Mathematisches Forschungsinstitut Oberwolfach

Report No. 27/2019

DOI: 10.4171/OWR/2019/27

Logarithmic Enumerative Geometry and Mirror Symmetry

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16 June – 22 June 2019

Abstract. The new field of log enumerative geometry has formed at the crossroads of mirror symmetry, Gromov-Witten theory and log geometry. This workshop has been the first to promote this field and bring together the junior and senior experts of this quickly evolving topic. Spontaneous exchange, unforeseen mutual benefit as well as having each participant give a presentation allowed for novel progress and insight.

Mathematics Subject Classification (2010): 14N35, 14J33, 14T05, 53D42, 53D45, 53D37.

Introduction by the Organizers

The workshop had the purpose to bring together experts from log geometry, enumerative geometry and mirror symmetry and has been as such the first workshop to promote the profound connections between these areas. We were positively surprised how fluent in log geometry the participants already were. Seemingly log geometry has become even more of a standard tool in enumerative geometry and mirror symmetry since the time when the proposal was made. The field has evolved quickly and found new application beyond improving the classical degeneration formula.

Besides establishing the new field of *log enumerative geometry*, the workshop has greatly helped junior participants to first meet senior experts and vice versa. Several participants reported their surprise about how many people they saw talks by for the first time. *Every* participant gave a talk, so in particular all young postdocs and PhD students had the opportunity to present their work to a large expert audience and to receive valuable feedback on it. The *lightning talks* on Tuesday evening, i.e. a sequence of short 5-minute presentations, led to a productive unscheduled impromptu hour long discussion. Two young researchers whose works build on top of one another met for the first time.

Several notable learnings would not have happened without the workshop. The concept of *idealized log structures* had not been much known before but several researchers came to see its value for their endeavors. *Punctured invariants* and their degeneration formula had not been on the program but got explained nonetheless in an informal session upon participant's initiative. New insights on how *quasimap invariants* interact with log geometry emerged from the workshop. Nonlog experts received log expert advice helping them overcome technical problems. Tropical experts got to talk to geometry experts and exchange views and concept, e.g. noticing unseen overlap in the context of *broken lines*. Young postdocs tested their ideas for calculating relative invariants in front of experienced researchers. Partipants learned that there also exists an analytic proof of the *topological vertex result*. The topology of *maximal degenerations* became a useful field of learning and exchange. Two teams with contradicting computations were able to sort out which one was correct. Some clarification on the differences of various versions of *log quantum cohomology/symplectic cohomology* was gained.

Towards the end of the workshop, a number of participants approached us and commented on how useful the workshop had been to their research. There was enthusiasm for holding similar workshops in the future and we committed ourselves to working towards it.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows." Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Jonathan Wise in the "Simons Visiting Professors" program at the MFO."'

Workshop: Logarithmic Enumerative Geometry and Mirror Symmetry

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Abstracts

The double ramification cycle

Rahul Pandharipande
(joint work with F. Janda, A. Pixton, D. Zvonkine)

1. Overview

I started the lecture with some aspects of the history of the double ramification cycle: Abel-Jacobi theory, relative Gromov-Witten theory, and the bChow approach of Holmes, Wise, and others. The main result presented was an explicit formula in tautological classes for the double ramification cycle conjectured earlier by Pixton and proven in [1].

2. Notation

Double ramification data for maps will be specified by a vector

$$A = (a_1, \ldots, a_n)\,, \quad a_i \in \mathbb{Z}$$

satisfying the balancing condition

$$\sum_{i=1}^{n} a_i = 0\,.$$

The integers a_i are the *parts* of A. The positive parts of A define a partition

$$\mu = \mu_1 + \ldots + \mu_{\ell(\mu)}\ .$$

The negatives of the negative parts of A define a second partition

$$\nu = \nu_1 + \ldots + \nu_{\ell(\nu)}\ .$$

Up to a reordering of the parts, we have

$$A = (\mu_1, \ldots, \mu_{\ell(\mu)}, -\nu_1, \ldots, -\nu_{\ell(\nu)})\,. \tag{1}$$

Since the parts of A sum to 0, the partitions μ and ν must be of the same size. Let

$$D = |\mu| = |\nu|$$

be the *degree* of A. We assume here for simplicity that there are no 0 parts (but they present no difficulty).

The double ramification cycle $\mathsf{DR}_g(A)$ represents (in a virtual sense) the locus in $\overline{\mathcal{M}}_{g,n}$ of curves which admit a degree D map to $\mathbb{P}^1$ with ramification profiles over 0 and ∞ given by μ and ν.

3. Weightings

Let $A = (a_1, \dots, a_n)$ be a vector of double ramification data. Let $\Gamma \in \mathsf{G}_{g,n}$ be a stable graph of genus g with n legs. A *weighting* of Γ is a function on the set of half-edges,

$$w : \mathsf{H}(\Gamma) \to \mathbb{Z},$$

which satisfies the following three properties:

(i) $\forall h_i \in \mathsf{L}(\Gamma)$, corresponding to the marking $i \in \{1, \dots, n\}$,

$$w(h_i) = a_i \ ,$$

(ii) $\forall e \in \mathsf{E}(\Gamma)$, corresponding to two half-edges $h, h' \in \mathsf{H}(\Gamma)$,

$$w(h) + w(h') = 0\,,$$

(iii) $\forall v \in \mathsf{V}(\Gamma)$,

$$\sum_{v(h)=v} w(h) = 0\,,$$

where the sum is taken over *all* $n(v)$ half-edges incident to v.

If the graph Γ has cycles, Γ may carry infinitely many weightings.

Let r be a positive integer. A *weighting mod* r of Γ is a function,

$$w : \mathsf{H}(\Gamma) \to \{0, \dots, r-1\},$$

which satisfies exactly properties (i-iii) above, but with the equalities replaced, in each case, by the condition of *congruence mod* r. For example, for (i), we require

$$w(h_i) = a_i \mod r\,.$$

Let $\mathsf{W}_{\Gamma,r}$ be the set of weightings mod r of Γ. The set $\mathsf{W}_{\Gamma,r}$ is finite, with cardinality $r^{h^1(\Gamma)}$.

4. Pixton's formula

Let $A = (a_1, \dots, a_n)$ be a vector of double ramification data. Let r be a positive integer. We denote by $\mathsf{P}_g^{d,r}(A) \in R^d(\overline{\mathcal{M}}_{g,n})$ the degree d component of the tautological class

$$\sum_{\Gamma \in \mathsf{G}_{g,n}} \sum_{w \in \mathsf{W}_{\Gamma,r}} \frac{1}{|\mathsf{Aut}(\Gamma)|} \frac{1}{r^{h^1(\Gamma)}} \xi_{\Gamma*} \Bigg[\prod_{i=1}^{n} \exp(a_i^2 \psi_{h_i}) \cdot \prod_{e=(h,h') \in \mathsf{E}(\Gamma)} \frac{1 - \exp(-w(h)w(h')(\psi_h + \psi_{h'}))}{\psi_h + \psi_{h'}} \Bigg].$$

in $R^*(\overline{\mathcal{M}}_{g,n})$.

Inside the push-forward in the above formula, the first product

$$\prod_{i=1}^{n} \exp(a_i^2 \psi_{h_i})$$

is over $h \in L(\Gamma)$ via the correspondence of legs and markings. The second product is over all $e \in E(\Gamma)$. The factor

$$\frac{1 - \exp(-w(h)w(h')(\psi_h + \psi_{h'}))}{\psi_h + \psi_{h'}}$$

is well-defined since

- the denominator formally divides the numerator,
- the factor is symmetric in h and h'.

No edge orientation is necessary.

The following fundamental polynomiality property of $\mathsf{P}_g^{d,r}(A)$ has been proven by Pixton.

Theorem [Pixton]. *For fixed g, A, and d, the class*

$$\mathsf{P}_g^{d,r}(A) \in R^d(\overline{\mathcal{M}}_{g,n})$$

is polynomial in r (for all sufficiently large r).

We denote by $\mathsf{P}_g^{d}(A)$ the value at $r = 0$ of the polynomial associated to $\mathsf{P}_g^{d,r}(A)$ by the above Theorem. In other words, $\mathsf{P}_g^{d}(A)$ is the *constant* term of the associated polynomial in r.

The main result of [1] is a proof of the formula for double ramification cycle $\mathsf{DR}_g(A)$ conjectured earlier by Pixton.

Theorem. *For $g \geq 0$ and double ramification data A, we have*

$$\mathsf{DR}_g(A) = 2^{-g}\,\mathsf{P}_g^{g}(A) \in R^g(\overline{\mathcal{M}}_{g,n}).$$

5. Target varieties

The double ramification cycle can be studied for stable maps to target varieties X. There is perfect generalization of Pixton's formula to the target variety setting. The full formula (with proof) can be found in [2].

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Tropical multiplicities from mirror polyvector fields and tropical quantum field theory

Travis Mandel
(joint work with Helge Ruddat)

1. Introduction

When relating counts of holomorphic curves to counts of tropical curves, it is necessary to count the tropical curves with certain non-trivial multiplicities. These multiplicities have been defined in terms of the index of a certain map of lattices in [5, 2], or in terms of intersections of tropical cycles in [6, 1]. Such descriptions tend to be impractical for computational purposes and for various applications (e.g., for application to the Gross-Siebert program). Simpler formulas have long been known to exist in various special cases, e.g. for planar tropical curves with point conditions as in [4], where multiplicities can be expressed as products of vertex multiplicities. My work with H. Ruddat [3] gives a practical description of tropical multiplicities in general.

Let us first review the relevant definitions. Let N be a lattice of finite rank r. Let $\overline{\Gamma}$ be the topological realization of a finite connected graph, and let Γ be the complement of the one-valent vertices of $\overline{\Gamma}$. Denote the sets of vertices, edges, compact edges, and non-compact edges of Γ by $\Gamma^{[0]}$, $\Gamma^{[1]}$, $\Gamma^{[1]}_c$, and $\Gamma^{[1]}_\infty$, respectively. Equip Γ with a weight function $w : \Gamma^{[1]} \to \mathbb{Z}_{\geq 0}$ and a marking $\epsilon : I \xrightarrow{\sim} \Gamma^{[1]}_\infty$ for some finite index-set I. For $i \in I$, we denote $\epsilon(i)$ by E_i. A parametrized marked tropical curve is data (Γ, w, ϵ) as above, plus the data of a continuous map $h : \Gamma \to N_\mathbb{R}$ such that:

- For each edge $E \in \Gamma^{[1]}$ with $w(E) > 0$, $h|_E$ is a proper embedding into an affine line with rational slope. For $E \in \Gamma^{[1]}$ with $w(E) = 0$, $h(E)$ is a point.
- For every $V \in \Gamma^{[0]}$, the balancing condition holds. I.e., for each edge $E \ni V$, denote by $u_{(V,E)}$ the primitive[1] integral vector emanating from $h(V)$ into $h(E)$ (or $u_{(V,E)} := 0$ if $h(E)$ is a point). Then $\sum_{E \ni V} w(E) u_{(V,E)} = 0$.

For a non-compact edge E_i and V the unique vertex of E_i, we may write $u_{(V,E_i)}$ as simply u_{E_i}. We may also write $u_{(V,E)}$ as u_E when the sign is unimportant. We will often abbreviate the data of a tropical curves as just Γ.

A (marked) tropical curve is then a parametrized marked tropical curve up to isomorphism (i.e., up to homeomorphism of Γ respecting w, ϵ, and h). The genus is then the first betti number of Γ. The degree of Γ is the data $\Delta : I \to N$, $\Delta(i) = w(E_i) u_{(\partial E_i, E_i)}$. The type is the data of Γ plus the data of $w(E) u_{(V,E)}$ for each flag (V, E).

[1] A nonzero vector v in a lattice L is called primitive if it is non a positive integral multiple of another element of L. We say nonzero $v \in L$ has index $|v| \in \mathbb{Z}_{>0}$ if v is equal to $|v|$ times a primitive vector.

Let $\mathbf{A} := (A_i)_{i\in I}$ denote a tuple of affine-linear subspaces of $N_{\mathbb{R}}$ with rational slope. We say that a tropical curve satisfies the constraint $\mathbf{A}$ if $h(E_i) \subset A_i$ for each $i \in I$.

Given a vertex V, let $\mathrm{val}(V)$ denote the valence of V, and let $\mathrm{ov}(V) := \mathrm{val}(V)-3$, called the overvalence of V. Let $I^\circ := \{i \in I | w(E_i) = 0\}$. Let $\Psi := (s_i)_{i\in I^\circ}$ be a tuple of non-negative integers $s_i \in \mathbb{Z}_{\geq 0}$. We say that Γ satisfies the ψ-class conditions Ψ if for each vertex V, we have $\sum_{E_i \ni V} s_i \leq \mathrm{ov}(V)$, where the sum is over all weight-0 non-compact edges E_i containing V.

Let $\mathfrak{T}_{g,\Delta}$ denote the space of tropical curves in $N_{\mathbb{R}}$ of genus g and degree Δ, and let $\mathfrak{T}_{g,\Delta}(\mathbf{A},\Psi)$ denote the subset which satisfy the conditions $\mathbf{A}$ and Ψ. Suppose that $\sum_{i\in I^\circ} s_i + \sum_{i\in I} \mathrm{codim}(A_i) = |I| + (r-3)(1-g)$, and that the spaces A_i are generically translated. We then call $\Gamma \in \mathfrak{T}_{g,\Delta}(\mathbf{A},\Psi)$ rigid if no nearby elements of $\mathfrak{T}_{g,\Delta}$ also live in $\mathfrak{T}_{g,\Delta}(\mathbf{A},\Psi)$. Otherwise, Γ is superabundant (an issue which only arises in higher genus).

We now define the multiplicity of a rigid tropical curve. Let Φ denote the map of lattices

$$\Phi : \prod_{V\in\Gamma^{[0]}} N \to \left(\prod_{E\in\Gamma_c^{[1]}} N/\mathbb{Z}u_E\right) \oplus \left(\prod_{i\in I} N/(A_i \cap N)\right)$$
$$(H_V)_V \mapsto ((H_{\partial^+ E} - H_{\partial^- E})_{E\in\Gamma_c^{[1]}}, (H_{\partial E_i})_{i\in I}).$$

Then rigidity implies that Φ has finite index [2, Lem/Def 2.16.], and we denote $\mathfrak{D}_\Gamma := \mathrm{index}(\Phi)$. Let $\mathrm{Mult}(\Gamma) := \mathfrak{D}_\Gamma \prod_{E\in\Gamma_c^{[1]}} w(E)$. For each vertex V, define $\langle V\rangle := \frac{(\mathrm{val}(V)-3)!}{\prod_{i\in I^\circ : E_i \ni V} s_i!}$. Then define $\langle\Gamma\rangle := \prod_V \langle V\rangle$. Finally, if every tropical curve in $\mathfrak{T}_{g,\Delta}(\mathbf{A},\Psi)$ is rigid, define

$$\mathrm{GW}^{\mathrm{trop}}_{g,\Delta}(\mathbf{A},\Psi) := \sum_{\Gamma\in\mathfrak{T}_{g,\Delta}(\mathbf{A},\Psi)} \frac{\langle\Gamma\rangle}{|\mathrm{Aut}(\Gamma)|} \mathrm{Mult}(\Gamma).$$

Theorem 1 ([2]). *If every tropical curve in $\mathfrak{T}_{g,\Delta}(\mathbf{A},\Psi)$ is rigid, then $\mathrm{GW}^{\mathrm{trop}}_{g,\Delta}(\mathbf{A},\Psi)$ is equal to the corresponding descendant log Gromov-Witten number.*

2. Tropical multiplicities from polyvector fields

Let $M := \mathrm{Hom}(N,\mathbb{Z})$. For the affine-linear space $A_i \subset N_{\mathbb{R}}$, let α_i be the unique-up-to-sign primitive element of $\Lambda^{\mathrm{codim}(A_i)} M \subset \Lambda M$ such that $\iota_n \alpha_i = 0$ for all $n \in N$ parallel to A_i. Fix a vertex $V_\infty \in \Gamma^{[0]}$ which we view as the sink for a flow on the graph. We use this flow to inductively associate an element $\zeta_E \in \mathbb{Z}[N]\otimes\Lambda M$ (well-defined up to sign) for each edge $E \in \Gamma^{[1]}$.

