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Combinatorics, Probability and Computing

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Abstract. The main theme of this workshop was the use of probabilistic methods in combinatorics and theoretical computer science. Although these methods have been around for decades, they are being refined all the time: they are getting more and more sophisticated and powerful. Another theme was the study of random combinatorial structures, either for their own sake, or to tackle extremal questions. The workshop also emphasized connections between probabilistic combinatorics and discrete probability.

Mathematics Subject Classification (2010): 05xx, 60xx.

Introduction by the Organizers

The meeting was very well attended with 53 participants from around the globe, including the US, Israel, Canada, Brazil, and many European countries. In addition, many excellent mathematicians who would have loved to participate could not be invited, for lack of space. The program consisted of 7 main lectures, 20 shorter talks, and a problem session, with plenty of time for discussion.

The workshop started appropriately with a talk by Deryk Osthus on the *Resolution of the Oberwolfach Problem* (joint work with Stefan Glock, Felix Joos, Jaehoon Kim and Daniela Kühn). The Oberwolfach problem refers to a problem posed by Ringel in Oberwolfach in 1967 that generalizes the Oberwolfach meal problem: assume n people are to be seated around round tables for $\frac{n-1}{2}$ meals, where the total number of seats is equal to n, but the tables may have different sizes. Is it possible to find a seating chart such that every person sits next to any other person exactly once?

Another spectacular highlight of the workshop was the presentation of Annika Heckel who solved a long-standing problem about the (anti-)concentration of the chromatic number of a random graph: she sketched a proof that shows that $\chi(G_{n,1/2})$ is w.h.p. not concentrated in an interval of size $n^{1/4-\varepsilon}$. In addition, Peter Keevash presented a seminal new result on sharp thresholds and expanded hypergraphs. In particular, he and his co-authors Noam Lifshitz, Eoin Long and Dor Minzer prove a slight variant of the Kahn–Kalai conjecture that characterizes properties with coarse thresholds resp. Boolean functions of small influence.

Richard Montgomery presented a beautiful and surprisingly short proof that the random directed graph $D(n,p)$ almost surely contains a copy of every oriented n-vertex cycle, whenever $p = (\log n + \omega(1))/n$. Frank Mousset (joint work with Matan Harel and Wojciech Samotij) presented a major step forward in understanding the probability that the number of triangles in a random graph $G_{n,p}$ exceeds its expectation by a factor of $1+\delta$. In particular, they pin down the dependence on δ for a large range of densities. Mohsen Ghaffari (joint work with Krzysztof Nowicki) presented an improvement of Karger's classical min-cut contraction algorithm that brings down the runtime to $O(m + n(\log n)^3)$, where m and n denote the number of edges resp. vertices. This is the first algorithm to achieve an optimal time complexity for sufficiently dense graphs.

As always, and on behalf of all participants, the organizers would like to thank the staff and the director of the Mathematisches Forschungsinstitut Oberwolfach for providing such a stimulating and inspiring atmosphere.

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Workshop: Combinatorics, Probability and Computing

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Abstracts

Sparse resilience results

Peter Allen
(joint work with Julia Böttcher)

We are interested in studying which graphs H are contained in random graphs $G(n,p)$, and how robustly. Resilience is one measure of how robustly a random graph contains a given H: we would like to know that a copy of H is present even after some adversarial edge deletion, or present monochromatically after adversarial edge colouring. It is also interesting to know universality results— to show that $G(n,p)$ is likely to contain not just one given H, but simultaneously all H from some large class of graphs. Finally, if we can prove that $G(n,p)$ is likely to contain some given H, but the argument is nonconstructive (such as the second moment method) we would like to have a constructive, ideally algorithmic proof.

Broadly, we know how to give constructive proofs, or universality results, provided a vertex-by-vertex greedy embedding does not obviously fail—that is, provided that when we want to embed a new vertex (or small collection of vertices) having fixed a partial embedding, the expected number of ways to continue the embedding is growing with n. In particular, letting $\mathcal{H}(cn,\Delta)$ denote the class of cn-vertex graphs with maximum degree at most Δ, and $\mathcal{H}(cn,D,\Delta)$ denote the subclass with degeneracy at most D, Conlon, Ferber, Nenadov and Škorić [3] proved that $G(n,p)$ is a.a.s. universal for $\mathcal{H}(cn,\Delta)$ for any fixed $c<1$ provided $p \gg n^{-1/(\Delta-1)}\log^5 n$, and Conlon and Nenadov [5] proved that $G(n,p)$ is a.a.s. universal for $\mathcal{H}(n,D,\Delta)$ provided $p \gg n^{-1/D}\text{polylog } n$. In the former result, the 'natural' $n^{-1/\Delta}$ barrier is broken by clever use of the observation that any graph with maximum degree Δ is 'almost' $(\Delta-1)$-degenerate (i.e. one can obtain a $(\Delta-1)$-degenerate graph by removing a relatively short induced cycle), but this observation cannot be pushed further. The first result is rather far from the best lower bounds we know: potentially $G(n,p)$ becomes $\mathcal{H}(n,\Delta)$-universal at the same p for which it contains a perfect $K_{\Delta+1}$-factor, which by the Johansson-Kahn-Vu Theorem [6] occurs at $p \approx n^{-2/(\Delta+1)}$ (ignoring log factors). But the latter result is almost tight—for $p \ll n^{-1/D}$ a simple first moment argument shows $G(n,p)$ typically does not even contain any given graph with close to Dn edges.

For resilience type results, we are not quite so advanced, because currently these results rely on maintaining regularity properties throughout an embedding process and these regularity properties cannot be guaranteed for small sets of vertices. To take one example, Kohayakawa, Rödl, Schacht and Szemerédi [7] proved that for any given r, if the constant $c>0$ is sufficiently small and $p \gg \left(\frac{\log n}{n}\right)^{-1/\Delta}$, then a.a.s. in any r-edge-colouring of $G(n,p)$, there is a colour class which simultaneously contains every graph of $G(cn,\Delta)$. We do not know how to reach the $n^{-1/(\Delta-1)}$ improvement corresponding to [3]. Similarly, Conlon and

Nenadov [5] could show that if $p \gg n^{-1/(D+2)}\text{polylog } n$ then a.a.s. in any r-edge-colouring of $G(n,p)$, there is a colour class which is $\mathcal{H}(cn, D, \Delta)$-universal, but they could not match their probability bound for universality of $G(n,p)$ itself.

In this talk, I outline an approach to the last of these, proving the following. Given r and $\gamma > 0$ there exists $c > 0$ such that if $p \geq n^{\gamma - 1/D}$, then a.a.s. in any r-edge-colouring of $G(n,p)$ there is a colour class which is $\mathcal{H}(cn, D, \Delta)$-universal. The key idea of the proof is not to attempt to preserve any regularity properties, but rather to identify before embedding $H \in \mathcal{H}(cn, D, \Delta)$ some 'bad sets' of vertices, and argue that since there are few such, one can perform a vertex-by-vertex embedding such that for any v in H, the vertices close to v in H and preceding it in degeneracy order are not embedded to a bad set; this allows the embedding of v. This general strategy comes from [1]. Here, we need to use a strengthened version of the Sparse Regularity Lemma (in the style of Alon, Fischer, Krivelevich and Szegedy [2]) and the counting KLR conjecture proved by Conlon, Gowers, Samotij and Schacht [4] to show that there are initially few bad sets, a theorem of Spencer [8] on counting rooted subgraphs of fixed size in $G(n,p)$ to show that one can avoid bad sets locally, and a beautiful method of Conlon and Nenadov [5] which shows that a large well-behaved collection of homomorphisms into $G(n,p)$ necessarily contains at least one injective map.

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On Erdős Covering systems

Paul Balister

(joint work with Béla Bollobás, Robert Morris, Julian Sahasrabudhe, and Marius Tiba.)

A *covering system* is a finite collection $A_1, \ldots, A_k$, of arithmetic progressions, $A_i = a_i + d_i\mathbb{Z}$, that cover the integers, i.e., that satisfy $\bigcup_{i=1}^{k} A_i = \mathbb{Z}$. The study of covering systems with distinct differences (or *moduli*) $d_1 < \cdots < d_k$ was initiated in 1950 by Erdős [3], who used them to answer a question of Romanoff, and posed a number of problems regarding their properties. For example, Erdős [3] asked

whether there exist such systems with minimum modulus d_1 arbitrarily large, Erdős and Selfridge (see, e.g., [5]) asked if there exists a covering system with all moduli distinct and odd, and Schinzel [9] conjectured that in any covering system there exist a pair of moduli, one of which divides the other. In 1980, Erdős and Graham [4] initiated the study of the density of the uncovered set; in particular, they conjectured that if the moduli $d_1 < \cdots < d_k$ all lie in the interval $[n, Cn]$, where $n \geq n_0(C)$ is sufficiently large, then the uncovered set has density at least ε for some $\varepsilon = \varepsilon(C) > 0$.

The first significant progress on these problems was made by Filaseta, Ford, Konyagin, Pomerance and Yu [6] in 2007, who proved (in a strong form) the conjecture of Erdős and Graham, and took an important step towards solving Erdős' minimum modulus problem by showing that the sum of reciprocals of the moduli of a covering system with distinct moduli grows quickly with the minimum modulus. Building on their work, and in a remarkable breakthrough, Hough [7] resolved the minimum modulus problem in 2015, showing that in every covering system with distinct moduli, the minimum modulus is at most 10^{16}. The method of [7] was further refined by Hough and Nielsen [8], who used it to prove that every covering system with distinct moduli contains a modulus that is divisible by either 2 or 3. However, Hough's method does not appear to be strong enough to resolve the problem of Erdős and Selfridge, and it moreover gives little information about the density of the uncovered set.

Our main result is a general technique for bounding the density of the uncovered set. Our method, which is based on that of Hough, but is actually somewhat simpler, turns out to be sufficiently powerful and flexible to allow us to also make further progress on the problem of Erdős and Selfridge, and to prove Schinzel's conjecture. Our starting point is the following natural and beautiful question of Filaseta, Ford, Konyagin, Pomerance and Yu [6].

Question. *Is it true that for each $C > 0$, there exist constants $M > 0$ and $\varepsilon > 0$ such that the following holds: for every covering system whose distinct moduli satisfy*

$$d_1, \ldots, d_k \geq M \qquad \text{and} \qquad \sum_{i=1}^{k} \frac{1}{d_i} < C, \tag{1}$$

the uncovered set has density at least ε?

We answer this question negatively for every $C \geq 1$, by constructing (a sequence of) families of arithmetic progressions with arbitrarily large moduli, for which the density of the uncovered set is arbitrarily small, and $\sum_{i=1}^{k} \frac{1}{d_i} < 1$. However, we can prove the following positive result [1].

Theorem 1. *Let $\varepsilon > 0$ and let μ be the multiplicative function defined by*

$$\mu(p^i) = 1 + \frac{(\log p)^{3+\varepsilon}}{p} \tag{2}$$

for all primes p and integers $i \geq 1$. There exists $M > 0$ so that if $A_1, \ldots, A_k$ are arithmetic progressions with distinct moduli $d_1, \ldots, d_k \geq M$, and

$$C = \sum_{i=1}^{k} \frac{\mu(d_i)}{d_i},$$

then the density of the uncovered set $R := \mathbb{Z} \setminus \bigcup_{i=1}^{k} A_i$ is at least $e^{-4C}/2$.

Note that Hough's theorem is an immediate consequence of Theorem 1. Our proof of Theorem 1 was inspired by that of Hough [7], but is simpler in various important ways (for example, we do not need to appeal to the Lovász Local Lemma, and do not need his notion of quasi-randomness), and as a result we obtain a somewhat simpler proof of his theorem, with a better bound on the minimum modulus (less than 10^6). We also prove that the theorem is close to best possible in the following much stronger sense. We show that if (2) is replaced by

$$\mu(p^i) = 1 + \frac{\lambda}{p}$$

for any fixed $\lambda > 0$, then there exists a constant $C = C(\lambda) > 0$ such that the following holds: for every $M > 0$ and $\varepsilon > 0$, there exists a finite collection of arithmetic progressions, with distinct moduli $d_1, \ldots, d_k \geq M$ satisfying $\sum_{i=1}^{k} \frac{\mu(d_i)}{d_i} \leq C$, such that the uncovered set has density less than ε. It would be extremely interesting to characterize the functions μ such that, under the conditions of Theorem 1, the density of the uncovered set is bounded from below by a constant $\delta(C) > 0$ depending only on C.

Although our sieve was developed to control the density of the uncovered set, it turns out that it can be used to prove a number of additional interesting results about covering systems. We will focus here on the two classical examples mentioned above: the question of Erdős and Selfridge, and the conjecture of Schinzel. Over 50 years ago, Erdős and Selfridge (see [5] or [9]) asked whether or not there exist covering systems with distinct odd moduli. Schinzel [9] showed that if no such covering system exists, then for every polynomial $f(x) \in \mathbb{Z}[X]$ with $f \not\equiv 1$, $f(0) \neq 0$ and $f(1) \neq -1$, there exists an (infinite) arithmetic progression of values of $n \in \mathbb{Z}$ such that $x^n + f(x)$ is irreducible over the rationals. He also showed that this would imply the following statement: in any covering system, one of the moduli divides another. We prove this latter statement, known as Schinzel's conjecture.

Theorem 2. *If $\mathcal{A}$ is a finite collection of arithmetic progressions that covers the integers, then at least one of the moduli divides another.*

Unfortunately, our method does not seem to be strong enough to resolve the Erdős–Selfridge problem. However, it does allow us to make some further progress towards a solution; in particular, we can prove that no such covering system exists under the additional constraint that the moduli are square-free [2].

Theorem 3. *If $\mathcal{A}$ is a finite collection of arithmetic progressions with distinct square-free moduli that covers the integers, then at least one of the moduli is even.*

A different strengthening of the condition in the Erdős–Selfridge problem was considered recently by Hough and Nielsen [8], who showed that in any covering system with distinct moduli, one of the moduli is divisible by either 2 or 3. Their proof required careful optimization of their techniques, and it seems difficult to use it to strengthen their result. Using our methods, we give a short proof of the following strenthening of their theorem [1].

Theorem 4. *Let $\mathcal{A} = \{A_d : d \in D\}$ be a finite collection of arithmetic progressions with distinct moduli that covers the integers, and let $Q = \mathrm{lcm}(D)$ be the least common multiple of the moduli. Then either $2 \mid Q$, or $9 \mid Q$, or $15 \mid Q$.*

In other words, either there is an even d, a d divisible by $3^2 = 9$, or there are $d_1, d_2 \in D$ (possibly equal) with $3 \mid d_1$ and $5 \mid d_2$. We remark that we are unable to prove that a *single* $d \in D$ has $15 \mid d$ in this last case.

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Perfectly packing degenerate graphs with many leaves

Julia Böttcher

(joint work with Peter Allen, Dennis Clemens, Anusch Taraz)

Let $\mathcal{G} = \{G_1, G_2, \dots, G_s\}$ be a collection of graphs, and H be a graph. We say that $\mathcal{G}$ *packs into* H if we can find pairwise edge-disjoint copies in H of the graphs $G_1, \dots, G_s$. If in addition we have $\sum_{i \in [s]} e(G_i) = e(H)$, we call the packing *perfect*. In this case, each edge of H is used in a copy of exactly one G_i.

We show the following result, obtaining a perfect packing of degenerate graphs with many leaves, where a graph G is *D-degenerate* if its vertices can be ordered so that each vertex has at most D neighbours among all the vertices that precede it in this order.

Theorem 1. *For all $\mu > 0$, $D \in \mathbb{N}$ there are $c > 0$ and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$. Let $G_1, \dots, G_t$ be a family of D-degenerate graphs such that for all $i \in [t]$*

- $v(G_i) \leq n$,
- $\sum_{i=1}^{t} e(G_i) \leq \binom{n}{2}$,
- $\Delta(G_i) \leq cn/\log n$,

and for all $i > t - \mu n$

- $v(G_i) \leq (1-\mu)n$,
- *G_i has at least μn leaves.*

Then $G_1, \dots, G_t$ pack into K_n.

This work is motivated by the following tree packing conjectures.

Conjecture 2 (Ringel [10], Gyárfás [5])**.**

- *$2n-1$ copies of any given n-vertex tree pack into K_{2n-1}.*
- *If $T_2, \dots, T_n$ is any sequence of trees such that $v(T_i) = i$, then $\{T_2, \dots, T_n\}$ packs into K_n.*

In both cases, the packing is necessarily perfect, which makes these conjectures difficult. Joos, Kim, Kühn and Osthus [6] proved that both conjectures hold when the trees have constant maximum degree Δ and n is large enough. Montgomery, Pokrovskiy and Sudakov [9] proved an approximate version of Ringel's conjecture, proving that K_{2n-1} can be packed with $2n-1$ copies of any tree T with $n - o(n)$ vertices. Previously, related results were obtained in [1, 2, 3, 4, 7, 8].

It is not difficult to show that if T is a uniform random labelled n-vertex tree, then for each $c > 0$, with probability $1 - e^{-O(n)}$ the tree T will have at least $n/100$ leaves and maximum degree at most $\frac{cn}{\log n}$. Hence, our result proves almost all cases of the tree packing conjectures

Corollary 3. *Let T be a uniform random n-vertex tree. With probability $1 - e^{-O(n)}$, there is a packing of $2n-1$ copies of T into K_{2n-1}.*

Let $T_2, \dots, T_n$ be chosen independently and uniformly at random such that T_i is an i-vertex tree for each $2 \leq i \leq n$. With probability $1 - e^{-O(n)}$, there is a packing of $\{T_2, \dots, T_n\}$ into K_n.

Further, we can replace the complete graph K_n in our theorem by any $(\xi, 4D+7)$-quasirandom graph H with density $p > 0$ (and with $\sum_{i=1}^{t} e(G_i) \leq \binom{n}{2}$ replaced by $\sum_{i=1}^{t} e(G_i) \leq e(H)$), where a graph H is (ξ, A)-quasirandom if for all $d \leq A$ every collection $v_1, \dots, v_d$ of distinct vertices in H satisfies $|N_H(v_1) \cap \dots \cap N_H(v_d)| = (1 \pm \xi)p^d v(H)$. For this variant of our result the degree bound $\Delta(G_i) \leq cn/\log n$ is optimal up to the constant.

We prove our result by using random embedding methods, and therefore obtain a polynomial-time randomised algorithm for embedding the given graphs.

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The rank of sparse random matrices

Amin Coja-Oghlan

(joint work with Alperen Ergür, Pu Gao, Samuel Hetterich, Maurice Rolvien)

We determine the rank of a random matrix A over an arbitrary field F with prescribed numbers of non-zero entries in each row and column. As an application we obtain a formula for the rate of low-density parity check codes. This formula verifies a conjecture of Lelarge [1]. The proofs are based on coupling arguments.

Specifically, let $\chi \neq 0$ be a random variable that takes values in a field F. Moreover, let $d \geq 1, k \geq 3$ be integer-valued random variables such that $E[d^r]+E[k^r] < \infty$ for a real $r > 2$ and set $\bar{d} = E[d]$, $\bar{k} = E[k]$. Let $n > 0$ be an integer divisible by the greatest common divisor of the support of k and let $m \sim \mathrm{Po}(dn/k)$. Further, let $(d_i, k_i, \chi_{i,j})_{i,j\geq 1}$ be copies of d, k, χ, respectively, mutually independent and independent of m. Given $\sum_{i=1}^{n} d_i = \sum_{i=1}^{m} k_i$, draw a simple bipartite graph G comprising a set $\{a_1, \ldots, a_m\}$ of *check nodes* and a set $\{x_1, \ldots, x_n\}$ of *variable nodes* such that the degree of a_i equals k_i and the degree of x_j equals d_j for all i, j uniformly at random. Then let A be the $m \times n$-matrix with entries

$$A_{ij} = 1\{a_i x_j \in E(G)\} \cdot \chi_{i,j}.$$

Since d, k have finite means the matrix A is sparse, i.e., the expected number of non-zero entries is $O(n)$.

The following theorem provides a formula for the asymptotic rank of A. Let $D(x)$ and $K(x)$ denote the probability generating functions of d and k, respectively.

Theorem 1. *Let*

$$\Phi(\alpha) = D\left(1 - K'(\alpha)/\bar{k}\right) + \frac{\bar{d}}{\bar{k}}(K(\alpha) + (1-\alpha)K'(\alpha) - 1).$$

Then

$$\lim_{n\to\infty} \frac{\mathrm{rk}(A)}{n} = 1 - \max_{\alpha\in[0,1]} \Phi(\alpha) \qquad \textit{in probability.}$$

The upper bound on the rank already follows from [1].

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Extremal numbers of subdivisions

David Conlon
(joint work with Oliver Janzer and Joonkyung Lee)

For a graph H, the *extremal number* $\mathrm{ex}(n, H)$ is the maximal number of edges in an H-free graph on n vertices. The celebrated Erdős–Stone–Simonovits theorem states that $\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right) n^2$, where $\chi(H)$ is the chromatic number of H. This determines the asymptotics of $\mathrm{ex}(n, H)$ for any H of chromatic number at least 3. However, for bipartite graphs H, it only gives $\mathrm{ex}(n, H) = o(n^2)$. One of the central problems in extremal combinatorics is to obtain more precise bounds in this case.

