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Arbeitsgemeinschaft: Elliptic Cohomology according to Lurie

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Abstract. In this collection we give an overview of Jacob Lurie's construction of elliptic cohomology and Lubin Tate theory. As opposed to the original construction by Goerss–Hopkins–Miller, which uses heavy obstruction theory, Lurie constructs these objects by a moduli problem in spectral algebraic geometry. A major part of this text is devoted to the foundations and background in higher algebra needed to set up this moduli problem (in the case of Lubin Tate theory) and prove that it is representable.

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Introduction by the Organizers

Twenty or more years ago, Mike Hopkins and his collaborators began to uncover a very strong connection between arithmetic algebraic geometry and stable homotopy theory. This revolutionized stable homotopy theory and gave impetus to the emergence of the field of derived algebraic geometry. We learned, in particular, that core objects in arithmetic geometry, such as the Lubin-Tate deformation space and the Deligne-Mumford moduli stack of elliptic curves, have canonical lifts to derived algebraic geometry. The original constructions used very difficult homotopy theory, but Lurie has recently found a way to much more directly access the classical geometry. The overall goal of the Arbeitsgemeinschaft is to give an exposition of this viewpoint and constructions.

If algebraic geometry studies geometric spaces with sheaves of commutative rings, then spectral algebraic geometry studies spaces with sheaves of rings in some category amenable to homotopy theory. If we would work in characteristic zero, then we could work with commutative differential graded algebras, but arithmetic geometry and homotopy theory both very much seek integral data, so the natural setting here is that of $\mathbb{E}_\infty$-ring spectra, or $\mathbb{E}_\infty$-rings for short. Stable homotopy theorists have long studied $\mathbb{E}_\infty$-rings as they serve as a rigid refinement of multiplicative cohomology theories – classical cohomology theories such as complex K-theory and ordinary cohomology admit canonical refinements as $\mathbb{E}_\infty$-rings.

The key insight is this. There are important objects in algebraic geometry which arise as solutions to moduli- or deformation theoretic problems. It turns out that appropriate variations of the same problems have solutions in derived algebraic geometry which lift the original object in algebraic geometry. Elliptic cohomology is one key example. In this collection of extended abstracts we focus on explaining this philosophy and its technical incarnation. All the material covered here is classical or taken from the work of Lurie, especially [1]. We recommend that the reader also consults this source, at least the introduction, to get an overview of the ideas.

Some words on the approach of Lurie

At the center of the connection between geometry and homotopy theory are formal groups. Every cohomology theory with a natural theory of Chern classes has an associated formal group and, starting in about 1970, Quillen and subsequent authors realized this connection was very rigid. In particular, there is a very checkable criterion, due to Landweber, which allows us to assign a cohomology theory to a formal group law.

There are two important examples of formal group laws for which this theory applies:

(1) The Lubin-Tate formal group is defined as a universal deformation of a formal group law of a given height n over a perfect field of characteristic p. Through Landweber's theorem one can assign a cohomology theory to the Lubin-Tate formal group that is called Morava E-theory (or Lubin-Tate theory) and denoted by E_n. By construction, E_n comes as a multiplicative cohomology theory, but not as an $\mathbb{E}_\infty$-ring. The classical story uses complicated obstruction theoretic argument due to Goerss, Hopkins and Miller to verify that E_n can be uniquely refined to an $\mathbb{E}_\infty$-ring.

(2) The formal completion of an elliptic curve at the marked point is a formal group. Under certain conditions on the elliptic curve (namely being étale over the moduli stack of elliptic curves) one can use Landweber's criterion to associate a cohomology theory with this elliptic curve which we refer to as elliptic cohomology. Another obstruction theoretic argument then shows that the elliptic cohomology theory obtained in this way admits a preferred refinement to an $\mathbb{E}_\infty$-ring which is even functorial in elliptic curves.

The approach sketched above is very indirect, at times ad hoc, and it relies on difficult and mysterious obstruction theory. The idea of Lurie is to describe E_n as a universal deformation of the (ordinary) height n formal group law in the world of $\mathbb{E}_\infty$-rings. This technique is quite flexible and can be extended to more global examples where the local deformation theory is governed by formal groups or, more generally, p-divisible groups with one-parameter formal subgroup. This includes elliptic cohomology.

References

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Abstracts

Cohomology theories and formal group laws

Dan Berwick-Evans

Given a cohomology theory with Chern classes, the tensor product formula for the tautological line bundle on $\mathbb{C}P^\infty$ often determines a formal group law. This comes from pulling back along the maps $p_1, p_2 \colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ where p_1, p_2 are the two projections and m is the map classifying the tensor product of complex line bundles. Then we can ask for a formula

$$m^*c = F(p_1^*c, p_2^*c) \in h(\mathbb{C}P^\infty \times \mathbb{C}P^\infty), \qquad c \in h(\mathbb{C}P^\infty),$$

where $c = c(\mathcal{O}(1))$ is the Chern class of the tautological line bundle on $\mathbb{C}P^\infty$. The fact that the tensor product of line bundles is unital, commutative, and associative (up to isomorphism) translates into F defining a formal group law.

For ordinary cohomology (for example, with rational coefficients) one can show that

$$F(p_1^*c, p_2^*c) = p_1^*c + p_2^*c,$$

recovering the formal additive group law where c is the usual first Chern class of a complex line bundle. In complex K-theory, the Chern class of a complex line bundle $\mathcal{L}$ is $c(\mathcal{L}) = [\mathcal{L}] - 1$, where $[\mathcal{L}]$ denotes the K-theory class underlying the line bundle. For $c = c(\mathcal{O}(1))$, one can show that

$$F(p_1^*c, p_2^*c) = p_1^*c + p_2^*c + p_1^*c \cdot p_2^*c$$

recovering the formal multiplicative group law.

The Lazard ring L encodes formal group laws via the following universal property [1]: a homomorphism $L \to R$ is the same data as a formal group law over a commutative ring R. Periodic complex cobordism MP has a canonical theory of Chern classes. Quillen proved [4] that the associated associated formal group law is the universal one: there is a canonical identification $MP^0(pt) \cong L$.

A multiplicative cohomology theory $h^\bullet$ is *even* if $h^k = 0$ for k odd, and *periodic* if there is some $\beta \in h^2(pt)$ that is invertible in $h^\bullet(pt)$. Any even periodic cohomology theory can be given a theory of Chern classes, and thereby defines a formal group law. Conversely, given a formal group law determined by a homomorphism $MP^0(pt) \cong L \to R$ one can consider the assignment

$$X \mapsto MP^\bullet(X) \otimes_{MP^0(pt)} R$$

where X is a finite CW complex. Landweber gives a criterion under which the above determines a cohomology theory [3]. Ordinary cohomology with rational coefficients can be recovered this way using the additive formal group law. Complex K-theory can be recovered this way using the multiplicative group law over $\mathbb{Z}$.

These ideas are overviewed in §1.1-1.2 of [2].

References

[1] M. Lazard, *Sur les groupes de Lie formels à un paramètre*, Bulletin de la Société Mathématique de France (1955).

[2] J. Lurie, *A Survey of Elliptic Cohomology*m In: Baas N., Friedlander E., Jahren B., Østvær P. (eds) Algebraic Topology. Abel Symposia, vol 4. Springer, Berlin, Heidelberg (2009).

[3] P. Landweber, *Homological properties of comodules over $MU_*(MU)$ and $BP_*(BP)$*. American Journal of Mathematics, 98 (3), (1976).

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Height and Landweber exact functor theorem

Irina Bobkova

In this expository talk we reviewed some aspects of the theory of 1-dimensional commutative formal groups and formal group laws, mostly following [1] and [2]. Let F be a formal group law over a commutative $\mathbb{F}_p$-algebra R. Then it is easy to see that for some n and some power series g we have

$$[p]_F(x) := x +_F x \ldots +_F x = g(x^{p^n})$$

Definition 1. We say that a formal group law F over an $\mathbb{F}_p$-algebra R has height at least n if there exists a power series $g(x) \in R[[x]]$, such that $[p]_F(x) = g(x^{p^n})$. We say that F has height exactly n, if $[p]_F(x)$ has leading term ax^{p^n} for $a \in R^\times$. If $[p]_F(x) = 0$, then we say that F has height ∞.

A formal group law is a formal group together with a choice of a coordinate. Below we define the notion of height for a formal group; for more details, see, for example, [2, 3.2]. Consider a formal group F over a scheme X over $\mathbb{F}_p$. Let $f : X \to X$ be the Frobenius morphism. We define the formal group $G^{(p)}$ as the pullback $G^{(p)} = f^*G$ and the relative Frobenius $F : G \to G^{(p)}$ as the unique morphism which makes the diagram

$$\begin{tikzcd}
G \arrow[r, "F"'] \arrow[rr, bend left, "f"] \arrow[dr] & G^{(p)} \arrow[r] \arrow[d] & G \arrow[d] \\
& X \arrow[r, "f"] & X
\end{tikzcd}$$

commute. If $\phi : G \to H$ is a homomorphism of formal groups over X for which $d\phi : \omega_H \to \omega_G$ vanishes, then ϕ factors through the relative Frobenius of G. This condition is satisfied for the endomorphism of G given by the $[p]$-series, and there is a unique homomorphism V, called the *Verschiebung*, making the following diagram commute

$$\begin{tikzcd}
G \arrow[r, "F"] \arrow[dr, "{[p]}"'] & G^{(p)} \arrow[d, "V"] \\
& G.
\end{tikzcd}$$

Then, if dV vanishes, we have a factoring

$$\begin{tikzcd} G^{(p)} \arrow[r, "F^{(p)}"] \arrow[dr, "V"'] & G^{(p^2)} \arrow[d, "V_2"] \\ & G \end{tikzcd}$$

and, if $dV_2 = 0$, we can continue. This leads us to the following definition.

Definition 2. Let G be a formal group over a scheme X over $\mathbb{F}_p$. We say that G has height at least n if there is a factoring

$$\begin{tikzcd} G \arrow[r, "F"] \arrow[drrrr, "{[p]}"'] & G^{(p)} \arrow[r, "F^{(p)}"] \arrow[drrr, "V"] & G^{(p^2)} \arrow[r, "F^{(p^2)}"] & \dots \arrow[r, "F^{(p^{n-1})}"] & G^{(p^n)} \arrow[d, "V_n"] \\ & & & & G \end{tikzcd}$$

We say G has height exactly n if $dV_n \neq 0$.

Proposition 3 (Lazard's Uniqueness Theorem, see, for example, [1, 5.24])**.** *Let k be a field of characteristic p and G_1 and G_2 two formal groups of height exactly n over k. Then there is a separable extension $f : k \to k'$ so that f^*G_1 and f^*G_2 are isomorphic. In particular, if k is separably closed, then G_1 and G_2 are isomorphic.*

Let v_n be the coeffient of x^{p^n} in the p-series of the universal formal group law over MU_*.

Theorem 4 (Landweber exact functor theorem)**.** *Let R be an MU_*-module. Suppose for each prime p and each n the sequence*

$$(p, v_1, \dots v_{n-1})$$

is a regular sequence for R. Then the functor

$$X \mapsto MU_*(X) \otimes_{MU_*} R$$

is a homology theory.

Given an elliptic curve E over $Spec(R)$, its completion along the identity section determines a formal group over $Spec(R)$ [3, Chapter IV].

Proposition 5. [3, Corollary IV. 7.5] *Let E be an elliptic curve over a field of positive characteristic. Let $\hat{E}$ be the associated formal group law. Then the height of $\hat{E}$ is either 1 or 2.*

If $ht(\hat{E}) = 1$, we say that E is ordinary and has Hasse invariant 1. If $ht(\hat{E}) = 2$, we say that E is supersingular and has Hasse invariant 0.

References

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Deformations of formal group laws

Alice Hedenlund

Introduction and preliminaries

Question 1. Let $\hat{\mathbf{G}}_0$ be a formal group law defined "at a point". What does $\hat{\mathbf{G}}_0$ look like after we deform it to an "infinitesimal neighborhood"?

Let us rephrase this question in a more mathematically rigorous way. Algebro-geometrically, we view fields k as "points"; indeed $\mathrm{Spec}(k)$ is a point. In this context, an "infinitesimal neighborhood" of k is a local Artin algebra over k.[1] We denote this category by Art_k. Indeed, $\mathrm{Spec}(A)$ for such a ring is also topologically a point, but it is non-trivial as a ringed space; we talk about "fat points". Let $\hat{\mathbf{G}}_0$ be a formal group law over a field k of finite height n. In this talk we cover the following.

- We give the definition of a deformation of $\hat{\mathbf{G}}_0$ to a local Artin k-algebra.
- We show Lubin-Tate's theorem, telling us, under mild conditions on the field k and the formal group $\hat{\mathbf{G}}_0$, that there is a universal deformation $\hat{\mathbf{G}}_{LT}$ of $\hat{\mathbf{G}}_0$.
- We show that the universal deformation $\hat{\mathbf{G}}_{LT}$ is Landweber exact.

We mainly follow [3, Lecture 21] with some extra details taken from [4].

1. Deformations of formal groups

We start by defining the notion of a deformation of $\hat{\mathbf{G}}_0$. Let A be local Artin algebra over k and note that the map $\rho_A : A \to k$ gives rise to a functor $\rho_A^* : \mathrm{FGL}(A) \to \mathrm{FGL}(k)$ between the corresponding small categories of formal group laws, per the previous talks.

[1] Recall that a local Artin k-algebra is a commutative ring A equipped with a surjective map $\rho_A : A \to k$ whose kernel $\mathfrak{m} = \ker(\rho_A)$ satisfies the following properties:

(1) The ideal $\mathfrak{m}$ is nilpotent; $\mathfrak{m}^a = 0$ for $a \gg 0$.
(2) Each quotient $\mathfrak{m}^a/\mathfrak{m}^{a+1}$ is a finite-dimensional vector space over k.

Definition 2. The category of deformations of $\hat{\mathbf{G}}_0$ to A is the pullback in the diagram

$$\begin{tikzcd} \mathrm{Def}_{\hat{\mathbf{G}}_0}(A) \arrow[r] \arrow[d] & \mathrm{FGL}(A) \arrow[d, "\rho_A^*"] \\ * \arrow[r] & \mathrm{FGL}(k) \end{tikzcd}$$

where $*$ denotes the discrete category with one object, and the lower horizontal functor is the functor that sends that one object to $\hat{\mathbf{G}}_0$.

Remark. Note that the diagram in the definition is a diagram of small categories. Now, Cat is really more that just a mere category: it is a 2-category. Usually we are interested in categories only up to equivalence, and not up to isomorphism, and so it makes sense to take this extra structure into account. The reader should view this as a warm-up to working higher-categorical, an approach that will be taken more seriously during the rest of the week. So to clarify: when we say that the diagram in the definition is a pullback, we mean this in the 2-categorical sense. That is, $\mathrm{Def}_{\hat{\mathbf{G}}_0}(A)$ is the essential fiber of the $\mathrm{FGL}(A) \to \mathrm{FGL}(k)$ over $\hat{\mathbf{G}}_0$.

If the last remark confuses you, know that you are not alone. It thus seems like a good idea to explicitly describe the category $\mathrm{Def}_{\hat{\mathbf{G}}_0}(A)$ in a way that is more digestible for people not well-versed in higher category theory. A priori, the objects in this category are pairs $(\hat{\mathbf{G}}, \alpha)$, where $\hat{\mathbf{G}}$ is a formal group law over A and $\alpha : \rho^*(A) \cong \hat{\mathbf{G}}_0$ is an isomorphism of formal group laws over k. However, one can show, and we leave this to the reader, that the category $\mathrm{Def}_{\hat{\mathbf{G}}_0}(A)$ can equivalently be described as follows.

- An object in the category is a formal group law $\hat{\mathbf{G}}$ over A that reduces to $\hat{\mathbf{G}}_0$ modulo $\mathfrak{m}$.
- A morphism $\varphi : \hat{\mathbf{G}} \to \hat{\mathbf{H}}$ is an isomorphism of formal group laws over A that reduces to the identity modulo the maximal ideal $\mathfrak{m}$. Such isomorphisms are referred to as $\star$-isomorphisms.

Although we could easily have defined $\mathrm{Def}_{\hat{\mathbf{G}}_0}$ by specifying the above data, the definition in terms of the pullback diagram directly gives us some formal properties of the category. Note for example that $\mathrm{Def}_{\hat{\mathbf{G}}_0}(A)$ depends functorially on A, so we have a functor

$$\mathrm{Def}_{\hat{\mathbf{G}}_0} : \mathrm{Art}_k \to \mathrm{Grpd}\ .$$

We will freely talk about homotopy groups of groupoids, with which we mean the homotopy groups of their classifying spaces. Recall that we can explicitly describe the homotopy groups as

$$\pi_0 \, \mathrm{Def}_{\hat{\mathbf{G}}_0}(A) = \mathrm{Iso}_\star(\mathrm{FGL}(A)) \quad \text{and} \quad \pi_1(\mathrm{Def}_{\hat{\mathbf{G}}_0}(A), \hat{\mathbf{G}}) = \mathrm{Aut}_\star(\hat{\mathbf{G}})\,,$$

where the $\star$ is supposed to remind us that we want to take isomorphism classes with respect to $\star$-isomorphisms, and automorphisms that are $\star$-isomorphisms. We first show that the groupoid of deformations is much simpler than one might guess.

Proposition 3. *The groupoid* $\mathrm{Def}_{\hat{\mathbf{G}}_0}(A)$ *is (homotopy) discrete.*

Proof. Let us explicitly write

$$\varphi(x) = c_1 x + c_2 x^2 + \cdots$$

for a $\star$-automorphism of $\hat{\mathbf{G}}$. We want to show that φ is actually the identity. Since the maximal ideal $\mathfrak{m}$ is nilpotent, it suffices to prove that

$$\varphi(x) \equiv x \mod \mathfrak{m}^a$$

for all positive integers a. We proceed by induction on a. The case $a = 1$ is trivial. For the induction step, let us assume that the assertion hold for $a - 1$. Let A' be the quotient of $A[c_1^{\pm}, c_2, \dots]$ classifying automorphisms of the given deformation $\hat{\mathbf{G}}$. We refer to the map $A' \to A$ classifying the automorphism φ as f, and the map $A' \to A$ classifying the identity as f_0. The two compositions

$$\begin{tikzcd} A' \arrow[r, shift left, "f"] \arrow[r, shift right, "f_0"'] & A \arrow[r] & A/\mathfrak{m}^{a-1} \end{tikzcd}$$

are the same by the induction hypothesis. We can hence view the difference

$$A' \xrightarrow{f - f_0} A \longrightarrow A/\mathfrak{m}^a$$

as an A-linear derivation $d : A' \to \mathfrak{m}^{a-1}/\mathfrak{m}^a$ where the A'-module structure on $\mathfrak{m}^{a-1}/\mathfrak{m}^a$ is given by f (or equivalently by f_0). The derivation factors as

$$A' \longrightarrow A' \otimes_A k \xrightarrow{d'} \mathfrak{m}^{a-1}/\mathfrak{m}^a$$

where d' is a k-linear derivation. Note that $A' \otimes_A k$ is the ring classifying automorphisms of our original formal group law $\hat{\mathbf{G}}_0$, which we know is formally étale over k, see for example [3, Lecture 14 Theorem 1]. It follows that $d' = 0$, and consequently that $d = 0$, which is what we wanted to show. $\square$

Due to the proposition above it makes sense to focus our attention on only the set of connected components. That is, we want to study the functor

$$\pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0} : \mathrm{Art}_k \to \mathrm{Set} \ .$$

2. The Lubin-Tate theorem

The fundamental theorem regarding deformation theory of formal group laws is due to Lubin-Tate [2]. It asserts, under mild hypotheses, that the functor

$$\pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0} : \mathrm{Art}_k \to \mathrm{Set}$$

is pro-representable. This means that the functor is representable, maybe not by an object of Art_k, but at least by an object of the slightly bigger category $\widehat{\mathrm{Art}}_k$ consisting of complete local Noetherian algebras over k. For reference, let us phrase the theorem in modern language.

Theorem 4 (Lubin-Tate)**.** *Let k be a perfect field of characteristic $p > 0$ and let $\hat{\mathbf{G}}_0$ be a height $n < \infty$ formal group law over k. Then the functor*

$$\pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0} : \operatorname{Art}_k \to \operatorname{Set}$$

is pro-representable. That is, there is a complete local Noetherian k-algebra A_{LT} and a deformation $\hat{\mathbf{G}}_{LT}$ over this ring such that extensions of scalars

$$\operatorname{Ring}_{/k}(A_{LT}, A) \cong \pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0}(A), \quad \varphi \mapsto \varphi^*(\hat{\mathbf{G}}_{LT})$$

is a bijection.

The ring pro-representing the deformation functor is non-canonically isomorphic to the complete local Noetherian ring

$$A_{LT} \cong \mathbb{W}(k)[[u_1, \ldots, u_{n-1}]]$$

where n is the height of the formal group law $\hat{\mathbf{G}}_0$ and $\mathbb{W}(k)$ denotes the p-typical Witt vectors. This is equipped with the canonical map $A_{LT} \to k$ whose kernel is the maximal ideal $(p, u_1, \ldots, u_{n-1})$. To give an indication to why this indeed must be the pro-representing object, recall that the p-typical Witt vectors $\mathbb{W}(k)$ of a perfect field k of characteristic p has the following universal property: If A is a complete Noetherian local algebra over k, then the diagram

$$\begin{tikzcd} \mathbb{W}(k) \arrow[rr, dotted] \arrow[dr] & & A \arrow[dl] \\ & k & \end{tikzcd}$$

can be supplied with a unique dotted map that makes it commute. This tells us, in particular, that the Lubin-Tate universal ring has to be a $\mathbb{W}(k)$-algebra. In fact, we know that the Lubin-Tate ring contains $\mathbb{W}(k)$, since there exists at least one deformation of $\hat{\mathbf{G}}_0$ to $\mathbb{W}(k)$. So it is reasonable to believe that the Lubin-Tate ring is a formal power series ring over $\mathbb{W}(k)$.

Now we can construct the universal deformation of $\hat{\mathbf{G}}_0$ in the following way. The formal group law $\hat{\mathbf{G}}_0$ is classified by some ring homomorphism $\varphi_0 : L_{(p)} \to k$ where $L_{(p)}$ denotes the p-localised Lazard ring. Note that

$$L_{(p)} = \mathbb{Z}_{(p)}[t_1, t_2, \ldots],$$

and that we can, without loss of generality, assume that $t_{p^i-1} = v_i$, where v_i denotes the Hasse invariants of the universal formal group law, as discussed in the previous talks. Since $\hat{\mathbf{G}}_0$ is of height n, we may also assume that φ_0 is such that

$$\varphi_0(v_i) = 0, \quad 1 \leq i \leq n-1\,.$$

We let $\varphi : L_{(p)} \to \mathbb{W}(k)[[u_1, \ldots, u_{n-1}]]$ be any ring homomorphism that lifts φ_0 and is such that

$$\varphi(v_i) = u_i, \quad 1 \leq i \leq n-1\,.$$

This map classifies a formal group law over $\mathbb{W}(k)[[u_1, \ldots, v_{n-1}]]$; this is the universal deformation. All such lifts are $\star$-isomorphic, so the specific lift is not so important.

3. Proof of the Lubin-Tate theorem

In this section we sketch the proof for Lubin-Tate's theorem following Schlessinger's more general theory on pro-representability of functors from Artin rings. In the article [5] the author gives conditions that guarantees when a general functor

$$F : \mathrm{Art}_k \to \mathrm{Set}$$

is pro-representable by some R. Basically the idea is that to show that F behaves similarly enough to $\mathrm{Hom}_{/k}(R, -)$ to allow for the question of pro-representability to be reduced to checking that the specific case

$$\mathrm{Hom}_{/k}(R, k[x]/(x^2)) \to F(k[x]/(x^2))$$

is a bijection. We outline how this works for the functor

$$\pi_0 \,\mathrm{Def}_{\hat{\mathbf{G}}_0} : \mathrm{Art}_k \to \mathrm{Set}$$

in two steps; we first show how some basic properties of $\pi_0 \,\mathrm{Def}_{\hat{\mathbf{G}}_0}$ allow us to reduce to the case $k[x]/(x^2)$, and we then show the Lubin-Tate theorem in that specific case.

3.1. Step 1: Reduction to the case $k[x]/(x^2)$.

We start with showing two "Schlessinger type condition" lemmas. These are the lemmas that show that $\pi_0 \,\mathrm{Def}_{\hat{\mathbf{G}}_0}$ behaves similarly enough to $\mathrm{Hom}_{/k}(A_{LT}, -)$ to allow for a reduction. Indeed, it is not hard to show that that the functor $\mathrm{Hom}_{/k}(A_{LT}, -)$ satisfies the properties below as well.

Lemma 5. *If $A \to B$ is a surjective map, then the induced map*

$$\pi_0 \,\mathrm{Def}_{\hat{\mathbf{G}}_0}(A) \to \pi_0 \,\mathrm{Def}_{\hat{\mathbf{G}}_0}(B)$$

is surjective.

Lemma 6. *For any pair of surjective maps $A \to B$ and $C \to B$ in Art_k, the canonical map*

$$\pi_0 \,\mathrm{Def}_{\hat{\mathbf{G}}_0}(A \times_B C) \to \pi_0 \,\mathrm{Def}_{\hat{\mathbf{G}}_0}(A) \times_{\pi_0 \,\mathrm{Def}_{\hat{\mathbf{G}}_0}(B)} \pi_0 \,\mathrm{Def}_{\hat{\mathbf{G}}_0}(C)$$

is a bijection.

For us, one of the most important consequenes of the last lemma is the following corollary.

Corollary 7. *The tangent space $\pi_0 \,\mathrm{Def}_{\hat{\mathbf{G}}_0}(k[x]/(x^2))$ is a group.*

Proof. Consider the "addition" map

$$k[x]/(x^2) \times_k k[x]/(x^2) \to k[x]/(x^2), \quad (a + bx, a + cx) \mapsto a + (b + c)x$$

and use Lemma 6 to obtain a map

$$\pi_0 \,\mathrm{Def}_{\hat{\mathbf{G}}_0}(k[x]/(x^2)) \times \pi_0 \,\mathrm{Def}_{\hat{\mathbf{G}}_0}(k[x]/(x^2)) \to \pi_0 \,\mathrm{Def}_{\hat{\mathbf{G}}_0}(k[x]/(x^2))\,.$$

This does indeed satisfy the necessary conditions for $\pi_0 \,\mathrm{Def}_{\hat{\mathbf{G}}_0}(k[x]/(x^2))$ to be a group. □

We now start showing how the above results allow us to reduce to checking the Lubin-Tate theorem in the case $k[x]/(x^2)$. We proceed by induction on the length of the Artinian ring A. If A has length 1, then $A = k$ and the map

$$\mathrm{Hom}_{/k}(A_{LT}, k) \to \pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0}(k)$$

is clearly a bijection since both sides consist of a single element. If the length of A is $a \geq 2$, we can pick an element x in A which is annihilated by the maximal ideal $\mathfrak{m}$. Note that A/x is an Artinian ring of length $a-1$, so by the induction hypothesis we know that

$$\mathrm{Hom}_{/k}(A_{LT}, A/x) \to \pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0}(A/x)$$

is a bijection. Let us study the relationship between deformations to A and deformations to A/x. First, note that we have a map

$$\lambda : k[x]/(x^2) \times_k A \to A\,, \quad (\rho_A(a) + bx, a) \mapsto bx + a$$

which fits into the commutative square

$$\begin{tikzcd} k[x]/(x^2) \times_k A \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\lambda"'] & A \arrow[d] \\ A \arrow[r] & A/x \end{tikzcd} .$$

One can check that this is a pullback square, and by Lemma 6 we hence get a pullback square

$$\begin{tikzcd} \pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0}(k[x]/(x^2)) \times \pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0}(A) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\lambda"'] & \pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0}(A) \arrow[d] \\ \pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0}(A) \arrow[r] & \pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0}(A/x) \end{tikzcd} .$$

Recall from Corollary 7 that $\pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0}(k[x]/(x^2))$ is a group, and the left hand vertical map in the diagram is indeed an action on $\pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0}(A)$ by this group. The fact that the above diagram is commutative shows that the projection map from A to A/x induces a map

$$\pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0}(A)/\pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0}(k[x]/(x^2)) \to \pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0}(A/x)\,.$$

The fact that the square is a pullback means that this map is injective, and Lemma 5 shows that it is also surjective. This exhibits $\pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0}(A)$ as a principal homogenous space for $\pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0}(k[x]/(x^2))$ over $\pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0}(A/x)$. All of the above discussion goes through for the functor $\mathrm{Hom}_{/k}(A_{LT}, -)$ as well. We conclude that to show that the Lubin-Tate theorem holds for A it suffices to show that

$$\mathrm{Hom}_{/k}(A_{LT}, k[x]/(x^2)) \to \pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0}(k[x]/(x^2))$$

is an isomorphism of groups.

3.2. **Step 2: The case $k[x]/(x^2)$.** To finish the proof for the Lubin-Tate theorem we explicitly construct an inverse for the functor

$$k^{n-1} \cong \mathrm{Hom}_{/k}(A_{LT}, k[x]/(x^2)) \to \pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0}(k[x]/(x^2))\,.$$

Let $\hat{\mathbf{G}}$ be a representative for a connected component of $\operatorname{Def}_{\hat{\mathbf{G}}_0}(k[x]/(x^2))$. In particular, $\hat{\mathbf{G}}$ is a formal group law over $k[x]/(x^2)$, and so it is classified by some local ring homomorphism $h : L_{(p)} \to k[x]/(x^2)$, where again $L_{(p)}$ is the p-localised Lazard ring. We have

$$h(v_i) = c_i x\,, \quad 1 \le i \le n-1$$

for some elements c_i in k. We now construct the map to k^{n-1} in the obvious way.

Lemma 8. *The map*

$$\theta : \pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0}(k[x]/(x^2)) \to k^{n-1}, \quad \hat{\mathbf{G}} \mapsto (c_1, \dots, c_{n-1})\,.$$

is a well-defined group homomorphism.

Proof. It is not hard to see that the map is a group homomorphism. Let $\hat{\mathbf{H}}$ be another representative for the isomorphism class determined by $\hat{\mathbf{G}}$. Since the two deformations are in the same isomorphism class we know that there is a $\star$-isomorphism f between them. Let us compare the p-series of the two deformations. They are related by the formula

$$[p]_{\hat{\mathbf{H}}}(t) = f([p]_{\hat{\mathbf{G}}}(f^{-1}(t)))\,.$$

Since $\hat{\mathbf{G}}_0$ has height n, the two p-series must be divisible by x modulo t^{p^n}. Since $f(t) \equiv t \pmod{x}$, we deduce that

$$[p]_{\hat{\mathbf{G}}}(t) \equiv [p]_{\hat{\mathbf{H}}}(t) \pmod{t^{p^n}}\,,$$

which proves that the inverse map is well-defined. □

All that remains is to show that the map in the lemma is indeed the inverse to the map in the Lubin-Tate theorem. The composition

$$k^{n-1} \longrightarrow \pi_0 \operatorname{Def}_{\hat{\mathbf{G}}_0}(k[x]/(x^2)) \longrightarrow k^{n-1}\ ,$$

is the identity, essentially by construction, which shows that the map of the Lubin-Tate theorem is injective. To show that θ is the inverse to the map of the Lubin-Tate theorem is suffices to show that θ is injective as well. Note that a formal group law $\hat{\mathbf{G}}$ is in the kernel of θ if and only if it has height exactly n over $k[x]/(x^2)$. We wish to show that $\hat{\mathbf{G}}$ is $\star$-isomorphic to the trivial deformation $\hat{\mathbf{G}}_{0,k[x]/(x^2)}$, that is, the deformation of $\hat{\mathbf{G}}_0$ obtained by extension of scalars along the counit $k \to k[x]/(x^2)$. We remember that that the $k[x]/(x^2)$-algebra R classifying isomorphisms between formal group laws of height n is formally étale, see [3, Lecture 14 Theorem 1]. It follows that the $k[x]/(x^2)$-algebra homomorphism $R \to k$ classifying the identity automorphism $\mathrm{id} : \hat{\mathbf{G}}_0 \to \hat{\mathbf{G}}_0$ lifts extends uniquely to a $k[x]/(x^2)$-algebra homomorphism $R \to k[x]/(x^2)$ classifying some $\star$-isomorphism $\hat{\mathbf{G}} \to \hat{\mathbf{G}}_{0,k[x]/(x^2)}$. This concludes the proof of the Lubin-Tate theorem.