For the non-compact edges E_i, we take

$$\zeta_{E_i} := z^{w(E_i)u_{E_i}} \alpha_i.$$

Then if k edges $E_1,\ldots,E_k$ flow into a vertex $V \neq V_\infty$ and E_{k+1} flows out of V, we inductively define

$$\zeta_{E_{k+1}} := l_k(\zeta_{E_1},\ldots,\zeta_{E_k}) := l_1(\zeta_{E_1}\cdots\zeta_{E_k}),$$

where $l_1(z^n\alpha) := z^n \iota_n(\alpha)$. Then at V_∞ we compute

$$\zeta_\Gamma := \prod_{E \ni V_\infty} \zeta_E.$$

Rigidity and balancing imply that $\zeta_\Gamma \in \Lambda^r M$. Despite the sign-ambiguity, the index $|\zeta_\Gamma|$ is well-defined.

Theorem 2 ([3])**.** $\mathrm{Mult}(\Gamma) = |\zeta_\Gamma|$.

We note that $\mathbb{Z}[N] \otimes \Lambda M$ can be viewed as the algebra of polyvector fields on the dual/mirror algebra torus $\mathbb{G}(M)$, with $m \in M$ corresponding to the derivation $\partial_m : z^n \mapsto \langle n, m\rangle z^n$. This algebra comes with a great deal of interesting structure; e.g., the the l_k's (with some sign-twisiting) make $\ker(l_1)$ into an L_∞-algebra, and $l_2|_{\ker(l_1)}$ agrees with the restriction of the Schouten-Nijenhuis Lie bracket. The action of wall-crossing on polyvector fields can be understood in terms of this Schouten-Nijenhuis bracket. This structure is mirror to structures in string topology/symplectic cohomology.

3. Tropical quantum field theory

Theorem 2 follows from a more general framework as we now sketch. We define a symmetric monoidal category Trop2Cob of $(N = 2)$ "tropical cobordisms." Let $\overline{N}$ denote N modulo the $\mathbb{Z}/2\mathbb{Z}$ action of negation (as a set). The objects of Trop2Cob are maps $\overline{\Delta} : I \to \overline{N}$. One should think of these objects as tropical degrees signs and without any balancing condition.

Given two objects $\overline{\Delta}_1$ and $\overline{\Delta}_2$, the morphisms of Trop2Cob are, roughly, the types (up to the $\mathbb{Z}/2\mathbb{Z}$ negation-action and without the balancing condition) of tropical curves of degree $\overline{\Delta}_1 \sqcup \overline{\Delta}_2$. Composition is defined via concatenation, and $\sqcup$ gives the monoidal operator.

A tropical quantum field theory is now defined as a symmetric monoidal functor from Trop2Cob to another symmetric monoidal category, e.g., to the category of super $\mathbb{Z}$-modules. Specifically, we consider a functor F defined on objects via

$$F(\overline{\Delta} : I \to \overline{N}) := \bigotimes_{i \in I} \Lambda^*(M_i \oplus M_i)$$

where $M_i := \Delta(i)^\perp \cap M$. Given a morphism $\Gamma \in \mathrm{Hom}(\overline{\Delta}, \emptyset)$, we determine $F(\Gamma) : F(\overline{\Delta}) \to \mathbb{Z}$ by choosing a flow on Γ and then working inductively, taking products/coproducts when flowing through vertices and then contracting by $\iota_{(n,0)\wedge(0,n)}$ when we traverse an edge of weighted direction n.

For any lattice L, $m_1, \ldots, m_k \in L$, and $\alpha = m_1 \wedge \cdots \wedge m_k \in \Lambda^k L$, define

$$\alpha^\square := (m_1, 0) \wedge (0, m_1) \wedge \cdots \wedge (m_k, 0) \wedge (0, m_k) \in \Lambda^{2k}(L \oplus L).$$

In particular, $\alpha_i^\square$ is defined without sign ambiguity, and we can define $\alpha_{\mathbf{A}} := \bigotimes_{i \in I} \alpha_i^\square \in F(\overline{\Delta})$.

Theorem 3 ([3])**.** $\mathrm{Mult}(\Gamma)^2 = F(\Gamma)(\alpha_{\mathbf{A}})$.

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A proof of a conjecture of N. Takahashi

Pierrick Bousseau

Let E be a smooth cubic curve in the complex projective plane $\mathbb{P}^2$. For every positive integer d, we consider degree d rational curves in $\mathbb{P}^2$ intersecting E in a unique point. More precisely, let $\overline{M}_0(\mathbb{P}^2/E, d)$ be the moduli space of genus 0 degree d stable maps to $\mathbb{P}^2$ relative to E, with maximal contact order with E in a unique point. The virtual dimension is zero and we denote

$$N_{0,d}^{\mathbb{P}^2/E} = \int_{[\overline{M}_0(\mathbb{P}^2/E,d)]^{virt}} 1 \in \mathbb{Q}\,,$$

the corresponding relative genus 0 Gromov-Witten invariant.

We fix p_0 one of the 9 flex points of E. Let L_{p_0} be the line tangent to E at p_0. If C is a degree d rational curve in $\mathbb{P}^2$, meeting E in a unique point p, then the cycle $C - dL_{p_0}$ intersects E in the cycle $3dp - 3dp_0 = 3d(p - p_0)$. As the cycle $C - dL_0$ has degree 0, it is linearly equivalent to 0, and so the cycle $3d(p - p_0)$ is linearly equivalent to zero in E.

It follows that if we denote $ev\colon \overline{M}_0(\mathbb{P}^2/E, d) \to E$ the evaluation at the contact point with E, and if p is in the image of ev, then $p - p_0$ is necessarily a $(3d)$-torsion point of the group $Pic^0(E)$ of degree 0 cycles on E up to linear equivalence. Thus, we have a decomposition

$$\overline{M}_0(\mathbb{P}^2/E, d) = \coprod_{p \in P_d} \overline{M}_0(\mathbb{P}^2/E, d)^p\,,$$

where the disjoint union is over the set P_d of $(3d)^2$ points p of E such that $p - p_0$ is a $(3d)$-torsion point in $Pic^0(E)$.

For every $p \in P_d$, we denote $[\overline{M}_0(\mathbb{P}^2/E,d)^p]^{virt}$ the restriction of $[\overline{M}_0(\mathbb{P}^2/E,d)]^{virt}$ to $\overline{M}_0(\mathbb{P}^2/E,d)$, and we define

$$N_{0,d}^{\mathbb{P}^2/E,p} = \int_{[\overline{M}_0(\mathbb{P}^2/E,d)^p]^{virt}} 1 \in \mathbb{Q}\,.$$

By definition, we have the equality

$$N_{0,d}^{\mathbb{P}^2/E} = \sum_{p \in P_d} N_{0,d}^{\mathbb{P}^2/E,p}\,.$$

The main question we wish to address is how the numbers $N_{0,d}^{\mathbb{P}^2/E,p}$ depend on the point p in P_d. It is not an obvious question as for different points $p \in P_d$, the geometry of multiple cover contributions to $N_{0,d}^{\mathbb{P}^2/E,p}$ can be quite different. For example, if $d=2$ and $p \in P_2$ is such that $p \notin P_1$, then $\overline{M}_0(\mathbb{P}^2/E,d)^p$ is a single point and $N_{0,2}^{\mathbb{P}^2/E,p} = 1$. But if $p \in P_2$ is such that $p \in P_1$, i.e. if p is a flex point, then contributions come from double covers of the tangent line to E at p, so $\overline{M}_0(\mathbb{P}^2/E,d)^p$ and the virtual class are non-trivial, and one finds after some computation that $N_{0,2}^{\mathbb{P}^2/E} = \frac{3}{4}$.

We introduce some notations in order to make some systematic study of this phenomenon. For p belonging to some P_d, we denote $d(p)$ the smallest integer $d>0$ such that $p \in P_d$. Remark that $p \in P_d$ if and only if d is a multiple of $d(p)$. For every k divisor of d, we denote $P_{d,k}$ the set of $p \in P_d$ such that $d(p)=k$.

The set $P_{d,d}$ is the set of "primitive" $3d$-torsion points, i.e. those which do not belong to any P_k with $k<d$ and so are the "simplest" from the point of view of multiple covers in Gromov-Witten theory. By contrast, $P_{d,1}$ is the set of the 9 flex points of E, which contribute to Gromov-Witten invariants in every degree d, and so are the most "complicated" from the point of view of multiple covers.

It is possible to prove that, for every positive integer d, the rational number $N_{0,d}^{\mathbb{P}^2/E,p}$ only depends on the point $p \in P_d$ through the integer $d(p)$. It follows that, for every positive integer d and for every positive integer k dividing d, it makes sense to denote $N_{0,d}^{\mathbb{P}^2/E,k}$ for the common value of the invariants $N_{0,d}^{\mathbb{P}^2/E,p}$, with $p \in P_d$ such that $d(p)=k$.

The question of the dependence of the point of contact is now reduced to the question of the dependence on k of the invariants $N_{0,d}^{\mathbb{P}^2/E,k}$. This question is completely solved our main result: for every positive integer d and for every positive integer k dividing d, we have

$$(-1)^{d-1} N_{0,d}^{\mathbb{P}^2/E,k} = \sum_{\substack{d' \\ k|d'|d}} \frac{1}{(d/d')^2} (-1)^{d'-1} N_{0,d'}^{\mathbb{P}^2/E,d'}\,.$$

This result was previously conjectured by N. Takahashi [Tak01].

The proof of our main result consists in two steps.

1) A tropical computation of the invariants $N_{0,d}^{\mathbb{P}^2/E}$ in terms of a scattering diagram. This was expected from the Gross-Siebert picture of mirror symmetry

[CPS10] and is proved by Gabele [Gab19] using an appropriate normal crossing degeneration.

2) The newest step is the interpretatin of the above scattering diagram in terms of space of Bridgeland stability conditions on the derived category of coherent sheave on local $\mathbb{P}^2$ and of Donaldson-Thomas invariants. It follows that the original question in Gromov-Witten theory can be translated into a question about the scattering diagram, and then into a question about sheaf counting on local $\mathbb{P}^2$, whose answer happens to be known.

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The logarithmic gauged linear sigma model

Qile Chen

(joint work with Felix Janda, Yongbin Ruan)

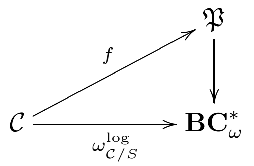
Background. In 1993, Witten gave a physical derivation of the Landau–Ginzburg (LG)/Calabi–Yau (CY) correspondence by constructing a family of theories, known as the *gauged linear sigma model* (GLSM) [22]. Since then, the mathematical theory of GLSM has been developed by many authors in various settings [2, 3, 4, 10, 11, 12, 13, 17, 20, 21].

As discovered in [3] and further developed in [5, 9, 18], GLSM can be viewed as a deep generalization of the hyper-plane property of Gromov–Witten (GW) theory for arbitrary genus. However, comparing to GW theory, a major difference as well as a main difficulty of GLSM is the appearance of an extra torus action on the target, called the *R-charge*, which makes the moduli stacks in consideration for defining the GLSM virtual cycles non-proper in general. This makes the powerful tool of virtual localization [14] difficult to apply.

In this talk, I introduced a recent work with Felix Janda and Yongbin Ruan on log R-maps aiming at providing a log compactification of the stack of GLSM in the case of *hybrid targets* [7]. This further extends our previous work with Adrien Sauvaget in the r-spin case [8].

A further localization calculation of log R-maps is currently a work in progress which in the GW setting will lead to a formula relating invariants of the ambient variety and the complete intersections. In many interesting examples including quintic 3-folds in $\mathbb{P}^4$, this should lead to an effective method for computing GW invariants of hyber-surfaces.

The log R-maps. Consider a proper log smooth morphism of log algebraic stacks $\mathfrak{P} \to \mathbf{BC}^*_\omega := \mathbf{BG}_m$ where $\mathbf{BG}_m$ is the stack with the trivial log structure parameterizing line bundles. A *log R-map* over a log scheme S is commutative diagram



where $\mathcal{C} \to S$ is a log orbifold curve with the log cotangent bundle $\omega^{\log}_{\mathcal{C}/S}$.

While studying log R-maps in the general setting above will be a long term goal, in this talk we focused on the hybrid situation below which covers many interesting cases including GW of complete intersections and the LG/FJRW theory. For this purpose, we consider a smooth Deligne-Mumford stack $\mathcal{X}$ with projective coarse moduli, a line bundle $\mathbf{L}$ and a finite rank vector bundle $\mathbf{E} = \bigoplus_{i \in \mathbb{Z}_{>0}} \mathbf{E}_i$ over $\mathcal{X}$ where i indicates the grading, and a positive integer r. We form the hybrid target $\mathfrak{P}$ in 2-steps. First, we form the Deligne-Mumford type morphism $\mathfrak{X} \to \mathbf{BC}^*_\omega$ depending on $(\mathcal{X}, \mathbf{L}, r)$. A morphism $\mathcal{C} \to \mathfrak{X}$ from a log curve $\mathcal{C} \to S$ amounts to a pre-stable map $g : \mathcal{C} \to \mathcal{X}$ and an r-th root $\mathcal{L}_{\mathcal{C}}$ of $g^*\mathbf{L}^\vee \otimes \omega^{\log}_{\mathcal{C}/S}$, called *the r-spin bundle*. Second, we carefully choose some weighted projective stacks bundle $\mathfrak{P} \to \mathfrak{X}$ compactifying $\mathfrak{P}^\circ := \bigoplus_{i>0} (\mathbf{E}_i^\vee|_{\mathfrak{X}} \otimes \mathcal{L}_{\mathfrak{X}}^{\otimes i})$ over $\mathcal{X}$ where $\mathcal{L}_{\mathfrak{X}}$ is the universal r-spin bundle over $\mathfrak{X}$. The log structure of $\mathfrak{P}$ is given by the divisorial log structure associated to the hyper-plane at infinity. Fixing an r-spin on $\mathcal{C} \to S$, a compatible morphism $p : \mathcal{C} \to \mathfrak{P}^\circ$ is equivalent to the so called *p-field* in GLSM. Thus log R-map can be viewed as a generalization where meromorphic p-fields are allowed.

The stacks and their canonical theory. After introducing a rather delicate stability condition on log R-maps, we define the stack $\mathscr{R}_{g,\vec{\varsigma}}(\mathfrak{P}, \beta)$ of stable log R-maps with genus g, curve class $\beta \in H_2(\mathcal{X})$, and a collection of contact orders $\vec{\varsigma}$ at markings. Using the theory of log maps [1, 6, 15], $\mathscr{R}_{g,\vec{\varsigma}}(\mathfrak{P}, \beta)$ is shown to be a proper log Deligne-Mumford stack carrying a *canonical perfect obstruction theory* hence a virtual fundamental class $[\mathscr{R}_{g,\vec{\varsigma}}(\mathfrak{P}, \beta)]^{vir}$.

Let $\mathscr{R}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P}, \beta) \subset \mathscr{R}_{g,\vec{\varsigma}}(\mathfrak{P}, \beta)$ be the closed sub-stack with the markings mapping into the zero section of $\mathfrak{P}^\circ \to \mathfrak{X}$. It contains an open sub-stack $\mathscr{R}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P}^\circ, \beta)$ with R-maps factoring through $\mathfrak{P}^\circ$. The canonical theory above restricts to a canonical perfect obstruction theories of both $\mathscr{R}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P}, \beta)$ and $\mathscr{R}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P}^\circ, \beta)$, hence canonical virtual cycles $[\mathscr{R}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P}, \beta)]^{vir}$ and $[\mathscr{R}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P}^\circ, \beta)]^{vir}$.

The super-potential and the reduced theory. A *super-potential* is a morphism of stacks $W\colon \mathfrak{P}^\circ \to \mathcal{L}_\omega$ over $\mathbf{BC}^*_\omega$ whose critical locus is proper over $\mathbf{BC}^*_\omega$. A super-potential induces a *cosection* $\sigma^\circ : \mathrm{Obs} \to \mathbb{C}$ from the canonical obstruction sheaf Obs over $\mathscr{R}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P}^\circ, \beta)$. Though $\mathscr{R}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P}^\circ, \beta)$ is non-compact in general, by [16] the canonical virtual cycle $[\mathscr{R}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P}^\circ, \beta)]^{vir}$ is represented by the cosection localized cycle $[\mathscr{R}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P}^\circ, \beta)]^{vir}_{\sigma^\circ}$ supported along the locus defined by $\sigma^\circ = 0$ which is proper. This cycle $[\mathscr{R}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P}^{\circ},\beta)]^{vir}_{\sigma^{\circ}}$ is shown to be the correct *GLSM virtual cycle*, and is differ from the canonical one $[\mathscr{R}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P},\beta)]^{vir}$ in general.