Our starting point here lies with one of the few general results in the area, first proved by Füredi [4] and later reproved by Alon, Krivelevich and Sudakov [1] using the celebrated dependent random choice technique.

Theorem 1 (Füredi, Alon–Krivelevich–Sudakov)**.** *Let H be a bipartite graph such that in one of the parts all the degrees are at most r. Then* $\mathrm{ex}(n, H) = O(n^{2-1/r})$.

This result is known to be tight, since, for s sufficiently large in terms of r, $\mathrm{ex}(n, K_{r,s}) = \Omega(n^{2-1/r})$. Moreover, it is conjectured that this should already hold when $s = r$. On the other hand, a recent conjecture of Conlon and Lee [3] says that containing $K_{r,r}$ as a subgraph should be the only reason why Theorem 1 is tight up to the constant.

Conjecture 2 (Conlon–Lee)**.** *Let H be a bipartite graph such that in one of the parts all the degrees are at most r and H does not contain $K_{r,r}$ as a subgraph. Then there exists some ¿0 such that* $\mathrm{ex}(n, H) = O(n^{2-1/r-\delta})$.

To say more, recall that the *k-subdivision* of a graph L is the graph obtained by replacing the edges of L by internally disjoint paths of length $k+1$. We shall write L^k for the k-subdivision of L and L' for the 1-subdivision. It is easy to see that any C_4-free bipartite graph in which every vertex in one part has degree at most

two is a subgraph of K'_t for some positive integer t. Conlon and Lee [3] verified their conjecture in the $r = 2$ case by proving the following result.

Theorem 3 (Conlon–Lee)**.** *For any integer $t \geq 3$,* $\mathrm{ex}(n, K'_t) = O(n^{3/2-1/6^t})$.

Our first result gives some small progress towards Conjecture 2 when $r > 2$.

Theorem 4. *Let H be a bipartite graph such that in one of the parts all the degrees are at most r and H does not contain C_4 as a subgraph. Then* $\mathrm{ex}(n, H) = o(n^{2-1/r})$.

The proof of this result relies on ideas of Janzer [5], who found a simpler proof of Theorem 3 with much improved bounds. Since $K'_3 = C_6$ and $\mathrm{ex}(n, C_6) = \Theta(n^{4/3})$, this result is tight up to the implied constant for $t = 3$ and it is plausible that it is also tight for all other t.

Theorem 5 (Janzer)**.** *For any integer $t \geq 3$,* $\mathrm{ex}(n, K'_t) = O(n^{3/2-\frac{1}{4t-6}})$.

Improving another result of Conlon and Lee [3], Janzer [5] also obtained the following bound for the extremal number of $K'_{s,t}$, the 1-subdivision of $K_{s,t}$.

Theorem 6 (Janzer)**.** *For any integers $2 \leq s \leq t$,* $\mathrm{ex}(n, K'_{s,t}) = O(n^{3/2-\frac{1}{4s-2}})$.

This theme was again taken up in a recent paper of Kang, Kim and Liu [7], where they made the following conjecture about the 1-subdivision of a general bipartite graph.

Conjecture 7 (Kang–Kim–Liu)**.** *Let H be a bipartite graph. If* $\mathrm{ex}(n, H) = O(n^{1+\alpha})$ *for some $\alpha > 0$, then* $\mathrm{ex}(n, H') = O(n^{1+\frac{\alpha}{2}})$.

In particular, as $\mathrm{ex}(n, K_{s,t}) = O(n^{2-\frac{1}{s}})$, they conjectured that $\mathrm{ex}(n, K'_{s,t}) = O(n^{3/2-\frac{1}{2s}})$, though they were only able to push their methods to give an alternative proof of Theorem 6. Our next result is a proof of this latter conjecture.

Theorem 8. *For any integers $2 \leq s \leq t$,* $\mathrm{ex}(n, K'_{s,t}) = O(n^{3/2-\frac{1}{2s}})$.

Moreover, this result is tight when t is sufficiently large compared to s.

Corollary 9. *For any integer $s \geq 2$, there exists some $t_0 = t_0(s)$ such that if $t \geq t_0$, then* $\mathrm{ex}(n, K'_{s,t}) = \Theta(n^{3/2-\frac{1}{2s}})$.

We now turn to another central conjecture in extremal graph theory. Following Kang, Kim and Liu [7], we say that $r \in (1, 2)$ is *realisable* (by H) if there exists a graph H such that $\mathrm{ex}(n, H) = \Theta(n^r)$. The rational exponents conjecture of Erdős and Simonovits states that every rational between 1 and 2 is realisable.

Conjecture 10 (Rational exponents conjecture)**.** *For every rational number $r \in (1, 2)$, there exists a graph H with* $\mathrm{ex}(n, H) = \Theta(n^r)$.

In a recent breakthrough, Bukh and Conlon [2] have proved that for any rational number $r \in (1, 2)$ there exists a finite family $\mathcal{H}$ of graphs such that $\mathrm{ex}(n, \mathcal{H}) =$

$\Theta(n^r)$, where $\mathrm{ex}(n, \mathcal{H})$ denotes the maximal number of edges in an n-vertex graph which does not contain any $H \in \mathcal{H}$ as a subgraph.

However, Conjecture 10 remains wide open. In fact, until very recently only a few realisable numbers were known, namely, $1 + \frac{1}{m}$ and $2 - \frac{1}{m}$ for $m \geq 2$. Just a few months ago, Jiang, Ma and Yepremyan [6] enlarged the class of realisable exponents by proving that $7/5$ and $2 - \frac{2}{2m-1}$ for $m \geq 2$ are also realisable. Subsequently, Kang, Kim and Liu [7] proved that for each $a, b \in \mathbb{N}$ with $a < b$ and $b \equiv \pm 1 \pmod{a}$, the number $2 - \frac{a}{b}$ is realisable, a result which then included all known examples of realisable exponents. Their main result was a tight upper bound on the extremal number of certain graphs from which the result just mentioned for $b \equiv -1 \pmod{a}$ follows fairly easily. We now define this family of graphs.

Consider a graph F with a set $R \subset V(F)$ of root vertices. The *ℓ-blowup* of this rooted graph is the graph obtained by taking ℓ vertex-disjoint copies of F and identifying the different copies of v for each $v \in R$. We let $H_{s,1}(r)$ be the graph consisting of vertices x_i $(1 \leq i \leq r-1)$, y, z_j $(1 \leq j \leq s)$ and $w_{j,k}$ $(1 \leq j \leq s, 1 \leq k \leq r-1)$ and edges $x_i y$ for all i, yz_j for all j and $z_j w_{j,k}$ for all j, k. Then $H_{s,t}(r)$ is the rooted t-blowup of $H_{s,1}(r)$, with the roots being $\{x_i : 1 \leq i \leq r-1\} \cup \{w_{j,k} : 1 \leq j \leq s, 1 \leq k \leq r-1\}$. The result of Kang, Kim and Liu [7, Lemma 3.2] is now as follows.

Theorem 11 (Kang–Kim–Liu)**.** *For any integers $s, t \geq 1$, $r \geq 2$, $\mathrm{ex}(n, H_{s,t}(r)) = O(n^{2 - \frac{s+1}{r(s+1)-1}})$.*

We give a new proof of this result which is significantly shorter than the original one. Combined with results of Bukh and Conlon [2], Theorem 11 easily implies that $2 - \frac{s+1}{r(s+1)-1}$ is realisable for every $s \geq 1, r \geq 2$. Therefore, following Kang, Kim and Liu, we see that $2 - \frac{1}{r}$ is a limit point of the set of realisable exponents for every integer $r \geq 2$.

To go further, we define $L_{s,t}(k)$ to be the graph which is the $(k-1)$-subdivision of $K_{s,t}$ with an extra vertex joined to all vertices in the part of size t. Put differently, this graph is the rooted t-blowup of $L_{s,1}(k)$, where $L_{s,1}(k)$ has vertices u, v, $w_{i,j}$ $(1 \leq i \leq k, 1 \leq j \leq s)$ and edges uv, $vw_{1,j}$ $(1 \leq j \leq s)$, $w_{i,j}w_{i+1,j}$ $(1 \leq i \leq k-1$, $1 \leq j \leq s)$, with roots $u, w_{k,1}, \ldots, w_{k,s}$. We have the following result.

Theorem 12. *For any integers $s, t, k \geq 1$, $\mathrm{ex}(n, L_{s,t}(k)) = O(n^{1 + \frac{s}{sk+1}})$.*

This result has several interesting corollaries. The first is a complete resolution of Problem 5.2 from [7].

Corollary 13. *For any integers $s, k \geq 1$, there exists some $t_0 = t_0(s,k)$ such that if $t \geq t_0$, then $\mathrm{ex}(n, L_{s,t}(k)) = \Theta(n^{1 + \frac{s}{sk+1}})$. In particular, the exponent $1 + 1/k$ is a limit point of the set of realisable numbers.*

Recall that H^k denotes the k-subdivision of the graph H. Conlon and Lee [3] have conjectured that the following estimate should hold.

Conjecture 14 (Conlon–Lee)**.** *Let $k \geq 2$ be an even integer and let H be a graph. Then there exists some $\delta > 0$ such that* $\mathrm{ex}(n, H^{k-1}) = O(n^{1+1/k-\delta})$.

Our Theorem 12 establishes this conjecture for bipartite H, since $K_{s,t}^{k-1}$ is a subgraph of $L_{s,t}(k)$.

Theorem 15. *For any integers $s, t, k \geq 1$,* $\mathrm{ex}(n, K_{s,t}^{k-1}) = O(n^{1+\frac{s}{sk+1}})$. *In particular, for any bipartite graph H, there exists $\delta > 0$ such that* $\mathrm{ex}(n, H^{k-1}) = O(n^{1+1/k-\delta})$.

Even for subdivisions of general graphs, we obtain a good estimate for Conjecture 14 using that $H^{k-1} = (H^1)^{k/2-1}$ and H^1 is bipartite.

Theorem 16. *Let $k \geq 2$ be an even integer and let H be a graph. Then there exists some ¿0 such that* $\mathrm{ex}(n, H^{k-1}) = O(n^{1+2/k-\delta})$.

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Online Ramsey Numbers

Jacob Fox

(joint work with David Conlon, Andrey Grinshpun, Xiaoyu He)

The *Ramsey number* $r(m, n)$ is the minimum integer N such that every red/blue-coloring of the edges of the complete graph K_N on N vertices contains either a red K_m or a blue K_n. Ramsey's theorem guarantees the existence of $r(m, n)$ and determining or estimating Ramsey numbers is a central problem in combinatorics. Classical results of Erdős–Szekeres and Erdős imply that $2^{n/2} \leq r(n, n) \leq 2^{2n}$ for $n \geq 2$. The only improvements to these bounds over the last seventy years have been to lower order terms (see [9, 26]), with the best known lower bound coming from an application of the Lovász local lemma [14].

Off-diagonal Ramsey numbers, where m is fixed and n tends to infinity, have also received considerable attention. In progress that has closely mirrored and often instigated advances on the probabilistic method, we now know that $r(3, n) = \Theta(n^2/\log n)$. The lower bound here is due to Kim [21] and the upper bound to Ajtai, Komlós and Szemerédi [1]. Recently, Bohman and Keevash [8] and, independently, Fiz Pontiveros, Griffiths and Morris [18] improved the constant in

Kim's lower bound via careful analysis of the triangle-free process, determining $r(3,n)$ up to a factor of $4+o(1)$.

More generally, for $m \geq 4$ fixed and n growing, the best known lower bound is $r(m,n) = \Omega_m(n^{\frac{m+1}{2}}/(\log n)^{\frac{m+1}{2}-\frac{1}{m-2}})$, proved by Bohman and Keevash [7] using the H-free process, while the best upper bound in this setting is $r(m,n) = O_m(n^{m-1}/(\log n)^{m-2})$, again due to Ajtai, Komlós and Szemerédi [1]. Here the subscripts denote the variable(s) that the implicit constant is allowed to depend on.

There are many interesting variants of the classical Ramsey problem. One such variant is the *size Ramsey number* $\hat{r}(m,n)$, defined as the smallest N for which there exists a graph G with N edges such that every red/blue-coloring of the edges of G contains either a red K_m or a blue K_n. It was shown by Chvátal (see Theorem 1 in the foundational paper of Erdős, Faudree, Rousseau and Schelp [13]) that $\hat{r}(m,n)$ is just the number of edges in the complete graph on $r(m,n)$ vertices, that is,

$$\hat{r}(m,n) = \binom{r(m,n)}{2}.$$

We will be concerned with a game-theoretic variant of the size Ramsey number, introduced independently by Beck [4] and by Kurek and Ruciński [25]. The (m,n)-online Ramsey game is a game between two players, Builder and Painter, on an infinite set of initially isolated vertices. Each turn, Builder places an edge between two nonadjacent vertices and Painter immediately paints it either red or blue. The *online Ramsey number* $\tilde{r}(m,n)$ is then the smallest number of turns N that Builder needs to guarantee the existence of either a red K_m or a blue K_n.

It is a simple exercise to show that $\tilde{r}(m,n)$ is related to the usual Ramsey number $r(m,n)$ by

$$\frac{1}{2}r(m,n) \leq \tilde{r}(m,n) \leq \binom{r(m,n)}{2}. \tag{1}$$

In the diagonal case, the upper bound in (1) has been improved by Conlon [10], who showed that for infinitely many n,

$$\tilde{r}(n,n) \leq 1.001^{-n}\binom{r(n,n)}{2}.$$

Our main result is a new lower bound for online Ramsey numbers.

Theorem 1. *If, for some $m,n,N \geq 1$, there exist $p \in (0,1)$, $c \leq \frac{1}{2}m$, and $d \leq \frac{1}{2}n$ for which*

$$p^{\binom{m}{2}-c(c-1)}(2N)^{m-c} + (1-p)^{\binom{n}{2}-d(d-1)}(2N)^{n-d} \leq \frac{1}{2},$$

then $\tilde{r}(m,n) > N$.

In particular, if $\tilde{r}(n) := \tilde{r}(n,n)$ is the diagonal online Ramsey number, Theorem 1 can be used to improve the classical bound $\tilde{r}(n) \geq 2^{n/2-1}$ by an exponential factor. Indeed, taking $p = \frac{1}{2}$ and $c = d \approx (1-\frac{1}{\sqrt{2}})n$ in Theorem 1, we get the following immediate corollary.

Corollary 2. *For the diagonal online Ramsey numbers $\tilde{r}(n)$,*

$$\tilde{r}(n) \geq 2^{(2-\sqrt{2})n - O(1)}.$$

As for the off-diagonal case, when m is fixed and $n \to \infty$, Theorem 1 can be also used to substantially improve the best-known lower bound. In this case, we take $c \approx (1 - \frac{1}{\sqrt{2}})m$, $d = 0$, and $p = C\frac{m \log n}{n}$ for a sufficiently large $C > 0$ to obtain the following corollary.

Corollary 3. *For fixed $m \geq 3$ and n sufficiently large in terms of m,*

$$\tilde{r}(m, n) \geq n^{(2-\sqrt{2})m - O(1)}.$$

For general m, Corollary 3 gives the best known lower bounds for the off-diagonal online Ramsey number. However, it is possible to do better for $m = 3$ by using a smarter Painter strategy which deliberately avoids building red triangles.

Theorem 4. *For $n \to \infty$,*

$$\tilde{r}(3, n) = \Omega\left(\frac{n^3}{\log^2 n}\right).$$

Roughly speaking, Painter's strategy is to paint every edge blue initially, but to switch to painting randomly if both endpoints of a freshly built edge have high degree. Also, when presented with an edge that would complete a red triangle, Painter always paints it blue. The bound given in Theorem 4 is n times the bound on the usual Ramsey number that comes from applying the Lovász Local Lemma [14]. However, our argument is closer in spirit to an earlier proof of the same bound given by Erdős [12] using alterations. This method for lower bounding $r(3, n)$ was later generalized to all $r(m, n)$ by Krivelevich [22] and we suspect that Theorem 4 can be generalized to $\tilde{r}(m, n)$ in the same way.

In the other direction, we prove a new upper bound on the off-diagonal online Ramsey number.

Theorem 5. *For any fixed $m \geq 3$,*

$$\tilde{r}(m, n) = O_m\left(\frac{n^m}{(\log n)^{\lfloor m/2 \rfloor - 1}}\right).$$

In particular, note that Theorems 4 and 5 determine the asymptotic growth rate of $\tilde{r}(3, n)$ up to a polylogarithmic factor, namely,

$$\Omega\left(\frac{n^3}{\log^2 n}\right) \leq \tilde{r}(3, n) \leq O\left(n^3\right).$$

Theorem 5 has a similar flavor to the improvement on diagonal online Ramsey numbers made by the first author [10] and work on the so-called vertex online Ramsey numbers due to Conlon, Fox and Sudakov [11]. It is obtained by adapting the standard Erdős–Szekeres proof of Ramsey's theorem to the online setting and applying a classical result of Ajtai, Komlós and Szemerédi [1] bounding $r(m, n)$.

In order to prove Theorem 1, we specialize to the case where Painter plays randomly. This is sufficient because Builder, who we may assume has unlimited computational resources, will always respond in the best possible manner to Painter's moves. Therefore, if a random Painter can stop this perfect Builder from winning within a certain number of moves with positive probability, an explicit strategy exists by which Painter can delay the game up to this point. This motivates the following key definition.

Definition 6. *For $m, n \geq 3$ and $p \in (0,1)$, define $\tilde{r}(m,n;p)$ to be the number of turns Builder needs to win the (m,n)-online Ramsey game with probability at least $\frac{1}{2}$ against a Painter who independently paints each edge red with probability p and blue with probability $1-p$. The* online random Ramsey number *$\tilde{r}_{rand}(m,n)$ is the maximum value of $\tilde{r}(m,n;p)$ over $p \in (0,1)$.*

We note that there is a rich literature on simplifying the study of various combinatorial games by specializing to the case where one or both players play randomly (see [5, 20, 23]). For example, a variant of the online Ramsey game with random Builder instead of random Painter was studied by Friedgut et al. [19].

We make the following conjectures about the growth rate of $\tilde{r}_{\mathrm{rand}}(m,n)$.

Conjecture 7. *The diagonal online random Ramsey numbers satisfy*

$$\tilde{r}_{\mathrm{rand}}(n,n) = 2^{(1+o(1))\frac{2}{3}n}.$$

The off-diagonal online random Ramsey numbers ($m \geq 3$ fixed and $n \to \infty$) satisfy

$$\tilde{r}_{\mathrm{rand}}(m,n) = n^{(1+o(1))\frac{2}{3}m}.$$

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Regular graphs with regular connected neighborhoods

Ehud Friedgut

As usual, we will say that a graph is (d_1)-regular if every vertex has degree precisely d_1. We will say that it is $(d_1, d_2, ...d_r)$- regular if it is d_1 regular and the graph induced on the neighborhood of every vertex is $(d_2, ...d_r)$-regular. The main challenge is constructing such graphs where the graphs themselves, and those induced on the neighborhoods (and so on, by recursion) are connected, and if possible also with good expansion. In principle one would like to fix $(d_1, ...d_r)$ and construct an infinite family of $(d_1, ...d_r)$-regular graphs with the number of vertices tending to infinity.

Following a method of Kaufman and Oppenheim we construct, for every positive integer s a tripartite $(2\binom{3s}{s}, \binom{2s}{s})$-regular graph which is the union of Cayley graphs coming from a symmetry group of triangulations of R^n. For every integer t and $\varepsilon > 1/t$ we give a construction of a $(n^{1-\varepsilon}, n^{1-2\varepsilon}, ..., n^{1-t\varepsilon}$- regular graph on n vertices using a generalization of the above method and subgroups of the symmetric group. Finally, we prove that one can take increasing finite quotients of a tessellation of the four-dimensional hyperbolic space to get an infinite family of $(120, 12, 5, 2)$-regular graphs.

It is an open question whether there exist infinite families of (a, b, c, d, e)-regular graphs, with connected links of all orders (the neighborhood of every vertex, edge, triangle, K_4 and K_5 is connected) for any integers (a, b, c, d, e).

Explicit construction of RIP matrices is Ramsey-hard

David Gamarnik

1. Background

An $n \times p$ real valued matrix Φ with unit norm column vectors is said to satisfy (s, δ) Restricted Isometry Property (RIP) for $s \leq p$ and $\delta \in (0, 1)$ if $|\|\Phi x\|_2^2 - 1| \leq \delta$ for all $x \in \mathbb{R}^p, \|x\|_2 = 1$ which are s-sparse, namely have at most s non-zero coordinates. In this case we simply say that the matrix Φ is RIP. The RIP is of fundamental importance in compressive sensing methods [FR13],[BVDG11]. Provided Φ is RIP, the linear programming based method enable efficient (polynomial time) unique recovery of the solution x of the linear system $y = \Phi x$, from y and Φ, whenever $x \in \mathbb{R}^p$ is $2s$-sparse. It is known that generating entries of Φ i.i.d. from a common distribution, satisfying minor properties such as sub-Gaussianity and then normalizing columns of Φ to unit norm, guarantees RIP provided $n = \Omega(s \log(p/s))$ [FR13]. Here $O(\cdot), \Omega(\cdot)$ and $\Theta(\cdot)$ are standard order of magnitude notations. This includes the case when $s = \log^{O(1)} p$. As a result it suffices to have poly-log values of n as well: $n = \log^{O(1)} p$.

