Mathematisches Forschungsinstitut Oberwolfach

Report No. 33/2019

DOI: 10.4171/OWR/2019/33

Mathematical Foundations of Isogeometric Analysis

Organized by
Annalisa Buffa, Lausanne
Tom Hughes, Austin
Angela Kunoth, Köln
Carla Manni, Roma

14 July – 20 July 2019

Abstract. Isogeometric analysis is a recent technology for numerical simulation, unifying computer aided design and finite element analysis. It offers a true design-through-analysis pipeline by employing the same representation models for both creating geometries and approximating the solution of partial differential equations defined on those geometries. This combined concept leads to improved convergence and smoothness properties of the solutions and dramatically faster overall simulations.

Even though substantial progress has been made in the isogeometric context over the last few years, there are several profound theoretical issues that are not yet well understood and that are currently investigated by researchers in numerical analysis, approximation theory, and applied geometry.

The workshop reported the substantial progress, both from the theoretical and applicative point of view, which has been made in the isogeometric context over the last three years. It offered a meeting point for leading scientists from isogeometric analysis and the mentioned mathematically relevant fields, and provided a rich and open ground of discussion within a diversified audience, profiting of different backgrounds and various perspectives.

Mathematics Subject Classification (2010): 65xx, 41xx, 51xx.

Introduction by the Organizers

Isogeometric analysis (IgA) is a recent paradigm for numerical simulation governed by partial differential equations (PDEs). Traditional approaches are based on modeling complex geometries by computer aided design (CAD) tools which are then converted into a computational mesh needed for the numerical solution of the corresponding PDEs by classical finite element methods (FEM). For decades, this process has presented a severe bottleneck in performing efficient numerical simulations.

IgA, initiated by the pioneering 2005 paper by one of the organizers (Hughes), bridges the gap between FEM and CAD concepts. The main idea of IgA is to exploit simulation methods that employ the same basis functions both for the representation of the underlying geometric computational domains and for the numerical simulations, allowing from the beginning the elimination of geometry errors. This is accomplished by abandoning traditional finite element approaches and by employing instead more general basis functions such as B-Splines and Non-Uniform Rational B-Splines (NURBS) for the PDE simulations. The combined concept of IgA leads to improved convergence and smoothness properties of the PDE solutions and dramatically faster overall simulations.

Traditionally, the employment of B-splines and NURBS in CAD allows to represent geometrical objects smoother than just piecewise linear objects. Consequently, the IgA paradigm has, most importantly, rejuvenated the study of higher order approximation methods for the solution of PDEs. By now, it is fair to say that this paradigm revolutionized the engineering communities as it triggered a vast number of new simulations and publications. In addition to the successful application in various areas, IgA is rapidly becoming a mainstream analysis methodology and also a new paradigm for geometric design due to the variety of open problems provided in the field of geometric modeling.

However, even though substantial progress has been made in the isogeometric context over the last few years, there are several profound theoretical issues that are not yet well understood and that are currently investigated by researchers in numerical analysis, approximation theory, and applied geometry. The workshop offered a unique opportunity of vivid discussion and presentation of new results for the following forefront issues.

- Efficiency in IgA computations. This requires optimized quadrature rules tailored for B-splines and NURBS, ad hoc assembly strategies (sum factorization and row-by-row assembly), and suitable solvers and preconditioners for the resulting linear systems, fully exploiting the characteristics of the discretization spaces.
- Transition from the surface representation of the boundary geometry to the volumetric representation for three dimensional problems, proper handling of trimmed and complex multivariate geometries and relation with immersed methods.
- Refinement (and coarsening) strategies: development of adaptive methods with respect to both the mesh and the spline degree/smoothness (h-p-k refinement), related a-posteriori error estimates, and superconvergence behavior of smooth spline spaces on local tensor-product structures.
- Modeling of complex geometries of arbitrary topologies by smooth spline representations on unstructured meshes consisting of quadrilaterals/hexahedrals. In order to profit as much as possible of the tensor-product

spline setting, the considered meshes maintain large structured regions. It is important that these spline spaces have full approximation power.
- Smooth spline spaces on triangulations: they offer an appealing alternative to (local) tensor-product structures for the modeling of complex geometries and for performing efficient refinements. The full theoretical understanding of these spaces is one of the most prominent open problems in approximation theory of the last half century.
- Kolmogorov n-widths, explicit Sobolev approximation inequalities for spline spaces, and their use for a theoretical comparison between IgA and FEM on a degree-of-freedom basis.
- Error analysis by spectral analysis: the mathematical foundation and theoretical explanations of advantages of IgA in spectral analysis w.r.t. high order FEMs.
- Isogeometric discretization for shape and topological optimization, space-time problems, boundary element methods, and variational inequalities.
- Physical, medical and financial applications. Isogeometric discretizations and results in several challenging contexts, including: turbines, shells and laminates, boiling, patient-specific and bioprosthetic heart valves, prostate cancer, pricing, and self-supporting structures.

The Workshop was the follow-up of the MFO Mini-Workshop with the same title "Mathematical Foundations of Isogeometric Analysis" (ID 1606c) which was held on February 7-13, 2016 (see Report No. 8/2016).

The Workshop was attended by 27 participants with broad geographic representation who were a nice blend of researchers with various backgrounds. The experienced participants were experts who are strong in spline theory (Lyche, Manni, Speleers), numerical analysis and multiscale methods (Buffa, Harbrecht, Kunoth, Langer, Sangalli), applied geometry and geometric design (Elber, Giannelli, Jüttler, Mantzaflaris, Mourrain, Peters), numerical linear algebra (Serra-Capizzano), together with researchers in the engineering sciences with a strong mathematical background in modeling and numerics (Evans, Hughes, Kvamsdal, Reali, Zhang). The experienced researchers were complemented by a congruous group of younger participants with already a solid reputation in approximation (Bressan, Sande), unstructured spline structures (Takacs, Toshniwal) and engineering sciences (Puppi, Wei). In addition, a PhD student (Boschert) presented promising first results for approximations of variational inequalities with application in finance.

The Workshop clearly reported the substantial progress, both from the theoretical and applicative point of view, which has been made in the isogeometric context over the last three years. In particular, from the theoretical side we would like to mention the estimates in Sobolev approximation inequalities with explicit "constants", providing the mathematical explanation of the numerically observed superior performance of IgA over FEM on a degree-of-freedom basis. From a more practical point of view, maximally-smooth, k-refinement IgA solver strategies were presented showing the conjectured high accuracy versus computational cost gains over classical FEM approaches.

The Workshop also pointed out several profound theoretical issues in numerical analysis, approximation theory, and applied geometry that need a deeper understanding and deserve future investigations. Among the others, we mention:

- Shape and topological optimization with trimming;
- Isogeometric discretizations for space-time problems, optimal control, variational inequalities and free boundary problems;
- Unstructured spline techniques and structure-preserving discretizations;
- Mathematical foundations of (multivariate isogeometric) collocation methods;
- Influence of parameterizations and physics-aware parameterizations;
- Fully robust and scalable solvers.

They will be the focus of several research activities in the next years and are candidate topics for a future Workshop.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting John Evans and Yongjie Jessica Zhang in the "Simons Visiting Professors" program at the MFO.

Workshop: Mathematical Foundations of Isogeometric Analysis

Table of Contents

Abstracts

B-spline based methods for variational inequalities and a special HJB equation with non-smooth initial data

Sandra Boschert
(joint work with Angela Kunoth)

For the numerical computation of solutions of elliptic variational inequalities on closed convex sets, error estimates, monotone multigrid solvers and adaptive methods employing *linear* basis functions have been investigated over the past decades. However, there are a number of problems which profit from *higher order* approximations. Among these are problems of pricing American put options, formulated as a parabolic variational inequality. Recall that, in finance, to determine optimal risk strategies, one is not only interested in the solution of the variational inequality, i.e., the option price, but also in its partial derivatives up to order two, the so-called Greeks. A special feature for these option prize problems is that initial conditions are typically given as piecewise linear continuous functions. Consequently, we have derived a spatial discretization based on cubic B-splines with coinciding knots at the points where the initial condition is not differentiable. Together with an implicit time stepping scheme, this enables us to achieve an accurate pointwise approximation of the partial derivatives up to order two.

The first problem we consider is the Black-Scholes model for an *American put option*. The option price $V := V(S,t)$ is assumed to depend on an underlying asset $S \in \mathbb{R}^+$, time t, constant volatility σ and is subject to a strike price K and payoff $\mathcal{H}(S) := \max\{0, K-S\}$. Although the model is very simple, a fundamental numerical difficulty arises from the fact that American options can be exercised at any time; thus, leading to a free boundary value problem. One can formulate this problem as the following parabolic variational inequality: Find $y(S,t) - \mathcal{H} := V(\tau - T, S) - \mathcal{H} \in \mathcal{K}$ with convex set $\mathcal{K} := \{\varphi \in L_2(\Omega) : \varphi \geq 0\} \cap \mathcal{V}$ and Sobolev space $\mathcal{V}$ for a.e. $\tau \in (0,T]$ such that

$$\left\langle \tfrac{\partial y}{\partial \tau}, \varphi - y \right\rangle + a^B(y, \varphi - y) \geq 0 \text{ for all } \varphi - \mathcal{H} \in \mathcal{K}, \tag{1}$$

with initial condition $y(0,S) = \mathcal{H}(S)$ and bilinear form $a^B(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ as

$$a^B(y,\varphi) := \int_\Omega \left(\tfrac{1}{2}\sigma^2 \tfrac{\partial y}{\partial S} \left(S^2 \tfrac{\partial \varphi}{\partial S} + 2S\varphi \right) - rS\tfrac{\partial y}{\partial S}\varphi + ry\varphi \right) dS.$$

Next we consider *Heston's model* which includes a stochastic volatility $\upsilon \in \mathbb{R}^+$ to better represent real market situations. Now the value of the American option $V := V(t,S,\upsilon)$ is supposed to satisfy the following parabolic variational inequality: Find $y(\tau,x,\upsilon) - g(x) := V(\tau - T, \log(S/K), \upsilon) - \mathcal{H}(\log(S/K))$ such that

$$\langle \tfrac{\partial y}{\partial \tau}, \varphi - y \rangle + a^H(y, \varphi - y) \geq 0 \text{ for all } \varphi - g \in \mathcal{K}, \tag{2}$$

with initial condition $y(0,x,\upsilon) = \mathcal{H}(x) := \mathcal{H}(\log(S/K))$ and bilinear form

$$a^H(y, \varphi - y) := \int_\Omega \Big(\tilde{A}\nabla y \cdot \nabla(\varphi - y) + (\tilde{\mathbf{b}} \cdot \nabla y + ry)(\varphi - y)\Big)\, d\Omega,$$

where $\tilde{A} := \frac{1}{2}\upsilon \begin{pmatrix} 1 & 2\rho\xi \\ 0 & \xi^2 \end{pmatrix}$ and $\tilde{\mathbf{b}} := \begin{pmatrix} \frac{1}{2}\upsilon - r \\ \kappa(\upsilon - \gamma) + \frac{1}{2}\xi^2 \end{pmatrix}$.

Recall that in order to obtain optimal convergence for higher order B-splines of order k in the spatial domain Ω, the solution must lie in $H^k(\Omega)$. For variational inequalities such smoothness of the solution on the whole domain Ω is not given: the solution for parabolic variational inequalities in Black-Scholes' and Heston's model are in the Bochner space $L_2(0,T;H^2(\Omega))$. However, numerical results show that the solution is smooth in space except at the free boundary. Recently, it was shown in [2] that for variational inequalities the error estimate in the energy space E is restricted by an additional term in comparison to the estimate for variational equations: the semidiscrete solution $y_h := y_h(t)$ in the discrete B-spline space $\mathcal{V}_h^k$ satisfies the error estimate

$$\|y - y_h\|_E^2 \lesssim \|y - \varphi_h\|_{L_2(0,T;L_2(\Omega))} + \|y - \varphi_h\|^2_{L_2(0,T;\mathcal{V})}, \tag{3}$$

where $\varphi_h \in \mathcal{V}_h^k$ is some test function. Therefore, it is reasonable to locally refine the grid near the free boundary.

The following elliptic test problem where the exact solution is known shows the expected convergence behaviour: Find $w(x) := w \in \mathcal{K}$ such that

$$b(w, \varphi - w) \geq \langle f, \varphi - w\rangle \text{ for all } \varphi \in \mathcal{K} \text{ with } b(w,\varphi) := \int_\Omega \frac{\partial w}{\partial x}\frac{\partial y}{\partial x}dx,$$

$f := (1.2 - g'(x))$, $g(x) := \max\{0, 10 - 10\exp(x)\}$ and exact free boundary x_f.

	locally refined grid at x_f				uniform grid size		
N	error in $H^1(\Omega)$	Rate	$\lvert x_f - x_f^h\rvert$	N	error in $H^1(\Omega)$	Rate	$\lvert x_f - x_f^h\rvert$
96	6.20e–5	–	1.41e–3	129	2.07e–3	–	1.51e–2
192	7.28e–6	3.09	4.35e–4	257	1.22e–3	0.76	1.51e–2
384	1.04e–6	2.81	6.86e–5	513	4.34e–4	1.50	7.27e–3
768	1.21e–7	3.10	7.61e–6	1025	1.49e–4	1.55	3.36e–3
1536	1.49e–8	3.02	7.61e–6	2049	3.98e–5	1.91	1.41e–3

Since in Black-Scholes' or Heston's model the volatility is often under- or overestimated, in practice the so-called *Black-Scholes-Barenblatt (BSB) equation*, which is of *Hamilton-Jacobi-Bellman (HJB) type*, is of particular importance to price European options exercised only at a fixed predetermined time T. Here the volatility is to lie in a set $\Sigma := [\sigma_{\min}, \sigma_{\max}]$. For a butterfly spread, this leads to a *nonlinear parabolic problem* which is in the worst case szenario of the form: Find $V(S,t)$ such that

$$V_t + \inf_{\underline{\sigma}\in\Sigma}(\mathcal{L}V) := V_t + \inf_{\underline{\sigma}\in\Sigma}\big(\tfrac{1}{2}S^2\underline{\sigma}^2 V_{SS}\big) + rSV_S - rV = 0 \;\text{ in } \Omega \times [0,T), \tag{4}$$

with zero boundary and end condition $V(0,S) = \begin{cases} \max(0, S-K_1) \text{ if } S < K, \\ \max(0, K_2 - S) \text{ if } S \geq K \end{cases}$ for some parameters $K_1 < K_2$. One difficulty is that no weak H^1-formulation in the continuous setting is known. By normalizing the B-splines in the L_1-norm, one can find a weak formulation in the discrete setting. For linear finite elements this idea stems from [3]. Since the BSB equation is a nonlinear problem with jumping diffusion coefficients, one has to find an analogue of the Jacobian. The principal requirement for the operator to fit in this framework is slant differentiability. Since the initial condition and the solution is only continuous where the volatlity jumps, we reduce the smoothness of the approximation by repeating knots at those points. Corresponding numerical results show that the convergence for cubic B-splines is optimal in the L_2-norm, when the placement where the volatility changes (therefore, internal free boundaries) is approximated with desired accuracy.

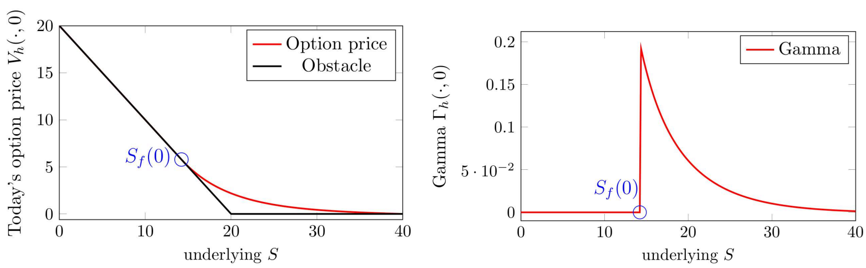


FIGURE 1. Numerical approximation of today's put option price (left) and its second derivative Gamma $\Gamma_h(S) := \frac{\partial^2 V_h(S)}{\partial S^2}$ (right) in Black-Scholes' model with $r = 0.04$, $\sigma = 0.2$ on $[0,T) \times \Omega = [0,5) \times (0,40)$.

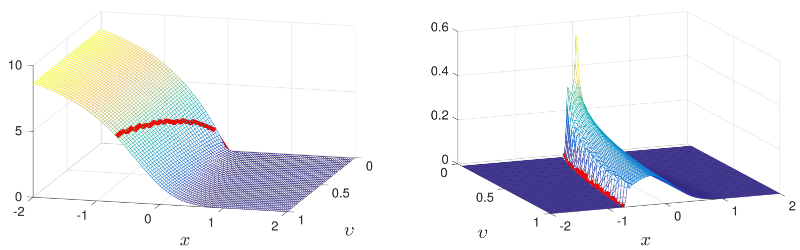


FIGURE 2. Numerical approximation of today's put option price and its second derivative in Heston's model with $K = 10$, $\varrho = 0.1$, $\xi = 0.9$, $\kappa = 5$, $\lambda = 0$, $r = 0.1$, $\gamma = 0.16$ on $[0,T) \times \Omega = [0, 0.25) \times [-2,2] \times [0.01,1]$.