4. Lubin-Tate theory

Recall that we have a way of producing a homology theory from a formal group law in certain cases, namely via Landweber's exact functor theorem. For example, this gives a purely algebraic way of producing complex topological K-theory by using the multiplicative formal group law $\hat{\mathbf{G}}_m$ over $\mathbb{Z}$. Given a finite height formal group law $\hat{\mathbf{G}}_0$ over a perfect field k of characteristic p, we might wonder whether the universal deformation $\hat{\mathbf{G}}_{LT}$ over the Lubin-Tate ring satisfies the Landweber condition. The answer is yes, and is an observation of Morava.

Proposition 9. *The universal deformation $\hat{\mathbf{G}}_{LT}$ satisfies the Landweber condition.*

Proof. The sequence $(p, v_1, \ldots, v_{n-1})$ is regular by construction and v_n is invertible since $\hat{\mathbf{G}}_0$ was assumed to be of height exactly n. $\square$

The proposition implies, via Landweber's exact functor theorem, that there is a homology theory E_n such that

$$\pi_* E_n \cong \mathbb{W}(k)[[u_1, \ldots, u_{n-1}]][\beta^{\pm 1}]$$

where $|\beta| = 2$. Despite the notational convention, E_n of course depends on both the field k and the formal group law $\hat{\mathbf{G}}_0$. The homology theory goes under many different names; Morava E-theory, Lubin-Tate theory, and completed Johnson-Wilson theory among them.

Note that the functor $\mathrm{Def}_{\hat{\mathbf{G}}_0}$ depends functorially on $\hat{\mathbf{G}}_0$. This gives us an action of the automorphism group $\mathrm{Aut}(\hat{\mathbf{G}}_0)$ on our functor. We refer to the automorphism group $\mathrm{Aut}(\hat{\mathbf{G}}_0)$ as the Morava stabiliser group.[2] The action extends to an action on the pro-representing object, which extends to an action of the Morava stabilier group on the Lubin-Tate spectrum E_n, at least in the homotopy category of spectra. We will return to this subtle point in a moment, but let us first look at an example.

Example 10. Consider the multiplicative formal group $\hat{\mathbf{G}}_m$ over the prime field $\mathbb{F}_p$. This is a formal group law of height 1 so the Lubin-Tate ring is in this case given by

$$A_{LT} = \mathbb{W}(\mathbb{F}_p) = \mathbb{Z}_p\,,$$

where $\mathbb{Z}_p$ denotes the p-adic integers. The universal deformation is also the multiplicative group $\hat{\mathbf{G}}_m$, but this time defined over the p-adics. In this case Lubin-Tate

[2] We really have more functoriality than just in the formal group law $\hat{\mathbf{G}}_0$. Although it is hidden in the notation, the deformation functor also depends on the field k that we started with, and we might want to vary this also. Consider the following category.

- Objects are pairs $(k, \hat{\mathbf{G}}_0)$ where k is a field and $\hat{\mathbf{G}}_0$ is a formal group law over k.
- A morphism $(k, \hat{\mathbf{G}}_0) \to (K, \hat{\mathbf{H}}_0)$ is a pair (φ, f) where $\varphi : k \to K$ is a field homomorphism and $f : \varphi^*(\hat{\mathbf{G}}_0) \to \hat{\mathbf{H}}_\mathbf{0}$ is a map of formal groups law over K.

By the same logic as before, we have an action of the automorphism group $\mathrm{Aut}(k, \hat{\mathbf{G}}_0)$ on the functor $\mathrm{Def}_{\hat{\mathbf{G}}_0}$. This is sometimes referred to as the extended Morava stabiliser group.

theory is homotopy equivalent to p-completed complex topological K-theory:

$$E_1 = \mathrm{KU}_p^\wedge \ .$$

The Morava stabiliser group $\mathrm{Aut}(\hat{\mathbf{G}}_m) = \mathbb{Z}_p^\times$ is the group of p-adic units. A p-adic unit ℓ acts on $\mathrm{KU}_p^\wedge$ via the p-adic Adams operations

$$\psi^\ell : \mathrm{KU}_p^\wedge \to \mathrm{KU}_p^\wedge \ .$$

5. Where do we go from here?

The construction of Lubin-Tate theory has a number of issues that need to be solved before we can go any further. The issue is not only localised to Lubin-Tate theory though; it is more related to the Landweber exact functor theorem, and the question of how much structure is actually obtained when going from a formal group law to a spectrum.

To explain the issue, and why it is of interest to rectify it, let us return a bit to previous talks, and recall the algebraic construction of complex topological K-theory. The proceedure, is simple enough; we start with the multiplicative formal group law $\hat{\mathbf{G}}_m$ over $\mathbb{Z}$, and use the Landweber exact functor theorem to conclude that there is an even periodic homotopy commutative ring spectrum, namely KU, with

$$\pi_0 \,\mathrm{KU} \cong \mathbb{Z} \quad \text{and} \quad \mathrm{Spf}(\mathrm{KU}^0(\mathbb{C}P^\infty)) \simeq \hat{\mathbf{G}}_m \, .$$

This constructs KU without any mention of complex vector bundles, and other topological nastinesses. However, there are certain limitations to constructing KU in this way. For example, we know than KU has more structure than just being commutative and associative up to homotopy; it is commutative and associative up to **coherent** homotopy. Briefly, this information is captured by the statement "KU is an $\mathbb{E}_\infty$-ring". This is a structure that cannot be seen on the level of the homology theory KU_*, and so it cannot be obtained from Landweber's exact functor theorem alone.

Why are we interested in the $\mathbb{E}_\infty$-structure of KU in the first place? Well, for one, it would allow us to construct real topological K-theory KO, which is a spectrum that cannot be obtained from Landweber's exact functor theorem directly, as it is not even periodic. The two-element group $\langle \pm 1 \rangle$ acts on the formal group law $\hat{\mathbf{G}}_m$, and so gives us an action on KU. In terms of complex vector bundles, this is the action on KU by complex conjugation. However, because of the above discussion, the group $\langle \pm 1 \rangle$ only acts on KU in the homotopy category hSp. It would be nice to have more; a strict action of $\langle \pm 1 \rangle$ on KU would allow us to construct KO as the homotopy fixed points:

$$\mathrm{KO} = \mathrm{KU}^{hC_2} \ .$$

We have the same type of issues when it comes to the Lubin-Tate theory E_n we constructed in this talk. As of now, we only have a construction of E_n as a spectrum satifying

$$\pi_0 E_n \cong \mathbb{W}(k)[[u_1, \ldots, u_{n-1}]] \quad \text{and} \quad \mathrm{Spf}(E_n^0(\mathbb{C}P^\infty)) \simeq \hat{\mathbf{G}}_{LT}$$

with an action of the automorphism group $\mathrm{Aut}(\hat{\mathbf{G}}_0)$, all on the level of the homotopy category hSp. It has more structure than that though; Goerss, Hopkins, and Miller showed that E_n carries an essentially unique $\mathbb{E}_\infty$-structure and that the action of the Morava stabiliser group can be strictified, see [1]. This, among more general discussions on $\mathbb{E}_\infty$-rings, is the topic of next talk.

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$\mathbb{E}_\infty$-Rings, Lubin-Tate Theory and Elliptic Cohomology

Lior Yanovski

In this talk we shall give an overview of the results of Hopkins, Goerss and Miller regarding the existence, uniqueness and functoriality of $\mathbb{E}_\infty$-ring structures for two important families of Landweber exact spectra:

(1) Lubin-Tate spectra.
(2) Elliptic Spectra.

We also review some applications of these results. Namely, that we can obtain a coherent action of the Morava stabilizer group on Lubin-Tate spectra and that we can construct various spectra of topological modular forms by gluing together elliptic spectra.

The development of the obstruction theory for $\mathbb{E}_\infty$-ring structures and its application to Lubin-Tate spectra can be found in [1, 2]. A detailed treatment of the application to elliptic cohomology and topological modular forms can be found in chapter 12 of [4]. An ∞-categorical account of the said obstruction theory can be found in [3].

1. Ring Spectra

There are several perspectives on what spectra are. For our purposes, the main manifestation of spectra is as the homotopical analogue of abelian groups. In particular, there is an ∞-category Sp of spectra and a diagram of functors

$$\begin{tikzcd} \mathrm{Set} \arrow[r, hook] \arrow[d, shift left, "\mathrm{Free}"] & \mathcal{S} \arrow[l, bend right, "\pi_0"'] \arrow[d, shift left, "\Sigma^\infty_+"] \\ \mathrm{Ab} \arrow[r, hook, "H"'] \arrow[u, shift left, "\mathrm{Forget}"] & \mathrm{Sp}\,. \arrow[l, bend right, "\pi_0"'] \arrow[u, shift left, "\Omega^\infty"] \end{tikzcd}$$

Just like Ab, the ∞-category Sp admits a canonical symmetric monoidal structure $\otimes$. It is characterized by the property that it distributes over colimits in each coordinate and the unit is the sphere spectrum $\mathbb{S} = \Sigma^\infty_+(\mathrm{pt})$. This makes the functor

$$\Sigma^\infty_+ : (\mathcal{S}, \times, \mathrm{pt}) \to (\mathrm{Sp}, \otimes, \mathbb{S})$$

symmetric monoidal. The symmetric monoidal structure allows us to consider "coherent" algebraic structures in Sp and in particular (commutative) rings with which we can do (commutative) algebra and even algebraic geometry. Roughly, a ring should be a spectrum R with a unit map $\mathbb{S} \to R$ and a multiplication map $\mu : R \otimes R \to R$, which is unital and associative in the homotopy coherent sense. There are several approaches for making this precise. Classically, one works in a model category which allows one to rectify the coherent structures into strict ones. But, it is also possible to work completely ∞-categorically using the language of ∞-operads. The structure of an "associative ring" is then a structure of an algebra over the ∞-operad $\mathbb{E}_1$ and the structure of a "commutative ring" is a structure of an algebra over the ∞-operad $\mathbb{E}_\infty$ (there are also the ∞-operads $\mathbb{E}_n$ which interpolate between them). Since the functor

$$\mathrm{Sp} \to h\,\mathrm{Sp}$$

is symmetric monoidal, any $\mathbb{E}_1$-ring (resp $\mathbb{E}_\infty$-ring) gives in particular a ring (resp. commutative ring) in the homotopy category of Sp, but giving this structure on the level of the ∞-category is much more than that (e.g. it induces power operations on $\pi_* R$). Since the structure of, say, an $\mathbb{E}_\infty$ (or even $\mathbb{E}_1$) ring is quite a lot of highly structured data, it is not easy to construct. There some examples that come from "formal considerations":

Example 1.

(1) $\mathbb{S}$ is an $\mathbb{E}_\infty$-ring (and so is any localization $\mathbb{S}_E$).
(2) HR for a commutative ring R is an $\mathbb{E}_\infty$-ring.
(3) The internal mapping spectrum $\underline{\mathrm{hom}}(E, E)$ is an $\mathbb{E}_1$-ring for any $E \in \mathrm{Sp}$.

But there are also examples which are not (obviously) of this sort. In particular, there are spectra for which we naturally have only a structure of a homotopy (commutative) ring for which it is not obvious if there even exists an $\mathbb{E}_\infty$-ring structure that induces it. One particular family of such examples comes for Landweber's theorem. Recall that for every commutative ring R with a formal group law $\mathbb{G}$, that is classified by a *flat* map $\mathrm{Spec}\, R \to \mathcal{M}_{\mathrm{FG}}$, Landweber's theorem provides an even periodic (hence complex orientable) *homotopy commutative ring* spectrum E_R with $\pi_0 E_R \simeq R$ and the associated formal group law isomorphic to $\mathbb{G}$. In fact, one can show a bit more:

Theorem 2. *The above construction yields a fully faithful functor from the category of such pairs* $(R, \mathbb{G})$ *to the category of commutative rings in the homotopy category of spectra.*

It is interesting to know which ones can be upgraded to $\mathbb{E}_\infty$-ring spectra (and in how many different ways).

Remark 3. Not every Landweber exact theory can be given the structure of an $\mathbb{E}_\infty$-ring. For example, by recent work of Tyler Lawson, the Brown-Peterson spectrum BP at $p = 2$ can not be given the structure of an $\mathbb{E}_\infty$-ring (and this was also extended to odd primes by Andrew Senger)

2. Lubin-Tate Theory

In light of the above, we specialize to a particular class of Landweber exact spectra. For a perfect field κ of characteristic p and a finite height n formal group law $\mathbb{G}_0$ over it, the pair $(\kappa, \mathbb{G}_0)$ is itself *not* flat over $\mathcal{M}_{\mathrm{FG}}$. Lubin-Tate theory constructs functorially a ring

$$R \simeq \mathbb{W}(\kappa)\,[[u_1, \ldots, u_{n-1}]]$$

and a formal group law $\mathbb{G}$ over R, such that $(R, \mathbb{G})$ is a universal deformation of $(\kappa, \mathbb{G}_0)$. The pair $(R, \mathbb{G})$ satisfies the conditions of Landweber's theorem, so we get functorially a homotopy commutative ring spectrum $E\,(\kappa, \mathbb{G}_0)$.

Corollary 4. *There is a fully faithful functor from the category of pairs* $(\kappa, \mathbb{G}_0)$ *to the category of commutative rings in the homotopy category of spectra.*

We are interested in upgrading (functorially) the homotopy commutative ring structure of $E\,(\kappa, \mathbb{G}_0)$ to an $\mathbb{E}_\infty$-ring structure. One of the key points of this workshop is to explain how this can be achieved by transporting the Lubin-Tate theory into the ∞-categorical world. But long before this, it was achieved by Goerss, Hopkins and Miller using a more "by hand" approach. Namely, they developed a general obstruction theory for constructing algebras over ∞-operads in spectra and showed that, in this particular case, the obstructions vanish.

Theorem 5 (Goerss-Hopkins-Miller)**.** *There is an essentially unique functor from pairs* $(\kappa, \mathbb{G}_0)$ *to* $\mathbb{E}_\infty$*-ring spectra that lifts the Lubin-Tate construction* $E\,(\kappa, \mathbb{G}_0)$*.*

One consequence of the functorially is that the Morava Stabilizer group

$$\Gamma = \mathrm{Aut}\,(\kappa, \mathbb{G}_0)$$

acts on $E_n = E\,(\kappa, \mathbb{G}_0)$ in the ∞-categorical sense (namely, "coherently" and not only up to homotopy) by $\mathbb{E}_\infty$-ring maps. In particular, for any subgroup $\Gamma_0 \subseteq \Gamma$ we can take the homotopy fixed points $E_n^{h\Gamma_0}$ and this is again an $\mathbb{E}_\infty$-ring spectrum.

Example 6. For the multiplicative formal group law over $\mathbb{F}_2$, we have $E_1 \simeq \widehat{KU}_2$ and $\Gamma = \mathbb{Z}_2^\times$. Taking $\Gamma_0 = \{\pm 1\} \subseteq \mathbb{Z}_2^\times$ we get $E_1^{h\Gamma_0} \simeq \widehat{KO}_2$, which is the 2-adic completion of the real K-theory spectrum KO.

For $n \geq 2$ and finite $\Gamma_0 \leq \Gamma$, we can think of $E_n^{h\Gamma_0}$ as higher analogues of (completed) real K-theory.

Remark 7. The group Γ is pro-finite and acts "continuously" on $\mathbb{E}_n$. For non-finite subgroups $\Gamma_0 \leq \Gamma$, one would actually like to take "continuous fixed points". Without going into details, this can be formalized and implemented. The most drastic (and very useful) case is $E_n^{h\Gamma} \simeq \mathbb{S}_{K(n)}$ where $K(n)$ is Morava K-theory.

It is important to note that the $\mathbb{E}_\infty$-ring spectra we get after taking fixed points are no longer complex orientable and therefore can not be constructed (even just as spectra) by Landweber's theorem.

3. Elliptic Spectra & tmf

Another important source for Landweber exact theories (i.e. formal group laws classified by a flat map) is the theory of elliptic curves. An elliptic curve C over a commutative ring R is in particular a commutative 1-dimensional algebraic group over R and hence its completion $\widehat{C}$ is a formal group over R (which turns out to always of height ≤ 2). If it is classified by a flat map to $\mathcal{M}_{\mathrm{FG}}$, by Landweber's theorem, it can be associated with a homotopy commutative ring spectrum E_C (such spectra are known as "elliptic cohomology"). Using a more refined version of the previously mentioned obstruction theory, Goerss, Hopkins and Miller proved:

Theorem 8 (Goerss-Hopkins-Miller)**.** *The spectrum E_C admits a structure of an $\mathbb{E}_\infty$-ring.*

Once again, there is also the question of functoriality, which in this case takes the form of "globalization". There is a Deligne-Mumford moduli stack $\mathcal{M}_{\mathrm{ell}}$ that classifies elliptic curves. For any commutative ring R, the collection of maps

$$f : \mathrm{Spec}\,(R) \to \mathcal{M}_{\mathrm{ell}}$$

is equivalent to the groupoid of elliptic curves over R. The map that associates to an elliptic curve its completion as a formal group is represented by a flat map of stacks

$$\Phi : \mathcal{M}_{\mathrm{ell}} \to \mathcal{M}_{\mathrm{FG}}.$$

Namely, for each f as above, the composition $\Phi \circ f$ classifies $\widehat{C}/R$. Hence, if f is flat, we can associate to it by Landweber's theorem a homotopy commutative ring spectrum. This can be done in particular to any étale map f in a functorial way. Thus, we get a presheaf $\mathcal{O}_0^{\mathrm{top}}$ of homotopy commutative ring spectra on the étale site of $\mathcal{M}_{\mathrm{ell}}$. With some considerable more work, one gets:

Theorem 9 (Goerss-Hopkins-Miller)**.** *There is a sheaf $\mathcal{O}^{top}$ of $\mathbb{E}_\infty$-rings on $\mathcal{M}_{ell}$, which lifts the presheaf $\mathcal{O}_0^{top}$.*

Having the sheaf $\mathcal{O}^{\mathrm{top}}$ allows us to take sections over non-affine étale open "subsets" of $\mathcal{M}_{\mathrm{ell}}$ obtaining new $\mathbb{E}_\infty$-ring spectra. In particular, we can take global sections:

$$\mathrm{TMF} = \mathcal{O}^{\mathrm{top}}\left(\mathcal{M}_{\mathrm{ell}}\right).$$

In fact, the entire discussion can be extended to a certain compactification $\mathcal{M}_{\overline{\mathrm{ell}}}$ of $\mathcal{M}_{\mathrm{ell}}$ that classifies "generalized elliptic curves" (allowing a nodal singularity) giving also:

$$\mathrm{Tmf} = \mathcal{O}^{\mathrm{top}}\left(\mathcal{M}_{\overline{\mathrm{ell}}}\right) \quad \text{and} \quad \mathrm{tmf} = \tau_{\geq 0}\left(\mathrm{Tmf}\right).$$

We reiterate that all versions of tmf above are not complex orientable.

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Overview of the Classic Theory of p-Divisible Groups

Catherine Ray

We will discuss a victory of the 50s-60s: Lie theory for abelian schemes over perfect fields of char p. For this lecture, until mentioned otherwise, we will fix a field k which is perfect and of characteristic p. Further, everything is commutative.

1. Basic Definitons

Definition 1. Affine group scheme is Spec A where A is a bicommutative Hopf k-algebra.

Definition 2. Finite group scheme is an affine group scheme represented by a finite k-vector space A.

1.1. **$\underline{\mathbb{Z}/p^n}$, μ_{p^n}, α_p and their Hopf algebras.** For example:

- $\underline{\mathbb{Z}/p} = \operatorname{Spec} \operatorname{Hom}_{\mathsf{Sets}}(\mathbb{Z}/p, k) = \operatorname{Spec} \prod_{\mathbb{Z}/p} k = \coprod \operatorname{Spec} k$;
- $\mu_{p^n} = \operatorname{Spec} k[x]/(x^{p^n} - 1)$ is the pth roots of unity; $\Delta(x) = x \otimes x$.
- $\alpha_p = \operatorname{Spec} k[x]/x^p$ is the additive pth roots of unity; $\Delta(x) = x \otimes 1 + 1 \otimes x$.
- $E[p] \simeq \mu_p \times \underline{\mathbb{Z}/p}$ (the kernel of multiplication by p on an ordinary elliptic curve)

1.2. **Definition and examples of p-divisible groups.** We give here examples of p-div groups, $\underline{\mathbb{Z}/p^n}$, μ_{p^n}, $A[p^n]$.

Definition 3. A **p-div group** of height h is an inductive system (i.e., an inductive limit before you take the limit) of commutative finite group schemes

$$G_1 \xrightarrow{i_1} G_2 \xrightarrow{i_2} G_3 \xrightarrow{i_3} \cdots$$

over k satisfying two properties:

(1) They must fit into an exact sequence

$$0 \to G_n \xrightarrow{i_n} G_{n+1} \xrightarrow{p^n} G_{n+1} \to 0$$

(that is, the kernel of the map p^n on G_{n+1} is the copy of G_n sitting inside of G_{n+1}).

(2) $\text{rank}(G_1) = p^h$ where h is an integer. (aka, $G_1 = \text{Spec}\, A_1$, and A_1 is a free k-algebra of dimension p^n) This is called **height**.

Often, we will work with the individual finite group schemes rather than the whole colimit, as they are easier to handle.

2. Formal groups are connected p-divisible groups

Definition 4. An affine finite group scheme is **connected** if its representing Hopf algebra is a local ring.

Definition 5. A **divisible formal group** F is such that this sequence is exact (as a sequence of formal group schemes)

$$F[p] \to F \xrightarrow{[p]} F$$

Note that we may make a p-divisible group out of a divisible formal group scheme F.

$$F \mapsto (F[p] \hookrightarrow F[p^2] \hookrightarrow ...)$$

Remark. It is important here that our formal groups are indeed smooth formal groups, which implies that they locally look like Spf of a power series ring quotiented by a closed ideal.

Theorem 6. *(Serre-Tate equivalence) There is an equivalence of categories between divisible smooth formal groups over k, and connected p-divisible groups over k. The functor sends:*

$$F \mapsto (F[p] \hookrightarrow F[p^2] \hookrightarrow ...)$$

Let's look at some finite flat group schemes, and see what their connected components look like.

- $\text{Spec}\, \mathbb{F}_p[x]/x^p$ is local, and thus connected.
- We see that for $\mu_{p/\mathbb{F}_p} = \text{Spec}\, \mathbb{F}_p[x]/(x^p - 1)$ is local, since $(x-1)^p$, so $G^0 = G$, and thus $G/G^0 = \text{Spec}\, \mathbb{F}_p$
- $\underline{\mathbb{Z}/p}$ is completely etale. (clearly disconnected)
- $E[p] \simeq \mu_p \times \underline{\mathbb{Z}/p}$

Definition 7. An affine group scheme G over k is **etale** if $G \times_k \bar{k} \simeq \text{Spec}\, \bar{k}[G(\bar{k})]$ (the coproduct of constant group schemes).

3. Connected-Etale Sequence and Splitting

Let G^0 be the connected component of the identity. Then, take $G^{ét} := G/G^0$.

$$0 \to G^0 \to G \to G^{ét} \to 0$$

This sequence in fact *always splits* (over a perfect field of char $p > 0$). And this splitting is natural!

We can think of this on the level of representing Hopf algebras, $A \simeq A_0 \otimes A_{ét}$.

Example 8. (analogy)

$$H^*(\Omega^\infty X;\mathbb{F}_p) \simeq \mathrm{Hom}(\pi_0 X,\mathbb{Z}/p)\otimes_{\mathbb{F}_p} H^*(\Omega^\infty X_0;\mathbb{F}_p)$$

(An example of a connected etale splitting of Hopf algebras, first part is "etale" (the only difference is that there could be an infinite number of connected components), second is connected)

Remark. Any map from G to an etale finite flat group scheme will factor through $G^{\acute{e}t}$.

Remark. The fact that G^0 is a sub-group scheme relies on the general fact that if X is connected and has a rational point over the base field, then it is geometrically connected. If X is geometrically connected, then $X\times_k X$ is geometrically connected.

Remark. We define this exact sequence for finite flat group schemes, then take colimit to get the exact sequence for p-divisible groups.

4. Basics of Dieudonne Theory over k

4.1. **Dieudonne Theory I: Classification up to Isomorphism.**

Moral: pDiv is equivalent to a category of modules, which one?

Where do these theorems come from? Manin proved them for a special case using the combination of a formal categorical statement of Gabriel, and some geometric input [2]. (Then he used descent and duality, discussed in the last section of [1], to prove the whole statement.)

We now discuss Gabriel's theorem on taking an abelian category and constructing a category of a modules.

4.1.1. *Gabriel's Theorem.*

Definition 9. An **injective hull** of an object c in abelian category $\mathcal{C}$ is a monomorphism $c\hookrightarrow I$ to an object $I\in\mathcal{C}$ such that:

- (injective) $\mathrm{Hom}(-,I)$ is exact
- (hull) $c\hookrightarrow I$ and there are no "smaller" I, that is, every other monomorphism from c to an injective object in $\mathcal{C}$ factors through this morphism.

Definition 10. Let a locally finite category be an abelian category with a finite set of generating objects, enough injectives, and "enough" colimits and limits.

Remark. (setup) Let C be a locally finite category, (S_α) the family of all simple objects of C, and I_α the injective hull of S_α. Let $I=\coprod I_\alpha$; the "universal" injective object. Let $E:=\mathrm{End}_C(I)$. Topologize E by taking as a base of neighborhoods of zero the system of all left ideals $I\subset E$ of finite colength. E is complete wrt this topology. We denote by M_E the category of complete topological left E-modules, whose topology is linear and has a base of neighborhoods of zero consisting of all sub-modules of finite colength.

Theorem 11. *The contravariant functor $C\to M_E$, $X\mapsto\mathrm{Hom}(X,I)$ is an antiequivalence between the categories.*

There is little hope to compute injective objects for a general category, nor the endomorphisms of an object in a category. The geometric input, and how we get Dieudonne theory from this formal statement, is that we know the generators of the category of certain finite flat group schemes. This category we will consider if Ind of locally-local finite group schemes, $\mathsf{fGrp}^{\ell,\ell}$:

Definition 12. A finite group scheme $\operatorname{Spec} A$ is **locally-local**, if A and A^* are both local rings.

The generator of $\mathsf{Ind}(\mathsf{fGrp}^{\ell,\ell})$ is α_p, and we know that the injective hull of α_p is the colim of truncated Witt-schemes.

Definition 13. The Witt scheme is a ring scheme whose k points are the rings of k-Witt vectors.

Remark. F and V act on a point of the Witt scheme as:

$$F : (x_0, x_1, x_2, ...) \mapsto (x_0^p, x_1^p, x_2^p, ...)$$
$$V : (x_0, x_1, x_2, ...) \mapsto (0, x_0, x_1, ...)$$

Definition 14.

$$W^r := \ker(W \xrightarrow{F^r} W)$$
$$W_s := \operatorname{coker}(W \xrightarrow{V^s} W)$$

$$W_s^r = \operatorname{Spec} k[x_0, ..., x_s]/(x_0^{p^r}, ..., x_s^{p^r})$$

$\operatorname{colim} W_s^r$ is an infinite dimensional formal group, an an object in $\mathsf{Ind}(\mathsf{fGrp}^{\ell,\ell})$. Now all that is left is to understand

$$\operatorname{End}_{\mathsf{Ind}(\mathsf{fGrp}^{\ell,\ell})}(\operatorname{colim}_{r,s} W_s^r)$$

It ends up being

$$W(k)\{F, V\}/(\sim)$$

These are formal variables, let's discuss the equivalence relations. We need Cartier duality to think of these relations properly, but suffice to say (for $a \in W(k)$, where σ is the Frobenius in Witt vectors):

$$F(a) = \sigma(a)F$$
$$V(\sigma(a)) = aV$$
$$FV = p$$

Theorem 15. *(finite Dieudonne) There is a categorical anti-equivalence between finite group schemes of order p^h; and E-modules with $W(k)$-length n.*

Corollary 16. *(Dieudonne up to isomorphism) There is a categorical anti-equivalence between* pDiv *of height h; and free E-modules which are free as $W(k)$-modules (of rank h).*

Proof. Taking a colimit over diagrams on one side, and a limit on the other. The freeness comes from the fact that with torsion, the length of the $W(k)$-module doesn't grow fast enough to make it into the limit. □

4.2. **Dieudonne Theory II: Classification up to Isogeny.** Can we understand this category of modules? Well, to understand it we must make some sacrifices.

Definition 17. An isogeny is a map whose kernel is a finite flat group scheme.

Let E_F be $W(k)[\frac{1}{p}]\{F\}/(Fa = \sigma(a)F)$.

Theorem 18. *(Dieudonne up to isogeny) The category of p-divisible groups over up to isogeny* $\mathsf{pDiv}^{\mathsf{isog}}$ *has a fully faithful embedding into the category of finitely generated E_F-modules.*

Theorem 19. *(Dieudonne-Manin Classification Theorem)*

- *The category of finitely generated E_F-modules is semi-simple.*
- *If $k = \bar{k}$ the simple objects are of the form $G_{s,r} := E_F/E_F(F^r - p^s)$. This s/r is called the slope.*

Remark.

$$E_F/E_F(F^r - p^s) \simeq E_F \otimes_E E/E(F^{r-s} - V^s)$$

Remark. A technical point on simple objects in the case of general k. Given a simple module M over $\bar{k}$ of slope s/r, then $\mathrm{Aut}(M) = D^*$, where D is a skew field over $\mathbb{Q}_p$ with invariant s/r. There are different simple objects of slope s/r over k, they are of the form $H^1(\mathrm{Gal}(\bar{k}/k), D^*)$, for different actions of $\mathrm{Gal}(\bar{k}/k)$ on D^*.

Example 20. Let's discuss some isocrystals of familiar p-divisible groups:

- $\mu_{p^\infty} = \mathbb{G}_m[p^\infty]$ is isogenous to $G_{1/1}$
- $\underline{\mathbb{Z}/p^\infty} \simeq \underline{\mathbb{Q}_p/\mathbb{Z}_p}$ is isogenous to $G_{0/1}$
- For an ordinary elliptic curve, $G_{0/1} \oplus G_{1/1}$.
- For a supersingular elliptic curve, $G_{1/2}$
- For the Honda formal group of height h over $\mathbb{F}_p$ is $G_{1/h}$, $E_F/(F^h - p)$, this has basis $\{1, F, ..., F^{(h-1)}\}$ and

$$VF^i = \begin{cases} F^{h-1}, & i = 0; \\ pF^{i-1}, & 1 \le i \le h-1 \end{cases}$$

Remark. The left category $E_F - \mathsf{Mod}^{f.g.}$ is sometimes called the category of isocrystals.

5. p-divisible groups over a more general base

We can get pretty far with fields.

Theorem 21. *(Reynaud-Tate) (Tate's rigidity theorem) Let R be a local DVR with residue characteristic p, and $K := \mathrm{Frac}(R)$ of characteristic 0. Then, the generic fiber functor is fully faithful.*

$$\mathsf{pDiv}_{/R} \to \mathsf{pDiv}_{/K}$$
$$G \mapsto G_K$$

This tells us that we can understand a p-divisible group over $\mathrm{Spec}\, R$ by its generic fiber. To do p-divisible groups over more general schemes, we repeat the original definition but make our constituent group schemes finite *flat*, rather than just finite. This was implicit before, but we were working over a field so everything was automatically flat.

