta(r))n\tau} \|u_0\|_Z .$$

In our main Theorem 5.3 of [3] we show an error estimate of the scheme.

**Theorem.** *Let the above assumptions be fulfilled. Let $u_0 \in B_Y(R/c_1) \cap B_Z(r)$, $T = \min\{T_0(r), T_1(r)\}$, $\tau \in (0, \tau_0(r)]$, and $n \in \mathbb{N}$ with $n \le T/\tau$. Let $u \in C([0,T], Z) \cap C([0,T], Y)$ be the solution of (1) and assume that $u^{(m+1)} \in L^2([0,T], D(A))$ and $u^{(m+2)} \in L^2([0,T], X)$. Then the error of the Runge–Kutta scheme (2) is bounded by*

$$\|e_n\|_X \le Cr^{1/2}\mathrm{e}^{CrT}\tau^{m+1}\Big(\int_0^T \big(\|u^{(m+1)}(t)\|_{D(A)}^2 + \|u^{(m+2)}(t)\|_X^2\big)\,\mathrm{d}t\Big)^{1/2}.$$

Theorem 6.3 of [3] also gives a variant of the result for convergence in $Y$. To treat (1) within Kato's approach, we first invert $\Lambda(u(t))$. Our analysis (and also that of [5, 8]) then crucially depends on the dissipativity of $\Lambda(u(t))^{-1}A$ with respect to the scalar product on $X$ with the (state-depending) weight $\Lambda(u(t))$. This fact is essential for the construction of one step of the scheme. Even more importantly, the dissipativity provides the main energy-type bounds for the numerical solution. On the other hand, these norms lead to substantial new challenges throughout the proofs since one is forced to switch between them within the estimates. Energy techniques for implicit Runge–Kutta methods which are algebraically stable and coercive have been successfully applied to analyze stiff ordinary differential equations. Our analysis is motivated by [7], where the algebraic stability was an essential tool to prove rigorous error bounds for quasi-linear parabolic problems.

As noted above, the setting of [4, 5, 8] does not work well for boundary value problems, in contrast to full space problems. This shortcoming is unfortunate since the more operator-theoretic approach in these works fits very well to the tools from numerical analysis used in [2, 3, 6]. In future work we want to combine the present approach with results and (more PDE type) methods of the very recent paper [9], and we will also study the space discretization error which is not considered here.

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## Non-trivial self-similar blowup for the focusing energy-supercritical wave equation

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(joint work with Irfan Glogić)

We consider the semilinear wave equation

$$(\partial_t^2 - \Delta)u = |u|^{p-1}u, \tag{1}$$

for $u : (t,x) \in I \times \mathbb{R}^d \to \mathbb{R}$, $I \subset \mathbb{R}$, $0 \in I$, $d \geq 3$, $p > 1$. The nonlinearity corresponds to the so-called focusing case. As a natural toy model for more involved problems, this equation has received a lot of attention in the recent past. Eq. (1) is invariant the scaling transformation $u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x)$. For $p = \frac{d+2}{d-2}$, rescaling leaves invariant the norm in the energy space $\dot{H}^1 \times L^2(\mathbb{R}^d)$, which defines the energy critical case. We are interested in supercritical nonlinearities $p > \frac{d+2}{d-2}$.

It is well-known that solutions to Eq. (1) can blow up in finite time for all $p > 1$ and $d \geq 1$. In view of the scale invariance it is natural to look for self-similar blowup solutions

$$u(t,x) = (T-t)^{-\frac{2}{p-1}} f(\tfrac{x}{T-t}), \quad T > 0.$$

A well-known example is given by the spatially constant ODE blowup profile $f_0 = (\frac{2(p+1)}{(p-1)^2})^{\frac{1}{p-1}}$. Apart from this trivial solution, countably many smooth, non-trivial radial profiles $\{f_n\}_{n\in\mathbb{N}_0}$ are known to exist at least for $d = 3$ and each odd $p \geq 7$, see [2]. Within these families, $f_0$ is the only profile that is known in closed form. Concerning the role of self-similar solutions in the generic time evolution, numerical experiments performed in [1] in the radial setting suggest the following picture: For small initial data, solutions disperse as $t \to \infty$. For large generic data, solutions blowup in finite time $T < \infty$ in a self-similar manner and approach $f_0$ locally around the blowup point as $t \to T^-$. For data fine-tuned to the threshold between these two basins of attraction, the evolution is described by the non-trivial self-similar profile $f_1$ that is approached for some intermediate period before one of the two above scenarios occurs.