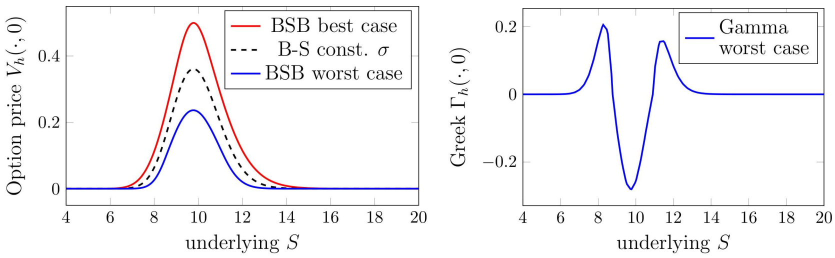


FIGURE 3. Numerical approximation of today's butterfly spread option price in the BSB model (left) and its second derivative (right) with $K_1 = 9, K = 10, K_2 = 11$ and $\Sigma = [0.15, 0.25]$, $r = 0.1$, $[0,T) \times \Omega = [0, 0.25) \times (4, 20)$.

REFERENCES

[1] S. Boschert, *B-spline Based Methods for Parabolic Free Boundary Value Problems and Special HJB Equations for Option Pricing Problems*, PhD Thesis, University of Cologne, in preparation.
[2] R.Z. Dautov and A.V. Lapin, *Sharp error estimate for implicit finite element scheme for American put option*, Russian Journal of Numerical Analysis and Mathematical Modelling **34**, 1–11 (2019).
[3] M. Jensen and I. Smears, *On the convergence of finite element methods for Hamilton-Jacobi-Bellman equations*, SIAM Journal on Numerical Analysis **51**, 137–162 (2013).
[4] A. Kunoth and M. Holtz, *B-spline-based monotone multigrid methods*, SIAM Journal on Numerical Analysis **45**, 1175–1199 (2007).

TBA: T-mesh B-spline Approximation

ANDREA BRESSAN
(joint work with Tom Lyche)

The approximation properties of a space of piecewise polynomials $\mathbb{S}$ of degree $\boldsymbol{d} := (d_1, \ldots, d_n)$ on the union of axis aligned boxes in $\mathbb{R}^n$ can be written using weighted Sobolev-like seminorms. Local approximation operators $\aleph : L^p(\Omega) \to \mathbb{S}$ are defined in terms of coefficient functionals and a basis Φ:

$$\aleph f := \sum_{\phi \in \Phi} \lambda_\phi(f)\phi. \tag{1}$$

Using them we obtain a priori approximation estimates of the form

$$\|\partial^{\boldsymbol{\sigma}}(f - \aleph f)\|_p \le C\mu(\Omega)^{\nu_-} \sum_{\boldsymbol{k} \in K} \|\rho_{\boldsymbol{k}} \partial^{\boldsymbol{k}} f\|_q, \tag{2}$$

where $\nu = 1/p - 1/q$, $K \subset \mathbb{N}^n$ is an index set of integers and the weights $\rho_{\boldsymbol{k}}$ are powers of the local resolution of $\mathbb{S}$. The precise form of the $\rho_{\boldsymbol{k}}$ depends on K, $\boldsymbol{k}$, p, q, $\boldsymbol{d}$ and $\boldsymbol{\sigma}$. In some cases, e.g., $K = \{\boldsymbol{k} \in \mathbb{N}^n : |\boldsymbol{k}| = d + 1\}$, $p \le q$ and $\boldsymbol{\sigma} = 0$, the weight $\rho_{\boldsymbol{k}}$ can be *anisotropic*, i.e. it takes into account the space resolution in the each coordinate direction. Otherwise the estimates are *isotropic* and $\rho_{\boldsymbol{k}}$ is at each point a power of the maximum of the diameters of the supports of the active functions. Common choices for K corresponds to Sobolev seminorms and *reduced* seminorms that involve a smaller set of partial derivatives. Estimate (2) can be achieved under different set of assumptions. Common assumptions in the univariate case are:

$\mathrm{H}^{\mathbf{1}}_{\mathbb{P}}$ $\aleph$ reproduces polynomials of degree $\boldsymbol{d}$,
$\mathrm{H}^{\mathbf{2}}_{\lambda}$ the local operator are continuous,
$\mathrm{H}^{\mathbf{3}}_{\phi}$ the basis functions are regular,
$\mathrm{H}^{\mathbf{4}}_{\mathbf{s}}$ the difference between $\operatorname{supp}\lambda_\phi$ and $\operatorname{supp}\phi$ is controlled,
$\mathrm{H}^{\mathbf{5}}_{\Pi}$ there are polynomial approximation estimates,
$\mathrm{H}^{\mathbf{6}}_{\mathrm{E}}$ the overlap of the support of the λ_ϕ is bounded,
$\mathrm{H}^{\mathbf{7}}_{\boxplus}$ the numbers of elements in the supports of the generators is bounded.

These require additional technicalities in the multivariate case. In particular the dependence of polynomial approximation estimates on the shape of the domain varies depending on p, q and the set of partial derivatives in K. In the multivariate case this dependence is controlled by the additional assumptions

$\mathrm{H}^{\mathbf{8}}_{\boxed{\omega}}$ the aspect ratio of the elements is bounded,
$\mathrm{H}^{\mathbf{9}}_{\boxed{\mathbf{S}}}$ the aspect ratio of the basis functions' supports is bounded,
$\mathrm{H}^{\mathbf{10}}_{\blacksquare}$ the supports of the basis functions are star shaped.

The dependence of C in (2) on the parameters of $\mathrm{H}^{\mathbf{1}}_{\mathbb{P}},\ldots,\mathrm{H}^{\mathbf{10}}_{\blacksquare}$ is explicit.

The abstract framework applies to many constructions of locally tensor product splines, e.g. to TPS, Analysis Suitable T splines (AST) [2, 1], truncated hierarchical splines (THB) [9] and Locally Refined splines (LR) [6].

References

[1] L. Beirão da Veiga, A. Buffa, D. Cho, and G. Sangalli, *Analysis-suitable T-splines are dual-compatible*, Computer Methods in Applied Mechanics and Engineering **249/252**, 42–51 (2012).

[2] L. Beirão da Veiga, A. Buffa, G. Sangalli, and R. Vázquez, *Analysis-suitable T-splines of arbitrary degree: definition, linear independence and approximation properties*, Mathematical Models and Methods in Applied Sciences **23**(11), 1979–2003 (2013).

[3] L. Beirão da Veiga, D. Cho, and G. Sangalli, *Anisotropic NURBS approximation in isogeometric analysis*, Computer Methods in Applied Mechanics and Engineering **209/212**, 1–11 (2012).

[4] A. Bressan, *Some properties of LR-splines*, Computer Aided Geometric Design **30**(8), 778–794 (2013).

[5] A. Buffa, E.M. Garau, C. Giannelli, and G. Sangalli, *On quasi-interpolation operators in spline spaces*, In: Building bridges: connections and challenges in modern approaches to numerical partial differential equations, Lecture Notes in Computational Science and Engineering **114**, pp. 73–91. Springer, Cham (2016).

[6] T. Dokken, T. Lyche, and K.F. Pettersen, *Polynomial splines over locally refined box-partitions*, Computer Aided Geometric Design **30**(3), 331–356 (2013).

[7] T. Dupont and R. Scott, *Polynomial approximation of functions in Sobolev spaces*, Mathematics of Computation **34**(150), 441–463 (1980).

[8] D.R. Forsey and R.H. Bartels, *Hierarchical B-spline refinement*, ACM SIGGRAPH Computer Graphics **22**(4), 205–212 (1988).

[9] C. Giannelli, B. Jüttler, and H. Speleers, *THB-splines: the truncated basis for hierarchical splines*, Computer Aided Geometric Design **29**(7), 485–498 (2012).

[10] T. Lyche and L.L. Schumaker, *Local spline approximation methods*, Journal of Approximation Theory **15**, 294–325 (1975).

[11] U. Reif and N. Sissouno, *Approximation with diversified B-splines*, Computer Aided Geometric Design **31**(7-8), 510–520 (2014).

[12] K. Scherer and A. Shadrin, *New upper bound for the B-spline basis condition number: II. a proof of de Boor's* 2^k*-conjecture*, Journal of Approximation Theory **99**(2), 217–229 (1999).

[13] T.W. Sederberg, D.L. Cardon, G.T. Finnigan, N.S. North, J. Zheng, and T. Lyche, *T-spline simplification and local refinement*, ACM Transactions on Graphics **23**, 276–283 (2004).

[14] T.W. Sederberg, J. Zheng, A. Bakenov, and A. Nasri, *T-splines and T-nurccs*, ACM Transactions on Graphics **22**, 477–484 (2003).

[15] H. Speleers, *Hierarchical spline spaces: quasi-interpolants and local approximation estimates*, Advances in Computational Mathematics **43**(2), 235–255 (2017).

[16] H. Speleers and C. Manni, *Effortless quasi-interpolation in hierarchical spaces*, Numerische Mathematik **132**(1), 155–184 (2016).

Volumetric representations (V-reps): design toward (isogeometric) analysis

Gershon Elber

The need for a tight coupling between design and analysis has been recognized as crucial almost since geometric modeling (GM) has been conceived, Unfortunately, even today, contemporary GM systems only offer a loose link between the two, if at all.

For about half a century, (trimmed) Non Uniform Rational B-spline (NURBs) surfaces has been the B-rep of choice for virtually all the GM industry. Fundamentally, B-rep GM has evolved little during this period. In this talk, we seek to examine an extended (trimmed) NURBs volumetric representation (V-rep) [4] that successfully confronts the existing and anticipated design, analysis, and manufacturing foreseen challenges. We extend all fundamental B-rep GM operations, such as primitive and surface constructors, and Boolean operations, to trimmed trivariate V-reps. This enables the much-needed tight link to (Isogeometric) analysis on one hand and the full support of (heterogeneous and anisotropic) additive manufacturing on the other.

Specific capabilities toward the support of Isogeometric analysis are also presented, that enable robust queries over the V-reps, including precise contact analysis, maximal penetration depth, and accurate integration over trimmed domains [3]. Examples and other applications of V-rep GM, including AM and lattice- and micro- structure synthesis (with heterogeneous materials) are also demonstrated [1, 2, 5].

In collaboration with many others, including Ben Ezair, Fady Massarwi, Boris van Sosin, Jinesh Machchhar, Annalisa Buffa, Giancarlo Sangalli, Pablo Antolin, and Massimiliano Martinelli.

References

[1] P. Antolin, A. Buffa, E. Cohen, J.F. Dannenhoffer, G. Elber, S. Elgeti, R. Haimes, and R. Riesenfeld, *Optimizing micro-tiles in micro-structures as a design paradigm*, Computer Aided Design **115**, 23–33 (2019).
[2] G. Elber, *Precise construction of micro-structures and porous geometry via functional composition*, In: Proceedings of the 9th International Conference on Mathematical Methods for Curves and Surfaces, pp. 108–125 (2016).
[3] F. Massarwi, P. Antolin, and G. Elber, *Volumetric untrimming: Precise decomposition of trimmed trivariates into tensor products*, Computer Aided Geometric Design **71**, 1–15 (2019).
[4] F. Massarwi and G. Elber, *A B-spline based framework for volumetric object modeling*, Computer Aided Design **78**, 36–47 (2016).
[5] F. Massarwi, J. Machchhar, P. Antolin, and G. Elber, *Hierarchical, random and bifurcation tiling with heterogeneity in micro-structures construction via functional composition*, Computer Aided Design **102**, 148–159 (2018).

Adaptive topology optimization with hierarchical B-splines

John Evans

(joint work with Kurt Maute, Christian Messe, Lise Noel, and Frits de Prenter)

Topology optimization has become immensely popular in recent years, largely driven by advances in rapid prototyping and additive manufacturing [1]. Topology optimization allows one to optimize material layout within a given design space given a set of loading criteria, boundary conditions, and constraints with the goal of maximizing performance. As opposed to shape optimization, topology optimization allows for topological changes in addition to shape changes.

In this talk, I discuss our recent work on developing an adaptive topology optimization framework based on three component technologies: (i) the level set method [2], (ii) the extended finite element method [3], and (iii) hierarchical B-splines [4, 5]. In our framework, the material layout of the domain is defined by one or more level set fields, each of which are defined using hierarchical B-splines. The response of the system is also described by a hierarchical B-spline discretization in conjunction with a generalized version of the extended finite element method. Our framework is general in that one may employ different hierarchical B-spline spaces to describe the level set fields and the system response. Our framework is adaptive in two ways. First, our framework allows for adaptivity of the hierarchical B-spline space describing the system response to better capture local solution features for a given design. Second, our framework allows for adaptivity of the hierarchical B-spline space describing the level set fields to adapt to emerging topological and geometric features as the topology optimization process continues. It should also be noted that since we use hierarchical B-splines to describe the level set fields as opposed to standard finite elements, our framework yields optimal designs which are smooth. An example design generated by our adaptive topology optimization framework is displayed in Figure 1. I conclude by discussing a geometric multigrid

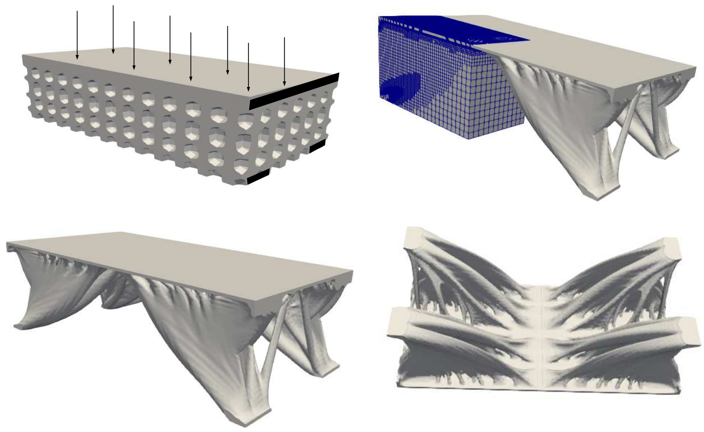


Figure 1. An example design generated by our adaptive topology optimization framework. The level set field is discretized with quadratic hierarchical B-splines and the state variable field is discretized with linear hierarchical B-splines on the same mesh. Top Left: The loading configuration and initial design. Top Right: The hierarchical mesh associated with the optimal design. Bottom Left: Top-down view of the optimal design. Bottom right: Bottom-up view of the optimal design.

methodology we have developed to solve the forward and adjoint problems arising in our adaptive topology optimization framework [6].

Numerical studies with our adaptive topology optimization framework have yielded several important observations:

(1) Topology optimization problems are typically nonconvex. As such, the optimal designs obtained from our topology optimization framework are highly dependent on the choice of initial design.
(2) In the two-dimensional setting, the optimal designs obtained from our topology optimization framework are highly dependent not only on the choice of initial design but on the resolution of the initial mesh and also the adaptive mesh refinement strategy.
(3) In the three-dimensional setting, the optimal designs obtained from our topology optimization framework are much less dependent on the resolution of the initial mesh. In fact, small features are able to emerge and evolve with adaptivity in the three-dimensional setting.
(4) Efficiency improvements are possible with our adaptive topology optimization framework, but said efficiency improvements are highly dependent on the volume fraction of the obtained optimal design. For three-dimensional structural topology optimization problems, efficiency improvements greater than 95% are observed when the volume fraction is less than 10%.

Our studies have also generated several important research questions that need to be addressed from a mathematical perspective:

(1) How often and in what manner should one refine and coarsen the level set field discretization in our adaptive topology optimization framework?
(2) How often and in what manner should one refine and coarsen the state variable field discretization in our adaptive topology optimization framework?
(3) Is it possible to construct a robust and efficient optimization routine for finding a global minimum rather than a local minimum?
(4) Can gradient-based optimization routines be accelerated using the multi-level structure of hierarchical B-splines?
(5) Is it possible to further improve performance in an efficient manner by fusing adaptive topology optimization with isogeometric shape optimization?
(6) What is the best way to extract a surface/volumetric parameterization from a level set field?

References

[1] O. Sigmund and K. Maute, *Topology optimization approaches*, Structural and Multidisciplinary Optimization **48**, 1031–1055 (2013).

[2] M. Yu, X. Wang, and D. Guo, *A level set method for structural topology optimization*, Computer Methods in Applied Mechanics and Engineering **192**, 227–246 (2003).

[3] S. Kreissl and K. Maute, *Level set based fluid topology optimization using the extended finite element method*, Structural and Multidisciplinary Optimization **46**, 311-326 (2012).

[4] D.R. Forsey and R.H. Bartels, *Hierarchical B-spline refinement*, ACM Siggraph Computer Graphics **22**, 205–212 (1988).

[5] C. Giannelli, B. Jüttler, and H. Speleers, *THB-splines: The truncated basis for hierarchical splines*, Computer Aided Geometric Design **29**, 485-498 (2012).