The extension of σ° to $\mathscr{R}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P},\beta)$ is not well-behaved. This issue can be fixed via a *log modification* $\mathcal{U}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P},\beta)\to\mathscr{R}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P},\beta)$ constructed in [8] which restricts to an identity over $\mathscr{R}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P}^{\circ},\beta)$, and virtually principalizes the boundary $\Delta := \mathcal{U}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P},\beta)\setminus\mathscr{R}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P}^{\circ},\beta)$. In [7], we established the following properties of the proper moduli stack $\mathcal{U}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P},\beta)$:

(1) The canonical theory of $\mathscr{R}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P}^{\circ},\beta)$ pulls back to a canonical theory of $\mathcal{U}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P},\beta)$, hence the canonical virtual cycle $[\mathcal{U}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P},\beta)]^{vir}$ which pushes forward to $[\mathscr{R}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P}^{\circ},\beta)]^{vir}$.

(2) The cosection σ° extends to a cosection $\sigma:\mathrm{Obs}\to\mathcal{O}(m\cdot\Delta)$ over $\mathcal{U}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P},\beta)$ where m is determined by the order of poles of W along the infinity of $\mathfrak{P}$.

(3) σ induces a *reduced perfect obstruction theory* of $\mathcal{U}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P},\beta)$, hence a reduced virtual cycle $[\mathcal{U}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P},\beta)]^{red}$ which recovers the GLSM virtual cycle:

$$[\mathcal{U}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P},\beta)]^{red} = [\mathscr{R}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P}^{\circ},\beta)]^{vir}_{\sigma^{\circ}}.$$

(4) The boundary Δ carries a *reduced virtual cycle* $[\Delta]^{red}$ such that

$$[\mathcal{U}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P},\beta)]^{vir} = [\mathcal{U}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P},\beta)]^{red} + m\cdot[\Delta]^{red}.$$

Finally, we emphasized that both the reduced and canonical theories of $\mathcal{U}^{cpt}_{g,\vec{\varsigma}}(\mathfrak{P},\beta)$ are equivariant with respect to the $\mathbb{C}^*$-scaling of the r-spin structure. This $\mathbb{C}^*$-action and its consequences will be studied in our subsequent work.

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Computation of logarithmic Gromov Witten invariants via degenerations

Lawrence Jack Barrott
(joint work with Navid Nabijou)

The computation of log Gromov Witten invariants given by Gross and Siebert gives a general construction of counts of curves meeting certain tangency conditions to a boundary. These are used in the Gross-Siebert program to produce smoothings of toric degenerations and so mirror families.

The invariants also naturally appear in the gluing formula, where a space degenerates to the union of several components with normal crossings boundaries.

In this talk we present a novel method to count these invariants in genus zero by degenerating a smooth anti-canonical boundary to a normal crossings boundary. In particular we study the case of a smooth conic in $\mathbb{P}^2$ degenerating to a union of three lines.

In this case any curve which meets the triangle of three lines at a single point must in fact be mapped into this triangle. We show that the two moduli spaces, of log stable maps and maps to the boundary are isomorphic. In particular there is a torus action on this space induced from the torus action on $\mathbb{P}^2$.

The techniques we use are explicit constructions of the obstruction theory, virtual pushforwards and pullbacks. In the end we can produce a formula for the logarithmic invariants in terms of the invariants of maps of fixed degree to the boundary. When there is a torus action we can reduce this further to a combinatorial sum over different curve types.

This should be compared to work by Pierrick Bousseau who uses an argument in mirror symmetry to prove that the invariants can be constructed by counts of curves in local $\mathbb{P}^2$. There is also work of Tim Gabele, who proved that the counts of curves on a degeneration of $\mathbb{P}^2$ give the counts of log curves.

This construction gives a refinement of the counts of P. Bousseau, they include data of how the curve degenerates as it approaches the boundary. In particular in degree two we see that there are eighteen smooth conics which degenerate to a reducible curve and nine which degenerate to double covers of a fixed line.

Reduced genus 1 invariants towards genus 2

Francesca Carocci
(joint work with Luca Battistella, Cristina Manolache)

The *Reduced Gromov–Witten invariants* introduced by Li and Zinger have the advantage of: satisfying a Lefscthez property, and thus they lead to computations of genus 1 Gromov–Witten invariants and they have a better enumerative meaning of the latter. The only unpleasant aspect of reduced invariants is that their modular interpretation is complicated. In this talk we will see two possible ways way to fix this issue by working with a different moduli space and speculate on how the second of tese approaches could be adapted to give a modular resolution of the moduli space of maps in genus 2.

Reduced invariants. The locus of maps from smooth elliptic curves to $\mathbb{P}^r$ is irreducible; we call its closure the *main component* of $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$. In [6] R. Vakil and A. Zinger construct a desingularisation $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)^{\text{main}}$ of the main component via an iterated blow-up construction. Let $\widetilde{\mathcal{C}} \xrightarrow{\widetilde{\pi}} \mathbb{P}^r$ denote the universal curve over $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)^{\text{main}}$, and let $\widetilde{f}$ be the universal map over the resolution of the main component. Then the sheaf $\widetilde{\pi}_*\widetilde{f}^*\mathcal{O}_{\mathbb{P}^r}(l)$ is a vector bundle for any $l \geq 1$. Given X_l a hypersurface of degree l in $\mathbb{P}^r$, *reduced genus* 1 *invariants* of X_l (see [6], [2]) are defined as

$$c_{\text{top}}(\widetilde{\pi}_*\widetilde{f}^*(\mathcal{O}_{\mathbb{P}^r}(l))) \cap [\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)^{\text{main}}].$$

Standard and reduced invariants of the quintic threefold $X_5 \subseteq \mathbb{P}^4$ are related by the Li–Zinger formula [4]:

$$N_1(X_5, d) = N_1^{\text{red}}(X_5, d) + \frac{1}{12}N_0(X_5, d). \tag{1}$$

Cuspidal Gromov–Witten invariants. Based on D.I. Smyth's work , M. Viscardi [7] has introduced a series of *alternate compactifications* of the moduli space of maps from smooth elliptic curves. The first instance of these alternate compactifications is $\overline{\mathcal{M}}^{(1)}_{1,n}(X, \beta)$, where elliptic tails are made unstable and cuspidal singularities are allowed in the source curve. These spaces carry perfect obstruction theories and lead to invariants; we call them cuspidal Gromov–Witten invariants.

We prove the following:

Theorem. Let $X_5 \subseteq \mathbb{P}^4$ be a generic smooth quintic threefold. Then:

$$N_1^{\text{red}}(X_5, d) = N_1^{\text{cusp}}(X_5, d),$$

Sketch of the proof. There exists a well-defined 1-stabilisation morphism at the level of weighted-stable curves:

$$\mathfrak{M}_{1,n}^{\mathrm{wt,st}} \to \mathfrak{M}_{1,n}^{\mathrm{wt,st}}(1)$$

replacing elliptic tails of weight 0 with cusps.

We then prove that $Z_{X_5} = \mathfrak{M}_{1,n}^{\mathrm{wt,st}} \times_{\mathfrak{M}_{1,n}^{\mathrm{wt,st}}(1)} \overline{\mathcal{M}}_{1,n}^{(1)}(X_5, d)$ is a substack of $\overline{\mathcal{M}}_{1,n}(X_5, d)$ that has no component with contracted elliptic tails. The fiber product endows Z_{X_5} with a virtual class which, by Costello's virtual push-forward formula , has the same enumerative content as $[\overline{\mathcal{M}}_{1,n}^{(1)}(X_5, d)]^{\mathrm{vir}}$

In order to compare the degree of $[Z_{X_5}]^{\mathrm{vir}}$ with the reduced Gromov–Witten invariants we follow in Chang and Li's footsteps. We introduce the moduli space of 1-stable maps with p-fields, which has a simpler geometry and admits a cosection-localised virtual class of the same degree of the cuspidal maps to the quintic. We then construct the space $Z_{X_5}^p$ analogous to Z_{X_5} but with p-fields, and perform a desingularisation of it via the study of local equations. In the end we analyse a splitting of the intrinsic normal cone [1].

Ranganathan-Santos–Parker-Wise moduli space. In [5] the authors give a modular interpretation to Vakil-Zinger resolution of the main component, which suggest the way to construct modular resolution also in higher genus. In a few, imprecise words, the idea behind the construction comes from the following observation:

Remark. Let $[f\colon C \to \mathbb{P}^r]$ be a stable map from a genus 1 curve C and let E be the maximal genus 1 subcurve contracted by f. Then f is smoothable if and only if it factors (not uniquely !) through a curve with a genus 1 singularity $f\colon C \xrightarrow{\phi} \widehat{C} \xrightarrow{\hat{f}} \mathbb{P}^r$, such that ϕ contracts $\mathrm{Exc}(f)$ and is an isomorphism outside it.

We then expect that in order to get a desingularization of the main component we should add to the moduli problem the information that will allow us to determine the contraction $\widehat{C}$ and at the same time ensure that the singularities appearing are only Goreinstein. Indeed once we can construct the contraction in families, we have a new obstruction theory defined via $\hat{f}$ and the non degeneracy of $\hat{f}$ implies un-obstructdness.

In [5] the authors elegantly describe the moduli problem with this extra information in terms of log structures.

They define $\mathrm{VZ}_{1,n}(Y, \beta)$ as the stack parametrising families of stable maps $C \to Y$, where C in endowed with a *radially aligned log structure*, and the following **factorization property** is satisfied: there exist a birational modification $\widetilde{C} \to C$ such that the composition $\widetilde{C} \to C \to Y$ factors through $\widehat{C} \xrightarrow{\hat{f}} Y$.

They showed that for Y the projective space this space is precisely Vakil-Zinger desingularisation . The data of the log structure is exactly what is needed (in genus 1) to construct a well defined birational modification and associated contraction with all the desired properties.

First step in genus 2. Also in genus 2, smoothability of those components in $\overline{\mathcal{M}}_{2,n}(\mathbb{P}^r, d)$ parametrising maps which contract a higher genus curve can be characterised in terms of factorisation through less degenerate maps from more singular curves. However, this time we need to be more careful:

Example. Let $Z \cup_p R$ be a genus 2 curve nodally attached at p to a single rational tail and f a map to $\mathbb{P}^r$ which contract Z. Then:

- If p is a Weierstrass point, f is smoothable if and only if it admits a factorization through a non degenerate map $\widehat{C} \xrightarrow{\hat{f}} \mathbb{P}^r$ where $\widehat{C}$ has a ramphoid cusp;
- If p is a Not Weierstrass point, f is smoothable if and only if it admits a factorization through a non degenerate map $\widehat{C} \xrightarrow{\hat{f}'} \mathbb{P}^r$ where $\widehat{C}'$ has a Non Goreinstein singularity with analytic local ring $\mathbb{C}[[t^3, t^4, t^5]]$.

  The latter is also equivalent admitting a factorization through a curve $\widehat{C}$ with an A_5 singularity in such a way that $\hat{f}$ is not constant on one of the two branches. We should thing that $\widehat{C}$ is obtained from C by first sprouting a $\mathbb{P}^1$ from the point $\bar{p}$ conjugate to p and then contracting the genus 2 curve.

The example tells us that this time around in order to determine the contraction, in addition to a log structure, we should add a section over the locus of tails telling us if the node to which the tail is attached is Weierstrass or not. We are now trying to figure out how to recall this extra information correctly and at the same time we are trying to compare our construction with the desingularisation by blow up recently constructed in [3].

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Logarithmic quasimaps and relative mirror symmetry

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(joint work with Luca Battistella)

Context. Over the past ten years, the theory of stable quasimaps [7, 1, 2] has emerged as a powerful framework for stating and proving mirror theorems. In this context, the "mirror theorem" is obtained as a combination of two logically independent results:

(1) *wall-crossing* formulae, which equate generating functions for quasimap invariants with generating functions for Gromov–Witten invariants, typically after a suitable change of variables called the "mirror map";
(2) *reconstruction* theorems, obtained via Birkhoff factorisation and other polynomiality results, which make it possible to write down explicitly the quasimap generating functions.

This program reached its denoument in a series of papers by Ciocan-Fontanine and Kim [3, 4], which established all-genus wall-crossing results as well as reconstruction theorems for a wide class of targets.

Separately, recent work of Fan–Tseng–You [5] exploits the equality between the genus-zero relative Gromov–Witten invariants of a smooth pair (X, D) and the orbifold Gromov–Witten invariants of the associated root stack $X_{D,r}$, in order to obtain a mirror theorem in the relative setting.

Our approach is to develop and apply a fully-fledged theory of *logarithmic quasimaps*, in order to strengthen the above relative mirror theorem and extend it beyond the case of a smooth divisor. The ultimate aim of this project is to obtain pleasant closed formulae for generating functions of logarithmic Gromov–Witten invariants.

Recap of absolute quasimaps. Unlike stable maps, moduli spaces of quasimaps are not defined for arbitrary smooth projective targets; they require some additional structure on the target space. As established in [2], the correct general context to work in is that of *affine GIT quotients*. That is, we consider smooth projective targets of the form

$$X = W /\!\!/_{\theta} G$$

where W is an affine variety acted on by a reductive algebraic group G, and θ is a character of G linearising the action. We assume that:

(1) there are no strictly semistable points in W;
(2) W^{s} is smooth;
(3) G acts freely on W^{s}.

This general setup incorporates a wide class of targets, including toric complete intersections and Nakajima quiver varieties. Under these assumptions, we have a closed embedding:

$$X = W /\!\!/_{\theta} G = [W^{\mathrm{s}}/G] \hookrightarrow [W/G].$$

A *stable quasimap* is then defined as a morphism to the full stack quotient $[W/G]$

$$u\colon C \to [W/G]$$

satisfying a certain *modified stability condition* [2]. This stability condition ensures in particular that only finitely many points of the curve can map into the unstable locus $[W^{\mathrm{us}}/G]$; these are referred to as the *basepoints* of the quasimap. Thus, every quasimap defines a *rational map*

$$C \dashrightarrow X$$

which is not defined on the finite locus of basepoints $B \subseteq C$. (Note, however, that the data of a quasimap is *not* equivalent to the data of a rational map to X; a quasimap contains strictly more information.)

A quasimap without any basepoints is nothing more than a stable map. Thus we may view the moduli spaces of quasimaps and of stable maps as open and closed substacks of a common larger moduli space (the space of all morphisms from a nodal curve to $[W/G]$), isolated by imposing different stability conditions.

Example. Consider $X = \mathbb{P}^m$ with the standard GIT presentation:

$$\mathbb{P}^m = \mathbb{C}^{m+1} /\!\!/ \mathbb{C}^\star.$$

A morphism to the full stack quotient $C \to [\mathbb{C}^{m+1}/\mathbb{C}^\star]$ consists of a line bundle L on C together with $m+1$ sections $u_0, \ldots, u_m \in \mathrm{H}^0(C, L)$. Then $p \in C$ is a basepoint if any only if $u_0(p) = \ldots = u_m(p) = 0$, and the expression $[u_0, \ldots, u_m]$ defines a morphism $C \setminus B \to \mathbb{P}^m$. Note that every basepoint has an associated *multiplicity*; the maximal order to which all of the u_i vanish.

Target spaces: logarithmic setting. When developing a theory of logarithmic quasimaps, the first question which needs to be addressed is: *what is the correct class of target spaces?* Our geometric setup is as follows: we have an affine logarithmic scheme

$$(W, M_W)$$

together with a (linearised) logarithmic action of the algebraic group G. Assumptions (1) and (3) above remain unchanged, while assumption (2) is modified to require that $(W^{\mathrm{s}}, M_W|_{W^{\mathrm{s}}})$ is logarithmically smooth. These assumptions mean that the logarithmic structure descends to the quotient $X = W /\!\!/_\theta G = [W^{\mathrm{s}}/G]$, and that X is logarithmically smooth.

As in the absolute case, this setup incorporates a wide class of targets. For divisorial targets, we have everything we might want: given any absolute quasimap target $X = W /\!\!/_\theta G$ and any simple normal crossings divisor $D \subseteq X$ we may consider the closure E of the preimage of D under the quotient map $W^{\mathrm{s}} \to X$, and equip W with the divisorial logarithmic structure corresponding to E. Then the action of G on W extends uniquely to a logarithmic action, and the induced logarithmic structure on the quotient is the divisorial logarithmic structure for the pair (X, D). Our setup also incorporates most semistable varieties which appear in the degeneration formula.

Logarithmic quasimaps: definition. Given such target data, we also obtain an induced logarithmic structure on the full stack quotient $[W/G]$, and a strict closed embedding of logarithmic stacks:

$$X = [W^{\mathrm{s}}/G] \hookrightarrow [W/G].$$

A *logarithmic quasimap* is then nothing but a logarithmic morphism

$$u\colon C \to [W/G]$$

where C is a logarithmically smooth curve. The stability condition is simply the one for the underlying scheme-theoretic morphism (as is the case for logarithmic stable maps).