At the same time, verifying whether a given matrix Φ satisfies the RIP is tricky, as the problem of certifying RIP of a matrix in the worst case is NP-hard [BDMS13]. Furthermore, even determining the RIP value δ up to a certain approximation factor is hard in the average case sense, as shown in [KZ14] by reduction from the Planted Clique Problem. Motivated by this complications, the researchers have sought to obtain explicit deterministic constructions of matrices satisfying the RIP. While there are constructions of RIP matrices when $s = O(\sqrt{n})$ [AHSC09], [DeV07],[FMT10], most methods however break down when s is at least some constant times $\sqrt{n}$, see Bandeira et al. [BFMW13] for a survey of the known methods, leaving this case as an open problem. This problem since then was popularly dubbed as the so-called "square root bottleneck" problem. It was also raised in Tao's blog [Taoa], as well as in a blog discussion by Moreira [Mor]. A notable exception to the square root bottleneck is the construction by Bourgain et al. [BDF$^+$11] which breaks this barrier by achieving $s = \Theta(n^{\frac{1}{2}+\varepsilon})$ for some small constant $\varepsilon > 0$, in the regime $n = \Omega(p^{1-\delta})$ and with matrix containing complex valued entries. Thus the cases of poly-logarithmic n and s, or even polynomial n and s with real valued entries remain open.

In this short note we give a very simple argument that this problem is "Ramsey-hard" so to speak, see [GGRS90] for a book reference on Ramsey theory and [CFS15] for a survey. Specifically, we show that an explicit construction of an RIP matrix when $n = O(\log^2 p)$ and $s \geq 2\sqrt{n} + 1$ implies an explicit construction of a 3-colored Ramsey graph on p nodes with monochromatic clique sizes bounded by

$O(\log^2 p)$. To elaborate, a complete graph K_p on p nodes with $p(p-1)/2$ edges colored using q colors is called $R(m_1,\ldots,m_q)$ Ramsey if the largest monochromatic clique with respect to color r (a subset of nodes with all induced edges colored r) is at most m_r for all $1 \le r \le q$. When all m_r are identically m we simply write $R(m;q)$. While the uniform random coloring provides an easy construction of $R(m;q)$ Ramsey graphs with $m = O(\log p)$ w.h.p. as $p \to \infty$, explicit construction of such graphs is a major open problem in the extremal combinatorics even for larger order of magnitude m. The best known construction for $q = 2$ gives $m = (\log p)^{\log\log\log^{O(1)} p}$ [Coh17]. The case $m = \log^{O(1)} p$ is believed to be out of reach using the known methods, and the problem of explicit construction of graphs satisfying this Ramsey type property has been open since the celebrated work of Erdös [Erd47].

2. Construction

Suppose $\Phi \in \mathbb{R}^{n\times p}$ is a matrix with unit norm column vectors, $\Phi = (u_i, 1 \le i \le p)$, $\|u_i\|_2 = 1$, which satisfies the (s,δ)-RIP with $\delta \in (0,1)$ and $s \ge 2\sqrt{n}+1$. Consider the complete graph $K_{[p]}$ on the set of nodes $[p] = \{1,2,\ldots,p\}$. We color all of the $\binom{p}{2}$ edges of this graph with three colors, white, blue and red, as follows. For each $1 \le i \ne j \le p$, the color of (i,j) is white if $|\langle u_i, u_j\rangle| \le 1/(2\sqrt{n})$, blue if $\langle u_i, u_j\rangle > 1/(2\sqrt{n})$ and red if $\langle u_i, u_j\rangle < -1/(2\sqrt{n})$. Here $\langle\cdot,\cdot\rangle$ denotes standard inner product in $\mathbb{R}^n$.

Theorem 1. *The graph $\mathbb{G}$ is $R(2n, 2\sqrt{n}+1, 2\sqrt{n}+1; 3)$ Ramsey.*

Proof. The following proposition is a simplified version of Kabatjanskii-Levenstein bound [KL78] discussed in Tao's blog [Taob]. We reproduce the proof from there for completeness.

Proposition 2. *For any set of unit norm vectors $u_1,\ldots,u_{2n} \in \mathbb{R}^n$, $\max_{1\le i\ne j\le 2n} |\langle u_i, u_j\rangle| > \frac{1}{2\sqrt{n}}$.*

Proof. Consider the symmetric matrix $U = (\langle u_i, u_j\rangle, 1 \le i,j \le 2n) \in \mathbb{R}^{2n\times 2n}$ of inner products. This is a rank-n matrix in $\mathbb{R}^{2n\times 2n}$ and as such $\bar{U} \triangleq U - I_{2n\times 2n}$ has an eigenvalue -1 with multiplicity at least n. Thus the trace of $\bar{U}^2$ which is $\sum_{1\le i\ne j\le 2n}(\langle u_i, u_j\rangle)^2$ is at least n, implying $\max_{i\ne j} |\langle u_i, u_j\rangle| \ge \frac{1}{\sqrt{2(2n-1)}}$. □

Now suppose $C \subset [p]$ is a white clique. Then by Proposition 2, $|C| < 2n$. Suppose $C \subset [p]$ is a blue clique which for the purposes of contradiction satisfies $|C| \ge 2\sqrt{n}+1$. Take any subset of C with cardinality $2\sqrt{n}+1$. For simplicity denote it by C as well. Let $x \in \mathbb{R}^p$ be the unit norm $\|x\|_2 = 1$ vector defined by $x_i = 1/\sqrt{|C|}, i \in C$ and $x_i = 0$ otherwise. This vector is $|C| \le s$-sparse. Since C supports a blue clique then

$$\|\Phi x\|_2^2 - \|x\|_2^2 = \frac{1}{|C|}\sum_{i\ne j\in C}\langle u_i, u_j\rangle \ge \frac{|C|-1}{2\sqrt{n}} = 1,$$

which contradicts RIP. The same argument applies for red cliques. We conclude that our graph is $R(2n, 2\sqrt{n}+1, 2\sqrt{n}+1; 3)$ Ramsey, and therefore crudely $R(2n; 3)$ Ramsey.

□

Now find $C > 0$ and $\delta \in (0, 1)$ so that for $n \geq Cs \log p$ the random $n \times p$ matrix Φ is RIP w.h.p. for all large enough p for sparsity s and parameter δ. Set $n = 9C^2 \log^2 p$ and $s = 9C \log p = 3\sqrt{n} > 2\sqrt{n} + 1$. Then $n = Cs \log p$. The explicit construction of RIP matrix $\Phi \in \mathbb{R}^{n \times p}$ for this choice of n, sparsity s, and parameter δ implies, by the above, an explicit construction of an $R(2n; 3)$ Ramsey graph on p nodes with $n = 9C^2 \log^2 p = O(\log^2 p)$ – the task which appears not yet reachable with known techniques.

We note that if the entries of the matrix Φ are assumed to be non-negative, and in fact some known explicit constructions are based on non-negative entries [BFMW13] (see below for one such construction), then we can restrict our construction above to just two colors, corresponding to the cases $0 \leq \langle u_i, u_j \rangle \leq 1/(2\sqrt{n})$ and $\langle u_i, u_j \rangle > 1/(2\sqrt{n})$. Thus in this case the explicit construction of a matrix Φ satisfying the stated RIP implies an explicit construction of a $R(2n; 2)$ graph - an explicit 2-coloring of edges of a complete graph on p nodes with largest monochromatic cliques of sizes at most $O(\log^2 p)$.

The case $n = p^{O(1)}$, covered by [BDF+11] remains open for general $s = n^{\frac{1}{2}+\varepsilon}$. It would be also interesting to see if, conversely, the Ramsey graphs can be used to construct the RIP matrices. It also would be interesting to see if the construction above can be extended to the complex valued matrices in the poly-logarithmic case.

We close this note by suggesting one additional open question. While the case of explicit constructions is solved in the regime $s \geq c\sqrt{n}$ for sufficiently small constant $c > 0$, all of the known constructions are in the polynomial regime $n = p^{O(1)}, s = p^{O(1)}$. This raises a potential issue as to whether the square root is even achievable in the polylogarithmic regime $n = \log^{O(1)} p$, or alternatively, in this regime the problem is even more challenging. We now show that the answer is *essentially* no, but closing this gap entirely is an open question. The following construction due to deVore [DeV07] leads to an example satisfying $s \geq n^{\frac{1}{2}-\varepsilon}$ for every $\varepsilon > 0$ in the polylogarithmic regime $n = \log^{O(1)} p$. Fix a prime number z and a positive integer r. Let $p = z^{r+1}$ be the number of all degree $\leq r$ polynomials with coefficients in $\mathbb{Z}_z = \{0, 1, \ldots, z-1\}$. Consider the $z^2 \times p$ matrix Φ constructed as follows. The rows are indexed by pairs $(x, y) \in \mathbb{Z}_z^2$. Thus the number of rows is $n = z^2$. The columns are indexed by polynomials described above. For every such polynomial P and every pair (x, y), the matrix entry corresponding to the location $((x, y), P)$ is set to $1/\sqrt{z}$ if $y = P(x)$ and $= 0$ otherwise. It is clear that the columns $u_j, 1 \leq j \leq p$ of this matrix are unit length $\|u\|_2 = 1$ as they contain precisely z non-zero entries, each valued at $1/\sqrt{z}$. For every two distinct columns $u_i, u_j, i \neq j$ we have $|\langle u_i, u_j \rangle| \leq r/z$. Indeed, the row (x, y) has non-zero (namely $1/\sqrt{z}$) entries in columns indexed by polynomials P and Q only if

$y = P(x) = Q(x)$, that is x is a root of $P - Q$, of which only r exist. Thus the coherence of this matrix, defined as $\max_{i \neq j} |\langle u_i, u_j \rangle|$ is at most r/z. As a result, the matrix satisfies the (s, δ) RIP when $sr/z \leq \delta$ (see [DeV07] for details).

Now we fix arbitrary $\varepsilon > 0$. For each sufficiently large z we set $r = z^{\varepsilon}$ and consider the $z^2 \times p$ matrix described above with $p = z^{r+1}$. We have $n = z^2$ and the sparsity level satisfying the RIP can be taken as

$$s = (1/2)(z/r) = (1/2)z^{1-\varepsilon} = (1/2)n^{\frac{1}{2}-\frac{\varepsilon}{2}}.$$

Namely, the construction almost achieves the square root barrier. At the same time

$$p = z^{r+1} > z^r = z^{z^{\varepsilon}} = \exp(z^{\varepsilon} \log z),$$

implying

$$n = z^2 = \left(\frac{\log p}{\log z}\right)^{\frac{2}{\varepsilon}} \leq (\log p)^{\frac{2}{\varepsilon}}.$$

Namely, the sample size n is indeed poly-logarithmic in the number of columns p. It would be interesting to obtain a construction which truly achieves the square root barrier $s \geq C\sqrt{n}$ in the polylogarithmic regime $n = \log^{O(1)} p$.

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Faster Algorithms for Minimum Cut via Radom Out Contractions

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(joint work with Krzysztof Nowicki)

Computing the minimum cut in a graph — i.e., the smallest number of edges whose removal disconnects the graph — is one of the classic problems in graph algorithms, and it has been studied extensively since the 1950s. We provide a new variant of Karger's celebrated random contraction approach [1], which leads to improved algorithms for sequential, distributed, and parallel models of computation.

The core new ingredient is a simple random contraction process that transforms any n-node simple undirected graph into a (multi-)graph with $O(n)$ edges, while preserving any particular non-trivial[1] minimum cut with a large *constant* probability. That is, even if the original graph has as many as $O(n^2)$ edges, we can easily turn it to one with only $O(n)$ edges, while preserving any particular non-trivial minimum cut with a constant probability. We can then solve the minimum cut problem on this sparsified graph, and use repetitions to amplify the success probability, if desired.

The claimed random process is as simple as this: First, each node samples three of its incident edges randomly (chosen with replacement, and independently among all vertices), and we contract all of these sampled edges (and erase self-loops). Then, we iteratively contract one random sampled edge from the entire remaining graph, one by one, until the number of remaining edges drops below $10n$. We can show that with a constant probability, any particular non-singleton minimum-cut is preserved in this contracted graph, that is, none of its edge is contracted and therefore it is a minimum cut of the new graph.

This random process, combined with some other algorithmic ideas, leads to improved algorithms for minimum cut in a number of computation models, including an $O(m + n \log^3 n)$ time *sequential* algorithm, where m denotes the number

[1]A trivial cut is one that is defined by removing all edges incident to one vertex. Since identifying the minimum degree is easy, the computational problem can focus on non-trivial cuts and assume that the minimum cut has size strictly less than the minimum degree.

of edges. This is the first algorithm to achieve an optimal time complexity for sufficiently dense graphs and it provides an improvement on a long line of developments ending with the state of the art $O(m \log^2 n (\log \log n)^2)$ algorithm of Henzinger, Rao, and Wang [2].

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Testing Graphs against an Unknown Distribution

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(joint work with Asaf Shapira)

Property testers are fast randomized algorithms whose goal is to distinguish (with high probability, say, 2/3) between objects satisfying some fixed property $\mathcal{P}$ and those that are ε-far from satisfying it. Here, ε-far means that an ε-fraction of the input object should be modified in order to obtain an object satisfying $\mathcal{P}$. Problems of this nature have been studied in so many areas that it will be impossible to survey them here. Instead, the reader is referred to the recent monograph [2] for more background and references.

The classical property testing model assumes that one can uniformly sample entries of the input. In *distribution-free* testing one assumes that the input is endowed with some *arbitrary and unknown* distribution $\mathcal{D}$, which also affects the way one defines the distance to satisfying a property. As discussed in [3], one motivation for this model is that it can handle settings in which one cannot produce uniformly distributed entries from the input. Another motivation is that the distribution $\mathcal{D}$ can assign higher weight/importance to parts of the input which we want to have higher impact on the distance to satisfying the given property.

In this work we study a distribution-free variant of the *adjacency matrix model*, also known as the *dense graph model*. The adjacency matrix model was first defined and studied in [4], where the area of property testing was first introduced. This model has been extensively studied in the past two decades, see Chapter 8 of [2].

Instead of defining the adjacency matrix model of [4], let us directly define its distribution-free variant which was introduced recently by Goldreich [3]. Since the distribution in this model is over the input's vertices, it is called the *Vertex-Distribution-Free* (VDF) model . The input to the algorithm is a graph G and some *arbitrary and unknown* distribution $\mathcal{D}$ on $V(G)$. We will thus usually refer to the input as the pair $(G, \mathcal{D})$. For a pair of graphs G_1, G_2 on the same vertex-set V, and for a distribution $\mathcal{D}$ on V, the (edit) distance between G_1 and G_2 with respect to $\mathcal{D}$ is defined as $\sum_{\{x,y\} \in E(G_1) \triangle E(G_2)} \mathcal{D}(x)\mathcal{D}(y)$. We say that $(G, \mathcal{D})$ is

ε-far from satisfying a graph property $\mathcal{P}$ if for every $G' \in \mathcal{P}$, the distance between G and G' with respect to $\mathcal{D}$ is at least ε. A tester for a graph property $\mathcal{P}$ is an algorithm that receives as input a pair $(G, \mathcal{D})$ and a proximity parameter ε, and distinguishes with high probability (say $\frac{2}{3}$) between the case that G satisfies $\mathcal{P}$ and the case that $(G, \mathcal{D})$ is ε-far from $\mathcal{P}$. The algorithm has access to a device that produces random vertices from G distributed according to $\mathcal{D}$. The only[1] other way the algorithm can access G is by performing "edge queries" of the form "is (u, v) an edge of G?". We say that property $\mathcal{P}$ is *testable* in the VDF model if there is a function $q(\varepsilon)$ and a tester for $\mathcal{P}$ that always performs a total number of at most $q(\varepsilon)$ vertex samples and edge queries to the input. We stress again that $\mathcal{D}$ is unknown to the tester, so (in particular) that q should be *independent* of $\mathcal{D}$. The function q is sometimes referred to as the sample (or query) complexity of the tester. A tester has 1-sided error if it always accepts an input satisfying $\mathcal{P}$. Otherwise it has 2-sided error.

If we restrict the distribution $\mathcal{D}$ to be the uniform distribution[2], then the resulting model is "basically" equivalent to the *adjacency matrix model*, which was introduced in [4]. Here we will refer to this model as the *standard model*. A very elegant result proved in [3], states that if $\mathcal{P}$ is testable in the VDF model then it is testable in the standard model with one-sided error. A natural follow-up question, raised by Goldreich in [3], asks whether every property which is testable with one-sided error in the standard model, is also testable in the VDF model. A characterization of the properties testable with one-sided error in the standard model was given in [1], where it was shown that these are precisely the *semi-hereditary* properties (see [1] for the definition of this term). We show that if $\mathcal{P}$ is testable in the VDF model then $\mathcal{P}$ is hereditary. Since there are properties which are semi-hereditary but not hereditary, this implies a negative answer to Goldreich's question. Thus, it is natural to ask the following revised version of Goldreich's question:

Problem 1. *Are all hereditary graph properties testable in the VDF model?*

It might be natural to guess[3] that every hereditary property is testable in the VDF model, the justification being that all lemmas that were used in [1] should also hold for weighted graphs. As it turns out, this is indeed the case. However, putting all these lemmas together does not seem to work in the VDF model. As our main result, Theorem 2 below, shows, it is no coincidence that the proof technique of [1] does not carry over as is to the weighted setting.

We start with an important definition. Let us say that a graph property $\mathcal{P}$ is *extendable* if for every graph G satisfying $\mathcal{P}$ there is a graph G' on $|V(G)| + 1$ vertices which satisfies $\mathcal{P}$ and contains G as an induced subgraph. In other words, $\mathcal{P}$ is extendable if whenever G is a graph satisfying $\mathcal{P}$ and v is a "new" vertex (i.e. $v \notin V(G)$), one can connect v to $V(G)$ in such a way that this larger graph will

[1]Note that the algorithm does not receive $|V(G)|$ as part of the input.

[2]In particular, G is ε-far from $\mathcal{P}$ if one needs to change at least εn^2 edges to turn G into a graph satisfying $\mathcal{P}$.

[3]This was at least our initial guess.

also satisfy $\mathcal{P}$. Note that if $\mathcal{P}$ is extendable then in fact for every graph $G \in \mathcal{P}$ and for every $n > |V(G)|$, there is an n-vertex graph satisfying $\mathcal{P}$ which contains G as an induced subgraph. Our main result is the following:

Theorem 2. *A graph property is testable in the VDF model if and only if it is hereditary and extendable.*

Let us mention some immediate consequences of Theorem 2. Since a graph cannot have both an isolated vertex and a vertex connected to all other vertices, the (hereditary) property of induced H-freess is extendable for every graph H. Hence:

Corollary 3. *The property of being induced H-free is testable in the VDF model for every fixed H.*

It is also clear that the property of being H-free is extendable if and only if H has no isolated vertices. We thus infer that:

Corollary 4. *The property of being H-free is testable in the VDF model if and only if H has no isolated vertices.*

We end this section with the following problem.

Problem 5. *Is it true that every extendable hereditary property $\mathcal{P}$ can be tested in the VDF model with the same (or close to the same) sample complexity as in the (standard) dense graph model?*

While we can give a positive answer to Problem 5 for some properties, there are many natural properties for which we cannot, such as H-freness for H which is not a clique and has no isolated vertices.

1. VDF models with restricted distributions

The proof of the "only if" part of Theorem 2, showing that if $\mathcal{P}$ is either non-extendable or non-hereditary then $\mathcal{P}$ is not testable in the VDF model, relies on allowing the input graph to have only $O(1)$ vertices (where the constant is independent of ε), as well as on having distributions $\mathcal{D}$ that assign to some vertices weight $\Theta(1)$ and to some vertices weight $o(1/|V(G)|)$. This raises the natural question of what happens if we only require the tester to work on sufficiently large graphs, or if we forbid $\mathcal{D}$ from assigning such very low or very high weights. As the following three theorems show, either one of these restrictions has a dramatic effect on the model, since it then allows *all* hereditary properties to be testable.

We start with the setting in which the input graph is guaranteed to be large enough. In a revised version of [3], Goldreich asked whether every hereditary property $\mathcal{P}$ is testable (in the VDF model) on graphs of order at least $M = M_{\mathcal{P}}$, for M which is independent of ε. While the answer to this question turns out to be negatvie, we can show that under the stronger assumption that the input size is at least $M_{\mathcal{P}}(\varepsilon)$ (where $M_{\mathcal{P}} : (0,1) \to \mathbb{N}$), *all* hereditary properties are testable.

Theorem 6. *If $|V(G)| \geq \Omega(1)$ then every hereditary property is testable in the VDF model.*

We now move on to settings in which we pose restrictions on the weights that the distribution $\mathcal{D}$ can assign.

Theorem 7. *If* $\max_{v \in V(G)} \mathcal{D}(v) = o(1)$ *then every hereditary property is testable in the VDF model.*

Theorem 8. *If* $\min_{v \in V(G)} \mathcal{D}(v) = \Omega\left(\frac{1}{n}\right)$ *then every hereditary property is testable in the VDF model.*

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Synchronization Strings: Codes for Insertions and Deletions Approaching the Singleton Bound

Bernhard Haeupler

(joint work with Amirbehshad Shahrasbi)

We introduce *synchronization strings*, which provide **a novel way of efficiently dealing with *synchronization errors***, i.e., insertions and deletions. Synchronization errors are strictly more general and much harder to deal with than more commonly considered *half-errors*, i.e., symbol corruptions and erasures. For every $\varepsilon > 0$, synchronization strings allow to index a sequence with an $\varepsilon^{-O(1)}$ size alphabet such that one can **efficiently transform k synchronization errors into $(1+\varepsilon)k$ half-errors**. This powerful new technique has many applications. In this paper, we focus on designing *insdel codes*, i.e., error correcting block codes (ECCs) for insertion-deletion channels.