Definition 22. A **p-div group of height h** is an inductive system (i.e., an inductive limit before you take the limit) of commutative flat group schemes (locally of finite presentation)

$$G_1 \xrightarrow{i_1} G_2 \xrightarrow{i_2} G_3 \xrightarrow{i_3} \cdots$$

over a base scheme S satisfying two properties:

(1) They must fit into an exact sequence

$$0 \to G_n \xrightarrow{i_n} G_{n+1} \xrightarrow{p^n} G_{n+1} \to 0$$

(that is, the kernel of the map p^n on G_{n+1} is the copy of G_n sitting inside of G_{n+1}).

(2) $\mathrm{rank}(G_1) = p^h$ where h is a locally constant function $h : S \to \mathbb{Z}$

Fact: h is invariant under base change $S' \to S$. So p-divisible groups are great for deformation theory.

6. Serre-Tate Theorem

Set up. Let R be a base ring where p is nilpotent. $p \in I \subset R$ is a nilpotent ideal. Let $\mathsf{AbSch}_{/R}$ be the category of abelian schemes over R. Let $\mathsf{Def}(R, R_0)$ be the category of triples (A_0, G, ε) consisting of:

- $A_0 \in \mathsf{AbSch}_{/R_0}$
- $G \in \mathsf{pDiv}_R$
- an isomorphism in $\mathsf{pDiv}_{/R_0}$;

$$\varepsilon : G \times_R R_0 \xrightarrow{\simeq} A[p^\infty]$$

Theorem 23. *Serre Tate (Katz 1.2.1* [3]*): Let R and R_0 be as above. Then the functor*

$$\mathsf{AbSch}_{/R} \to \mathsf{Def}(R, R_0)$$
$$A \mapsto (A_0, A_0[p^\infty], \varepsilon)$$

is an equivalence of categories.

In other words, we can completely understand the deformations of an abelian scheme in terms of the deformations of its associated p divisible group.

Remark. Note that this does not depend on the choice of $G_0 := G \times_R R_0 \in \mathsf{pDiv}_{/R_0}$. That is, we are not fixing a group which we are lifting. This is a more general statement. We can take a fiber and recover the case that Betram and Alice discuss, where G_0 is fixed.

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The main theorem of Lurie's Elliptic Cohomology II: Orientations

Bertram Arnold

Recall from the previous talk the following theorems:
(Lubin-Tate). Let k be a perfect field, $\hat{G}$ a formal group law of height $n < \infty$. Then there is a universal deformation $\hat{G}_{LT}$ of $\hat{G}$ defined over $A_{LT}(k, \hat{G}) \cong W(k)[[u_1, \dots, u_{n-1}]]$.
(Morava). The formal group law $\hat{G}_{LT}$ is Landweber exact, hence defines a homotopy commutative complex oriented ring spectrum $E(k, F)$.
(Goerss-Hopkins). There is a unique E_∞-ring structure on $E(k, F)$ compatible with the homotopy ring structure, which in addition is functorial in the pair (k, F).

We generalize this by replacing k by a $\mathbb{F}_p$-algebra R_0 and allowing deformations over (connective) E_∞-rings. To do this, we need to replace formal groups by *p-divisible groups*, which have a better-behaved deformation theory.

Theorem 1 (0.0.8)**.** *Let R_0 be a* F-finite *Noetherian $\mathbb{F}_p$-algebra and $G_0 \in BT^p(R_0)$ a* nonstationary *1-dimensional p-divisible group over R_0. Then there is a universal deformation G^{cl} defined over a complete adic ring R^{cl} and a complex periodic E_∞-ring spectrum $R^{un}_{G_0}$ such that $\pi_{\mathrm{odd}} R^{un}_{G_0} = 0, \pi_0(R^{un}_{G_0}) \cong R^{cl}_{G_0}$ and the formal group on $R^{cl}_{G_0}$ defined via the complex orientation is isomorphic to $(G_{cl})^\circ$.*

Furthermore, both of these are functorial in (R_0, G_0).

Here is what the two conditions mean:

F-finite means that the Frobenius $\phi_{R_0} : R_0 \to R_0, x \mapsto x^p$ is a finite algebra map, i.e. there are elements $x_1, \dots, x_n \in R_0$ such that $\{\sum_i x_i r_i^p \mid r_i \in R_0\} = R_0$. This is automatic if R_0 is *semiperfect*, i.e. $R = R^p$.

Nonstationary means that the morphism $\operatorname{Spec} R_0 \to \mathcal{M}_{BT^p}$ is unramified, i.e. infinitesimal deformations admit at most one lift. More explicitly, for any $x \in |\operatorname{Spec} R_0|$ with residue field $R_0 \xrightarrow{\beta_0} \kappa(x)$, the base change map $\mathrm{Der}(R_0, \kappa) = \mathrm{CAlg}^{\heartsuit}_{/\kappa(x)}(R_0, \kappa(x)[\epsilon]/\epsilon^2) \to BT^p(\kappa(x)[\epsilon]/\epsilon^2)/\cong$ which sends $\beta = \beta_0 + \epsilon \mathrm{d}$ to $\beta_* G_0$

is injective. If R_0 is semiperfect, the calculation $\mathrm{d}(x^p) = p\beta_0(x)\mathrm{d}x = 0$ shows that the source of this map is zero, so this condition is vacuous.

Complex periodic means that the spectrum $E = R^{or}_{G_0}$ is complex orientable and weakly periodic, i.e. the multiplication map $\pi_n(E) \otimes_{\pi_0(E)} \pi_2(E) \to \pi_{n+2}(E)$ is an isomorphism for all $n \in \mathbb{Z}$. In particular, $\pi_2(E)$ is a rank 1 projective module over $\pi_0(E)$, and the even homotopy groups of E are given by its powers.

Example 2. If $G_0 = \hat{G}[p^\infty]$ for a divisible formal group of finite height over a perfect field, $R^{cl}_{G_0} \cong A_{LT}(k, \hat{G})$ and $R^{or}_{G_0} \cong E(k, \hat{G})$.

Recall that we proved the Lubin-Tate theorem by using formal smoothness to lift the universal first-order deformation to $A_{LT}(k, \hat{G})$ and then used induction over the lengths of Artinian algebras to show that this deformation is universal since the lifts of a map from $A_{LT}(k, \hat{G})$ and a deformation of $\hat{G}$ along a square-zero extension are both torsors over the space of first-order deformations. For the more general theorem, a deformation doesn't necessarily have to lift over a square-zero extension, so it's already not clear what $R^{cl}_{G_0}$ should be. Furthermore, it's in general quite hard to write down an E_∞-ring explicitly in terms of generators and relations. Instead we first define the *unoriented* deformation ring $R^{un}_{G_0}$ via its corepresented functor $\mathrm{Map}_{\mathrm{CAlg}^{ad}_{cpl}}(R^{un}_{G_0}, A) = Def_{G_0}(A)$, using an abstract theorem to guarantee its representability. The two conditions are needed to control the cotangent complex of the functor $Def_{G_0}(A)$: F-finiteness guarantees it is almost perfect and connective, while nonstationarity means its zero-truncation vanishes, i.e. that it is 1-connective.

Then, we build $R^{or}_{G_0}$ by adjoining and inverting a "Bott element", which means that maps out of it also have an explicit description. After defining it this way, it is not at all clear how to control maps into either of these rings, i.e. their homotopy groups! Once existence has been established, the hard work in the proof of Theorem 0.0.8 is in showing that $\pi_{odd}(R^{or}_{G_0}) = 0$ and that the map $R^{un}_{G_0} \to R^{or}_{G_0}$ is an isomorphism on π_0. The advantage of introducing $R^{un}_{G_0}$ is that it maps to both R_0 (classifying the trivial deformation) and $R^{or}_{G_0}$ (classifying the fact that we can forget the orientation), so that these statements can be checked flat-locally on $R^{un}_{G_0}$.

Definition 3 (0.0.11)**.** An *adic E_∞-ring* is an E_∞-ring A equipped with an adic topology τ on $\pi_0(A)$, i.e. such that there is a finitely generated ideal $I \subset \pi_0(A)$ with $\{x + I^n\}$ a basis of τ. Such an I is called an *ideal of definition.* The adic E_∞-ring (A, τ) is *complete* if A is I-complete for some ideal of definition I, i.e. if for $x \in \pi_0(A)$ topologically nilpotent we have $\lim(\cdots \xrightarrow{\cdot x} A \xrightarrow{\cdot x} A) \simeq *$. Morphisms of adic E_∞-rings are E_∞-ring morphisms which are continuous on π_0, i.e. map an ideal of definition to an ideal of definition.

There is a generalization of the notions of p-divisible and formal group to E_∞-rings which recovers the usual notions when restricted to $\mathrm{CAlg}^\heartsuit$. We will see the precise definitions in Talks 7 through 9. For now, we take for granted that for

$A \in \mathrm{CAlg}$ there is an ∞-category $BT^p(A)$ which furthermore depends functorially on A.

Definition 4 (3.1.4)**.** Let R_0 be a commutative ring, $G_0 \in BT^p(R)$, A an adic E_∞-ring. A *deformation of* G_0 *over* A consists of $G \in BT^p(A)$, a morphism $\mu : R_0 \to \pi_0(A)/I$ for some ideal of definition I, and an isomorphism $\alpha : \mu_* G_0 \cong G|_{\pi_0(A)/I}$, up to equivalence. Formally:

$$\mathrm{Def}_{G_0}(A) := \varinjlim_I BT^p(A) \times_{BT^p(\pi_0(A)/I} \mathrm{Map}(R_0, \pi_0(A)/I)\ .$$

Note that for $f : R_0 \to S_0$, there's a canonical map $\mathrm{Def}_{f_* G_0}(A) \to \mathrm{Def}_{G_0}(A)$.

Remark 5 (3.1.8)**.** Suppose $f : R_0 \twoheadrightarrow R_1$ with finitely generated nilpotent kernel J and let $(G, I, \mu, \alpha) \in \mathrm{Def}_{G_0}(A)$, in particular I is an ideal of definition. Then the preimage J' of $\mu(J)\pi_0(A)/I$ is contained in $\sqrt{I}$, so $I + J'$ is also an ideal of definition, and $(G, I + J', \mu/J, \alpha|_{\pi_0(A)/(I+J')}) \in \mathrm{Def}_{f_* G_0}(A)$. This construction provides a homotopy inverse to the above map; in particular, if R_0 is Noetherian, we may replace R_0 by its reduction and obtain an equivalent functor.

Theorem 6 (3.1.15)**.** *Let R_0 be a F-finite $\mathbb{F}_p$-algebra and $G_0 \in BT^p(R_0)$ a nonstationary p-divisible group. Then there is a complete adic E_∞-ring $R^{un}_{G_0}$ corepresenting $\mathrm{Def}_{G_0}(-)$, i.e. there is $(G^{un}, \dots) \in \mathrm{Def}_{G_0}(R^{un}_{G_0})$ such that for A complete adic, the map $\mathrm{Map}_{\mathrm{CAlg}^{ad}}(R^{un}_{G_0}, A) \to \mathrm{Def}(G_0)$ is a weak equivalence.*

It is connective and Noetherian, the map $\rho : R^{un}_{G_0} \to R_0$ classifying the trivial deformation is surjective on π_0, and its kernel is an ideal of definition.

Remark 7 (3.1.16)**.** If all these conditions are satisfied, R_0 must be F-finite and G_0 nonstationary.

Example 8. If $G_0 \in BT^p(\mathbb{F}_p)$ is étale, the deformation functor is given by

$$A \mapsto \begin{cases} * & \text{if } p \in \pi_0(A) \text{ topologically nilpotent} \\ \emptyset & \text{else} \end{cases}$$

Thus $R^{un}_{G_0} \simeq S^\wedge_{(p)}$ is the p-completed sphere.

Example 9. Cartier duality works for general E_∞-rings, in particular it induces a natural isomorphism $\mathrm{Def}_{G_0}(A) \simeq \mathrm{Def}_{G_0^\vee}(A)$. In particular, since the Cartier dual $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ of $\mu_{p^\infty} \in BT^p(\mathbb{F}_p)$ is étale, we obtain $R^{un}_{\mu_{p^\infty}} \simeq S^\wedge_{(p)}$. Contrast this with Theorem 0.0.8's assertion that $R^{or}_{\mu^{p^\infty}} \simeq K^\wedge_{(p)}$.

Example 10. Let $G_0 \in BT^p(R_0)$ and G_1 its image in $BT^p(R_0[\epsilon]/\epsilon^2)$. We obtain a commutative diagram

$$\begin{array}{ccc} R^{un}_{G_0} & \xrightarrow{\simeq} & R^{un}_{G_1} \\ \downarrow & & \downarrow \\ R_0 & \longrightarrow & R_0[\epsilon]/\epsilon^2 \end{array}$$

where the top horizontal map is a weak equivalence since the map between corepresented functors is. It follows that the right-hand vertical map is not surjective on π_0 since it factors through R_0. Of course, G_1 is not nonstationary, since it is "constant along the fibers of the projection to R_0".

Theorem 11 (6.3.1)**.** *Suppose in addition to R_0 F-finite and G_0 nonstationary that G_0° is 1-dimensional. Then $R_{G_0}^{un}[p^{-1}] \in \mathrm{CAlg}^\heartsuit$, i.e. all higher homotopy groups are p-torsion.*

We now turn our attention to $R_{G_0}^{or}$. To define it, we first need to define the *Quillen formal group* G_A^Q which is defined for any complex periodic E_∞-ring A. It induces a formal group on $\pi_0(A)$ which is just the formal group we get from the complex orientation. Given any formal group $G \in FG(R)$, we can define its *orientation classifier* $Or(G)$, which is initial amongst complex periodic E_∞-rings receiving a map $f : R \to Or(G)$ together with an isomorphism $f_*G \cong G_A^Q$.
Construction (6.0.1). $R_{G_0}^{or}$ is the orientation classifier of the identity component of the universal deformation of G_0. It satisfies

$$\mathrm{Map}_{\mathrm{CAlg}_{cpl}^{ad}}(R_{G_0}^{or}, A) = \begin{cases} \mathrm{fib}_{\{G_A^Q\}}(\mathrm{Def}_{G_0}(A) \xrightarrow{(G,\cdots)\mapsto G^\circ} FG(A)) & A \text{ complex periodic} \\ \emptyset & \text{else} \end{cases}$$

If $G_0 = G_0^\circ$ is connected, its universal deformation G^{un} is formally connected, and since formally connected p-divisible groups form a full subcategory of formal groups, the space of maps $\mathrm{Map}_{\mathrm{CAlg}_{cpl}^{ad}}(R_{G_0}^{or}, A)$ for some complex periodic A is equivalent to the *set* of equivalence classes of maps $f : R_0 \to \pi_0(A)/I$, together with an isomorphism between the formal group f_*G_0 and the image of the classical Quillen formal group on $\pi_0(A)$.

Formal groups over $\mathbb{E}_\infty$-rings

Johannes Anschütz

1. Description of the talk

The aim of this talk is to present a definition of a formal group over $\mathbb{E}_\infty$-rings, which extends the classical definition over discrete commutative rings.

2. Conventions

Let R be a (discrete) commutative ring and let A be an adic R-algebra, i.e., A is a complete topological R-algebra whose topology is I-adic for a finitely generated ideal $I \subseteq A$. Then the *formal spectrum* of A is defined as the functor

$$\mathrm{Spf}(A)\colon (R-\mathrm{Alg}) \to (\mathrm{Sets}),\ B \to \mathrm{Hom}_{R,\mathrm{cts}}(A,B) = \varinjlim_n \mathrm{Hom}_R(A/I^n, B)$$

on the category $(R-\mathrm{Alg})$ of (discrete) commutative R-algebras. In this talk all formal schemes we consider will be of this form. Moreover, the functor $A \to \mathrm{Spf}(A)$ is fully faithful. Given $\mathrm{Spf}(A)$ we set $\mathcal{O}_{\mathrm{Spf}(A)}(\mathrm{Spf}(A)) := A$.

We denote by (Sets) resp. (Ab) the categories of sets resp. of abelian groups. We denote furthermore by $\mathcal{S}$ the ∞-category of spaces, i.e., Kan complexes.

3. Formal groups over discrete commutative rings

Let k be a field.

Definition 1. A *formal group* over k is a functor

$$G\colon (k-\mathrm{Alg}) \to (\mathrm{Ab})$$

whose underlying functor to sets is isomorphic to $\mathrm{Spf}(k[[t_1,\dots,t_n]])$ for some $n \geq 0$. The (uniquely determined) integer n is called the dimension of G.

Remark 2.

- More precisely, such a G should be called a "commutative, smooth, connected" formal group. For simplicity we stick with "formal group".
- If $n = 1$, i.e., $G \cong \mathrm{Spf}(k[[t]])$ is one-dimensional (with zero section automatically given by the locus $t = 0$), then (as in talk 2) the multiplication

  $$\mu\colon G \times G \cong \mathrm{Spf}(k[[t_1,t_2]]) \to G \cong \mathrm{Spf}(k[[t]])$$

  is determined by the power series $f(t_1,t_2) := \mu^*(t) \in k[[t_1,t_2]]$, which satisfies the relations

  $$\begin{aligned} f(0,t) = t &= f(t,0), \\ f(t_1,t_2) &= f(t_2,t_1), \\ f(t_1,f(t_2,t_3)) &= f(f(t_1,t_2),t_3), \end{aligned}$$

  i.e., f is a formal group law over k (in the sense of talk 1). Conversely, for a formal group law $f(t_1,t_2)$ defining μ by $\mu^*(t) := f(t_1,t_2)$ enhances the functor $G := \mathrm{Spf}(k[[t]])$ to a formal group.[1] Thus formal group laws over k are equivalent to formal groups over k together with the choice of a coordinate.[2]

Let R be a (discrete) commutative ring. The direct analog of a formal group over R in the sense of Definition 1, i.e., just replacing the field k by the ring R, is not reasonable: the requirement that as formal schemes $G \cong \mathrm{Spf}(R[[t_1,\dots,t_n]])$ is too strong, because this requirement is not Zariski local on $\mathrm{Spec}(R)$. The following lemma explains how this condition can be corrected.

Lemma 3. *Let R be a (discrete) commutative ring and let $\mathrm{Spf}(A) \to \mathrm{Spec}(R)$ be a formal scheme over R. The following are equivalent:*

(1) there exists a covering $\mathrm{Spec}(R) = \bigcup_{i=1}^{m} \mathrm{Spec}(R_i)$ by affine open subsets such that for all i there is an isomorphism

$$\mathrm{Spf}(A) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R_i) \cong \mathrm{Spf}(R_i[[t_1,\dots,t_{n_i}]])$$

for some $n_i \geq 0$.

[1] The only thing to check is that f automatically admits an inverse, i.e., there exists a power series $\iota \in k[[t]]$ such that $f(t,\iota(t)) = t$.

[2] And the morphisms of formal groups don't respect the coordinate.

(2) there exists a finite projective R-module M and an isomorphism

$$A \cong \widehat{\mathrm{Sym}^{\bullet}_R(M)},$$

where the completion is (M)-adic.

Proof. (Sketch) That (2) implies (1) is clear. Assume the converse. It suffices to show that $\mathrm{Spf}(A) \to \mathrm{Spec}(R)$ has a section

$$s\colon \mathrm{Spec}(R) \to \mathrm{Spf}(A)$$

as we then can set $M := R \otimes_A J$ where $J \subseteq A$ is the ideal of elements $a \in A$ such that the pull back $s^*(a)$ vanishes. The module M will be finite projective and the canonical morphism $\widehat{\mathrm{Sym}^{\bullet}_R(M)} \to A$ will be an isomorphism. It is clear that there exists a unique section $s_0\colon \mathrm{Spec}(R_{\mathrm{red}}) \to \mathrm{Spf}(A)$ over the reduction $R_{\mathrm{red}} := R/\mathrm{Nil}(R)$ as the sections $R_i[[t_1, \ldots, t_n]] \to R_{i,\mathrm{red}}$ must all send t_i to 0 and thus glue to a global one. By formal smoothness of $\mathrm{Spf}(A) \to \mathrm{Spec}(R)$ the section s_0 extends then to some section $s\colon \mathrm{Spec}(R) \to \mathrm{Spf}(A)$ as the pair $(R, \mathrm{Nil}(R))$ is henselian. □

Remark 4. The R-module M in Lemma 3 is not determined by $\mathrm{Spf}(A)$ but as the proof shows is determined by the choice of a section $s\colon \mathrm{Spec}(R) \to \mathrm{Spf}(A)$.

We introduce a name for formal schemes isomorphic to the one appearing in Lemma 3.

Definition 5. Let R be a (discrete) commutative ring and let $\mathrm{Spf}(A) \to \mathrm{Spec}(R)$ be a formal scheme over R. We call $\mathrm{Spf}(A)$ a formal hyperplane over R if it satisfies the conditions in Lemma 3. We denote by

$$\mathrm{Hyp}(R)$$

the category of formal hyperplanes over R.

If $A \cong \widehat{\mathrm{Sym}^{\bullet}_R(M)}$, then we call the function $\mathrm{rk}_R(M)$ on $\mathrm{Spec}(R)$ the dimension of $\mathrm{Spf}(A)$ (which might be non-constant) .

Remark 6. It is technically convenient (and equivalent) to describe formal hyperplanes over R via the cospectrum of a coalgebra (cf. [2, Section 1.1]). Namely, define a coalgebra C over R to be *smooth* if it is isomorphic to the (topological) R-module dual

$$\mathrm{Hom}_{R,\mathrm{cts}}(\widehat{\mathrm{Sym}^{\bullet}_R(M)}, R)$$

of the completed symmetric algebra of some finite projective R-module M.[3] For a morphism $R \to R'$ of (discrete) commutative rings, the base change $C \otimes_R R'$ is a smooth coalgebra over R' and one can define the cospectrum

$$\mathrm{cSpec}(C)\colon (R-\mathrm{Alg}) \to (\mathrm{Sets}),\ R' \mapsto \mathrm{Hom}_{R'-\mathrm{coalg}}(R', R' \otimes_R C)$$

of C. By dualizing one obtains a (covariant) equivalence

$$\{\text{ smooth coalgebras over } R\ \} \cong \mathrm{Hyp}(R),\ C \mapsto \mathrm{cSpec}(C).$$

[3]These coalgebras can also be described via divided power algebras, cf. [2, Definition 1.1.14.].

A technical advantage is thatthere is no need to consider the topology of adic R-algebras A anymore when passing to coalgebras. Moreover, smooth coalgebras are flat over R while this need not be true for completed symmetric algebras (if R is not noetherian).

We can now give the correct definition of a formal group over $\mathrm{Spec}(R)$.

Definition 7. Let R be a (discrete) commutative ring. A *formal group* G over R is a functor $G\colon (R-\mathrm{Alg}) \to (\mathrm{Ab})$ such that the underlying functor to sets is a formal hyperplane.

Remark 8.
- Equivalently, a formal group can be defined to be an abelian group object in the category $\mathrm{Hyp}(R)$ of formal hyperplanes over R.[4]
- If G is a formal group over R, then automatically
$$G \cong \mathrm{Spf}(\widehat{\mathrm{Sym}^\bullet_R(\mathfrak{g}^\vee)})$$
where $\mathfrak{g} := \mathrm{Lie}(G)$ is the Lie algebra of G. This follows from Remark 4 and the definition of the Lie algebra as the tangent space along the zero section of G.
- Similarly to Remark 2 one-dimensional formal groups G over R with the choice of an isomorphism $R[[t]] \cong \mathcal{O}_G(G)$ (which might not exist for a general G) are equivalent to formal group laws over R.

4. Formal groups over $\mathbb{E}_\infty$-rings

We want to extent Definition 7 to a definition of a formal group over an arbitrary $\mathbb{E}_\infty$-ring (cf. [2, Section 1.6]). For the definition of an $\mathbb{E}_\infty$-ring we refer to talk 4.

In order to obtain this we are facing two problems:

- What should replace the category $\mathrm{Hyp}(R)$ of formal hyperplanes over R?
- What should replace the notion of an abelian group in an ∞-categorical setting?

The features of $\mathbb{E}_\infty$-rings we use are mostly formal. Most importantly, we need that over an $\mathbb{E}_\infty$-ring there is a well-behaved symmetric monoidal ∞-category Mod_R of R-modules (aka R-module spectra) via the tensor (or smash) product $-\otimes_R -$. In particular, we can define an ∞-category cCAlg_R of commutative[5] coalgebras over R.[6]

We recall that an R-module (or coalgebra) M is *flat* if $\pi_0(M)$ is a flat $\pi_0(R)$-module and for all $n \in \mathbb{Z}$ the natural morphism $\pi_n(R) \otimes_{\pi_0(R)} \pi_0(M) \to \pi_n(M)$ is an isomorphism (cf. [3, Definition 7.2.2.10.]).

[4]It suffices to note that the product of two formal hyperplanes over R is a again a formal hyperplane.

[5]Strictly speaking, these should be called "cocommutative", but we stick with this lighter terminology.

[6]More formally, $\mathrm{cCAlg}_R := \mathrm{CAlg}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ where $\mathcal{C} := \mathrm{Mod}_R$ and $\mathrm{CAlg}(\mathcal{D})$ denotes the ∞-category of commutative algebra objects in a symmetric monoidal ∞-category $\mathcal{D}$.

We want to present a good definition of a formal hyperplane over an $\mathbb{E}_\infty$-ring R. The most naive one would be to imitate the description of these as the formal spectra

$$\mathrm{Spf}(\widehat{\mathrm{Sym}_R^\bullet(M)})$$

of completed symmetric algebras of a finite projective R-module M. However, over a general $\mathbb{E}_\infty$-ring R the free symmetric algebra

$$\mathrm{Sym}_R^\bullet(M) = \bigoplus_{n=0}^{\infty}(M^{\otimes_R n})_{h\Sigma_n}$$

on a finite projective R-module M need not be flat over R because of the occurence of group homology of the symmetric groups Σ_n.

The correct definition will proceed via Remark 6.

Definition 9. Let R be an $\mathbb{E}_\infty$-ring and let $C \in \mathrm{cCAlg}_R$ be a commutative coalgebra over R. Then C is called *smooth* if it is flat over R and $\pi_0(C)$ is a smooth coalgebra over $\pi_0(R)$ in the sense of Remark 6.

Remark 10.
- The $\pi_0(R)$-module $\pi_0(C)$ is naturally a coalgebra because the functor $M \mapsto \pi_0(M)$ commutes with the tensor product for *flat* R-modules (cf. this follows from (Equation (2))).
- The flatness condition in Definition 9 ensures that if R is discrete the categories of smooth coalgebras in the sense of Definition 9 and Remark 6 are equivalent. This will later ensure that the definition of a formal group over an $\mathbb{E}_\infty$-ring R recovers the classical one if R is discrete.

Flat modules over an $\mathbb{E}_\infty$-ring R are equivalent to flat modules over its connective cover $\tau_{\geq 0}R$ (cf. [3, Proposition 7.2.2.16.]). The proof relies on the spectral sequence

$$E_2^{pq} = \mathrm{Tor}_p^{\pi_*(R)}(\pi_*(M), \pi_*(N))_q \Rightarrow \pi_*(M \otimes_R N) \tag{1}$$

from [3, Proposition 7.2.1.19.], which computes the homotopy of a tensor product via graded Tor-groups. If N is flat over R, then the spectral sequence (Equation (1)) degenerates and yields isomorphisms

$$\pi_n(M) \otimes_{\pi_0(R)} \pi_0(N) = \mathrm{Tor}_0^{\pi_0(R)}(\pi_n(M), \pi_0(N)) \cong \pi_n(M \otimes_R N) \tag{2}$$

for $n \in \mathbb{Z}$.

This leads to the desired equivalence.

Proposition 11. *Let R be an $\mathbb{E}_\infty$-ring and $\tau_{\geq 0}R$ its connective cover. Then the base change functor*

$$\mathrm{Mod}_{\tau_{\geq 0}R}^\flat \to \mathrm{Mod}_R^\flat,\ M \to R \otimes_{\tau_{\geq 0}R} M$$

is an equivalence between flat R- and $\tau_{\geq 0}R$-modules with inverse functor $N \mapsto \tau_{\geq 0}N$.

Proof. Cf. [3, Proposition 7.2.2.16.(3)]. ∎

Thus it is reasonable to define formal hyperplanes, formal groups etc. on an arbitrary $\mathbb{E}_\infty$-ring R via formal hyperplanes, formal groups on its connective cover $\tau_{\geq 0}R$.

¿From now we will concentrate on $\mathbb{E}_\infty$-rings which are connective.

Definition 12. Let R be a connective $\mathbb{E}_\infty$-ring and let $C \in \mathrm{cCAlg}_R^{\mathrm{sm}}$ be a smooth coalgebra. We define the *cospectrum* $\mathrm{cSpec}(C)$ of C to be the functor

$$\mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S},\ R' \mapsto \mathrm{Map}_{\mathrm{cCAlg}_{R'}}(R', R' \otimes_R C)$$

from connective $\mathbb{E}_\infty - R$-algebras to spaces.

Remark 13.
- The base change $R' \otimes_R C$ is again a smooth coalgebra over R' because by flatness of C there is a canonical isomorphism $\pi_0(R' \otimes_R C) \cong \pi_0(R') \otimes_{\pi_0(R)} \pi_0(C)$ (cf. (Equation (2))).
- If R' is discrete, then the mapping space $\mathrm{Map}_{\mathrm{cCAlg}_{R'}}(R', R' \otimes_R C)$ is discrete and thus, if R is discrete, the restriction of $\mathrm{cSpec}(C)$ to discrete rings recovers the definition of the cospectrum from Remark 6.
- The functor $C \mapsto \mathrm{cSpec}(C)$ from smooth coalgebras over R to the ∞-category $\mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathcal{S})$ of functors from connective R-algebras[7] to spaces is fully faithful (cf. [2, Proposition 1.5.9.]).

Thanks to remark Remark 13 we can now define a category of formal hyperplanes.

Definition 14. Let R be a connective $\mathbb{E}_\infty$-ring. Then we define the category $\mathrm{Hyp}(R)$ of formal hyperplanes over R to be the essential image of the (fully faithul) cospectrum functor

$$\mathrm{cCAlg}_R^{\mathrm{sm}} \to \mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathcal{S})$$

from smooth coalgebras over R to $\mathcal{S}$-valued functors on connective R-algebras (cf. Definition 12).

Remark 15.
- From Remark 13 we can conclude that Definition 14 agrees (up to equivalence) with Definition 5 in the case where R is discrete.
- Let A be a flat R-algebra such that $\pi_0(A)$ is smooth over $\pi_0(R)$, i.e., A is fiber smooth over R in the sense of [4, Section 11.2.3.] (cf. [4, Remark 11.2.3.5.]). Then for any morphism $s\colon A \to R$ of R-algebras the formal completion of A along s (cf. [4, Definition 8.1.6.1.]) is a formal hyperplane over R (cf. [2, Proposition 1.5.15.]).
- It is possible to give a definition of formal hyperplanes in terms of a formal spectrum of a "formally smooth" adic $\mathbb{E}_\infty$-ring (cf. [2, Section 1.4.] and [2, Section 1.5.]). First, define an adic $\mathbb{E}_\infty - R$-algebra to be an $\mathbb{E}_\infty - R$-algebra A together with the I-adic topology on $\pi_0(A)$ for some finitely generated ideal $I \subseteq \pi_0(A)$.[8]. If A is a connective adic $\mathbb{E}_\infty - R$-algebra,

[7]Recall that we assumed R to be connective.