[6] F. de Prenter, C.V. Verhoosel, E.H. van Brummelen, J.A. Evans, C. Messe, J. Benzaken, and K. Maute, *Scalable multigrid methods for immersed finite element methods and immersed isogeometric analysis*, arXiv preprint arXiv:1903.10977 (2019).

Adaptive refinement and coarsening with (T)HB-splines and extensions on two-patch geometries

Carlotta Giannelli

Isogeometric analysis is a recent paradigm for the numerical solution of partial differential equation which considers (smooth) spline functions both for the representation of the computational domain and for the description of the approximation space. Unluckily, the tensor-product structure of standard spline models, commonly used in computer aided design, prevents not only the possibility of representing flexible shapes, but also to perform local mesh refinement.

The hierarchical spline model is an adaptive spline technology that enables the possibility to properly deal with local problems. Based on the multi-level concept of hierarchical splines, truncated hierarchical B-splines (THB-splines) were introduced as an effective tool to perform hierarchical refinement while reducing the interactions between different refinement levels [4]. Adaptive schemes based on (truncated) hierarchical B-splines have been successfully applied in different problems related to computer aided design and isogeometric analysis.

Thanks to the possibility of easily handling local mesh refinements, hierarchical spline structures provide an effective tool to design and analyze adaptive isogeometric methods. The requirements of applications additionally demand automatic refinement and coarsening algorithms which suitably provide effective discretizations and improve computational efficiency. While several papers investigated refinement schemes for hierarchical isogeometric methods in the last years, see e.g., [1] for a recent overview on adaptive methods with THB-splines, only very recently, few authors also focused on the study of coarsening algorithms [7, 8].

A complete set of algorithms to perform adaptive refinement and coarsening with THB-splines defined on certain suitably graded hierarchical meshes, indicated as admissible meshes, was recently presented in [3]. The proposed algorithms are applied to linear heat transfer problems with localized moving heat source, as simplified models for additive manufacturing applications. The numerical examples show that THB-spline admissible solutions deliver effective discretizations. In addition, they also confirm that our algorithms strongly improve computational efficiency.

Adaptive techniques should be combined with multi-patch constructions to design efficient computational schemes that enable to perform isogeometric simulations on complex geometries. Due to the benefits of higher continuity in isogeometric methods, the construction of C^1 isogeometric spline spaces defined on two or more patches was recently addressed in the tensor-product setting, see e.g., [5, 6]. A first step to combine adaptive hierarchical spline construction with smooth spline spaces on general multi-patch configurations is to address the two-patch case. Obviously, the possibility of defining smooth hierarchical splines on more than one patch enlarge the capabilities of the considered adaptive scheme by removing the restriction to rectangular topologies. The construction of C^1 continuous hierarchical splines on two-patch domains was recently presented in [2]. The generated hierarchical spaces were there used to numerically solve the laplacian and bilaplacian equations on two-patch geometries, demonstrating the potential of C^1 hierarchical constructions for applications in isogeometric analysis.

References

[1] C. Bracco, A. Buffa, C. Giannelli, and R. Vázquez, *Adaptive isogeometric methods with hierarchical splines: an overview*, Discrete and Continuous Dynamical Systems **39**, 241–262 (2019).

[2] C. Bracco, C. Giannelli, M. Kapl, and R. Vázquez, *Isogeometric analysis with C^1 hierarchical functions on planar two-patch geometries*, arXiv preprint arXiv:1901.09689 (2019).

[3] M. Carraturo, C. Giannelli, A. Reali, and R. Vázquez, *Suitably graded THB-spline refinement and coarsening: Towards an adaptive isogeometric analysis of additive manufacturing processes*, Computer Methods in Applied Mechanics and Engineering **348**, 660–679 (2019).

[4] C. Giannelli, B. Jüttler, and H. Speleers, *THB-splines: The truncated basis for hierarchical splines*, Computer Aided Geometric Design **29**, 485–498 (2012).

[5] M. Kapl, G. Sangalli, and T. Takacs, *Dimension and basis construction for analysis-suitable G^1 two-patch parameterizations*, Computer Aided Geometric Design **52–53**, 75–89 (2017).

[6] M. Kapl, G. Sangalli, and T. Takacs, *An isogeometric C^1 subspace on unstructured multi-patch planar domains*, Computer Aided Geometric Design **69**, 55–75 (2019).
[7] G. Lorenzo, M. Scott, K. Tew, T. J. R. Hughes, and H. Gomez, *Hierarchically refined coarsened splines for moving interface problems, with particular application to phase-field models of prostate tumor growth*, Computer Methods in Applied Mechanics and Engineering **319**, 515–548 (2017).
[8] P. Hennig, M. Ambati, L. De Lorenzis, M. Kästner, *Projection and transfer operators in adaptive isogeometric analysis with hierarchical B-splines*, Computer Methods in Applied Mechanics and Engineering **334**, 313–336 (2018).

About a fast isogeometric boundary element method

Helmut Harbrecht
(joint work with Jürgen Dölz, Michael Multerer, Stefan Kurz, Sebastian Schöps, and Felix Wolf)

1. Introduction

In the search of a method incorporating simulation techniques into the design workflow of industrial development, [8] proposed the concept of *Isogeometric Analysis (IGA)* to unite *Computer Aided Design (CAD)* and *Finite Element Analysis (FEA)*. It enables to perform simulations directly on geometries described by volumetric NURBS parametrizations. Nonetheless, many CAD systems use boundary representations only. Thus, volumetric parametrizations often have to be constructed solely for the purpose of simulation. The boundary parametrization, however, can be easily exported from CAD. Therefore, an approach via isogeometric boundary element methods seems to be natural.

2. Isogeometric boundary element methods

The utilization of parametric mappings in numerical implementations of the boundary element method is not new, going back further than the introduction of the isogeometric concept, see [6] for example. Parametric mappings avoid the problem of a slow convergence of the geometry due to the limited polynomial approximation of the geometry. Thus, they encourage the application of higher order Galerkin schemes. Through the parametric mappings, a tensor product structure on the geometries is induced, making it possible to define patchwise tensor product B-spline bases of high order and regularity.

One of the major downsides of the application of boundary element methods is that the integral operators involved yield dense discrete systems. To counteract the dense matrices, so-called *fast methods* must be employed for compression and efficiency. As shown in [3, 7], the tensor product structure induced by the mappings can be exploited to achieve an efficient implementation of compression techniques such as $\mathcal{H}$-matrices or the fast multipole method [5]. An isogeometric boundary element method promises hence runtimes which can compete with classical discretization methods.

Therefore, we developed the software library Bembel, **B**oundary **E**lement **Me**thod **B**ased **E**ngineering **L**ibrary, which is written in C and C++ [1]. It solves boundary value problems governed by the Laplace, Helmholtz or electric wave equation within the isogeometric framework. The development of the software started in the context of *wavelet Galerkin methods* on parametric surfaces, see [6], where the integration routines for the Green's function of the Laplacian have been developed and implemented. It was then extended to *hierarchical matrices* ($\mathcal{H}$-matrices) in [7] and to $\mathcal{H}^2$*-matrices* and higher order B-splines in [3]. With support of B-splines and NURBS for the geometry mappings, the Laplace and Helmholtz code became isogeometric in [2]. Finally, in [4], it has been extended to the electric field integral equation.

3. Numerical example

We shall present numerical results for the Laplace equation $\Delta U = 0$ inside the gear worm geometry Ω found in the left plot of Figure 1, whose surface $\Gamma = \partial\Omega$ is represented by 290 patches. The harmonic polynomial $U(\mathbf{x}) = 4x_1^2 - 3x_2^2 - x_3^2$ is used to prescribe either Dirichlet boundary conditions $f = U|_\Gamma$ or Neumann boundary conditions $g = \langle \nabla U, \mathbf{n}\rangle$ on Γ.

Making the single layer potential ansatz

$$U(\mathbf{x}) = \int_\Gamma \frac{u(\mathbf{y})}{4\pi\|\mathbf{x}-\mathbf{y}\|_2} \mathrm{d}\sigma_\mathbf{y}, \quad \mathbf{x} \in \Omega, \tag{1}$$

leads to a Fredholm integral equation of the first kind

$$\mathcal{S}u(\mathbf{x}) = \int_\Gamma \frac{u(\mathbf{y})}{4\pi\|\mathbf{x}-\mathbf{y}\|_2} \mathrm{d}\sigma_\mathbf{y} = f(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \tag{2}$$

for the unknown density u in case of the Dirichlet problem. Whereas, making a double layer potential ansatz

$$U(\mathbf{x}) = \int_\Gamma \frac{\langle \mathbf{x}-\mathbf{y}, \mathbf{n}_\mathbf{y}\rangle u(\mathbf{y})}{4\pi\|\mathbf{x}-\mathbf{y}\|_2^3} \mathrm{d}\sigma_\mathbf{y}, \quad \mathbf{x} \in \Omega, \tag{3}$$

amounts to a Fredholm integral equation of the first kind

$$\mathcal{W}u(\mathbf{x}) = \frac{\partial}{\partial \mathbf{n}_\mathbf{x}} \int_\Gamma \frac{\langle \mathbf{x}-\mathbf{y}, \mathbf{n}_\mathbf{y}\rangle u(\mathbf{y})}{4\pi\|\mathbf{x}-\mathbf{y}\|_2^3} \mathrm{d}\sigma_\mathbf{y} = g(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \tag{4}$$

for the unknown density u in case of the Neumann problem.

Since the density u is unknown, the error of the potential U is measured on the 115'241 vertices of a grid of 83'437 cubes fitted inside the domain. A visualization of these cubes together with the computed potential for the single layer ansatz can be found in the right plot of Figure 1. In view of having only a Lipschitz continuous boundary, the theoretical convergence rates are limited to at most h^3 for the single layer ansatz and to h^1 for the hypersingular ansatz. Figure 2 illustrates that these convergence rates are achieved for all polynomial degrees under consideration. In fact, the higher order ansatz functions even seem to produce a convergence rate of up to h^5 for both, the single layer ansatz and the hypersingular ansatz. Note that the dashed lines correspond to the convergence rates h^3 and h^5 while the

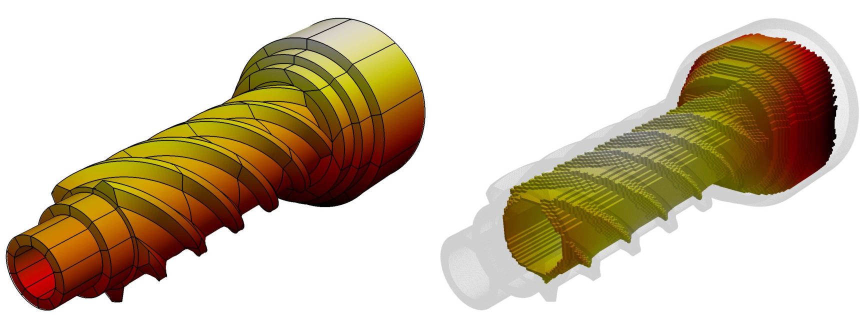


FIGURE 1. The gear worm geometry and the approximate potential in case of the Dirichlet problem for the Laplacian vs. level of uniform refinement.

accompanying numbers are the polynomial degrees of the interpolation in the fast multipole method.

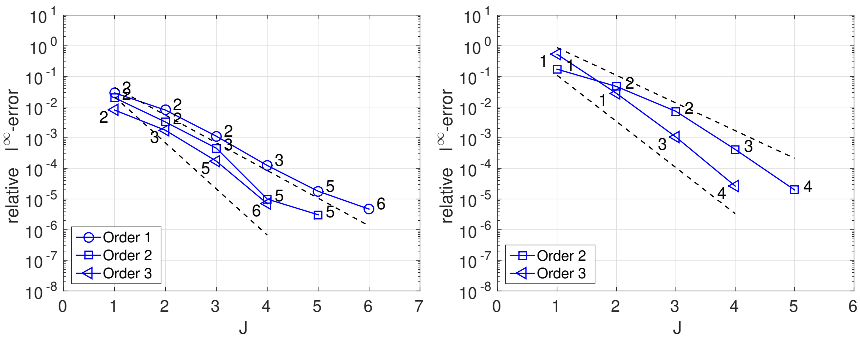


FIGURE 2. Relative errors of the potentials in case of the single layer ansatz (*left*) and the hypersingular ansatz (*right*).

REFERENCES

[1] J. Dölz, H. Harbrecht, S. Kurz, M.D. Multerer, S. Schöps, and F. Wolf, *Bembel: The fast isogeometric boundary element C++ library for Laplace, Helmholtz, and electric wave equation*, arXiv preprint arXiv:1906.00785 (2019).

[2] J. Dölz, H. Harbrecht, S. Kurz, S. Schöps, and F. Wolf, *A fast isogeometric BEM for the three dimensional Laplace- and Helmholtz problems*, Computer Methods in Applied Mechanics and Engineering **330**, 83–101 (2018).

[3] J. Dölz, H. Harbrecht, and M. Peters, *An interpolation-based fast multipole method for higher order boundary elements on parametric surfaces*, International Journal for Numerical Methods in Engineering **108**, 1705–1728 (2016).

[4] J. Dölz, S. Kurz, S. Schöps, and F. Wolf, *Isogeometric boundary elements in electromagnetism: Rigorous analysis, fast methods, and examples*, arXiv preprint arXiv:1807.03097 (2018).

[5] L. Greengard and V. Rokhlin, *A fast algorithm for particle simulations*, Journal of Computational Physics **135**, 280–292 (1997).
[6] H. Harbrecht, *Wavelet Galerkin schemes for the boundary element method in three dimensions*, PhD thesis, Technische Universität Chemnitz (2001).
[7] H. Harbrecht and M. Peters, *Comparison of fast boundary element methods on parametric surfaces*, Computer Methods in Applied Mechanics and Engineering **261–262**, 39–55 (2013).
[8] T.J.R. Hughes, J.A. Cottrell, and Y. Bazilevs, *Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement*, Computer Methods in Applied Mechanics and Engineering **194**, 4135–4195 (2005).

Mathematics of isogeometric analysis and applications: a status report

Thomas J.R. Hughes

I presented a sampling of the state-of-the-art in Isogeometric Analysis (IGA) with emphasis on mathematical developments. The field has become so enormous and broad based that it is impossible to even briefly mention all areas of activity. This is our second Oberwolfach workshop on IGA. The first was held in February of 2016 and was smaller than the present one. In my talk I tried to identify progress that has been made in the almost 3 ½ years since the first workshop. During that time, according to Web of Science, there have been approximately 1300 papers published on IGA in archival research journals. I began my talk with a comparison of the publication history of the first 30 years of the Finite Element Method (FEM) with that of IGA, which began in 2005. It is striking how quickly publications and citations in IGA have grown. In the first 10 years of IGA the numbers are much larger than in the first 30 years of FEM.

I presented a few applications with the FEM, specifically, automobile crash dynamics, full-body patient-specific, fluid-structure analysis of the cardiovascular system, and a HeartFlow, Inc., analsysi of blood flow in human coronary arteries. These illustrate the breadth and success of the FEM. I know a lot about all the applications because I developed many of the technologies employed. I observed that in each case the lowest order finite elements were utilized. Why not higher-order finite elements? From the academic research literature one would think the higher-order elements exhibit superior accuracy and efficiency. An academic answer might be that complex practical problems do not enjoy the solution regularity necessary to obtain higher-order convergence rates, but that is only a small part of the reason. The sad truth is that higher-order C^0-continuous finite elements are not robust and fail in many practical applications. Later in my talk, I used spectral analysis to reveal why this is the case and at the same time why spline-based approaches, such as IGA, do not suffer the same deficiencies. Indeed, one can show that maximally smooth C^{p-1}-continuous smooth splines exhibit a unique combination of accuracy and robustness. In fact, the higher the p, the more robust, the exact opposite of C^0-continuous finite elements.

Here are the main topics I covered in the rest of my talk: Basics technologies, such as B-splines and NURBS; approximation estimates in Sobolev norms; error analysis by spectral analysis techniques; Kolmogorov n-widths; efficient formation and assembly based on weighted quadrature, sum factorization and row-by-row assembly; collocation; thin shells; trim and immersion; and physical applications to wind turbines, boiling, patient-specific and bioprosthetic heart valves, and prostate cancer.

It was very exciting to discover later on in the workshop that our 2009 Kolmogorov n-width study comparing IGA and FEM (Evans-Bazilevs-Babuška-Hughes, 2009) prompted Espen Sande and Andrea Bresson to pursue the subject, which led to their beautiful analytical estimates of the "constants" in Sobolev approximation inequalities, explicitly establishing the superiority of IGA over FEM on a degree-of-freedom basis. This is what many of us have repeatedly experienced in solving practical problems. In my talk I concluded my brief description of n-widths with the statement that "smooth splines are always good." Sande and Bresson have certainly made that mathematically precise.

It was also gratifying to see that the IGA maximally-smooth, k-refinement solver strategy presented in the talk of Giancarlo Sangalli showed enormous accuracy versus computational cost gains over C^0-continuous finite element approaches, confirming another conjecture made in the early days of IGA.