While ECCs for both half-errors and synchronization errors have been intensely studied, the later has largely resisted progress. As Mitzenmacher puts it in his 2009 survey: "*Channels with synchronization errors ... are simply not adequately understood by current theory. Given the near-complete knowledge we have for channels with erasures and errors ... our lack of understanding about channels with synchronization errors is truly remarkable.*" Indeed, it took until 1999 for the first insdel codes with constant rate, constant distance, and constant alphabet size to be constructed and only since 2016 are there constructions of constant rate insdel codes for asymptotically large noise rates. Even in the asymptotically large or small noise regime these codes are polynomially far from the optimal rate-distance tradeoff. This makes the understanding of insdel codes up to this work equivalent to what was known for regular ECCs after Forney introduced concatenated codes in his doctoral thesis 50 years ago.

A straight forward application of our synchronization strings based indexing method gives a simple black-box construction which **transforms any ECC into**

an equally efficient insdel code with only a small increase in the alphabet size. This instantly transfers much of the highly developed understanding for regular ECCs into the realm of insdel codes. Most notably, for the complete noise spectrum we obtain efficient "near-MDS" insdel codes which get arbitrarily close to the optimal rate-distance tradeoff given by the Singleton bound. In particular, for any $\delta \in (0,1)$ and $\varepsilon > 0$ we give insdel codes achieving a rate of $1 - \delta - \varepsilon$ over a constant size alphabet that efficiently correct a δ fraction of insertions or deletions.

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Non-concentration of $\chi(G_{n,\frac{1}{2}})$

Annika Heckel

The question of the chromatic number of a random graph was raised in one of the seminal papers of Erdős and Rényi [6]. Grimmett and McDiarmid [8] first found the order of magnitude of $\chi(G_{n,\frac{1}{2}})$ and in a breakthrough paper in 1987 [2], Bollobás established the asymptotic value: whp $\chi(G_{n,\frac{1}{2}}) \sim \frac{n}{2\log_2 n}$.

Several improvements of these bounds were obtained [11, 12, 7], most recently in [9]: whp,

$$\chi(G_{n,\frac{1}{2}}) = \frac{n}{2\log_2 n - 2\log_2\log_2 n - 2} + o\left(\frac{n}{\log^2 n}\right). \tag{1}$$

While these bounds give an explicit interval of length $o\left(\frac{n}{\log^2 n}\right)$ which contains $\chi(G_{n,\frac{1}{2}})$ whp, much narrower concentration is known to hold. The classic result of Shamir and Spencer [14] states that for *any* sequence $p = p(n)$, $\chi(G_{n,p})$ is whp contained in a (non-explicit) sequence of intervals of length about $\sqrt{n}$. For $p = \frac{1}{2}$, this can be improved slightly to about $\frac{\sqrt{n}}{\log n}$ (see [13]).

For sparse random graphs, much more is known: Shamir and Spencer [14] showed that for $p < n^{-\frac{5}{6}-\varepsilon}$, $\chi(G_{n,p})$ is whp concentrated on *five* consecutive values; Łuczak [10] improved this to two consecutive values and finally Alon and Krivelevich [1] showed that two point concentration holds for $p < n^{-\frac{1}{2}-\varepsilon}$.

However, while there are numerous sharp concentration results, there have been no non-trivial cases where $\chi(G_{n,p})$ is known *not* to be extremely narrowly concentrated. (Alon and Krivelevich [1] note that it is trivial that $\chi(G_{n,p})$ is not concentrated on fewer than $\Theta(\sqrt{n})$ values for $p = 1 - 1/(10n)$.)

In 2004, Bollobás [4] asked for any non-trivial examples of non-concentration, specifically suggesting the dense random graph $G_{n,m}$ with $m = \lfloor n^2/4 \rfloor$ (which corresponds to $p = \frac{1}{2}$) and noting that even the weakest results claiming non-concentration would be of interest.

We sketch an argument which shows that $\chi(G_{n,\frac{1}{2}})$ is not whp concentrated on fewer than $n^{\frac{1}{4}-\varepsilon}$ consecutive values.

Theorem 1. *For any constant $\varepsilon > 0$, there is no sequence of intervals of length $n^{\frac{1}{4}-\varepsilon}$ which contain $\chi(G_{n,\frac{1}{2}})$ whp.*

Independence number. The chromatic number of $G_{n,\frac{1}{2}}$ is closely linked to its independence number $\alpha(G_{n,\frac{1}{2}})$, which is very well understood (see [5]). Let

$$\alpha_0 := 2\log_2 n - 2\log_2\log_2 n - 1 + \frac{2}{\log 2} \quad \text{and} \quad a := \lfloor \alpha_0 \rfloor,$$

then whp $\alpha(G_{n,\frac{1}{2}}) = \lfloor \alpha_0 + o(1) \rfloor$, and for most values n, whp $\alpha(G_{n,\frac{1}{2}}) = a$. Furthermore, if we let X_a denote the number of independent sets of size a in $G_{n,\frac{1}{2}}$, then the distribution of X_a is known to be approximately Poisson with mean $\mu_a = \mathbb{E}[X_a]$ (see Theorem 11.9 in [3]). A straightforward calculation shows

$$\mu_a := \binom{n}{a} 2^{-\binom{a}{2}} = n^x \quad \text{where} \quad o(1) \leqslant x(n) \leqslant 1 + o(1).$$

In particular, X_a is not whp contained in any sequence of intervals of length shorter than $\sqrt{\mu_a} = n^{x/2}$.

It is plausible that an optimal colouring of $G_{n,\frac{1}{2}}$ contains all or almost all independent a-sets, because such colourings maximise the expectation for a fixed number of colours. Therefore, intuitively $\chi(G_{n,\frac{1}{2}})$ should vary at least by about $\sqrt{\mu_a}$ (up to a log-factor). We show that this is indeed the case for *some* values n.

Proof sketch for Theorem 1. Suppose we have a sequence of intervals $[s_n, t_n]$ so that whp $\chi(G_{n,p}) \in [s_n, t_n]$. In light of (1), we may assume

$$s_n = \frac{n}{2\log_2 n - 2\log_2\log_2 n - 2} + o\left(\frac{n}{\log^2 n}\right). \tag{2}$$

It is not hard to show that there are infinitely many values n where $x = x(n)$ is slightly less than $\frac{1}{2}$, so that μ_a is slightly less than $n^{\frac{1}{2}}$. Fix such an n. Then by a first moment argument, whp all independent a-sets in $G_{n,\frac{1}{2}}$ are *disjoint*.

Let $r = \lfloor \sqrt{\mu_a} \rfloor \sim n^{x/2}$ be roughly the standard deviation of X_a, let $n' = n + ar$, denote by X'_a the number of independent a-sets in $G_{n',\frac{1}{2}}$ and let $\mu'_a = \mathbb{E}[X'_a]$. Then $\mu'_a = \mu_a + o(1) = n^x + o(1)$, so X'_a and X_a both have essentially the same distribution, namely Poi_{μ_a}, and whp all a-sets are disjoint in $G_{n,\frac{1}{2}}$ and $G_{n',\frac{1}{2}}$.

Fix some $A \approx \mu_a$ such that

$$\mathbb{P}\left(\chi(G_{n,\frac{1}{2}}) \in [s_n, t_n] \;\middle|\; X_a = A, \text{ all } a\text{-sets disjoint}\right) = 1 - o(1), \tag{3}$$

$$\mathbb{P}\left(\chi(G_{n',\frac{1}{2}}) \in [s_{n'}, t_{n'}] \;\middle|\; X'_a = A + r, \text{ all } a\text{-sets disjoint}\right) = 1 - o(1). \tag{4}$$

(Both of these statements hold for almost all A under Poi_{μ_a}.)

The proof now relies on constructing a coupling of essentially the two conditional distributions in (3) and (4), so that the conditional $G_{n,\frac{1}{2}}$ lies within the conditional $G_{n',\frac{1}{2}}$ and their difference consists of exactly r independent a-sets.

For this, let $V' = [n']$, fix some arbitrary disjoint vertex subsets $S_1, \ldots, S_{A+r}$ of size a, and let $V = V' \setminus \bigcup_{i=1}^{r} S_i$. Include every edge between vertices in V' independently with probability $\frac{1}{2}$, and then condition on the sets $S_1, \ldots, S_{A+r}$ being independent and no other independent a-sets being present. Call the resulting random graph G', and let G be the induced subgraph of G' on the vertex set V.

Now note that, if we obtain $\hat{G}'$ by randomly permuting the vertex labels of G', then by symmetry $\hat{G}'$ has exactly the conditional distribution in (4) . This is because, if we condition $G_{n',\frac{1}{2}}$ to contain exactly $A + r$ disjoint a-sets, all possible fixed such collections of a-sets are equivalent. Therefore, whp $\chi(G') \in [s_{n'}, t_{n'}]$.

If we obtain $\hat{G}$ by randomly permuting the vertex labels of G, then $\hat{G}$ does *not* have exactly the conditional distribution in (3). This is because the distribution of G is also conditional on the event that no independent a-sets with some vertices from $\bigcup_{i=1}^{r} S_i$ and some vertices from V are present. As the expected number of such independent sets is $o(1)$, however, it is possible to show that the random graph distributions are *contiguous*. Therefore, whp $\chi(G) \in [s_n, t_n]$.

Since any colouring of G can be extended to a colouring of G' by adding $S_1, \ldots, S_r$ as colour classes, we have $\chi(G') \leqslant \chi(G) + r$. So whp,

$$s_{n'} \leqslant \chi(G') \leqslant \chi(G) + r \leqslant t_n + r = s_n + (t_n - s_n) + r. \tag{5}$$

Recall the estimate for s_n given in (2). Note that

$$\frac{n'}{2\log_2 n' - 2\log_2\log_2 n' - 2} - \frac{n}{2\log_2 n - 2\log_2\log_2 n - 2} = r + \Theta\left(\frac{r}{\log n}\right).$$

Without the error term $o\left(\frac{n}{\log^2 n}\right)$ in (2), together with (5) this would imply

$$t_n - s_n \geqslant \Theta\left(\frac{r}{\log n}\right) = \Theta\left(n\frac{x/2}{\log n}\right) \geqslant n^{\frac{1}{4}-\varepsilon}.$$

To beat the error term, we need to repeat the argument for a sequence $n_1 < n_2 < n_3 < \ldots$, to get $\sum_i (t_{n_i} - s_{n_i}) \geqslant o\left(\frac{n}{\log^2 n}\right) + \sum_i \Theta\left(\frac{r_i}{\log n_i}\right)$. Carefully checking we only need to increase n so much that the assumption that μ_a is less than $n^{\frac{1}{2}}$ remains valid throughout, we find some $n^* \geqslant n$ such that $t_{n^*} - s_{n^*} \geqslant \Theta\left(\frac{r^*}{\log n^*}\right) \geqslant (n^*)^{\frac{1}{4}-\varepsilon}$.

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Sparse hypercontractivity and sharp thresholds

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(joint work with Noam Lifshitz, Eoin Long, Dor Minzer)

The sharp threshold phenomenon has led to fundamental insights spanning a variety of areas of Mathematics, including Combinatorics (pioneered by Margulis in Random Graphs), and Probability (by Russo in Percolation). The interpretation of the sharpness of a threshold as the influence of a Boolean function has led to a further wealth of connections through the Analysis of Boolean functions and its many applications in Mathematics (Gaussian Geometry and Isoperimetry), Computer Science (Computational Complexity) and Economics (Social Choice).

A prominent open problem at the interface of Geometry, Analysis and Combinatorics is to understand the stability of isoperimetric problems. The meta-problem is to characterise sets whose boundary is close to the minimum possible given their volume. The relevant setting for our paper is that of the cube $\{0,1\}^n$, endowed with the p-biased measure μ_p, in which we choose a vector by letting its coordinates be independent random variables each taking the value 1 with probability p. We identify any subset of $\{0,1\}^n$ with its characteristic Boolean function $f : \{0,1\}^n \to \{0,1\}$, and define its 'boundary' as the (total) influence $\mathrm{I}[f] = \sum_{i=1}^n \mathrm{I}_i[f]$, where each $\mathrm{I}_i[f] = \Pr_{\boldsymbol{x}\sim\mu_p}[f(\boldsymbol{x} \oplus e_i) \neq f(\boldsymbol{x})]$, i.e. the *ith influence* $\mathrm{I}_i[f]$ of f is the probability that f depends on bit i at a random input according to μ_p. In the case $p = 1/2$ this is equivalent to the classical combinatorial notion of edge-boundary,[1] for which stability results showing that any

[1]For the vertex boundary, stability results showing that approximately isoperimetric sets are close to Hamming balls were obtained independently by Keevash and Long [8] and by Przykucki and Roberts [11].

approximately isoperimetric set is close to a union of few subcubes were obtained independently by Keller and Lifshitz [9] and Keevash and Long [7]. In particular, the characteristic function is a *junta* (meaning that it depends on few co-ordinates).

However, the case of general p seems significantly more challenging. The problem is somewhat understood in the *dense regime* (where $\mu_p(f)$ is bounded away from 0 and 1) for Boolean functions f that are *monotone* (satisfy $f(x) \leq f(y)$ whenever all $x_i \leq y_i$). Roughly speaking, most results in the dense regime say that monotone Boolean functions of small influence have some 'junta-like' behaviour (see the seminal works of Friedgut–Kalai [3], Friedgut [1, 2], Bourgain [2, Appendix], and Hatami [4]). In particular, Bourgain [2, Appendix] showed that for any monotone Boolean function f with $\mu_p(f)$ bounded away from 0 and 1 and $I[f] \leq \frac{K}{p(1-p)}$ there is a set J of $O(K)$ co-ordinates such that $\mu_p(f_{J\to 1}) \geq \mu_p(f) + e^{-O(K^2)}$, where $f_{J\to 1}$ denotes the function on $\{0,1\}^{\overline{J}}$ obtained from f by setting all bits in J equal to 1. This was strengthened by Hatami [4] who showed that one can even obtain $\mu_p(f_{J\to 1}) \geq 1 - o(1)$, where $|J|$ may be considerably larger (but still dimension-free). Our first result strengthens Bourgain's Theorem in a different way: we replace the term $e^{-O(K^2)}$ by $e^{-O(K)}$, which is sharp.

Theorem 1. *Let $f\colon \{0,1\}^n \to \{0,1\}$ be a monotone Boolean function with $\mu_p(f)$ bounded away from 0 and 1 and $\mathrm{I}[f] \leq \frac{K}{p(1-p)}$. Then there is a set J of $O(K)$ co-ordinates such that $\mu_p(f_{J\to 1}) \geq \mu_p(f) + e^{-O(K)}$.*

On the other hand, the *sparse regime* (allowing any value of $\mu_p(f)$), seemed out of reach of previous methods in the literature (bar [9, 7] for the case $p = 1/2$). Here Kahn and Kalai [5] made a series of interconnected conjectures that have been very influential on subsequent research. One of these (Conjecture 4.1(a)) is that for any $C > 0$ there are $K, \delta > 0$ such that if $p\mathrm{I}[f] \leq C\mu_p(f)\log\mu_p(f)^{-1}$ then there is a set J of $\leq K\log\mu_p(f)^{-1}$ co-ordinates such that $\mu_p(f_{J\to 1}) \geq (1+\delta)\mu_p(f)$. Our second theorem establishes a variant of this conjecture without the log factors (which is also a natural question alluded to by remarks in [5]). Moreover, our lower bound on $\mu_p(f_{J\to 1})$ is significantly stronger than that conjectured in the sparse regime $\mu_p(f) = o(1)$, and it is sharp.

Theorem 2. *Let f be a monotone Boolean function with $p(1-p)\mathrm{I}[f] < K\mu_p(f)$. Then there is a set J of $\leq CK$ co-ordinates, where C is an absolute constant, such that $\mu_p(f_{J\to 1}) \geq e^{-CK}$.*

The results of Friedgut and Bourgain mentioned above also had the striking consequence that any 'global' Boolean function has a 'sharp threshold', which was a breakthrough in the understanding of this phenomenon, as it superceded many results for specific functions. Let f be a monotone Boolean function. We say that f has a *coarse threshold* in an interval $[p,q]$ if $\mu_q(f) = O(\mu_p(f))$; otherwise, we say f has a *sharp threshold* in $[p,q]$. The classical approach for understanding sharp thresholds is based on the Margulis-Russo formula $\frac{d\mu_p(f)}{dp} = \mathrm{I}_{\mu_p}(f)$. Here one notes that if $q = \Theta(p)$, $\mu_p(f) = \Theta(1)$ and $\mu_q(f) < K\mu_p(f)$ then by the

Mean Value Theorem there is some $p' \in [p,q]$ where $p'\mathrm{I}_{\mu_{p'}}(f) = \Theta(1)$. Thus Bourgain's Theorem implies that there is a set J of $O(K)$ co-ordinates such that $\mu_{p'}(f_{J\to 1}) \geq \mu_{p'}(f) + e^{-O(K^2)}$, and Hatami's Theorem that there is a set J of $O_K(1)$ co-ordinates such that $\mu_{p'}(f_{J\to 1}) \geq 1 - o(1)$. Our main sharp threshold result is as follows.

Theorem 3. *For any $\zeta > 0$ there is $C > 0$ so that for any $p, q \in (0, 1/2)$ with $q = (1+\zeta)p$, if f is a monotone Boolean function with $\mu_q(f) < K\mu_p(f)$, where $K > 2$, then there is a set J of $\leq C \log K$ co-ordinates such that $\mu_p(f_{J\to 1}) > K^{-C}$.*

The most significant new feature of our theorem in comparison to the previous sharp threshold results is that it applies in the sparse regime (we allow any $\mu_q(f)$), but nevertheless we can (e.g.) obtain an absolute constant bump in the measure by fixing only a constant number of co-ordinates. Furthermore, the theorem is effective for much larger values of K, even $\mu_p(f)^{-\Theta(1)}$, which will be important for our combinatorial applications in [6]. We also remark that J need grow only logaritheoremically in K, and that our conclusion applies to $\mu_p(f_{J\to 1})$, not just to $\mu_{p'}(f_{J\to 1})$ for some unknown $p' \in [p,q]$. The latter point reflects our use of the recent new approach of Lifshitz [10] for obtaining sharp threshold results from noise sensitivity.

Many of the fundamental results in the Analysis of Boolean functions, including the characterisations of sharp thresholds discussed above, are based on a classical hypercontractive inequality for the noise operator obtained independently by Bonami, Gross and Beckner. The results were first proved for the uniform (1/2-biased) measure, but can be extended to any p, and so imply sharp threshold results in intervals $[p,q]$ where p and q are bounded away from 0 and 1. These hypercontractive inequalities are necessarily ineffective in the sparse regime, but our key insight is that the only obstructions to a better inequality are junta-like functions. The most fundamental contribution of this paper is a new hypercontractive theorem for quasiregular functions that is applicable for establishing noise sensitivity, and so sharp thresholds, in the sparse regime. We also apply our results to obtain a version of the Invariance Principle of Mossel, O'Donnell and Oleszkiewicz that is effective in the sparse regime.

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The maximum length of K_r-Bootstrap Percolation

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(joint work with József Balogh, Alexey Pokrovskiy, and Tibor Szabó)

Weak saturation of graphs was introduced by Bollobás in 1968 [8]. A graph G on n vertices is *weakly saturated* with respect to a graph H, if G has no copies of H, but there exists an ordering of $E(K_n)\setminus E(G) = \{e_1, \dots, e_t\}$ such that the addition of e_i to $G \cup \{e_1, \dots, e_{i-1}\}$ will create a new copy of H, for every $i \in [t]$. It was later noticed by Balogh, Bollobás and Morris [2] that weak saturation is strongly related to the so-called *bootstrap percolation process*, which is a type of cellular automata introduced in 1979 by Chalupa, Leath, and Reich [10].

For our setting, we first redefine the notion of a weakly saturated graph in terms of an infection process, known as the *graph-bootstrap percolation*. For graphs F, H we describe the (F, H)*-bootstrap process* as follows. We start with an initial *infected set of edges* $E_0 \subseteq E(F)$ and write $G_0 := (V(F), E_0)$ (sometimes called the *starting graph*). At each step, an edge of F becomes infected if it completes an infected copy of H. More formally, denote by $n_H(G)$ the number of copies of H in a graph G. Let

$$G_t = G_{t-1} \cup \{e \in E(F) \mid n_H(G_{t-1} \cup \{e\}) > n_H(G_{t-1})\}$$

and $E_t = E(G_t)$. We say that the *running time* of the (F, H)-bootstrap process is t, if t is the minimum integer such that $G_{t+1} = G_t$. In this case we say that $\langle G_0\rangle_{(F,H)} := G_t$ is the *final graph* (also called the *closure* of the starting graph G_0 under the (F, H)-bootstrap percolation). We say that E_0 *percolates* if every edge of F is eventually infected, that is, if the final graph $\langle G_0\rangle_{(F,H)} = F$. In the special case when $F = K_n$, we refer to the (K_n, H)-bootstrap percolation process as the H-process.