[8]Following [2, Definition 0.0.11.] we don't require $\pi_0(A)$ to be I-adically complete. However, duals of smooth coalgebras will automatically complete.

then its formal spectrum $\mathrm{Spf}(A)$ (cf. [2, Section 1.5.]) is defined to be the functor

$$\mathrm{Spf}(A)\colon \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S},\ B \mapsto \mathrm{Map}_R^{\mathrm{cont}}(A, B)$$

on connective R-algebras. Here a morphism $A \to B$ is called continuous if $\pi_0(A) \to \pi_0(B)$ is continuous (for the discrete topology on $\pi_0(B)$). For a smooth coalgebra C over R the R-linear dual $C^\vee := \mathrm{Hom}_R(C, R)$ is naturally an adic $\mathbb{E}_\infty - R$-algebra (cf. [2, Proposition 1.3.10.]) and the formal spectrum $\mathrm{Spf}(C^\vee)$ of $C^\vee$ is isomorphic to the cospectrum $\mathrm{cSpec}(C)$ of C (cf. [2, Proposition 1.5.8.]). Moreover, an adic $\mathbb{E}_\infty - R$-algebra A is isomorphic to the dual $C^\vee$ of some smooth coalgebra C over R if and only if there exists an isomorphism of graded $\pi_*(R)$-algebras

$$\pi_*(A) \cong \prod_{n \geq 0} (\mathrm{Sym}^n_{\pi_0(R)}(M) \otimes_{\pi_0(R)} \pi_*(R))$$

for some finite projective $\pi_0(R)$-module M (cf. [2, Proposition 1.4.11.]). If M is finite free, i.e., $\pi_*(A) \cong \pi_*(R)[[t_1, \ldots, t_n]]$ for some $n \geq 0$, such an adic $\mathbb{E}_\infty - R$-algebra is isomorphic *as an* $\mathbb{E}_1 - R$*-algebra* to the adic $\mathbb{E}_\infty - R$-algebra $R[[t_1, \ldots, t_n]]$ (cf. [2, Proposition 1.4.5.]).[9]

After having defined a reasonable definition of formal hyperplanes over an $\mathbb{E}_\infty$-ring R we pass to the second question of how to define an abelian group object in an ∞-categorical context.

Unfortunately, there is not a unique answer, but there are (at least) two reasonable. For spaces the distinction would be between topological abelian groups (or simplicial abelian groups) and (grouplike) $\mathbb{E}_\infty$-spaces, i.e., whether commutativity and associativity are required to hold "on the nose" or "up to coherent homotopy".

Formal groups will be defined as abelian group objects (and not via $\mathbb{E}_\infty$-spaces) and thus the following definition is the one relevant to us.

Definition 16. (cf. [1, Definition 1.2.4.]) Let $\mathcal{C}$ be an ∞-category which admits finite products. An abelian group object of $\mathcal{C}$ is a functor $A\colon \mathrm{Lat}^{\mathrm{op}} \to \mathcal{C}$ from (the nerve) of the category of finite, free abelian groups Lat to $\mathcal{C}$ which commutes with finite products.

We denote by $\mathrm{Ab}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{Lat}^{\mathrm{op}}, \mathcal{C})$ the full subcategory of abelian group objects in the ∞-category $\mathcal{C}$.

Remark 17.
- If $\mathcal{C}$ is (the nerve) of the category (Sets) of sets, then the abelian group objects in the sense of Definition 16 are equivalent to classical abelian groups by evaluating a functor $A\colon \mathrm{Lat}^{\mathrm{op}} \to$ (Sets) on the cogroup object $\mathbb{Z}$ (cf. [1, Proposition 1.2.7.]).

[9] The reason that one gets only an isomorphism of $\mathbb{E}_1 - R$-algebras is that the polynomial algebra $R[t]$ is a free $\mathbb{E}_1 - R$-algebra on one generator, but not a free $\mathbb{E}_\infty - R$-algebra unless R is a $\mathbb{Q}$-algebra (cf. [2, Warning 1.4.2.] and [2, Remark 1.4.3.]).

- Similarly, topological abelian groups (or equivalently simplicial abelian groups) are equivalent to abelian group objects in spaces. More precisely, the ∞-category $\mathrm{Ab}(\mathcal{S})$ of abelian group objects in spaces is equivalent to the ∞-category $\mathrm{Ab}^{\Delta^{\mathrm{op}}}$ of simplicial abelian groups (cf. [1, Example 1.2.9.]). Using the Dold-Kan correspondece the ∞-category $\mathrm{Ab}^{\Delta^{\mathrm{op}}}$ of simplicial abelian groups is equivalent to the ∞-category $\mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}}$ of connective $\mathbb{Z}$-module spectra (cf. [1, Remark 1.2.10.]).

As a concrete example, by the Dold-Thom theorem the free simplicial abelian group on the 2-sphere S^2 has homotopy given by the reduced homology $\tilde{H}^*(S^2)$ and is thus a $K(\mathbb{Z}, 2)$, i.e., equivalent to $\mathbb{C}P^\infty$. In this case the corresponding abelian group object in spaces is given by the functor $\mathrm{Lat}^{\mathrm{op}} \to \mathcal{S}$, $\Lambda \mapsto K(\Lambda^\vee, 2)$.

Finally we are able to give the definition of a formal group over an $\mathbb{E}_\infty$-ring.

Definition 18. Let R be a connective $\mathbb{E}_\infty$-ring. A formal group over R is a functor

$$G\colon \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}}$$

such that the underlying functor to spaces is a formal hyperplane in the sense of Definition 14.

Remark 19.

- If R is an arbitrary $\mathbb{E}_\infty$-ring, then we define the ∞-category $\mathrm{FGroup}(R)$ of formal groups over R as the category $\mathrm{FGroup}(\tau_{\geq 0} R)$ of formal groups[10] over its connective cover $\tau_{\geq 0} R$.
- Examples (and non-examples) of formal groups will be given in talk 8.

References

[1] Jacob Lurie. Elliptic cohomology I: Spectral abelian varieties. Available at `http://www.math.harvard.edu/~lurie/`.

[2] Jacob Lurie. Elliptic cohomology II: Orientations. Available at `http://www.math.harvard.edu/~lurie/`.

[3] Jacob Lurie. Higher algebra. Available at `http://www.math.harvard.edu/~lurie/`.

[4] Jacob Lurie. Spectral algebraic geometry. Available at `http://www.math.harvard.edu/~lurie/`.

[10] which is defined as the full subcategory $\mathrm{FGroup}(\tau_{\geq 0} R) \subseteq \mathrm{Fun}(\mathrm{CAlg}_{\tau_{\geq 0} R}^{\mathrm{cn}}, \mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}})$ spanned by formal groups

Examples of formal groups

Achim Krause

In this talk, we gave examples for the previously introduced notion of formal groups over a connective $\mathbb{E}_\infty$-ring R. These are given by functors

$$\mathrm{CAlg}_R \to \mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}},$$

whose underlying space-valued functor, given by the composite

$$\mathrm{CAlg}_R \longrightarrow \mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}} \xrightarrow{\Omega^\infty} \mathscr{S}$$

is a formal hyperplane, i.e. of the form $\mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_R}(A, -)$ for an adic R-algebra A which is dual to a smooth coalgebra. These adic algebras A are also characterized by their homotopy: A is the dual of a smooth coalgebra if there is a finitely generated projective $\pi_0(R)$-module M and a continous isomorphism

$$\pi_*(A) \cong \prod_{n\geq 0} (\mathrm{Sym}^n_{\pi_0(R)}(M) \otimes_{\pi_0(R)} \pi_*(R)).$$

For M free of rank k, this is precisely the statement that $\pi_*(A)$ has the form $\pi_*(R)[[x_1, \ldots, x_k]]$ with $x_1, \ldots, x_k$ in degree 0.

Contrary to the classical situation, where formal hyperplanes over an ordinary ring R are, Zariski-locally, classified by their dimension, and the interesting part of a formal group is definitely the group structure, over an $\mathbb{E}_\infty$-ring R there are typically many nonequivalent ways to realize the same power series algebra over $\pi_*(R)$ as homotopy groups of an adic R-algebra.

Example 1: The multiplicative formal group

In ordinary rings, the multiplicative group $\mathbb{G}_m$ is given by the functor of points $\mathrm{CAlg}^\heartsuit \to \mathrm{Ab}$, $R \mapsto (R^\times, \cdot)$, i.e. the functor that sends a ring to its unit group. This is corepresented by the ring $\mathbb{Z}[t^{\pm 1}]$, and the group structure is induced by the Hopf algebra structure $\Delta t = t \otimes t$. Observe that we can also write this as a group ring $\mathbb{Z}[\mathbb{Z}]$, where the Hopf algebra structure is just induced by the diagonal $\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$.

The formal completion of $\mathbb{G}_m$, i.e. the formal multiplicative group, is given by

$$\widehat{\mathbb{G}}_m(R) = \mathrm{fib}(\mathbb{G}_m(R) \to \mathbb{G}_m(R^{\mathrm{red}})).$$

Explicitly, it is given by sending $R \mapsto (1+N, \cdot)$, with $N \subseteq R$ the nilradical of R. This functor is corepresented by the completion $\mathbb{Z}[t^{\pm 1}]^\wedge_{(t-1)}$, in the sense that

$$\mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}^\heartsuit}(\mathbb{Z}[t^{\pm 1}]^\wedge_{(t-1)}, R) = 1 + N.$$

The adic algebra $\mathbb{Z}[t^{\pm 1}]^\wedge_{(t-1)}$ is of course isomorphic to $\mathbb{Z}[[u]]$, with $u = t - 1$. In terms of this generator, we see that the Hopf algebra structure $\Delta t = t \otimes t$ turns into $\Delta u = u \otimes 1 + 1 \otimes u + u \otimes u$, the usual multiplicative formal group law.

This story carries over completely to the world of $\mathbb{E}_\infty$-algebras, but one has to be careful about the notion of units in a ring. For R an $\mathbb{E}_\infty$-ring spectrum, $\Omega^\infty R$ inherits a second, "multiplicative" $\mathbb{E}_\infty$-algebra structure due to the fact that Ω^∞ is lax symmetric monoidal. This $\mathbb{E}_\infty$-space is not grouplike, since π_0 of it is the monoid $\pi_0(R)$ with multiplication. But we can define a grouplike sub-$\mathbb{E}_\infty$-space $\mathrm{GL}_1(R)$ by passing to the full subspace on the units $\pi_0(R)^\times$ (i.e. the union of the corresponding connected components). $\mathrm{GL}_1(R)$ is the space of units of R, and its $\mathbb{E}_\infty$-structure provides us with a lift to a connective spectrum $\mathrm{gl}_1(R)$.

However, $\mathrm{gl}_1(R)$ is very rarely a $\mathbb{Z}$-module spectrum. We thus neeed a stricter notion of units. To that end, we let ourselves be guided by the previously discussed classical multiplicative group, which turned out to be corepresented by the group ring of $\mathbb{Z}$. We define

$$\mathbb{G}_m(R) := \mathrm{Map}_{\mathrm{CAlg}}(\mathbb{S}[\mathbb{Z}], R).$$

This functor $\mathrm{CAlg} \to \mathscr{S}$ actually admits a natural lift $\mathrm{CAlg} \to \mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}}$, since we can extend $\mathrm{Map}_{\mathrm{CAlg}}(\mathbb{S}[\mathbb{Z}], R)$ to a functor $\mathrm{Lat}^{\mathrm{op}} \to \mathscr{S}$, sending a finitely generated free abelian group M to $\mathrm{Map}_{\mathrm{CAlg}}(\mathbb{S}[M], R)$.

Note that, by a chain of adjunctions,

$$\begin{aligned}
\mathbb{G}_m(R) &= \mathrm{Map}_{\mathrm{CAlg}}(\mathbb{S}[\mathbb{Z}], R)\\
&\simeq \mathrm{Map}_{\mathbb{E}_\infty}(\mathbb{Z}, \Omega^\infty R)\\
&\simeq \mathrm{Map}_{\mathbb{E}_\infty}(\mathbb{Z}, \mathrm{GL}_1(R))\\
&\simeq \mathrm{Map}_{\mathrm{Sp}}(\mathbb{Z}, \mathrm{gl}_1(R)).
\end{aligned}$$

The third line of this chain of equivalences demonstrates that $\mathbb{G}_m(R)$ should be thought of as some kind of "space of units of R that commute coherently with themselves". Also, the last line shows that in general, $\mathbb{G}_m(R)$ is much more mysterious than $\mathrm{gl}_1(R)$, since maps out of $H\mathbb{Z}$ are difficult to compute.

The formal multiplicative group $\widehat{\mathbb{G}}_m(R)$ of a connective ring R is now defined as the fiber, taken in connective $\mathbb{Z}$-modules,

$$\widehat{\mathbb{G}}_m(R) = \mathrm{fib}(\mathbb{G}_m(R) \to \mathbb{G}_m(R^{\mathrm{red}})),$$

where R^{red} denotes the ordinary ring $\pi_0(R)^{\mathrm{red}}$, consistent with the slogan that a connective $\mathbb{E}_\infty$ ring R should be thought of as a fancy nil-thickening of its underlying ordinary ring $\pi_0(R)$. One can verify that

$$\widehat{\mathbb{G}}_m(R) = \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{cont}}}(\mathbb{S}[\mathbb{Z}]^\wedge_{t-1}, R),$$

where t is the generator of $\mathbb{Z}$.

The homotopy groups of $\mathbb{S}[\mathbb{Z}]^\wedge_{t-1}$ are of the form $\pi_*(\mathbb{S})[[u]]$, with $u = t-1$, so $\widehat{\mathbb{G}}_m$ is a formal hyperplane. Note however that $\mathbb{S}[\mathbb{Z}]^\wedge_{t-1}$ is not equivalent as an $\mathbb{E}_\infty$-ring to the most immediate candidate with those homotopy groups, namely $\mathbb{S}[u]^\wedge_u$ (where $\mathbb{S}[u] := \mathbb{S}[\mathbb{N}]$), see Remark 4.

(Non-)example 2: The additive formal group

The additive group in ordinary rings is given by the functor $\mathbb{G}_a : \mathrm{CAlg}^{\heartsuit} \to \mathrm{Ab}$ sending a ring R to its additive group $(R, +)$. This is corepresented by the free ring $\mathbb{Z}[t]$, with group structure corepresented by the Hopf algebra structure $\Delta t = t \otimes 1 + 1 \otimes t$. Its formal completion

$$\widehat{\mathbb{G}}_a(R) := \mathrm{fib}(\mathbb{G}_a(R) \to \mathbb{G}_a(R^{\mathrm{red}}))$$

sends a ring R to the abelian group $(N, +)$, where N is the nilradical of R. One therefore sees that $\widehat{\mathbb{G}}_a$ is corepresented by the adic algebra $\mathbb{Z}[[t]]$, with group structure corepresented by the Hopf algebra structure given by $\Delta t = t \otimes 1 + 1 \otimes t$.

The adic algebra $\mathbb{Z}[[t]]$ admits a reasonable lift to $\mathbb{S}$, given by $\mathbb{S}[t]^{\wedge}_t$, where $\mathbb{S}[t]$ denotes $\mathbb{S}[\mathbb{N}]$. However, it is not obvious how to put a $\mathbb{Z}$-module structure on the functor $\mathrm{CAlg} \to \mathscr{S}$ given by $\mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}}(\mathbb{S}[t]^{\wedge}_t, -)$. One could try to first tackle the less structured problem of finding a lift of the comultiplication on $\mathbb{Z}[[t]]$, i.e. an $\mathbb{E}_\infty$ ring map $\mathbb{S}[t]^{\wedge}_t \to \mathbb{S}[t_1, t_2]^{\wedge}_{(t_1,t_2)}$ which on homotopy sends $t \mapsto t_1 + t_2$. The problem one runs into is that $\mathbb{S}[t]$ is not free as an $\mathbb{E}_\infty$-ring, for similar reasons as the ones that came up during the discussion of the different notions of units for $\mathbb{E}_\infty$-rings in the previous section. So we cannot define a map $\mathbb{S}[t] \to \mathbb{S}[t_1, t_2]$ by just defining it on t. And in fact, there is no map with the desired properties, as the proof of Proposition 3 shows.

The nonexistence will follow from properties of a certain kind of power operation. Observe first that for any spectrum X, we can form the extended power $(X \otimes X)_{hC_2}$. Given an element $x \in \pi_0(X)$, we obtain a map $\mathbb{S} \to X$, which the extended power construction turns into a map $\mathbb{S}_{hC_2} \to (X \otimes X)_{hC_2}$. As $\mathbb{S}_{hC_2}$ is the suspension spectrum $\Sigma^\infty_+(BC_2)$, we have a distinguished element in $\pi_1(\mathbb{S}_{hC_2})$ coming from the generator of $\pi_1(BC_2) = \mathbb{Z}/2$.

Definition 1. For an $\mathbb{E}_\infty$ ring R, we define an operation $\eta_m : \pi_0(R) \to \pi_1(R)$ by sending $x : \mathbb{S} \to R$ to the composite

$$\mathbb{S}^1 \to \mathbb{S}_{hC_2} \to (R \otimes R)_{hC_2} \to R.$$

This has an interpretation in terms of unstable homotopy: The multiplicative $\mathbb{E}_\infty$ structure of $\Omega^\infty R$ gives us a natural homotopy

$$\begin{tikzcd} \Omega^\infty R \times \Omega^\infty R \arrow[r, "\mu"] \arrow[d, "\mathrm{flip}"'] & \Omega^\infty R \arrow[d, "\mathrm{id}"] \arrow[dl, phantom, "\Swarrow"] \\ \Omega^\infty R \times \Omega^\infty R \arrow[r, "\mu"] & \Omega^\infty R. \end{tikzcd}$$

Evaluated at any point $x \in \Omega^\infty R$, this produces a loop γ_x at $\mu(x, x) \in \Omega^\infty R$. Under the isomorphisms $\pi_0(\Omega^\infty R) \cong \pi_0 R$ and $\pi_1(\Omega^\infty R, \mu(x, x)) \cong \pi_1 R$, this construction corresponds to η_m.

Lemma 2. *The operation η_m satisfies*

$$\eta_m(x + y) = \eta_m(x) + \eta_m(y) + \eta xy.$$

Proof. Given two elements $x, y \in \pi_0(R)$, we get a map $\mathbb{S} \oplus \mathbb{S} \to R$ representing them. We can thus form the following commutative diagram

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r] \arrow[d, "\mathrm{id}"] & \mathbb{S}_{hC_2} \arrow[r] \arrow[d, "\mathrm{id}"] & ((\mathbb{S} \oplus \mathbb{S}) \otimes (\mathbb{S} \oplus \mathbb{S}))_{hC_2} \arrow[d] & \\ \mathbb{S}^1 \arrow[r] & \mathbb{S}_{hC_2} \arrow[r] & (R \otimes R)_{hC_2} \arrow[r] & R \end{tikzcd}$$

Now we have a splitting as follows:

$$((\mathbb{S}_x \oplus \mathbb{S}_y) \otimes (\mathbb{S}_x \oplus \mathbb{S}_y))_{hC_2} = (\mathbb{S}_{xx} \oplus \mathbb{S}_{xy} \oplus \mathbb{S}_{yx} \oplus \mathbb{S}_{yy})_{hC_2} = (\mathbb{S}_{xx})_{hC_2} \oplus (\mathbb{S}_{yy})_{hC_2} \oplus \mathbb{S}_{xy},$$

where the subscripts are just to distinguish the roles of the various summands. The map $\eta_m(x+y) : \mathbb{S}^1 \to R$ can thus be written as sum of three maps

$$\mathbb{S}^1 \to (\mathbb{S}_{xx})_{hC_2} \to R,$$
$$\mathbb{S}^1 \to (\mathbb{S}_{yy})_{hC_2} \to R,$$
$$\mathbb{S}^1 \to \mathbb{S}_{xy} \to R.$$

The first two are, by naturality considerations, given by $\eta_m(x)$ and $\eta_m(y)$. The last one is of the form $c \cdot xy$, with $c \in \pi_1(\mathbb{S}_{xy})$ some fixed element, which we need to identify with η. To do so, we observe that $\pi_1(\mathbb{S}) = \mathbb{Z}/2 = \{0, \eta\}$, and so we either globally have

$$\eta_m(x+y) = \eta_m(x) + \eta_m(y) + \eta xy,$$

or

$$\eta_m(x+y) = \eta_m(x) + \eta_m(y).$$

To preclude the second option, it is sufficient to find some $\mathbb{E}_\infty$-ring R where η_m is not linear. We turn to the sphere spectrum, and analyze for some $x \in \pi_0\mathbb{S}$, the path in $\pi_1(\Omega^\infty\mathbb{S}, x \cdot x)$ given by the natural homotopy

$$\begin{tikzcd} \Omega^\infty\mathbb{S} \times \Omega^\infty\mathbb{S} \arrow[r, "\mu"] \arrow[d, "\mathrm{flip}"] \arrow[dr, phantom, "\swarrow"] & \Omega^\infty\mathbb{S} \arrow[d, "\mathrm{id}"] \\ \Omega^\infty\mathbb{S} \times \Omega^\infty\mathbb{S} \arrow[r, "\mu"] & \Omega^\infty\mathbb{S}. \end{tikzcd}$$

For positive x, we can compare this to the natural homotopy given by the symmetric-monoidal structure on the groupoid $\mathrm{Fin}^\sim$ of finite sets:

$$\begin{tikzcd} N(\mathrm{Fin}^\sim) \times N(\mathrm{Fin}^\sim) \arrow[r, "\otimes"] \arrow[d, "\mathrm{flip}"] \arrow[dr, phantom, "\swarrow"] & N(\mathrm{Fin}^\sim) \arrow[d] \\ N(\mathrm{Fin}^\sim) \times N(\mathrm{Fin}^\sim) \arrow[r, "\otimes"] & N(\mathrm{Fin}^\sim) \end{tikzcd}$$

For a finite set x, this produces the automorphism $x \times x \to x \times x$ given by interchanging the two coordinates. As the map $N(\mathrm{Fin}^\sim) \to \Omega^\infty\mathbb{S}$ is on $\pi_1(N(\mathrm{Fin}^\sim), y)$

$= \mathrm{Aut}(y)$ given by the sign homomorphism $\mathrm{Aut}(y) \to \mathbb{Z}/2$, and the flip automorphism of $x \times x$ has sign $(-1)^{\binom{|X|}{2}}$, we see that, for $x \in \pi_0(\mathbb{S})$ with $x \geq 0$:

$$\eta_m(x) = \begin{cases} 0 & \text{if} \quad x = 0, 1 \bmod 4 \\ \eta & \text{if} \quad x = 2, 3 \bmod 4 \end{cases}$$

In particular, η_m is not additive. So our previous discussion proves the lemma. □

Proposition 3. *The formal additive group $\mathbb{G}_a$ over $\mathbb{Z}$ does not admit a lift along the map $\mathbb{S} \to \tau_{\leq 0}\mathbb{S} \cong \mathbb{Z}$.*

Proof. Suppose $\widehat{\mathbb{G}}$ were such a lift. Then the formal hyperplanes $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{G}} \times \widehat{\mathbb{G}}$ are corepresented by adic algebras A and B with $\pi_0(A) = \mathbb{Z}[[t]]$, $\pi_0(B) = \mathbb{Z}[[t_1, t_2]]$, and higher homotopy groups given by $\pi_*(A) = \pi_*(\mathbb{S}) \otimes_{\pi_0(\mathbb{S})} \otimes \pi_0(A)$ and $\pi_*(B) = \pi_*(\mathbb{S}) \otimes_{\pi_0(\mathbb{S})} \otimes \pi_0(B)$. In addition, the map $\mu : \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to \widehat{\mathbb{G}}$ is corepresented by a map $A \to B$ which, on π_0, sends $t \mapsto t_1 + t_2$. Since η_m is natural, we get the following commutative diagram on π_0 and π_1:

$$\begin{tikzcd} \pi_0(A) \arrow[r] \arrow[d, "\eta_m"] & \pi_0(B) \arrow[d, "\eta_m"] \\ \pi_1(A) \arrow[r] & \pi_1(B) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z}[[t]] \arrow[r, "t \mapsto t_1+t_2"] \arrow[d, "\eta_m"] & \mathbb{Z}[[t_1, t_2]] \arrow[d, "\eta_m"] \\ \mathbb{Z}/2[[t]] \arrow[r, "t \mapsto t_1+t_2"] & \mathbb{Z}/2[[t_1, t_2]] \end{tikzcd}$$

Now let $f \in \mathbb{Z}/2[[t]]$ be characterized by $\eta_m(t) = f(t)$. So the commutative diagram expresses that $\eta_m(t_1 + t_2) = f(t_1 + t_2)$. By naturality applied to the both maps $A \to B$ corepresenting the two projections $\widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to \widehat{\mathbb{G}}$, which on homotopy send $t \mapsto t_1$ and $t \mapsto t_2$ respectively, we see that also $\eta_m(t_1) = f(t_1)$ and $\eta_m(t_2) = f(t_2)$. Lemma 2 shows that

$$f(t_1 + t_2) = f(t_1) + f(t_2) + t_1 t_2.$$

This is absurd, since the $t_1 t_2$-coefficient of $f(t_1 + t_2) - f(t_1) - f(t_2)$ is always 0 mod 2. □

Note that this proof also shows that $\widehat{\mathbb{G}}_a$ doesn't even admit a lift to $\tau_{\leq 1}\mathbb{S}$.

Remark 4. The same techniques apply to show that the adic algebra $\mathbb{S}[t^{\pm 1}]^{\wedge}_{t-1}$ corepresenting the underlying formal hyperplane of $\widehat{\mathbb{G}}_m$ is not equivalent to $\mathbb{S}[[t]] = \mathbb{S}[t]^{\wedge}_t$. To see this, note that any equivalence would have to send $t-1$ to a generator $g(t)$ of the ideal of definition of $\mathbb{S}[[t]]$, i.e. a power series of the form $g(t) = a_1 t + a_2 t^2 + \dots$ with a_1 a unit, i.e. $a_1 = \pm 1$. Now one checks that on both sides, $\eta_m(t^k) = 0$, since the elements t^k come from the unstable $\pi_0\mathbb{Z}$. Using the additivity relation from Lemma 2, one sees

$$\eta_m(t-1) = -t\eta = -(t-1)\eta - \eta \quad \text{in } \pi_1(\mathbb{S}[t^{\pm 1}]^{\wedge}_{t-1}).$$

and

$$\eta_m(-t) = -\eta t^2 \quad \text{in } \pi_1(\mathbb{S}[[t]]).$$

Naturality of η_m now implies that $\eta_m(g(t)) = -g(t)\eta - \eta$ in $\pi_1(\mathbb{S}[[t]])$. However, Lemma 2 together with $\eta_m(t) = 0$ and $\eta_m(-t) = -\eta t^2$ implies that $\eta_m(g(t))$ has vanishing linear term, whereas the right hand side has a nontrivial linear term.

Example 3: The Quillen formal group

The most important example for a formal group over a ring spectrum is given by the Quillen formal group, which is defined over any complex-orientable weakly even-periodic ring spectrum R. This is well-known on homotopy, for R strongly even-periodic: The cohomology $R^0\mathbb{C}P^\infty$ is then noncanonically just a power series ring $\pi_0(R)[[t]]$, and the multiplication map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ gives rise to a formal group law in $\pi_0(R)[[t_1, t_2]]$. We want to quickly outline how this actually gives a formal group in this more highly structured sense discussed here.

Proposition 5. *For R weakly even-periodic and complex-orientable, the adic R-algebra given by the mapping spectrum $R^{\mathbb{C}P^\infty}$ corepresents a formal hyperplane. This formal hyperplane admits a natural formal group structure.*

Proof. $R^{\mathbb{C}P^\infty}$ is the R-dual of $R\otimes\mathbb{C}P^\infty$. This splits, due to complex-orientability, into

$$R \otimes \mathbb{C}P^\infty \simeq \bigoplus_{n\geq 0} \Sigma^{2n} R,$$

so

$$R^{\mathbb{C}P^\infty} \simeq \prod_{n\geq 0} \Sigma^{-2n} R.$$

R being weakly 2-periodic means that $\pi_2(R) \otimes_{\pi_0(R)} \pi_n(R) = \pi_{n+2}(R)$. In particular, $\pi_2(R)$ is an invertible projective module over $\pi_0(R)$, and $\pi_{2n}(R)$ is its n-th tensor power. Also, all the $\Sigma^{-2n}(R)$ are flat over R. It follows that

$$\pi_*(R^{\mathbb{C}P^\infty}) \simeq \prod_{n\geq 0} \mathrm{Sym}^n_{\pi_0(R)}(\pi_2(R)) \otimes_{\pi_0(R)} \pi_*(R),$$

which is exactly saying that $R^{\mathbb{C}P^\infty}$ corepresents a formal hyperplane.

The formal group structure comes from the fact that $\mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_R}\left(R^{\mathbb{C}P^\infty}, A\right)$ admits extra functoriality in lattices: For a finitely generated free abelian group M, consider the space

$$\mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_R}\left(R^{K(M^\vee,2)}, A\right).$$

This defines a functor $\mathrm{Lat}^{\mathrm{op}} \to \mathscr{S}$, whose underlying object (i.e. value at $M = \mathbb{Z}$) agrees with $\mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_R}\left(R^{\mathbb{C}P^\infty}, A\right)$, since $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$ $\square$

Remark 6. Both the nonexistence result for the additive formal group (Proposition 3) as well as the nonequivalence of the formal hyperplane underlying the multiplicative formal group and the "additive" formal hyperplane $\mathbb{S}[[t]]$ (Remark 4) rely crucially on the nontriviality of $\eta \in \pi_1(\mathbb{S})$. Over $\mathbb{Z}$ the two formal hyperplanes agree, and the additive formal group exists. One can ask whether there are more interesting spectra where this still happens. A convincing candidate is given by the periodic bordism spectrum MP. Concretely, we can ask:

(1) Does there exist a formal group over MP with underlying formal hyperplane $MP[[t]]$, for which the corresponding formal group over $\pi_0(MP)$ is the additive one?
(2) Do the three formal hyperplanes considered over MP, namely the ones corepresented by $MP[t^{\pm 1}]^{\wedge}_{t-1}$, $MP[[t]]$, and $MP^{\mathbb{C}P^\infty}$, agree?

References

[1] J. Lurie, *Elliptic cohomology II*, preprint, http://www.math.harvard.edu/~lurie/papers/Elliptic-II.pdf.

p-divisible groups over E_∞-rings

Viktoriya Ozornova

In this talk, we discussed the basics of p-divisible groups over E_∞-rings. It turns out to be convenient to switch to the functor of points perspective, introduced in Talk 7 on formal groups. Moreover, instead of the collection of group schemes, which could be thought of as p^n-torsion, we consider the colimit of all such. In total, a **p-divisible group** over a connective E_∞-ring R is defined to be a functor $\mathbf{G}\colon \mathrm{CAlg}_R^{cn} \to \mathrm{Mod}_{H\mathbb{Z}}^{cn}$ with the following properties:

- $\mathbf{G}(A)$ is p-nilpotent for all connective E_∞-rings A, i.e., $\mathbf{G}(A)\left[\frac{1}{p}\right] \simeq 0$,
- for every finite abelian p-group M, the space-valued functor $\mathrm{Map}_{\mathrm{Mod}_{H\mathbb{Z}}^{cn}}(M, \mathbf{G}(-))$ is corepresentable by a finite flat R-algebra,
- multiplication by p is surjective in finite flat topology, i.e., for every $x \in \pi_0\mathbf{G}(A)$ there is a finite flat algebra morphism $A \to B$ so that the image of x is divisible by p in $\pi_0\mathbf{G}(B)$ and the induced map $\mathrm{Spec}\,\pi_0 B \to \mathrm{Spec}\,\pi_0 A$ is a surjective map of sets.

We note that this definition is generalizing the definition of Talk 5, although it looks different at the first glance. The individual G_n considered before correspond to p^n-torsion of the 'colimit' $\mathbf{G}$, and the former can be recovered as $\mathrm{Map}_{\mathrm{Mod}_{H\mathbb{Z}}^{cn}}(\mathbb{Z}/p^n, \mathbf{G}(-))$. Starting from the classical definition, it is easy to see that the first two conditions of the new definition are satisfied. The last condition is, roughly speaking, encoding the notion of 'p-divisibility', and the local nature of this condition makes it slightly more subtle.