Naturally, the aim is to describe these observations rigorously. In $d = 3$, the stability of the ODE blowup for Eq. (1) in the supercritical case as been established in [6], [7]. Generalizations to higher space dimensions are given in [8], [3]. Concerning the threshold conjecture, little is known so far. One difficulty here is that the conjectured *critical* solution $f_1$ is not known in closed form. In this respect, the situation seems to improve in higher space dimensions: For $d \geq 5$ and $p = 3$ we found an *explicit* non-trivial blowup solution, see [9], given by

$$u_T^*(t,x) = (T-t)^{-1} f^*\left(\tfrac{|x|}{T-t}\right), \quad f^*(\rho) = \frac{2\sqrt{2(d-1)(d-4)}}{d-4+3\rho^2}.$$

To our knowledge, this solution was not known before. By exploiting further symmetries of Eq. (1), namely the invariance under spatial translations and Lorentz boosts, where the latter are parametrized in terms of the rapidity $a \in \mathbb{R}^d$, we obtain the $(2d+1)$-parameter family of explicit solutions $u^*_{T,x_0,a}$. In [9], we restrict ourselves to $d = 7$ and prove that this family is *codimension one stable* in a backward lightcone under small perturbations in a suitably high Sobolev space. The proof is based on the methods of [7], which rely on the reformulation of the problem in similarity coordinates and the use of semigroup theory and operator analysis. Apart from obvious modifications due to the codimension one nature of our result, the main difficulty is the analysis of the spectral problem arising in the linearization. This is due to the fact that $f^*$ is spatially non-trivial. By refining the techniques of [4], [5] we were able to prove that the corresponding linearized operator has exactly one genuine unstable eigenvalue $\lambda^* > 0$ apart from symmetry eigenvalues that can be controlled in the nonlinear time evolution by modulating the blowup parameters $(T, x_0, a)$. For this, it was crucial that $\lambda^* = 3$ in seven dimensions and that the corresponding eigenfunction is known explicitly. This is not the case in other dimensions and the investigation of the spectral problem for $d \geq 5$ will be the subject of further research.

In view of our result, we conjecture that $f^*$ is a critical self-similar profile for the focusing cubic wave equation in $d \geq 5$, describing the behavior of solutions at and, at least for some intermediate time, close to the threshold between stable blowup and global existence.

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## Low regularity integrators for the Schrödinger equation

Katharina Schratz
(joint work with Marvin Knöller, Alexander Ostermann, Frédéric Rousset)

### 1. A first-order low regularity scheme

In the recent work [6] a new low regularity Fourier integrator for the periodic cubic Schrödinger equation

$$i\partial_t u = -\Delta u + \mu|u|^2 u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^d \tag{1}$$

(with $\mu \in \mathbb{R}$) was introduced. The main benefit of the new scheme [6] lies in the fact that its first-order convergence only requires the boundedness of one additional derivative of the solution (compared to classical numerical schemes which require the boundedness of two additional derivatives of the solution). More precisely, estimating the global error in $H^r$ only requires solutions in $H^{r+1}$ ($r > d/2$) compared to the required regularity of $H^{r+2}$ solutions $u(t)$ imposed by classical schemes.

The construction of the new scheme ([6])

$$u^{n+1} = \mathrm{e}^{i\tau\Delta}\left[u^n - i\mu\tau\big(u^n\big)^2\Big(\varphi_1(2i\tau\Delta)\overline{u^n}\Big)\right] \quad \text{with} \quad \varphi_1(z) = \frac{\mathrm{e}^z - 1}{z} \tag{2}$$

is based on Duhamel's formula looking at the *twisted* variable $v(t) = \mathrm{e}^{-it\Delta}u(t)$ and treating the dominant nonlinear frequency interactions (triggered by $\overline{u}$) in an exact way in the scheme while neglecting the lower order terms. This idea of twisting the variable is widely used in the analysis of partial differential equations in low regularity spaces.

The analysis in [6] is restricted to periodic Schrödinger equations (that is $x \in \mathbb{T}^d$). The benefit of periodicity lies in the fact that the nonlinear frequency interactions and the corresponding resonance structure can be analysed explicitly with Fourier expansion techniques. More precisely, iterating Duhamel's formula of (1) yields (cf. [4])

(3)

$$\begin{aligned} u(t_n+\tau) &= \mathrm{e}^{i\tau\Delta}u(t_n) - i\mu\mathrm{e}^{i\tau\Delta}\int_0^\tau \mathrm{e}^{-i\xi\Delta}\left(\left|u(t_n+\xi)\right|^2 u(t_n+\xi)\right)\mathrm{d}\xi \\ &= \mathrm{e}^{i\tau\Delta}u(t_n) - i\mu\mathrm{e}^{i\tau\Delta}\int_0^\tau \mathrm{e}^{-i\xi\Delta}\left(\left|\mathrm{e}^{i\xi\Delta}u(t_n)\right|^2 \mathrm{e}^{i\xi\Delta}u(t_n)\right)\mathrm{d}\xi + \mathcal{R}(\tau,t_n,u), \end{aligned}$$

where the remainder satisfies for $r > d/2$ the bound

$$\|\mathcal{R}(\tau, t_n, u)\|_r \leq c\tau^2 \sup_{t_n \leq t \leq t_{n+1}} \|u(t)\|_r^5. \tag{4}$$