Much work has been done on the extremal properties of the K_r-process. Alon [1], Frankl [11], and Kalai [12], showed that the smallest percolating set of edges in the K_r-process in K_n has size $\binom{n}{2} - \binom{n-r+2}{2}$, thus confirming a conjecture of Bollobás [8]. This question was also studied for other graphs F and H, see [1, 3, 4, 14, 15].

Despite missing almost all the $\binom{n}{2}$ edges of K_n, the smallest percolating starting graph in the K_r-process percolates very fast: every non-edge is the sole missing edge from a copy of K_r, so is added simultanously in the very first step of the process. Nevertheless, in applications the speed of percolation is quite relevant.

In this direction Bollobás raised the extremal problem of determining the **slowest** percolating set in the bootstrap process (i.e. the one that has the maximum running time). Benevides and Przykucki [6, 7, 16] studied this problem in the related setting of neighborhood percolation. The question for the K_r-process on the edges was investigated independently by Bollobás, Przykucki, Riordan and Sahasrabudhe [9] and Matzke [13]. They defined

$$M_r(n) = \max\{t \mid \exists G_0 \subseteq K_n \text{ such that } G_t \neq G_{t-1} \text{ in the } K_r\text{-bootstrap process}\}$$

to be the maximum running time for the K_r-bootstrap percolation on n vertices until it stabilizes, taken over all starting graphs. Again it is easy to see that in the K_3-process the diameter of the infected graph decreases at least by a factor of two in each step, and hence $M_3(n) = \lceil \log_2(n-1) \rceil$. For the K_4-process the precise answer was found and turned out to be linear in n.

Theorem 1 ([13, 9])**.** *$M_4(n) = n - 3$ for all $n \geq 3$.*

For $r \geq 5$ Bollobás, Przykucki, Riordan and Sahasrabudhe gave subquadratic polynomial lower bounds with the exponents tending to 2 as r tends to infinity.

Theorem 2 (Theorem 2 in [9])**.** *For each fixed $r \geq 5$, we have $M_r(n) \geq n^{2-\frac{1}{\lambda(r)}-o(1)}$ as $n \to \infty$, where $\lambda(r) = \frac{\binom{r}{2}-2}{r-2}$.*

As for an upper bound, in [9] it was conjectured that the running time of any K_r-bootstrap percolation process should be subquadratic for $r \geq 5$.

Conjecture 3. [9] *For all $r \geq 5$ we have $M_r(n) = o(n^2)$.*

In our first main result, we disprove Conjecture 3 for all $r \geq 6$.

Theorem 4. *For every $r \geq 6$ and large enough n, we have $M_r(n) \geq \frac{n^2}{2500}$.*

Our construction of the starting graph for the slow K_r-process does not obviously extend to $r = 5$. Nevertheless, some of the ideas can be salvaged by utilizing sets of integers having no arithmetic progression of length three. Using the relevant constructions from additive number theory allows us to improve the lower bound of [9] for $M_5(n)$ to almost quadratic.

A set B of numbers is called 3-*AP-free* if for any $b_1, b_2, b_3 \in B$ with $2b_1 = b_2 + b_3$, we have $2b_1 = b_2 + b_3$. Let $r_3(n)$ denote the largest cardinality of a 3-AP-free subset of $[n]$. The estimation of $r_3(n)$ (and its generalization for k-AP-free subsets) is a cornerstone of additive number theory, with a rich history that also involves inspiring a significant portion of modern combinatorics. Behrend [5] showed that there are 3-AP-free subsets of $[n]$ of size $n^{1-1/O(\sqrt{\log n})}$. From the other side the function $r_3(n)$ was shown to be $o(n)$ by Roth in 1953 using analytic number theory. Later this was also proved by various other methods, including combinatorics, ergodic theory, and non-standard analysis. The current best upper bound is due to Bloom.

Here we connect $r_3(n)$ to the extremal function $M_5(n)$ of slow K_5-bootstrap percolation.

Theorem 5. $M_5(n) \geq \frac{nr_3(n)}{1200}$. *In particular,* $M_5(n) \geq n^{2-O(1/\sqrt{\log n})}$.

Note that Conjecture 3 is still open for $r = 5$ and we tend to agree with the authors of [9] about its validity. Our optimism stems from our combinatorial Key Lemma, which connects $M_5(n)$ to the maximum number of edges of a graph with certain combinatorial properties. There are several other combinatorial upper bounds on $r_3(n)$ of similar nature, which were succesfully bounded from above. For example, the maximum number of edges in an n-vertex graph with every edge participating in exactly one triangle is known to be $o(n^2)$ by the Regularity Lemma. The positive resolution of Conjecture 3 would closely tie the classic additive number theoretic function $r_3(n)$ to percolation.

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Anticoncentration for subgraph counts in random graphs

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(joint work with Jacob Fox, Lisa Sauermann)

Fix a graph H and some $p \in (0,1)$, and let X_H be the number of copies of H in a random graph $\mathbb{G}(n,p)$. Subgraph-counting random variables of this form are central objects of study in the theory of random graphs, going back to the foundational work of Erdős and Rényi [5].

In the setting where $p \in (0,1)$ is a constant, X_H has an asymptotically normal distribution (this was proved by Nowicki and Wierman [12] and Ruciński [13]). Further work by Barbour, Karoński and Ruciński [1] provided quantitative bounds on the rate of convergence to the normal distribution. However, this asymptotic normality only characterises the "large-scale" behaviour of the distribution of X_H, and is basically due to the fact that X_H closely correlates with the number of edges in $\mathbb{G}(n,p)$.

Recently, there has been more attention on "local" aspects of the distribution of X_H. Following work by Loebl, Matoušek and Pangrác [10] for the case where H is a triangle, it was proved by Kolaitis and Kopparty [9] (see also [4]) that if we fix some $p \in (0,1)$, some prime $q \in \mathbb{N}$ and some connected graph H with at least one edge, then $X_H \bmod q$ has an asymptotically uniform distribution on $\{0,\dots,q-1\}$. Even more recently, *local central limit theorems* have begun to emerge, giving asymptotic formulas for the point probabilities $\Pr(X_H = x)$ in terms of a normal density function. Such a theorem was first proved for the case where H is a triangle by Gilmer and Kopparty [7] (see also [2]), and this was extended by Berkowitz [3] to the case where H is any clique.

A somewhat looser question was suggested by Meka, Nguyen and Vu [11], and is the subject of this abstract: what can be said about the *anticoncentration* behaviour of X_H? Roughly speaking, this is asking for uniform upper bounds on the point probabilities $\Pr(X_H = x)$, or more generally on the small ball probabilities $\Pr(X_H \in I)$, where I is an interval of prescribed length. As one of the main applications of the polynomial anticoncentration inequalities they developed in their paper, Meka, Nguyen and Vu used the polynomial structure of X_H to prove the bound $\Pr(X_H = x) \le n^{-1+o(1)}$ for constant $p \in (0,1)$ and any H that contains at least one edge. However, this seems to be far from optimal. A natural conjecture is as follows.

Conjecture 1. *Fix $p \in (0,1)$ and fix a graph H with no isolated vertices. Then*

$$\max_{x \in \mathbb{N}} \Pr(X_H = x) = O\left(n^{1-v(H)}\right).$$

The motivation for Conjecture 1 is that if p is fixed then $\operatorname{Var} X_H = \Theta\left(n^{2v(H)-2}\right)$, so X_H is concentrated on an interval of length $\Theta\left(n^{v(H)-1}\right)$. Provided that the distribution of X_H is sufficiently "smooth", we should expect each value in this interval to have comparable probability. Note that this line of reasoning implies that Conjecture 1, if true, is best possible: any stronger bound would contradict

the central limit theorem known to hold for X_H. Also, observe that the assumption that H has no isolated vertices is necessary: if H' is obtained from H by removing isolated vertices then X_H is a deterministic multiple of $X_{H'}$, so inherits its point probabilities.

Recently, we made two new contributions to this area. First, we were able to use ideas related to Erdős' combinatorial proof of the Erdős–Littlewood–Offord theorem (see [6]) to give a simple proof of the following general bound, improving on the aforementioned result of Meka, Nguyem and Vu.

Theorem 2. *Fix $p \in (0,1)$ and fix a graph H with at least one edge. Then*

$$\max_{x \in \mathbb{N}} \Pr\left(|X_H - x| \leq n^{v(H)-2}\right) = O(1/n).$$

Second, we were able to prove an approximate version of Conjecture 1 for all connected H.

Theorem 3. *Fix $p \in (0,1)$ and fix a connected graph H. Then*

$$\max_{x \in \mathbb{N}} \Pr\left(X_H = x\right) = n^{1-v(H)+o(1)}.$$

The proof of Theorem 3 proceeds via an inductive decomposition of X_H into random variables which are "almost" independent and which fluctuate at "different scales". There are several other ingredients; one that is perhaps worth highlighting is a combinatorial anticoncentration inequality for vector-valued random variables that behave "almost linearly", in the spirit of some anticoncentration theorems due to Halász [8].

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Distinct degrees in induced subgraphs

Eoin Long

(joint work with Peter Keevash, Matthew Jenssen and Liana Yepremyan)

Given a graph G we write

$$\hom(G) := \max\big\{|U| : U \subset V(G) \text{ with } G[U] \text{ complete or empty}\big\}.$$

The quantitative form of Ramsey's theorem due to Erdős and Szekeres [6] shows that any N-vertex graph G satisfies $\hom(G) \geq \frac{1}{2}\log_2 N$. On the other hand, Erdős [5] showed that N-vertex graphs exist with $\hom(G) \leq 2\log_2 N$. In this context, G is said to be C-Ramsey if $\hom(G) \leq C\log|V(G)|$; simply a 'Ramsey graph' if $\hom(G) = O(\log|V(G)|)$.

One challenge in narrowing the gap between the upper and lower bounds in the Ramsey problem is a lack of examples. Indeed, all known examples of graphs with $\hom(G) = O(\log|V(G)|)$ are inexplicit, arising as the result of some random process. It is an old problem of Erdős to explicitly construct such graphs and much effort has gone into furthering our understanding here (see [1], [8]).

Motivated in part by the difficulty in providing such explicit constructions, a focus of study has been on understanding the intrinsic properties of Ramsey graphs. Since all known Ramsey graphs are randomly generated – often according to the Erdős-Renyi random graph $G(N,p)$ – these random models provide natural benchmarks. This indirect study has been very fruitful, and it is known that N-vertex Ramsey graphs and $G(N,1/2)$ display similar behaviour in a number of respects: the edge density by Erdős and Szemerédi [7]; universality of small induced subgraphs by Prömel and Rödl [13]; the number of non-isomorphic induced subgraphs by Shelah [14]; the sizes and orders of induced subgraphs by Kwan and Sudakov [9, 10], and Narayanan, Sahasrabudhe and Tomon [12].

Here we focus on another parameter that has been studied in this context. Given a graph G, let

$$f(G) := \max\big\{k \in \mathbb{N} : G \text{ has an induced subgraph with } k \text{ distinct degrees}\big\}.$$

A simple argument gives that $G(N,1/2)$ typically has $\Omega(N^{1/2})$ distinct degrees, proving that $f\big(G(N,1/2)\big) = \Omega(N^{1/2})$ and it was conjectured by Erdős, Faudree and Sós [4] that this bound also holds for Ramsey graphs. This was confirmed by Bukh and Sudakov [2], who showed that an N-vertex graph G with $\hom(G) \leq C\log N$ has $f(G) = \Omega_C(N^{1/2})$.

Although $\Omega(N^{1/2})$ gives a natural lower bound for $f\big(G(N,1/2)\big)$, Bukh and Sudakov noted in [2] that no known upper bound of this order is known, and provided an upper bound $f\big(G(N,1/2)\big) = O(N^{2/3})$. An unpublished result of Conlon,

Morris, Samotij and Saxton [3] shows that this is in fact correct for $G(N,1/2)$; whp $f\big(G(N,1/2)\big) = \Theta(N^{2/3})$.

Our first theorem shows that Ramsey graphs essentially match this behaviour.

Theorem 1. *Let G be an N-vertex C-Ramsey graph. Then $f(G) = \Omega_C\big(\frac{N^{2/3}}{\log^{1/3} N}\big)$.*

The relationship between $\hom(G)$ and $f(G)$ was also investigated by Narayanan and Tomon [11]. For example, they proved that if given $\varepsilon \in (0,1/2)$ then every N-vertex G with $\hom(G) \le N^{\varepsilon}$ satisfies $f(G) = \Omega(N^{1/2-\varepsilon})$. They also proved that the relationship between $\hom(G)$ and $f(G)$ significantly simplifies provided $\hom(G) \gg f(G)$: given $k \in \mathbb{N}$ and $\varepsilon > 0$, every N-vertex graph G with $f(G) < k$ has $\hom(G) \ge N/(k-1+\varepsilon)$, provided $N \ge N_0(k,\varepsilon) = 2^{\Omega_\varepsilon(k)}$. This is almost optimal, as for $k < \ell$ then the $(k-1)$-partite Turán graph on $N = (k-1)(n-1)$ vertices has $f(G) \le k-1$ and $\hom(G) \le n-1 = N/(k-1)$.

Narayanan and Tomon conjectured that in fact this Turán graph determines the optimal relationship between $\hom(G)$ and $f(G)$, provided $f(G) \ll |V(G)|$. Our second theorem confirms this conjecture.

Theorem 2. *Suppose G is an N-vertex graph with $N > (n-1)(k-1)$ where $n = \Omega(k^9)$. Then either $\hom(G) \ge n$ or $f(G) \ge k$.*

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Spanning cycles in random digraphs

Richard Montgomery

A foundational result in the study of random graphs is the determination of the threshold for a Hamilton cycle to appear in the binomial random graph $G(n,p)$, achieved independently by Pósa [7] and Korshunov [5] in 1976. Independently, Bollobás [1], and Komlós and Szemerédi [4] showed in 1983 that, if $p = (\log n + \log\log n + \omega(1))/n$, then $G(n,p)$ is almost surely Hamiltonian. This is the same probability needed to ensure that, almost surely, $\delta(G(n,p)) \geq 2$, and thus the obstruction to Hamiltonicity in the random graph is that every vertex need have at least 2 neighbours. The key method behind these results is *Pósa rotation*, introduced in [7], which has been crucial in many subsequent results concerning Hamiltonicity in random graphs.

Questions involving Hamiltonicity have also been studied in the directed random graph, $D(n,p)$, where each possible directed edge among n vertices is chosen independently at random with probability p. In the directed setting, the most natural cycle to consider is the directed Hamilton cycle, where the edge directions run consistently around the cycle, but we can consider any orientation of an n-vertex cycle, and study its appearance in $D(n,p)$.

McDiarmid [6] gave a beautiful coupling argument which demonstrates that, for any p, $D(n,p)$ is at least as likely to contain a directed Hamilton cycle as $G(n,p)$ is to contain an (undirected) Hamilton cycle. However, the natural local obstruction to Hamiltonicity in $D(n,p)$ is that every vertex need have both an in- and an out-neighbour. This property almost surely occurs in $D(n,p)$ if $p = (\log n + \omega(1))/n$, and almost surely does not occur if $p = (\log n - \omega(1))/n$. Frieze [3] confirmed that this is also when Hamiltonicity is likely to occur in $D(n,p)$. That is, if $p = (\log n + \omega(1))/n$, then $D(n,p)$ almost surely contains a directed Hamilton cycle.

When studying arbitrarily oriented directed cycles, Ferber and Long [2] observed that McDiarmid's coupling argument shows that, given any oriented n-vertex cycle C, a copy of C is likely to appear in $D(n,p)$ if $p = (\log n + \log\log n + \omega(1))/n$. Furthermore, they conjectured that this should be true if $p = (\log n + \omega(1))/n$. Another interesting direction of enquiry begins by asking when the random digraph $D(n,p)$ is likely to contain a copy of *every* possible oriented n-vertex cycle.

In this talk, we discussed how to use McDiarmid's coupling in conjuction with constructive methods, including Pósa rotation, to confirm the conjecture of Ferber and Long, while finding all such cycles simultaneously. These methods can, moreover, be used to give a more accurate result in the directed random graph process (generalising a similar result for directed Hamilton cycles by Frieze [3]). However, this talk concentrated on showing that, if $p = (\log n + \omega(1))/n$, then $D(n,p)$ almost surely contains a copy of every oriented n-vertex cycle. The key method is to show that a pseudorandom digraph with additional pairs of random edges added between random vertex pairs is likely to contain any specific n-vertex cycle, with probability at least $1 - \exp(-n)$. Using McDiarmid's coupling and

a union bound over all possible orientations of an n-vertex cycle then gives our result.

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The upper tail for triangles in random graphs

Frank Mousset

(joint work with Matan Harel, Wojciech Samotij)

This talk is based on a recent joint paper [8] with M. Harel and W. Samotij. Let $X = X_{n,p}$ be the number of triangles in the random graph $G_{n,p}$. Perhaps the simplest question that can be asked about the typical behaviour of X is whether it is concentrated around its expectation. Once this is established, it is natural to ask for quantitative estimates of the tail probabilities $\Pr(X \leq (1-\delta)\mathbb{E}[X])$ and $\Pr(X \geq (1+\delta)\mathbb{E}[X])$. As it turns out, these two probabilities are governed by very different phenomena. On the one hand, using a combination of Harris's inequality [9] and Janson's inequality [10], one can show that

$$-\log \Pr\big(X \leq (1-\delta)\mathbb{E}[X]\big) = \Theta_\delta(\min\{n^2p, n^3p^3\}) \tag{1}$$

On the other hand, there are no comparably simple tools that allow one to easily obtain similar estimates on the logarithm of the upper tail probability, which is the topic of this talk.

An important reason for this difficulty is that, unlike the lower tail, the upper tail is susceptible to the influence of small subgraphs whose appearance increases the value X atypically, a phenomenon that we refer to as *localisation*. To give a somewhat extreme example, observe that there are graphs with n vertices and $O_\delta(n^2p^2)$ edges that contain contain $(1+\delta)\mathbb{E}[X]$ triangles (consider a clique of size $C_\delta np$ for a large enough C_δ). The upper tail probability is at least the probability that $G_{n,p}$ contains such a graph, that is,

$$-\log \Pr\big(X \geq (1+\delta)\mathbb{E}[X]\big) \leq O_\delta(n^2p^2\log(1/p)). \tag{2}$$

A sequence of papers [11, 12, 13, 15], culminating in the work of Chatterjee [2] and DeMarco–Kahn [6], established that

$$-\log \Pr\big(X \geq (1+\delta)\mathbb{E}[X]\big) = \Theta_\delta(\min\{\mathbb{E}[X], n^2p^2\log(1/p)\}),$$

which matches the upper bound (2) in the range $p \geq n^{-1}\log n$. This leaves open the problem of determining the dependence of the upper tail on the constant δ.

In order to quantify this dependence, define

$$\Phi_X(\delta) = \min\Big\{e_G \log(1/p) : G \subseteq K_n \text{ and } \mathbb{E}[X \mid G \subseteq G_{n,p}] \geq (1+\delta)\mathbb{E}[X]\Big\}. \tag{3}$$

It is not difficult to show that

$$-\log \Pr\big(X \geq (1+\delta)\mathbb{E}[X]\big) \leq (1+o(1))\cdot\Phi_X(\delta). \tag{4}$$

The last decade has seen the development of an increasingly powerful theory of 'nonlinear large deviations' [1, 3, 4, 5, 7] that provides a variational description of the dependence of the upper tail probability on δ in a certain range of p. In particular, these results show that the upper bound (4) is asymptotically optimal in the range $n^{-1/2}(\log n)^{O(1)} \ll p \ll 1$ [1, 5, 14]. Our main results for triangles pin down the dependence on δ in a much larger range of densities.

Theorem 1. *Let X denote the number of triangles in $G_{n,p}$. Then, for every fixed positive constant δ and all $p = p(n)$ satisfying $n^{-1}\log n \ll p \ll 1$,*

$$-\log \Pr\big(X \geq (1+\delta)\mathbb{E}[X]\big) = (1\pm o(1))\cdot\Phi_X(\delta).$$

Theorem 2. *Let X denote the number of triangles in $G_{n,p}$. Then, for every fixed positive constant δ and all $p = p(n)$ satisfying $n^{-1} \ll p \ll n^{-1}\log n$,*

$$\lim_{n\to\infty} \frac{-\log \Pr\big(X \geq (1+\delta)\mathbb{E}[X]\big)}{\mathbb{E}[X]} = (1+\delta)\log(1+\delta) - \delta.$$

Together, these theorems resolve the upper tail problem for X nearly completely. At the heart of the proof of Theorem 1 lies a general method for proving bounds on the upper tail probability of 'low-complexity' functions of i.i.d. Bernoulli variables. The proof uses an adaptation of the classical moment argument of Janson, Oleszkiewicz, and Ruciński [11], which we use to formalise the intution that the upper tail event is dominated by the appearance of near-minimisers of the combinatorial optimisation problem (3). Roughly speaking, we say that a graph $G \subseteq K_n$ is a *core* if it is a feasible set for the above optimisation problem, its size is $O\big(\Phi_X(\delta)\big)$, and it satisfies a certain natural rigidity condition. We show that the upper tail probability is approximately equal to the probability of the appearance of a core. In particular, when the number of cores of size m is $(1/p)^{o(m)}$, a property we term *entropic stability*, then a union bound implies that $-\log \Pr\big(X \geq (1+\delta)\mathbb{E}[X]\big)$ is well-approximated by $\Phi_X(\delta)$; this turns out to be the case precisely when $n^{-1}\log n \ll p \ll 1$. The proof of Theorem 2 involves a change of measure and factorial moment estimates.