With some work, the existing proofs for logarithmic stable maps and absolute quasimaps can be adapted to show that the resulting moduli space is a finite-type, proper, Deligne–Mumford stack with a natural perfect obstruction theory. The resulting invariants are referred to as *logarithmic quasimap invariants*.

Example. Returning to the example of $X = \mathbb{P}^m$, suppose that we equip $W = \mathbb{C}^{m+1}$ with the logarithmic structure corresponding to the hyperplane $\{z_0 = 0\}$. The induced logarithmic structure on X is the one corresponding to the smooth pair $(\mathbb{P}^m, H_0)$, and a logarithmic quasimap to this target consists of an ordinary quasimap to $\mathbb{P}^m$ together with an enhancement of the section $u_0 \in \mathrm{H}^0(C, L)$ to a logarithmic morphism

$$u_0\colon C \to \mathrm{Tot}(L)$$

where $\mathrm{Tot}(L)$ is equipped with the divisorial logarithmic structure corresponding to the zero section.

The role of basepoints. A natural question to ask at this point is: should basepoints be allowed to acquire logarithmic structure? The answer to this question, informed by a number of different viewpoints, is a resounding "no." For a very straightforward justification, observe that in the space of quasimaps, basepoints may pop in and out of existence, while marking-type logarithmic structures on logarithmic curves always persist in an open neighbourhood.

This has consequences. For one, it means that the tropicalisation of a logarithmic quasimap does not record the basepoints, and consequently the loci of quasimaps with basepoints do not form logarithmic strata in the moduli space. The degree of the quasimap on each component of the source curve, which appears in the modified balancing condition [6, §1.4], is computed in terms of the degree of the relevant line bundles on the source curve, and as such accounts for both the "residual degree" of the induced rational map *and* the multiplicity of any basepoints which appear.

More importantly, the fact that the logarithmic structure on C is locally constant near any basepoint means that the logarithmic structure on W cannot "jump" at the unstable locus. This imposes severe restrictions on the geometry of logarithmic quasimaps containing basepoints (which may also be observed on the stable map side, by examining when rational tails can or cannot appear). For instance, we have:

Lemma. If $(X, \partial X)$ is a smooth toric variety with divisorial logarithmic structure corresponding to the toric boundary, then a quasimap to $(X, \partial X)$ has no basepoints.

Thus, the theories of logarithmic quasimaps and logarithmic stable maps coincide in this case. This might seem surprising, because $(X, \partial X)$ is a log Calabi–Yau geometry, and we expect Calabi–Yau geometries to induce non-trivial mirror maps (and hence non-trivial wall-crossing formulae). However, the absence of basepoints here is not really a consequence of the log Calabi–Yau property, but rather of the maximal degeneracy property of the boundary. For instance, if we consider a Fano variety relative a smooth anticanonical section (another example of log Calabi–Yau geometry), then we expect that basepoints will appear, and consequently that the logarithmic quasimap and stable map theories will diverge.

Wall-crossing and reconstruction. Having established the rudiments of the theory, the next step is to apply it to derive relative mirror formulae. The two steps outlined at the beginning – wall-crossing and reconstruction – are obtained using two techniques, which can be traced back to Givental's original proof of the mirror theorem:

(1) **reconstruction:** graph space localisation to derive polynomiality properties of generating functions;
(2) **wall-crossing:** torus localisation on the target space, to derive recursion formulae for generating functions.

In the logarithmic context, graph space localisation behaves in much the same way as it does in the absolute setting. A novelty is that we are forced to introduce *punctured invariants* into the generating functions, in order to obtain a formalism which works well (this same phenomenon has been observed by others in other contexts). The result is powerful Birkhoff factorisation and polynomiality properties, which in the semi-positive case can be used to derive closed formulae for appropriate generating functions.

For technical but fundamental reasons, the usual proof of wall-crossing does not carry over to the logarithmic setting. New ideas are required here in order to extend the results, and we are in the process of pursuing these.

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Reduced Gromov-Witten invariants in genus one: the absolute and relative theory of smooth hyperplane sections

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(joint work with Navid Nabijou and Dhruv Ranganathan)

Whereas the rational Gromov-Witten theory of $\mathbb{P}^N$ - and hypersurfaces therein - is well understood thanks to the recursive structure of the boundary of the moduli space of stable maps, torus localisation and the Lefschetz hyperplane theorem [Kon95]; the higher genus theory is much complicated by the onset of degenerate contributions from boundary components. This problem has been studied extensively in genus one: Vakil and Zinger [VZ08] (and, later, Hu and Li) proposed a natural desingularisation of the main component of $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^N, d)$ - the closure of the locus of maps with a smooth source -, which allowed them to introduce a notion of *reduced* invariants for projective complete intersections; Zinger [Zin09] calculated the reduced theory of Calabi-Yau hypersurfaces by torus localisation; Li and Zinger [LZ09] (and, later, Chang and Li) related reduced and ordinary Gromov-Witten invariants, completing the first mathematical proof of the BCOV predictions from mirror symmetry; Ranganathan, Santos-Parker and Wise reinterpreted the Vakil-Zinger desingularisation in terms of logarithmic geometry and elliptic singularities, providing it with a modular meaning [RSW17a], further studying the relative problem in the toric case, and answering a long-standing question about the realisability of tropical maps [RSW17b].

In [BNR19] we develop one missing piece of technology in this framework, namely a kind of degeneration formula. We focus on the case of smooth very ample pairs (X, D), which easily reduces to a hyperplane in projective space $(\mathbb{P}^N, H)$. We construct a logarithmic desingularisation of the main component of $\overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H, d)$, for any choice of a contact order $\alpha \vdash d$. A natural Cartier divisor of tropical origin - expressing the condition that the contact order degenerates at a fixed marking - singles out some components of the boundary of this space, which admit a tautological description in terms of moduli spaces of maps with lower numerical data, thus allowing for a recursive reconstruction algorithm. We repurpose here a scheme put forward by Gathmann in genus zero [Gat02], simplifying his proofs by advocating a fully tropical approach; novel technical difficulties are encountered in describing the strata where an explicit interplay between the geometry of elliptic singularities and the relative stable map is displayed.

In order to simplify two technical points, we choose to build on Kim's moduli space of logarithmic stable maps to expanded targets. A *centrally aligned map* over a base S - whose underlying scheme is a geometric point - is then a logarithmic map $C \to \mathbb{P}^N[a]$ (the central fiber of the a-iterated deformation to the normal cone of $H \subseteq \mathbb{P}^N$), together with a *radius* $\delta \in \Gamma(S, \overline{M}_S)$, such that the distance from any component of C to the minimal subcurve of genus one (*core*) is comparable to δ, and two such distances are comparable between themselves if they are less than δ; this describes a *logarithmic modification* of Kim's moduli space. Any disc of radius δ' less than δ around the core can be canonically contracted - possibly after destabilising - to yield a curve with an *elliptic singularity* $C \to \overline{C}_{\delta'}$ [RSW17a]. The radius δ is required to pass through a component of positive horizontal degree, and the alignment to be compatible with the logarithmic map. A centrally aligned map *factorises completely* if: (i) $f\colon C \to \mathbb{P}^N[a]$ factors through $\overline{C}_{\delta_E}$, where $\delta_E \le \delta$ is determined by the closest component to the core that is not contracted by the tropical map; and (ii) the collapsed map $f_B\colon C_B \to \mathbb{P}^N$ factors through $\overline{C}_\delta$. Roughly, the latter requirement guarantees that the underlying stable map is smoothable in the absolute sense, while the former implies that there is no obstruction to lifting this smoothing to a transverse map to the expansion.

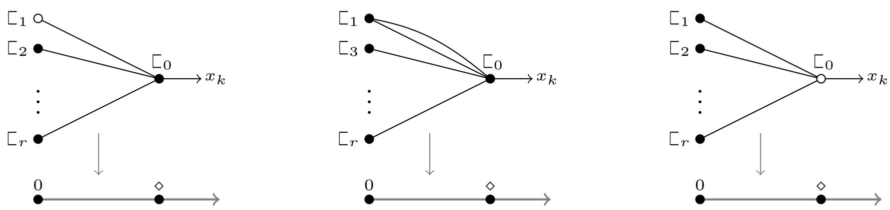
Theorem 1. *The moduli space of centrally aligned maps that factorise completely $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$ is logarithmically smoooth and of expected dimension.*

Logarithmic smoothness ensures that we can read off the codimension of logarithmic strata from the dimension of the corresponding cones in the tropicalization.

Lemma 1. *Consider the piecewise linear function on the tropicalization $\phi(v_k)$, where ϕ is the tropical map, and the vertex v_k corresponds to the component supporting the marking x_k. The associated line bundle $\mathcal{L}_k$ satisfies $c_1(\mathcal{L}_k) = \alpha_k\psi_k + \mathrm{ev}_k^*(H)$, where ψ_k is a* collapsed *psi-class. It follows formally that:*

$$(\alpha_k\psi_k + \mathrm{ev}_k^*(H)) \cap [\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)] = \sum_{\mathcal{D}} \lambda_{\mathcal{D}}[\mathcal{D}]. \tag{1}$$

The sum is over all boundary divisors such that the component containing x_k is mapped to higher level (or into H by the collapsed map). $\lambda_{\mathcal{D}}$ is a combinatorial factor (!). We enumerate the corresponding rays in the tropicalization.



It is understood that we sum over all compatible splittings of the curve class, markings, and contact order; the core is represented by a white disc; in the rightmost type, the core is assumed to have positive horizontal degree. These three types are patently expressible as fibered products of (reduced) relative and rubber moduli spaces of maps with lower numerical data (genus, degree, and number of markings) and already accounted for in Vakil's enumeration [Vak00]. There is a fourth type †, which is a priori similar to the rightmost above, but the core is assigned zero horizontal degree instead. In this case, the interplay with the elliptic singularities cannot be swept under the carpet. Indeed, type † strata are contained in the exceptional divisors of the logarithic modification $\overline{\mathcal{M}}^{\mathrm{cen}}_{1,\alpha}(\mathbb{P}^N|H,d) \to \overline{\mathcal{M}}^{\mathrm{Kim}}_{1,\alpha}(\mathbb{P}^N|H,d)$, and their image under this map has higher (virtual) codimension. To construct them, we first build an appropriately compactified torus bundle over the corresponding locus in Kim's space (the combinatorics of the compactification is encoded in the subdivision procedure dictated by the alignment), and then express the factorisation condition as a tautological integral. Roughly, the torus fiber corresponds to the moduli space of *attaching data* for the elliptic singularity [Smy11b]; generically, the compactification is given by the projective bundle $\mathcal{P} = \mathbb{P}\left(\bigoplus_{i=1}^r T_{q_i} C_i\right)$ - where C_i are the rational tails lying on the circle of radius δ, and q_i are the nodes joining them with the core -, while the locus of maps that factorise is cut by the composition:

$$\mathcal{O}_{\mathcal{P}}(-1) \to p^*\left(\bigoplus_{i=1}^r T_{q_i} C_i\right) \xrightarrow{\sum \mathrm{d} f} \mathrm{ev}^*_{C_0} TH.$$

In order to be made dimensionally transverse to the boundary, the latter needs to be twisted after a modification of $\mathcal{P}$ along logarithmic substrata. To sum up, every factor in the right-hand side of formula (1) can be analysed combinatorially and related to tautological integrals on moduli spaces of maps with lower numerics. Once the formula is established in the unobstructed case $(\mathbb{P}^N, H)$, it can be painlessly pulled back to any smooth, very ample pair.

Theorem 2. *For (X, D) smooth and very ample, a recursive algorithm calculates*

(1) *the (restricted) reduced genus one Gromov–Witten invariants of D;*
(2) *the reduced genus one relative Gromov–Witten invariants of (X, D);*
(3) *the (restricted) reduced genus one rubber invariants of $\mathbb{P} = \mathbb{P}_D(\mathrm{N}_{D|X} \oplus \mathcal{O}_D)$*

from the full genus zero and reduced genus one theory of the ambient variety X.

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Tropical correspondence for $(\mathbb{P}^2, E)$

Tim Gabele

A smooth projective surface Y over an algebraically closed field of characteristic zero together with a reduced effective anticanonical divisor D forms a *log Calabi-Yau pair* (Y, D), meaning that $K_Y + D$ is numerically trivial. The case where $D = D_1 + \ldots + D_{m+1}$ is a cycle of smooth rational curves has been studied in [1]. In [2] it was shown that generating functions of relative Gromov-Witten invariants of (Y, D) in this case can be read off from a certain *scattering diagram*. This statement was generalized in [3] to q-refined scattering diagrams and generating functions of higher genus Gromov-Witten invariants with insertions of λ-classes.

I describe a similar correspondence for the case of $\mathbb{P}^2$ with a smooth anticanonical divisor, that is, an elliptic curve E. One can explicitly write down a toric degeneration of the pair $(\mathbb{P}^2, E)$ and construct its *dual intersection complex* $(B, \mathcal{P}, \varphi)$. This is an integral tropical manifold $(B, \mathcal{P})$ together with a *polarization* φ. The affine manifold with singularities B does not have a boundary.

In [4] it is described how to construct from such a triple $(B, \mathcal{P}, \varphi)$ a toric degeneration of a Landau-Ginzburg model – the mirror of $\mathbb{P}^2$. The construction involves the scattering calculations described in [5], leading to a *consistent structure* $\mathcal{S}_\infty$. This is a collection of 1-dimensional polyhedral subsets of B (*walls*) with attached functions describing the gluings of affine pieces to form the toric degeneration.

Let N_d be the logarithmic Gromov-Witten invariant of $\mathbb{P}^2$ of genus 0 and degree d stable log maps with full tangency with E at a single unspecified point. Let f_{out} be the product of all functions attached to unbounded walls in $\mathcal{S}_\infty$, regarded as elements of $\mathbb{C}[[t]][x]$ for $x := z^{(m_{\text{out}},1)}, t := z^{(0,1)} \in \mathbb{C}[\Lambda \oplus \mathbb{Z}]$. Here Λ is the sheaf of integral tangent vectors on B and $m_{\text{out}} \in \Lambda$ is the primitive vector pointing in the (unique) unbounded direction of B. Then

$$\log f_{\text{out}} = \sum_{d=1}^{\infty} 3d \cdot N_d \cdot t^{3d} x^{-3d}.$$

This correspondence respects the torsion points on E. Further, a q-refinement of the wall structure leads to generating functions for higher genus invariants with inserstions of λ-classes.

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Intrinsic mirror symmetry

Daniel Pomerleano

The introduction of log geometric techniques in mirror symmetry has given rise to enormous progress in the construction of mirror manifolds [6, 4]. These efforts have culminated in the recent announcement [7], where Gross and Siebert have proposed an **intrinsic mirror construction** which is expected to unify and generalize both of these mirror constructions. The goal of this talk is to describe work-in-progress [8] which aims to give a symplectic interpretation of the intrinsic construction. The chief advantage of our approach is that the objects involved in our interpretation (Floer cohomology and wrapped Fukaya categories) connect directly to Kontsevich's Homological Mirror Symmetry (HMS) conjecture.

We begin by recalling a few details concerning the construction of [7]. A **maximally degenerate pair** will denote a pair $(M, \mathbf{D})$ such that M is a smooth projective variety and $\mathbf{D}$ is an snc anticanonical divisor which has a zero dimensional stratum. For any maximally degenerate pair $(M, \mathbf{D})$, Gross and Siebert construct a mirror partner by taking $\mathrm{Spec}(A)$ of an explicit commutative ring of **theta functions**, A. The algebra A comes with a **canonical basis** which is indexed by integral points $B_{\mathbb{Z}}(M, \mathbf{D})$ of a certain affine manifold with singularities $B(M, \mathbf{D})$. In this canonical basis $(\theta_{\mathbf{v}})_{\mathbf{v} \in B_{\mathbb{Z}}(M,\mathbf{D})}$, the structure constants of the multiplication are defined by certain punctured log Gromov-Witten invariants.

Discussions with Mark Gross have suggested that the algebra A is isomorphic to the degree zero piece of a certain Floer theoretic invariant of the complement $X = M \setminus \mathbf{D}$ known as **symplectic cohomology**(see e.g. [9]). In order to simplify the construction of this invariant, we restrict to positive pairs $(M, \mathbf{D})$[1]— those pairs for which $\mathbf{D}$ supports an ample line bundle $\mathcal{L}$ (and the Kahler class is taken to agree with $c_1(\mathcal{L})$).