Some remarks on integration by interpolation and look-up

Bert Jüttler
(joint work with Alessandro Giust and Maodong Pan)

Methods for the efficient generation of system matrices in isogeometric analysis have been investigated in a substantial number of publications during the last years. Mass and stiffness matrices possess $\mathcal{O}(Np^d)$ non-zero elements, where N is the dimension of the space of test functions. Their assembly via Gauss quadrature requires $\mathcal{O}(Np^{3d})$ floating point operations (flops), thus requiring substantial computation time, especially for larger values of the polynomial degree p. Several techniques have been developed in order to reduce the computational costs associated with matrix generation. We summarize some of the existing results in the following table:

reference	# flops	method and remarks
[1] (2015)	$\mathcal{O}(Np^{2d+1})$	sum factorization (SF)
[2] (2015)	$\mathcal{O}(Np^{2d})$	integration by interpolation and look-up (IIL)
[3] (2017)	$\mathcal{O}(RNp^d)$	tensor decomposition (TD); truncated tensor rank R
[4] (2017)	$\mathcal{O}(Np^{d+1})$	weighted quadrature; not symmetry-preserving
[5] (2018)	$\mathcal{O}(rNp^d)$	partial TD; uses SVD; truncated matrix rank r ($< R$)
[6] (2019)	$\mathcal{O}(Np^{d+2})$	improved SF
this paper	$\mathcal{O}(Np^{d+1})$	combines IIL and SF; symmetry preserving

It has been observed that GQ requires $(p+1)^d$ Gauss nodes per element to preserve the overall accuracy of the numerical simulation. This implies that the individual matrix elements are evaluated with accuracy of order h^{2p+3}.

We use the lemma of Strang to analyze the effect of the quadrature error for the isogeometric discretization of the weak form of the Poisson problem on an interval $[\alpha, \beta]$, which is (artificially) represented by a spline parameterization over the unit interval. More precisely, we consider the pull-back of this problem to the unit interval, where we perform a discretization based on spline functions. The consistency error caused by using the approximation $\tilde{a}$ (generated by GQ) of the exact bilinear form a takes the form

$$\sup_{v_h \in V_h} \frac{|a(u_h^*, v_h) - \tilde{a}(u_h^*, v_h)|}{\|v_h'\|_{2,[0,1]}} , \tag{1}$$

where u_h^* is the best approximation of the solution in the discretization space. The quadrature error can be estimated as

$$|a(u_h^*, v_h) - \tilde{a}(u_h^*, v_h)| \leq C h^{q+2} \sum_{i=1}^{n} \|({u_h^*}' A v_h')^{[q+1]}\|_{\infty,[\frac{i-1}{n},\frac{i}{n}]} ,$$

where we assume that the unit interval is subdivided uniformly into n elements of size $h = 1/n$ and q is the degree of exactness provided by GQ. The weight function A is determined by the domain parameterization.

The individual terms in the sum on the right-hand sided can be bounded by

$$\|({u_h^*}' A v_h')^{[q+1]}\|_{\infty,[\frac{i-1}{n},\frac{i}{n}]} \leq \sum_{j=0}^{p-1} \binom{q+1}{i} \|({u_h^*}' A)^{[q+1-j]}\|_{\infty,[\frac{i-1}{n},\frac{i}{n}]} \|v^{[j+1]}\|_{\infty,[\frac{i-1}{n},\frac{i}{n}]} .$$

Note that we do not need to consider derivatives $v^{[j+1]}$ for $j \geq p$, since v_h is a polynomial of degree p on each element. Markov- and Bernstein-type inequalities confirm that

$$\|v^{[j+1]}\|_{\infty,[\frac{i-1}{n},\frac{i}{n}]} \leq C' h^{-\frac{1}{2}} \|v^{[j+1]}\|_{2,[\frac{i-1}{n},\frac{i}{n}]} \leq C'' h^{-p+\frac{1}{2}} \|v'\|_{2,[\frac{i-1}{n},\frac{i}{n}]} ,$$

since the index j does not exceed $p-1$. Finally we use the Cauchy-Schwarz inequality in order to combine the individual H^1 semi-norms on the elements to the corresponding norm $\|v'\|_{2,[0,1]}$ on the unit interval.

In view of (1) we conclude that the consistency error possesses order $h^{q-p+\frac{5}{2}}$, meaning that at least p Gauss nodes (where $q = 2p-1$) are needed to achieve the optimal rate p of convergence with respect to the H^1 semi-norm. This is in perfect agreement with the experimental results reported in [2, Example 3].

The method of integration by interpolation and look-up (IIL) relies on spline approximations of degree q of the weighting functions in the integrals and uses look-up tables of B-spline tri-product integrals to integrate the resulting approximate integrands. It was shown that degree $q = p$ suffices to preserve the overall accuracy of the numerical simulation [2], even though the individual matrix elements are evaluated only with accuracy of order h^{p+2}, which is much lower than for GQ. This leads us to conclude that IIL is better suited to obtain a low-order consistency error than GQ.

The remainder of this talk describes our recent work on the combination of IIL with the technique of sum factorization (SF). The resulting method is based on the observation that the B-spline tri-product integrals can be factorized into d univariate integrals. We obtain a matrix assembly algorithm that requires $\mathcal{O}(Np^{d+1})$ flops for the matrix assembly. Thus, it achieves the same performance as weighted quadrature [4], while preserving the symmetry of the matrices. The direct evaluation of matrix-vector products (needed for matrix-free methods) requires $\mathcal{O}(Np^{d})$ flops.

We also present a detailed analysis in the low degree cases ($p \leq 3$), which shows that the new method compares favorably with the existing techniques. Finally we note that the idea of combining IIL with SF admits a generalization to the assembly of isogeometric discretizations that are based on hierarchical B-splines. This leads to a similar results about the computational costs, provided that the meshes satisfy certain admissibility assumptions.

References

[1] P. Antolin, A. Buffa, F. Calabrò, M. Martinelli, and G. Sangalli, *Efficient matrix computation for tensor-product isogeometric analysis: The use of sum factorization*, Computer Methods in Applied Mechanics and Engineering **285**, 817–828 (2015).

[2] A. Mantzaflaris and B. Jüttler, *Integration by interpolation and look-up for Galerkin-based isogeometric analysis*, Computer Methods in Applied Mechanics and Engineering **284**, 373–400 (2015).

[3] A. Mantzaflaris, B. Jüttler, B.N. Khoromskij, and U. Langer, *Low rank tensor methods in Galerkin-based isogeometric analysis*, Computer Methods in Applied Mechanics and Engineering **316**, 1062–1085 (2017).

[4] F. Calabrò, G. Sangalli, and M. Tani, *Fast formation of isogeometric Galerkin matrices by weighted quadrature*, Computer Methods in Applied Mechanics and Engineering **316**, 606–622 (2017).

[5] F. Scholz, A. Mantzaflaris, and B. Jüttler, *Partial tensor decomposition for decoupling isogeometric Galerkin discretizations*, Computer Methods in Applied Mechanics and Engineering **336**, 485–506 (2018).

[6] A. Bressan and S. Takacs, *Sum-factorization techniques in Isogeometric Analysis*, Computer Methods in Applied Mechanics and Engineering **352**, 437–460 (2019).

Recovery based error estimates for isogeometric analysis

Trond Kvamsdal

(joint work with Mukesh Kumar and Kjetil A. Johannessen)

In this talk, we explore that Isogeometric analysis (IGA) based on B-splines or Non-uniform rational B-splines (NURBS) produces structured tensor product meshes within each patch [3] and facilitates superconvergence behavior in the Galerkin discretization [6]. Adaptively refined LR B-splines [2] based on structured adaptive mesh refinement [4] facilitates superconvergence behavior on adaptive meshes as they produce local tensor product meshes. The aim of the present talk is to present the underlying idea and the efficiency of using superconvergent patch recovery and a posteriori error estimation technique in adaptive isogeometric analysis. The details of the presented methodology are given in [5].

We start out by addressing the existence of derivative superconvergence points in the computed finite element solution based on B-splines and LR B-splines for our elliptic model problem (1D and 2D Poisson). Inspired by earlier theoretical work presented in Wahlbin [6] and computer based proof of Babuska [1], we demonstrate that we are able to compute true derivative superconvergence points by means of local Neumann projection of a set of proper monomials for both uniform B-splines as well as for general (non-uniform) adapted LR B-splines. For uniform B-splines the true derivative superconvergent points are located at different location than the case of classical C^0 Lagrange elements. Thus, the continuity of the underlying finite element basis plays an important role for the location of true derivative superconvergent points. For the case of quadratic C^1 B-splines, on uniform mesh partition, they share the same location given by the (2×2)-Gauss Legendre points (or Barlow points) for classical quadratic C^0 Lagrange elements. While in case of cubic C^2 and C^1 B-spline spaces, on uniform mesh partition, the derivative superconvergence points will be at (3×3)-Gauss Lobatto points within each elements in contrary to the (3×3)-Gauss Legendre points (or Barlow points) for classical cubic C^0 Lagrange elements.

In [5] a study of three different gradient recovery techniques were performed for the purpose of enabling effective adaptive refinement in isogeometric analysis: Continuous L^2-projection (CL2P), Discrete least square fitting (DLSF), and Superconvergent Patch Recovery (SPR).

The main findings were:

- The difference between using true superconvergent derivative points and (2×2)-Gauss Legendre points for p=2 is noticeable but not pronounced for the accuracy of the recovered gradient field and the corresponding global effectivity indices.
- Adaptive refinement using all the three recovery based a posteriori error estimates provides optimal convergence rate
- The obtained global effectivity indices are for all the three recovery techniques remarkable close to one - This is in contrast with the basic explicit residual based error estimates.
- The local elementwise effectivity indices for all the three recovery techniques are close to one after some initial refinement steps to take care of any possible pollution effect.
- The main difference between the three recovery methods is that for both CL2P and DLSF one have to solve a global (mass matrix) problem, whereas SPR only involve solution of a local problem.

Another interesting use of the superconvergence property is to do collocation by using the superconvergent points as the collocation points. Recent studies shows that one may for certain problems achieve optimum convergence rates. However, here it is important to compute true superconvergent points for non-uniform meshes.

References

[1] I. Babuška and T. Strouboulis, *The finite element method and its reliability*, Numerical Mathematics and Scientific Computation, The Clarendon Press, Oxford University Press, New York (2001).

[2] T. Dokken, T. Lyche, and K. F. Pettersen, *Polynomial splines over locally refined box-partitions*, Computer Aided Geometric Design **30**, 331–356 (2013).

[3] T.J.R. Hughes, J.A. Cottrell, and Y. Bazilevs, *Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement*, Computer Methods in Applied Mechanics and Engineering **194**, 4135–4195 (2005).

[4] K.A. Johannessen, T. Kvamsdal, and T. Dokken, *Isogeometric analysis using LR B-splines*, Computer Methods in Applied Mechanics and Engineering **269**, 471–514 (2014).

[5] M. Kumar, T. Kvamsdal, and K.A. Johannessen, *Superconvergent patch recovery and a posteriori error estimation technique in adaptive isogeometric analysis*, Computer Methods in Applied Mechanics and Engineering **316**, 1086–1156 (2017).

[6] L.B. Wahlbin, *Superconvergence in Galerkin finite element methods*, Lecture Notes in Mathematics **1605**, Springer-Verlag, Berlin (1995).

Adaptive space-time isogeometric analysis of parabolic equations

Ulrich Langer

(joint work with Svetlana Kyas and Sergey Repin)

We introduce and investigate new locally stabilized space-time Isogeometric Analysis (IgA) approximations to initial-boundary value parabolic problems of the form

$$\partial_t u - \operatorname{div}_x(\nu \nabla_x u) = f \text{ in } Q, \quad u = 0 \text{ on } \Sigma, \quad u = 0 \text{ on } \Sigma_0, \tag{1}$$

where $Q := \Omega \times (0,T)$, $\Sigma := \partial\Omega \times (0,T)$, and $\Sigma_0 := \Omega \times \{0\}$. We assume that $f \in L_2(Q)$ and that $\nu = \nu(x,t)$ (non-autonomous case) is uniformly positive and bounded almost everywhere in Q. Under weak additional conditions imposed on ν, one can show maximal parabolic regularity (MPR) [1], i.e., $\partial_t u \in L_2(Q)$ and $Lu := -\operatorname{div}_x(\nu \nabla_x u) \in L_2(Q)$. Therefore, the parabolic PDE (1) makes sense in $L_2(Q)$. Previously, similar schemes (but weighted with a global mesh parameter h) were presented and studied in [5]. The current work devises a locally stabilized version of this scheme, which is suited for adaptive mesh refinement based on a posteriori error estimators. Such locally stabilized versions are obtained by multiplying the PDE (1) in $K \in \mathcal{K}_h = \Phi(\hat{\mathcal{K}}_h)$ with a locally scaled time-upwind test function $v_h + \theta_K h_K \partial_t v_h$, where Φ maps the parameter domain $\hat{Q} = (0,1)^{d+1}$ to the physical space-time cylinder $Q = \Phi\hat{Q}$, and the mesh $\hat{\mathcal{K}}_h$ of $\hat{Q}$ to the mesh $\mathcal{K}_h$ of Q. After integration over K, summation over all $K \in \mathcal{K}_h$, and integration by parts, we obtain the locally stabilized space-time IgA scheme: find u_h from the finite-dimensional IgA space V_{0h} such that

$$a_h(u_h, v_h) = \ell_h(v_h) \quad \forall v_h \in V_{0h}, \tag{2}$$

where the bilinear form is given by $a_h(u_h,v_h) := \sum_{K\in\mathcal{K}_h} \int_K (\partial_t u_h v_h + \nu \nabla_x u_h \cdot \nabla_x v_h) d(x,t) + \sum_{K\in\mathcal{K}_h} \theta_K h_K \int_K (\partial_t u_h - \operatorname{div}_x(\nu \nabla_x u_h)) \partial_t v_h d(x,t)$, whereas the linear form is defined by $l_h(v_h) := \sum_{K\in\mathcal{K}_h} \int_K f(v_h + \theta_K h_K \partial_t v_h) d(x,t)$. The IgA scheme (2) is obviously consistent, i.e., $a_h(u,v_h) = \ell_h(v_h)$ for all $v_h \in V_{0h}$ at the solution u of (1). Consistency yields Galerkin orthogonality $a_h(u-u_h,v_h) = \ell_h(v_h)$. Galerkin orthogonality easily implies the best-approximation estimate

$$\|u-u_h\|_h \le \Big(1+\frac{\mu_b}{\mu_c}\Big) \inf_{v_h \in V_{0h}} \|u-v_h\|_{h,*}. \tag{3}$$

provided that $a_h(v_h,v_h) \ge \mu_c \|v_h\|_h$ (V_{0h}-coercivity) and $a_h(v,v_h) \le \mu_b \|v\|_{h,*}\|v_h\|_h$ ($V_{h,*} \times V_{0h}$-boundedness) for all $v_h \in V_{0h}$ and $v \in V_{h,*} := V_{0h} + H^{L,1}_{0,\underline{0}}(Q)$, where $H^{L,1}_{0,\underline{0}}(Q)$ is the Sobolev space to which the solution u belongs in the MPR setting. V_{0h}-coercivity and $V_{h,*} \times V_{0h}$-boundedness of the bilinear form $a_h(\cdot,\cdot)$ of the IgA scheme (2) can be shown for $\theta_K = O(h_K)$. We refer to [4] for the precise definition of θ_K and the norms. From (3), one can easily derive optimal convergence rate estimates for solutions u from $H^{1+s}(Q)$ for some $s > 0$. The best-approximation estimate (3) can be generalized to distributional right-hand sides of the form $f + \mathrm{div}_x(\mathbf{f})$, where the vector-function $\mathbf{f}$ belongs to $H(\mathrm{div}_x, Q_i)$ for some non-overlapping decomposition of Q into subdomains Q_i. This setting is typical for 2d eddy-current problems such as electrical machines, where $\mathbf{f}$ models permanent magnets, f describes the current impression, and ν is the reluctivity.

The local error indicators used for mesh refinement in our adaptive space-time IgA scheme are based on Repin's functional a posteriori error estimates (2002), see also monograph [6]. The simplest one has the form

$$\|e\|^2 \le (1+\beta)\|\mathbf{y}+\mathbf{f}-\nu\nabla_x u_h\|^2_{L_2(Q)} + (1+\frac{1}{\beta})C^2\|f+\mathrm{div}_x(\mathbf{y})-\partial_t u_h\|^2_{L_2(Q)}, \tag{4}$$

where $\|e\|^2 := \|\nabla_x e\|^2_{L_2(Q)} + \|e\|^2_{L_2(\Sigma_T)}$, $e = u - u_h$ denotes the discretization error, $\Sigma_T := \Omega \times \{T\}$, β is a positive scaling parameter, C denotes the Friedrichs constant of Ω, and $\mathbf{y}$ is the reconstructed flux. In the single-patch case, one can use $\nabla_x u_h$ as $\mathbf{y}$ provided that u_h is C^1 smooth. In order to obtain good efficiency indices close to 1, we minimize the quadratic functional on the right-hand side of (4) on a coarser spline space with a higher polynomial degree and the highest smoothness exploiting the superior approximation properties of these splines. The numerical results presented in [4] confirm the efficiency of this approach for several benchmark examples with different features in terms of the efficiency index and the computational overhead for the flux reconstruction.