The asymptotics of $\Phi_X(\delta)$ are given by the following theorem, which extends an earlier result due to Lubetzky–Zhao [14]. For positive reals δ and c, we define

$$\varphi(\delta, c) = \min\left\{\frac{\delta^{2/3}}{2}, \frac{\lfloor \delta c/3 \rfloor + \{\delta c/3\}^{1/2}}{c}, \frac{\lfloor \delta c/3 \rfloor}{c} + \frac{(r\{\delta c/3\}/c)^{2/r}}{2}\right\}.$$

Theorem 3. *Let X denote the number of triangles in $G_{n,p}$. Then, for every fixed positive constant δ and all $p = p(n)$ satisfying $n^{-1} \ll p \ll 1$,*

$$\lim_{n\to\infty} \frac{\Phi_X(\delta)}{n^2p^2\log(1/p)} = \begin{cases} \delta^{2/3}/2 & \text{if } np^2 \to 0, \\ \varphi(\delta, c) & \text{if } np^2 \to c \in (0,\infty), \\ \min\left\{\delta^{2/3}/2, \delta/3\right\} & \text{if } np^2 \to \infty. \end{cases}$$

The different possible values for this limit correspond to different types of subgraphs achieving the minimum in the definition of $\Phi_X(\delta)$. Our methods also allow us to show that if one conditions $G_{n,p}$ on the upper tail event, then with high probability, $G_{n,p}$ contains a graph closely resembling one of these minimisers. The results mentioned above extend to the case where one replaces triangles by larger cliques and, to some extent, by nonbipartite regular graphs.

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Resolution of the Oberwolfach Problem

Deryk Osthus
(joint work with Stefan Glock, Felix Joos, Jaehoon Kim, and Daniela Kühn)

A central theme in Combinatorics and related areas is the decomposition of large discrete objects into simpler or smaller objects. In graph theory, this can be traced back to the 18th century, when Euler asked for which orders orthogonal Latin squares exist (which was finally answered by Bose, Shrikhande, and Parker [1]). This question can be reformulated as the existence question for resolvable triangle decompositions in the balanced complete tripartite graph. (Here a resolvable triangle decomposition is a decomposition into edge-disjoint triangle factors.) In the 19th century, Walecki proved the existence of decompositions of the complete graph K_n (with n odd) into edge-disjoint Hamilton cycles and Kirkman formulated the school girl problem. The latter triggered the question for which n the complete graph on n vertices admits a resolvable triangle decomposition, which was finally resolved in the 1970s by Ray-Chaudhuri and Wilson [8] and independently by Jiaxi. This topic has developed into a vast area with connections e.g. to statistical design and scheduling, Latin squares and arrays, graph labellings as well as combinatorial probability.

A far reaching generalisation of Walecki's theorem and Kirkman's school girl problem is the following problem posed by Ringel in Oberwolfach in 1967.

Problem 1 (Oberwolfach problem)**.** *Let $n \in \mathbb{N}$ and let F be a 2-regular graph on n vertices. For which (odd) n and F does K_n decompose into edge-disjoint copies of F?*

Addressing conference participants in Oberwolfach, Ringel fittingly formulated his problem as a scheduling assignment for diners: assume n people are to be seated around round tables for $\frac{n-1}{2}$ meals, where the total number of seats is equal to n, but the tables may have different sizes. Is it possible to find a seating chart such that every person sits next to any other person exactly once?

We answer this affirmatively for all sufficiently large n. Note that the Oberwolfach problem does not have a positive solution for every odd n and F (indeed, there are four known exceptions).

A further generalisation is the Hamilton-Waterloo problem; here, two cycle factors are given and it is prescribed how often each of them is to be used in the decomposition. We also resolve this problem in the affirmative (for large n) via the following even more general result. We allow an arbitrary collection of types of cycle factors, as long as one type appears linearly many times. This immediately implies that the Hamilton-Waterloo problem has a solution for large n for any

bounded number of given cycle factors. (In [5], we actually state and prove an even more general result.)

Theorem 2 ([5]). *For every $\alpha > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all odd $n \geq n_0$ the following holds. Let $F_1, \dots, F_k$ be 2-regular graphs on n vertices and let $m_1, \dots, m_k \in \mathbb{N}$ be such that $\sum_{i \in [k]} m_i = (n-1)/2$ and $m_1 \geq \alpha n$. Then K_n admits a decomposition into graphs $H_1, \dots, H_{(n-1)/2}$ such that for exactly m_i integers j, the graph H_j is isomorphic to F_i.*

The Oberwolfach problem and its variants have attracted the attention of many researchers, resulting in more than 100 research papers covering a large number of partial results. Most notably, Bryant and Scharaschkin [2] proved it for infinitely many n. Most classical results in the area are based on algebraic approaches, often by exploiting symmetries. More recently, major progress for decomposition problems has been achieved via absorbing techniques in combination with approximate decomposition results (often also in conjunction with probabilistic ideas). In [5], at a very high level, we also pursue such an approach. As approximate decomposition results, we exploit a hypergraph matching argument due to Alon and Yuster (which in turn is based on the Rödl nibble via the Pippenger-Spencer theorem) and a bandwidth theorem for approximate decompositions due to Condon, Kim, Kühn, and Osthus [3]. Our absorption procedure utilizes as a key element a very special case of a recent result of Keevash on resolvable designs [6].

Earlier, Kim, Kühn, Osthus, and Tyomkyn [7] considered approximate decompositions into graphs of bounded degree in host graphs G satisfying weaker quasirandom properties (namely, ε-superregularity). Their resulting blow-up lemma for approximate decompositions was a key ingredient for [3] (and thus for Theorem 2). It already implies that an approximate solution to the Oberwolfach problem can always be found (the latter was obtained independently by Ferber, Lee, and Mousset [4]).

While considerably more general than the Oberwolfach problem, Theorem 2 may be just the tip of the iceberg, and it seems possible that the following is true.

Conjecture 3. *For all $\Delta \in \mathbb{N}$, there exists an $n_0 \in \mathbb{N}$ so that the following holds for all $n \geq n_0$. Let $F_1, \dots, F_t$ be n-vertex graphs such that F_i is r_i-regular for some $r_i \leq \Delta$ and $\sum_{i \in [t]} r_i = n - 1$. Then there is a decomposition of K_n into $F_1, \dots, F_t$.*

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Asymptotic Enumeration and Limit Laws for Multisets

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(joint work with Leon Ramzews)

Let $\mathcal{C}$ be a combinatorial class, that is, a countable set endowed with a size function $|\cdot| : \mathcal{C} \to \mathbb{N}_0$ such that $\mathcal{C}_n := \{C \in \mathcal{C} : |C| = n\}$ contains only finitely many objects for all $n \in \mathbb{N}_0$. Assume that $\mathcal{C}_0 = \emptyset$. Then the combinatorial class of multisets $\mathcal{G} = \mathsf{Mset}(\mathcal{C})$ is given by the collection of all objects of the form

$$\{(C_1, d_1), \dots, (C_k, d_k)\}, \quad C_i \in \mathcal{C}, d_i \in \mathbb{N}, k \in \mathbb{N}$$

for some $k \in \mathbb{N}$, where $(C_i)_{1 \le i \le k}$ are pairwise distinct objects from $\mathcal{C}$ and d_i describes the multiplicity of C_i in the multiset. Given $G = \{(C_1, d_1), \dots, (C_k, d_k)\} \in \mathcal{G}$ we denote by

$$|G| := \sum_{1 \le i \le k} d_i |C_i| \quad \text{and} \quad \kappa(G) := \sum_{1 \le i \le k} d_i$$

the *size* and the number of *components* of G. Let further $\mathcal{G}_n := \{G \in \mathcal{G} : |G| = n\}$ and $\mathcal{G}_{n,N} := \{G \in \mathcal{G}_n : \kappa(G) = N\}$ for $n, N \in \mathbb{N}$. We associate to $\mathcal{C}$ and $\mathcal{G}$ the (ordinary) generating series in two formal variables x and y

$$C(x) := \sum_{k \in \mathbb{N}} |\mathcal{C}_k| x^k \quad \text{and} \quad G(x, y) := \sum_{k, \ell \in \mathbb{N}} |\mathcal{G}_{k,\ell}| x^k y^\ell.$$

We thus have $|\mathcal{G}_{n,N}| = [x^n y^N] G(x, y)$ for all $n, N \in \mathbb{N}$. These two power series are known to fulfil the fundamental relation, see for example [5],

$$G(x, y) = \exp\left\{ \sum_{j \ge 1} y^j \frac{C(x^j)}{j} \right\}. \tag{1}$$

The described setting encompasses a variety of situations, and in particular the (asymptotic) magnitude of $g_n := |\mathcal{G}_n| = [x^n] G(x) := [x^n] G(x, 1)$ has been studied intensively in various works. For example, the most basic case that $\mathcal{C} = \mathbb{N}$, where $\mathcal{G}$ is then the class of integers partitions, was already studied in 1918 by Hardy and Ramanujan [4]. Since then, the asymptotic value of g_n for general $\mathcal{C}$ has been the topic of study in several works, and we refer to [2, 3, 6, 7] for some more recent developments.

Here we will study a situation that is frequently encountered in asymptotic enumeration. In particular, we assume that $C(x)$ is *subexponential* with radius of convergence $0 < \rho < 1$, that is, setting $(c_k)_{k\in\mathbb{N}} := (|\mathcal{C}_k|)_{k\in\mathbb{N}}$, we assume that

$$\frac{c_{n-1}}{c_n} \sim \rho \quad \text{and} \quad c_n^{-1} \sum_{1\le k\le n} c_k c_{n-k} \sim 2C(\rho) < \infty, \quad n \to \infty.$$

A crucial assumption here is that C converges at ρ. A prototypical situation where this is satisfied is when $c_n \sim c \cdot L(n) \cdot n^{-\alpha} \cdot \rho^{-n}$, for some $c > 0$, some slowly varying function L, and some $\alpha > 1$. Such counting sequences are ubiquitous in enumerative combinatorics, with examples being the number the number of (unlabeled) trees or more general the number of graphs with n vertices in some subcritical class [8].

An important result in the asymptotic enumeration of multisets with subexponential $C(x)$ was obtained in [1]. There it was shown that the number of multisets of size n with N components behaves like

$$g_{n,N} \sim \frac{[y^{N-1}]G(\rho, y)}{G(\rho, 1)} g_n, \quad \text{as} \quad N \to \infty.$$

Here we resolve the counting problem for (essentially) all values of N. Let $m \equiv m(C) \in \mathbb{N}$ be such that $c_m > 0$ and $c_1 = \cdots = c_{m-1} = 0$. This allows for restricting the class $\mathcal{C}$ to only have elements of size at least m.

THEOREM 1. *As $n, n - N \to \infty$,*

$$[x^n y^N]G(x, y) \sim A \cdot N^{c_m - 1} \cdot c_{n-m(N-1)},$$

where

$$A = \frac{1}{\Gamma(c_m)} \exp\left\{ \sum_{j\ge 1} \frac{C(\rho^j) - c_m \rho^{jm}}{j\rho^{jm}} \right\}.$$

Let us take a closer look at this formula. Suppose for simplicity that $m = c_m = 1$. Then the theorem asserts that the number of multisets of size n with N components is (asymptotically) proportional to c_{n-N}. On the other hand, there is a very simple way of creating such a multiset: take an object of size $n - N + 1$ (c_{n-N+1} ways) an $N - 1$ objects of size 1 (one way, as $c_1 = 1$). Then Theorem 1 says that this is essentially the only possible way; the next result makes this more precise.

We consider an object $G_{n,N}$ drawn uniformly at random from $\mathcal{G}_{n,N}$, such that $\Pr[G_{n,N} = G] = |\mathcal{G}_{n,N}|^{-1}$ for $G \in \mathcal{G}_{n,N}$. Let $L_{n,N}$ denote the size of a largest component in $G_{n,N}$.

THEOREM 2. As $n \to \infty$ we have $L_{n,N} = n - mN + O_p(1)$.

In other words, a typical object in $\mathcal{G}_{n,N}$ looks as follows: it contains a large component of size $n - mN + O(1)$, $N + O(1)$ components of (smallest possible) size m, and another $O(1)$ components of size $> m$ but still $O(1)$. It is also possible to study the exact shape of the remainder, but this is omitted here.

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Approximate counting and sampling at low temperatures via the cluster expansion

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(joint work with Tyler Helmuth, Guus Regts, Matthew Jenssen, Peter Keevash)

We give efficient algorithms to approximate the partition function of and sample from certain statistical physics models on lattices and expander graphs at sufficiently low temperatures. As an example, consider the q-state, ferromagnetic Potts model, a probability distribution on q-colorings of a graph G given by

$$\mu_G(\sigma) = \frac{e^{-\beta m(G,\sigma)}}{Z_G(\beta)}$$

where $m(G,\sigma)$ is the number of bichromatic edges of G under the coloring σ, $\beta > 0$ is the 'inverse temperature', and

$$Z_G(\beta) = \sum_{\sigma} e^{-\beta m(G,\sigma)}$$

is the partition function, or normalizing constant of the probability distribution. The approximate counting problem is to output a number $\hat{Z}$ so that $e^{-\varepsilon}\hat{Z} \leq Z_G(\beta) \leq e^{\varepsilon}\hat{Z}$, and the approximate sampling problem is to output a coloring σ with distribution $\hat{\mu}$ so that $\|\hat{\mu} - \mu_G\|_{TV} < \varepsilon$. We say an approximate counting or sampling algorithm is efficient if it runs in time polynomial in $|V(G)|$ and $1/\varepsilon$.

In general there are several algorithmic approaches to these problems, including Markov Chain Monte Carlo (MCMC), the method of correlation decay, and polynomial interpolation. These approaches generally work efficiently in the *high temperature* regime (weak interactions, or small β in the Potts example). However, for certain families of graphs (including lattices and expander graphs), these

approaches provably fail at low temperatures due to the phase transition phenomenon.

We develop an efficient algorithmic framework for approximate counting and sampling that exploits the phase transition phenomenon for models with a bounded number of *ground states* (e.g. the ferromagnetic Potts model, the hard-core model on bipartite graphs, the Widom-Rowlinson model). Our approach is based on rewriting a given model as an *abstract polymer model* or *contour model*, expressing the partition function in terms of deviations from the ground states. This reformulation transforms a low-temperature model into an auxiliary high-temperature model.

Probabilistically the most detailed understanding of spin models is obtained when the cluster expansion, a power series for the log partition function, converges. The cluster expansion is the multivariate Taylor series for the log partition function in the weights of polymers in an abstract polymer model, and has many connections to combinatorics (see e.g. [6]).

A convenient sufficient condition for convergence of the cluster expansion is given by Kotecký and Preiss [5]. Inspired by the polynomial interpolation method of Barvinok [1], in [3] we show how truncating the cluster expansion leads to efficient counting and sampling algorithms when the Kotecký-Preiss condition holds, and we apply this to obtain efficient algorithms at low temperatures on $\mathbb{Z}^d$. In [4], we apply this to expander graphs. A representative result is the following. Recall that an α-expander is a graph G so that $|\partial_E S| \geq \alpha|S|$ for all $S \subset V(G)$ with $|S| \leq |V(G)|/2$.

Theorem 1. *For all $\beta \geq \frac{5\log(q\Delta)}{\alpha}$, there are efficient approximate counting and sampling algorithms for the q-state ferromagnetic Potts model at inverse temperature β on all α-expander graphs of maximum degree Δ.*

Several directions related to low-temperature algorithms remain wide open. First, one would like to develop Markov chain based algorithms at low temperatures, since MCMC algorithms tend to be much faster and simpler than algorithms based on series truncations. Some initial steps in this direction are taken in [2], but analyzing the most natural algorithms remains open.

Second, the worst-case computational complexity of approximate counting and sampling in models such as the ferromagnetic Potts model and the hard-core model on bipartite graphs is an open problem: there is no polynomial-time algorithm known but no NP-hardness known either. These problems are captured by the complexity class #-BIS (bipartite independent set) and resolving its complexity is a major open problem in the field of approximate counting.

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The Structure and Number of Erdős Covering Systems

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(joint work with Paul Balister, Béla Bollobás, Robert Morris, Marius Tiba)

A *covering system* is a finite collection of arithmetic progressions that covers[1] the integers. Erdős [4] initiated the study of covering systems in 1950, and since numerous beautiful questions have been asked about their properties (see, for example, [4, 5, 6, 7, 8, 9, 10, 11, 12, 19, 20]). Until recently, little progress had been made on many of these problems, but following the groundbreaking work of Filaseta, Ford, Konyagin, Pomerance and Yu [12] in 2007, a fundamental result was obtained by Hough [15], who resolved a problem from the original paper of Erdős [4] by proving that there do not exist covering systems with distinct moduli and arbitrarily large minimum modulus. Building on his work, the authors of this paper [1, 2] recently made further progress on several related open problems.

In this talk we turn our attention to a problem of a somewhat different flavor, whose study was initiated by Erdős [5] in 1952 (and re-asked, for example, in 1980 [9]):

How many minimal covering systems of size n are there?

Erdős [5] gave a simple proof that there are only finitely many minimal[2] covering systems of size n, but the bound he obtained on their number was doubly exponential. A more reasonable upper bound follows from a result of Simpson [21], who proved in 1985 that the largest modulus in a minimal covering system of size n is at most 2^{n-1}. Indeed, it is not hard to see that this gives an upper bound of $2^{O(n^2)}$. We answer Erdős' question on the number of minimal covering systems, up to lower order terms in the exponent.

Theorem 1. *The number of minimal covering systems of $\mathbb{Z}$ of size n is*

$$\exp\left(\left(\frac{4\sqrt{\tau}}{3}+o(1)\right)\frac{n^{3/2}}{(\log n)^{1/2}}\right) \tag{1}$$

[1] We emphasize that we do *not* require the progressions to be disjoint - although this systems with this additional condition has been studied [13, 14, 18, 22].

[2] A covering system $\mathcal{A}$ is *minimal* if no proper subset of it covers $\mathbb{Z}$. Without this restriction there are infinitely many covering systems of size 2, since we can take $\mathcal{A} = \{\mathbb{Z}, A\}$ for any arithmetic progression A.

as $n \to \infty$, *where*

$$\tau = \sum_{t=1}^{\infty} \left(\log \frac{t+1}{t} \right)^2 .$$

In order to motivate the form of the formula (1), let us begin by describing a simple construction that gives a slightly weaker lower bound. Let $p_1 < \ldots < p_k$ be the first k primes and, for each $i \in [k]$, choose $p_i - 1$ arithmetic progressions $A_1^{(i)}, \ldots, A_{p_i-1}^{(i)}$ with the following properties: for each $j \in [p_i - 1]$, the modulus of $A_j^{(i)}$ is divisible by p_i and divides $Q_i := p_1 \cdots p_i$, and $A_j^{(i)}$ contains $j \cdot Q_{i-1}$. It is not hard to see that, for each such choice, by adding the progression $\{0 \pmod{Q_k}\}$ we obtain a distinct minimal covering system of size $n = \sum_{i=1}^{k} (p_i - 1) + 1 \approx k^2 \log k$. Since we have 2^{i-1} choices for the progression $A_j^{(i)}$ for each $i \in [k]$ and $j \in [p_i - 1]$, this implies that there are at least

$$\prod_{i=1}^{k} 2^{(i-1)(p_i-1)} \;=\; \exp\Big(\Omega\big(k^3 \log k\big)\Big) \;=\; \exp\left(\frac{\Omega\big(n^{3/2}\big)}{(\log n)^{1/2}}\right)$$

minimal covering systems of $\mathbb{Z}$ of size n. To obtain the lower bound in Theorem 1 we need a more complicated construction; however, the construction is still of this general form.

To prove our upper bound on the number of minimal coverings our approach is structural: we show that almost all covering systems must have a very particular structure. Indeed, we are motivated by the following (imprecise) conjecture

"Almost every minimal covering system of size n is close to a frame."

While we don't prove exactly this, we show a result that is sufficient for our purposes and is in some ways stronger as it pertains to all covering systems. Roughly speaking it says that every "efficient" covering is close to a frame-like-object. Below we state the main structural theorem without explaining precisely the main structure in the conclusion: the *δ-generalized frame*. The reader should understand this to be a more general notion of "frame".

For the statement of the theorem, we define the *least common multiple* $\operatorname{lcm}(\mathcal{A})$ of a covering system $\mathcal{A}$ to be the least common multiple of the differences of the progressions in $\mathcal{A}$.

Theorem 2. *For every $C, \varepsilon > 0$ there exists $\delta = \delta(C, \varepsilon)$ so that every minimal covering system $\mathcal{A}$ with $\operatorname{lcm}(\mathcal{A}) = p_1^{\gamma_1} \cdots p_k^{\gamma_k}$, which satisfies*

$$|\mathcal{A}| \le C \sum_{i=1}^{k} \gamma_i (p_i - 1), \tag{2}$$

contains a δ-generalized frame $(\mathcal{F}_1, \ldots, \mathcal{F}_k)$, with

$$\sum_{i=1}^{k} |\mathcal{F}_i| \ge (1 - \varepsilon) \sum_{i=1}^{k} \gamma_i (p_i - 1). \tag{3}$$

We note that the covering systems that we constructed in our example above have least common multiple $p_1 \cdots p_k$ and

$$|\mathcal{A}| = 1 + \sum_{i=1}^{k} (p_i - 1).$$

One can think of Theorem 2 as saying that any covering covering system that covers roughly as efficiently as a "frame" (that is, up to a large constant factor) must contain a large (δ-generalized) frame.