With the aid of the Fourier expansion we observe with $w = u(t_n)$ that

$$\int_0^\tau \mathrm{e}^{-i\xi\Delta}\left(\left|\mathrm{e}^{i\xi\Delta}w\right|^2 \mathrm{e}^{i\xi\Delta}w\right)\mathrm{d}\xi \;=\; \int_0^\tau \sum_{k=-k_1+k_2+k_3} \mathrm{e}^{i\xi \mathrm{R}(k)}\mathrm{d}\xi\, \overline{\hat{w}}_{k_1}\hat{w}_{k_2}\hat{w}_{k_3}\mathrm{e}^{ikx},$$

where the underlying resonance structure is given by all $k_1, k_2, k_3 \in \mathbb{Z}^d$ and $k = -k_1 + k_2 + k_3$ such that

$$\mathrm{R}(k) = k^2 + k_1^2 - k_2^2 - k_3^2 = 2k_1^2 - 2k_1(k_2 + k_3) + 2k_2k_3. \tag{5}$$

Note that $k_1^2$ corresponds to two derivatives acting on $\overline{w}$, i.e., a term of type $w^2\Delta\overline{w}$, while the lower order products $k_1k_2, k_1k_3$ and $k_2k_3$ correspond to terms of type $w\nabla\overline{w}\cdot\nabla w$ and $\overline{w}\nabla w\cdot\nabla w$, respectively. Therefore, we select $k_1^2$ as the dominant term of the resonance structure (5) and treat it exactly in our scheme. On the other hand, we approximate the lower order terms $k_\ell k_j$ with $\ell \neq j$ and neglect them in our scheme. This procedure leads to the scheme (2)(see [6] for details).

## 2. A second-order low regularity scheme

In the recent preprint [4] a second-order Fourier integrator for the Schrödinger equation (1) was introduced. The new schemes allows us to lower classical regularity assumptions of higher-order methods and nevertheless obtain a convergence order $\nu > 1$. The idea is again based on iterating Duhamel's formula in the twisted variable $v(t) = \mathrm{e}^{-it\Delta}u(t)$ (for the second-order scheme up to higher-order terms) and controlling the dominant terms in the nonlinear frequency interactions. The construction, however, is much more involved due to the complicated structure of resonances. In particular, a straightforward higher-order extension of the ideas for the first-order scheme developed in [6] would only lead to an unstable scheme.

Based on a rigorous higher-order resonance analysis we constructed in [4] a second-order Fourier integrator which in dimension $d = 1$ takes the form

$$u^{n+1} = \mathrm{e}^{i\tau\partial_x^2}\Big(\mathrm{e}^{i\mu\tau|u^n|^2}u^n - i\mu\big(J_{1,x}^\tau(u^n) + J_{2,x}^\tau(u^n)\big)\Big). \tag{6}$$

For the precise structure of the correction terms see [4]. The new scheme (6) allows second-order approximations of (1) in $H^r$ with $r \geq 0$ for solutions in $H^{r+2}$ in the one dimensional setting $d = 1$, see [4, Theorem 3.2] for the precise convergence estimate. This in particular includes the important class of classical solutions in $L^2$ (set $r = 0$) and extends the first-order bound in a natural and expected way. However, due to the complicated structure of resonances we face an order reduction down to $3/2$ in the higher-dimensional setting $d > 1$.

## 3. Error estimates in low regularity spaces

While the ideas in [6, 4] allowed us to improve the local error produced by the low regularity scheme, the error analysis was restricted to classical tools in the analysis of nonlinear partial differential equations. Due to Sobolev embedding our error estimates could only be established in higher-order (smooth) Sobolev spaces $H^r$ with $r > d/2$. The latter requires solutions (at least) in $H^{\delta+d/2}$ to allow convergence rates of order $\tau^\delta$.