Some evidence for this conjecture is provided by the following result, which will be proved in [8] building on techniques developed in [2, 3]:

Theorem 1. *Let* $\mathbf{k}$ *be a field containing* $\mathbb{Q}$. *For any positive maximally degenerate pair, there is an additive basis* $(\theta_{\mathbf{v}})_{\mathbf{v} \in B_{\mathbb{Z}}(M,\mathbf{D})}$ *for* $SH^0(X, \mathbf{k})$ *indexed by the set of integral points* $B_{\mathbb{Z}}(M, \mathbf{D})$.

We conjecture that this additive identification gives a ring isomorphism between the ring of theta functions A and symplectic cohomology. In [8], we will prove this in the case when $\dim_{\mathbb{C}}(X) = 2$. Proving this in general would involve developing a symplectic analogue of punctured invariants.

We will also discuss a second result which gives some indication that the intrinsic construction is natural from the point of view of Kontsevich's conjecture. To state

[1] We expect, however, that our results can be extended to all anticanonical pairs.

our result, let $\mathcal{W}(X)$ denote the wrapped Fukaya category of Abouzaid and Seidel [1] and let $HH^*(\mathcal{W}(X))$ denote its Hochschild cohomology. Recall that there is a closed-open isomorphism:

$$\mathcal{CO} : SH^*(X) \xrightarrow{\cong} HH^*(\mathcal{W}(X)). \tag{1}$$

For any two objects $E, F \in \mathrm{Ob}(\mathcal{W}(X))$, the group $\mathrm{Hom}^*(E, F)$ inherits the structure of a module over $R = SH^0(X)$ via (1).

Theorem 2. *For any two objects E, F of $\mathcal{W}(X)$, $\mathrm{Hom}^*(E, F)$ is a finitely generated module over R.*

To see how this statement has implications for Kontsevich's conjecture, suppose that Y is a smooth (log) Calabi-Yau variety such that

$$\mathcal{W}(X) \cong \mathrm{Perf}(Y) \tag{2}$$

Then by (1), (2) and the HKR isomorphism,

$$R \cong HH^0(\mathrm{Perf}(Y)) \cong \Gamma(\mathcal{O}_Y).$$

Theorem 2 would imply that the natural morphism $\phi_a : Y \to \mathrm{Spec}(R)$ is proper morphism of $n = \dim_{\mathbb{C}}(M)$ dimensional varieties, and hence by Zariski's main theorem a birational morphism. Thus Theorem 2 implies:

If Y is a smooth (log) Calabi-Yau mirror partner to X, ϕ_a is a proper birational map and Y is necessarily a crepant resolution of $\mathrm{Spec}(R)$.

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Topological realization over $\mathbb{C}(\!(t)\!)$ via Kato–Nakayama spaces

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(joint work with Piotr Achinger)

In complex algebraic geometry, the analytification functor, that associates to a locally finite type scheme $X/\mathbb{C}$ the set of closed points $X_{\mathrm{an}} = X(\mathbb{C})$ equipped with the analytic topology, plays a central role. In particular it gives access to topological invariants associated to X, such as singular (co)homology, the fundamental group and more generally its homotopy type. For a general locally noetherian scheme X, not necessarily defined over $\mathbb{C}$, one can recover a profinite version of such invariants from the so-called "étale homotopy type" $\Pi(X)$ [2].

For such schemes X defined over the field of formal Laurent series $\mathbb{C}(\!(t)\!)$, we define a functorial topological realization $\Psi(X)$ which refines the étale homotopy type $\Pi(X)$ to something closer to an "analytification". Our construction produces a topological space equipped with a fibration to the circle $\mathbb{S}^1$, and in order to achieve functoriality we have to work with the homotopy category (or even better with the ∞-category) of such objects.

Denote by $\mathcal{S}/\mathbb{S}^1$ the ∞-category of spaces fibered over $\mathbb{S}^1$.

Theorem 1 ([1])**.** *There is a "Betti realization" functor $\Psi\colon \boldsymbol{Sch}^{\mathrm{lft}}_{\mathbb{C}(\!(t)\!)} \to \mathcal{S}/\mathbb{S}^1$ with the following properties.*

- *If X is separated and finite type, then $\Psi(X)$ is a fibration in finite CW complexes over $\mathbb{S}^1$.*
- *If $X = X_0 \times_{\mathbb{C}} \mathbb{C}(\!(t)\!)$, then $\Psi(X) \simeq ((X_0)_{\mathrm{an}} \times \mathbb{S}^1 \xrightarrow{\pi_2} \mathbb{S}^1)$.*
- *If X is the germ of a family $f\colon \mathcal{X} \to \Delta^*$ of complex varieties over a punctured disk, then $\Psi(X) \simeq (f\colon f^{-1}(\mathbb{S}^1) \to \mathbb{S}^1)$ for a small loop $\mathbb{S}^1 \subseteq \Delta^*$ winding around the puncture once.*
- *If $Y_\bullet \to X$ is a hypercovering in the h-topology, then the induced map* $\mathrm{hocolim}\,\Psi(\mathrm{Y}_\bullet) \to \Psi(\mathrm{X})$ *is an equivalence.*
- *There is a canonical map $\Psi(X) \to \widehat{\Pi}(X)$ inducing an isomorphism upon profinite completion.*
- *The fiber $\widetilde{\Psi}(X)$ of $\Psi(X)$ carries a canonical mixed Hodge structure with a monodromy operator.*

If X is smooth, separated and finite-type over $\mathbb{C}(\!(t)\!)$, and $(\mathscr{X}, \mathscr{D})$ is a "weakly" semistable model (i.e. with possibly non-reduced central fiber) over the formal disk $\operatorname{Spec} \mathbb{C}[\![t]\!]$, then the morphism of pairs $(\mathscr{X}, \mathscr{D}) \to (\operatorname{Spec} \mathbb{C}[\![t]\!], \{0\})$ is a *log smooth morphism*, and $\Psi(X)$ can be constructed as the map of *Kato–Nakayama spaces* [3] $(\mathscr{X}, \mathscr{D})_{0,\log} \to 0_{\log} \simeq \mathbb{S}^1$ in the central fiber. The value of $\Psi(X)$ for general X is then dictated by the descent property for the h-topology.

In [1], this construction is complemented by an alternative one that approximates the given $X/\mathbb{C}(\!(t)\!)$ with some finite-type $\mathcal{X}/\operatorname{Spec} R$, where R is a smooth finite-type $\mathbb{C}[t]$-algebra, and applies analytification over $\mathbb{C}$. Moreover, the result can be extended to locally finite-type schemes over a discretely valued non-archimedean field K, equipped with an embedding of its residue field k into $\mathbb{C}$, and to smooth rigid analytic spaces over a complete K as above.

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Tautological relations on the moduli space of stable maps

Younghan Bae

Let X be a nonsingular projective variety over $\mathbb{C}$. Given topological data $\beta \in H_2(X,\mathbb{Z})$, $g \geq 0$, $n \geq 0$, there exists a Deligne–Mumford stack $\overline{\mathcal{M}}_{g,n,\beta}(X)$ which parametrizes stable maps to the target X. In most cases, these spaces are singular and intersecting two cycles is not well defined. Nevertheless, there is a ring structure on the space of tautological classes.

Theorem 1. [1] *The subspace of $A_*(\overline{\mathcal{M}}_{g,n,\beta}(X))_{\mathbb{Q}}$ spanned by tautological classes has a natural ring structure.*

The existence of the ring structure was first predicted in [4]. Our main examples of tautological relations on $\overline{\mathcal{M}}_{g,n,\beta}(X)$ follows from the *Double Ramification Cycle Relations* developed in [3]. Let S be a line bundle over X and let $(a_1,\cdots,a_n)$ be a vector of integers satisfy $\sum a_i = \int_\beta c_1(S)$. Let $\mathsf{P}^{d,r}_{g,A,\beta}(X,S)$ be the polynomial associated to Pixton's double ramification cycle formula, see [3]. Our main result is the vanishing of this polynomial in higher degrees.

Theorem 2. [1] *If $d > g$, then the constant term of the polynomial $\mathsf{P}^{d,r}_{g,A,\beta}(X,S)$ vanishes.*

This result extends similar vanishing result on $\overline{\mathcal{M}}_{g,n}$ established in [2].

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Deformations of semi-smooth varieties

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(joint work with Marco Franciosi, Rita Pardini)

A quasi-projective variety X over the complex numbers is called *semi-smooth* if it is smooth locally isomorphic to the pinch point $uv^2 = w^2$. Equivalently, X is obtained by gluing a smooth variety $\bar{X}$ to itself along a smooth divisor $\bar{Y}$, via an involution ι of $\bar{Y}$ with fixed locus $\bar{B}$ in codimension 1. The singular locus of X is a smooth variety $Y = \bar{Y}/\iota$, with a nontrivial natural scheme structure Y' (given by the Jacobian ideal) along the image B of $\bar{B}$.

We describe the sheaves T_X and $\mathcal{E} := \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)$ in terms of $(\bar{X}, \bar{Y}, \iota)$. We relate them to some sufficient conditions that guarantee the existence of a smoothing of X (see definition below). As an application, we aim to prove smoothability for all semismooth surfaces constructed in [FPR18]; this part of the paper is still in progress.

A *one-parameter deformation* of X is a flat projective family $\mathcal{X}$ over a pointed irreducible nonsingular quasiprojective curve (C, c_0) together with an isomorphism $\phi : \mathcal{X}_{c_0} \to X$; the one-parameter deformation is called a *smoothing* if the general fiber $\mathcal{X}_c$ is nonsingular. Note that to such a one-parameter deformation $\mathcal{X} \to C$ we can associate a first-order deformation of X, corresponding to its restriction to the first infinitesimal neighbourhood of c_0 in C; it is unique up to multiplication by a nonzero complex number.

Even in the case of codimension one nodes $xy = 0$, our results partially extend those of Friedmann [Fr83], while using algebraic and not analytic methods; we expect to eventually replace the complex numbers in the assumption by an algebraically closed field of characteristic different from 2.

Acknowledgements: This material is partly based upon work supported by the National Science Foundation under Grant No. DMS-1440140 while the second and third author were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2019 semester. This research was partly supported by PRIN 2015 "Geometry of algebraic varieties".

1. Sheaves computation

Let $f : \bar{X} \to X$ be the natural morphism (which is also the normalization of X) and $g : \bar{Y} \to Y$ its restriction. Let L be the line bundle on Y such that $L^\vee$ is the antiinvariant part of $g_*(\mathcal{O}_{\bar{Y}})$, so that $2L = \mathcal{O}_Y(B)$. Let M be the normal line bundle of $\bar{Y}$ in $\bar{X}$.

Proposition. *There is an isomorphism of line bundles on $\bar{Y}$ between $g^*(\mathcal{E}|_Y)$ and $g^*B + M + \iota^*M$.*

Sketch of proof. By using degeneration to the normal cone, or rather normal bundle in this case, we reduce to the case where $\bar{X}$ is the total space of M on $\bar{Y}$. We find an explicit embedding of X as a hypersurface in a rank 2 bundle on Y. □

Proposition

(1) *There is a natural injective morphism $\alpha : T_X \to f_* T_{\bar{X}}$ of coherent sheaves on X;*

(2) *The cokernel $\mathcal{G}$ of α is a coherent sheaf on Y, isomorphic to the cokernel of the natural map $(g_* T_{\bar{Y}})^{inv} \to g_*(T_{\bar{X}}|_{\bar{Y}})$.*

Sketch of proof. The proof is obtained by a combination of defining morphims of sheaves globally and computing their properties in local coordinates. For instance the first step is showing that there is a natural map $T_{\bar{X}} \to f^*(\Omega_X)^\vee$ (global) and that it is an isomorphism (local). □

2. Deformations and local deformations

For a scheme X, let D_X be the functor of infinitesimal deformations of X; it associates to each A (local Artnian algebra with residue field $\mathbb{C}$) the set of isomorphism classes of deformations X_A over (the spectrum of) A. We define a functor of local deformations LD_X by associating to each Artinian local algebra A a collection of deformations U_A over (the spectrum of) A where U varies over all open affines in X; the collection has to be consistent, ie if $V \subset U$, then the deformation U_A must induce V_A. If X is a variety with isolated singularities, then LD_X is the product over the singular points of the infinitesimal deformations of each singularity.

A detailed exposition of these elementary basic facts is the object of a separate work in progress, we only need the following elementary properties.

Proposition *Assume X is a variety with hypersurface singularities. Then*

(1) *the tangent space to LD_X is $H^0(\mathcal{E})$, an obstruction space is $H^1(\mathcal{E})$;*

(2) *the sheaf $\mathcal{E}$ is a line bundle on the singular locus Y' in X;*

(3) *the natural morphism $D_X \to LD_X$ is formally smooth if $H^2(X, T_X) = 0$.*

3. Smoothability criteria

We are currently still working on stating smoothability in its natural generality. Since the original focus of the paper was the question of smoothability of specific surfaces, we will concentrate on the case of interest.

Theorem. *Let X be a projective semi-smooth surface such that the dualizing sheaf K_X (or its dual) is ample. Assume that $H^2(X, T_X) = 0$, $\mathcal{E}|_Y$ is generated by global sections, and $H^1(\mathcal{E}|_Y) = 0$. Then X is smoothable.*

Sketch of proof We choose a first order local deformation X_1 of X corresponding to a section of $\mathcal{E}$ which doesn't vanish at any pinch point.

Since the functor LD_X is unobstructed (because $H^1(\mathcal{E}) \to H^1(\mathcal{E}|_Y)$ is an isomorphism), we can extend X_1 to a sequence of infinitesimal deformations X_n of X over $\mathbb{C}[t]/(t^{n+1})$ such that each X_n induces X_{n-1} when restricted to $t^n = 0$.

The assumption on K_X guarantees that we can apply Artin's approximation theorem, hence we can find a one-parameter deformation of X inducing X_1 as first-order deformation.

We then prove that the general fibre must have at most rational double points. We conclude by applying results of Burns and Wahl on the smoothability of canonical singularities for surfaces. □

The applicability of the theorem to the semismooth surfaces in [FPR18] is work in progress.

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Torus localization for logarithmic stable maps

Tom Graber

When X is a smooth projective variety with an action of a torus $T \cong \mathbb{G}_m^r$, one can compute integrals of cycle classes over X which admit equivariant lifts by a reduction to the fixed point locus. This fact (and especially its virtual generalization from [3]) has been the basis for many results in enumerative geometry over the past 20 years. However, that virtual localization theorem does not apply to the space of logarithmic stable maps, because the perfect obstruction theory for this space is defined relative to a singular base stack. The purpose of this talk is to outline both a general result on torus localization for the virtual fundamental class for relative perfect obstruction theories over possibly singular bases, and to begin to spell out what this yields in the particular case of the virtual fundamental class on the space of log stable maps. In work in progress I show that while I cannot give an explicit formula at the level of that obtained for ordinary stable maps, one can control the terms enough to prove that the log GW invariants of toric varieties can be expressed in terms of tautological classes together with some higher double ramification cycles related to work of Holmes, Pixton, and Schmitt.

1. Localization for relative perfect obstruction theories

We suppose that M is a proper DM stack with an action of a torus T, that $\pi : M \to \mathcal{Y}$ is a morphism from M to an Artin stack $\mathcal{Y}$ which is equivariant with respect to the trivial action of T on $\mathcal{Y}$. Finally, we suppose that M is equipped with a T-equivariant perfect obstruction theory relative to π. In other words, there is an element E in the T-equivariant derived category of M which is perfect of amplitude $[-1, 0]$ together with a morphism $E \to L_{M/\mathcal{Y}}$ satisfying the conditions from [2]. If $\mathcal{Y}$ has a fundamental class, then the virtual pullback defined by Manolache in [5]gives rise to a virtual fundamental class $[M]^{\text{vir}} = \pi^!_E([\mathcal{Y}])$. We would like to write down a formula – as explicitly as possible – for a class supported on the fixed locus M^T which pushes forward to $[M]^{\text{vir}}$.

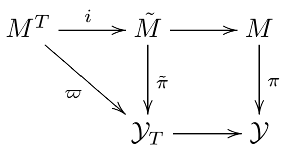
First, we recall the form of the result in the absolute case, where $\mathcal{Y} = \operatorname{Spec} \mathbb{C}$. Here, we find that M^T has its own absolute perfect obstruction theory given by the natural map from $E^f \to L_{M_T/\mathbb{C}}$. Here we use the notational convention that when we restrict a T-equivariant sheaf or complex to the fixed locus, we let E^f denote the subsheaf on which T acts trivial, and we let E^m denote the "complement", the direct sum of the subsheaves associated to all the nonzero characters of T.

Then we see that each connected component of the fixed locus thus inherits its own virtual fundamental class $[M_i^T]^{\mathrm{vir}}$, and the virtual localization theorem from [3] states

$$[M]^{\mathrm{vir}} = \sum_i e(N_i^{\mathrm{vir}})^{-1} \cap [M_i^T]^{\mathrm{vir}}$$

where $N_i^{\mathrm{vir}} = (E^m_{M_i})^\vee$. This equality holds in an appropriate localization of the Chow group of M.