After giving the definition and recovering the discrete examples, we discussed two further examples. The first one is the *multiplicative* p-divisible group μ_{p^∞}, which is defined over the sphere spectrum and thus, by extension of scalars, over any E_∞-ring, and determined by the formula

$$A \mapsto \mathrm{fib}_{\mathrm{Mod}_{H\mathbb{Z}}^{cn}}(\mathbf{G}_m(A) \to \mathbf{G}_m(A)\left[\tfrac{1}{p}\right]),$$

where $\mathbf{G}_m$ denotes the strict multiplicative group. The second example was given by the constant p-divisible group $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$, determined by the formula

$$A \mapsto \{\text{locally constant maps } |\mathrm{Spec}\,\pi_0 A| \to \mathbb{Q}_p/\mathbb{Z}_p\}.$$

Next, we discussed a connection between formal groups and p-divisible groups, which is originally due to Messing in the classical case:

Theorem. *Let R be a (p)-adically complete connective E_∞-ring and let $\mathbf{G}$ be a p-divisible group over R. Then the restriction of the functor*

$$A \mapsto \mathrm{fib}_{\mathrm{Mod}_{H\mathbb{Z}}^{cn}}(\mathbf{G}(A) \to \mathbf{G}(A^{red}))$$

to the subcategory of CAlg_R^{cn} spanned by truncated, p-nilpotent R-algebras is also the restriction of an essentially unique formal group $\mathbf{G}^\circ$ over R, called the identity component *of $\mathbf{G}$.*

It is not hard to see that the identity component of the multiplicative p-divisible group is, as the name suggests, the multiplicative formal group. In the second example, we observe that the identity component of $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ is trivial.

It turns out that the relationship between the p-divisible group and its identity component is the closest under certain connectedness assumptions. One defines a p-divisible group over a connective E_∞-ring R to be *connected* if for the R-algebra A corepresenting $\mathrm{Map}_{\mathrm{Mod}_{H\mathbb{Z}}^{cn}}(\mathbb{Z}/p, \mathbf{G}(-))$ the induced map of underlying sets $\mathrm{Spec}\,\pi_0 A \to \mathrm{Spec}\,\pi_0 R$ is a bijection. However, this first notion of connectivity generalizing the classical one is too restrictive. The existence of a non-trivial connected p-divisible group over R (with a further mild assumption) would already imply that p is nilpotent in $\pi_0 R$, which is too limiting for our purposes. Instead, the more flexible notion of a *formally connected* p-divisible group over an adic E_∞-ring R with an ideal of definition I is defined as being connected after extension of scalars to $\pi_0 R/I$.

On the full subcategory (or more precisely, ∞-subcategory) of formally connected p-divisible groups in the category $\mathrm{BT}^p(R)$ of p-divisible groups the identity component functor is fully faithful, and the formal groups in the essential image of this subcategory are called *p-divisible formal groups.*

A notion of p-divisible groups being as far from connected as possible is that of an étale p-divisible group. A p-divisible group over R is called *étale* if the algebras corepresenting $\mathrm{Map}_{\mathrm{Mod}_{H\mathbb{Z}}^{cn}}(M, \mathbf{G}(-))$ for finite abelian p-groups M are in addition étale. It can be shown that a p-divisible group is étale if and only if its identity component is trivial, and in particular, $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ is an example of an étale p-divisible group.

Finally, we briefly discussed connected-étale exact sequences for p-divisible groups. We remarked that the notion of exact sequences for p-divisible groups is slightly subtle, cf. [2, §2.4]. In many situations, it is possible to 'decompose' a p-divisible group $\mathbf{G}$ into formally connected and étale pieces, more precisely, to find an exact sequence

$$0 \to \mathbf{G}' \to \mathbf{G} \to \mathbf{G}'' \to 0$$

with $\mathbf{G}'$ formally connected and $\mathbf{G}''$ étale. If such exact sequence exists, it is necessarily unique. A special case of situations where any p-divisible group $\mathbf{G}$ admits a connected-étale exact sequence is over a complete local noetherian E_∞-ring R whose residue field is of characteristic p.

References

[1] J. Lurie, *Elliptic Cohomology I: Spectral Abelian Varieties*, preprint available at `http://www.math.harvard.edu/~lurie/papers/Elliptic-I.pdf`, last accessed Apr 2019.

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The cotangent complex

Gijs Heuts

The aim of this talk was threefold:

(1) Introduce the (algebraic) cotangent complex $\mathbf{L}_A^{\mathrm{alg}}$ of a simplicial ring A, as well as the topological cotangent complex $\mathbf{L}_A$ which exists for any $\mathbf{E}_\infty$-ring A.
(2) Discuss some finiteness properties of $\mathbf{L}_A$ in the case where A is an F-finite Noetherian $\mathbb{F}_p$-algebra.
(3) Introduce the notion of cotangent complex for a functor $X\colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, where $\mathrm{CAlg}^{\mathrm{cn}}$ is the ∞-category of connective $\mathbf{E}_\infty$-algebras.

The general philosophy is mostly Quillen's [3]. Here we follow Sections 17 and 25 of [1] and Section 3.3 of [2].

1. The cotangent complex

If A is a commutative ring and M an A-module, the trivial square-zero extension of A by M is the abelian group $A \oplus M$ equipped with the ring structure $(a, m)(a', m') = (aa', am' + a'm)$. There is an evident projection $A \oplus M \to A$. Sections of this ring homomorphism correspond precisely to derivations $A \to M$. Derivations out of A are corepresented by the A-module Ω_A of Kähler differentials. Writing $\mathrm{CAlg}^{\heartsuit}$ for the category of (ordinary) commutative rings, we find an adjoint pair of functors

$$\mathrm{CAlg}^{\heartsuit}_{/A} \underset{A\oplus -}{\overset{}{\rightleftarrows}} \mathrm{Mod}_A,$$

where the left adjoint sends a morphism $B \to A$ of rings to the A-module $A \otimes_B \Omega_B$.

The cotangent complex arises from deriving this construction. Write CAlg^{Δ} for the ∞-category of simplicial commutative rings and, for a given simplicial ring A, write Mod_A^{Δ} for the ∞-category of simplicial A-modules. The construction of trivial square-zero extensions $A \oplus M$ still makes sense in this context.

Definition 1. The space of derivations of A into M is

$$\mathrm{Der}(A, M) := \mathrm{Map}_{\mathrm{CAlg}^{\Delta}_{/A}}(A, A \oplus M).$$

We now observe the following:

(1) The functor $\mathrm{Der}(A, -)$ is corepresentable by an A-module $\mathbf{L}_A^{\mathrm{alg}}$, which we refer to as the *algebraic cotangent complex* of A. This corepresentability follows from the adjoint functor theorem; alternatively, we will describe an explicit construction of $\mathbf{L}_A^{\mathrm{alg}}$ shortly.

(2) For a polynomial ring P on a set of generators S, the space $\mathrm{Der}(P, M)$ is easily identified with $\mathrm{Map}(S, M)$. Thus $\mathbf{L}_P^{\mathrm{alg}} \cong \Omega_P$ is the (discrete) free simplicial P-module generated by S.

(3) There is an adjoint pair of functors

$$\mathrm{CAlg}^{\Delta}_{/A} \underset{A\oplus -}{\overset{}{\rightleftarrows}} \mathrm{Mod}^{\Delta}_{A},$$

where the left adjoint sends $B \to A$ to the simplicial A-module $A \otimes_B \mathbf{L}_B^{\mathrm{alg}}$.

Any commutative ring A admits a simplicial resolution $P_\bullet \to A$ by polynomial rings. An immediate consequence of (2) and (3) above is that there is an equivalence of simplicial A-modules

$$\mathbf{L}_A^{\mathrm{alg}} \simeq A \otimes_{P_\bullet} \Omega_{P_\bullet}.$$

Definition 2. For a morphism $A \to B$ of simplicial rings, the *relative algebraic cotangent complex* $\mathbf{L}_{B/A}^{\mathrm{alg}}$ is the cofiber of

$$B \otimes_A \mathbf{L}_A^{\mathrm{alg}} \to \mathbf{L}_B^{\mathrm{alg}}$$

in $\mathrm{Mod}^{\Delta}_{B}$.

The simplicial B-module $\mathbf{L}_{B/A}^{\mathrm{alg}}$ classifies A-linear derivations out of B. For morphisms $A \to B \to C$ of simplicial rings, one easily constructs the *transitivity sequence*

$$C \otimes_B \mathbf{L}_{B/A}^{\mathrm{alg}} \to \mathbf{L}_{C/A}^{\mathrm{alg}} \to \mathbf{L}_{C/B}^{\mathrm{alg}},$$

which is a cofiber sequence in $\mathrm{Mod}^{\Delta}_{C}$.

There is a topological version of these constructions. Consider the 'forgetful functor'

$$\mathrm{CAlg}^{\Delta} \to \mathrm{CAlg}\colon A \mapsto A^\circ$$

from the ∞-category of simplicial rings to the ∞-category of $\mathbf{E}_\infty$-rings. It assigns to an ordinary ring the corresponding Eilenberg–MacLane spectrum and is characterized by this and the fact that it preserves colimits. (The terminology 'forgetful' is justified by the fact that this functor is conservative; it is even monadic and comonadic.)

Definition 3. The space of $\mathbf{E}_\infty$-derivations of A into M is

$$\mathrm{Der}_{\mathbf{E}_\infty}(A, M) := \mathrm{Map}_{\mathrm{CAlg}_{/A^\circ}}(A^\circ, (A \oplus M)^\circ).$$

Of course the above definition makes sense more generally; one can replace A° by a general $\mathbf{E}_\infty$-ring R. There is an adjunction

$$\mathrm{CAlg}_{/R} \underset{R\oplus -}{\overset{}{\rightleftarrows}} \mathrm{Mod}_R,$$

which can be constructed by exploiting universal properties; the ∞-category Mod_R is the stabilization of $\mathrm{CAlg}_{/R}$ and the trivial square-zero functor $R \oplus -$ plays the role of the right adjoint $\Omega^\infty_{\mathrm{CAlg}_{/R}}$. As before, the functor $\mathrm{Der}_{\mathbf{E}_\infty}(A, -)$ is corepresentable by an A-module $\mathbf{L}_A$ which we refer to as the *cotangent complex* of A. There are evident analogs of observations (2) and (3) above for this construction as well.

Remark 4. For A a simplicial ring, we will tacitly identify the equivalent ∞-categories Mod^Δ_A and $\mathrm{Mod}^{\mathrm{cn}}_{A^\circ}$, so that $\mathbf{L}^{\mathrm{alg}}_A$ and $\mathbf{L}_A$ can be considered as objects of the same ∞-category. The constructions above can be summarized by saying that the cotangent complex $\mathbf{L}_A$ arises from *stabilization* in the ∞-category $\mathrm{CAlg}_{/A}$, whereas the algebraic cotangent complex $\mathbf{L}^{\mathrm{alg}}_A$ arises from *derived abelianization.* These are analogous, but one is not a special case of the other (unless A is a $\mathbb{Q}$-algebra).

The relation between $\mathbf{L}^{\mathrm{alg}}_A$ and $\mathbf{L}_A$ can be understood by studying the following diagram:

$$\begin{tikzcd} \mathrm{CAlg}^\Delta_{/A} \arrow[r] \arrow[d, "\Sigma^\infty_+"] & (\mathrm{CAlg}_{H\mathbb{Z}})_{/A^\circ} \arrow[d, "\Sigma^\infty_+"] \\ \mathrm{Sp}(\mathrm{CAlg}^\Delta_{/A}) \arrow[r, "u"] & \mathrm{Mod}_{A^\circ}. \end{tikzcd}$$

The vertical arrows are stabilization; in particular, the one on the right gives $\Sigma^\infty_+ A^\circ = \mathbf{L}_{A^\circ/H\mathbb{Z}}$ by construction. The top horizontal arrow is the forgetful functor $(-)^\circ$ and the bottom one u is its stabilization (or 'linearization'). The augmentation ideal functor

$$\mathrm{CAlg}^\Delta_{/A} \to \mathrm{Sp}\colon (B \to A) \mapsto \mathrm{fib}(B^\circ \to A^\circ)$$

is part of a monadic adjunction; similarly, its linearization

$$\psi\colon \mathrm{Sp}(\mathrm{CAlg}^\Delta_{/A}) \to \mathrm{Sp}$$

exhibits $\mathrm{Sp}(\mathrm{CAlg}^\Delta_{/A})$ as monadic over Sp for some colimit-preserving monad T on Sp. Such a monad is necessarily of the form $T(X) = A^+ \otimes X$ for some $\mathbb{E}_1$-ring spectrum A^+. Thus we may identify

$$\mathrm{Sp}(\mathrm{CAlg}^\Delta_{/A}) \simeq \mathrm{LMod}_{A^+}.$$

Since ψ factors over u, the functor u must be given by restriction of scalars along a map $\alpha\colon A^\circ \to A^+$ of $\mathbf{E}_1$-rings. Summarizing, we see that for a simplicial commutative ring A, the cotangent complex $\mathbf{L}_{A^\circ/H\mathbb{Z}}$ is canonically a left A^+-module. Its usual A°-module structure is retrieved by restriction along α.

We aim to give an explicit formula relating $\mathbf{L}^{\mathrm{alg}}_A$ and $\mathbf{L}_{A^\circ/H\mathbb{Z}}$. To do this, note that the assignment $M \mapsto A \oplus M$ factors through a functor θ on stabilizations as follows:

$$\mathrm{Mod}^{\mathrm{cn}}_{A^\circ} \xrightarrow{\theta} \mathrm{Sp}(\mathrm{CAlg}^{\Delta}_{/A}) \xrightarrow{\Omega^\infty} \mathrm{CAlg}^{\Delta}_{/A}.$$

The composition $u\theta$ gives the identity. It follows that θ is induced by restriction along a morphism of $\mathbf{E}_1$-rings $\gamma\colon A^+ \to A^\circ$ such that $\gamma\alpha$ is homotopic to the identity of A°. Taking left adjoints in the composition of arrows above (once of the composite, once of the two arrows individually), we conclude that

$$\mathbf{L}^{\mathrm{alg}}_A \simeq A^\circ \otimes_{A^+} \mathbf{L}_{A^\circ/H\mathbb{Z}}.$$

Here $\mathbf{L}_{A^\circ/H\mathbb{Z}}$ has the left A^+-module structure described in the previous paragraph and A° is regarded as a right A^+-module via γ.

Finally, we will need the following result of Schwede (cf. Proposition 25.3.4.2 of [1]). It is important to note that this is a description of the underlying left A°-module of A^+, rather than of its ring structure.

Theorem 5 (Schwede)**.** *There is an equivalence of left A°-modules*

$$A^+ \simeq A^\circ \otimes_{\mathbb{S}} \mathbb{Z}.$$

2. The cotangent complex of an F-finite ring

Fix a Noetherian $\mathbb{F}_p$-algebra R that is F-finite (i.e., the Frobenius φ_R exhibits R as a finitely generated module over its subring of pth powers). Throughout this section we will consistently use the term R-module to mean an object of Mod_{HR}, i.e. a module for the Eilenberg–MacLane spectrum HR. The following plays an important role in later lectures:

Proposition 6. *The absolute cotangent complex $\mathbf{L}_R$ is an almost perfect R-module.*

Recall that an R-module M is *almost perfect* if for all $n \geq 0$ and all filtered diagrams $\{N_\alpha\}_{\alpha\in I}$ of n-truncated R-modules, the natural map

$$\varinjlim_{\alpha} \mathrm{Map}(M, N_\alpha) \to \mathrm{Map}(M, \varinjlim_{\alpha} N_\alpha)$$

is an equivalence. For Noetherian R this can be simplified as follows: M is almost perfect if and only if it is bounded below and each homotopy group $\pi_n M$ is a finitely presented (discrete) R-module. Roughly, the proof of Proposition 6 proceeds through the following steps:

(1) The $\mathbf{E}_\infty$-ring $H\mathbb{Z}$ is *almost of finite presentation* over the sphere spectrum $\mathbb{S}$, meaning that for every $n \geq 0$ there exists a finitely presented commutative $\mathbb{S}$-algebra A such that $H\mathbb{Z}$ is a retract of the Postnikov truncation $\tau_{\leq n}A$. As a consequence, $L_{H\mathbb{Z}/\mathbb{S}}$ is an almost perfect $H\mathbb{Z}$-module. By the transitivity sequence, the absolute cotangent complex L_R is an almost perfect R-module if and only if the relative one $L_{R/\mathbb{Z}}$ is.

(2) It follows from Theorem 5 that R^+ is an almost perfect R-module. To prove that $L_{R/\mathbb{Z}}$ is an almost perfect R-module, it therefore suffices to prove that it is almost perfect as an R^+-module. Using that $\gamma\colon R^+ \to R^\circ$ is a map of connective $\mathbb{E}_1$-rings that induces an isomorphism on π_0, it is not hard to see that any connective left R^+-module M is almost perfect if and only if $R^\circ \otimes_{R^+} M$ is an almost perfect R°-module. In particular, the equivalence $\mathbf{L}_R^{\mathrm{alg}} \simeq R^\circ \otimes_{R^+} \mathbf{L}_{R^\circ/H\mathbb{Z}}$ shows that it suffices to prove that $\mathbf{L}_R^{\mathrm{alg}}$ is an almost perfect R-module.

(3) We write $R^{1/p}$ for R regarded as an R-algebra via the Frobenius φ_R. The conditions that R be Noetherian and F-finite clearly imply that $R^{1/p}$ is an almost perfect R-module. This guarantees that in fact $R^{1/p}$ is almost of finite presentation over R as an $\mathbb{E}_\infty$-algebra (cf. Corollary 5.2.2.2 of [1]). It follows that the relative cotangent complex $L_{R^{1/p}/R}$ is an almost perfect R-module. Switching back and forth between the algebraic and topological cotangent complex as in items (1) and (2), we conclude that also $L^{\mathrm{alg}}_{R^{1/p}/R}$ is an almost perfect $R^{1/p}$-module.

(4) The Frobenius of R induces the map $\overline{\varphi}_R$ in the following transitivity sequence:

$$R^{1/p} \otimes_R L^{\mathrm{alg}}_{R/\mathbb{F}_p} \xrightarrow{\overline{\varphi}_R} L^{\mathrm{alg}}_{R^{1/p}/\mathbb{F}_p} \to L^{\mathrm{alg}}_{R^{1/p}/R}.$$

Since $d(x^p) = 0$ for any derivation d out of an $\mathbb{F}_p$-algebra, the map $\overline{\varphi}_R$ is canonically null. Hence $L^{\mathrm{alg}}_{R^{1/p}/\mathbb{F}_p}$ is a direct summand of $L^{\mathrm{alg}}_{R^{1/p}/R}$ as an $R^{1/p}$-module and therefore an almost perfect $R^{1/p}$-module. Regarded as an $\mathbb{F}_p$-algebra $R^{1/p}$ is just R, so that $L^{\mathrm{alg}}_{R/\mathbb{F}_p}$ is an almost perfect R-module.

(5) Finally, consider the transitivity sequence

$$R \otimes_{\mathbb{F}_p} L^{\mathrm{alg}}_{\mathbb{F}_p} \to \mathrm{L}^{\mathrm{alg}}_R \to \mathrm{L}^{\mathrm{alg}}_{R/\mathbb{F}_p}.$$

The algebraic cotangent complex $L^{\mathrm{alg}}_{\mathbb{F}_p}$ of $\mathbb{F}_p$ is quite easily seen to be $\Sigma\mathbb{F}_p$. Hence the first term can be identified with ΣR and it follows that the middle term $\mathrm{L}^{\mathrm{alg}}_R$ is indeed an almost perfect R-module.

3. The cotangent complex of a functor

The definition of the cotangent complex of a ring can be globalized to give a construction of the cotangent complex of a scheme or stack. It is a quasi-coherent sheaf on such and reduces to the previous construction in the affine case. However, what will be needed for future lectures is a formalism of cotangent complexes for general functors

$$X\colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$$

of which we do not (yet) know that they are representable by a reasonable geometric object. Recall that here $\mathrm{CAlg}^{\mathrm{cn}}$ denotes the ∞-category of connective $\mathbf{E}_\infty$-rings.

Fix X as above and let

$$\overline{X} \to \mathrm{CAlg}^{\mathrm{cn}}$$

be a left fibration classified by X. In particular, the fiber of $\overline{X}$ over $A \in \mathrm{CAlg}^{\mathrm{cn}}$ is equivalent to the space $X(A)$. A *quasi-coherent sheaf* $\mathcal{F}$ on X is a commutative diagram

$$\begin{tikzcd} \overline{X} \arrow[rr, "\mathcal{F}"] \arrow[dr] & & \mathrm{Mod} \arrow[dl] \\ & \mathrm{CAlg}^{\mathrm{cn}} & \end{tikzcd}$$

with the horizontal map preserving coCartesian edges. Here the right slanted arrow is the coCartesian fibration classified by the functor $A \to \mathrm{Mod}_A$. In particular, the objects of Mod can be identified with pairs (A, M) of $A \in \mathrm{CAlg}^{\mathrm{cn}}$ and $M \in \mathrm{Mod}_A$. Informally, a quasi-coherent sheaf consists of the following:

- For every pair $(A, \eta \in X(A))$ an A-module $\mathcal{F}_\eta$.
- For every morphism $f\colon A \to B$ with $f_*(\eta) = \eta'$, an equivalence
$$B \otimes_A \mathcal{F}_\eta \simeq \mathcal{F}_{\eta'}.$$
- Homotopies expressing the coherence of these data.

Of course the last item consists of a lot of data; the definition given above is an efficient way to package it.

Definition 7. Write

$$\mathrm{Mod}_X^{\mathrm{cn}} := \overline{X} \times_{\mathrm{CAlg}^{\mathrm{cn}}} \mathrm{Mod}^{\mathrm{cn}}$$

and $q\colon \mathrm{Mod}_X^{\mathrm{cn}} \to \overline{X}$ for projection onto the first factor.

In particular, q is a coCartesian fibration. The objects of $\mathrm{Mod}_X^{\mathrm{cn}}$ can be identified with triples (A, η, M) with $\eta \in X(A)$ and M a connective A-module. The following is the point of this section. We will explain the terminology below.

Definition 8. Consider two functors $X, Y\colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ and a natural transformation $\alpha\colon X \to Y$. Define $F\colon \mathrm{Mod}_X^{\mathrm{cn}} \to \mathcal{S}$ by

$$F(A, \eta, M) := \mathrm{fib}_\eta\big(X(A \oplus M) \to X(A) \times_{Y(A)} Y(A \oplus M)\big).$$

Then α *admits a cotangent complex* if F is *locally almost corepresentable with respect to q.*

Concretely the condition that F be locally almost corepresentable means the following:

(a) For every A and $\eta \in X(A)$, the functor
$$F(A, \eta, -)\colon q^{-1}(A, \eta) \simeq \mathrm{Mod}_A^{\mathrm{cn}} \to \mathcal{S}$$
is *almost corepresentable*, meaning that there exists an A-module M_η which is *almost connective* (i.e. bounded below) and an equivalence
$$F(A, \eta, N) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(M_\eta, N).$$
natural in N.

(b) The modules M_η are natural in η, in the sense that for a morphism $f\colon A \to B$ the induced map

$$B \otimes_A M_\eta \to M_{f_*\eta}$$

is an equivalence.

Observe that if (a) and (b) are satisfied, then the modules M_η define a quasi-coherent sheaf on X. This sheaf is denoted $L_{X/Y}$ and called the relative cotangent complex of α.

Remark 9. In the special case $Y = *$ and $X = \mathrm{Spec}(R)$, the functor F of Definition 8 is given by

$$F(A, \eta\colon R \to A, M) \simeq \mathrm{Der}_{\mathbf{E}_\infty}(R, M).$$

This is a locally corepresentable functor; the quasi-coherent sheaf L_X on $\mathrm{Spec}(R)$ corresponds precisely to the R-module L_R of previous sections.

We conclude with two useful elementary properties of the cotangent complex.

Proposition 10. *A natural transformation $\alpha\colon X \to Y$ admits a cotangent complex if and only if for every corepresentable functor Y' and $\varphi\colon Y' \to Y$, the natural transformation $Y' \times_Y X \to Y'$ admits a cotangent complex.*

Proposition 11. *Consider natural transformations $X \xrightarrow{f} Y \xrightarrow{g} Z$. If g and gf admit cotangent complexes, then so does f. In this case there is a cofiber sequence*

$$f^* L_{Y/Z} \to L_{X/Z} \to L_{X/Y}$$

of quasi-coherent sheaves on X.

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Formal Thickenings

Lukas Brantner

Overview. In this talk, we will discuss Lurie's generalisation of Schlessinger's criterion, which specifies conditions for when a functor from connective $\mathbb{E}_\infty$-rings to spaces can be represented by an adic $\mathbb{E}_\infty$-ring.

Background. Recall the following classical result in algebraic geometry (cf. [6]):

Theorem 1 (Schlessinger's criterion, weak form). *Let k be a perfect field of characteristic p. Assume that F : {Local Artin rings with residue field k} $\to$ Set is a functor satisfying the following properties:*

(1) Normalisation. The set $F(k) \simeq *$ consists of a point.
(2) Gluing. Given a pair of surjections $A' \to A$ and $A'' \to A$, the natural map $F(A' \times_A A'') \xrightarrow{\simeq} F(A') \times_{F(A)} F(A'')$ is a bijection.
(3) Formal smoothness. If $A' \to A$ is surjective, then so is $F(A') \to F(A)$.
(4) Finiteness. The k-vector space $F(k[\epsilon]/\epsilon^2)$ is finite-dimensional.

Then F is prorepresented by $W(k)[[x_1, \dots, x_d]]$, where $d = \dim_k(F(k[\epsilon]/\epsilon^2))$.

In Lecture 3, we have applied this criterion to the deformation functor attached to a height n formal group law, thereby constructing Lubin-Tate space.

Formal representability. In order to later construct spectral deformation rings for p-divisible groups (cf. [5, Theorem 6.0.3]), we shall need to generalise Schlessinger's criterion in two ways. First, we will work in the context of functors defined on (connective) $\mathbb{E}_\infty$-rings; second, we will allow the base to be a ring (rather than just a perfect field). The following appears as [5, Theorem 18.2.3.2]:

Theorem 2 (Lurie). *Let B be a connective $\mathbb{E}_\infty$-ring. Suppose that*

$$f : \mathrm{Spec}(B) \longrightarrow Y$$

is a transformation of functors $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ from connective $\mathbb{E}_\infty$-rings to spaces satisfying the following properties:

(1) The functor Y is nilcomplete (cf. Definition 17.3.2.1 in [5]), that is, the canonical map $Y(R) \xrightarrow{\simeq} \varprojlim_n Y(\tau_{\leq n} R)$ is an equivalence for all $R \in \mathrm{CAlg}^{\mathrm{cn}}$.
(2) The functor Y is infinitesimally cohesive (cf. Definition 17.3.1.5 in [5]). This means that whenever $A \simeq B \times^h_D C$ is a (homotopy) pullback in $\mathrm{CAlg}^{\mathrm{cn}}$ with $\pi_0(B) \twoheadrightarrow \pi_0(D)$ and $\pi_0(C) \twoheadrightarrow \pi_0(D)$ surjective, applying Y gives a (homotopy) pullback $Y(A) \simeq Y(B) \times^h_{Y(D)} Y(C)$.
(3) The functor Y admits a cotangent complex (cf. Definition 17.2.4.2 in [5]).
(4) The relative cotangent complex $L_{\mathrm{Spec}(B)/Y}$ is almost perfect and 1-connective.
(5) The functor Y is formally complete along f (cf. Definition 18.2.1.6 in [5]). This means that for any ordinary commutative ring R, the canonical map $\varinjlim_I \mathrm{Spec}(B)(R/I) \to \varinjlim_I Y(R/I)$ is an equivalence, where the colimits range over all nilpotent ideals $I \subset R$.

Then there is a connective $\mathbb{E}_\infty$-ring A and a π_0-surjective map $\rho : A \to B$ such that B is almost perfect as an A-module. Moreover, the kernel of $\pi_0(\rho)$ makes A into a complete adic $\mathbb{E}_\infty$-ring, and f is equivalent to $\mathrm{Spec}(B) \longrightarrow \mathrm{Spf}(A) \simeq Y$.

The key technical tool in the proof of Theorem 2 is the cotangent complex formalism, which was discussed in the preceding lecture. It will serve as our main tool for constructing new $\mathbb{E}_\infty$-rings: given an $\mathbb{E}_\infty$-ring B and a map $\eta : L_{\mathrm{Spec}(B)} \to M$ from its cotangent complex to another B-module M, we can form the homotopy pullback

$$\begin{tikzcd} B^\eta \arrow[r] \arrow[d] & B \arrow[d, "g_\eta"] \\ B \arrow[r, "g_0"] & B \oplus M \end{tikzcd} .$$

Here g_η and g_0 denote the maps classified by $\eta, 0 : L_{\mathrm{Spec}(B)} \to M$, respectively. We say that the $\mathbb{E}_\infty$-ring B^η is a *square-zero extension* of B by $\Sigma^{-1}M$; this terminology is justified by [4, Proposition 7.4.1.14].

In the context of the above theorem, we can iterate this construction to canonically factor the transformation $f : \mathrm{Spec}(B) \to Y$ through an infinite chain of morphisms

$$\mathrm{Spec}(B) = \mathrm{Spec}(B_0) \xrightarrow{f_0} \mathrm{Spec}(B_1) \xrightarrow{f_1} \quad \dots \quad \longrightarrow Y. \tag{1}$$

Indeed, assume that we have constructed the first n stages of the above diagram, with $L_{\mathrm{Spec}(B_i)/Y}$ being 1-connective and almost perfect for all $i = 0, \dots, n$. We then define B_{n+1} as the square-zero extension of B_n by $\Sigma^{-1}L_{\mathrm{Spec}(B_n)/Y}$ corresponding to the canonical map

$$\eta_n : L_{\mathrm{Spec}(B_n)} \longrightarrow L_{\mathrm{Spec}(B_n)/Y}.$$

In order to proceed with the recursive definition of sequence (1), and later prove Theorem 2, we will need several basic facts.

Factorisation. Since $L_{\mathrm{Spec}(B_n)/Y}$ is 1-connective, the infinitesimal cohesiveness of Y gives rise to a homotopy pullback

$$\begin{tikzcd} Y(B_{n+1}) \arrow[r] \arrow[d] & Y(B_n) \arrow[d, "Y(g_{\eta_n})"] \\ Y(B_n) \arrow[r, "Y(g_0)"] & Y(B_n \oplus L_{\mathrm{Spec}(B_n)/Y}) \end{tikzcd} .$$

The images of $f_n \in Y(B_n)$ under $Y(g_0)$ and $Y(g_{\eta_n})$ both lie in the same path component of $Y(B_n \oplus L_{\mathrm{Spec}(B_n)/Y})$, and we may therefore deduce that the morphism $\mathrm{Spec}(B_n) \to Y$ admits a factorisation

$$\mathrm{Spec}(B_n) \xrightarrow{f_n} \mathrm{Spec}(B_{n+1}) \longrightarrow Y.$$

Vanishing of transition maps. Unravelling the definitions, we can prove that the identity map on $L_{\mathrm{Spec}(B_n)/Y}$ factors through $L_{\mathrm{Spec}(B_n)/Y} \to L_{\mathrm{Spec}(B_n)/\mathrm{Spec}(B_{n+1})}$, which in turn implies that the following morphism is null:

$$f_n^*(L_{\mathrm{Spec}(B_{n+1})/Y}) \longrightarrow L_{\mathrm{Spec}(B_n)/Y}. \tag{2}$$

1-*Connectivity.* Informally speaking, Theorem 7.4.3.12 of [4] bounds the difference between cofibre and relative cotangent complex for an arbitrary morphism of connective $\mathbb{E}_\infty$-rings. In our situation, this means that the following map has 2-connective cofibre:

$$B_n \otimes_{B_{n+1}} \mathrm{cofib}(B_{n+1} \to B_n) \longrightarrow L_{\mathrm{Spec}(B_n)/\mathrm{Spec}(B_{n+1})}.$$

By a further argument, this implies that the relative cotangent complex $L_{\mathrm{Spec}(B_{n+1})/Y}$ is again 1-connective.
Almost perfectness. Since B_{n+1} is a square-zero extension of B_n by the almost perfect B_n-module $L_{\mathrm{Spec}(B_n)/Y}$, Corollary 5.2.2.5 and Proposition 2.7.3.2 in [5] together imply that $L_{\mathrm{Spec}(B_{n+1})/Y}$ is an almost perfect B_{n+1}-module.