In [5]we considered (inspired by [1, 2, 3]) a filtered version of (2)

$$u^{n+1} = \mathrm{e}^{i\tau\Delta}\left(u^n - i\tau\Pi_K\left(\left(\Pi_K u^n\right)^2 \varphi_1(-2i\tau\Delta)\Pi_K\overline{u}^n\right)\right) \tag{7}$$

with $u^0 = \Pi_K u(0)$ and the projection operator defined by the Fourier multiplier

$$\Pi_K = \chi^2\left(\frac{-i\nabla}{K}\right),$$

where $\chi$ is a smooth radial nonnegative function supported in $B(0,2)$. With the aid of general discrete Strichartz estimates

$$\left\|\mathrm{e}^{in\tau\Delta}\Pi_K f\right\|_{l^p_\tau L^q} \le C(K\tau^{\frac{1}{2}})^{\frac{2}{p}}\|f\|_{L^2}$$

we could prove the following $L^2$ error estimates for $H^1$ solutions (cf. Theorem 2.1 in [5]): For every $T > 0$ and $u_0 \in H^1$, let us denote by $u \in \mathcal{C}([0,T], H^1)$ the exact solution of (1) with initial datum $u_0$ and by $u^n$ the sequence defined by scheme (7). Then, there exist $\tau_0 > 0$ and $C_T > 0$ such that for every step size $\tau \in (0, \tau_0]$, we have the following global error estimates:

- if $d = 1$, with the choice $K = 1/\tau^{\frac{5}{6}}$,
$$\|u^n - u(t_n)\|_{L^2} \le C_T\tau^{\frac{5}{6}}, \quad 0 \le n \le N,$$
- if $d = 2$, with the choice $K = 1/\tau^{\frac{3}{4}}$,
$$\|u^n - u(t_n)\|_{L^2} \le C_T\tau^{\frac{3}{4}}, \quad 0 \le n \le N,$$
- if $d = 3$, with the choice $K = 1/\tau^{\frac{2}{3}}$,
$$\|u^n - u(t_n)\|_{L^2} \le C_T\tau^{\frac{2}{3}}\left|\log\tau\right|^{\frac{2}{3}}, \quad 0 \le n \le N,$$

where $N$ is such that $N\tau \le T$.

This in particular allowed us to break the natural order barrier of $\tau^{1/2}$ imposed by classical schemes for $H^1$ solutions.

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## The gyrokinetic approximation of the Vlasov-Maxwell equations

Eric Sonnendrücker

Magnetic fusion research is being actively pursued in order to provide a clean energy available on demand. The principle is to confine a plasma, which is a gas of charged particles at a very large temperature, around $100000^oC$, so that the fusion reaction can generate energy with a positive balance. At such a large temperature the plasma needs to be completely isolated from the wall of the reactor. This can be achieved in toroidal devices, thanks to a very large magnetic field.

An appropriate model for the nonlinear evolution of the plasma is the Vlasov-Maxwell system, based on the Vlasov equation which describes the evolution of the particle density in phase space for each species

$$\frac{\partial f_s}{\partial t} + v \cdot \nabla_x f_s + \frac{q_s}{m_s}(E + v \times B) \cdot \nabla_v f_s = 0.$$

nonlinearly coupled to Maxwell's equations

$$\frac{\partial E}{\partial t} - \mathrm{curl} B = -J, \qquad \frac{\partial B}{\partial t} + \mathrm{curl} E = 0, \tag{1}$$

$$\mathrm{div} E = \rho, \qquad \mathrm{div} B = 0. \tag{2}$$

The sources for the Maxwell equations are obtained from the Vlasov equations as follows: The charge density is $\rho = \sum_s q_s \int f_s dv$, and the current density is $J = \sum_s q_s \int f_s v \, dv$. The so-called particle distribution functions $f_s$ depend on the 6 phase-space variables $(x, v)$ and on time.

In a large slowly varying magnetic field $\mu = \frac{1}{2} m v_\perp^2 / |B|$ is an adiabatic invariant of the particles equations of motion

$$\frac{\mathsf{d}X}{\mathsf{d}t} = V, \quad \frac{\mathsf{d}V}{\mathsf{d}t} = \frac{q_s}{m_s}(E + V \times B)$$

which are the characteristics of the Vlasov equation, which is a transport equation in phase space (it is an exact invariant for a constant magnetic field). This property can be used to transform the equations order by order in the small parameter (corresponding to the slow variation of B), using a change of coordinates obtained by a near identity Lie transform, to find a new equation of motion where the new $\mu$ variable is exactly conserved and the fast motion restricted to its conjugate angle $\theta$, that can be averaged out. One finds reduced equations of motion, which are dynamical only in 4 variables instead of 6, $\mu$ being an invariant and $\theta$ being averaged

out. The Vlasov type equation constructed with this new set of characteristics is called the *gyrokinetic* equation.

In this talk we recalled the development of gyrokinetic theory, starting with the reduction of the single particles equation of motion. This started with the asymptotic theory of Hamiltonian systems by Kruskal [1]. Another seminal paper was written by Littlejohn [2], where the reduction was directly performed on the particle Lagrangian, which allowed to preserve the variational structure of the original model. A full mathematical proof, with a different kind of averaging technique, was recently proposed by Possanner [3].