The situation is somewhat more complicated in the relative case. In what follows we assume that T acts trivially on the fixed locus M^T (which can be achieved by taking a finite cover of T). In this case, we see that the morphism $M^T \to \mathcal{Y}$ factors as $M^T \to \mathcal{Y}_T \to \mathcal{Y}$ where $\mathcal{Y}_T$ is the *toric inertia stack* $\mathrm{Hom}(BT, \mathcal{Y})$. The fixed part E^f admits a natural morphism to $L_{M^T/\mathcal{Y}_T}$, and this gives a relative perfect obstruction theory. Unfortunately, both in general, and in the particular case of interest for logarithmic stable maps, the stack $\mathcal{Y}_T$ does not have a fundamental class – it may have components of different dimensions which intersect. Thus, there is not really a virtual fundamental class on the connected components of the fixed locus – at least not in as strong a sense as in the absolute case. We obtain two analogues of the virtual localization theorem. To state them, consider the following diagram where $\tilde{M}$ is defined to make the right hand square Cartesian.



$\tilde{M}$ aquires a relative perfect obstruction theory over $\mathcal{Y}_T$ by pullback whose associated virtual cotangent bundle is the pullback of E, and we obtain the following theorem (suppressing the sum over components to avoid even worse notation)

Theorem 1.

$$\tilde{\pi}^!_E(\alpha) = e(N^{\mathrm{vir}})^{-1} \cap \varpi^!_{E^f}(\alpha)$$

In the general setting, one can choose an appropriate α to which to apply this theorem by considering the T action on the normal cone of the morphism $\mathcal{Y}_T \to \mathcal{Y}$ but for the application to log stable maps I described instead a global approach for the relevant $\mathcal{Y}$.

2. Application to logarithmic stable maps

The space of logarithmic stable maps has a relative perfect obstruction theory over Olsson's stack parametrizing log structures, or the appropriate finite type approximation to that stack called an Artin fan in work of Abramovich, Chen, Marcus, Ulirsch, and Wise [1]. These stacks are etale locally described as the quotient of a toric variety by the action of the torus. The toric inertia stack in this case is given locally as the closed substack corresponding to the fixed locus of the action of a subtorus. It is possible to find an appropriate class α supported on the toric inertia stack so that the class $\tilde{\pi}^!_E(\alpha)$ from the lefthand side of Theorem 1 will pushforward to the virtual fundamental class of M. Moreover, the class α can be computed in principle by essentially toric methods – for example, taking barycentric subdivisions of the cone complex used to construct the Artin fan. In practice however, it seems quite complicated to compute this class except in extremely favorable situations.

Finally, one needs to actually identify the fixed locus of the space of maps (with its virtual structure) for a reasonable class of targets. If we look at the space of log maps to a toric variety X with its standard log structure, and work equivariantly with respect to the natural torus action, then one sees that where the fixed points of the space of ordinary stable maps to the underlying scheme of X are indexed by decorated graphs, the components of the fixed locus of the space of log maps will be indexed by equivalence classes of certain tropical curves living in the fan defining X. The class α can be written (noncanonically) as a sum over the irreducible components of $\mathcal{Y}_T$. The preimages of these irreducible components are then the fundamental geometric objects over which one needs to compute virtual integrals in order to apply torus localization to the space of stable maps. These loci have explicit descriptions as spaces of pointed curves where the collections of points are required to satisfy equations in the Picard group. If there is a single equation (which would happen for a one dimensional target) the resulting space is known as a double ramification cycle, and its virtual class has been the subject of extensive study in recent years. When there are multiple equations, it is tempting to believe that the resulting virtual class is just the product of several of these cycles, but work of Holmes, Pixton, and Schmitt [4] shows that the natural virtual structure on the locus of pointed curves satisfying several equations is different from the product of the cycle classes corresponding to the equations separately. As a result it is still unknown whether the classes arising here lie in the tautological ring.

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There and back again: a tale of expansions in Gromov–Witten theory

Dhruv Ranganathan

In this talk, I described a theory of logarithmic Gromov–Witten invariants with expansions, and a proof of the degeneration formula for Gromov–Witten invariants in simple normal crossings (or toroidal) degenerations. The construction is based on "virtual semistable reduction" – for the universal curve, universal map, and (constant) target family – over moduli space of logarithmic stable maps. The technical ingredients that go into this are weak semistable reduction for toroidal morphisms, as developed by Abramovich and Karu in the late 1990's, and the modern perspective on the relationship between logarithmic modifications, subdivisions of tropical moduli spaces, and the effect they have on the virtual fundamental class [3].

The main results. Let X be a smooth projective algebraic variety and let $D \subset X$ be a simple normal crossings divisor with irreducible components $D_1, \dots, D_k$. Relative Gromov–Witten theory is concerned with maps of pairs

$$(C, p_1, \dots, p_n) \to (X, D),$$

where C is a smooth genus g curve in a homology class β, meeting the component D_i at p_j with contact order $c_{ij} \in \mathbb{N}$. For each i, the contact order c_{ij} is nonzero for at most one index i. These numerical data are packaged in the symbol Γ and the divisor D is understood as part of the notation X. There is a Deligne–Mumford stack $\mathsf{K}^{\circ}_{\Gamma}(X)$, with quasi-projective coarse moduli space, parameterizing such maps. The main geometric construction is the following.

Theorem A (Logarithmic maps with expansions). *There exists a proper moduli stack $\mathsf{K}_{\Gamma}(X)$ containing $\mathsf{K}^{\circ}_{\Gamma}(X)$, and, for a logarithmic point S, whose S-points are flat families*

$$\mathcal{C} \to \mathcal{X} \to X \times S,$$

of maps from nodal curves to logarithmic modifications $\mathcal{X} \to X \times S$. The map $\mathcal{C} \to \mathcal{X}$ is logarithmically transverse – the curve is disjoint from the codimension 2 strata and the generic points of components of $\mathcal{C}$ map to the generic points of $\mathcal{X}$.

This space is built as a logarithmically étale modification of the space of logarithmic stable maps constructed by Abramovich–Chen–Gross–Siebert. It comes equipped with a virtual fundamental class, and the canonical morphism

$$\mathsf{K}_{\Gamma}(X) \to \mathsf{ACGS}_{\Gamma}(X)$$

identifies virtual fundamental classes, so the invariants of the former determine invariants of the latter.

A word about the proof. The moduli space constructed by Abramovich–Chen and Gross–Siebert encode "transversality" in logarithmic and tropical data, and our goal is simply to make this visible geometrically. Over the moduli space $\mathsf{ACGS}_{\Gamma}(X)$, there exists a universal map

$$\mathcal{C} \to X \times \mathsf{ACGS}_{\Gamma}(X).$$

After passing to the associated family of Artin fans, one can use toroidal geometry to make this map *weakly semistable*. That is, we blowup the target in such a way that the curve is disjoint from the codimension 2 strata. The resulting target family $\mathcal{X}$ will no longer be flat over $\mathsf{ACGS}_\Gamma(X)$, but a modification of the latter produces $\mathsf{K}_\Gamma(X)$. These modifications are all done using tropical geometry, following the path laid out by Abramovich and Karu [2]. A further modification to $\mathcal{C}$ produces the desired moduli space. A key insight required here is that logarithmically, modifications are subfunctors/subcategories, lending them a tautological modular interpretation. The arguments concerning obstruction theories developed by Abramovich and Wise quickly give rise to the space we require [3].

Using the spaces. The main advantage of the new spaces is the logarithmic transversality. This allows one to generalize well-known arguments, used by Jun Li and others, to relate the virtual fundamental class of the space of maps to a variety, with spaces of maps to components of the central fiber of a degeneration.

Theorem B (The degeneration formula). *Let $\mathcal{Y} \to \mathbb{A}^1$ be a toroidal degeneration without self intersections, general fiber Y_η, and special fiber Y_0. There exist moduli spaces $\mathsf{K}_\Gamma(\cdot)$ of maps to expansions with the following properties.*

(1) **Virtual deformation invariance.** *There is an equality of virtual classes*

$$[\mathsf{K}_\Gamma(Y_0)] = [\mathsf{K}_\Gamma(Y_\eta)]$$

in the Chow group of the space of maps $\mathsf{K}_\Gamma(\mathcal{Y})$.

(2) **Decomposition.** *The virtual class of maps to Y_0 decomposes as a sum over combinatorial splittings of the discrete data (i.e. tropical stable maps)*

$$[\mathsf{K}_\Gamma(Y_0)] = \sum_\rho m_\rho [\mathsf{K}_\rho(Y_0)]$$

where $m_\rho \in \mathbb{Q}$ are explicit combinatorial multiplicities depending on the splitting ρ. Each space $\mathsf{K}_\rho(Y_0)$ is a space of maps to expansions, marked by splitting type.

(3) **Gluing.** *For each splitting ρ with graph type G, there are moduli spaces $\mathsf{K}_\rho(X_v)$ of maps to expansions of components X_v of Y_0. There is a virtual birational model of their product*

$$\widetilde{\prod_v} \mathsf{K}_\rho(X_v) \to \prod_v \mathsf{K}_\rho(Y_v),$$

and an explicit formula relating the virtual class $[\mathsf{K}_\rho(Y_0)]$ with the virtual class $[\widetilde{\prod}_v \mathsf{K}_\rho(X_v)]$ and the class of the relative diagonal of the universal divisor expansion.

Some things are the same, some things are not. For those who are fond with Jun Li's theory of stable maps to expansions of smooth pairs, much of this theory will be familiar [6, 7]. In particular, the transversality ensures that logarithmic structures play a minimal role, and when the target is a smooth pair, the expanded theory (essentially) returns Jun Li's theory.

The main change from the old theory comes from the fact that certain statements are only after a virtual birational modification. The precise nature of this modification is fairly explicit, but certainly adds a large degree of combinatorial complexity. The need for such modifications in part (3) above is clear – in the product geometry, the degeneration of one of the targets does not force the degeneration of the targets to which it is glued. As a consequence, there is no well-defined evaluation space for boundary markings to produce a morphism to. In Part (3) above, this can be fixed by "tying the fates" of the different components of a degeneration together. This amounts to a tropical subdivision, which produces a virtual birational model.

The result of the birational modifications however, is that the familiar numerical degeneration formula, appearing as a convolution, is not immediately visible in the logarithmic setting. Indeed it appears too much to expect it. The diagonal classes above are strict transforms of the usual diagonal under birational modifications, and consequently do not satisfy a Künneth splitting. A solution to this issue nevertheless seems possible. While the theory is still calculable in principle, it awaits developments in logarithmic intersection theory.

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The Tropological Vertex

Brett Parker

(joint work with Norm Do)

Complex manifolds with normal crossing divisors, and normal crossing degenerations, have a natural large scale with a tropical, or piecewise integral-affine structure. Moreover, relative Gromov-Witten invariants of these spaces correspond to counts of tropical curves in this tropical large scale. I will draw some pictures, and explain some beautiful aspects of this tropical correspondence in the Calabi-Yao 3-fold setting, related to the StromingerÑYaoÑZaslow approach to mirror symmetry: In this case, the large scale is a 3-dimensional integral affine manifold with singularities along a 1-dimensional graph. The vertices of this singular graph come in two types: positive and negative, and the relative GromovÑWitten invariants around the positive vertices contain the topological vertex of Aganagic, Klemm, Marino and Vafa. In this setting the GromovÑWitten invariants in all genus are determined by a tropical gluing formula.

The Logarithmic Picard Group

Sam Molcho
(joint work with Jonathan Wise)

Let $X \to S$ be a family of nodal curves with smooth fibers over some open subscheme $U \subset S$. The Jacobian $\mathrm{Pic}^0(X \times_S U/U)$ is an abelian variety over U, but in general there is no way to construct an abelian variety over all of S whose restriction to U agrees with $\mathrm{Pic}^0(X \times_S U/U)$. One must choose to either sacrifice properness and work with the generalized Jacobian, which is only a semiabelian scheme, or, as in [1] and [2], to compactify the generalized Jacobian in some way and sacrifice the group structure in the process.

Following ideas of Kato and Illusie, we describe a solution to this problem in the setting of logarithmic geometry: given a family of logarithmically smooth curves $\pi : X \to S$, we construct the log Picard group $\mathrm{LogPic}(X/S)$ over S. Let $\mathbb{G}_m^{\log}$ denote the functor on log schemes defined by $\mathbb{G}_m^{\log}(Y) = \Gamma(Y, M_Y^{\mathrm{gp}})$. We then define $\mathbf{LogPic}(X/S)$ to be the stack $(\pi_* B\mathbb{G}_m^{\log})^\dagger$ and $\mathrm{LogPic}(X/S)$ its sheaf of isomorphism classes. The † indicates that we are only taking $\mathbb{G}_m^{\log}$-torsors that satisfy a certain combinatorial condition on the dual graph of X, which we call bounded monodromy. We refer to the bounded monodromy torsors as log line bundles. The degree 0 log line bundles $\mathbf{LogPic}^0(X/S)$ form a proper group stack, which coincides with the usual stack $\mathbf{Pic}^0$ on the locus of S where the log structure is trivial, i.e. over the smooth locus of $X \to S$. For instance, applying this construction to the universal curve $\overline{C}_{g,n} \to \overline{M}_{g,n}$ provides a compactification $\mathrm{LogPic}^0(\overline{C}_{g,n}/\overline{M}_{g,n})$ of the universal Jacobian $\mathrm{Pic}^0(C_{g,n}/M_{g,n})$. On the other hand, $\mathbf{LogPic}$ is not representable by an algebraic stack with a logarithmic structure. It is only a log algebraic stack – the analogue of an algebraic stack over the category of log schemes. It nevertheless has rich structure that allows one to study it; the log Jacobian LogPic^0 is, for instance, a log abelian variety in the sense of [3].

As a first step, $\mathbf{LogPic}(X/S)$, a priori a stack in the étale topology, is, in fact, invariant under logarithmic blowups and under extracting roots of the log structure. It is thus a stack in the full log étale topology. A consequence of this observation is that $\mathbf{LogPic}(X/S)$ has a cover by the stacks $\mathbf{Pic}(Y)$ as Y ranges over all log blowups of X, and that log line bundles on X can thus be understood as actual line bundles on semistable models of X, up to some elaborate equivalence relation.

Additional structure of the log Picard group is revealed by studying its tropicalization. To a log curve $X \to S$, one associates its tropicalization $\mathfrak{X}$. This is a collection of dual graphs, which have edge lengths valued in the characteristic monoid of M_S, and are compatible with generization. The tropical curve $\mathfrak{X}$ carries a sheaf

$\mathcal{L}$ of linear functions, which allows us to define the tropical Picard group $\mathrm{TroPic}(\mathfrak{X})$ as the sheaf of isomorphism classes of the stack $\mathbf{TroPic}(\mathfrak{X})$: this is the stack of tropical line bundles on $\mathfrak{X}$, that is, the $\mathcal{L}$-torsors on $\mathfrak{X}$ which, again, have bounded monodromy. The stack $\mathbf{TroPic}(\mathfrak{X})$ is a combinatorial object that can be explicitly calculated. More importantly for our purposes, it coincides with the tropicalization of $\mathbf{LogPic}(X/S)$. Furthermore, $\mathbf{LogPic}(X/S)$ (resp. $\mathrm{LogPic}(X/S)$) contains the stack of multidegree 0 line bundles $\mathbf{Pic}^{[0]}(X)$ (resp. the generalized Jacobian) as a subgroup, and there are exact sequences of group stacks and sheaves respectively:

$$0 \to \mathbf{Pic}^{[0]}(X/S) \to \mathbf{LogPic}(X/S) \to \mathbf{TroPic}(\mathfrak{X}) \to 0$$
$$0 \to \mathrm{Pic}^{[0]}(X/S) \to \mathrm{LogPic}(X/S) \to \mathrm{TroPic}(\mathfrak{X}) \to 0$$

In particular, the failure of representability of $\mathrm{LogPic}(X/S)$ by a scheme with a logarithmic structure is entirely due to the failure of representability of $\mathrm{TroPic}(\mathfrak{X})$ by a polyhedral complex. The tropical Picard group, though not a polyhedral complex itself, has subdivisions which are polyhedral complexes. By pulling back subdivisions of $\mathrm{TroPic}(\mathfrak{X})$ under the tropicalization map, one obtains log blowups of $\mathrm{LogPic}(X)$, which in fact are representable by schemes. Restricting to the log Jacobian, one obtains by this procedure toroidal compactifications of the generalized Jacobian.