Time-parallel and space-time methods, in particular, space-time finite element methods have a long history. In particular, we mention [3] where a Galerkin/Least-Squares stabilized time-slice finite element method was proposed and analyzed for parabolic and other problems. The revival of space-time methods is certainly connected with the availability of massively parallel computers with thousands or even millions of cores. We refer the interested reader to the survey articles [2] and [7] for a comprehensive review of time-parallel and space-time methods, respectively.

References

[1] D. Dier, *Non-autonomous maximal regularity for forms of bounded variation*, Journal of Mathematical Analysis and Applications **425**, 33–54 (2015).
[2] M. Gander, *50 years of time parallel time integration*, In: T. Carraro, M. Geiger, S. Körkel, and R. Rannacher (eds.), Multiple Shooting and Time Domain Decomposition, pp. 69–114, Springer-Verlag, Heidelberg (2015).
[3] T.J.R. Hughes, L.P. Franca, and G.M. Hulbert, *A new finite element formulation for computational fluid dynamics: VIII. The Galerkin/least-squares method for advection-diffusive equations*, Computer Methods in Applied Mechanics and Engineering **73**, 173–189 (1989).
[4] U. Langer, S. Matculevich, and S. Repin, *Adaptive space-time Isogeometric Analysis for parabolic evolution problems*, In: U. Langer and O. Steinbach (eds.) Space-Time Methods: Applications to Partial Differential Equations, Radon Series on Computational and Applied Mathematics **25**, pp. 141–183, de Gruyter, Berlin (2019).
[5] U. Langer, S. Moore, and M. Neumüller, *Space-time finite element methods for parabolic of evolution equations*, Computer Methods in Applied Mechanics and Engineering **306**, 342–336 (2016).
[6] S. Repin, *A posteriori estimates for partial differential equations*, Radon Series on Computational and Applied Mathematics **4**, de Gruyter, Berlin (2008).
[7] O. Steinbach, and H. Yang, *Space-time finite element methods for parabolic evolution equations: Discretization, a posteriori error estimation, adaptivity and solution*, In: U. Langer and O. Steinbach (eds.) Space-Time Methods: Applications to Partial Differential Equations, Radon Series on Computational and Applied Mathematics **25**, pp. 207–248, de Gruyter, Berlin (2019).

Interesting splits

Tom Lyche
(joint work with Carla Manni and Hendrik Speleers)

Splines on triangulations have widespread applications in many areas ranging from finite element analysis and physics/engineering applications to computer graphics and entertainment industry.

For many of these applications, C^0 piecewise linear surfaces do not offer sufficient smoothness. However, obtaining spline surfaces of higher smoothness on arbitrary triangulations may require assumptions on the polynomial degree. Low degrees, which are appealing from a practical point of view, are very problematic in this context. For example, it is known that the space of C^1 quadratic splines has no stable dimension, meaning that the dimension can depend on the exact geometry of the triangulation and not only on combinatoric and topological quantities as number of vertices, edges, etc. The dimension of C^1 cubic splines is still an open problem for general triangulations [5].

To obtain a local and stable construction of the elements of a spline space with global C^1, C^2 or C^3 smoothness over an arbitrary triangulation, one must use polynomials of degree 5, 9, 13 respectively on each triangle. An alternative is to use lower-degree macro-elements, that is, by subdividing each triangle into a number of subtriangles. Popular splits are the Powell-Sabin 6 (or 12) split that subdivides each triangle in 6 (or 12) subtriangles, and the Clough-Tocher split where each triangle is subdivided in 3 subtriangles. The minimum degree to get C^2 smoothness is 5 on the Powell-Sabin splits, while piecewise polynomials of degree 6 are needed to achieve C^2 smoothness on the Clough-Tocher split. Of course, C^2 cubic splines are very appealing because they couple the lowest possible degree with a smoothness that is sufficient to efficiently address many problems in practice. However, to obtain such a low degree, one must rely on more complicated macro-structures [3, 7].

Locally supported basis functions are necessary both for efficient computation and for optimal approximation power of the space. Simplex splines are one of the most elegant generalizations of univariate B-splines to the multivariate setting. They can be interpreted as the density function of a simplex shadow. For the univariate setting this nice interpretation dates back to Schoenberg in the middle sixties and has been later used by several authors to define multivariate B-splines and investigate their properties. In particular, the beautiful geometric construction of simplex splines allows to immediately see or to derive in a straightforward manner, properties such as smoothness and recursion, the knot insertion and degree elevation formulas [1, 2, 4].

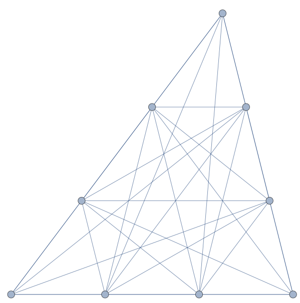


FIGURE 1. C^2 cubic split.

In this talk we present the construction of a simplex-spline basis for a C^2 cubic spline space that can be defined on any given triangulation suitably refined [6]; see Figure 1. This space ensures full approximation power and any element of the space admits a local construction. The provided simplex-spline basis possesses many important properties we wish for when dealing with both geometric modelling and approximation. More precisely, our basis enjoys the following properties:

- nonnegative partition of unity and minimal support,
- computational efficiency via stable recurrence relation and differentiation formula,
- representation in terms of a geometrically meaningful control polygon,
- meaningful geometric interpretation of smoothness conditions between adjacent triangles,

- explicit dual functionals giving representation of polynomials via a Marsden like identity,
- well-conditioned collocation matrices for Lagrange and Hermite interpolation using certain sites.

The constructed simplex basis allows us to globally represent in each subpatch the elements of the considered C^2 cubic spline space without taking care of the complicate geometry of the refinement illustrated in Figure 1.

References

[1] E. Cohen, T. Lyche, and R.F. Riesenfeld, *A B-spline-like basis for the Powell-Sabin 12-split based on simplex splines*, Matematics of Computation **82**, 1667–1707 (2013).
[2] C.A. Micchelli, *On a numerically efficient method for computing multivariate B-splines*, In: W. Schempp and K. Zeller (eds.) Multivariate Approximation Theory, pp. 211–248, Birkhauser Verlag, Basel (1979).
[3] M.-J. Lai and L.L. Schumaker, *Spline Functions on Triangulations*, Cambridge University Press (2007).
[4] H. Prautzsch, W. Boehm, and M. Paluszny, *Bézier and B-spline Techniques*, Springer (2002).
[5] H. Schenck, *Algebraic methods in approximation theory*, Computer Aided Geometric Design **45**, 14–31 (2016).
[6] R.-H. Wang and X.-Q. Shi, $S^{\mu}_{\mu+1}$*surface interpolations over triangulations*, In: A.G. Law and C.L. Wang (eds.) Approximation, Optimization and Computing: Theory and Applications, pp. 205–208, Elsevier Science Publishers B.V., North-Holland (1990).
[7] A. Zenisek, *A general theorem on triangular finite* C^m*-elements*, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge **8**, no. R-2 (1974).

Isogeometric design-through-analysis of self-supporting structures

Angelos Mantzaflaris
(joint work with Ping Hu, Bert Jüttler, Hao Pan, Wenping Wang, and Yang Xia)

The design and computation of self-supporting structures is a challenging problem in contemporary architecture. In this work we propose a spline-based approach for the construction of smooth self-supporting surfaces. The equilibrium state of the surface is expressed in terms of control variables, using membrane shell theory. Airy stresses within the surface are used as tunable variables in the design pipeline. The self-supporting shapes corresponding to the stress states are calculated by a nonlinear isogeometric analysis (IGA) method. Self-supporting surfaces have been treated previously by means of thurst network analysis, which discretizes the surface into a network and assigns forces on the nodes [3, 5]. Recently in [2] a NURBS representation of masonry structures was introduced.

Three-dimensional shell structures are generally represented by their middle surfaces [6]. Based on Monge's description (Figure 1), a surface

$$S = \{(x, y, h(x, y) : (x, y) \in U)\}, \qquad (1)$$

without overhangs can be represented as a height function $h(x, y)$ over $U \subset \mathbb{R}$.

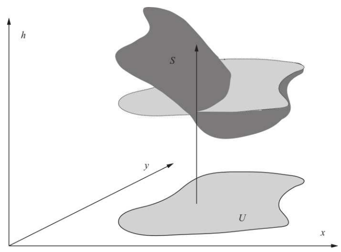


Figure 1. Monge's description of a masonry shell surface.

With IGA both U and h are described using NURBS basis functions:

$$(x,y) = G(\xi,\eta) = \sum_{i=1}^{n}\sum_{j=1}^{m}(x_{i,j},y_{i,j})R_{i,j}(\xi,\eta) \quad , \quad G([0,1]^2) = U, \tag{2}$$

$$h = \sum_{i=1}^{n}\sum_{j=1}^{m} h_{i,j}R_{i,j}(\xi,\eta) = \sum_{i=1}^{n}\sum_{j=1}^{m} h_{i,j}\,(R_{i,j}\circ G^{-1})(x,y), \tag{3}$$

where $(\xi,\eta) \in [0,1]^2$, $R_{i,j}$ are bivariate NURBS basis functions of certain polynomial degree and $(x_{i,j}, y_{i,j}, h_{i,j})$ denote the control points. The mapping (2) defines a parameterization of U over the parameter domain $[0,1]^2$.

Membrane theory is used to describe the balance conditions relating the surface geometry and the stress field on the surface which resists external loads [4, 7]. The stress per surface point, projected onto the horizontal plane, is encoded by a symmetric 2×2 stress matrix

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}, \tag{4}$$

and equilibrium can be expressed by the second-order differential equations [8]

$$\mathrm{div}(\sigma) = 0, \tag{5}$$

$$-\mathrm{div}(\sigma\nabla h) = f(h), \tag{6}$$

where $f(h)$ is the gravitational load on the surface, $f(z) = \rho g t\sqrt{1+\|\nabla z\|^2}$. Here $\rho g t$ yields the unit area load, where ρ is the density of the material, g is the gravitational acceleration, and t is the thickness of the surface.

For a simply connected domain, there exists a global function $\phi : U \to \mathbb{R}$ s.t.

$$\sigma = \begin{bmatrix} \partial_y^2\phi & -\partial_x\partial_y\phi \\ -\partial_x\partial_y\phi & \partial_x^2\phi \end{bmatrix}. \tag{7}$$

Note that the divergence free property of σ (Eq. 5) is automatically satisfied. The function $\phi(x,y)$ is called the Airy stress function [9].

We assume that the material of masonry reacts elastically to arbitrary compression but cannot bear the slightest traction. Therefore, the elasticity properties of the masonry can be characterized by the requirement that, in addition to the above equilibrium equations, the resultant stress tensor σ be negative semi-definite. Equivalently, we require that ϕ is convave [4], namely:

(8) $\det H_\phi \geq 0$ and $\operatorname{tr} H_\phi \geq 0$, where H_ϕ is the Hessian matrix of ϕ.

The problem to be solved is formulated as follows: given the planar boundary curves of a shape, find a smooth surface with a specific inner stress state, such that the self-supporting equilibrium under the given loads and boundary conditions is fulfilled. We discretize (6) using IGA and obtain the weak form

$$\int_\Omega \nabla h\, \sigma\, \nabla v \, \mathrm{d}x = \int_\Omega f(h) v \, \mathrm{d}x \quad , \qquad \forall v \in \mathcal{V}, \tag{9}$$

where $\mathcal{V}$ is the usual isogeometric spline space spanned by $R_{ij} \circ G^{-1}$, see also (3). An Airy stress function is prescribed with uniform stresses, so that all points in the structure are equally strong [8]:

$$\phi = \frac{1}{2}\sigma_{22}x^2 + \frac{1}{2}\sigma_{11}y^2 - \sigma_{12}xy \quad , \quad \sigma_{11}, \sigma_{22}, \sigma_{12} \in \mathbb{R}. \tag{10}$$

Linear terms do not affect the stress, and are therefore omitted. The concave condition of the Airy stress reads $\sigma_{11} + \sigma_{12} \leq 0$ and $\sigma_{11}\sigma_{12} - {\sigma_{22}}^2 \geq 0$, cf. (8). A self-supporting surface generally has fixed or free boundary conditions, which are implemented in IGA by enforcing Dirichlet or Neumman boundary conditions. We solve the boundary value problem for the unknown height function using a Newton iterative method.

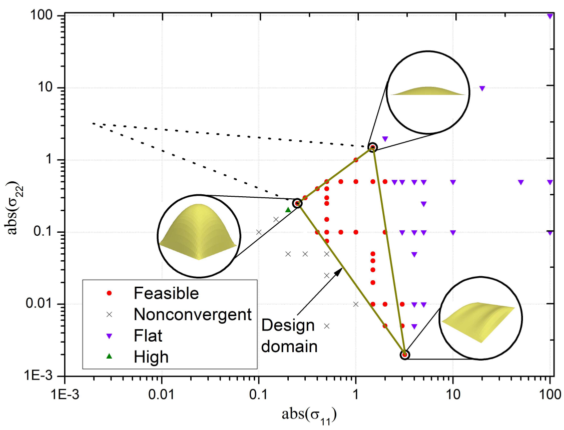


FIGURE 2. The feasible design domain of Airy stresses.

We explore the effect of uniform Airy stress functions which have zero shear stress ($\sigma_{12} = \sigma_{21} = 0$). We set $\sigma_{11} > \sigma_{22}$, since models with $\sigma_{11} < \sigma_{22}$ can be obtained by rotation. Our results show that the feasible design domain of Airy stresses is located within a triangle, see Figure 2. When both stresses are tiny, which means the structure is in a loose state and there is not enough compression force to hold the pieces together, Newton's iteration will not converge. When stress increases, the height of the obtained shape generally decreases. This is due to the fact that equilibrium with smaller plane stresses can be obtained with steeper shapes, until reaching the extreme case when walls are built straight up with blocks. In the opposite direction, when inner stress increases, the height of the obtained shape decreases, until it eventually becomes planar. For further experiments and details, the interested reader is kindly referred to [1].

References

[1] Y. Xia, A. Mantzaflaris, B. Jüttler, H. Pan, P. Hu, and W. Wang, *Design of self-supporting surfaces with isogeometric analysis*, Computer Methods in Applied Mechanics and Engineering **353**, 328–347 (2019).

[2] M. Miki, T. Igarashi, and P. Block, *Parametric self-supporting surfaces via direct computation of airy stress functions*, ACM Transactions on Graphics **34**(4), art. 89 (2015).

[3] D. Panozzo, P. Block, and O. Sorkine-Hornung, *Designing unreinforced masonry models*, ACM Transactions on Graphics **32**(4), art. 91 (2013).

[4] M. Angelillo, E. Babilio, and A. Fortunato, *Singular stress fields for masonry-like vaults*, Continuum Mechanics and Thermodynamics **25**(2-4), 423–441 (2013).

[5] E. Vouga, M. Hobinger, J. Wallner, and H. Pottmann, *Design of self-supporting surfaces*, ACM Transactions on Graphics **31**(4), art. 87 (2012).

[6] V.S. Kelkar and R.T. Sewell, *Fundamentals of the Analysis and Design of Shell Structures*, Prentice Hall (1987).

[7] M. Giaquinta and E. Giusti, *Researches on the equilibrium of masonry structures*, Archive for Rational Mechanics and Analysis **88**(4), 359–392 (1985).

[8] J.J. Connor, J.P. Wolf, and R.M. Miller, *Automatic solution of Pucher's Equation*, MIT (1966).

[9] S. Timoshenko, J.N. Goodier, *Theory of Elasticity*, McGraw-Hill, New York (1951).

Geometrically smooth splines, dimensions, bases, projections

Bernard Mourrain

(joint work with Ahmed Blidia, Nelly Villamizar, and Gang Xu)

We analyse the space of geometrically continuous piecewise polynomial functions, or G^1 splines, on general topological surfaces. Our construction is based on transition maps or gluing data attached to the edges shared by faces. We present compatibility conditions on the transition maps which yield interesting G^1 spline spaces. We provide dimension formula for the space of G^1 spline functions on topological surfaces with quadrangular pieces and polynomial or b-spline patches, under a separability condition. We show that the separability condition allows to deduce bases from the analysis of the G^1 conditions along an edge. An explicit and efficient algorithm to construct such bases is presented, in which basis functions attached to vertices, edges and faces are defined.

We illustrate the basis construction in different projection problems. The first illustration is a new and explicit scheme for the construction of geometrically smooth spline surfaces from a coarse quadrangular mesh. The resulting surface is G^1 everywhere and C^2 except around extraordinary vertices. Each face of the quadrangular mesh is associated to a bi-quintic spline patch. An approximate Catmull-Clark subdivision scheme is used to compute the control points of b-spline patches associated to the faces of the quadrangular mesh, which are then projected on the space of G^1 splines.

The second illustration is a fitting problem for the reconstruction of smooth surfaces of arbitrary topology from a point cloud. A basis of a G^1 spline space is used in the construction of the G^1 parametric surface, which is the closest to a set of points.

