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Homotopy Theory

Organized by
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Abstract. The workshop *Homotopy Theory* was organized by Jesper Grodal (Copenhagen), Michael Hill (Los Angeles), and Birgit Richter (Hamburg). It covered a wide variety of topics in homotopy theory, from foundational questions to particular computational techniques, and it explored connections to related fields.

Mathematics Subject Classification (2010): 55xx.

Introduction by the Organizers

The workshop *Homotopy Theory*, organised by Jesper Grodal (Copenhagen), Michael Hill (Los Angeles), and Birgit Richter (Hamburg) was well attended, with over fifty participants representing a number of countries around Europe and the world. Participants from all career stages attended, ranging from advanced graduate students to senior faculty, and the workshop also represented almost all research areas in homotopy theory. The workshop consisted of 23 talks, ranging in length from 30 minutes to an hour. Talks on the first day included more introductory material, serving as a broad overview of the respective area, and all talks described cutting-edge research in homotopy theory. Two evenings also included scheduled sessions for people to bring up research or expository questions in an informal setting.

Homotopy theory covers a wide-swath of algebraic topology, exploring everything from particular algebraic invariants of spaces or spectra to fundamental, structural questions in homotopical or higher category theory. Much of the work discussed during the workshop draws from many different classical approaches, using tools of equivariant, motivic, and chromatic homotopy to study problems in topology and number theory, and a common thread through all the talks was the uses of higher categories.

1.1. **Chromatic homotopy.** Starting with work of Quillen, homotopy theory has had a close connection with algebraic geometry. Lazard's height stratification of 1-dimensional, commutative formal groups gives rise to a filtration of the stable homotopy category: the chomatic filtration. This filtration provides a systematic way to understand maps between finite complexes by breaking them into periodic families corresponding to various chromatic strata, and chromatic homotopy techniques are the primary way we can carry out explicit computations.

Piotr Pstragowski described his recent work which shows that chromatic homotopy is equivalent to a category of comodules in a particular range. When we instead look at arbitrary heights and primes, we still have additional unexpected structure when we look at the individual layers (the $K(n)$-local stable homotopy categories). Lior Yanovski described joint work with Carmeli and Schlank where they showed that Hopkins–Lurie ambidexterity holds more generally than just $K(n)$-locally, using instead telescopic localizations. Working unstably, Lukas Brantner explained how to model $K(n)$-local homotopy with an emphasis on describing loop spaces.

Spectacular work of Devinatz–Hopkins–Smith in the early 1980s explicitly described how we can understand the chromatic filtration when restricted to finite spectra. Part of this was the determination of the "thick" subcategories of finite spectra, showing these are all describable in chromatic terms. Balmer showed that much of their analysis works in an arbitrary tensor-triangulated category, showing how to associated to a tensor-triangulated category a space analogous to the Zariski spectrum of a ring. Markus Hausmann spoke about joint work with T. Barthel, J. Greenlees, N. Naumann, T. Nikolaus, J. Noel, and N. Stapleton analyzing the Balmer spectrum for finite G-spectra and resolving a conjecture of Balmer–Sanders for abelian Lie groups. Nick Kuhn described joint work with Chris Lloyd on how classical constructions in homotopy theory could be used to build explicit examples of finite G-spectra with particular properties, working towards the non-abelian case. Drew Heard reported on work with Barthel, Castellana, and Valenzuela on understanding the thick subcategories of the categories of modules over a commutative Noetherian ring spectrum.

1.2. **Derived algebraic geometry.** Goerss–Hopkins–Miller produced the first example of a derived algebraic geometry object, showing how the Lubin–Tate universal deformation of a formal group law in characteristic p lifts to commutative ring spectra. The resulting commutative ring spectrum has an action of the automorphism group of the original formal group law, and the homotopy fixed points of this action recovers the $K(n)$-localization of the sphere spectrum. In general, this automorphism group is a p-adic Lie group of dimension n^2, and cohomological information about it can be difficult to access. Dustin Clausen described a general framework to understand a generalization of the Lazard–Serre analysis of the continuous cohomology of p-adic Lie groups. Restricting the action of the automorphism groups to finite subgroups gives various finitary approximations to the $K(n)$-local sphere. Mingcong Zeng talked about recent work with Lennart Meier on how constructions in equivariant homotopy theory can be used to study related problems.