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Universal coefficients for logarithmic curves

Jonathan Wise

(joint work with Samouil Molcho, Martin Ulirsch)

Suppose that X is a smooth, proper algebraic curve over a base S. Then there is a canonical symmetric biextension of its Jacobian, which we notate as a bilinear pairing:

$$\mathrm{Pic}^0(X/S) \times \mathrm{Pic}^0(X/S) \to \mathrm{B}\mathbf{G}_{m,S} \tag{1}$$

If X is a flat, proper curve over S with nodal fibers then the Jacobian will fail to extend to an abelian variety if the dual graph of a fiber of X is not a tree. Furthermore, the Jacobian contains an algebraic torus in this case, and a theorem of Grothendieck says that there is no nontrivial biextension between an algebraic torus and *any* smooth, connected algebraic group [3]. It follows that there can be no extension of (1) to the case of nodal curves.

Theorem 1 (Molcho–W)**.** *Let X be a proper, vertical family of logarithmic curves over S. Then the logarithmic Picard group* $\mathbf{LogPic}(X/S)$ *is logarithmically smooth and proper over S and has a logarithmically smooth cover by a logarithmic scheme. The diagonal of the sheaf of isomorphism classes* $\mathrm{LogPic}(X/S)$ *is representable by algebraic spaces with logarithmic structures.*

Theorem 2 (Molcho-Ulirsch-W)**.** *The logarithmic Picard group stack has a canonical symmetric principal polarization:*

$$\mathbf{LogPic}(X/S) \times \mathbf{LogPic}(X/S) \to \mathrm{B}\mathbf{G}_{\log,S}$$

The proof follows Deligne's construction [1, 2] of the principal polarization of the group stack $\mathbf{Pic}(X/S)$ when X is smooth and proper over S. As in that case, the theorem follows from a universal coefficients theorem. In order to state that theorem, we first describe the coefficient groups to which it applies.

We consider the following properties of a category $\mathcal{G}$ fibered in strictly commutative 2-groups over logarithmic schemes:

(1) $\mathcal{G}$ is a stack in the strict fppf topology and is invariant under logarithmic modification.
(2) If $Z \to S$ is a strict, finite, flat morphism of degree d then the morphism $\mathcal{G}(\mathrm{Sym}^d_S(Z)) \to G([\mathrm{Sym}^d_S(Z)])$ is an isomorphism, where $[\mathrm{Sym}^d_S(Z)]$ is the d-th stack symmetric pwer of Z over S and $\mathrm{Sym}^d_S(Z)$ is its coarse moduli space. This implies that there is a trace homomorphism $\mathrm{trace}_{Z/S} : \mathcal{G}(Z) \to \mathcal{G}(S)$.
(3) If $P \to S$ is a logarithmic degeneration of projective spaces with its vertical logarithmic structure then any section $S \to P$ splits the pullback map $\mathcal{G}(S) \simeq \mathcal{G}(P)$. This splitting must be independent of the choice of section.

Theorem 3 (Molcho–Ulirsch–W)**.** *Let $\mathcal{G}$ be as above and let X be a proper, vertical logarithmic curve over S. Then there is a perfect pairing:*

$$\mathbf{LogPic}(X/S) \times \mathcal{G}(X) \to \mathcal{G}(S)$$

Applying this with $\mathcal{G}$ being $\mathrm{B}\mathbf{G}_{\log}$ (and restricting to the bounded monodromy subgroup), we obtain Theorem 2. The theorem also relates cohomology of X with coefficients in certain groups to extensions of the logarithmic Picard group:

Corollary 4. *The following equality holds for any proper, vertical logarithmic curve X over S and any $r \geq 0$:*

$$H^1(X, \mu_r) \simeq \mathrm{Ext}^1(\mathbf{LogPic}(X/S), \mu_r)$$

Sketch of the proof of Theorem 3. To describe the pairing in Theorem 3, let L be a logarithmic line bundle on X and let $K \in \mathcal{G}(X)$. After fppf localization and logarithmic modification, we can assume that L is the logarithmic line bundle associated to a divisor Z in the strict locus of X over S. We then define $\langle L, K\rangle = \mathrm{trace}_{Z/S}(K)$. We then prove the independence of the choice of Z by finding a homotopy, parameterized by a logarithmic degeneration of projective spaces, between any such choices. □

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Towards a logarithmic compactification of strata of abelian differentials

Martin Ulirsch
(joint work with Martin Möller and Annette Werner)

Let $g \geq 2$. The Hodge bundle $\Omega\mathcal{M}_g$ is a vector bundle over the moduli space $\mathcal{M}_g$ whose fiber over a point $[X]$ in $\mathcal{M}_g$ is the space $H^0(X, \omega_X)$ of abelian differential on X. Its projectivization $\mathbb{P}\Omega\mathcal{M}_g$ parameterizes pairs (X, K_X) consisting of a smooth projective algebraic curve X and an effective canonical divisor K_X on X. It is naturally stratified according to the multiplicity profile $\mu = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ of K_X (with $m_1 + \ldots + m_n = 2g - 2$). Marking the points in the support of K_X we are lead to study a *stratum* $\mathbb{P}\Omega\mathcal{M}_g(\mu)$ of the Hodge bundle, which parameterizes triples $(X, K_X, p_1, \ldots, p_n)$ consisting of a smooth curve X an effective canonical divisor K_X on X and marked points $p_1, \ldots, p_n$ such that $K_X = m_1 p_1 + \ldots + m_n p_n$.

It is an open problem to find a "good" compactification of $\mathbb{P}\Omega\mathcal{M}_g(\mu)$, i.e. one that is proper and smooth, and which admits a natural modular interpretation. In [BCGGM18] the authors give a description of the points of the closure of $\mathbb{P}\Omega\mathcal{M}_g(\mu)$ in $\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}$, the extension of the projectivized Hodge bundle to $\overline{\mathcal{M}}_{g,n}$ by considering sections of the relative dualizing sheaf. This compactification is unfortunately highly singular and it also does not admit a modular interpretation. In a follow-up project, the authors of [BCGGM18] construct a smooth orbilfold compactification using methods from complex analytic geometry.

We are developing a different approach towards the construction of a compactification of a stratum using logarithmic geometry. In particular, we use new methods that connect it with tropical geometry, as e.g. [RSPW17a, RSPW17b], which in turn are based on the theory of Artin fans, as e.g. in [CCUW17, Part 2]. Our approach proceeds as follows:

- Construct a tropical analogue of a stratum $\mathbb{P}\Omega\mathcal{M}_g^{trop}(\mu)$ in the tropical Hodge bundle, as introduced in [LU17], but endowed with its "stack-theoretically" correct structure as a *cone stack*, a notion that has been introduced in [CCUW17].

- Using our realizability result in [MUW17] we can determine the realizability locus $\mathbb{P}\Omega\mathcal{M}_g^{trop,real}(\mu)$ in $\mathbb{P}\Omega\mathcal{M}_g^{trop}(\mu)$, i.e. the image of the natural tropicalization map from $\mathbb{P}\Omega\mathcal{M}_g^{an}(\mu)$ to $\mathbb{P}\Omega\mathcal{M}_g^{trop}(\mu)$.
- We further show that the realizability locus canonically carries the structure of a cone stack so that the natural map to $\mathbb{P}\Omega\mathcal{M}_g^{trop}(\mu)$ is a (not necessarily proper) subdivision. This map induces a toroidal modification of $\overline{\mathcal{M}}_{g,n}$.

In the spirit of Tevelev's theory of tropical compactifications [Tev07], we then define our logarithmic compactification $\mathbb{P}\Omega\mathcal{M}_g^{log}(\mu)$ of a stratum as the closure in the modified $\overline{\mathcal{M}}_{g,n}$. A logarithmic version of Tevelev's Lemma, as in [Uli15], immediately shows that $\mathbb{P}\Omega\mathcal{M}_g^{log}(\mu)$ is proper. A modular interpretation follows using the theory of Artin fans, which allows us to combine tropical and algebraic data within the same functor. We are currently working on a proof that $\mathbb{P}\Omega\mathcal{M}_g^{log}(\mu)$ is represented by a smooth Deligne-Mumford stack, whose logarithmic structure is associated to a normal crossing boundary.

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Toric degenerations and tropical enumeration of real curves with Welschinger signs

Hülya Argüz
(joint work with Pierrick Bousseau)

The work of Mikhalkin [5] in dimension two, expressing counts of rational curves in the complex projective space as tropical curve counts in the real plane is a groundbreaking result in tropical enumerative geometry. This correspondence theorem is generalized also to higher dimensional toric varieties using log geometric and degeneration techniques by Nishinou-Siebert [4].

A real version of the tropical correspondence theorem of Mikhalkin has also been carried out for the projective plane in [5], and for toric del Pezzo surfaces in [3, 6]. We prove a real analogue of this correspondence theorem in higher dimensions for toric varieties, using the approach of Nishinou-Siebert. This method provides a way to enumerate real counts tropically, not only in higher dimensions, but potentially also in non-toric situations.

We define real structures on log schemes as follows [1, §1].

Definition. Given a log scheme $(X, \mathcal{M}_X)$ over $\mathbb{C}$ with a real structure $\iota_X : X \to X$ on the underlying scheme (that is, an anti-holomorphic involution $\iota : X \to X$), a *real structure* on $(X, \mathcal{M}_X)$ is an involution

$$\tilde{\iota}_X = (\iota_X, \iota_X^\flat) : (X, \mathcal{M}_X) \longrightarrow (X, \mathcal{M}_X)$$

Let f be a marked log smooth curve and let $(X_0, \mathcal{M}_{X_0})$ be the central fiber of a toric degeneration, endowed with the log structure induced by the embedding of X_0 into the total space. A morphism of log schemes

$$f : (C, \mathcal{M}_C) \to (X_0, \mathcal{M}_{X_0})$$

is called a *real log map* if f is a real morphism of real log schemes as defined in [1, Defn 1.2].

We describe a recipe to enumerate real log maps in the central fiber of a toric degeneration with appropriate Welschinger signs [We, We2]. For this, as a first step, we define the real analogue of the lattice index in [4, Prop 5.7], by choosing real point constraints. For a fixed graph Γ, together with an associated weight function w on its set of edges, the real lattice index equals to the count of maximally degenerate real curves with dual graph Γ. The next step is then to enumerate real log curves with given underlying maximally degenerate real curve. For this, we define *real weights* on edges of Γ as follows.

$$w_{\mathbb{R}}(e) = \begin{cases} 2, & \text{if } w \text{ is even} \\ 1, & \text{if } w \text{ is odd} \end{cases}$$

The final step is to determine signs associated to real log curve from the tropical data. In dimension two, this amounts to a local analysis to deduct the number of elliptic nodes of the real curve in the smoothing of the central fiber. The details of this will appear in work with Pierrick Bousseau. This analysis is recorded in the data of the log structure. The signs we obtain are then described combinatorially, in terms of counts of integral points in the interior of certain subdivisons of a given polygon, similar to [6, Eqn (2.12)]. We apply the analogous combinatorial algorithm to determine signs in dimension three. Our main theorem is the following.

Theorem. The count of tropical curves with real weights and signs, coincides with the count of real curves with signs in a "real" fiber of the Nishinou-Siebert degeneration close enough of the special fiber.

We enumerate real log curves with signs in some particular examples, such as the projective space in dimension three, which gives compatible results with [2], where this case is analysed using the techniques of floor diagrams. The tropical invariance of these counts can be shown using similar techniques as in [GMS]. However, so far there is no known method to show invariance in the complex-geometric setup. It is still mysterious what these counts with associated signs will correspond to in higher dimensions. In a recent work of Solomon and Tukachinsky [ST], real invariants in higher dimensions, which are proposed to generalize Welschinger invarinats, are introcuced using A_∞ relations. Our computations in projective spaces, agree with the results obtained in [ST].

A particular objective we have, is to use the tropical enumeration of Welschinger invariants in the construction of scattering diagrams, which are the main ingredient to construct mirror families, in the sense of [GPS].

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Remarks on group actions and sections of vector bundles.

Martin Olsson

Inspired by Mumford's theory of the theta group and algebraic theta functions [1], as well as recent work of Gross, Hacking, Keel, Kontsevich, and Siebert on Mirror Symmetry, we study the problem of constructing canonical sections of vector bundles on moduli spaces along the formal completion of the space at a highly degenerate point.

Let k be a field and let $\mathcal{S}/k$ be an algebraic stack. Let $x \in \mathcal{S}(k)$ be a k-point with stabilizer group scheme $\mathcal{G}$ a linearly reductive group scheme, so we have a closed immersion

$$B\mathcal{G} \hookrightarrow \mathcal{S}.$$

Let $\mathcal{I} \subset \mathcal{O}_\mathcal{S}$ be the ideal sheaf of this closed immersion. Denote by $\mathcal{S}_n \subset \mathcal{S}$ the closed substack defined by $\mathcal{I}^{n+1}$ so we have closed immersions defined by nilpotent ideals

$$i_n : B\mathcal{G} \hookrightarrow \mathcal{S}_n.$$

Definition 1. An *n-th order infinitesimal presentation of* $\mathcal{S}$ is a lifting $P_n \to \mathcal{S}_n$ of the tautological $\mathcal{G}$-torsor $\mathrm{Spec}(k) \to B\mathcal{G}$ to $\mathcal{S}_n$.

The n-th order infinitesimal presentations of $\mathcal{S}$ form a category which we denote by $\mathrm{Pres}^n(B\mathcal{G} \hookrightarrow \mathcal{S})$. Note that for such a torsor $P_n \to \mathcal{S}_n$, the underlying space P_n is necessarily a scheme and we have

$$\mathcal{S}_n \simeq [P_n/\mathcal{G}],$$

which justifies the terminology "presentation".

For $n < m$ there is a natural reduction functor

$$\mathrm{Pres}^m(B\mathcal{G} \hookrightarrow \mathcal{S}) \to \mathrm{Pres}^n(B\mathcal{G} \hookrightarrow \mathcal{S}),$$

and we define

$$\mathrm{Pres}^\infty(B\mathcal{G} \hookrightarrow \mathcal{S}) := \varprojlim_n \mathrm{Pres}^n(B\mathcal{G} \hookrightarrow \mathcal{S}).$$

We refer to objects of $\mathrm{Pres}^\infty(B\mathcal{G} \hookrightarrow \mathcal{S})$ as *formal presentations of* $\mathcal{S}$.

Let I^n/I^{n+1} denote the representation of $\mathcal{G}$ corresponding to the sheaf on $B\mathcal{G}$ given by the restriction of $\mathcal{I}^n$. Also view the Lie algebra $\mathrm{Lie}(\mathcal{G})$ is a $\mathcal{G}$-representation with the adjoint action.

Theorem 2. *(i) The category* $\mathrm{Pres}^\infty(B\mathcal{G} \hookrightarrow \mathcal{S})$ *is nonempty.*

(ii) Suppose that

$$H^0(G, \mathrm{Lie}(G) \otimes I^n/I^{n+1}) = 0$$

for all $n \geq 1$. *Then the category* $\mathrm{Pres}^\infty(B\mathcal{G} \hookrightarrow \mathcal{S})$ *is equivalent to the one point set. In other words, there exists a formal presentation of* $\mathcal{S}$, *unique up to unique isomorphism.*

Fix now a vector bundle $\mathcal{E}$ on $\mathcal{S}$ with pullback $\mathcal{E}_0$ to $B\mathcal{G}$ as well as a formal presentation of $\mathcal{S}$, which we can view as a compatible (in a suitable 2-categorical sense) collection of morphisms

$$r_n : \mathcal{S}_n \to B\mathcal{G}.$$

Let E_0 denote the representation of $\mathcal{G}$ corresponding to $\mathcal{E}_0$.

For each n let $\mathcal{F}_n$ denote the lifting of $\mathcal{E}_0$ given by $r_n^*\mathcal{E}_0$. We then get two systems

$$\{\mathcal{E}_n\}_{n\geq 0}, \quad \{\mathcal{F}_n\}_{n\geq 0} \tag{1}$$

of vector bundles on the $\mathcal{S}_n$, together with an isomorphism $\mathcal{E}_0 \simeq \mathcal{F}_0$.

Theorem 3. *(i) The two systems* (1) *are isomorphic by an isomorphism which agrees with the given isomorphism for $n = 0$.*

(ii) If

$$H^0(\mathcal{G}, \mathrm{End}(E_0) \otimes I^n/I^{n+1}) = 0$$

for all $n \geq 1$ then there exists a unique isomorphism of systems

$$\{\sigma_n : \mathcal{E}_n \to \mathcal{F}_n\}$$

inducing the given isomorphism for $n = 0$.