The third illustration is an application to Isogeometric Analysis. The G^1 spline basis functions are used to approximate the solution of an elliptic equation with boundary conditions. We present numerical results for different levels of refinement of an initial mesh, using Catmull-Clark subdivision and the G^1 basis construction.

References

[1] A. Blidia, B. Mourrain, and N. Villamizar, *G^1-smooth splines on quad meshes with 4-split macro-patch elements*, Computer Aided Geometric Design **52-53**, 106–125 (2017).

[2] A. Blidia, B. Mourrain, and G. Xu, *Geometrically smooth spline bases for data fitting and simulation*, Preprint (2019).

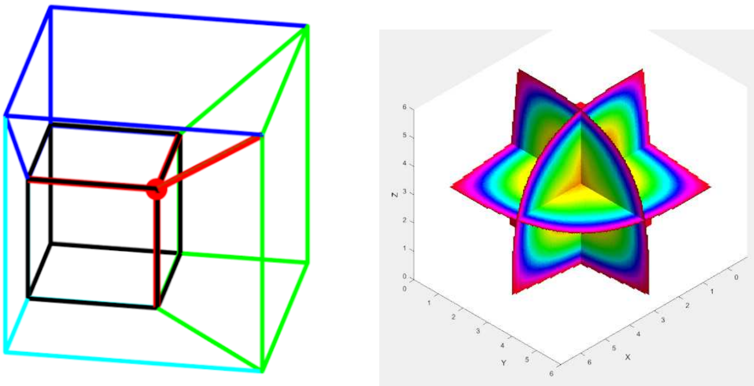
Refinable tri-variate C^1 splines for box-complexes including irregular points and irregular edges

Jörg Peters

The grid points of a regular partition of 3-space into boxes can be interpreted as the control points of a tri-variate tensor-product spline with one polynomial piece per cube. The theory of such splines is well-understood [3, 4]. By contrast, for box-complexes where the tensor-grid gives way to an irregular arrangement of boxes including irregular points and irregular edges, there is to date no simple prescription to join the corresponding polynomial pieces with more than C^0 continuity. Efficiently modeling C^1 fields over general box-complexes is of interest in areas ranging from scientific data visualization to solving higher-order differential equations. For example, to visualize a flow computed by the Discontinuous Galerkin approach currently requires substantial post-processing to extract stream lines that the theory predicts to be smooth [17].

Already in two variables – where the box-complex is a quad mesh and the only irregularities are points where more or fewer than four quadrilaterals meet – associating one or more bi-cubic polynomial pieces with each quad and joining them to form a C^1 space is far from trivial. The main options developed over the past 30 years are: geometric continuity (change of variables between pieces) [7, 13, 9, 8], generalized subdivision [6, 15] or introducing removable singularities into the parameterization [14, 11, 16, 12]. Each option encounters challenges. Geometric continuity requires increased polynomial degree near irregularities and careful book-keeping to adjust reparameterizations under refinement. Subdivision creates an infinite sequence of nested piecewise polynomial rings that complicate engineering analysis, e.g. integration, near irregularities. Singular corner parameterizations are often shape deficient and must ensure that the singularity is locally removable so the resulting space can be certified C^1 despite the vanishing of partial derivatives at irregularities.

In three variables, there are few C^1 constructions for box-complexes with irregularities. The impressive work of the meshing community to generate unstructured hex-meshes and the improved understanding of fields and their singularity graph is not met with corresponding progress in more flexible spline representations. Trivariate subdivision rules analogous to Catmull-Clark subdivision [6] have been proposed in [10] and used in engineering applications in [5] but come without guarantee of smoothness and approximation order. Geometric continuity in three variables, although well-understood in principle, is in practice barely explored: [1] analyze one pair of face-adjacent boxes.



(a) box-complex with irregular point ($n = 4$) and irregular edges ($n_e = 3$)

(b) solution of $\Delta(u \circ \mathbf{x}) = 1$

FIGURE 1. Modeling and computing with refinable tri-variate C^1 splines for box-complexes including (a) four irregular edges of valence 3 and one irregular vertex of valence 4. (b) The four domains map to curved boxes partitioning an octant of a ball and Poisson's equation is solved on the octant, by collocation.

This paper introduces a trivariate C^1 space with singular parameterization, the third option. Wherever possible, the vertices of the box-complex are interpreted as B-spline coefficients [3, 4]. Then, at each irregularity, a well-behaved linear function is determined and composed with a local singular expansion $\check{\mathbf{x}}$ that is specially crafted to be consistent with the local layout of the box-complex and based on the intersection of edge-dual planes within each box. All first derivatives of the expansion $\check{\mathbf{x}}$ are continuous, albeit zero across irregularities. Apart from the irregularities, its Jacobian is positive definite. This is the key ingredient to show that the inverse $\breve{\mathbf{x}}^{-1}$ is well defined and the local expansion of the linear function composed with $\breve{\mathbf{x}}$ can be evaluated at $\breve{\mathbf{x}}^{-1}$, removing the singularity. The polynomial pieces of the spline space therefore join not just nominally C^1, but smoothly over the whole box-complex. The spline space has a basis of $2 \times 2 \times 2$ independent functions per hexahedral input box (one per sub-box after a dyadic split in each dimension), can reproduce linear functions and is nestedly refinable.

Figure 1a shows a box complex with irregular point ($n = 4$) and four irregular edges ($n_e = 3$). The corresponding piecewise tri-cubic map is smooth across the irregularities and parameterizes an octant of a ball. To test the construction as physical domain, the Poisson equation is solved on the octant, with zero boundary conditions. Figure 1b shows slices colored by the resulting scalar field satisfying the Poisson equation in the sense of collocation.

References

[1] K. Birner, B. Jüttler, and A. Mantzaflaris, *Bases and dimensions of C^1-smooth isogeometric splines on volumetric two-patch domains*, Graphical Models **99**, 46–56 (2018).

[2] H. Bohl and U. Reif, *Degenerate Bézier patches with continuous curvature*, Computer Aided Geometric Design **14**, 749–761 (1997).

[3] C. de Boor, *A Practical Guide to Splines*, Springer (1978).

[4] C. de Boor, *B-form basics*, In: G. Farin (Ed.) Geometric Modeling: Algorithms and New Trends, pp. 131–148, SIAM (1987).

[5] D. Burkhart, B. Hamann, and G. Umlauf, *Iso-geometric finite element analysis based on Catmull-Clark subdivision solids*, Computer Graphics Forum **29**, 1575–1584 (2010).

[6] E. Catmull and J. Clark, *Recursively generated B-spline surfaces on arbitrary topological meshes*, Computer-Aided Design **10**, 350–355 (1978).

[7] T. DeRose, *Necessary and sufficient conditions for tangent plane continuity of Bézier surfaces*, Computer Aided Geometric Design **7**, 165–180 (1990).

[8] K. Karčiauskas and J. Peters, *Minimal bi-6 G^2 completion of bicubic spline surfaces*, Computer Aided Geometric Design **41**, 10–22 (2016).

[9] C.T. Loop and S. Schaefer, *G^2 tensor product splines over extraordinary vertices*, Computer Graphics Forum **27**, 1373–1382 (2008).

[10] R. MacCracken and K.I. Joy, *Free-form deformations with lattices of arbitrary topology*, In: Proceedings of the ACM, Conference on Computer Graphics, pp. 181–188, ACM (1996).

[11] M. Neamtu and P.R. Pfluger, *Degenerate polynomial patches of degree 4 and 5 used for geometrically smooth interpolation in 3*, Computer Aided Geometric Design **11**, 451–474 (1994).

[12] T. Nguyen and J. Peters, *Refinable C^1 spline elements for irregular quad layout*, Computer Aided Geometric Design **43**, 123–130 (2016).

[13] J. Peters, *Geometric continuity*, In: Handbook of Computer Aided Geometric Design, pp. 193–229, Elsevier (2002).

[14] J. Peters, *Parametrizing singularly to enclose vertices by a smooth parametric surface*, In: S. MacKay, E.M. Kidd (eds.) Graphics Interface '91, Calgary, Alberta, 3–7 June 1991: proceedings, pp. 1–7, Canadian Information Processing Society (1991).

[15] J. Peters and U. Reif, *Subdivision Surfaces*, Geometry and Computing **3**, Springer-Verlag (2008).

[16] U. Reif, *A refinable space of smooth spline surfaces of arbitrary topological genus*, Journal of Approximation Theory **90**, 174–199 (1997).

[17] D. Walfisch, J.K. Ryan, R.M. Kirby, and R. Haimes, *One-sided smoothness-increasing accuracy-conserving filtering for enhanced streamline integration through discontinuous fields*, Journal of Scientific Computing **38**, 164–184 (2009).

Multi-mesh isogeometric analysis with minimal stabilization

Riccardo Puppi and Xiaodong Wei

Boolean operations represent one of the most fundamental tool in CAD environments, where complex geometries are built using boolean operations. At the same time this construction constitutes a source of difficulty in the interplay between the geometry and the numerical analysis of PDEs. When parts of the physical object are cut away or being overlapped, its visualization changes, while its mathematical description does not. It turns out that we have to deal with elements unfitted with the boundary which cause instability of the discrete problem.

We present a novel method for isogeometric analysis (IGA) to directly work on geometries constructed by Boolean operations. Particularly, this work focuses on the union and intersection (or trimming) operations, which involve multiple independent, generally non-conforming and trimmed spline patches.

A minimal stabilization technique based on a modification of the variational formulation is presented in the context of the Poisson problem in order to recover the well-posedness of the discrete formulation of the problem. A simple preconditioning coupled with our stabilization technique is proposed to solve the issue of the ill-conditioning of the stiffness matrix.

Moreover, we show in theory that our proposed method recovers optimal error estimates. In the end, we numerically verify the theory by solving the Poisson problem in various geometries that are obtained by union and trimming operations.

References

[1] P. Antolín, A. Buffa, and M. Martinelli, *Isogeometric Analysis on V-reps: first results*, arXiv preprint (2019).

[2] A. Buffa, R. Puppi, and R. Vázquez, *A minimal stabilization procedure for isogeometric methods on trimmed geometries*, arXiv preprint (2019).

[3] A. Johansson, B. Kehlet, M.G. Larson, and A. Logg, *Multimesh finite element methods: Solving PDEs on multiple intersecting meshes*, Computer Methods in Applied Mechanics and Engineering **343**, 672–689 (2019).

[4] B. Marussig and T.J.R. Hughes, *A review of trimming in isogeometric analysis: challenges, data exchange and simulation aspects*, Archives of Computational Methods in Engineering **25**(4), 1059-1127 (2018).

Advanced isogeometric modeling and applications with a focus on shells and laminates

Alessandro Reali

(joint work with Alessia Patton, John-Eric Dufour, Pablo Antolín, Josef Kiendl, Giancarlo Sangalli, and Ferdinando Auricchio)

Isogeometric Analysis (IGA) is a recent simulation framework originally proposed by T.J.R. Hughes and coworkers [1] with the aim of bridging the gap between Computational Mechanics and Computer Aided Design (CAD) towards a cost-saving simplification of the typically expensive mesh generation and refinement processes required by standard finite element analysis. Thanks to the high-regularity properties of its basis functions, IGA has shown a better accuracy per-degree-of-freedom and an enhanced robustness with respect to standard finite elements in a number of applications ranging from solids and structures to fluids and fluid-structure interaction, opening also the door to geometrically flexible discretizations of higher-order partial differential equations in primal form, as well as to highly efficient (strong-form) collocation methods [2]. In particular, this higher regularity gave "new life" to shell modeling and applications, making it possible to easily and efficiently implement (rotation-free) Kirchhoff-Love shells (see, e.g., [3] and references therein). Within this context, this lecture mainly focuses on some recent advances on modeling and applications of shell structures allowed by the unique IGA features (see, e.g., [4]), with special attention to the accurate and inexpensive simulation technique for laminates proposed in [5]. In particular, this approach takes advantage of the favorable properties of IGA discretizations to efficiently simulate the behavior of laminated structures comprising a large number of layers using only a single element through the thickness and a post-processing technique able to recover an accurate out-of-plane stress state by direct integration of the equilibrium equations in strong form (cf., e.g., Figure 1). The results of the recent and successful extension of this approach to IGA collocation [6] are also shown, as well as convincing preliminary results for the case of Kirchhoff plates and curved structures.

References

[1] T.J.R. Hughes, J.A. Cottrell, and Y. Bazilevs, *Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement*, Computer Methods in Applied Mechanics and Engineering **194**, 4135–4195 (2005).

[2] J.A. Cottrell, T.J.R. Hughes, and Y. Bazilevs, *Isogeometric Analysis: Toward integration of CAD and FEA*, Wiley (2009).

[3] J. Kiendl, M.-C. Hsu, M.C.H. Wu, and A. Reali, *Isogeometric Kirchhoff-Love shell formulations for general hyperelastic materials*, Computer Methods in Applied Mechanics and Engineering **291**, 280–303 (2015).

[4] F. Xu, S. Morganti, R. Zakerzadeh, D. Kamensky, F. Auricchio, A. Reali, T.J.R. Hughes, M.S. Sacks, and M.-C. Hsu, *A framework for designing patient-specific bioprosthetic heart valves using immersogeometric fluid-structure interaction analysis*, International Journal for Numerical Methods in Biomedical Engineering **34**, e2938 (2018).

[5] J.-E. Dufour, P. Antolin, G. Sangalli, F. Auricchio, and A. Reali, *A cost-effective isogeometric approach for composite plates based on a stress recovery procedure*, Composites Part B: Engineering **138**, 12–18 (2018).

[6] A. Patton, J.-E. Dufour, P. Antolin, and A. Reali, *Fast and accurate elastic analysis of laminated composite plates via isogeometric collocation and an equilibrium-based stress recovery approach*, Composite Structures **225**, 111026 (2019).

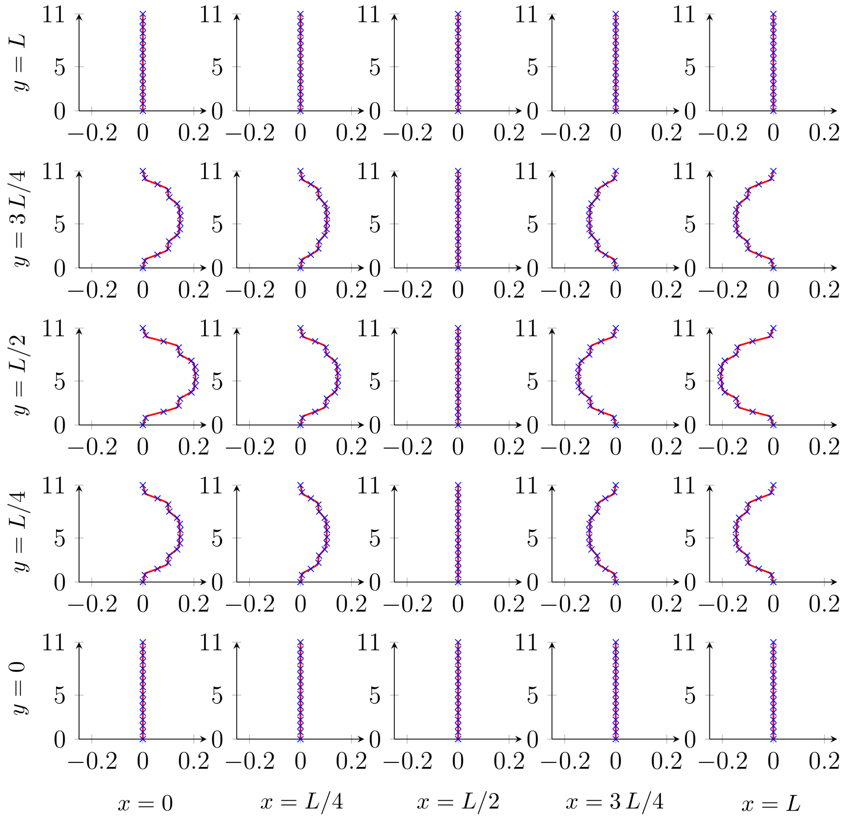


Figure 1. Recovered (red solid line) σ_{13} stress component compared to the analytical one (blue crosses) for the famous Pagano example, at several in-plane positions. L is the total length of the plate, that in this case is L = 110 mm (being L = S t with t = 11 mm and S = 10), while the number of layers is 11.

n-Widths and error estimates for k-refinement

Espen Sande

(joint work with Andrea Bressan, Michael Floater, Carla Manni, and Hendrik Speleers)

Splines are piecewise polynomial functions that are glued together in a certain smooth way. When using them in an approximation method, the availability of sharp error estimates is of utmost importance. Classical error estimates in Sobolev (semi-)norms for spline approximation are expressed in terms of

(a) a certain power of the maximal grid spacing (this is the approximation power),
(b) an appropriate derivative of the function to be approximated, and
(c) a "constant" which is independent of the previous quantities but usually depends on the spline degree.

An explicit expression of the constant in (c) is not always available in the literature, because it is a minor issue in the most standard approximation analysis. They are mainly interested in the approximation power of spline spaces of a given degree.