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Lower tails via relative entropy

Wojciech Samotij
(joint work with Gady Kozma)

Suppose that $X_1, \dots, X_N$ are i.i.d. (independent, identically distributed) random variables and let $X = X_1 + \cdots + X_N$. A problem of interest to both theoretical and applied mathematicians is to estimate, for every $\delta > 0$, the logarithmic tail probabilities

$$(1) \qquad -\log \Pr\big(X \geq (1+\delta)\mathbb{E}[X]\big) \qquad \text{and} \qquad -\log \Pr\big(X \leq (1-\delta)\mathbb{E}[X]\big).$$

This was achieved by the famous result of Cramér, which determines these two quantities asymptotically (as $N \to \infty$) under mild assumptions on the distribution of the X_is. The proof of Cramér's theorem essentially uses the assumption that $X_1, \dots, X_N$ are independent, whereas the assumption that these variables share the same distribution is less important. In particular, the theory of large deviations has developed many general tools that allow one to estimate the quantities in (1) when X is a linear functional of a vector of independent random variables. In spite of this progress, developing a unified approach that could handle the case where X depends on $X_1, \dots, X_N$ in a nonlinear way remains a formidable challenge.

A natural class of 'simple' nonlinear functionals that serve as a 'test case' are low-degree polynomials. The following concrete challenge arises from the theory of random graphs.

Problem 1. *Determine the asymptotics of the logarithmic tail probabilities in* (1) *when* $X = N_{K_3}(G_{n,p})$ *is the number of triangles in the random graph* $G_{n,p}$.

Observe that $N_{K_3}(G_{n,p})$ can be expressed as a degree-three polynomial in $\binom{n}{2}$ independent Bernoulli(p) random variables. A relaxation of Problem 1 was first studied almost exclusively by the random graphs community. The challenge was to determine the logarithmic tail probabilities in (1) up to a multiplicative constant. This was eventually achieved, but not without significant effort.

The subsequent work of Chatterjee and Varadhan [3] solved Problem 1 in the dense regime, that is, when the edge density p is independent of the number of vertices n; it also popularised the problem in the large deviations community. As for the sparse regime, $p \to 0$ as $n \to \infty$, it had been been known for some time that the lower and the upper tail problems exhibit very different phenomenology. Here, we focus exclusively on the lower tail and only remark that the upper tail problem has a very rich history.

Let $X = N_{K_3}(G_{n,p})$. Janson, Łuczak, and Ruciński [7] were the first to prove that, when $p \leq 1 - \Omega(1)$,

$$-\log \Pr(X = 0) = \Theta\big(\min\{n^2 p, n^3 p^3\}\big).$$

The difficult part of the above two-sided estimate is the lower bound; the upper bound is an easy consequence of Harris's inequality. Soon afterwards, Janson [6] proved a general inequality that implies that, for every $\delta \in (0, 1]$,

$$-\log \Pr\big(X \leq (1-\delta)\mathbb{E}[X]\big) = \Theta\big(\min\{n^2 p, n^3 p^3\}\big), \tag{2}$$

provided that $p \leq 1 - \Omega(1)$. As the right-hand side of (2) suggests, the lower tail problem undergoes a 'phase transition' at $p = \Theta(n^{-1/2})$. Here, we restrict our attention to the case $n^{-1/2} \ll p \leq 1 - \Omega(1)$, where the logarithmic tail probability has order $n^2 p$.

In the extreme case $\delta = 1$, the function implicit in the asymptotic notation in the right-hand side of (2) was determined by Łuczak [8]; in fact, Łuczak obtained a structural characterisation of the tail event $\{X = 0\}$ that easily implies

$$-\log \Pr(X = 0) = \left(\frac{1}{4} + o(1)\right) n^2 \log \frac{1}{1-p}.$$

Chatterjee and Varadhan [3] and Dembo (unpublished) were the first to seek similar asymptotic estimates of the logarithmic tail probabilities for arbitrary $\delta \in (0, 1]$. For $p \in (0, 1)$ and $q \in [0, 1]$, let

$$I_p(q) = q \log \frac{q}{p} + (1-q) \log \frac{1-q}{1-p}.$$

In other words, $I_p(q)$ is the Kullback–Leibler divergence (also called relative entropy) of the Bernoulli(q) distribution from the Bernoulli(p) distribution. Further, given a function $q\colon E(K_n) \to [0, 1]$, let $G_{n,q}$ be the random subgraph of K_n obtained by independently retaining each edge $e \in E(K_n)$ with probability q_e and let

$$\Phi_{K_3}(\delta, n, p) = \min\left\{ \sum_{e \in E(K_n)} I_p(q_e) : \mathbb{E}[N_{K_3}(G_{n,q})] \leq (1-\delta)\mathbb{E}[N_{K_3}(G_{n,p})] \right\}.$$

A standard measure-tilting argument can be used to show that

$$-\log \Pr\big(X = (1-\delta)\mathbb{E}[X]\big) \leq (1 + o(1)) \cdot \Phi_{K_3}(\delta, n, p). \tag{3}$$

(In the more general context of subgraph counts in $G_{n,p}$, this was first observed by Dembo.) Chatterjee and Varadhan [3] proved that (3) holds with equality when $p \in (0, 1)$ is a constant.

The method of [3] relies on a combination of Szemerédi's regularity lemma and the associated counting lemma and therefore the assumption that p is a constant cannot be relaxed significantly. It is therefore of great interest to develop methods that would extend the results of [3] to the sparse regime. This was first achieved by Chatterjee and Dembo [2], who showed that (3) holds with equality as long as

$p \geq n^{-\alpha}$ for some (small) constant $\alpha > 0$. Subsequent works of Eldan [5], Cook–Dembo [4], and Augeri [1] refined the methods of [4], allowing for larger and larger values of α. However, the real focus of these works was the upper tail problem and thus the exact assumptions on p that make (3) hold with equality were not explicitly stated. We resolve the lower tail problem under the optimal assumption $p \gg n^{-1/2}$.

Theorem 2. *If $n^{-1/2} \ll p \leq 1 - \Omega(1)$, then, for every $\delta \in (0, 1]$,*

$$-\log \Pr\left(N_{K_3}(G_{n,p}) = (1-\delta)\mathbb{E}[N_{K_3}(G_{n,p})]\right) = (1 + o(1)) \cdot \Phi_{K_3}(\delta, n, p).$$

In fact, we prove a much more general theorem in the setting of bounded-degree polynomials of independent Bernoulli random variables that establishes an analogous large deviation principle under natural assumptions on the partial derivatives of the respective polynomial. One corollary of our general theorem is the resolution of the lower tail problem for a general nonempty graph H. Let $N_H(G)$ denote the number of copies of H in a graph G and let $m_2(H)$ denote the so-called 2-density of H, that is, the maximum ratio $(e_J - 1)/(v_J - 2)$ over all subgraphs J of H with at least three vertices.

Theorem 3. *Let H be a nonempty graph. If $n^{-1/m_2(H)} \ll p \leq 1 - \Omega(1)$, then, for every $\delta \in (0, 1]$,*

$$-\log \Pr\left(N_H(G_{n,p}) = (1-\delta)\mathbb{E}[N_H(G_{n,p})]\right) = (1 + o(1)) \cdot \Phi_H(\delta, n, p),$$

where

$$\Phi_H(\delta, n, p) = \min\left\{ \sum_{e \in E(K_n)} I_p(q_e) : \mathbb{E}[N_H(G_{n,q})] \leq (1-\delta)\mathbb{E}[N_H(G_{n,p})] \right\}.$$

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The inducibility problem for random Cayley graphs of abelian groups

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(joint work with Jacob Fox, Fan Wei)

1. Background

In 1975 Pippenger and Golumbic [7] asked the following question: Given a k-vertex graph H, what is the maximum number of k-vertex subsets in an n-vertex graph that induce a copy of H? Denoting this number by $\mathrm{ind}(H,n)$ we have $0 \leq \mathrm{ind}(H,n) \leq \binom{n}{k}$. Pippenger and Golumbic defined the *inducibilty* of H as $\mathrm{ind}(H) = \lim_{n\to\infty}(\mathrm{ind}(H,n)/\binom{n}{k})$. This limit exists because $\mathrm{ind}(H,n)/\binom{n}{k}$ is monotone decreasing in n (for $n \geq k$). Note that $0 \leq \mathrm{ind}(H) \leq 1$.

In general, determining $\mathrm{ind}(H)$ is very hard and the precise value is only known for a few (explicit) classes of graphs H. Pippenger and Golumbic [7] showed that for every k-vertex graph H the lower bound

$$\mathrm{ind}(H) \geq \frac{k!}{k^k - k} \tag{1}$$

holds. This lower bound can be obtained from considering balanced iterated blow-ups of H, which we will formally define below. Fox, Huang, and Lee [4] proved that for a randomly chosen graph H this bound is almost surely tight. In fact, for a random graph H, they proved that the n-vertex graphs with the maximum number of induced copies of H are precisely the balanced iterated blow-ups of H. Independently, Yuster [8] obtained the latter result for $n \leq 2^{\sqrt{k}}$ and concluded that almost surely $\mathrm{ind}(H) \leq (1 + o_k(1)) \cdot k!/(k^k - k)$ for a random k-vertex graph H.

Let us now define balanced iterated blow-ups, see also [4]. Given a graph H, a *blow-up* of H is a graph Γ whose vertex set can be partitioned into non-empty subsets W_i for $i \in V(H)$ such that for distinct $i, j \in V(H)$ the graph Γ is complete between W_i and W_j if i and j are adjacent in H and otherwise Γ is empty between W_i and W_j. The graph Γ is a *balanced blow-up* of H, if the sets W_i can be chosen in such a way that their sizes differ by at most one. We call Γ a *balanced iterated blow-up* of H, if $|V(\Gamma)| < |V(H)|$ or if it is a balanced blow-up of H where for each of the subsets W_i the induced subgraph on W_i is again a balanced iterated blow-up of H.

The problem of determining $\mathrm{ind}(H)$ when H is a path or cycle has received much attention. For cycles C_k with $k \geq 5$, Pippenger and Golumbic [7] conjectured that (1) is sharp, in other words $\mathrm{ind}(C_k) = k!/(k^k - k)$. For $k \geq 6$, the currently best known upper bound is $\mathrm{ind}(C_k) \leq 2k!/k^k$ proved by Král', Norin, and Volec [6] improving on earlier bounds by Pippenger and Golumbic [7] and by Hefetz and Tyomkyn [5]. For $k = 5$, Balogh, Hu, Lidický, and Pfender [1] proved $\mathrm{ind}(C_k) = 5!/(5^5 - 5)$ using the Flag algebra method.

For paths, however, the situation is significantly different. Exoo [3] observed that for a path P_k on $k \geq 4$ vertices, for large n an n-vertex balanced iterated

blow-up of the cycle C_{k+1} contains more induced copies of P_k than an n-vertex balanced iterated blow-up of P_k itself. This is because in the blow-up of the cycle one can "rotate" the path in different ways (and this overcompensates the fact that the parts of the blow-up are slightly smaller). In particular, the lower bound (1) is not sharp for paths P_k on $k \geq 4$ vertices. For $k = 4$ and $k = 5$, even better constructions were obtained by Even-Zohar and Linial [2], building upon a construction of Exoo [3] for $k = 4$. For large k, the balanced iterated blow-up of C_{k+1} is the best known construction. The question of determining $\mathrm{ind}(P_k)$ is still open for all $k \geq 4$.

Note that a path P_k on k vertices can be obtained from deleting one vertex from a cycle C_{k+1} on $k+1$ vertices. The reason for balanced iterated blow-ups of C_{k+1} having more induced copies of P_k than balanced iterated blow-ups of P_k is that the cycle C_{k+1} has a lot of symmetries (and so there are many ways to embed P_k into C_{k+1}). Put in a different way, C_{k+1} is a Cayley graph of the abelian group $\mathbb{Z}/(k+1)\mathbb{Z}$ with generator 1. Let us consider more generally the situation where H is a graph obtained by deleting a few vertices from a Cayley graph $\widetilde{H}$ of an abelian group. Then, if H is prime, which is usually the case, the n-vertex balanced iterated blow-ups of $\widetilde{H}$ contain more induced copies of H than n-vertex balanced iterated blow-ups of H (as long as we deleted sufficiently few vertices).

2. Results

In this talk, we discuss the case where $\widetilde{H}$ is a random Cayley graph of an abelian group and H is obtained from $\widetilde{H}$ by deleting a few vertices. Our main result is that in this case, almost surely all n-vertex graphs maximizing the number of induced copies of H are balanced iterated blow-ups of $\widetilde{H}$ (see Theorem 3 below).

Formally, a Cayley graph of an abelian group is defined as follows.

Definition 1. *Given an abelian group G and a subset $\Lambda \subseteq G \setminus \{0\}$ with $\Lambda = -\Lambda$, the* Cayley graph $\mathrm{Cayley}(G, \Lambda)$ *is the graph with vertex set G in which two vertices $x, y \in G$ are connected if and only if $x - y \in \Lambda$.*

Note that due to $\Lambda = -\Lambda$ and $0 \notin \Lambda$, this is indeed a well-defined undirected graph (without loops).

For a given abelian group G (additively written) and $0 < p < 1$, we construct a random Cayley graph with vertex set G by choosing a subset $\Lambda \subseteq G \setminus \{0\}$ randomly as follows.

Procedure 2. *Let us choose a random subset $\Lambda \subseteq G \setminus \{0\}$ by including each $\{g, -g\} \subseteq G \setminus \{0\}$ into Λ independently with probability p.*

Note that the random set Λ by construction always satisfies $\Lambda = -\Lambda$.

Now, we are ready to state our main result. Roughly speaking, it states that for a graph H which is obtained from a random Cayley graph of an abelian group by deleting a few vertices, the n-vertex graphs Γ maximizing the number of induced copies of H are balanced iterated blow-ups of the Cayley graph $\widetilde{H}$.

Theorem 3. *Let G be an abelian group with $\tilde{k}$ elements and assume that $0 < p < 1$ satisfies $\min(p, 1-p) \geq 10^6(\log \tilde{k})^{6/5}\tilde{k}^{-1/5}$. If $\Lambda \subseteq G$ is chosen according to Procedure 2, then with probability $1 - o(1)$ the Cayley graph $\widetilde{H} = \mathrm{Cayley}(G, \Lambda)$ satisfies the following: For every induced subgraph H of $\widetilde{H}$ on $k \geq \tilde{k} - \frac{1}{4}\log \tilde{k}$ vertices and for all n, every n-vertex graph Γ with the maximum number of induced copies of H is a balanced iterated blow-up of $\widetilde{H}$.*

Here, the $o(1)$-term tends to zero as $\tilde{k} \to \infty$, independently of p.

If $\widetilde{H}$ is as in Theorem 3, but H is an induced subgraph on $k = \tilde{k} - \lceil \log \tilde{k} \rceil - 1$ vertices, then almost surely for large n, the n-vertex balanced iterated blow-ups of H contain more induced copies of H than the n-vertex balanced iterated blow-ups of $\widetilde{H}$. Therefore, the assumption $k \geq \tilde{k} - \frac{1}{4}\log \tilde{k}$ in Theorem 3 is tight up to the constant factor $\frac{1}{4}$.

When taking $H = \widetilde{H}$ in Theorem 3, we obtain that for $\min(p, 1-p) \geq 10^6(\log \tilde{k})^{6/5}\tilde{k}^{-1/5}$ almost surely the n-vertex graphs Γ maximizing the number of induced copies of $\widetilde{H}$ are the balanced iterated blow-ups of $\widetilde{H}$. This implies the following corollary.

Corollary 4. *Under the assumptions of Theorem 3, with probability $1 - o(1)$ the Cayley graph $\widetilde{H} = \mathrm{Cayley}(G, \Lambda)$ has inducibility $\mathrm{ind}(\widetilde{H}) = \tilde{k}!/(\tilde{k}^{\tilde{k}} - \tilde{k})$.*

We remark that the results for random graphs of Fox, Huang, and Lee [4] do not apply for the graphs H we are considering in the theorems above. This is because they only consider graphs that are far from having any symmetries, but in our case by construction H is close to having symmetries coming from symmetries of $\widetilde{H}$. In fact, even the answer is different, because in our case the optimal structures are not balanced iterated blow-ups of H, but of the Cayley graph $\widetilde{H}$.

Our proof uses some of the ideas in [4]. However, many of the techniques in [4] do not apply in our situation. Therefore a new approach is required here.

We remark that we did not optimize the absolute constants in our statements.

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Covering random graphs by monochromatic trees and Helly-type results for hypergraphs

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(joint work with M. Bucić and D. Korándi)

1. Introduction

Given an r-edge-coloured graph G, how many monochromatic paths, cycles or general trees does one need to cover all vertices of G? The study of such problems has a very rich history going back to the 1960's for examples and detailed history, we refer the reader to a recent survey by Gyárfás [4].

We study the problem of covering graphs using monochromatic connected components. Let us denote by $\mathrm{tc}_r(G)$ the minimum m such that in any r-edge-colouring of G, there is a collection of m monochromatic trees that cover the vertices of G. We write $\mathrm{tc}_r(G) = \infty$ if such an m does not exist. The question of covering graphs with monochromatic components was first considered by Lovász in 1975 [9] and Ryser in 1970 [5], who conjectured that $\mathrm{tc}_r(K_n) = r-1$, or in other words, given any r-edge-colouring of K_n we can cover its vertices using at most $r-1$ monochromatic components. It is easy to see that $\mathrm{tc}_r(K_n) \le r$ by fixing a vertex and taking the r monochromatic components containing it in each of the colours. On the other hand, it is not hard [1] to construct classes of graphs that only miss very few edges but admit no cover with a number of monochromatic components that is even bounded by a function of r.

A common theme in the combinatorics of recent years is to obtain sparse random analogues of extremal or Ramsey-type results. For some examples, see Conlon and Gowers [2] and Schacht [10] and references therein. With this in mind, Bal and DeBiasio [1] initiated the study of covering random graphs by monochromatic components. They proved that the number of components needed becomes bounded when p is somewhere between $\left(\frac{r\log n}{n}\right)^{1/r}$ and $\left(\frac{r\log n}{n}\right)^{1/(r+1)}$.

Theorem 1 (Bal, DeBiasio)**.** *Let r be a positive integer. Then for $G \sim \mathcal{G}(n,p)$,*

(a) if $p \ll \left(\frac{r\log n}{n}\right)^{1/r}$, then w.h.p. $tc_r(G) \to \infty$, and

(b) if $p \gg \left(\frac{r\log n}{n}\right)^{1/(r+1)}$, then w.h.p. $tc_r(G) \le r^2$.

They also made the conjecture that $\mathrm{tc}_r(\mathcal{G}(n,p)) \le r$ when $p \gg \left(\frac{r\log n}{n}\right)^{1/r}$. This was subsequently proved by Kohayakawa, Mota and Schacht [6] for $r = 2$. On the other hand, [6] presents a far from trivial construction, due to Ebsen, Mota, and Schnitzer, showing that $\mathrm{tc}_r(\mathcal{G}(n,p)) \ge r+1$ for $p \ll \left(\frac{r\log n}{n}\right)^{1/(r+1)}$, which disproves the conjecture for $r \ge 3$. Since this example forces just one additional component and only applies for slightly larger values of edge probability, it was still generally believed that the conjecture is close to being true. In fact, Kohayakawa,

Mota and Schacht ask if r components are enough to cover $\mathcal{G}(n,p)$ when p is slightly larger than $\left(\frac{r\log n}{n}\right)^{1/(r+1)}$.

We show that the conjecture is quite spectacularly wrong in both parameters, and obtain a good understanding of the behaviour of $\mathrm{tc}_r(\mathcal{G}(n,p))$ throughout the probability range. In particular, we find that $\mathrm{tc}_r(G)$ only becomes equal to r when the density is exponentially larger than conjectured.

Theorem 2. *Let r be a positive integer. There are constants c, C such that for $G \sim \mathcal{G}(n,p)$,*

(a) if $p < \left(\frac{c\log n}{n}\right)^{\sqrt{r}/2^{r-2}}$, then w.h.p. $\mathrm{tc}_r(G) > r$, and

(b) if $p > \left(\frac{C\log n}{n}\right)^{1/2^r}$, then w.h.p. $\mathrm{tc}_r(G) \le r$.

It is easy to see that $\mathrm{tc}_r(G) \ge r$ holds whenever $\alpha(G) \ge r$ (see e.g. [1]). So the second part of the theorem actually implies that $\mathrm{tc}_r(\mathcal{G}(n,p)) = r$ for all larger values of p, so long as $\alpha(\mathcal{G}(n,p)) \ge r$.

Moreover, we show that near the threshold where $\mathrm{tc}_r(G)$ becomes bounded (and for quite some time after that), its value is not linear, as had been conjectured, but is of order $\Theta(r^2)$.