In order to obtain a variational form of the self-consistent gyrokinetic model, coupled with the field equations, the Vlasov-Maxwell action principle can be used

$$\mathcal{S} = \sum_{\mathrm{s}} \int f_s(\mathbf{z}_0, t_0) L_s(\mathbf{X}(\mathbf{z}_0, t_0; t), \dot{\mathbf{X}}(\mathbf{z}_0, t_0; t), t) d\mathbf{z}_0 \, dt + \frac{\epsilon_0}{2} \int |\nabla\phi + \frac{\partial \mathbf{A}}{\partial t}|^2 d\mathbf{x} \, dt - \frac{1}{2\mu_0} \int |\nabla \times \mathbf{A}|^2 d\mathbf{x} \, dt.$$

the single particle Lagrangian being given by

$$L_s = (\mathbf{x}, \mathbf{v}, \dot{\mathbf{x}}, t) = (m_s \mathbf{v} + q_s \mathbf{A}) \cdot \dot{\mathbf{x}} - (\frac{1}{2} m_s |\mathbf{v}|^2 + q_s \phi).$$

As proposed by Sugama [4], the single particle Lagrangian can then be replaced by the reduced single particle Lagrangian derived before.

This has become the base of modern gyrokinetic theory, see [5] for a review for physicists, which is widely used for self-consistent plasma simulations in magnetic fusion devices like Tokamaks or Stellarators.

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## Local wellposedness of quasilinear Maxwell equations on domains


Martin Spitz

(joint work with Roland Schnaubelt)


The Maxwell equations are the mathematical formulation of electromagnetism. In the presence of matter they are given in their macroscopic version by

$$\begin{aligned} \partial_t \boldsymbol{D} &= \operatorname{curl} \boldsymbol{H} - \boldsymbol{J}, & \operatorname{div} \boldsymbol{D} &= \rho, \\ \partial_t \boldsymbol{B} &= -\operatorname{curl} \boldsymbol{E}, & \operatorname{div} \boldsymbol{B} &= 0, \end{aligned} \tag{1}$$

where $\boldsymbol{E}(t,x), \boldsymbol{D}(t,x) \in \mathbb{R}^3$ denote the electric and $\boldsymbol{H}(t,x), \boldsymbol{B}(t,x) \in \mathbb{R}^3$ the magnetic fields, $\rho(t,x) \in \mathbb{R}$ the charge density, and $\boldsymbol{J}(t,x) \in \mathbb{R}^3$ the current density. These equations have to be complemented by constitutive equations between the electric and magnetic fields. There are several models for these so called material laws. Here we look at the instantaneous ones. We take $(\boldsymbol{E}, \boldsymbol{H})$ as state variable and assume that $\boldsymbol{D}$ and $\boldsymbol{B}$ are given by

$$(\boldsymbol{D}, \boldsymbol{B}) = \theta(x, \boldsymbol{E}, \boldsymbol{H}), \tag{2}$$

where $\theta\colon G \times \mathbb{R}^6 \to \mathbb{R}^6$ is sufficiently regular and we consider system (1) on a domain $G \subseteq \mathbb{R}^3$. A typical example is the Kerr nonlinearity arising in nonlinear optics, where $\boldsymbol{D} = \boldsymbol{E} + \vartheta(x)|\boldsymbol{E}|^2\boldsymbol{E}$ and $\boldsymbol{B} = \boldsymbol{H}$. Our main assumption on $\theta$ is that its derivative $\chi = \partial_{(\boldsymbol{E},\boldsymbol{H})}\theta$ is symmetric and uniformly positive definite.

If $G \neq \mathbb{R}^3$, this system has to be equipped with suitable boundary conditions. Typical ones are those of a perfect conductor

$$\boldsymbol{E} \times \nu = 0, \quad \boldsymbol{B} \cdot \nu = 0 \quad \text{on } \partial G \tag{3}$$

or absorbing ones

$$\boldsymbol{H} \times \nu + (\zeta(x, \boldsymbol{E} \times \nu)\boldsymbol{E} \times \nu) \times \nu = 0 \quad \text{on } \partial G, \tag{4}$$

where $\nu$ denotes the outer unit normal vector of $\partial G$. In the case of absorbing boundary conditions, the symmetric and positive definite function $\zeta\colon \partial G \times \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ models a conductivity at the boundary.

In this talk we present recent local wellposedness results for the quasilinear first order initial boundary value problem arising from the Maxwell system (1) equipped with the instantaneous material law (2), the perfectly conducting boundary conditions (3) or the absorbing boundary conditions (4), and suitable initial conditions. Moreover, we address the same question for the Maxwell interface problem.