We can therefore proceed with our recursive definition of sequence (1) above; a detailed treatment of this construction can be found in [5, Construction 18.2.5.5]. The $\mathbb{E}_\infty$-ring A appearing in the conclusion of Theorem 2 above is then given by the inverse limit

$$A := \varprojlim_n (B_n),$$

and the finitely generated ideal $\ker(\pi_0(A) \to \pi_0(B))$ equips A with an adic topology. To verify that the formal spectrum $\mathrm{Spf}(A)$ is indeed equivalent to the functor Y, we make use of the following detection criterion (cf. [5, Theorem 18.2.5.3]):

Theorem 3 (Lurie)**.** *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ which are both nilcomplete and infinitesimally cohesive. Moreover, assume that $L_{X/Y}$ exists and vanishes, and that Y is formally complete along f. Then f is an equivalence.*

The key condition in this criterion is the vanishing of the relative cotangent complex $L_{X/Y}$, which is satisfied in the situation of interest because the transition maps appearing in expression (2) above vanish (cf. [5, Lemma 18.2.5.6]).

The proof of Theorem 3 relies on the fact that any connective $\mathbb{E}_\infty$-ring R can be written as an inverse limit of its Postnikov tower

$$R \longrightarrow \dots \longrightarrow \tau_{\leq 2}R \longrightarrow \tau_{\leq 1}R \longrightarrow \tau_{\leq 0}R\ ,$$

with each stage $\tau_{\leq n}R$ being a square-zero extension of the preceding stage $\tau_{\leq n-1}R$ (cf. Corollary 7.4.1.28 in [4]). This last observation can be traced back to the classical work of Kriz [2], which was developed further by Basterra [1].

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The unoriented deformation ring

Allen Yuan

Let R_0 be an $\mathbb{F}_p$-algebra and let $\mathbb{G}_0$ be a p-divisible group over R_0 classified by a map $\operatorname{Spec} R_0 \xrightarrow{\eta} \mathcal{M}_{BT}$. We saw in previous lectures that one can write down a functor $\operatorname{Def}_{\mathbb{G}_0}(-) : \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}} \to \mathcal{S}$ which roughly corresponds to the formal completion of $\mathcal{M}_{BT}$ along η. The goal of this lecture is to prove the following theorem, which says that under certain conditions, this deformation functor is corepresented by an $\mathbb{E}_\infty$-ring spectrum $R^{\mathrm{un}}_{\mathbb{G}_0}$, known as the unoriented spectral deformation ring:

Theorem 1 ([1], Theorem 3.1.15)**.** *Suppose that R_0 is F-finite and Noetherian, and $\mathbb{G}_0$ is nonstationary. Then, the functor $\operatorname{Def}_{\mathbb{G}_0}(-)$ is corepresented by a complete adic Noetherian $\mathbb{E}_\infty$-ring $R^{\mathrm{un}}_{\mathbb{G}_0}$. Moreover, there is a canonical map $\rho : R^{\mathrm{un}}_{\mathbb{G}_0} \to R_0$ which is surjective on π_0 and for which the kernel on π_0 is an ideal of definition for $R^{\mathrm{un}}_{\mathbb{G}_0}$.*

We will review the meaning of the hypotheses and describe how they are used in the proof in the course of the talk.

The strategy for the proof is to apply the following theorem (stated here informally) which was the main result of the previous lecture:

Theorem 2 ([2], Theorem 18.2.5.1)**.** *Let $\operatorname{Spec} R_0 \xrightarrow{\eta} Y$ be a natural transformation of functors $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Suppose that:*

(1) Y is nilcomplete, infinitesimally cohesive, and admits a cotangent complex.
(2) Y is formally complete along η.
(3) The cotangent complex $L_{\operatorname{Spec} R_0/Y}$ is 1-connective and almost perfect.

Then Y is corepresented by a complete connective adic $\mathbb{E}_\infty$-ring A which is Noetherian if R_0 is.

We will apply this in the case $Y = \operatorname{Def}_{\mathbb{G}_0}|_{\mathrm{CAlg}^{\mathrm{cn}}}$ and $\operatorname{Spec} R_0 \to Y$ being the tautological deformation. We will not discuss the first condition of the theorem as it has to do with setting up the theory of spectral p-divisible groups, and the second condition follows more or less by construction of the functor $\operatorname{Def}_{\mathbb{G}_0}(-)$. We therefore focus on the condition on the cotangent complex. In fact, since Y was obtained by formally completing the classifying map $\operatorname{Spec} R_0 \xrightarrow{\eta} \mathcal{M}_{BT}$, it suffices to show that $L_{\operatorname{Spec} R_0/\mathcal{M}_{BT}}$ is 1-connective and almost perfect. There is a transitivity sequence

$$\eta^* L_{\mathcal{M}_{BT}} \to L_{\operatorname{Spec} R_0} \to L_{\operatorname{Spec} R_0/\mathcal{M}_{BT}}.$$

The conclusion will therefore follow from the following three statements:

(1) The cotangent complex L_{R_0} is almost perfect when R_0 is a Noetherian F-finite $\mathbb{F}_p$-algebra.
(2) The relative cotangent complex $L_{\operatorname{Spec} R_0/\mathcal{M}_{BT}}$ is 1-connective when $\mathbb{G}_0$ is non-stationary and L_{R_0} is almost perfect.
(3) The R_0-module $\eta^* L_{\mathcal{M}_{BT}}$ is almost perfect when p is nilpotent in R_0.

In the remainder of the talk, we will first make a digression on finiteness conditions in $\mathbb{E}_\infty$-rings, and then prove these three statements in order.

0.1. **Finiteness conditions for $\mathbb{E}_\infty$-algebras and modules.** The finiteness conditions of *almost perfect* and *almost of finite presentation* were already discussed more precisely in the previous two talks, but in this talk, we discuss what they mean on a more heuristic level. The goal is to get a feeling for them and understand why they are the appropriate replacement for the corresponding finiteness restrictions in the classical case. The reader is invited to consult [3, Section 7.2.4] for a more detailed discussion.

For the remainder of this subsection, let R be a connective $\mathbb{E}_\infty$-ring, let M be a module over R, and let $R \to A$ be a connective R-algebra.

Then M is *perfect* roughly when it is built up from finitely many shifted copies of R under extensions and retracts. The analog of this condition for algebras is that the algebra $R \to A$ is *finitely presented*, meaning that it is built out of finite colimits by free $\mathbb{E}_\infty$-algebras over R.

These conditions are often too strong in practice. For instance, if R and M are discrete, and M is finitely generated as a discrete R-module, then M need not be perfect in the above sense. The problem is that one can imagine creating a resolution of M by free R-modules; the condition that M is finitely generated ensures that the first stage of this resolution consists of finitely many free R-modules. However, two things could go wrong:

(1) The resolution may not terminate at a finite stage.
(2) The resolution could at some stage involve an infinite number of R-modules.

In many of the situations in higher algebra, the first issue is not so bad; each successive stage in the resolution involves higher and higher cells, and so if we want to know about a particular homotopy group, only finitely many stages of the resolution will contribute to it.

The notion of *almost perfect* is exactly designed so that the second issue does not happen. Namely, a module M is *almost perfect* roughly if it is bounded below, and it can be built from shifted copies of R using only a finite number of $\Sigma^i R$ for each particular i.

Example 3. For instance, one can show that if M is almost perfect and $\cdots \to N_2 \to N_1 \to N_0$ is an inverse system of connective R-modules, then the natural map

$$M \otimes (\varprojlim N_i) \to \varprojlim (M \otimes N_i)$$

is an equivalence.

Analogously, one says the connective algebra A is *almost of finite presentation* if it is built from R by attaching finitely many free $\mathbb{E}_\infty$ R-cells in each nonnegative dimension.

Example 4. It is easy to see from these descriptions that if $R \to A$ is almost of finite presentation, then the cotangent complex $L_{A/R}$ is an almost perfect A-module.

This concludes the digression and we shall proceed to prove the three statements outlined above.

0.2. L_{R_0} **is almost perfect.** We have already seen this in the previous talk; we saw that if R was a Noetherian F-finite $\mathbb{F}_p$-algebra, then $L_{\operatorname{Spec} R}$ is an almost perfect R-module.

In this talk, we review the proof briefly so that we can make one additional remark (Remark 5). For general reasons, it was enough to see that $L^{\mathrm{alg}}_{R/\mathbb{F}_p}$ was almost perfect. We observe two things about the situation:

(1) The Frobenius $\varphi : R \to R^{1/p}$ always induces 0 on cotangent complexes roughly because $d(f^p)$ is divisible by p and $p = 0$ in R.
(2) When R is a perfect $\mathbb{F}_p$-algebra in the sense that the Frobenius is an equivalence, then φ also induces an equivalence on cotangent complexes.

As a result, if R is a perfect $\mathbb{F}_p$-algebra, then the Frobenius map φ simultaneously induces 0 and an equivalence on cotangent complexes, so we learn that $L^{\mathrm{alg}}_{R/\mathbb{F}_p} = 0$. If R is not perfect, then by the transitivity sequence on the map φ, the failure of this argument to go through is measured by $L^{\mathrm{alg}}_{R^{1/p}/R}$. The condition that R is Noetherian and F-finite implies that $\varphi : R \to R^{1/p}$ is almost of finite presentation, and by Example 4, this means that $L^{\mathrm{alg}}_{R^{1/p}/R}$ is almost perfect, and so $L^{\mathrm{alg}}_{R/\mathbb{F}_p}$ remains almost perfect.

Remark 5 ([1], Remark 3.4.2)**.** In the proof of the Theorem 1, the condition of "Noetherian and F-finite" is used only in the above way to see that $L_{\operatorname{Spec} R_0}$ is almost perfect. We have just seen that this holds when R_0 is any perfect $\mathbb{F}_p$ algebra. Hence, it follows that $R^{\mathrm{un}}_{\mathbb{G}_0}$ exists for *any* p-divisible group $\mathbb{G}_0$ over a perfect $\mathbb{F}_p$-algebra R_0. For instance, one could input a ring like $R_0 = \mathbb{F}_p[t^{1/p^\infty}]$.

0.3. $L_{\operatorname{Spec} R_0/\mathcal{M}_{BT}}$ **is** 1**-connective.** This relative cotangent complex arises as the cofiber of a map $\eta^* L_{\mathcal{M}_{BT}} \to L_{\operatorname{Spec} R_0}$. The latter term is certainly connective, and the first term is connective for general reasons that we will not discuss. Therefore, $L_{\operatorname{Spec} R_0/\mathcal{M}_{BT}}$ is connective, and it suffices to show that the map above is surjective on π_0.

We first explain what is happening heuristically. One has a map $\eta : \operatorname{Spec} R_0 \to \mathcal{M}_{BT}$ and we are attempting to build the formal neighborhood of $\operatorname{Spec} R_0$ by iterated infinitesimal extensions. One way for the map induced by η on differentials to fail to be surjective is if the dual map on tangent vectors fails to be injective. In other words, if η sends some tangent vector in $\operatorname{Spec} R_0$ to zero. The condition of $\mathbb{G}_0$ being *nonstationary* is exactly saying that this does not happen.

Definition 6. A p-divisible group $\mathbb{G}_0$ over R_0 is nonstationary if for any point $x \in |\operatorname{Spec} R_0|$ and any tangent vector at x, given by $p : \operatorname{Spec} \kappa(x)[\epsilon]/\epsilon^2 \to \operatorname{Spec} R_0$, the restriction $p^*\mathbb{G}_0$ of $\mathbb{G}_0$ to the tangent vector is a nontrivial deformation of $(\mathbb{G}_0)_x$.

The desired statement now follows from:

Claim 7. *If $\mathbb{G}_0$ is nonstationary and $\pi_0 L_{R_0}$ is finitely generated (e.g., if L_{R_0} is an almost perfect R_0-module), then the natural map $\pi_0\eta^* L_{\mathcal{M}_{BT}} \to \pi_0 L_{R_0}$ is surjective.*

To see this, we would like to see that the cokernel is zero. We observe that the cokernel is, by assumption, finitely generated, and thus, by Nakayama's lemma, it suffices to check surjectivity at each residue field. This is exactly dual to the definition of $\mathbb{G}_0$ being nonstationary.

0.4. $\eta^* L_{\mathcal{M}_{BT}}$ **is almost perfect.** Motivated by Example 4, we might hope to prove this by showing that $\mathcal{M}_{BT}$ is almost of finite presentation. The following example shows that this is not true:

Example 8 ([1], Warning 3.1.9)**.** Let $A = \mathbb{F}_p[x^{1/p^\infty}]/(x-1)$. The ring A has all pth power roots of unity, so there is a canonical map of p-divisible groups

$$\gamma : \left(\underline{\mathbb{Q}_p/\mathbb{Z}_p}\right)_A \to (\mu_{p^\infty})_A$$

given by sending $1/p^n$ to x^{1/p^n}. This map has the feature that it is nonzero in A/I for any finitely generated nilpotent ideal I because I cannot contain all the pth power roots of unity, but the map is zero in $A^{\mathrm{red}} \simeq \mathbb{F}_p$.

The claim is that this means that the functor $R \mapsto BT^p(R)$ does not commute with filtered colimits of ordinary rings, and thus, is not almost of finite presentation. Strictly speaking, the functor $BT^p(-)$ takes values in spaces, and thus, only sees equivalences of p-divisible groups. We therefore take $\mathbb{G} = \underline{\mathbb{Q}_p/\mathbb{Z}_p} \oplus \mu_{p^\infty}$ and consider the transformation defined by the matrix $\begin{pmatrix} \mathrm{id} & \gamma \\ 0 & \mathrm{id} \end{pmatrix}$. It is the identity in A^{red} but not in A/I for any finitely generated nilpotent ideal I.

Nevertheless, in our situation, $\eta^* L_{\mathcal{M}_{BT}}$ is still almost perfect by the following proposition:

Proposition 9 ([1], Proposition 3.2.5)**.** *Let R be a connective $\mathbb{E}_\infty$-ring, and let $\mathbb{G}$ be a p-divisible group over R classified by a map $\eta : \operatorname{Spec} R \to \mathcal{M}_{BT}$. If p is nilpotent in $\pi_0(R)$, then $\eta^*\mathcal{M}_{BT}$ is connective and almost perfect.*

The R-module $\eta^* L_{\mathcal{M}_{BT}}$ has something to do with the infinitesimal automorphisms of $\mathbb{G}$. The idea will be that when $p^k = 0$ in R, these will be essentially controlled by things involving the p^k-torsion in $\mathbb{G}$, and thus will have nice finiteness properties.

Proof sketch. Let $\underline{\mathrm{Aut}}(\mathbb{G})$ be the functor which assigns to a connective $\mathbb{E}_\infty$-ring A the space of pairs (u, f) where $u : R \to A$ is a map of $\mathbb{E}_\infty$-rings and $f : \mathbb{G}_A \to \mathbb{G}_A$ is an automorphism of $\mathbb{G}$ defined over A. One can easily reduce to showing that $L_{\underline{\mathrm{Aut}}(\mathbb{G})/\operatorname{Spec} R}$ is 1-connective and almost perfect. In fact, one can replace $\underline{\mathrm{Aut}}(\mathbb{G})$ by the related functor $\underline{\mathrm{Map}}_{\mathbb{Z}}(\mathbb{G}, \mathbb{G})$, where the map f is no longer required to be an automorphism, but only a natural transformation of functors valued in connective $\mathbb{Z}$-modules.

Suppose that $p^k = 0$ in $\pi_0(R)$. Consider the short exact sequence

$$\mathbb{G}[p^k] \to \mathbb{G} \xrightarrow{[p^k]} \mathbb{G}$$

of groups. This induces a pullback square

$$\begin{tikzcd}
\underline{\mathrm{Map}}_{\mathbb{Z}}(\mathbb{G}, \mathbb{G}) \arrow[r, "{[p^k]}"] \arrow[d] & \underline{\mathrm{Map}}_{\mathbb{Z}}(\mathbb{G}, \mathbb{G}) \arrow[d] \\
\operatorname{Spec} R \arrow[r] & \underline{\mathrm{Map}}_{\mathbb{Z}}(\mathbb{G}[p^k], \mathbb{G}).
\end{tikzcd}$$

For ease of notation, let $Z := \underline{\mathrm{Map}}_{\mathbb{Z}}(\mathbb{G}, \mathbb{G})$ and $Z_{p^k} := \underline{\mathrm{Map}}_{\mathbb{Z}}(\mathbb{G}[p^k], \mathbb{G})$. In fact, Z_{p^k} can be expressed as $\underline{\mathrm{Map}}_{\mathbb{Z}/|\urcorner}(\mathbb{G}[p^k], \mathbb{G}[p^k])$, which depends only on finite flat group schemes. We then have the following fact which we will not prove:

Lemma 10 ([1], Lemma 3.2.11)**.** *$L_{Z_{p^k}}$ is connective and almost perfect.*

We would like to check that for any A-point of Z, the restriction $L_Z|_A$ is 1-connective and almost perfect. Without loss of generality, we may choose this point to correspond to the 0-map $\mathbb{G}_A \to \mathbb{G}_A$. Then, one obtains a transitivity sequence

$$L_{Z_{p^k}}|_A \to L_Z|_A \xrightarrow{q} L_Z|_A$$

where q is induced by $[p^k]$. One can check that q is in fact the map of A-modules given by multiplication by p^k. Since $p^k = 0$ in A, we conclude that $q = 0$. It follows that we have a splitting:

$$L_{Z_{p^k}}|_A \simeq L_Z|_A \oplus \Sigma^{-1} L_Z|_A.$$

Because the left-hand side is connective and almost perfect by the above lemma, we conclude that $L_Z|_A$ is 1-connective and almost perfect.

□

References

[1] J. Lurie, *Elliptic Cohomology II: Orientations*, available online at *http://www.math.harvard.edu/ lurie/index.html.*
[2] J. Lurie, *Spectral Algebraic Geometry*, available online at *http://www.math.harvard.edu/ lurie/index.html.*
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Orientations of formal groups

Christian Wimmer

In this talk, we discussed the theory of *orientations* of formal groups over E_∞-rings. This adresses the basic

Question. *Let $\widehat{G} : \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geq 0}R} \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}}$ be a (1-dimensional) formal group over a complex periodic E_∞-ring R. When does $\widehat{G}$ admit a morphism $\widehat{G}^Q_R \to \widehat{G}$ from the Quillen formal group associated with R and when is it an equivalence?*

In the following we will assume that all formal hyperplanes and formal groups are one-dimensional, although many of the definitions and statements make sense in any dimension.

We first generalize the classical concept of *dualizing lines* to the spectral setting, which will allow us to detect equivalences. Let $X : \mathrm{CAlg} \to S_*$ be a pointed formal hyperplane with coordinate ring $\mathcal{O}_X$ and basepoint $\eta \in X(\tau_{\geq 0}R) \simeq \mathrm{Map}_{\mathrm{CAlg}_R}(\mathcal{O}_X, R)$ classified by $\eta : \mathcal{O}_X \to R$.

Definition. Let $\mathcal{O}_X(-\eta) \in \mathrm{Mod}_{\mathcal{O}_X}$ be the fiber of η. The *dualizing line* $\omega_{X,\eta}$ of X at the point η is defined as the R-module $R \otimes_{\mathcal{O}_X} \mathcal{O}(-\eta)$.

Since the induced map $\pi_*(\eta)$ on homotopy groups is locally of the form (after translating by nilpotents)

$$(\pi_* R)[\![t]\!] \to R, t \mapsto 0,$$

the dualizing line is a locally free R-module of rank 1. The fact that this map detects units is the main non-formal input used to show

Proposition. *Let $f : X \to X'$ be a morphism of pointed formal hyperplanes over an E_∞-ring R. Then f is an equivalence if and only if the induced map $f^* : \omega_{w',\eta'} \to \omega_{w,\eta}$ is an equivalence of R-modules.*

The dualizing line can also be described as the desuspension of the fiber of the multiplication map in the sequence

$$\Sigma(\omega_{w,\eta}) \to R \otimes_{\mathcal{O}_X} R \to R.$$

If $\widehat{G}$ is a formal group, we write $\omega_{\widehat{G}}$ for the dualizing line of the underlying pointed formal hyperplane $\Omega^\infty \widehat{G}$ with the basepoint η corresponding to the unit in the group structure.

Example. Let R be a complex periodic E_∞-ring. The coordinate ring of the Quillen formal group $\widehat{G} = \widehat{G}^Q_R$ is given by the mapping spectrum $C^*(C\mathbb{P}^\infty, R)$ and we have a fiber sequence

$$C^*_{\mathrm{red}}(C\mathbb{P}^\infty, R) \to C^*(C\mathbb{P}^\infty, R) \to R.$$

From this we obtain an identification of the dualizing line $\omega_{\widehat{G}} \simeq C^*_{\mathrm{red}}(C\mathbb{P}^1, R) \simeq \Sigma^{-2}R$ with the 2-fold desuspension of R.

Definition. Let X be a pointed formal hyperplane. A *preorientation* is a map

$$e : S^2 \to X(\tau_{\geq 0}R)$$

of based spaces. The space of preorientations is the twofold loop space $\mathrm{Pre}(X) = \Omega^2 X(\tau_{\geq 0}R)$.

Given a preorientation $e \in \mathrm{Pre}(X)$, we can associate with it a *Bott map*

$$\beta_e : \omega_{w,\eta} \to \Sigma^{-2}R.$$

It is constructed as the image of e under the composition

$$\Omega^2 X(\tau_{\geq 0}R) \simeq \Omega\,\mathrm{Map}_{\mathrm{CAlg}}(R \otimes_{\mathcal{O}_X} R, R) \longrightarrow$$
$$\longrightarrow \Omega\,\mathrm{Map}_{\mathrm{Mod}_R}(\Sigma\omega_{w,\eta}, R) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(\omega_{X,\eta}, \Sigma^{-2}R),$$

where the map in the middle is induced by precomposition with $\Sigma\omega_{X,\eta} \to R \otimes_{\mathcal{O}_X} R$.

Definition. A preorientation $e \in \mathrm{Pre}(X)$ is called an *orientation* if the Bott map $\beta_e : \omega_{w,\eta} \to \Sigma^{-2}R$ is an equivalence. The space of orientations $\mathrm{OrDat}(X) \subset \mathrm{Pre}(X)$ is the subspace given by the corresponding components.

Now let R be complex periodic and $\widehat{G}$ a formal group. Then preorientations of $\widehat{G}$ classify maps from the Quillen formal group:

Proposition. *There is an equivalence* $\mathrm{Pre}(\widehat{G}) \simeq \mathrm{Map}_{\mathrm{FG(R)}}(\widehat{G}^Q_R, \widehat{G})$.

Under the identification $\omega_{\widehat{G}^Q_R} \simeq \Sigma^{-2}R$ the Bott map is given by the 'derivative' of the associated morphism of formal groups, so that orientations correspond to equivalences. More precisely:

Proposition. *Let R be an E_∞-ring, $\widehat{G}$ a formal group over R, and $e \in \mathrm{Pre}(\widehat{G})$ a preorientation. Then e is an orientation if and only if the following conditions are satisfied:*

(i) *R is complex periodic.*
(ii) *The morphism $\widehat{G}^Q_R \to \widehat{G}$ classified by e is an equivalence.*

Let X be a pointed formal hyperplane. Then it is possible find an R-algebra $R \to R'$ together with a universal preorientation of the basechange $X_{R'}$ of X to R'. More precisely, we have:

Proposition. *The functor* $\mathrm{CAlg}_R \to \mathcal{S}, R' \mapsto \mathrm{Pre}(X_{R'})$ *is corepresentable by $A = R \otimes_B R$, where $B = R \otimes_{\mathcal{O}_X} R$.*

To obtain a universal orientation, we need to 'invert' the Bott map $\beta_e : \omega_{X_A,\eta} \to \Sigma^{-2}A$ of the tautological preorientation coming with the R-algebra A. More generally, for any map $u : L \to L'$ between locally free A-modules of rank 1 we can form the localization $A[u^{-1}]$. As a commutative A-algebra it is characterized by the following universal property: For every $B \in \mathrm{CAlg}_A$, the mapping space $\mathrm{Map}_{\mathrm{CAlg}_A}(A[u^{-1}], B)$ is contractible if $B \otimes_A u$ is an equivalence and empty otherwise. Moreover, the underlying A-module can be identified with the colimit of the sequential system

$$A \xrightarrow{u} L'^{-1} \otimes_A L \xrightarrow{u} (L'^{-1})^{\otimes 2} \otimes_A L^{\otimes 2} \longrightarrow \cdots$$

Corollary. *The functor* $\mathrm{CAlg}_R \to S, R' \mapsto \mathrm{OrDat}(X_{R'})$ *is corepresentable by the localization* $\mathfrak{D}_X = A[\beta_e^{-1}]$.

We will also refer to $\mathfrak{D}_X$ as an *orientation classifier* of X. If $\widehat{G}$ is a formal group, we write $\mathfrak{D}_{\widehat{G}}$ for the orientation classifier of the underlying formal hyperplane $\Omega^\infty \widehat{G}$.

In the larger context of this Arbeitsgemeinschaft, this was used to define oriented deformation rings:

Definition. Let R_0 be a Noetherian F-finite $\mathbb{F}_p$-algebra and $G \in \mathrm{BT}^p(R_0)$ a non-stationary p-divisible group. Let $G \in \mathrm{BT}^p(R^{\mathrm{un}}_{G_0})$ be the universal deformation over the spectral deformation ring. The *oriented deformation ring* of G_0 is defined as an orientation classifier $R^{\mathrm{or}}_{G_0} = \mathfrak{D}_{\widehat{G}}$ of the identity component $\widehat{G} = G^\circ$.

At this point, it is not even clear that oriented deformation rings are non-trivial. We conclude by stating one of the main results, which was discussed in a later talk:

Theorem. *In the above situation, the following holds:*

(i) *The map* $R^{\mathrm{un}}_{G_0} \to R^{\mathrm{or}}_{G_0}$ *induces an isomorphism on* π_0 .
(ii) *The homotopy groups* $\pi_*(R^{\mathrm{or}}_{G_0})$ *are concentrated in even degrees.*

References

[1] J. Lurie, *Elliptic cohomology II.*

The eye of the storm: Lubin-Tate spectra

Charles Rezk

We describe the theorem proved by Goerss-Hopkins-Miller, which asserts the existence of an $\mathbb{E}_\infty$-ring E called *Morava E-theory.* This object is associated to a pair $(\kappa, \widehat{G}_0)$ consisting of a perfect field κ and a one-dimensional commutative formal group $\widehat{G}_0$ of finite height over κ, and is equipped with an isomorphism

$$\alpha\colon (\kappa, \widehat{G}_0) \xrightarrow{\sim} (\pi_0 E/I_n, \widehat{G}_E^{\mathcal{Q}_n})$$

of formal groups, where $\widehat{G}_E^{\mathcal{Q}}$ is the *Quillen formal group* (a formal group over the $\mathbb{E}_\infty$-ring E), we write $\widehat{G}_E^{\mathcal{Q}_0}$ for the induced formal group over the ordinary ring $\pi_0 E$, and $\widehat{G}_E^{\mathcal{Q}_n}$ for its base-change to the quotient by $\pi_0 E$ by the nth Landweber ideal. and is characterized by either of the following properties:

(1) The spectrum E is *even periodic* (i.e., $\pi_2 E \otimes_{\pi_0 E} \pi_{-2} E \approx \pi_0 E$ and $\pi_{\mathrm{odd}} E = 0$), and the induced map $(\pi_0 E, \widehat{G}_E^{\mathcal{Q}_0}) \to (\kappa, \widehat{G}_0)$ exhbits the (classical) universal deformation of the formal group $(\kappa, \widehat{G}_0)$.
(2) The spectrum E is $K(n)$*-local*, and for any complex periodic $K(n)$-local $\mathbb{E}_\infty$-ring A the evident map induces an equivalence

$$\mathrm{Map}_{\mathrm{CAlg}}(E, A) \approx \mathrm{Hom}_{\mathrm{FG}}((\kappa, \widehat{G}_0),\ (\pi_0 A/I_n, \widehat{G}_A^{\mathcal{Q}_n}).$$

We describe how the existence of E and α satisfying both properties implies that it is uniquely characterized by both properties. Then we outline how the existence of E and α satisfying property (2) is proved, using the theory of the universal oriented p-divisible groups. More precisely, we have

$$E = L_{K(n)} R^{\mathrm{or}}_{G_0},$$

where G_0 is the connected p-divisible group over κ with associated formal group $\widehat{G}_0$, and R^{or} is the $\mathbb{E}_\infty$-ring which carries the universal deformation of G_0.

References

[1] J. Lurie, *Elliptic Cohomology II: Orientations*, available online at *http://www.math.harvard.edu/ lurie/index.html.*

[2] J. Lurie, *Spectral Algebraic Geometry*, available online at *http://www.math.harvard.edu/ lurie/index.html.*

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Lubin Tate Spectra and the Goerss Hopkins Miller Theorem

Hood Chatham

Let R_0 be a perfect $\mathbb{F}_p$-algebra, $\widehat{\mathbb{G}}_0$ a formal group of strict height n over R_0, $\mathbb{G}_0$ a p-divisible group with $\mathbb{G}_0^\circ = \widehat{\mathbb{G}}_0$, and $E = L_{K(n)} R^{\mathrm{or}}_{\mathbb{G}_0}$.

Construction 1 (5.1.4)**.** Corollary 4.4.25 gives a map $\pi_0(R^{\mathrm{un}}_{\mathbb{G}_0}) \to R_0$ with kernel $\mathcal{J}_n^{\widehat{\mathbb{G}}}$. From this we get a map of formal groups $\alpha\colon (R_0, \widehat{\mathbb{G}}_0) \to (\pi_0(E)/\mathcal{J}_n^E, \widehat{\mathbb{G}}_E^{\mathcal{Q}_n})$ as the following composite:

$$\begin{aligned}
(R_0, \widehat{\mathbb{G}}_0) &\longrightarrow \left(\pi_0(R^{\mathrm{un}})/\mathcal{J}_n, \left(\widehat{\mathbb{G}}_0\right)_{\pi_0(R^{\mathrm{un}})/\mathcal{J}_n^{\widehat{\mathbb{G}}_0}}\right) \\
&\longrightarrow \left(\pi_0(R^{\mathrm{or}})/\mathcal{J}_n, \left(\widehat{\mathbb{G}}_0\right)_{\pi_0(R^{\mathrm{or}})/\mathcal{J}_n^{\widehat{\mathbb{G}}_0}}\right) \\
&\xrightarrow{\ \cong\ } \left(\pi_0(R^{\mathrm{or}})/\mathcal{J}_n^{R^{\mathrm{or}}}, \widehat{\mathbb{G}}_{R^{\mathrm{or}}}^{\mathcal{Q}_n}\right) \\
&\longrightarrow \left(\pi_0(E)/\mathcal{J}_n^E, \widehat{\mathbb{G}}_E^{\mathcal{Q}_n}\right)
\end{aligned}$$

Our goal in this section is to prove that our oriented deformation ring is E-theory. We want to reprove the theorem of Goerss-Hopkins-Miller:

Theorem 2 (5.0.2). *The spectrum E and the map α have the following properties:*

(i) The map

$$(\pi_0 E, \widehat{\mathbb{G}}_E^{\mathcal{Q}_0}) \longrightarrow (\pi_0 E/\mathcal{J}_n^E, \widehat{\mathbb{G}}_E^{\mathcal{Q}_n}) \xrightarrow{\alpha^{-1}} (k, \widehat{\mathbb{G}}_0)$$

exhibits $(\pi_0 E, \widehat{\mathbb{G}}_E^{\mathcal{Q}_0})$ as a universal deformation of $(R_0, \widehat{\mathbb{G}}_0)$. In particular, $\pi_0(E) \cong R_{\mathrm{LT}}(\widehat{\mathbb{G}}_0)$

(ii) The ring E is $K(n)$ local and for a $K(n)$-local E_∞ ring A, the composition

$$\begin{tikzcd}[column sep=small] & \mathrm{CAlg}(E, A) \arrow[r] & \mathcal{FG}((\pi_0 E, \widehat{\mathbb{G}}_E^{\mathcal{Q}_0}), (\pi_0 A, \widehat{\mathbb{G}}_A^{\mathcal{Q}_0})) \arrow[dl] \\ \mathcal{FG}((\pi_0 E/\mathcal{J}_n^E, \widehat{\mathbb{G}}_E^{\mathcal{Q}_n}), (\pi_0 A/\mathcal{J}_n^A, \widehat{\mathbb{G}}_A^{\mathcal{Q}_n})) \arrow[rr, "\alpha^*"] & & \mathcal{FG}((R_0, \widehat{\mathbb{G}}_0), (\pi_0 A/\mathcal{J}_n^A, \widehat{\mathbb{G}}_A^{\mathcal{Q}_n})) \end{tikzcd}$$

is an equivalence. In particular, the mapping space $\mathrm{CAlg}(E, A)$ is discrete.