As an application we study the following example. Assume that 6 is invertible in k and let $\mathcal{S}$ be the stack whose fiber over a k-scheme S is the groupoid of pairs $(Y, \mathcal{L})$ as follows:

(i) $f : Y \to S$ is a proper flat morphism of schemes.

(ii) $\mathcal{L}$ is a relatively ample invertible sheaf on Y such that the sheaf $f_*\mathcal{L}$ is locally free of rank 3 and its formation commutes with arbitrary base change $S' \to S$.

(iii) The sheaf $f_*\mathcal{L}$ is base point free and the induced map

$$Y \to \mathbf{P}(f_*\mathcal{L})$$

is a closed immersion which étale locally on S identifies Y with a surface of degree 4 in $\mathbf{P}^3$.

Let $x \in \mathcal{S}(k)$ be the point given by

$$V(X_0X_1X_2X_3) \subset \mathbf{P}^3.$$

Then the automorphism group scheme of x is the semi-direct product $\mathbf{G}_m^4 \rtimes S_4$ acting in the natural way on the coordinates.

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Smoothing Normal Crossing Spaces

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(joint work with Matej Filip, Helge Ruddat)

What does this mean? A complex space X is a *normal crossing space* if it is locally isomorphic to $\{x_1 \cdot ... \cdot x_n = 0\} \subset \mathbb{C}^n$, the union of the coordinate hyperplanes. It is *simple*, if all components are smooth. Given a compact normal crossing space, a *smoothing* is a cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \big\downarrow & & \big\downarrow{\scriptstyle F} \\ \{0\} & \longrightarrow & \Delta \end{array}$$

where $F : \mathcal{X} \to \Delta$ is a proper flat map to a small disk such that the generic fiber $\mathcal{X}_t$ is nonsingular. If X is a nc *variety* (where local means étale local), then by a smoothing we mean a family over Spec $\mathbb{C}[[t]]$.

Absolute Spectral Sequences. If X is a compact Kähler manifold, then by classical Hodge theory we obtain a decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^q(X, \Omega_X^p)$$

of the cohomology. From an algebro-geometric perspective, a more conceptual viewpoint is the degeneracy at E_1 of the spectral sequence

$$E_1^{p,q} = H^q(X, \Omega_X^p) \Rightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet)$$

for a proper smooth scheme X/k where k is a field of characteristic 0. Classically proven with analytic methods, algebraic proofs go back to Faltings 1985 and Deligne-Illusie 1987. If $D \subset X$ is a normal crossing divisor, then we can form the differentials $\Omega_X^\bullet(\log D)$ with logarithmic poles in D. They fit into a spectral sequence

$$E_1^{p,q} = H^q(X, \Omega_X^p(\log D)) \Rightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log D)) \cong H^{p+q}(X \setminus D, \mathbb{C})$$

which computes (ordinary) cohomology on $X \setminus D$. This spectral sequence degenerates at E_1.

Definition 1. *A* toroidal pair (X, D) *consists of a variety X and a divisor $D \subset X$ such that the pair is étale locally isomorphic to a toric variety with a toric divisor. Let $j : U \subset X$ be the locus where X is smooth and D is normal crossing. Its complement has codimension ≥ 2, and we define differential forms $\Omega_X^\bullet(\log D) := j_*\Omega_U^\bullet(\log D|_U)$.*

Theorem 1 (F-Ruddat)**.** *(Danilov's conjecture) Let (X, D) be a proper toroidal pair. Then the spectral sequence*

$$E_1^{p,q} = H^q(X, \Omega_X^p(\log D)) \Rightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log D))$$

degenerates at E_1.

Proof. Adapt the methods of Deligne-Illusie in [7]. □

Remark 1. *This has been previously known in some special cases.*

- *If D is locally the entire toric boundary, the pair is log smooth, so degeneration follows from results of Illusie-Kato-Nakayama* [1].
- *For $D = \emptyset$ it has been solved by Danilov* [2].
- *In* [3], *Steenbrink proves related results. E.g. if X is a projective orbifold, the spectral sequence degenerates.*

Relative Spectral Sequences. For $\mathbb{N} \subset P$ a saturated injection of sharp toric monoids, we obtain an induced map of log schemes with divisorial log structure

$$F_P : (\text{Spec } \mathbb{C}[P], F^{-1}(0)) \to (\mathbb{A}^1, 0)$$

which is log smooth on the nonsingular locus $U \subset \text{Spec } \mathbb{C}[P]$.

Definition 2. *Let $S \to (\mathbb{A}^1, 0)$ be a strict morphism of log schemes. A* log toroidal family *$f : X \to S$ is a log morphism which is étale locally isomorphic to* Spec $\mathbb{C}[P] \times_{\mathbb{A}^1} S \to S$ *for various P.*

We define the de Rham complex $\Omega^\bullet_{X/S} := j_*\Omega^\bullet_{U/S}$ for some locus $U \subset X$ such that $f|_U : U \to S$ is log smooth.

Theorem 2 (F-Ruddat)**.** *Let $S =$ Spec $\mathbb{C}[t]/(t^{k+1})$, and let $f : X \to S$ be a log toroidal family. Then the spectral sequence associated to the Hodge filtration on $\Omega^\bullet_{X/S}$ degenerates at E_1.*

Smoothing Normal Crossing Spaces. If $\pi : \mathcal{Y} \to \Delta$ is an algebraic degeneration, then by the semistable reduction theorem, there is a map $\Delta \to \Delta, z \mapsto z^k$ and a diagram

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{\phi} & \tilde{\mathcal{Y}} & \longrightarrow & \mathcal{Y} \\ & \searrow^{f} & \downarrow & & \downarrow \pi \\ & & \Delta & \xrightarrow{z\mapsto z^k} & \Delta \end{array}$$

with cartesian square such that $\mathcal{X}$ is nonsingular, $\phi : \mathcal{X} \to \tilde{\mathcal{Y}}$ is a blow-up isomorphic over Δ^* and $f^{-1}(0)$ is a simple normal crossing space, in particular reduced. The morphism $f : \mathcal{X} \to \Delta$ is called a semistable degeneration. Conversely, when is a simple normal crossing space X the central fiber of a semistable degeneration?

Theorem 3 (Friedman Annals '83)**.** [4] *If a snc space X is the central fiber of a flat and proper semistable degeneration, then $\mathcal{T}^1_X := \mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X) \cong \mathcal{O}_D$ where $D = X_{\text{sing}}$. Conversely, if X is a snc surface with Kähler components such that $\mathcal{T}^1_X \cong \mathcal{O}_D$ and $K_X = 0$ and $H^1(X, \mathcal{O}_X) = 0$, then X is the central fiber of some semistable degeneration.*

Normal crossing spaces with $\mathcal{T}^1_X \cong \mathcal{O}_{X_{\text{sing}}}$ are called *d-semistable*. The smoothing has been generalized by Kawamata-Namikawa.

Theorem 4 (Kawamata-Namikawa Inventiones '94)**.** [5] *Let X be a d-semistable snc space with Kähler components of dimension d with $K_X = 0$, $H^{d-1}(X, \mathcal{O}_X) = 0$ and $H^{d-2}(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ where $\tilde{X} \to X$ is the normalization. Then X is the central fiber of a semistable degeneration.*

Proof. Use mixed Hodge structures. □

The condition $\mathcal{T}^1 \cong \mathcal{O}_D$ is necessary, if we want X to be the central fiber of a semistable degeneration. If we relax the requirement that the total space $\mathcal{X}$ should be nonsingular, then we obtain more general smoothings as defined above.

Theorem 5 (F-Filip-Ruddat)**.** *Let X be a projective (algebraic) snc space with $H^2(X, \mathcal{O}_X) = 0$ and $K_X = 0$. Assume that $\mathcal{T}^1_X$ is generated by global sections. Then there is a smoothing.*

Proof. Taking a sufficiently nice global section $s \in \mathcal{T}^1_X$, it induces a structure of log toroidal family on X which is log smooth outside $Z := \{s = 0\} \subset D$. Putting log toroidal deformation theory and the degeneration of the spectral sequence into the machinery of [6], we obtain the result. The smoothing that we produce is a semistable degeneration outside Z. □

Example 1. *We take $X = \{xy = 0\} \subset \mathbb{C}^3$ two planes intersecting in a line $D \cong \mathbb{C}$ and a section $s \in \mathcal{T}^1_X$ such that $Z = \{0\} \in D$. Then a (local) smoothing is given by $\mathcal{X} = \{xy - ts = 0\} \to \Delta_t$. The singular locus of the total space is $\mathcal{X}_{\mathrm{sing}} = Z$, so it is not semistable though the central fiber is a snc space.*

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The Topology of Log Degenerations

Arthur Ogus

(joint work with Piotr Achinger)

This talk was a summary of the paper [1]. Let $f\colon X \to Y$ be a smooth, proper, vertical, and saturated morphism of fine saturated log analytic spaces, and let y be a point of Y. Our goal is to extract information about the germ of the underlying topology of f at y from the log fiber $f_y\colon X_y \to y$. An important tool at our disposal is the "Betti realization" of a log analytic space constructed by Kato and Nakayama [3], which provides a commutative diagram:

$$\begin{array}{ccc} X_{log} & \xrightarrow{f_{log}} & Y_{log} \\ {\scriptstyle \tau_X}\downarrow & & \downarrow{\scriptstyle \tau_Y} \\ X_{top} & \xrightarrow[f_{top}]{} & Y_{top}. \end{array}$$

The "relative rounding theorem" [4] implies that the map f_{log} is a locally trivial fibration of topological manifolds.

Our first result asserts that, if Y is the standard log disc, then f_{top} can be reconstructed explicitly from f_y. In this case there is a commutative diagram:

$$\begin{array}{ccc} Cyl(\tau_{X_y}) & \xrightarrow{\sim} & X_{top} \\ \downarrow & & \downarrow{\scriptstyle f_{top}} \\ Cyl(\tau_y) & \xrightarrow{\sim} & Y_{top}. \end{array}$$

Here the mapping cylinder $Cyl(\tau)$ of a map $\tau\colon X' \to X$ is defined as the pushout in the diagram:

$$\begin{array}{ccc} X' \times 0 & \longrightarrow & X' \times [0, \infty) \\ {\scriptstyle \tau}\downarrow & & \downarrow \\ X \times 0 & \longrightarrow & Cyl(\tau). \end{array}$$

The paper [1] contains a conjecture describing a possible generalization of this result to the case of a more general log disc.

For an illustrative example, let n be an integer, let Q_n be the monoid given by generators q_1, q_2, q and relation $q_1 + q_2 = nq$, and let $f\colon X \to Y$ be the morphism of log schemes corresponding to the monoid homomorphism $\mathbf{N} \to Q_n$ sending 1 to q. Then f_{top} is the map

$$X_{top} \to Y_{top} : \{(z_1, z_2, z) \in \mathbf{C}^3 : z_1 z_2 = z^n\} \mapsto z \in \mathbf{C}.$$

There is a natural homeomorphism from $\{(r_1, r_2) \in \mathbf{R}^2_{\geq} : r_1 r_2 = 0\}$ to $\mathbf{R}$ sending (r_1, r_2) to s=r_1 if $r_2 = 0$ and to $s = -r_2$ if r_1 is zero. With this identification, we can write

$$X_y^{log} = \{s, \zeta_1, \zeta_2, \zeta_3) \in \mathbf{R} \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 : \zeta_1 \zeta_2 = \zeta^n).$$

Then $f_{y,log}(s, \zeta_1, \zeta_2, \zeta) = (\sqrt[n]{|s|}, \zeta)$, and $\tau_{X,y}(s, \zeta_1, \zeta_2, \zeta)$ is $(\zeta_1 s, 0, 0)$ if $s \geq 0$ and is $(0, -\zeta_2 s, 0)$ if $s \leq 0$. One checks easily that the map $X_y \times \mathbf{R}_{\geq} \to X_{top}$:

$$(s, \zeta_1, \zeta_2, \zeta_3, r) \mapsto \left(2^{-1}\zeta_1(s + \sqrt{s^2 + 4r^n}), 2^{-1}\zeta_2(-s + \sqrt{s^2 + 4r^n}), \zeta r\right)$$

factors through a homeomorphism $Cyl(\tau_y) \to X_{top}$, compatible with the maps to Y_{top}.

With this motivation, we focus on morphisms $f\colon X \to S$, where S is a log point. In this context, the formalism of idealized log spaces becomes useful. Recall [5] that an *idealized log space* is a log space (X, α_X) together with a sheaf of ideals $K_X \subseteq M_X$ such that $\alpha_X(K_X) = \{0\}$. If K is an ideal in an fs monoid Q, then the idealized log scheme $A_{Q,K}$ defined by $\mathbf{C}[Q]/\mathbf{C}[K]$ with the log structure defined by Q and the idealized structure by K is a typical example. Morphisms in the category of idealized log spaces are defined in the obvious way, and smoothness is defined using Grothendieck's infinitesimal lifting property for strict (and ideally strict) thickenings. It turns out that an idealized log scheme over $\mathbf{C}$ is smooth if and only if it is locally of the form $A_{Q,K}$ described above. We show that an fs idealized log space on which K_X is reduced, can be cut nicely into pieces. Namely, let $X'' \to X$ be the normalization of X, with the induced idealized log structure, and let $U'' := \{x \in X'' : K_{X'',x} = M^+_{X'',x'}\}$. Then U'' is open and dense in X'' and its underlying space $\underline{U}''$ is smooth. Let X' be the space $\underline{X}''$ endowed with the compactifying log structure coming from U''. Then X' is fine and saturated, and there is a unique morphism of log spaces $h\colon X'' \to X'$ such that $\underline{h}$ is the identity map.

Now if $f\colon X \to S$ is smooth and S is the log point associated to an fs and sharp monoid P, then X and S become smooth if endowed with the idealized structure defined by P^+. Applying the result from the previous paragraph, we find a morphism $X'' \to X'$, where X' is now an fs and smooth log scheme endowed with a compactifying log structure. This morphism induces a map $X'' \to X' \times S$, which, although it is not an isomorphism in general, induces an isomorphism on Betti realizations. Thus we find a surjective and finite morphism $X'_{log} \times S_{log} \to X_{log}$, compatible with the maps to S_{log}. This construction gives a canonical way of breaking X_{log} into pieces, each one of which is given a canonical trivialization over S_{log}. The gluing data which recovers X_{log} from this map can often be computed; for curves it involves Dehn twists.

We apply these results to obtain a combinatorial description of the E_2-page of the nearby cycle spectral sequence of a log degeneration. Since S is a saturated log point, its Betti realization S_{log} is a torus, and its universal cover $\tilde{S}$ is a Euclidean space. Let $\tilde{X} := X_{log} \times_S \tilde{S}$ and let $\tilde{\tau}\colon \tilde{X} \to X_{top}$ be the obvious map. If $X \to S$ is the log fiber of a log degeneration g at a point y, the Leray spectral sequence of $\tilde{\tau}$ corresponds to the "nearby cycle" spectral sequence of g near y [2]. It abuts to the cohomology of $\tilde{X}$, which is isomorphic to the cohomology of the general fiber of g. Following ideas of [3], we find an isomorphism

$$R^q\tilde{\tau}_*(\mathbf{Z}) \cong \Lambda^q M_{X/S}(-q),$$

so that this spectral sequence can be rewritten:

$$E_2^{p,q} = H^p(X_{top}, \Lambda^q_{X/S}(-q)).$$

The fundamental group of S_{log} is $I_P := Hom(P, \mathbf{Z}(1))$, and its action on the the cohomology of $\tilde{X}$ and on the spectral sequence is unipotent. In fact, the action ρ_γ of an element γ on each $E_2^{p,q}$ is trivial, but $\rho_\gamma - \mathrm{id}$ induces maps $\kappa_\gamma \colon E_2^{p,q} \to E_2^{p+1,q-1}$. We show that these maps are given by cup product with the pushout along γ of the extension class defined by the exact sequence:

$$0 \to P \to \overline{M} \to M_{X/S} \to 0.$$

This formula reduces to the Picard-Lefschetz formula in the case of curves. We also show that that the $d_2^{p,q}$ differential is given, up to a factor of $q!$, by cup product with the extension class of the sequence

$$0 \to \mathbf{Z}(1) \to \mathcal{O}_X \to M_{X/P} \to M_{X/S} \to 0.$$

The monodromy formula also holds in arithmetic settings when reformulated using logarithmic étale cohomology. Its proof involves some rather elaborate homological algebra. However, if one is willing to work analytically and with complex coefficients, the formula can be deduced by studying the action of the monodromy on the complex $\tilde{\tau}_* \Omega^{\cdot log}_{\tilde{X}/\mathbf{C}}$, an analog of the Steenbrink complex.

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