We start with the case of a perfect conductor. In a first step we show that the macroscopic Maxwell equations with instantaneous material laws, perfectly conducting boundary conditions, and appropriate initial conditions can be reformulated as the first order quasilinear hyperbolic initial boundary value problem

$$\begin{aligned} \chi(u)\partial_t u + \textstyle\sum_{j=1}^3 A_j \partial_j u &= f, && x \in G, && t \geq t_0; \\ Bu &= 0, && x \in \partial G, && t \geq t_0; \\ u(t_0) &= u_0, && x \in G, \end{aligned} \tag{5}$$

with additional conditions for the initial values. Here $u = (\boldsymbol{E}, \boldsymbol{H})$ is the new state variable, $A_1, A_2$ and $A_3$ are constant symmetric $6 \times 6$-matrices, and $B(x) \in \mathbb{R}^{3\times 6}$ for every $x \in \partial G$. We refer to [4, 5] for an overview of results related to (5).

We present a complete local wellposedness theory for (5) showing the existence of a unique maximal solution, characterizing finite maximal existence times, and establishing the continuous dependence of the solutions on the data. The proof relies on precise energy-type a priori estimates and a detailed regularity theory for the corresponding linear problem which arises by freezing a function $\tilde{u}$ in the nonlinearity $\chi$. Here we start from the $L^2$-theory in [1], which provides a unique weak solution of the linear problem in $C([t_0, T], L^2(G))$ for a fixed time $T > t_0$ and

a corresponding a priori estimate. As the linear $L^2$-theory requires coefficients in $W^{1,\infty}((t_0,T)\times G)$, we are naturally led to develop the nonlinear theory in $H^m(G)$ with $m \geq 3$. We thus have to show a priori estimates for and existence of solutions of the linear problem in these spaces. However, we note that the existence of solutions in $H^m$ implies compatibility conditions between the coefficients and the data on the boundary both in the linear and the nonlinear situation.

The main difficulty in the derivation of the a priori estimates and the regularity of the solutions arises from the fact that the Maxwell problem has a characteristic boundary, i.e., $\sum_{j=1}^3 A_j \nu_j$ is singular. In general, there can be a loss of derivatives for characteristic initial boundary value problems and it is crucial to exploit the special structure of the Maxwell equations to avoid this loss of regularity. The precise results of the linear theory can be found in [5].

Building on this linear theory, we then derive the following local wellposedness theorem.

**Theorem 1** ([4]). *Fix $m \in \mathbb{N}$ with $m \geq 3$. Let $G \subseteq \mathbb{R}^3$ be a domain with compact $C^{m+2}$-boundary. Let $\chi \in C^m(G \times \mathbb{R}^6, \mathbb{R}^{6\times 6})$ be symmetric and uniformly positive definite. Choose $f \in H^m((t_0,T)\times G)$ for all $T > t_0$ and $u_0 \in H^m(G)$ such that $(\chi, f, u_0)$ satisfy nonlinear compatibility conditions. Then*

*(1) there exists a unique maximal solution $u \in \bigcap_{j=0}^m C^j([t_0,T_+), H^{m-j}(G))$ of the nonlinear problem (5) and thus of the Maxwell system,*

*(2) if $T_+ < \infty$, then $\limsup_{t \nearrow T_+} \|u(t)\|_{W^{1,\infty}(G)} = \infty$,*

*(3) we have continuous dependence on the data, i.e., $(f, u_0) \mapsto u(\,\cdot\,; f, u_0)$ is continuous in $H^m(G)$.*

In fact, this theorem applies to more general situations. We can also treat zeroth order nonlinearities (arising from a nonlinear conductivity), inhomogeneous boundary conditions, and material laws wich are only locally positive definite.

In view of applications, Theorem 1 has certain limitations. On the one hand the regularity assumption for $\chi$ in the spatial variable $x$ excludes the treatment of composite materials where the material law has a discontinuity across the interface. On the other hand, the modeling leading to the boundary conditions of a perfect conductor (3) assumes to know the fields outside of $G$.

Both restraints can be overcome by studying interface problems. The Maxwell equations themselves imply the interface conditions

$$[\boldsymbol{D}\cdot\nu] = \rho_\Sigma, \quad [\boldsymbol{B}\cdot\nu] = 0, \quad [\boldsymbol{E}\times\nu] = 0, \quad [\boldsymbol{H}\times\nu] = -\boldsymbol{J}_\Sigma \tag{6}$$

on an interface $\Sigma$ between two components of the domain. Here $\nu$ is a unit normal on $\Sigma$, the brackets denote the jumps across the interface, and $\rho_\Sigma$ and $\boldsymbol{J}_\Sigma$ are charge and current densities on the interface, respectively. If we denote the two components of $G$ by $G_+$ and $G_-$ and consider the Maxwell system on $G_+$ and $G_-$ separately, complemented by the interface conditions (6), we obtain the following result.