These estimates are perfectly suited to study approximation under h-refinement, i.e., refining the mesh by the insertion of new knots. On the other hand, one of the most interesting features in Isogeometric Analysis (IGA) is k-refinement, which denotes degree elevation with increasing interelement smoothness (and requires the use of splines of high degree and maximal smoothness). The above mentioned error estimates are not sufficient to explain the benefits of approximation under k-refinement as long as it is not well understood how the degree of the spline affects the whole estimate, including the "constant" in (c).

In this talk we provide a priori error estimates for k-refinement on arbitrary grids with an explicit constant that is, in many cases, sharp [2]. These a priori estimates are actually good enough to cover convergence to eigenfunctions of classical differential operators. This forms a theoretical foundation for the outperformance of smooth spline discretizations of eigenvalue problems that has been numerically observed in the literature, and for optimality of geometric multigrid solvers in the isogeometric analysis context.

Moreover, we discuss how these a priori error estimates can be used to mathematically justify the benefits of spline approximation under k-refinement by isogeometric discretization methods. Specifically, by comparing the constant for spline approximation of maximal smoothness with a lower bound on the constant for continuous and discontinuous spline approximation, we show that k-refinement provides better approximation in degrees of freedom in almost all cases of practical interest [1].

The key tools to get these results are the theory of Kolmogorov L^2 n-widths and the representation of the considered Sobolev spaces in terms of integral operators.

References

[1] A. Bressan and E. Sande, *Approximation in FEM, DG and IGA: A theoretical comparison*, Numerische Mathematik (in press).

[2] E. Sande, C. Manni, and H. Speleers, *Sharp error estimates for spline approximation: Explicit constants, n-widths, and eigenfunction convergence*, Mathematical Models and Methods in Applied Sciences **29**, 1175–1205 (2019).

A solver for the isogeometric *k*-method

Giancarlo Sangalli
(joint work with Mattia Tani, Francesco Calabrò, René R. Hiemstra, Thomas J.R. Hughes, Gabriele Loli, Monica Montardini)

The concept of k-refinement, or k-method, was proposed as one of the key features of isogeometric analysis, "a new, more efficient, higher-order concept", in the seminal work [1]. The idea of using high-degree and continuity splines (or NURBS, etc.) as a basis for a new high-order method appeared very promising from the beginning, and received confirmations from the next developments. The k-refinement leads to several advantages: higher accuracy per degree-of-freedom, improved spectral accuracy, the possibility of structure-preserving smooth discretizations are the most interesting features that have been studied actively in the community. At the same time, the k-refinement brings significant challenges at the computational level: using standard finite element routines, its computational cost grows with respect to the degree, making degree raising computationally expensive. However, recent ideas allow a computationally efficient k-refinement: I present in this talk the results of [2] and more recent ones in [4]. We propose a matrix-free strategy combined with weighted quadrature, which is an ad-hoc strategy to compute the integrals of the Galerkin system. Matrix-free weighted quadrature (MF-WQ) speeds up matrix operations, and, perhaps even more important, greatly reduces memory consumption. Our strategy also requires an efficient preconditioner for the linear system iterative solver: we deal with an elliptic model problem, and adopt a preconditioner based on the Fast Diagonalization method, an old idea to solve Sylvester-like equations. an old idea to solve Sylvester-like equations. Our numerical tests show that the isogeometric solver based on MF-WQ is faster than standard approaches (where the main cost is the matrix formation by standard Gaussian quadrature) even for low degree. But the main achievement is that, with MF-WQ, the k-method gets orders of magnitude faster by increasing the degree, given a target accuracy. Therefore, we are able to show the superiority, in terms of computational efficiency, of the high-degree k-method with respect to low-degree isogeometric discretizations. What we present here is applicable to more complex and realistic differential problems, but its effectiveness will depend on the preconditioner stage, which is as always problem-dependent. This situation is typical of modern high-order methods: the overall performance is mainly related to the quality of the preconditioner.

References

[1] T.J.R. Hughes, J.A. Cottrell, and Y. Bazilevs, *Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement*, Computer Methods in Applied Mechanics and Engineering **194**, 4135–4195 (2005).

[2] G. Sangalli and M. Tani, *Matrix-free isogeometric analysis: the computationally efficient k-method*, arXiv preprint arXiv:1712.08565 (2017).

[3] F. Calabrò, G. Sangalli, and M. Tani, *Fast formation of isogeometric Galerkin matrices by weighted quadrature*, Computer Methods in Applied Mechanics and Engineering **316**, 606–622 (2017).

[4] R.R. Hiemstra, G. Sangalli, M. Tani, F. Calabrò, and T.J.R. Hughes, *Fast formation and assembly of finite element matrices with application to isogeometric linear elasticity*, Computer Methods in Applied Mechanics and Engineering **355**, 234–260 (2019).

Approximated infinite dimensional operators and their spectral analysis: what the GLT analysis can say

Stefano Serra-Capizzano

(joint work with Carlo Garoni, Marco Donatelli, Sven-Erik Ekström, Carla Manni, and Hendrik Speleers)

In the last fifteen years, the class of Generalized Locally Toeplitz (GLT) sequences [13, 14] has been introduced [17, 18] as a generalization both of classical Toeplitz sequences and of variable coefficient differential operators and, for every sequence of the class, it has been demonstrated that it is possible to give a rigorous description of the asymptotic spectrum [2, 20] in terms of a function (the symbol) that can be easily identified; see also [19].

This generalizes the notion of a symbol for differential operators (discrete and continuous) or for Toeplitz sequences for which it is identified through the Fourier coefficients and is related to the classical Fourier Analysis.

The GLT class has nice algebraic properties and indeed it has been proven that it is stable under linear combinations, products, and inversion when the sequence which is inverted shows a sparsely vanishing symbol (sparsely vanishing symbol = a symbol which vanishes at most in a set of zero Lebesgue measure). Furthermore, the GLT class virtually includes any approximation of partial differential equations (PDEs) and fractional differential equations (FDEs) by local methods (Finite Difference, Finite Element, Isogeometric Analysis, etc.) and, based on this, we demonstrate that our results on GLT sequences can be used in a PDE/FDE setting in various directions, including preconditioning, multigrid, spectral detection of branches, fast 'matrix-less' computation of eigenvalues, stability issues (see the list of references for few representative contributions). We discuss specifically the impact and the further potential of the theory with special attention to the IgA setting.

References

[1] F. Ahmad, E.S. Al-Aidarous, D. Abdullah Alrehaili, S.-E. Ekström, I. Furci, and S. Serra-Capizzano, *Are the eigenvalues of preconditioned banded symmetric Toeplitz matrices known in almost closed form?*, Numerical Algorithms **78**(3), 867–893 (2018).

[2] A. Böttcher and B. Silbermann, *Introduction to Large Truncated Toeplitz Matrices*, Springer-Verlag, New York (1999).

[3] M. Donatelli, C. Garoni, C. Manni, S. Serra-Capizzano, and H. Speleers, *Robust and optimal multi-iterative techniques for IgA Galerkin linear systems*, Computer Methods in Applied Mechanics and Engineering **284**, 230–264 (2015).

[4] M. Donatelli, C. Garoni, C. Manni, S. Serra-Capizzano, and H. Speleers, *Robust and optimal multi-iterative techniques for IgA collocation linear systems*, Computer Methods in Applied Mechanics and Engineering **284**, 1120–1146 (2015).

[5] M. Donatelli, C. Garoni, C. Manni, S. Serra-Capizzano, and H. Speleers, *Spectral analysis and spectral symbol of matrices in isogeometric collocation methods*, Mathematics of Computation **85**(300), 1639–1680 (2016).

[6] M. Donatelli, C. Garoni, C. Manni, S. Serra-Capizzano, and H. Speleers, *Symbol-based multigrid methods for Galerkin B-spline isogeometric analysis*, SIAM Journal on Numerical Analysis **55**(1), 31–62 (2017).

[7] M. Donatelli, M. Mazza, and S. Serra-Capizzano, *Spectral analysis and preconditioning for variable coefficient fractional derivative operators*, Journal of Computational Physics **307**, 262–279 (2016).

[8] A. Dorostkar, M. Neytcheva, and S. Serra-Capizzano, *Spectral analysis of coupled PDEs and of their Schur complements via Generalized Locally Toeplitz sequences in 2D*, Computer Methods in Applied Mechanics and Engineering **309**, 74–105 (2016).

[9] S.E. Ekström, C. Garoni, and S. Serra-Capizzano, *Are the eigenvalues of banded symmetric Toeplitz matrices known in close form?* Experimental Mathematics **27**(4), 478–487 (2018).

[10] C. Garoni, C. Manni, F. Pelosi, S. Serra-Capizzano, and H. Speleers, *On the spectrum of stiffness matrices arising from isogeometric analysis*, Numerische Mathematik **127**, 751–799 (2014).

[11] C. Garoni, C. Manni, S. Serra-Capizzano, D. Sesana, and H. Speleers, *Spectral analysis and spectral symbol of matrices in isogeometric Galerkin methods*, Mathematics of Computation **86**(305), 1343–1373 (2017).

[12] C. Garoni, C. Manni, S. Serra-Capizzano, D. Sesana, and H. Speleers, *Lusin theorem, GLT sequences and matrix computations: an application to the spectral analysis of PDE discretization matrices*, Journal of Mathematical Analysis and Applications **446**(1), 365–382 (2017).

[13] C. Garoni and S. Serra-Capizzano, *The theory of Generalized Locally Toeplitz sequences: theory and applications - Vol I*, Springer Monographs in Mathematics, Springer, Cham (2017).

[14] C. Garoni and S. Serra-Capizzano, *The theory of Generalized Locally Toeplitz sequences: theory and applications - Vol II*, Springer Monographs in Mathematics, Springer, Cham (2018).

[15] C. Garoni and S. Serra-Capizzano, *Generalized Locally Toeplitz sequences: a spectral analysis tool for discretized differential equations*, In: T. Lyche, C. Manni, and H. Speleers (eds.) Splines and PDEs: From Approximation Theory to Numerical Linear Algebra, Lecture Notes in Mathematics **2219**, pp. 161–236, Springer International Publishing (2018).

[16] C. Garoni, S. Serra-Capizzano, and D. Sesana, *Spectral analysis and spectral symbol of d-variate Q_p Lagrangian FEM stiffness matrices*, SIAM Journal on Matrix Analysis and Applications **36**(3), 1100–1128 (2015).

[17] S. Serra-Capizzano, *Generalized Locally Toeplitz sequences: spectral analysis and applications to discretized partial differential equations*, Linear Algebra and its Applications **366**, 371–402 (2003).

[18] S. Serra-Capizzano, *The GLT class as a generalized Fourier analysis and applications*, Linear Algebra and its Applications **419**, 180–233 (2006).

[19] P. Tilli, *Locally Toeplitz sequences: spectral properties and applications*, Linear Algebra and its Applications **278**, 91–120 (1998).

[20] E.E. Tyrtyshnikov, *A unifying approach to some old and new theorems on distribution and clustering*, Linear Algebra and its Applications **232**, 1–43 (1996).

Smooth B-spline representations on Powell-Sabin triangulations

Hendrik Speleers

Isogeometric analysis (IgA) is a simulation paradigm aiming to reduce the gap between the worlds of finite element analysis (FEA) and computer-aided design (CAD). The main idea is to use the CAD representations not only to model physical domains but also to approximate the solution of differential problems [1]. Tensor-product B-splines and non-uniform rational B-splines (NURBS) are common tools in CAD, and so they are in IgA.

Adaptive local mesh refinement is an important ingredient for obtaining efficiently an accurate solution of differential problems. In the context of classical FEA, local mesh refinement strategies are a well established procedure. Unfortunately, the tensor-product structure of NURBS spaces precludes strictly localized refinements. This motivates the interest in alternative structures for IgA that permit local refinements.

In this talk we discuss the construction of a suitable B-spline representation for smooth splines on general triangulations [3, 5]. The considered splines have smoothness r and degree $d \geq 3r - 1$, and are defined over a special refinement of the given triangulations, called Powell-Sabin refinement. In such a refinement every triangle of the triangulation is split into six subtriangles. The C^1 quadratic Powell-Sabin B-splines are the most known member in this family [2]. The B-spline construction can be geometrically interpreted as determining a set of triangles that must contain a specific set of points. The B-spline functions possess several interesting properties:

- local support,
- linear independence,
- nonnegative partition of unity.

This B-spline representation exhibits a natural definition of control points and an intuitive control structure in terms of local triangular nets. These triangular nets locally mimic the shape of the spline surface, and hence they can be used in the geometric design of smooth surfaces. On the other hand, such representation also presents interesting properties for engineering analysis. In particular, the representation allows for:

- stable evaluation and differentiation,
- efficient triangular Bézier extraction,
- optimal approximation and convergence,
- adaptive local mesh refinement.

Furthermore, we describe a general construction of quasi-interpolants based on this B-spline representation [6]. Such a quasi-interpolation operator can be defined by providing a collection of polynomials (one for each degree of freedom), which can be constructed based on local data sites through any standard polynomial approximation method. The full spline scheme inherits the same approximation order of the local approximation schemes (up to $d + 1$). Next, we discuss some strategies to reduce further the degrees of freedom while keeping optimal approximation power [4, 6]. The availability of the quasi-interpolation framework can be of help here by carefully selecting the local polynomials.

Finally, we demonstrate the applicability of Powell-Sabin B-splines, and their rational version (so-called NURPS splines), for solving differential problems. Thanks to their structure based on triangulations and their B-spline properties, they constitute a natural bridge between classical FEA and NURBS-based IgA. We illustrate the use of PS/NURPS splines in IgA with several numerical examples [7, 8, 9].

References

[1] J.A. Cottrell, T.J.R. Hughes, and Y. Bazilevs, *Isogeometric Analysis: Toward Integration of CAD and FEA*, John Wiley & Sons (2009).

[2] P. Dierckx, *On calculating normalized Powell-Sabin B-splines*, Computer Aided Geometric Design **12**, 61–78 (1997).

[3] J. Grošelj, *A normalized representation of super splines of arbitrary degree on Powell-Sabin triangulations*, BIT Numerical Mathematics **56**, 1257–1280 (2016).

[4] J. Grošelj and H. Speleers, *Three recipes for quasi-interpolation with cubic Powell-Sabin splines*, Computer Aided Geometric Design **67**, 47–70 (2018).

[5] H. Speleers, *Construction of normalized B-splines for a family of smooth spline spaces over Powell-Sabin triangulations*, Constructive Approximation **37**, 41–72 (2013).

[6] H. Speleers, *A family of smooth quasi-interpolants defined over Powell-Sabin triangulations*, Constructive Approximation **41**, 297–324 (2015).

[7] H. Speleers and C. Manni, *Optimizing domain parameterization in isogeometric analysis based on Powell-Sabin splines*, Journal of Computational and Applied Mathematics **289**, 68–86 (2015).

[8] H. Speleers, C. Manni, and F. Pelosi, *From NURBS to NURPS geometries*, Computer Methods in Applied Mechanics and Engineering **255**, 238–254 (2013).

[9] H. Speleers, C. Manni, F. Pelosi, and M.L. Sampoli, *Isogeometric analysis with Powell-Sabin splines for advection-diffusion-reaction problems*, Computer Methods in Applied Mechanics and Engineering **221-222**, 132–148 (2012).

C^1 smooth multi-patch isogeometric spaces

Thomas Takacs

(joint work with Annabelle Collin, Mario Kapl, Giancarlo Sangalli, and Pascal Weinmüller)

In isogeometric analysis globally C^1-smooth isogeometric spaces over unstructured quadrilateral meshes allow the direct solution of fourth order partial differential equations on complex geometries via their Galerkin discretization. While the design of smooth spaces is trivial for single patch geometries, it is a challenging task in the case of multi-patch or manifold geometries. In this talk we focus on multi-patch domains, which give sufficient flexibility for geometric design without exhibiting the difficulties of fully unstructured quad meshes. We mostly discuss the results developed in [6, 10, 11, 12, 13].

We consider multi-patch parametrizations that are regular everywhere, parametrically (at least) C^1 within every patch, and meet C^0 at the patch interfaces.

On such a parametrization, the C^1 condition across the patch interfaces has to be enforced separately. It is mainly based on the observation that an isogeometric function is C^1 if and only if the associated graph surface is G^1 (that is, geometrically continuous of order 1), cf. [7].

As a possible alternative, one may also consider constructions that are parametrically C^1 (almost) everywhere. In those cases the patches either become singular at extraordinary vertices or remain C^0 in small regions around extraordinary vertices. In both cases, additional geometric continuity conditions have to be satisfied (close to the EVs) in order to obtain C^1 isogeometric spaces, see [15, 18, 19, 20].

In [6] we study the reproduction properties of the C^1-smooth subspaces along an interface for arbitrary B-spline patches. From the presented results, bounds for the dimension of the C^1-smooth subspaces of arbitrary geometries can be derived. These dimension bounds are consistent with the theory developed in [4, 14, 17]. Moreover, we identify the class of analysis-suitable G^1 parametrizations. The underlying condition states that the gluing data (determining the C^1 constraints) has to be linear and the regularity r within the patches has to satisfy $r \leq p- 2$, where p is the spline degree. AS-G^1 parametrizations cover exactly those geometries which allow the design of C^1 isogeometric spline spaces with optimal approximation properties. This class includes but is not limited to (mapped) bilinear multi-patch parametrizations, see [3, 9, 14, 16].