Charles Rezk proved our E theory satisfies (ii) and showed that given the existence of E theory, either property (i) or property (ii) classifies E theory. If all we wanted was to show that our E theory is the Goerss-Hopkins-Miller E-theory, we could be done. Our goal for this talk is to reprove the Goerss-Hopkins-Miller theorem, so we need to show that our E theory satisfies property (i).

We will prove the following restated version of part (i):

Theorem 3 (5.4.1). *Let R_0 be a perfect $\mathbb{F}_p$-algebra, let $\widehat{\mathbb{G}}_0$ be a 1-d formal group of exact height n over R_0 and let E be the Lubin Tate spectrum. Then*

(a) The map α induces an isomorphism $R_0 \to \pi_0 E/\mathcal{J}_n^E$.

(b) The homotopy groups of E are in even degrees.

(c) Choose a sequence of elements $p = \overline{v}_0, \dots, \overline{v}_{n-1}$ lifting $v_m \in \pi_* E/\mathcal{J}_m^E$. Then $\overline{v}_0, \dots, \overline{v}_{n-1}$ is a regular sequence in $\pi_* E$.

This implies (i): since α is an isomorphism, there is a map $R_{\mathrm{LT}} \to \pi_0 E$ corresponding to the deformation $(\pi_0(E), \widehat{\mathbb{G}}_E^{\mathcal{Q}_0}) \to (R_0, \widehat{\mathbb{G}}_0)$. Theorem 5.4.1 shows that there is a diagram

$$\begin{tikzcd} R_{\mathrm{LT}} \arrow[rr] \arrow[dr] & & \pi_0(E) \arrow[dl] \\ & R_0 & \end{tikzcd}$$

where the maps $R_{\mathrm{LT}} \to R_0$ and $\pi_0(E) \to R_0$ are complete, the kernels are generated by regular sequences of the same length, and the map $R_{\mathrm{LT}} \to \pi_0(E)$ takes the regular sequence generating the kernel of $R_{\mathrm{LT}} \to R_0$ to the regular sequence generating the kernel of $\pi_0(E) \to R_0$. This implies that they are isomorphic, and that $(\pi_0(E), \widehat{\mathbb{G}}_E^{\mathcal{Q}_0}) \to (R_0, \widehat{\mathbb{G}}_0)$ is the universal deformation.

So what's our approach? First we base change to the universal coordinatized formal group over R_0 (this is the same as trivializing the dualizing line $\omega_{\widehat{\mathbb{G}}_0}$). Later, we will use faithfully flat descent to deduce our results about $\widehat{\mathbb{G}}_0$ from similar results on the universal coordinatized formal group over $\widehat{\mathbb{G}}_0$. Let $R_0 \to \widetilde{R}_0$ classify the universal coordinate on $\widehat{\mathbb{G}}_0$. If $\widehat{\mathbb{G}}_0$ has a coordinate t then $\widetilde{R}_0 \cong R_0[b_0^{\pm}, b_1, b_2, \ldots]$ where the universal coordinate is $\sum b_i t^{i+1}$. The descent coalgebra $\widetilde{R}_0 \otimes_{R_0} \widetilde{R}_0 \cong \widetilde{R}_0[x_0^{\pm}, x_1, x_2, \ldots]$ looks like $\widetilde{R}_0 \otimes_{MP_0} MP_0 MP \otimes_{MP_0} \widetilde{R}_0$.

There is a canonical map $\rho_0 \colon L/I_n \to \widetilde{R}_0$ classifying our formal group law. We show:

Proposition 4. *There is a pushout square:*

$$\begin{tikzcd} MP_{\geq 0} \arrow[r] \arrow[d] & \tau_{\geq 0} L_{K(n)}(MP \wedge E) \arrow[d] \\ L/I_n \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \widetilde{R}_0 \end{tikzcd}$$

From this, 5.4.1 will be pretty easy. We will prove the proposition by recognizing $\tau_{\geq 0} L_{K(n)}(MP \wedge E)$ as a "thickening" of the map $\rho_0 \colon L/I_n \to \widetilde{R}_0$.

Definition 5. Let S be a commutative E_∞-ring, let $I \subseteq \pi_0 S$ a finitely generated ideal, and set $S_0 = \pi_0(S)/I$. Suppose given a diagram:

$$\sigma\colon \quad \begin{tikzcd} S \arrow[r, "f"] \arrow[d] & T \arrow[d] \\ S_0 \arrow[r, "f_0"] & T_0 \end{tikzcd}$$

where T_0 is discrete. We say that σ *exhibits f as an S-thickening of f_0* if:

(a) T is I-complete
(b) σ is a pushout diagram.
(c) $\mathrm{CAlg}_S(T, U) \simeq \mathrm{CAlg}^{\heartsuit}{}_{S_0}(T_0, U_0)$.

We'll use (a) to show that T is $K(n)$-local, (b) to demonstrate our pushout proposition, and (c) to produce a map $\tau_{\geq 0} MP \wedge E \to T$. We need a way to know that thickenings exist.

Theorem 6 (5.2.5 – existence of thickenings)**.** *Let S be a commutative E_∞ ring, $I \subseteq \pi_0(S)$ a finitely generated ideal, and $S_0 = \pi_0(S)/I$. Suppose*

(a) S_0 is a perfect S-module.
(b) S_0 is an $\mathbb{F}_p$-algebra and ϕ is flat.
(c) $f \colon S_0 \to T_0$ is a relatively perfect map of $\mathbb{F}_p$-algebras.

Then there is a diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & T \arrow[d] \\ S_0 \arrow[r, "f_0"] & T_0 \end{tikzcd}$$

exhibiting f as an S-thickening of f_0.

Proposition 7 (Prop 5.4.6)**.** *There exists a diagram*

$$\begin{array}{ccc} MP_{\geq 0} & \xrightarrow{\rho} & \widetilde{A} \\ \downarrow & & \downarrow \\ L/I_n & \xrightarrow{\rho_0} & \widetilde{R}_0 \end{array}$$

exhibiting ρ as an $MP_{\geq 0}$ thickening of ρ_0.

Proof. We need to check the conditions of the existence theorem. (a) and (b) are easy and Niko Naumann will prove that (c) is satisfied in his talk. □

Let $A = \widetilde{A}[u^{-1}]$ where u is the image of $u \in \pi_2 MP$ under ρ. The ring A is even periodic. We know that $\pi_0(A)$ is $\mathcal{J}_n^A$-complete. The following proposition tells us that A is $K(n)$-local:

Theorem 8 (4.5.2)**.** *Let A be a p-local complex periodic E_∞-ring and n a positive integer. Then A is $K(n)$-local if and only if:*

(a) A is complete with respect to $\mathcal{J}_n^A \subseteq \pi_0 A$
(b) The $(n+1)$st Landweber ideal $\mathcal{J}_{n+1}^A$ is the unit ideal. Equivalently, $\widehat{\mathbb{G}}_0$ has height at most n.

Now we'll show:

Theorem 9. *$L_{K(n)} MP \wedge E \simeq A$.*

This will complete the proof of the pushout proposition which is the hard part of proving Theorem 5.4.1. First we need maps $MP \to A$ and $E \to A$. We have the map $MP \to A$ by construction. Because E satisfies theorem 5.0.2(ii) already, we know that $\mathrm{CAlg}(E, A) \simeq \mathcal{FG}(\widehat{\mathbb{G}}_0, \widehat{\mathbb{G}}_A^{\mathcal{Q}_n})$. There is a natural map $\widehat{\mathbb{G}}_0 \to \widehat{\mathbb{G}}_A^{\mathcal{Q}_n}$ coming from the fact that $\widehat{\mathbb{G}}_A^{\mathcal{Q}_n}$ is the universal coordinatized formal group on $\widehat{\mathbb{G}}_0$.

Proof. Let U be a $K(n)$-local complex period E_∞ ring and $U_0 = \pi_0(U)/\mathcal{J}_n^U$. There is a restriction map $\mathrm{CAlg}_{MP}(A, U) \to \mathrm{CAlg}(E, U)$. We will show that this is an isomorphism. We compare:

$$\begin{aligned} \mathrm{CAlg}_{MP}(A, U) &\simeq \mathrm{CAlg}_{MP_{\geq 0}}(A_{\geq 0}, U_{\geq 0}) \\ &\simeq \mathrm{CAlg}^{\heartsuit}(\widetilde{R}_0, U_0) \\ &\simeq \mathcal{FG}\left(\widehat{\mathbb{G}}_0, \widehat{\mathbb{G}}_U^{\mathcal{Q}_n}\right) \end{aligned}$$

$$\mathrm{CAlg}(E, U) \simeq \mathcal{FG}\left(\widehat{\mathbb{G}}_0, \widehat{\mathbb{G}}_U^{\mathcal{Q}_n}\right)$$

□

Now we have a pushout diagram

$$\begin{tikzcd} MP_{\geq 0} \arrow[r] \arrow[d] & \widetilde{A} \arrow[d] \\ L/I_n \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \widetilde{R}_0 \end{tikzcd}$$

where $\widetilde{A} = \tau_{\geq 0} L_{K(n)}(MP \wedge E)$. For a complex periodic ring S, let $S(m) = \bigotimes_{i=0}^{m-1} C(\overline{v}_m)$ where $\overline{v}_m$ is any lift of the Hasse invariant in $\pi_0(S)/J_m^S$ to $\pi_0(S)$. Our pushout diagram tells us that $\widetilde{A}(n) \cong \widetilde{R}_0[u]$. Inducting downwards on m and using the $J_n^{\widetilde{A}}$ completeness of $\widetilde{A}$, we deduce that $(p, \overline{v}_1, \ldots, \overline{v}_{n-1})$ is a regular sequence in $\pi_0(\widetilde{A})$ so it's also a regular sequence in $\pi_0(A)$. Inducting back up on m, we deduce that $A(n) \simeq \widetilde{R}_0[u^{\pm}]$. By descent we deduce that $E(n)$ is even and 5.4.1(a). Inducting down on m and using $\mathcal{J}_n^E$ completeness shows that $E(m)$ is even for each m and the $\overline{v}_i$'s are regular in $\pi_0(E)$ which is 5.4.1(b) and 5.4.1(c).

Spectral Witt vectors

Niko Naumann

We presented a result of Lurie's allowing to lift certain discrete algebras to connective $\mathbb{E}_\infty$-ring spectra and applied it to support a technical step in the computation of the homotopy of Lubin-Tate spectra. The result is the following.

Theorem 1 (Lurie). *Assume that A is a connective $\mathbb{E}_\infty$-ring and $I \subseteq \pi_0(A)$ is a finitely generated ideal such that the map*

$$A \longrightarrow A_0 := \pi_0(A)/I$$

makes A_0 an almost finitely presented A-module.[1] *Then, given any A_0-algebra*

$$A_0 \longrightarrow B_0$$

with $\mathbb{L}_{B/A} = 0$, there is a unique connective A-albegra

$$A \longrightarrow B$$

which is I-adically complete, has $\pi_0(B)/I \cdot \pi_0(B) \simeq B_0$ and is such that for every connective I-adically complete A-algebra R, the canonical map

$$map_{\mathbb{E}_\infty-A-alg}(B, R) \longrightarrow map_{A_0-alg}(B_0, \pi_0(R)/I \cdot \pi_0(R))$$

is an equivalence. In particular, the space $map_{\mathbb{E}_\infty-A-alg}(B, R)$ is homotopy discrete.

[1] This holds automatically if A is coherent.

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Proof of a main theorem

Dylan Wilson

Let k be a perfect field of characteristic p and Γ a height n formal group over k. We have seen that Morava E-theory $E(k,\Gamma)$ arises as the $K(n)$-localization of the $\mathbb{E}_\infty$-ring which carries the universal, oriented deformation of the pair (k,Γ). In this talk we compute the homotopy groups of oriented deformation rings in general. As a consequence, we will see that the $K(n)$-localization above is unnecessary.

Before stating the theorem, we recall some notation.

Notation 1.

- We will denote by $(R_0, \mathbf{G}_0)$ a pair consisting of a Noetherian, F-finite $\mathbb{F}_p$-algebra R_0 and a nonstationary, one-dimensional p-divisible group $\mathbf{G}_0$ over R_0.
- We denote by $R^{\mathrm{un}}(\mathbf{G}_0)$ the **unoriented deformation ring** of the pair $(R_0, \mathbf{G}_0)$, and by $\mathbf{G}$ the corresponding universal deformation.
- We denote by $R^{\mathrm{or}}(\mathbf{G}_0)$ the **oriented deformation ring**, or the **orientation classifer** of the connected component $\mathbf{G}^\circ$. That is:
$$R^{\mathrm{or}}(\mathbf{G}_0) \simeq \left(R^{\mathrm{un}}(\mathbf{G}_0) \otimes_{R^{\mathrm{un}}(\mathbf{G}_0)\otimes_{\mathcal{O}_{\mathbf{G}^\circ}} R^{\mathrm{un}}(\mathbf{G}_0)} R^{\mathrm{un}}(\mathbf{G}_0)\right)[\beta^{-1}]$$
where β is the Bott element.

Theorem 2. [1, 6.0.3,6.4.6] *$\mathbf{G}^\circ$ is balanced over $R^{\mathrm{or}}(\mathbf{G}_0)$. In other words:*
$$\pi_{\mathrm{odd}} R^{\mathrm{or}}(\mathbf{G}_0) = 0$$
$$R^{\mathrm{cl}}(\mathbf{G}_0) := \pi_0 R^{\mathrm{un}}(\mathbf{G}_0) \xrightarrow{\simeq} \pi_0 R^{\mathrm{or}}(\mathbf{G}_0)$$

The talk will proceed by first explaining some general dévissage techniques for deformation rings, using these to reduce to the case when R_0 is a field, and then proving the result in that case by combining the $K(n)$-local result with the dévissage techniques. To motivate this procedure, let's examine an example.

Example 3. Let C be the family of elliptic curves defined by
$$C: \; y^2 = 4x^3 + u_1x^2 + 2x$$
over the ring $\mathbb{W}(\overline{\mathbb{F}}_3)[\![u_1]\!]$. Modulo 3, the 3-series is given by
$$[3](t) \equiv u_1t^3 + 2(1+u_1^2)t^9 + \cdots \mod 3$$
In particular, we learn that C deforms a supersingular elliptic curve
$$C_0 := C_{\overline{\mathbb{F}}_3} : \; y^2 = x^3 - x,$$

and hence that the 3-divisible group $C[3^\infty]$ deforms a height 2 formal group over $\overline{\mathbb{F}}_3$. In fact, this is the universal such deformation. But $C[3^\infty]$ is also a deformation of a (non-connected) 3-divisible group over $\overline{\mathbb{F}}_3[\![u_1]\!]$, and it is *also* the universal such- though, viewing it this way privileges the 3-adic topology on $\mathbb{W}(\overline{\mathbb{F}}_3)[\![u_1]\!]$ instead of the $(3, u_1)$-adic topology. At the generic point, e.g. over $\overline{\mathbb{F}}_3(\!(u_1)\!)$ or its algebraic closure, the connected component of our 3-divisible group has height one, as we can see by looking at the 3-series above. The universal deformation of this generic member of the family turns out to live over $\mathbb{W}(\overline{\mathbb{F}_3(\!(u_1)\!)})[\![x_1]\!]$, and we get a natural zig-zag:

$$\begin{tikzcd}
\mathbb{W}(\overline{\mathbb{F}}_3)[\![u_1]\!] \arrow[r] \arrow[d, equal] & \mathbb{W}(\overline{\mathbb{F}_3(\!(u_1)\!)})[\![x_1]\!] \arrow[d, equal] & \mathbb{W}(\overline{\mathbb{F}_3(\!(u_1)\!)}) \arrow[l] \arrow[d, equal] \\
R^{\mathrm{cl}}(\mathbf{G}_{\overline{\mathbb{F}}_3[\![u_1]\!]}) \arrow[r] & R^{\mathrm{cl}}(\mathbf{G}_{\overline{\mathbb{F}_3(\!(u_1)\!)}}) & R^{\mathrm{cl}}(\mathbf{G}^{\circ}_{\overline{\mathbb{F}_3(\!(u_1)\!)}}) \arrow[l]
\end{tikzcd}$$

The key observations are: (i) the first map is faithfully flat after localizing at 3, (ii) the second map is faithfully flat, and (iii) the connected component of the 3-divisible group lying over the middle term is pulled back from the connected component over the right-hand term. Thus: questions about the connected component of the p-divisible group over the left hand term can be related to the same questions for the deformation of a p-divisible group *of smaller height*.

We want to generalize the lessons we learned in the previous example. The main technique is summarized in the following result:

Theorem 4. *Let $\mathfrak{p} \subseteq R^{\mathrm{cl}}(\mathbf{G}_0)$ be a prime and $\mathfrak{m} \subseteq R_0$ a maximal ideal. Suppose that the residue field $k(\mathfrak{p})$ has characteristic p. Then we have two natural zig-zags:*

$$R^{\mathrm{un}}(\mathbf{G}_0) \xleftarrow{A} R^{\mathrm{un}}(\mathbf{G}_{R^{\mathrm{cl}}(\mathbf{G}_0)/p}) \xrightarrow{B} R^{\mathrm{un}}(\mathbf{G}_{\overline{k(\mathfrak{p})}}) \xleftarrow{C} R^{\mathrm{un}}(\mathbf{G}^{\circ}_{\overline{k(\mathfrak{p})}})$$

$$R^{\mathrm{un}}(\mathbf{G}_0) \xrightarrow{D} R^{\mathrm{un}}((\mathbf{G}_0)_{\overline{k(\mathfrak{m})}}) \xleftarrow{E} R^{\mathrm{un}}((\mathbf{G}_0)^{\circ}_{\overline{k(\mathfrak{m})}})$$

with the following properties:

(i) *A is an equivalence of $\mathbb{E}_\infty$-rings (but not of adic $\mathbb{E}_\infty$-rings in general.)*
(ii) *B is flat and becomes faithfully flat after localization at $\mathfrak{p}$.*
(iii) *D is flat and becomes faithfully flat after localization at $\mathfrak{m}$.*
(iv) *C and E are flat and the connected components of the p-divisible groups on their targets are obtained by extension of scalars along the respective maps.*

We will refer to the first of these zig-zags as the **height reduction zig-zag** because it relates a given deformation ring to the deformation ring for a formal group of smaller height (in the case that $\mathfrak{p}$ is non-maximal). We refer to the second zig-zag as the **Lubin-Tate reduction zig-zag** because it relates a general deformation ring to the deformation ring for a formal group over an algebraically closed field.

Proof sketch. Claim (i) is not hard to prove from the definitions of the associated deformation functors. The flatness claims (ii) and (iii) follow from a more general result [1, 6.1.2] which states that, whenever $R_0 \to R_0'$ has the property that Ω_{R_0'/R_0} vanishes, then we get an induced map $R^{\mathrm{un}}(\mathbf{G}_0) \to R^{\mathrm{un}}((\mathbf{G}_0)_{R_0'})$ which is flat. Ultimately this boils down to a general criteria for detecting flatness of a map between p-complete, Noetherian $\mathbb{E}_\infty$-rings in terms of the relative cotangent complex [1, 6.1.8,3.5.5]. This, in turn, eventually reduces to the claim [1, 3.5.4] that, if R is a Noetherian $\mathbb{E}_\infty$-ring over a field k and $L_{R/k} = 0$, then R is discrete and regular. That claim is proved by comparison with the algebraic cotangent complex where an explicit calculation can be made.

Finally, claim (iv) rests on an understanding of the relationship between deforming p-divisible groups and deforming their associated connected component [1, 6.2.4]. The intuition is that finite étale group schemes do not deform, so a deformation of a p-divisible group should be controlled by deforming its connected component together with some an extension class. The extension class parameter is 'free', by an explicit computation of Ext groups, so the ring governing deformations of the p-divisible group is flat over the ring governing deformations of its connected component. One should compare Example 3 to see this in action. □

With these tools in hand, we can embark on a proof of Theorem 2.

Recall that if X is a one-dimensional, pointed hyperplane over an $\mathbb{E}_\infty$-ring R, we say that it is **balanced** if the orientation classifier $\mathrm{Or}(X)$ has no odd homotopy and the map $R \to \mathrm{Or}(X)$ is an isomorphism on π_0. Here is an outline of the proof of Theorem 2:

- Step 1: Reduce to the case $R_0 = k$ is a perfect field and $\mathbf{G}_0$ is connected.
- Step 2: In that case, show by induction on the height that, for every non-maximal ideal $\mathfrak{p} \subseteq R^{\mathrm{cl}}(\mathbf{G}_0)$ with residue field of characteristic p, the hyperplane $(\mathbf{G}^\circ)_\mathfrak{p}$ is balanced.
- Step 3: Do the same in the case that $k(\mathfrak{p})$ has characteristic zero.

Proof that these steps imply the theorem. By Step 1 we are reduced to the case when $R_0 = k$ is a perfect field and $\mathbf{G}_0$ is connected. We have already seen that the completion $R^{\mathrm{or}}(\mathbf{G}_0)^\wedge_{I_n}$ has even homotopy and the correct π_0, where I_n is the Landweber ideal. Thus, $\mathbf{G}_0$ is balanced if and only if the map $R^{\mathrm{or}}(\mathbf{G}_0) \to R^{\mathrm{or}}(\mathbf{G}_0)^\wedge_{I_n}$ is an equivalence. It suffices to check this is an equivalence after localization at each non-maximal ideal $\mathfrak{p} \subseteq R^{\mathrm{cl}}(\mathbf{G}_0)$, since the fiber of the map is I_n-nilpotent and I_n is the maximal ideal in $R^{\mathrm{cl}}(\mathbf{G}_0)$. This, in turn, is equivalent to showing that $(\mathbf{G}^\circ)_\mathfrak{p}$ is balanced for all non-maximal primes, which is the content of Steps 2 and 3. □

Proof of Step 1. A hyperplane is balanced if and only if it becomes balanced after localizing at every maximal ideal, and, moreover, the property of being balanced is stable under flat extension of scalars. So this step follows immediately from the Lubin-Tate reduction zig-zag applied to each maximal ideal in R_0. □

Proof of Step 2. Argue by induction on n, and use the height reduction zig-zag and the comments about balanced hyperplanes in the previous proof. □

Step 3 is more complicated, and breaks up into two further steps:

Step 3a**:** Every hyperplane over a *discrete* $\mathbb{Q}$-algebra R is balanced [1, 6.4.4].
Step 3b**:** $R^{\mathrm{un}}(\mathbf{G}_0)[p^{-1}]$ is discrete [1, 6.3.1].

The first of these is a pleasant exercise.

Proof of Step 3a. The question is local on R so we may assume $X = \mathrm{Spf}(R[\![t]\!])$. (Recall that, in general, the $\mathbb{E}_\infty$-rings of functions on formal hyperplanes can be quite complicated, but in characteristic zero things are much nicer). Now, the preorientation classifer can be computed in several steps:

$$R[\![t]\!] \rightsquigarrow R \otimes_{R[\![t]\!]} R = R \wedge S^1_+ \rightsquigarrow R \otimes_{R \wedge S^1_+} R = R \wedge \mathbb{C}P^\infty_+$$

In general we know that $R_*\mathbb{C}P^\infty \simeq \Gamma\{\beta\}$ is a divided power algebra on the class corresponding to $[\mathbb{C}P^1]$, i.e. the Bott element. In characteristic zero, this is the same as a polynomial algebra, so we deduce that the orientation classifier has homotopy $R[\beta^{\pm 1}]$ as desired. □

The last step is harder.

Proof sketch for Step 3b. First, using the Lubin-Tate reduction zig-zag, one reduces to the case $R_0 = k$ is algebraically closed of characteristic p and $\mathbf{G}_0$ is connected. Now we examine the support K of the module $\pi_k R^{\mathrm{un}}(\mathbf{G}_0)$, where $k \neq 0$. We want to show that K is contained in the locus $\{p = 0\}$. The idea is roughly that there are two possibilities: either K only intersects $\{p = 0\}$ at the origin, or $K \cap \{p = 0\}$ has a non-closed point x. We can rule out the latter possibility using induction on height and the height-reducing zig-zag to relate the problem to the Lubin-Tate ring of a formal group of smaller height over the residue field $k(x)$. In the former case, we now have a bizarre subscheme of Lubin-Tate space which is preserved under the action of the Morava stabilizer group $\Gamma = \mathrm{Aut}(\mathbf{G}_0)$. Now, restrict attention to an irreducible component K' of K with generic point the prime ideal $\mathfrak{p}$, and to the finite index subgroup Γ' of Γ which fixes this component. This subgroup now acts faithfully on $\mathbf{G}_{R^{\mathrm{cl}}(\mathbf{G}_0)/\mathfrak{p}}$, and still acts faithfully after inverting p. Finally, this produces a faithful action on the Lie algebra, i.e. an injection $\Gamma' \to (R^{\mathrm{cl}}(\mathbf{G}_0)/\mathfrak{p})[p^{-1}]^\times$. But there are no finite index abelian subgroups of the Morava stabilizer group at heights greater than one, so we get a contradiction. □

This completes the sketch of the proof of Theorem 2.

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Spectral Elliptic Curves

Andrew Senger

The goal of this talk is to set up the spectral algebraic preliminaries for the construction of elliptic cohomology. We will describe the construction of the moduli stack of (strict) spectral elliptic curves and state the Serre-Tate theorem for spectral elliptic curves.

Elliptic cohomology will be constructed as the structure sheaf on the moduli stack of oriented spectral elliptic curves. To see that this gives the desired result, one must show that the formal group associated to the universal elliptic curve is balanced in the sense of Definition 6.4.1 of [2]. The Serre-Tate theorem for spectral elliptic curves reduces this to the main result of Dylan's talk.

1. Definitions in Spectral Algebraic Geometry

We begin with a series of definitions leading up to the definition of a spectral Deligne-Mumford stacks. Instead of defining ∞-topoi in their full generality, we will restrict ourselves to the case of 1-localic ∞-topoi, which correspond to spectral Deligne-Mumford 1-stacks. All of the spectral Deligne-Mumford stacks that we will need in this talk will be spectral Deligne-Mumford 1-stacks.

Definition 1. Given a site S and an ∞-category $\mathcal{C}$, we define the category of sheaves on S with values in $\mathcal{C}$ to be the full subcategory $\mathrm{Shv}_{\mathcal{C}}(S) \subset \mathrm{Fun}(S^{op}, \mathcal{C})$ consisting of those functors F which satisfy the following condition: given any cover $U \to X$ in S, the augmented cosimplicial diagram

$$F(X) \longrightarrow F(U) \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} F(U \times_X U) \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} F(U \times_X U \times_X U) \cdots$$

produced by applying F to the augmented Čech nerve of $U \to X$ is a limit diagram in $\mathcal{C}$.

Remark 2. In the case that $\mathcal{C}$ is a 1-category, then the above definition recovers the classical definition of a sheaf on a site.

We let $\mathcal{S}$ denote the ∞-category of spaces.

Definition 3. A 1-localic ∞-topos is an ∞-category of the form $\mathrm{Shv}_{\mathcal{S}}(S)$ for some site S.

Definition 4. Given ∞-topoi $\mathcal{X}$ and $\mathcal{Y}$, a geometric morphism

$$f_* : \mathcal{X} \to \mathcal{Y}$$

of ∞-topoi is a functor f_* which admits a left adjoint f^* that preserves finite limits.

Definition 5. Let $\mathcal{C}$ denote an ∞-category with limits and let $\mathcal{X}$ be an ∞-topos. Then we define $\mathrm{Shv}_{\mathcal{C}}(\mathcal{X})$ to be the full subcategory of $\mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C})$ consisting of limit-preserving functors.

Remark 6. When $\mathcal{X} = \mathrm{Shv}_{\mathcal{S}}(S)$ for a site S, then $\mathrm{Shv}_{\mathcal{C}}(\mathcal{X}) \cong \mathrm{Shv}_{\mathcal{C}}(S)$.

Definition 7. A spectrally ringed ∞-topos is a pair $X = (\mathcal{X}, \mathcal{O}_X)$ where $\mathcal{X}$ is an ∞-topos and $\mathcal{O}_X$ is a sheaf of E_∞-rings on $\mathcal{X}$. A morphism $X \to Y$ of spectrally ringed ∞-topoi is a geometric morphism $f_* : \mathcal{X} \to \mathcal{Y}$ and a morphism of sheaves of E_∞ rings $f^*\mathcal{O}_Y \to \mathcal{O}_X$.

Definition 8. Given an E_∞-ring R, we set

$$\operatorname{Spec} R = (\operatorname{Shv}_{\mathcal{S}}(R_{\mathrm{et}}), \mathcal{O}_R)$$

where R_{et} is the étale site of R and $\mathcal{O}_R$ is the tautological sheaf of E_∞ rings on R_{et} which on an étale algebra S takes value S: $\mathcal{O}_R(S) = S$.

Remark 9. While R_{et} is a priori an ∞-site, the invariance of the étale site implies that $R_{\mathrm{et}} \cong (\pi_0 R)_{\mathrm{et}}$ and is in particular a 1-category.

Definition 10. Let $\mathcal{X}$ be an ∞-topos. Then a collection of morphisms $\{U_i \to X\}_{i \in I}$ is a covering of X if the augmented Čech nerve

$$\cdots \begin{smallmatrix} \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{smallmatrix} \coprod_{(i_1,i_2) \in I \times I} U_{i_1} \times_X U_{i_2} \begin{smallmatrix} \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{smallmatrix} \coprod_{i \in I} U_i \longrightarrow X$$

is a colimit diagram.

Definition 11. A spectral Deligne-Mumford stack is a spectrally ringed ∞-topos $X = (\mathcal{X}, \mathcal{O}_X)$ for which there exists a covering $\{U_i \to \star\}_{i \in I}$ for which $(X_{/U_i}, \mathcal{O}_X|_{U_i}) \cong \operatorname{Spec} R_i$[1] for some R_i.

A morphism $X \to Y$ of spectral Deligne-Mumford stacks is a morphism of spectrally ringed ∞-topoi which satisfies the following condition: for each point p of $\mathcal{X}$, the map $\pi_0 \mathcal{O}_{Y, f(p)} \to \pi_0 \mathcal{O}_{X,p}$ is a local homomorphism of local rings.

Fact 1. *The functor* $\operatorname{Spec} : \mathrm{CAlg} \to \mathrm{SpDM}$ *is fully faithful.*

Fact 2. *The functor* $\mathrm{SpDM} \to \operatorname{Fun}(\mathrm{CAlg}, \mathcal{S})$ *which takes a spectral Deligne-Mumford stack its functor of points is fully faithful.*

Definition 12. A spectral Deligne-Mumford 1-stack is a spectral DeligneMumford stack $X = (\mathcal{X}, \mathcal{O}_X)$ for which $\mathcal{X}$ is 1-localic. This is equivalent to asking that $X(R) = \operatorname{Map}(\operatorname{Spec} R, X)$ is equivalent to a 1-groupoid for all discrete commutative rings R.