**Theorem 2** ([3]). *Fix $m \in \mathbb{N}$ with $m \geq 3$. Let $G \subseteq \mathbb{R}^3$ be a domain with compact $C^{m+2}$-interface. Let $\chi_\pm \in C^m(G_\pm \times \mathbb{R}^6, \mathbb{R}^{6\times 6})$ be symmetric and uniformly*

*positive definite. We then obtain a local wellposedness theory for the Maxwell interface problem corresponding to Theorem 1 but with $H^m$ replaced by the space of piecewise $H^m$-functions.*

Finally, we consider another common type of boundary conditions for the Maxwell equations, the absorbing boundary conditions (4). They arise if one assumes a nonlinear Ohm's law for the current density $\boldsymbol{J}_\Sigma$ in (6) and that the fields vanish outside of $G$. In the analysis of the Maxwell system with an instantaneous material law and the absorbing boundary conditions, new error terms appear at the boundary. If $\zeta$ does not depend on $\boldsymbol{E} \times \nu$, these are manageable. Otherwise, there are certain highest order terms requiring an additional smallness condition at the boundary. Nevertheless, we obtain a satisfying small data theory.

**Theorem 3** ([2])**.** *Fix $m \in \mathbb{N}$ with $m \geq 3$. Let $G \subseteq \mathbb{R}^3$ be a domain with compact $C^{m+2}$-boundary. We then obtain a local wellposedness theory for the quasilinear Maxwell system with absorbing boundary conditions corresponding to Theorem 1*

*(1) for $\zeta$ independent of $\boldsymbol{E} \times \nu$,*
*(2) for nonlinear $\zeta$ if $\partial_{\boldsymbol{E}}\zeta$ or $|\boldsymbol{E}|$ are small at the boundary.*

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### Dispersive decay of small data solutions for the KdV equation


Daniel Tataru

(joint work with Mihaela Ifrim, Herbert Koch)


This work is concerned with the Korteweg-de Vries equation (KdV)

$$\begin{cases} u_t + u_{xxx} - 6uu_x = 0 \\ u(0) = u_0, \end{cases} \tag{1}$$

on the real line. Assuming that the initial data is small and localized, we seek to understand the long time dispersive properties of the solution.

The fundamental solution for the corresponding linear KdV flow is

$$K(t,x) = t^{-\frac{1}{3}} \mathrm{Ai}(x/t^{\frac{1}{3}}).$$

where Ai denotes the classical Airy function. To describe the decay of solutions with localized data it is natural to split the space-time into three regions:

(1) The hyperbolic region

$$H := \{x \lesssim -t^{\frac{1}{3}}\},$$

where one sees an oscillatory, Airy type behavior for the solution, with dispersive decay.

(2) The self-similar region

$$S := \{|x| \lesssim t^{\frac{1}{3}}\},$$

where the solution essentially looks like a bump function with $t^{-\frac{1}{3}}$ decay.

(3) The elliptic region,

$$E := \{x \gtrsim t^{\frac{1}{3}}\},$$

which is eventually left by each oscillatory component of the solution, and consequently we have better decay.

Defining the expression $\langle x\rangle$ in a time dependent fashion as

$$\langle x\rangle := (x^2 + |t|^{\frac{2}{3}})^{\frac{1}{2}},$$

the following scale invariant result describes the dispersive decay of linear KdV waves:

**Proposition 1.** *Assume that the initial data $u_0$ for linear KdV satisfies*

$$\|u_0\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}} + \|x^2 u_0\|_{\dot{H}^{\frac{1}{2}}} \leq \epsilon. \tag{2}$$

*Then the corresponding solution satisfies the bound*

$$t^{\frac{1}{4}}\langle x\rangle^{\frac{1}{4}}|u(t,x)| + t^{\frac{3}{4}}\langle x\rangle^{-\frac{1}{4}}|u_x(t,x)| \lesssim \epsilon. \tag{3}$$

*Furthermore, in the elliptic region $E$ we have the better bound*

$$\langle x\rangle|u(t,x)| + t^{\frac{1}{2}}\langle x\rangle^{\frac{1}{2}}|u_x(t,x)| \lesssim \epsilon \ln(t^{-\frac{1}{3}}\langle x\rangle). \tag{4}$$

The question one might ask is whether the solutions to the nonlinear KdV equation satisfy a similar bound globally in time. But the answer to this is clearly no, due to two nonlinear phenomena which may occur even for small data, simultaneously or separately:

- **Solitons.** These are solutions with a fixed profile which move with constant velocity. Unlike dispersive waves which move two the left, the KdV solitons move to the right and may be arbitrarily small.
- **Dispersive shocks.** If one neglects the third order derivative in the KdV equation then what is left is the Burgers equation, which develops shocks in finite time. The third order derivative adds dispersion to the mix, sending the high frequencies to the left as a dispersive tail. This guarantees that shocks as a jump discontinuity cannot form. Nevertheless, a smooth version of a shock still appears due to the strong Burgers type effects near the self-similar region.