In [13] we propose to define a suitable subspace of the full AS-G^1 multi-patch isogeometric space by enforcing C^2 at all vertices. It turns out that one can then describe the resulting subspace in terms of Argyris-like degrees of freedom: By C^2 data at the vertices, point and normal derivative data at the interfaces as well as point data inside every patch. This facilitates the definition of a basis.

If the geometry parametrization does not satisfy the AS-G^1 conditions, the developed basis construction cannot be applied. The full C^1 space yields suboptimal convergence behavior or, in the worst case, full locking of the solution. There are several possibilities to handle this issue.

In [11] we propose a reparametrization strategy. This reparametrization keeps the boundary of a given planar multi-patch domain fixed and interpolates its vertices C^1. Using this information, one can construct analysis-suitable gluing data, which can then be used to fit an AS-G^1 parametrization onto the initial non-AS-G^1 domain.

Alternatively, one may construct a C^1 smooth space using higher degree basis function locally near the interface to circumvent the suboptimal behavior. This was tested numerically in [5] and discussed in more detail in [12].

Instead of enforcing the C^1-continuity across the patch interfaces in a strong sense, the C^1-smoothness could also be achieved by coupling the neighboring patches in a weak sense. This can be done using a Nitsche-like approach, penalizing the jump of the normal derivative, or using Lagrange multipliers, see e.g. [1, 2, 8]. Alternatively, we present a new approach that is based on approximating the underlying gluing data and using that approximate gluing data to construct a set of basis functions for each interface. These basis functions are by definition C^0, but not C^1 across the interfaces. One can however ensure that the jump of the normal derivative goes to zero when refining. First numerical tests suggest that this can be done efficiently and optimal orders of convergence are observed for certain configurations.

References

[1] A. Apostolatos, M. Breitenberger, R. Wüchner, and K.-U. Bletzinger, *Domain decomposition methods and Kirchhoff-Love shell multipatch coupling in isogeometric analysis*, In: B. Jüttler and B. Simeon (eds.) Isogeometric Analysis and Applications 2014, pp. 73–101, Springer (2015).

[2] A. Benvenuti, *Isogeometric Analysis for C^1-continuous Mortar Method*, PhD Thesis, Università degli Studi di Pavia (2017).

[3] M. Bercovier and T. Matskewich, *Smooth Bézier Surfaces over Unstructured Quadrilateral Meshes*, Lecture Notes of the Unione Matematica Italiana, Springer (2017).

[4] A. Blidia, B. Mourrain, and N. Villamizar, *G^1-smooth splines on quad meshes with 4-split macro-patch elements*, Computer Aided Geometric Design **52–53**, 106–125 (2017).

[5] C.L. Chan, C. Anitescu, and T. Rabczuk, *Isogeometric analysis with strong multipatch C^1-coupling*, Computer Aided Geometric Design **62**, 294–310 (2018).

[6] A. Collin, G. Sangalli, and T. Takacs, *Analysis-suitable G^1 multi-patch parametrizations for C^1 isogeometric spaces*, Computer Aided Geometric Design **47**, 93–113 (2016).

[7] D. Groisser and J. Peters, *Matched G^k-constructions always yield C^k-continuous isogeometric elements*, Computer Aided Geometric Design **34**, 67–72 (2015).

[8] Y. Guo and M. Ruess, *Nitsche's method for a coupling of isogeometric thin shells and blended shell structures*, Computer Methods in Applied Mechanics and Engineering **284**, 881–905 (2015).

[9] M. Kapl, F. Buchegger, M. Bercovier, and B. Jüttler, *Isogeometric analysis with geometrically continuous functions on planar multi-patch geometries*, Computer Methods in Applied Mechanics and Engineering **316**, 209–234 (2017).

[10] M. Kapl, G. Sangalli, and T. Takacs, *Dimension and basis construction for analysis-suitable G^1 two-patch parameterizations*, Computer Aided Geometric Design **52-53**, 75–89 (2017).

[11] M. Kapl, G. Sangalli, and T. Takacs, *Construction of analysis-suitable G^1 planar multi-patch parameterizations*, Computer-Aided Design **97**, 41–55 (2018).

[12] M. Kapl, G. Sangalli, and T. Takacs, *Isogeometric analysis with C^1 functions on unstructured quadrilateral meshes*, arXiv preprint arXiv:1812.09088 (2018).

[13] M. Kapl, G. Sangalli, and T. Takacs, *An isogeometric C^1 subspace on unstructured multi-patch planar domains*, Computer Aided Geometric Design **69**, 55–75 (2019).

[14] M. Kapl, V. Vitrih, B. Jüttler, and K. Birner, *Isogeometric analysis with geometrically continuous functions on two-patch geometries*, Computers and Mathematics with Applications **70**(7), 1518–1538 (2015).

[15] K. Karčiauskas, T. Nguyen, and J. Peters, *Generalizing bicubic splines for modeling and IGA with irregular layout*, Computer-Aided Design **70**, 23–35 (2016).

[16] T. Matskewich, *Construction of C^1 surfaces by assembly of quadrilateral patches under arbitrary mesh topology*, PhD Thesis, Hebrew University of Jerusalem (2001).

[17] B. Mourrain, R. Vidunas, and N. Villamizar, *Dimension and bases for geometrically continuous splines on surfaces of arbitrary topology*, Computer Aided Geometric Design **45**, 108–133 (2016).

[18] T. Nguyen, K. Karčiauskas, and J. Peters, *A comparative study of several classical, discrete differential and isogeometric methods for solving Poisson's equation on the disk*, Axioms **3**(2), 280–299 (2014).

[19] T. Nguyen, K. Karčiauskas, and J. Peters, *C^1 finite elements on non-tensor-product 2d and 3d manifolds*, Applied Mathematics and Computation **272**, 148–158 (2016).

[20] D. Toshniwal, H. Speleers, and T.J.R. Hughes, *Smooth cubic spline spaces on unstructured quadrilateral meshes with particular emphasis on extraordinary points: Geometric design and isogeometric analysis considerations*, Computer Methods in Applied Mechanics and Engineering **327**, 411–458 (2017).

Dimension of bi-degree splines on T-meshes

Deepesh Toshniwal

(joint work with Thomas J.R. Hughes, Bernard Mourrain, and Nelly Villamizar)

Polynomial splines on triangulations and quadrangulations have myriad applications and are ubiquitous, especially, in the fields of computer aided geometric design and computational mechanics. Meaningful use of splines for these purposes requires the construction and analysis of a suitable set of basis functions for the spline spaces. The dimension of these spaces can depend on an interplay between geometry, topology and combinatorics, and a theoretical understanding of its computation (or estimation) can be a useful tool when assessing constructive approaches. This talk discusses this problem on T-meshes.

In 1D, the problem of dimension computation is tractable and dimension formulas follow from classical arguments. The study of multivariate splines, and bivariate splines on T-meshes in particular, poses an interesting challenge as the spline space dimension can depend on the geometric embedding of the mesh, e.g., see [1]. In practice, identifying meshes where the dimension is *stable* – i.e., free from this geometric dependence – is useful for avoiding cases where spline spaces on combinatorially and topologically equivalent meshes can have different dimensions. Several techniques have been used for studying this problem. We proceed using the homology-based approach first introduced for multivariate splines by Louis Billera [2], and in the process generalize the results presented by Bernard Mourrain [3] in two directions.

The first part of this talk focuses on splines that have *mixed smoothness*, i.e., splines that are allowed to have different orders of smoothness across different mesh edges [4]. Usually, it is customary to work with splines that are C^r smooth across all mesh edges for a fixed $r \in \mathbb{Z}_{\geq -1}$; the choice of this r depends on the intended application. However, several applications also require working with splines for which smoothness can be reduced across an arbitrary subset of the mesh edges; e.g., to model non-smooth or even discontinuous geometric features. Given a spline space with stable dimension, we specify sufficient conditions that guarantee that the smoothness across a subset of the edges can be reduced without introducing any geometric dependence into the dimension formula.

In the second part of this talk, we consider splines of *mixed bi-degree*, i.e., bi-degree splines that incorporate the idea of local polynomial degree adaptivity [5, 6]. The notion of mixed bi-degree splines can be very powerful in the contexts of both geometric modeling and isogeometric analysis. In particular, the resulting flexibility would allow design of complex shapes with fewer control points, i.e., cleaner and simpler designs, while for isogeometric analysis the same would lead to more efficient analysis. We derive combinatorial lower and upper bounds on the dimension of mixed bi-degree splines and specify sufficient conditions for the bounds to coincide.

References

[1] X. Li and F. Chen, *On the instability in the dimension of splines spaces over T-meshes*, Computer Aided Geometric Design **28**(7), 420–426 (2011).

[2] L.J. Billera, *Homology of smooth splines: generic triangulations and a conjecture of Strang*, Transactions of the American Mathematical Society **310**(1), 325–340 (1988).

[3] B. Mourrain, *On the dimension of spline spaces on planar T-meshes*, Mathematics of Computation **83**(286), 847–871 (2014).

[4] D. Toshniwal and N.Y. Villamizar, *Dimension of polynomial splines of mixed smoothness on T-meshes*, in preparation (2019).

[5] D. Toshniwal and T.J.R. Hughes, *Polynomial splines of non-uniform degree on planar triangulations: Combinatorial bounds on the dimension*, Computer Aided Geometric Design, accepted (2019).

[6] D. Toshniwal, B. Mourrain, and T.J.R. Hughes, *Polynomial spline spaces of non-uniform bi-degree on T-meshes: Combinatorial bounds on the dimension*, arXiv preprint arXiv:1903.05949 (2019).

Convergence rate study using hybrid non-uniform subdivision basis functions

Yongjie Jessica Zhang

(joint work with Xin Li and Xiaodong Wei)

We present a new hybrid non-uniform subdivision surface (HNUSS) representation [1], and study its convergence rate in isogeometric analysis applications [3, 4]. We focus on defining a non-uniform subdivision scheme such that it can be used for both design (shape quality around extraordinary nodes) and analysis (with optimal convergence rate). The HNUSS is constructed through two steps. We first insert a set of edges around the parametric lines connecting to extraordinary nodes to create a dual mesh. In this way, we introduce a band of elements for each of such parametric lines and convert a valence-n extraordinary point into a valence-n face with zero knot interval assigned to all its edges. Then, we define a new subdivision scheme which combines the primal and dual subdivision rules in the subsequent levels. The main features of this new subdivision scheme include:

(1) The HNUSS generalizes bi-cubic NURBS to arbitrary topology, supporting valence-n extraordinary points;

(2) The HNUSS limit surface is proved to be G^1-continuous for any valence extraordinary points and any non-negative knot intervals. It has satisfactory geometric quality for non-uniform parameterization. The HNUSS limit surface has comparable shape quality as non-uniform subdivision via

eigen-polyhedron [2] and has better shape quality than all the other subdivision schemes. To the authors' best knowledge, this is the first paper which can prove G^1 continuity for non-uniform extraordinary points; and

(3) Numerical experiments show that HNUSS based isogeometric analysis yields improved convergence rates compared to any existing non-uniform subdivision schemes. By introducing a single parameter to control the changing rate for irregular regions, HNUSS can be analysis-suitable with optimal convergence rates achieved.

References

[1] X. Li, X. Wei, and Y.J. Zhang, *Hybrid non-uniform recursive subdivision with improved convergence rates*, Computer Methods in Applied Mechanics and Engineering **352**, 606–624 (2019).

[2] X. Li, T. Finnigin, and T. W. Sederberg, *G^1 non-uniform Catmull-Clark surfaces*, ACM Transactions on Graphics **35**, 1–8 (2016).

[3] T.J.R. Hughes, J.A. Cottrell, and Y. Bazilevs, *Isogeometric analysis: CAD, finite elements, NURBS, exact geometry, and mesh refinement*, Computer Methods in Applied Mechanics and Engineering **194**, 4135–4195 (2005).

[4] X. Wei, Y.J. Zhang, D. Toshniwal, H. Speleers, X. Li, C. Manni, J.A. Evans, and T.J. Hughes, *Blended B-spline construction on unstructured quadrilateral and hexahedral meshes with optimal convergence rates in isogeometric analysis*, Computer Methods in Applied Mechanics and Engineering **341**, 608–639 (2018).

Reporters: Carla Manni and Hendrik Speleers

Participants

Sandra Boschert
Mathematisches Institut
Universität zu Köln
Weyertal 86-90
50931 Köln
GERMANY

Prof. Dr. Andrea Bressan
Dipartimento di Matematica
Università di Pavia
Via Ferrata, 1
27100 Pavia
ITALY

Prof. Dr. Annalisa Buffa
EPFL SB MATH MNS
Bâtiment MA C2 573
Station 8
1015 Lausanne
SWITZERLAND

Prof. Dr. Gershon Elber
Department of Computer Science
TECHNION
Israel Institute of Technology
Haifa 3200003
ISRAEL

Prof. Dr. John A. Evans
Smead Aerospace Engineering
Sciences Department, Rm 361
University of Colorado at Boulder
3775 Discovery Drive
Boulder, CO 80303
UNITED STATES

Prof. Dr. Carlotta Giannelli
Dipartimento di Matematica e
Informatica
"U. Dini"
Università di Firenze
Viale Morgagni 67/A
50134 Firenze
ITALY

Helmut Harbrecht
Departement Mathematik und
Informatik
Universität Basel
Spiegelgasse 1
4051 Basel
SWITZERLAND

Prof. Dr. Thomas J.R. Hughes
Institute for Computational Engineering
and Sciences
The University of Texas at Austin
C0200
201 East 24th Street
Austin TX 78712-1229
UNITED STATES

Prof. Dr. Bert Jüttler
Institut für Angewandte Geometrie
Johannes-Kepler-Universität
Altenberger Straße 69
4040 Linz
AUSTRIA

Prof. Dr. Angela Kunoth
Departement Mathematik/Informatik
Universität zu Köln
Weyertal 86-90
50931 Köln
GERMANY

Prof. Trond Kvamsdal
Department of Mathematical Sciences
NTNU
7491 Trondheim
NORWAY

Prof. Dr. Ulrich Langer
Johann Radon Institute for Comput.
and Applied Mathematics (RICAM)
Austrian Academy of Sciences
Altenbergerstrasse 69
4040 Linz
AUSTRIA

Prof. Dr. Tom Lyche
Department of Mathematics
University of Oslo
Blindern
P.O. Box 1053
0316 Oslo
NORWAY

Prof. Dr. Carla Manni
Dipartimento di Matematica
Università degli Studi di Roma "Tor
Vergata"
Via della Ricerca Scientifica, 1
00133 Roma
ITALY

Dr. Angelos Mantzaflaris
INRIA Sophia Antipolis
Université Cote d'Azur
Byron Building
BP 93
2004 route des Lucioles
06902 Sophia-Antipolis Cedex
FRANCE

Prof. Dr. Bernard Mourrain
AROMATH
INRIA Sophia Antipolis
2004 route des Lucioles
06902 Sophia-Antipolis Cedex
FRANCE

Prof. Dr. Jörg Peters
Department C.I.S.E.
University of Florida
CSE Building
Gainesville FL 32611-6120
UNITED STATES

Riccardo Puppi
EPFL SB MATH MNS
Bâtiment MA B2 505
Station 8
1015 Lausanne
SWITZERLAND

Prof. Dr. Alessandro Reali
Department of Civil Engineering and
Architecture
Università di Pavia
Via Ferrata, 3
27100 Pavia
ITALY

Espen Sande
Dipartimento di Matematica
Università degli Studi di Roma II
"Tor Vergata"
Via della Ricerca Scientifica, 1
00133 Roma
ITALY

Prof. Dr. Giancarlo Sangalli
IMATI - CNR
Via Ferrata, 1
27100 Pavia
ITALY

Prof. Dr. Stefano Serra Capizzano
Dipartimento di Scienze
Università degli Studi dell'Insubria
Via Bossi 5
22100 Como
ITALY

Prof. Dr. Hendrik Speleers
Dipartimento di Matematica
Universita di Roma "Tor Vergata"
Via della Ricerca Scientifica, 1
00133 Roma
ITALY

Dr. Thomas Takacs
Institut für Angewandte Geometrie
Johannes-Kepler-Universität
Altenbergerstrasse 69
4040 Linz
AUSTRIA

Dr. Deepesh Toshniwal
Faculty of Mathematics and Informatics
Delft University of Technology
P.O. Box 356
2628 BL Delft
NETHERLANDS

Dr. Xiaodong Wei
Institut de Mathématiques
École Polytechnique Fédérale de
Lausanne
MA-Ecublens
Station 8
1015 Lausanne
SWITZERLAND

Prof. Dr. Yongjie Jessica Zhang
Department of Mechanical Engineering
Carnegie Mellon University
318 Scaife Hall
5000 Forbes Avenue
Pittsburgh, PA 15213-3890
UNITED STATES