Goerss–Hopkins–Miller lifted the classial Landweber–Ravenel–Stong result to a sheaf of commutative ring spectra lifting the structure sheaf of the moduli stack of elliptic curves, giving a concrete instance of algebraic geometry objects in homotopy theory. Recent work of Lurie provided a vast generalization of this, providing a framework for one to simply do many classical algebraic geometry constructions in commutative ring spectra. Lennart Meier spoke on Lurie's equivariant elliptic cohomology, describing recent work with David Gepner on explicit computations of various fixed point spectra for S^1-equivariant topological modular forms. John Rognes continued the discussion of topological modular forms, describing joint work with Bruner on the Anderson and Brown–Comenetz duals of tmf at the prime 2, originally described by Stojanoska.

1.3. **Higher categories and geometry.** A feature of Lurie's work is a vast literature on quasicategories, a particular model of higher categories (in particular "$(\infty, 1)$-categories"). This reflects decades of work on models of higher categories and tools to work with them. Julie Bergner described joint work with Nick Kuhn and Inna Zakharevich on a study of 2-Segal spaces, a 2-dimensional generalization of the usual Segal spaces, together with applications to Hall algebras. Viktoriya Ozornova explained recent work with Martina Rovelli on stratified simplicial sets as a model for (∞, n)-categories, focusing on the particular case of $(\infty, 2)$-categories.

Peter Haine described an application of higher categories to the study of stratified spaces. Mixing homotopy theory and geometry, Sam Nariman explained how Thurston's fragmentation technique in foliation theory can be used to show a homology h-principle for certain sheaves on manifolds. Alexander Kupers described joint work with Manuel Krannich in which they study the cohomology groups of the space diffeomorphisms of certain $2n$-manifolds outside of the classically studied stable range.

1.4. **Algebraic K-theory.** Quillen's algebraic K-theory of a ring is a fundamental invariant, recording deep and subtle number-theoretic information about the ring. Computations of algebra K-groups are notoriously difficult, but trace methods, pioneered by Goodwillie and Bökstedt–Hsiang–Madsen, show that topological Hochschild homology, as a cyclotomic spectrum, can approximate these groups in a variety of cases. Thomas Nikolaus described an application of this, talking about a vast generalization of Beilinson's fiber sequence in K-theory obtained via topological cyclic homology. Markus Land then discussed recent work with Meier and Tamme that shows that the algebraic K-theory functor preserves telescopic equivalences. On a computational side, Achim Krause talked about joint work with Nikolaus on an extension of Bökstedt's classical periodicity for THH of a perfect field in characteristic p to discrete valuation rings, giving a way to understand the topological Hochschild homology of DVRs.

1.5. **Equivariant and motivic homotopy.** Tom Bachmann described joint work with Jeremy Hahn on normed motivic spectra, a motivic analogue of an equivariant commutative ring spectrum, exploring some of the basic examples and some of the motivic analogues of classical equivariant results. Kyle Ormsby reported on recent work with Röndigs wherein they use the motivic slice spectral sequence to compute the homotopy groups of the η-local sphere spectrum, working over an arbitrary base field. On the equivariant front, Clover May described work with Dugger and Hazel on a classification of compact objects in the category of chain complexes of modules over the constant Mackey functor $\mathbb{F}_2$ for the group C_2.

1.6. **Operads and loop spaces.** Joana Cirici talked about formality for algebraic structures, explaining how the presence of a weight decomposition can be used to obtain new formality results e.g., for configuration spaces, as in her joint work with Geoffroy Horel. Anna Marie Bohmann talked on joint work with Angelica Osorno. They show a a multiplicative comparison of Segal and Waldhausen K-theory which yields a comparison of the ring spectra built from these.