Definition 13. We say that a spectral Deligne-Mumford stack X is a spectral algebraic space if $X(R)$ is equivalent to a set for every discrete commutative ring R.

Definition 14. Given a spectral Deligne-Mumford stack $X = (\mathcal{X}, \mathcal{O}_X)$, we define the underlying Deligne-Mumford stack of X to be $X_0 = (\mathcal{X}^\heartsuit, \pi_0 \mathcal{O}_X)$, where $\mathcal{X}^\heartsuit$ is the underlying topos of $\mathcal{X}$.

Remark 15. In the case that $\mathcal{X} = \operatorname{Shv}_{\mathcal{S}}(S)$ is 1-localic, $\mathcal{X}^\heartsuit = \operatorname{Shv}_{\mathrm{Sets}}(S)$.

[1] $\mathcal{O}_X|_{U_i}$ is defined to be the composition $\mathcal{X}^{op}_{/U_i} \to \mathcal{X}^{op} \to \mathrm{CAlg}$.

2. Spectral Elliptic Curves

We will define spectral elliptic curves to be spectral abelian varieties of dimension one.

Warning 16. They are **not** spectral genus 1 curves with a marked point: there is no guarantee that a product structure will exist on such a curve, nor will such a product structure be uniquely determined if it does exist.

In general, one needs to be careful: there are several notable differences between the theory of spectral abelian varieties and the classical theory of abelian varieties. For example, the dual spectral abelian variety $A^\vee$ doesn't classify line bundles on A. Moreover, spectral elliptic curves are not canonically self-dual, though oriented spectral elliptic curves are.

In the following, let R denote an E_∞ ring.

Definition 17. A variety over R is a morphism

$$X \to \operatorname{Spec} R$$

of spectral algebraic spaces which is flat and locally almost of finite presentation such that

$$X_0 \to \operatorname{Spec} \pi_0 R$$

is proper, geometrically connected and geometrically reduced. We let $\mathrm{Var}(R)$ denote the ∞-category of varieties over R.

In the above definition, we say that a morphism of spectral Deligne-Mumford stacks $X \to \operatorname{Spec} R$ is flat (resp. locally almost of finite presentation) if and only if for one (and hence every) étale covering $\coprod_i \operatorname{Spec} S_i \to X$, the maps $R \to S_i$ are flat (resp. almost of finite presentation).

Definition 18. A strict abelian variety over a R is an abelian group object of $\mathrm{Var}(R)$:

$$\mathrm{AVar}^s(R) = \mathrm{Ab}(\mathrm{Var}(R)) = \mathrm{Fun}^\times(\mathrm{Lat}^{op}, \mathrm{Var}(R)).$$

Remark 19. When R is discrete, the category of strict abelian varieties over R is equivalent to the category of abelian schemes over $\operatorname{Spec} R$. This is not a priori obvious, since in the above definition we only require A to be an abelian algebraic space over $\operatorname{Spec} R$. However, it is a theorem due to Raynaud that every abelian algebraic space is equivalent to an abelian scheme.

Definition 20. A strict elliptic curve over R is a strict abelian variety A over R such that its underlying abelian scheme A_0 over $\pi_0 R$ is of relative dimension one.

We let $\mathrm{Ell}^s(R)$ denote the full subcategory of $\mathrm{AVar}^s(R)$ spanned by the strict elliptic curves.

3. The Moduli Stack of Spectral Elliptic Curves

Theorem 21 (Theorem 2.4.1 in [1])**.** *The functor*

$$\mathrm{Ell}^s(-)^{\simeq} : \mathrm{CAlg} \to \mathcal{S}$$

is representable by a Deligne-Mumford 1*-stack* M^s_{ell}*.*

This may be proved by invoking the following theorem, which proof is similar to that of the representability theorem proved in Lukas's talk.

Theorem 22 (Theorem 18.1.0.2 in [3])**.** *A functor* $X : \mathrm{CAlg}^{cn} \to \mathcal{S}$ *is representable by a connective Deligne-Mumford* 1*-stack if and only if*

(1) $X|_{\mathrm{CAlg}^{\heartsuit}}$ *is representable by a classical Deligne-Mumford stack.*
(2) X *is nilcomplete.*
(3) X *is infinitesimally cohesive.*
(4) X *admits a cotangent complex.*

Since (1) is classical and may be proved, for example, by working explicitly with Weierstrass equations for elliptic curves, to prove Theorem 21 we just need to verify (2), (3) and (4) by studying the deformation theory of strict elliptic curves.

In fact, the key point is to study the deformation theory of (spectral) varieties: it is not very difficult to get to the deformation theory of strict elliptic curves (or, more generally, strict abelian varieties) from there.

Unfortunately, it is outside the scope of this talk to get into the details of the proof of (2), (3) and (4) for $X = \mathrm{Var}(-)$. Nevertheless, we will at least interpret what they mean.

Condition (2) just comes down to the fact that a spectral Deligne-Mumford stack $(\mathcal{X}, \mathcal{O}_X)$ with $\mathcal{O}_X$ connectie is the limit of the truncated spectral Deligne-Mumford stacks $(\mathcal{X}, \tau_{\leq n}\mathcal{O}_X)$. This may be checked locally, and so comes down to the convergence of Postnikov towers for connective E_∞ rings.

Condition (3) follows from the fact that varieties may be glued along closed subvarieties: this is Theorem 16.3.0.1 and Proposition 19.4.2.1 of [3].

Condition (4) comes down to two things: the existence of cotangent complexes for spectral Deligne-Mumford stacks, which is easy, and Grothendieck duality for varieties, which is Chapter 6 of [3].

4. The Serre-Tate Theorem

Finally, we wish to conclude by stating the Serre-Tate theorem for strict abelian varieties.

Let A denote a strict abelian variety. Then we define

$$A[p^n] = \mathrm{fib}(A \xrightarrow{p^n} A).$$

As n varies, these fit together into a p-divisible group $A[p^\infty]$.

It is a fact of the classical theory of abelian schemes that if A is of dimension g, then $A[p^\infty]$ has height $2g$.

This construction defines a natural transformation

$$\mathrm{AVar}_g^s(R) \to \mathrm{BT}_{2g}^p(R).$$

The spectral Serre-Tate theorem then states that in the p-complete context the deformation theory of a strict abelian variety is equivalent to that of its associated p-divisible group.

Theorem 23 (Theorem 7.0.1 of [1]). *Let R be a connective E_∞ ring and let $\overline{R}$ denote a square-zero extension of R by a p-complete connective R-module. Then the following square is a pullback of ∞-categories:*

$$\begin{tikzcd} \mathrm{AVar}_g^s(\overline{R}) \arrow[r] \arrow[d] & \mathrm{AVar}_g^s(R) \arrow[d] \\ \mathrm{BT}_{2g}^p(\overline{R}) \arrow[r] & \mathrm{BT}_{2g}^p(R). \end{tikzcd}$$

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Applications

Jonas McCandless

1. Introduction

The goal of this talk is to explain some applications of the machinery developed in [1] and [2]. Concretely, we will show that Snaith's theorem on the structure of KU is a formal consequence of classical Bott periodicity combined with the observation that the formal multiplicative group over the sphere spectrum is balanced. Secondly, we show that the formal completion of a strict elliptic curve over a connective $\mathbb{E}_\infty$-ring is balanced. Consequently, we deduce the Goerss–Hopkins–Miller theorem using the methods of [1, 2] bypassing complicated obstruction theoretic arguments. We will rely heavily on the following result of [2] which we recall for convenience.

Theorem 1 ([2, Theorem 6.4.6]). *Let $\mathbb{G}_0$ be a non-stationary p-divisible group of dimension 1 over an F-finite Noetherian $\mathbb{F}_p$-algebra R_0, and let $\mathbb{G}$ denote its universal deformation. Then the identity component of $\mathbb{G}$ is a balanced formal group over the spectral deformation ring of $\mathbb{G}_0$.*

2. Snaith's theorem on the structure of KU

The connective complex K-theory spectrum ku admits the structure of an $\mathbb{E}_\infty$-ring and there exists a map of $\mathbb{E}_\infty$-rings $\Sigma^\infty_+(\mathbf{CP}^\infty) \to \mathrm{ku}$. The composite map

$$S^2 \simeq \mathbf{CP}^1 \hookrightarrow \mathbf{CP}^\infty \to \Omega^\infty \Sigma^\infty_+ \mathbf{CP}^\infty$$

determines an element $\beta \in \pi_2(\Sigma^\infty_+(\mathbf{CP}^\infty))$ which we refer to as the Bott element. By inverting the Bott element we obtain a commutative square of $\mathbb{E}_\infty$-rings

$$\begin{tikzcd} \Sigma^\infty_+(\mathbf{CP}^\infty) \arrow[r] \arrow[d] & \mathrm{ku} \arrow[d] \\ \Sigma^\infty_+(\mathbf{CP}^\infty)[\beta^{-1}] \arrow[r] & \mathrm{KU}, \end{tikzcd}$$

where KU denotes the periodic complex K-theory spectrum. We have the following result.

Theorem 2 (Snaith, [4])**.** *The map $\Sigma^\infty_+(\mathbf{CP}^\infty)[\beta^{-1}] \to \mathrm{KU}$ is an equivalence of $\mathbb{E}_\infty$-rings.*

The first goal of this talk will be to show that Theorem 2 is a formal consequence of the classical Bott periodicity theorem combined with the observation that the formal multiplicative group over the sphere spectrum is a balanced. The latter will be a consequence of Theorem 1 above. Let us recall the definition of a balanced formal group.

Definition 3 ([2, Definition 6.4.1])**.** Let R be a connective $\mathbb{E}_\infty$-ring. Let X be a 1-dimensional pointed formal hyperplane over R, and let $\mathfrak{O}_X$ denote the orientation classifier of X ([2, Definition 4.3.14]). We will say that X is balanced if the following two conditions are satisfied:

(1) The map of $\mathbb{E}_\infty$-rings $R \to \mathfrak{O}_X$ which exhibits the orientation classifier $\mathfrak{O}_X$ of X as an $\mathbb{E}_\infty$-algebra over R induces an isomorphism of commutative rings $\pi_0(R) \to \pi_0(\mathfrak{O}_X)$.
(2) The homotopy groups of $\mathfrak{O}_X$ are concentrated in even degrees.

A 1-dimensional formal group $\widehat{\mathbb{G}}$ over R is balanced if the underlying formal hyperplane with basepoint given by the identity section of $\widehat{\mathbb{G}}$ is balanced.

Remark 4. Let $f\colon R \to R'$ be a morphism of connective $\mathbb{E}_\infty$-ring, and let X be a 1-dimensional pointed formal hyperplane over R. Let X' be the pointed formal hyperplane over R' obtained from X by extension of scalars along f. It follows that

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d] & R' \arrow[d] \\ \mathfrak{O}_X \arrow[r] & \mathfrak{O}_{X'} \end{tikzcd}$$

is a pushout square of $\mathbb{E}_\infty$-rings. An immediate consequence of this is the following two assertions:

(1) If f is flat and X is balanced, then X' is balanced.
(2) If f is faithfully flat and X' is balanced, then X is balanced.

In particular, if $\mathfrak{m}$ is a maximal ideal of $\pi_0(R)$, then the canonical map of $\mathbb{E}_\infty$-rings $R \to R_\mathfrak{m}$ is faithfully flat. It follows that X is balanced if $X_\mathfrak{m}$ is balanced for every maximal $\mathfrak{m}$ of $\pi_0 R$, where $X_\mathfrak{m}$ denotes the pointed formal hyperplane over the localization $R_\mathfrak{m}$.

Proposition 5 ([2, Proposition 6.5.2])**.** *The formal multiplicative group $\widehat{\mathbb{G}}_m$ is balanced over the sphere spectrum.*

Proof. Let p be a prime number. It suffices to show that $\widehat{\mathbb{G}}_m$ is balanced over the p-local sphere $S_{(p)}$ by Remark 4. Since the canonical map of $\mathbb{E}_\infty$-rings $S_{(p)} \to S^\wedge_{(p)}$ is faithfully flat, it suffices to show that $\widehat{\mathbb{G}}_m$ is balanced over the (p)-complete sphere spectrum by Remark 4. Recall that the (p)-complete sphere spectrum is equivalent to the spectral deformation ring of the p-divisible group μ_{p^∞} over $\mathbb{F}_p$ by ([2, Corollary 3.1.19]). Moreover, it follows from ([2, Proposition 2.2.12]) that the identity component of the p-divisible group μ_{p^∞} over $S^\wedge_{(p)}$ is equivalent to $\widehat{\mathbb{G}}_m$. Thus, the formal multiplicative group $\mathbb{G}_m$ is balanced over the p-complete sphere spectrum by virtue of Theorem 1. □

Proof of Theorem 2. It follows from the classical Bott periodicity theorem that inverting the Bott element β gives an isomorphism of graded rings $\mathbb{Z}[\beta^{\pm 1}] \to \pi_*(\mathrm{KU})$. The localization $\Sigma^\infty_+ \mathbf{CP}^\infty[\beta^{-1}]$ is an orientation classifier of the formal multiplicative group $\widehat{\mathbb{G}}_m$ over the sphere spectrum ([2, Corollary 4.3.27]). There is an isomorphism of graded commutative rings $\mathbb{Z}[\beta^{\pm 1}] \to \pi_*(\Sigma^\infty_+(\mathbf{CP}^\infty)[\beta^{-1}])$ since $\widehat{\mathbb{G}}_0$ is balanced over the sphere spectrum by Proposition 5. Finally, there is a commutative diagram

$$\begin{tikzcd} \mathbb{Z}[\beta^{\pm 1}] \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \pi_*(\mathrm{KU}) \\ \pi_*(\Sigma^\infty_+(\mathbf{CP}^\infty)[\beta^{-1}]) \arrow[ur] & \end{tikzcd}$$

which shows the wanted. □

3. Elliptic cohomology and topological modular forms

The goal of the second part of this talk is to reprove the Goerss–Hopkins–Miller theorem using the machinery developed in [1] and [2]. The formulation of this result will require some preliminaries.

Remark 6. Let $\mathcal{U}$ denote the category whose objects are pairs (R, E), where R is a commutative ring and E is an elliptic curve over R which is classified by an étale morphism $\mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{Ell}}$. A morphism from (R, E) to (R', E') in $\mathcal{U}$ is given by a pullback diagram of schemes

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & E' \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] & \mathrm{Spec}(R') \end{tikzcd}$$

with the property that f induces an isomorphism $E \to \mathrm{Spec}(R) \times_{\mathrm{Spec}(R')} E'$ of elliptic curves over R. The underlying étale topos of the Deligne–Mumford stack $\mathcal{M}_{\mathrm{Ell}}$ is equivalent to the category of Set-valued sheaves on the category $\mathcal{U}$. We equip the category $\mathcal{U}$ with the structure of a site whose coverings are given by the étale coverings. The structure sheaf $\mathcal{O}_{\mathcal{M}_{\mathrm{Ell}}}$ of the moduli stack of elliptic curves can be viewed as a sheaf of commutative rings on the category $\mathcal{U}$ given by $\mathcal{O}_{\mathcal{M}_{\mathrm{Ell}}}(R, E) = R$.

Remark 7. Let E be an elliptic curve over a commutative ring R classified by an étale morphism $\mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{Ell}}$. In this case we can extract a 1-dimensional formal group $\widehat{E}$ by formally completing E along its identity section and one can show that the formal group $\widehat{E}$ is Landweber exact. Consequently, there exists an essentially unique even periodic homotopy commutative ring spectrum A_E which is characterized by $R \simeq \pi_0(A_E)$ and $\widehat{E} \simeq \mathrm{Spf} A^0_E(\mathbf{CP}^\infty)$. Moreover, the construction $(R, E) \mapsto A_E$ determines a presheaf

$$\mathcal{O}^h_{\mathcal{M}_{\mathrm{Ell}}} \colon \mathcal{U}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{hSp}).$$

which is a refinement of the structure sheaf $\mathcal{O}_{\mathcal{M}_{\mathrm{Ell}}}$ of the moduli stack of elliptic curves in the sense that $\mathcal{O}_{\mathcal{M}_{\mathrm{Ell}}} = \pi_0(\mathcal{O}^h_{\mathcal{M}_{\mathrm{Ell}}})$.

Theorem 8 (Goerss–Hopkins–Miller)**.** *There exists a functor* $\mathcal{O}^{\mathrm{top}}_{\mathcal{M}_{\mathrm{Ell}}} \colon \mathcal{U}^{\mathrm{op}} \to \mathrm{CAlg}$ *such that the following diagram of* ∞*-categories commutes*

$$\begin{tikzcd} & \mathrm{CAlg}(\mathrm{Sp}) \arrow[d] \\ \mathcal{U}^{\mathrm{op}} \arrow[ur, dashed, "\mathcal{O}^{\mathrm{top}}_{\mathcal{M}_{\mathrm{Ell}}}"] \arrow[r, "\mathcal{O}^h_{\mathcal{M}_{\mathrm{Ell}}}"'] & \mathrm{CAlg}(\mathrm{hSp}) \end{tikzcd}$$

Moreover, the functor $\mathcal{O}^{\mathrm{top}}_{\mathcal{M}_{\mathrm{Ell}}}$ *is a* CAlg*-valued sheaf with respect to the étale topology on* $\mathcal{U}$*.*

Definition 9. Let TMF denote the $\mathbb{E}_\infty$-ring of global sections of the sheaf $\mathcal{O}^{\mathrm{top}}_{\mathcal{M}_{\mathrm{Ell}}} \colon \mathcal{U}^{\mathrm{op}} \to \mathrm{CAlg}$ concretely given by the formula

$$\mathrm{TMF} \simeq \varprojlim_{(R,E)\in\mathcal{U}} \mathcal{O}^{\mathrm{top}}_{\mathcal{M}_{\mathrm{Ell}}}(R, E).$$

The rest of this talk will be devoted to the proof of Theorem 8. We will start by showing that every strict elliptic curve X over an $\mathbb{E}_\infty$-ring admits a formal completion $\widehat{\mathsf{X}}$ which is a 1-dimensional formal group over R. Next, we construct the moduli stack of oriented elliptic curves. This is a variant of the moduli stack of strict elliptic curves which classifies strict elliptic curves X together with an orientation of the formal completion $\widehat{\mathsf{X}}$. The structure sheaf of the moduli stack of oriented elliptic curves determines a CAlg-valued sheaf $\mathcal{O}^{\mathrm{top}}_{\mathcal{M}_{\mathrm{Ell}}}$ on the category $\mathcal{U}$. We will show that the underlying presheaf of homotopy commutative ring spectra agrees with $\mathcal{O}^{h}_{\mathcal{M}_{\mathrm{Ell}}}$. We will deduce this from the Serre–Tate theorem ([1, Theorem 7.0.1]) and Theorem 1 above.

Definition 10 ([2, Construction 7.1.1])**.** Let R be an $\mathbb{E}_\infty$-ring and let X be a strict elliptic curve over R which we identity with its functor of points $\mathsf{X}\colon \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geq 0}R} \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}}$. The formal completion of X is the functor $\widehat{\mathsf{X}}\colon \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geq 0}R} \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}}$ determined by the construction

$$A \mapsto \mathrm{fib}(\mathsf{X}(A) \to \mathsf{X}(A^{\mathrm{red}})).$$

Remark 11. If X is a strict elliptic curve over an $\mathbb{E}_\infty$-ring, then the formal completion of X is a formal group over R. See [2, Proposition 7.1.2] for a proof of this fact.

We will need to adapt the notion of (pre)orientations to the setting of strict elliptic curves.

Definition 12. Let X be a strict elliptic curve over an $\mathbb{E}_\infty$-ring R.

(1) A preorientation of X is a preorientation of the formal group $\widehat{\mathsf{X}}$.
(2) An orientation of X is an orientation of the formal group $\widehat{\mathsf{X}}$.

Let $\mathrm{Pre}(\mathsf{X}) = \mathrm{Pre}(\widehat{\mathsf{X}})$ denote the space of preorientations of X.

Construction 13. Let X be a strict elliptic curve over an an $\mathbb{E}_\infty$-ring R. The construction $\mathsf{X} \mapsto \mathrm{Pre}(\mathsf{X})$ determines a functor $\mathrm{Ell}^s(R) \to \mathcal{S}$ which is classified by a left fibration of ∞-categories $\mathrm{Ell}^{\mathrm{pre}}(R) \to \mathrm{Ell}^s(R)$. We will refer to $\mathrm{Ell}^{\mathrm{pre}}(R)$ as the ∞-category of preoriented elliptic curves over R. Note that an object of $\mathrm{Ell}^{\mathrm{pre}}(R)$ can be identified with a pair (X, e), where X is a strict elliptic curve over R and e is a preorientation of X. We let $\mathrm{Ell}^{\mathrm{or}}(R)$ denote the full subcategory of $\mathrm{Ell}^{\mathrm{pre}}(R)$ spanned by those pairs (X, e) where e is an orientation of X. We will refer to $\mathrm{Ell}^{\mathrm{or}}(R)$ as the ∞-category of oriented elliptic curves.

Remark 14. If R is an $\mathbb{E}_\infty$-ring, then the functor determined by the construction $R \mapsto \mathrm{Ell}^{\mathrm{or}}(R)^{\simeq}$ is representable by a nonconnective spectral Deligne–Mumford stack $\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}$. Moreover, the canonical map of nonconnective spectral Deligne–Mumford stacks $\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}} \to \mathcal{M}^{s}_{\mathrm{Ell}}$ is affine. See [2, Proposition 7.2.10] for details.

The proof of Theorem 8 will be a consequence of the following result.

Theorem 15 ([2, Theorem 7.3.1])**.** *Let R be a connective $\mathbb{E}_\infty$-ring. If X is a strict elliptic curve over R classified by an étale morphism $\mathrm{Spec}(R) \to \mathcal{M}^{s}_{\mathrm{Ell}}$, then the formal completion $\widehat{\mathsf{X}}$ of X is a balanced formal group over R.*

Proof of Theorem 8 from Theorem 15. We regard the moduli stack of strict elliptic curves $\mathcal{M}^{s}_{\mathrm{Ell}}$ as a spectral Deligne–Mumford stack with underlying ∞-topos

$\mathcal{X}$ and structure sheaf $\mathcal{O}_{\mathcal{M}^s_{\mathrm{Ell}}}$. The underlying 0-truncated spectral Deligne–Mumford stack $(\tau_{\leq 0}\mathcal{X}, \pi_0(\mathcal{O}_{\mathcal{M}^s_{\mathrm{Ell}}}))$ is equivalent to the classical moduli stack of elliptic curves $\mathcal{M}_{\mathrm{Ell}}$. It follows that the category $\mathcal{U}$ is equivalent to the full subcategory of $\mathcal{X}$ spanned by the affine objects by virtue of ([3, Corollary 1.4.7.3]). Let $\phi\colon \mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}} \to \mathcal{M}^s_{\mathrm{Ell}}$ denote the canonical map of nonconnective spectral Deligne–Mumford stacks. The pushforward $\phi_*\mathcal{O}_{\mathcal{M}^{\mathrm{or}}_{\mathrm{Ell}}}$ is a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, which defines a functor $\mathcal{O}^{\mathrm{top}}_{\mathcal{M}_{\mathrm{Ell}}}\colon \mathcal{U}^{\mathrm{op}} \to \mathrm{CAlg}$ which is a sheaf for the étale topology on $\mathcal{U}$ since ϕ is affine. We will show that the underlying presheaf of homotopy commutative ring spectra coincides with the presheaf described in Remark 7. Let U be an object of $\mathcal{U}$ given by a commutative ring $R = \mathcal{O}_{\mathcal{M}_{\mathrm{Ell}}}(U)$ together with an elliptic curve X over R classified by an étale morphism $f\colon \mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{Ell}}$. Set $R' = \mathcal{O}_{\mathcal{M}^s_{\mathrm{Ell}}}(U)$ and note that R' is a connective $\mathbb{E}_\infty$-ring with $\pi_0(R') \simeq R$ equipped with an étale map $f'\colon \mathrm{Spec}(R') \to \mathcal{M}^s_{\mathrm{Ell}}$ classifying a lift of X to a strict elliptic curve X' over R'. In other words, the elliptic curve X is obtained by extension of scalars of X' along the canonical map $R' \to \tau_{\leq 0}R' \simeq R$. Set $A = \mathcal{O}^{\mathrm{top}}_{\mathcal{M}_{\mathrm{Ell}}}(U)$ and note that A is an orientation classifier of $\widehat{\mathsf{X}'}$. It follows from [Proposition 4.3.23] that A is complex periodic and its Quillen formal group $\widehat{\mathbb{G}}^{\mathcal{Q}}_A$ is equivalent to the formal completion of X'_A. Consequently, the classical Quillen formal group $\widehat{\mathbb{G}}^{\mathcal{Q}_0}_A$ is obtained from the formal completion $\widehat{\mathsf{X}}$ of X be extending scalars along the composite $R \simeq \pi_0(R') \to \pi_0(A)$ where the map $\pi_0(R') \to \pi_0(A)$ is induced by the $\mathbb{E}_\infty$-R'-algebra structure of A. We will complete the proof by showing that $\pi_0(R') \to \pi_0(A)$ is an isomorphism and that the homotopy groups of A are concentrated in even degrees. This follows readily by Definition 3 since Theorem 15 ensures that $\widehat{\mathsf{X}'}$ is balanced. □

The remainder of this talk will be devoted to proving Theorem 15 which will be an immediate consequence of the following result.

Proposition 16 ([2, Proposition 7.4.2])**.** *Let R be a connective $\mathbb{E}_\infty$-ring and let X be a strict elliptic curve over R classified by an étale morphism $\mathrm{Spec}(R) \to \mathcal{M}^s_{\mathrm{Ell}}$. Let $\mathfrak{m}$ be any maximal ideal of R. Then:*

(1) The residue field $\kappa = \pi_0(R)/\mathfrak{m}$ is finite.
(2) Let $\widehat{R}$ denote the completion of R with respect to the maximal ideal $\mathfrak{m}$ and let p be the characteristic of the residue field κ. Then the p-divisible group $\mathsf{X}[p^\infty]_{\widehat{R}}$ is a universal deformation of $\mathsf{X}[p^\infty]_\kappa$.

Proof. We first prove (1). The spectral Deligne–Mumford stack $\mathcal{M}^s_{\mathrm{Ell}}$ is locally almost of finite presentation over the sphere spectrum by virtue of ([1, Theorem 2.4.1]). It follows that the residue field κ is finite since the quotient $\pi_0(R)/\mathfrak{m}$ is a field which is a finitely generated $\mathbb{Z}$-algebra. We now prove (2). The completion $\widehat{R}$ of R with respect to the maximal ideal $\mathfrak{m}$ is a complete local Noetherian $\mathbb{E}_\infty$-ring with residue field κ, so the deformation $\mathsf{X}[p^\infty]_{\widehat{R}}$ of $\mathbf{G}_0$ is classified by a map of connective $\mathbb{E}_\infty$-rings $f\colon R^{\mathrm{un}}_{\mathbf{G}_0} \to \widehat{R}$ which is the identity on residue fields. We wish to show that f is an equivalence. It suffices to show that for every complete local Noetherian $\mathbb{E}_\infty$-ring A equipped with a map $\rho\colon A \to \kappa$ which exhibits κ as the residue field of A, composition with f induces an equivalence

$$\mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}(\widehat{R}, A) \to \mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}(R^{\mathrm{un}}_{\mathbf{G}_0}, A).$$

Using that A is a complete local Noetherian $\mathbb{E}_\infty$-ring we may assume that A is truncated and that $\pi_0(A)$ is Artinian. Hence, there exists a finite sequence of maps

$$A = A_m \to A_{m-1} \to \cdots \to A_0 = \kappa$$

such that $A_{i+1} \to A_i$ exhibits A_{i+1} as a square-zero extension of A_i by an almost perfect A_i-module for every i. Consequently, we have reduced to showing the following claim:

(∗) Let A be as above and suppose that $\widetilde{A}$ is a square-zero extension of A by a connective A-module which is almost perfect over A, then the following diagram is a pullback:

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}(\widehat{R}, \widetilde{A}) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}(R^{\mathrm{un}}_{\mathbf{G}_0}, \widetilde{A}) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}(\widehat{R}, A) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}(R^{\mathrm{un}}_{\mathbf{G}_0}, A) \end{array}$$

Let $\widetilde{\rho}$ denote the composite $\widetilde{A} \to A \xrightarrow{\rho} \kappa$. Combining [2, Theorem 3.0.11] and the universal property of completions, it suffices to show that the following commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}(R, \widetilde{A}) & \longrightarrow & \mathrm{Def}_{\mathbf{G}_0}(\widetilde{A}, \widetilde{\rho}) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}(R, A) & \longrightarrow & \mathrm{Def}_{\mathbf{G}_0}(A, \rho) \end{array}$$

is a pullback. The map $\mathrm{Map}_{\mathrm{CAlg}/\kappa}(R, \widetilde{A}) \to \mathrm{Def}_{\mathbb{G}_0}(\widetilde{A}, \widetilde{\rho})$ is equivalent to the following composite

$$\mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}(R, \widetilde{A}) \longrightarrow \mathrm{Ell}^s(\widetilde{A}) \times_{\mathrm{Ell}^s(\kappa)} \{\mathsf{X}_\kappa\} \longrightarrow \mathrm{Def}_{\mathbf{G}_0}(\widetilde{A}, \widetilde{\rho}),$$

where the first map is given by sending $\alpha\colon R \to \widetilde{A}$ to the strict elliptic curve $\mathsf{X}_{\widetilde{A}}$ obtained from X by extension of scalars along α. The second map is given by sending a strict elliptic curve X to the p-divisible group $\mathsf{X}[p^\infty]$. Consequently, we can identity the diagram above with the outer square in the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}(R, \widetilde{A}) & \longrightarrow & \mathrm{Ell}^s(\widetilde{A}) \times_{\mathrm{Ell}^s(\kappa)} \{\mathsf{X}_\kappa\} & \longrightarrow & \mathrm{Def}_{\mathbf{G}_0}(\widetilde{A}, \widetilde{\rho}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}(R, A) & \longrightarrow & \mathrm{Ell}^s(A) \times_{\mathrm{Ell}^s(\kappa)} \{\mathsf{X}_\kappa\} & \longrightarrow & \mathrm{Def}_{\mathbf{G}_0}(A, \rho) \end{array}$$

where the horizontal composites are given as described above. The left square in this diagram is a pullback since the strict elliptic curve X over R is classified by an étale morphism $\mathrm{Spec}(R) \to \mathcal{M}^s_{\mathrm{Ell}}$. The right square is a pullback by the Serre–Tate theorem ([1, Theorem 7.0.1]). This ends the proof. $\square$

Proof of Proposition 15. Let X be a strict elliptic curve over R classified by en étale morphism $f\colon \mathrm{Spec}(R) \to \mathcal{M}^s_{\mathrm{Ell}}$. By Remark 4 it suffices to show that the formal group $\widehat{\mathsf{X}}_{\mathfrak{m}}$ is balanced over the localization $R_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of $\pi_0(R)$. Let p be the characteristic of the residue field $\kappa = \pi_0(R)/\mathfrak{m}$ and let $\widehat{R}$ denote the completion of R with respect to the maximal ideal $\mathfrak{m}$ as in Proposition 16 above. Since $\widehat{R}$ is faithfully flat over $R_{\mathfrak{m}}$ it suffices to show that $\widehat{\mathsf{X}}_{\widehat{R}}$ is balanced over $\widehat{R}$ by Remark 4. We can identify $\widehat{\mathsf{X}}_{\widehat{R}}$ with the identity component of the p-divisible group $\mathsf{X}[p^\infty]_{\widehat{R}}$ by ([2, Proposition 7.4.1]) since $\widehat{R}$ is (p)-complete. Finally, since $\mathsf{X}[p^\infty]_{\widehat{R}}$ is a universal deformation of $\mathsf{X}[p^\infty]_\kappa$ by Proposition 16 we conclude the wanted by virtue of Theorem 1. $\square$

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