Given these considerations, the better question to ask is

**Question:** *If $\epsilon \ll 1$ is the initial data size, then we ask what is the time scale up to which the solution will satisfy linear dispersive decay bounds?*

In this work we show that for initial data of size $\epsilon$, the correct time scale is the quartic one, $T_\epsilon = \epsilon^{-3}$. Precisely our main result is as follows:

**Theorem 2.** *Assume that the initial data $u_0$ for KdV satisfies*

$$\|u_0\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}} + \|xu_0\|_{\dot{H}^{\frac{1}{2}}} \leq \epsilon \ll 1. \tag{5}$$

*Then for the quartic lifespan*

$$|t| \ll \epsilon^{-3}, \tag{6}$$

*we have the dispersive bounds*

$$|u(t,x)| \lesssim \epsilon t^{-\frac{1}{4}} \langle x \rangle^{-\frac{1}{4}} \qquad |u_x(t,x)| \lesssim \epsilon t^{-\frac{3}{4}} \langle x \rangle^{\frac{1}{4}}. \tag{7}$$

*Furthermore, in the elliptic region $E$ we have the better bound*

$$\langle x \rangle |u(t,x)| + t^{\frac{1}{2}} \langle x \rangle^{\frac{1}{2}} |u_x(t,x)| \lesssim \epsilon \ln(t^{-1/3} \langle x \rangle). \tag{8}$$

*The implicit constants are independent of $\varepsilon$ and $u$.*

This is a scale invariant result, and the choice of the function spaces is optimal and consistent with scaling. This is also an optimal result, as inverse scattering computations show that solitons may emerge from the self-similar region exactly at quartic time.

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## Long time behaviour for Stochastic wave equation

Leonardo Tolomeo

In this talk, we consider this defocusing wave equation with damping and cubic nonlinearity with space-time white noise forcing, posed on $\mathbb{T}^2$.

$$u_{tt} + u_t - \Delta u + u^3 = \xi. \tag{SNLW}$$

The space-time white noise is defined to be the only (in law) random variable which is independent and equidistributed in every point of space and time, i.e.

$$\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y).$$

This is a quite common physical hypothesis, hence our interest in studying the behaviour of (SNLW). However, it forces $\xi$ to be very rough, and indeed $\xi$ is just defined as a space-time distribution belonging to the space $C_t^{-\frac{1}{2}-\epsilon}C_x^{-1-\epsilon}$. Therefore, a perturbative analysis suggests that $u \in H^{-\epsilon} \setminus L^2$, so it is not clear how to make sense of the term $u^3$. Indeed, in a series of works, Albeverio, Haba,

Oberguggenberger, and Russo [1, 4, 5, 6] showed that the nonlinearity $u^3$ *cannot* make sense.

To overcome this issue, in [2], Gubinelli, Koch, and Oh introduced a suitable renormalisation, formally expressed as

$$u_{tt} + u_t - \Delta u + u^3 - 3\infty \cdot u = \xi.$$

In their work, they showed local well posedness for this equation, starting from smooth enough enough initial data (in $H^{\frac{1}{4}+\epsilon}$).

In a joint work with Gubinelli, Koch, Oh [3], we show global existence for these solutions, through a very tight argument that involves application of the $I$-method to the stochastic setting and precise large deviation estimates regarding the stochastic terms that appear in the perturbative analysis. While this argument is very tight, it does not exploit fully the damping term in the equation, and it provides a bound on the growth of solutions which is double exponential in time.

On the other hand, as proven in [3], (SNLW) admits an invariant measure, formally given by

$$\rho := \exp\Big( -\frac{1}{2}\int u_t^2 - \frac{1}{2}\int |\nabla u|^2 - \frac{1}{4}\int (u^4 - 6\infty \cdot u^2)\Big) \mathrm{d}u\mathrm{d}u_t,$$

and one finds that, for almost every initial data (according to this invariant measure), the solution has logarithmic growth in time. However, this result does not provide any information about smooth initial data, since almost every initial data according to this measure is not in $L^2(\mathbb{T}^2)$.

In a work in preparation, we show that this is also the case for every initial data. Moreover, for smooth initial data, we show that when the time is going to $\infty$, the law of $u(t)$ is converging to $\rho$.

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