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Workshop: Homotopy Theory

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Abstracts

Nilpotence in normed MGL-modules

Tom Bachmann
(joint work with Jeremy Hahn)

Motivic stable homotopy theory. For a scheme S, we have the motivic stable ∞-category

$$\mathcal{SH}(S) = L_{\mathbb{A}^1, Nis} P(Sm_S)_*[(\mathbb{P}^1)^{-1}].$$

It is clear that if $f : S' \to S$ is a morphism of schemes, then there is an induced symmetric monoidal cocontinuous functor $f^* : \mathcal{SH}(S) \to \mathcal{SH}(S')$. It is less clear, but established in [BHo17] that for a finite étale morphism $p : S' \to S$, the Weil restriction [CGP15, §A.5] induces a symmetric monoidal functor $p_\otimes : \mathcal{SH}(S') \to \mathcal{SH}(S)$. It only preserves sifted colimits, however.

Normed spectra. The functorialities f^* and $p_\otimes$ can be used to define the category $\mathrm{NAlg}(\mathcal{SH}(S))$ [BHo17, §7] of so-called *normed spectra*. This comes with a forgetful functor $\mathrm{NAlg}(\mathcal{SH}(S)) \to \mathrm{CAlg}(\mathcal{SH}(S))$, i.e. is an enhancement of the category of E_∞-ring spectra in $\mathcal{SH}(S)$. Intuitively, an object $E \in \mathrm{NAlg}(\mathcal{SH}(S))$ consists of an underlying spectrum $E \in \mathcal{SH}(S)$, for every finite étale map $p : X \to Y \in Sm_S$ a morphism

$$p_\otimes(E|_X) \to E|_Y,$$

and an infinite amount of coherences among these data.

Some examples.

(1) Many of the motivic spectra analogous to classical E_∞-rings are normed spectra. This is true for example for the sphere spectrum S^0, the algebraic cobordism spectrum MGL, the algebraic K-theory spectrum KGL, and the motivic cohomology spectrum $H\mathbb{Z}$.
(2) Recall that the geometric Hopf map $\mathbb{A}^2 \setminus 0 \to \mathbb{P}^1$ induces a non-nilpotent endomorphism η of the motivic sphere spectrum S^0. One may show that the mapping telescope $S^0[\eta^{-1}]$ is not normed [BHo17, Example 12.11], even though it is an E_∞-ring.
(3) More is true: if $E \in \mathrm{NAlg}(\mathcal{SH}(S))$ is any normed spectrum on which η acts invertibly, then $E \simeq 0$ [BHo17, Example 12.12].
(4) On the other hand, if $p : S' \to S$ is any finite étale morphism (with $S \neq \emptyset$), then $p_\otimes(\eta) \neq 0$.

Normed nilpotence. In general, given $E \in \mathrm{NAlg}(\mathcal{SH}(S))$ and $x \in \pi_{**}(E)$, we call x *normed nilpotent* if the analog of (3) holds: for any $F \in \mathrm{NAlg}(\mathcal{SH}(S))_{E/}$ such that x acts invertibly on F, we have $F \simeq 0$. It seems to be an important problem to develop criteria for deciding if some $x \in \pi_{**}E$ is normed nilpotent. In a lot of ways, this is equivalent to finding criteria for deciding if $E \simeq 0$. Inspired by analogous results about classical E_∞-ring spectra, we establish the following.

Theorem (motivic May nilpotence theorem [BHa19])**.** *Let $E \in \mathrm{NAlg}(\mathcal{SH}(S))$ (where S is noetherian, finite dimensional, and whenever $1/p \notin S$ then $1/p \in E$). If $E \wedge H\mathbb{Z} \simeq 0$, then also $E \wedge MGL \simeq 0$.*

Some questions.

(1) Let $E \in \mathrm{NAlg}(\mathcal{SH}(S))$. If $E \wedge MGL \simeq 0$, is then also $E \simeq 0$?
(2) Recall the motivic "cofiber τ" ring spectrum $S^0/\tau \in \mathrm{CAlg}(\mathcal{SH}(\mathbb{C})^{\wedge}_2)$ [Ghe17]. Can we lift $S^0/\tau \in \mathrm{NAlg}(\mathcal{SH}(\mathbb{C})^{\wedge}_2)$?