Theorem 3. *Let r be a positive integer, $d > 1$ a constant and $G = \mathcal{G}(n,p)$. There are constants c, C such that if $\left(\frac{C\log n}{n}\right)^{\frac{1}{r}} < p < \left(\frac{c\log n}{n}\right)^{\frac{1}{d(r+1)}}$ then w.h.p. $\mathrm{tc}_r(G) = \Theta(r^2)$.*

Note that Theorems 1 and 3 together establish a threshold of $\left(\frac{\log n}{n}\right)^{1/r}$ for the property of having tc_r bounded by a function of r.

The lower bound in Theorem 3 answers a question of Lang and Lo [8], who following in the footsteps of Korándi, Mousset, Nenadov, Škorić and Sudakov [7], considered the problem of partitioning $\mathcal{G}(n,p)$ into cycles, and ask if $o(r^2)$ monochromatic cycles are enough for $p = O\left(\left(\frac{\log n}{n}\right)^{1/r}\right)$. Our result shows that not only is the answer no, it is not even possible for much larger values of p, even if we only need to *cover* the vertices and are allowed to use trees instead of cycles.

The above two theorems describe the value of $\mathrm{tc}_r(\mathcal{G}(n,p))$ when p is quite small or quite large. We also obtain a very good understanding of the behaviour of $\mathrm{tc}_r(\mathcal{G}(n,p))$ in the range between.

2. The connection to covering partite hypergraphs

Our proofs of the above results, in most of the regimes, rely on a surprising connection to a problem of independent interest about hypergraph covers. Loosely speaking, the question asks how big a cover of a hypergraph H can be if any subgraph of H with few edges has a small cover. Here by a (vertex) cover of a hypergraph H, we mean a set of vertices that has a non-empty intersection with

all edges of H. The minimum size of such a cover is called the cover number of H, and is denoted by $\tau(H)$. In our case, we consider a variant for r-partite r-graphs (r-uniform hypergraphs). A *transversal cover* is then defined as a cover containing exactly one vertex in each part of the r-partition.

We denote by $\mathrm{hp}_r(k)$ the largest possible $\tau(H)$ for an r-partite r-graph H for which any subgraph of H with at most k edges has a transversal cover, if such a maximum exists, and set $\mathrm{hp}_r(k) = \infty$ otherwise. The following result describes the aforementioned relation between these two seemingly unrelated problems.

Theorem 4. *Let $k > r \geq 2$ be integers, and let $G \sim \mathcal{G}(n,p)$. There are constants $C, c > 0$ such that:*

(1) If $np^k > C \log n$ then w.h.p $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.
(2) If $np^{k+1} < c \log n$ then w.h.p $\mathrm{tc}_{r+1}(G) \geq \mathrm{hp}_r(k) + 1$.

What this is saying is that for $\left(\frac{\log n}{n}\right)^{1/k} \ll p \ll \left(\frac{\log n}{n}\right)^{1/(k+1)}$ the value of $\mathrm{tc}_r(\mathcal{G}(n,p))$ is essentially determined by $\mathrm{hp}_r(k)$. We determine $\mathrm{hp}_r(k)$ up to a $O(\log r)$ factor, allowing us to obtain the results mentioned in the previous section.

2.1. **Covering by components of different colours.** The connection mentioned above does not only apply to random graphs. For example part (1) of Theorem 4 also applies to graphs with large minimum degree. Covering such graphs using monochromatic components was first considered in [1]. A particularly nice conjecture they raised is that any graph G with minimum degree $\delta(G) \geq (1 - 1/2^r)n$ can be covered by monochromatic components of *distinct* colours. They gave an example showing that if true, this conjecture is best possible. Girão, Letzter and Sahasrabudhe [3] proved the conjecture for $r \leq 3$. Our connection enables us to prove this conjecture for all values of r.

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Large girth approximate Steiner triple systems

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(joint work with Tom Bohman)

A *Steiner triple system* is a 3-uniform hypergraph $\mathcal{H}$ with the property that every 2-element subset of its vertex-set $V(\mathcal{H})$ is contained in exactly one triple of $\mathcal{H}$ (so $\mathcal{H}$ decomposes the complete graph with vertex-set $V(\mathcal{H})$ into edge-disjoint triangles). Steiner triple systems and their many natural generalizations are central to combinatorics, and have been studied since the work of Plücker, Kirkman, and Steiner in the mid-nineteenth century (see [27, 17, 12, 13, 19]).

We shall consider a 'high-girth' generalization of Steiner triple systems proposed by Erdős [8]. We define the *girth* of a 3-uniform hypergraph to be the smallest $g \geq 4$ for which there is a set of g vertices that spans at least $g-2$ triples. All known constructions of Steiner triple systems have small girth (see, e.g., [15, 28, 11, 10]), and there seems to be no simple reason to believe that this should be necessary. It thus is natural to ask whether or not there are Steiner triple systems of arbitrarily large girth (also called 'locally sparse').

Question (Erdős, 1973)**.** *Let $\ell \geq 4$. Does there exist $n_0 = n_0(\ell)$ such that there are n-vertex Steiner triple systems with girth greater than ℓ for every $n \geq n_0$ with $n \equiv 1,3 \mod 6$?*

This question remains largely open (the partial results [15, 28, 11, 10] answer this question only for $\ell \leq 6$). Erdős [8] also asked an approximate version of Question . A 3-uniform hypergraph $\mathcal{H}$ is called a *partial Steiner triple system* if every 2-element vertex-subset is contained in at most one triple of $\mathcal{H}$. Note that any partial Steiner triples system on n vertices has at most $\frac{1}{3}\binom{n}{2} = n^2/6 - \Theta(n)$ triples.

Question (Erdős, 1973)**.** *For which $\ell \geq 4$ and $c \in (0, 1/6)$ are there n-vertex partial Steiner triple systems with at least cn^2 triples and girth larger than ℓ for all $n \geq n_0(\ell, c)$?*

In 1993 Lefmann, Phelps, and Rödl [21] showed that for any $\ell \geq 4$ one can take $c = c_\ell > 0$ with $c_\ell \to 0$ as $\ell \to \infty$, and raised the question whether one can take a constant $c > 0$ that does not depend on ℓ. This natural question was also formulated more recently by Ellis and Linial [7]; see also [23].

We answer Erdős' Question from 1973, by showing existence of approximate Steiner triple systems with arbitrary high girth. Regarding the above-mentioned questions from [21, 7], this implies that one can take $c_\ell \sim 1/6$ for any $\ell \geq 4$. *These results were obtained independently by Glock, Kühn, Lo, and Osthus* [14].

Theorem 1 (Bohman and Warnke [6])**.** *For every $\ell \geq 4$ there are $n_\ell, \beta_\ell > 0$ such that, for all $n \geq n_\ell$, there exists an n-vertex partial Steiner triple system with at least $\big(1 - n^{-\beta_\ell}\big)n^2/6$ triples and girth larger than ℓ.*

In [6] we prove Theorem 1 by showing that the following natural (see [20]) constrained random process is very likely to produce the desired object for fixed $\ell \geq 4$. Beginning with the empty 3-uniform hypergraph $\mathcal{H}_0$ on n vertices we sequentially

set $\mathcal{H}_{i+1} := \mathcal{H}_i + e_{i+1}$, where the added triple e_{i+1} is chosen uniformly at random from the collection of triples $xyz \notin \mathcal{H}_i$ with the property that the girth of $\mathcal{H}_i + xyz$ remains larger than ℓ (i.e., that $\mathcal{H}_i + xyz$ contains no set of $4 \leq a \leq \ell$ vertices that spans at least $a-2$ triples). This process terminates with a maximal partial Steiner triple system with girth larger than ℓ.

Our differential equation method based analysis of this random process is motivated by a pseudo-random heuristic for divining the trajectories that govern the evolution of various key parameters. Such heuristics play a central role in our understanding of several other constrained random processes that produce interesting combinatorial objects (such as the triangle-free process [1, 4, 9, 16], the triangle removal process [2, 3], and the H-free process [5, 24, 25, 26, 22]). A surprising consequence of our proof is that the general case only introduces minor modifications of the trajectories compared to the $\ell = 4$ case, i.e., adding the arbitrary high-girth constraint does not affect the evolution significantly. See [6] for more details.

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A reverse Sidorenko inequality

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(joint work with Ashwin Sah, Mehtaab Sawhney, and David Stoner)

Let $i(G)$ denote the number of independent sets of G, $c_q(G)$ the number of proper vertex-colorings of G using q labeled colors, and $\hom(G, H)$ the number of graph homomorphisms from G to H (we allow H to have loops as well as vertex- and edge-weights). Note that $i(G) = \hom(G, \bigcirc\!\!-\!\!\bullet)$ and $c_q(G) = \hom(G, K_q)$. Consider the following questions:

Q1. Fix d. Among d-regular graphs, which G maximizes $i(G)^{1/|V(G)|}$?
Q2. Fix d and q. Among d-regular graphs, which G maximizes $c_q(G)^{1/|V(G)|}$?
Q3. Fix d and H. Among d-regular graphs, which G maximizes $\hom(G, H)^{1/|V(G)|}$?

Question 1 was answered by Kahn [3] and Zhao [6]. The maximizing G is $K_{d,d}$ (or a disjoint union of copies of $K_{d,d}$). In other words, for all n-vertex d-regular graph G,

$$i(G) \le i(K_{d,d})^{n/(2d)}.$$

A partial answer to Questions 2 and 3 had been given by Galvin and Tetali [2], who showed that the maximizing G *among bipartite graphs* is also $K_{d,d}$. One of our main results is that the bipartite hypothesis can be dropped for Question 2 on colorings, i.e., for all n-vertex d-regular graph G,

$$c_q(G) \le c_q(K_{d,d})^{n/(2d)}. \tag{1}$$

More generally, for Question 3 without the bipartite assumption on G, the problem is wide open, lacking even good conjectures for most H. We show that the *bipartite* hypothesis on G can be replaced by *triangle-free*, proving a conjecture of

Cohen, Csikvári, Perkins, and Tetali [1], i.e., for all triangle-free n-vertex d-regular G, and all H (allowing loops and weights)

$$\hom(G,H)^{1/|V(G)|} \le \hom(K_{d,d},H)^{1/(2d)}. \tag{2}$$

Furthermore, the triangle-free hypothesis on G is best possible for general H.

We also prove analogous inequalities for irregular graphs G, initially conjectured by Kahn [3]. For example, for the number of independent sets, we prove that for every graph G without isolated vertices, letting d_v denote the degree of vertex v in G, one has

$$i(G) \le \prod_{uv\in E(G)} i(K_{d_u,d_v})^{1/(d_u d_v)},$$

and similarly for the number of colorings $c_q(G)$. We establish a more general list-coloring version of the inequality that plays an essential role as a stronger induction hypothesis in our proof of (1), which starts by removing a maximum degree vertex of G.

Allowing weights on H, we conjecture that the triangle-free hypothesis in (2) can be dropped if H is *anti-ferromagnetic*, i.e., all eigenvalues of the edge-weight matrix of H except the top one are nonpositive. We also prove that if H is *ferromagnetic*, i.e., the edge-weight matrix of H is positive semidefinite, then the answer to Question 3 is $G = K_{d+1}$.

More details can be found in [4, 5].

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Problem Session

(1) (Nati Linial) *Girth vs. Diameter of Graphs*

We consider here *only graphs without vertices of degree* 1 *or* 2.

It is an easy observation that in every such graph G there holds

$$diam(G) \geq \lfloor girth(G)/2 \rfloor$$

I have asked many times what is

$$\limsup_{diam(G)\to\infty} \frac{girth(G)}{diam(G)}$$

By the above this number is at most 2. It is also at least 1 as first shown in [1], but I know nothing beyond that.

However, it has recently occurred to me that I do not even know what is

$$\limsup_{diam(G)\to\infty} girth(G) - diam(G).$$

It is hard to believe that this lim sup is finite, but that remains unknown at present.

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(2) (Yufei Zhao) *Does vertex percolation preserve max cut?*

Let $G = G_n$ be graphs with $e(G)/v(G) \to \infty$. Let S be a random subset of $V(G)$ where every vertex is included with probability $1/2$ independently.

Prove or disprove: with probability $1 - o(1)$,

$$\frac{\mathsf{maxcut}(G[S])}{e(G[S])} = \frac{\mathsf{maxcut}(G)}{e(G)} + o(1).$$

Here $G[S]$ is the subgraph induced by S and $\mathsf{maxcut}(G)$ is the maximum number of edges of a bipartite subgraph of G.

Remarks:

(a) The $e(G)/v(G) \to \infty$ hypothesis is necessary since a disjoint union of n triangles does not have the above property.

(b) The statement is true if G is sufficiently dense. A regularity lemma argument proves the statement for constant density graphs. More advanced results on approximating max-cut via sampling (e.g., Alon–Fernandez de la Vega–Kannan–Karpinski) implies the statement for $e(G) \geq v(G)^{2-c}$, where $c > 0$ is some constant.

(3) (Lior Gishboliner)

For a pair of graphs H_1, H_2 and $\varepsilon \in (0,1)$, let $\delta = \delta_{H_1,H_2}(\varepsilon)$ be the maximal $\delta \in (0,1)$ such that for every graph G, if G is δ-close to being induced H_i-free for each $i = 1, 2$, then G is ε-close to being induced $\{H_1, H_2\}$-free (recall that an n-vertex graph G is ε-close to a graph property $\mathcal{P}$ if one can turn G into a graph satisfying $\mathcal{P}$ by adding/deleting at most εn^2 edges).

A theorem of Alon-Shapira [1] implies that $\delta_{H_1,H_2}(\varepsilon)$ is well-defined. The problem is to understand the "correct" dependence of $\delta_{H_1,H_2}(\varepsilon)$ on ε. More precisely:

Problem 1. *Is it true that for every H_1, H_2, $\delta_{H_1,H_2}(\varepsilon) = poly(\varepsilon)$?*

Problem 2. *Is it true that for every $a > 0$ there are graphs H_1, H_2 for which $\delta_{H_1,H_2}(\varepsilon) \leq O(\varepsilon^a)$?*

Comments:
(a) The result of [1] gives a tower-type dependence of $\delta_{H_1,H_2}(\varepsilon)$ on ε (due to use of regularity lemma).
(b) The answer to Problem 1 is positive in cases where the removal lemma for induced $\{H_1, H_2\}$-freeness has polynomial dependence.
(c) The answer to Problem 2 is positive for $a = 2$.

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(4) (Matthew Kwan) *Hypergraph cuts above the average*

Fix $k \geq r \geq 2$, and let G be a k-uniform hypergraph. An *r-cut* of G is a partition of the vertex set into r parts, and the size of the cut is the number of edges of G which have at least one vertex in each part. Erdős and Kleitman [2] observed via a simple averaging argument that if G has m edges, then it has an r-cut of size at least $\alpha_{r,k}m$, where $\alpha_{r,k} = S(k,r)r!/r^k$ and $S(k,r)$ is the number of unlabelled partitions of $\{1, \ldots, k\}$ into r nonempty sets.

It is not hard to improve this to $\alpha_{r,k}m + \Omega(\sqrt{m})$, noting that the standard deviation of the size of a random r-cut is $\Omega(\sqrt{m})$. This bound turns out to be best-possible if $(k,r) \in \{(2,2),(3,2)\}$, but with David Conlon, Jacob Fox and Benny Sudakov [1] we proved that in all other cases there is actually an r-cut of size at least $\alpha_{r,k}m + \Omega(m^{5/9})$. We conjectured the following stronger bound.

Conjecture 1. *Fix $k \geq r \geq 2$ such that $r \geq 3$ or $k \geq 4$. Then every k-uniform hypergraph G with m edges has an r-cut of size at least $\alpha_{r,k}m + \Omega(m^{2/3})$.*

If true, Conjecture 1 is best-possible, as can be seen by considering an appropriate random hypergraph.

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(5) (Alexey Pokrovskiy)

Problem 1. *Prove or disprove: There is some $\varepsilon > 0$, such that every ordered graph G has an increasing path of length $\Omega((e(G)/|G|)^{\frac{1}{2}+\varepsilon})$.*

This problem was asked by Bucić, Kwan, Pokrovskiy, Sudakov, Tran, Wagner in [1]. In that paper this problem is solved when G is reasonably dense. Because of this the interesting case is when $e(G)/|G| = d$ for a fixed number d and n is very large.

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(6) (Tibor Szabó) *List Ramsey number of cliques*

For a graph H and list assignment $L : E(H) \to \{K \subseteq \mathbb{N}\}$ of colors to the edges, we call a function $c : E(H) \to \mathbb{N}$ an *L-edge-coloring* if $c(e) \in L(e)$ for every edge $e \in E(H)$. For a graph H and positive integer k the *k-color list Ramsey number* $R_L(H, k)$ of H is the smallest number n of vertices such that there exists a list assignment L on $E(K_n)$ such that every L-coloring of $E(K_n)$ contains a monochromatic copy of H. The list Ramsey number is clearly at most the ordinary Ramsey number.

Question 1: Is $R_L(K_k, 2) = R(K_k, 2)$ or not?
Question 2: Is $R_L(K_3, r) = R(K_3, k)$ or not?

Literature: Alon, Bucić, Kalvari, Kuperwasser, Szabó: List Ramsey numbers, submitted.

(7) (Lutz Warnke) *Phase transition in random graphs evolving by degrees*

As suggested by Erdős and Rényi [2] and Lovász [4], it is natural to study random graph processes where the probability of joining v and w in the next step depends on the current degrees d_v and d_w. To be more concrete, given a function $f = f_n : \mathbb{N} \to [0, \infty)$, we write $(G^f_{n,m})_{m \geq 0}$ for the random graph process with fixed vertex set $[n] = \{1, \dots, n\}$ where new edges are added one-by-one (starting with no edges) such that the

next edge connects two currently non-adjacent v and w with probability proportional to $f(d_v)f(d_w)$. Note that $G^f_{n,m}$ has m edges. Similarly, we write $(G^{f,*}_{n,m})_{m\geq 0}$ for the multigraph variant where the next edge connects distinct v and w with probability proportional to $2f(d_v)f(d_w)$, and forms a loop at v with probability proportional to $f(d_v)f(d_v+1)$.

This general class of dynamic network models not only contains the classical Erdős–Rényi process via $f(k) := 1$, but also the configuration model for d-regular graphs and the Steger–Wormald algorithm [6] via $f(k) := \max\{d-k, 0\}$, a 'fixed vertex-set' preferential attachment process [4, 1, 3] via $f(k) := k+\alpha$, and the well-known random d-process [5, 7] via $f(k) := \mathbb{1}_{\{k<d\}}$; see [3] for more details.

We are interested in the emergence of a 'linear' size giant component in such models, which is one of the most important and fascinating phase transitions in random graph theory. This is well-understood in the Erdős–Rényi reference model $f(k) = 1$, where around $m \approx n/2$ many steps the largest component typically changes from size $\Theta(\log n)$ to size $\Theta(n)$. Similar results have also been recently established for the above-mentioned preferential attachment process $f(k) = k+\alpha$ by Pittel [4] and Janson–Warnke [3], and for the random d-process $f(k) = \mathbb{1}_{\{k<d\}}$ by Warnke–Wormald [7]. In all of the aforementioned cases the corresponding function f is approximately linear (by which we here mean a polynomial of degree at most one in k), which motivates the following conceptually interesting research problem; see also Section 5.2.2 and Problem 5.1 in [3].

Problem 1. *Study the giant component problem for $G^f_{n,m}$ or $G^{f,*}_{n,m}$ when f is 'truly' non-linear.*

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(8) (Jacob Fox) *Edge-ordered Ramsey numbers*

An edge-ordered graph is a graph with a linear ordering of its edges. Two edge-ordered graphs are *equivalent* if there is an isomorphism between them preserving the ordering of the edges. The *edge-ordered Ramsey number* $r_{\text{edge}}(H)$ of an edge-ordered graph H is the smallest N for which

there exists an edge-ordered graph G on N vertices such that, for every two-coloring of the edges of G, there is a monochromatic subgraph of G equivalent to H. Recently, Balko and Vizer proved that $r_{\text{edge}}(H)$ exists, and Fox and Li proved $r_{\text{edge}}(H) = 2^{n^{O(1)}}$ for every edge-ordered graph on n vertices. The following question asks if this can be improved.

Question. *Does every edge-ordered graph H on n vertices satisfy*

$$r_{edge}(H) = 2^{O(n)}?$$

The Burr-Erdős conjecture, proved recently by Lee, states that every d-degenerate graph has Ramsey number linear in the number of vertices. Fox and Li also proved that every d-degenerate edge-ordered graph H on n vertices satisfies $r_{\text{edge}}(H) \leq n^{C(d)}$ for some $C(d)$ only depending on d. The following question asks if the bound can be improved to linear as in the Burr-Erdős conjecture.

Question. *Does every d-degenerate edge-ordered graph H on n vertices satisfy*

$$r_{edge}(H) = O(C(d)n)?$$

We conjecture the answer is no.

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(9) (Asaf Ferber)

Let $H(n)$ be the number of $n \times n$ Hadamard matrices. It is a relatively simple exercise to show that $H(n) \leq 2^{n^2/2+o(n^2)}$, and it is a notoriously hard conjecture to show $H(n) > 0$ for all n which is divisible by 4.

Following an attempt by Jain, Zhao and myself I propose the following problem:

Show that $H(n) = 2^{o(n^2)}$.

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