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2-Segal sets, algebraic K-theory, and Hall algebras

Julie Bergner
(joint work with Nick Kuhn and Inna Zakharevich)

The notion of 2-Segal space was defined by Dyckerhoff and Kapranov [2], and independently under the name of decomposition space by Gálvez-Carrillo, Kock, and Tonks [3]. The structure given by a 2-Segal space is something like that of a topological category up to homotopy, but for which composition need not be defined, or be unique even when it is, but still satisfies associativity. Although both sets of authors give many different examples arising from different perspectives, two key themes emerge. First, 2-Segal spaces can be obtained via $S_\bullet$-constructions, and hence have a close relationship with algebraic K-theory. Second, 2-Segal spaces satisfying certain finiteness assumptions give rise to Hall algebra constructions.

A natural question, then, is what these two themes have to do with one another. Indeed, it has long been thought by experts that there should be a close relationship between Hall algebras and algebraic K-theory. However, many of the classical examples in both fields tend to be somewhat large; for example, most familiar Hall algebras are infinite-dimensional. In this talk, we consider 2-Segal sets, or discrete 2-Segal spaces, and some explicit examples whose Hall algebras have very concrete descriptions.

The family of examples we consider arises from finite graphs. Given a finite graph G, we construct a simplicial set X_G with a single 0-simplex, 1-simplices given by the subgraphs of G, and higher simplices given by subgraphs equipped with ordered partitions of the vertices of the subgraph into (possibly empty) subsets. That this simplicial set is a 2-Segal set was shown in joint work with Angélica Osorno, Viktoriya Ozornova, Martina Rovelli, and Claudia Scheimbauer [1], following the example of the decomposition space of all graphs as described by Gálvez-Carrillo, Kock, and Tonks [3].

Although their construction can be defined for more general 2-Segal objects, we consider Dyckerhoff and Kapranov's definition of the Hall algebra associated to a reduced 2-Segal set [2]. Its underlying vector space is generated by the 1-simplices of the 2-Segal set, and the multiplication essentially counts the 2-simplices which encode the different ways to "compose" a given pair of 1-simplices. This definition makes sense so long as this counting always produces a finite sum, so we restrict to 2-Segal sets with finitely many nondegenerate simplices.

In particular, this construction can be applied to the 2-Segal set X_G associated to a finite graph G. The resulting Hall algebra has several interesting properties. It is always commutative, and the multiplication can be described very explicitly in terms of geometric properties of the graph. For example, the product of the basis elements associated to two subgraphs of G is nonzero precisely when those two subgraphs are disjoint.

Because the Hall algebras of this particular family of 2-Segal sets can be described so explicitly, we can try to identify them with other familiar algebras. Working over the field $\mathbb{F}_2$ for simplicity, for many examples of this kind we can impose a natural grading so that these Hall algebras agree with the cohomology of explicitly described topological spaces with coefficients in $\mathbb{F}_2$. For example, for a graph with n vertices and no edges, the Hall algebra has n generators. If we say that these generators are in degree 1, then this algebra coincides with the cohomology of the n-dimensional torus. When G is the graph with two vertices and a single edge between them, the associated Hall algebra agrees with the cohomology of the wedge of a torus with a 2-sphere.

While we have worked out many special cases on a small number of vertices, the general formula for the associated topological space is still conjectural. Still more intriguing is that fact that, for many of these small examples it can be verified that this assocated space is homotopy equivalent to the geometric realization of the simplicial set X_G. Thus, a more general question is when the Hall algebra of a 2-Segal set agrees with the cohomology of its geometric realization. We know, however, that this relationship does not hold in general. A counterexample is given by the 2-Segal set given by the nerve of a discrete group G, whose associated Hall algebra is the group algebra of G over the underlying field, which does not coincide with the group cohomology algebra of G.

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A multiplicative comparison of Segal and Waldhausen K-theory

Anna Marie Bohmann
(joint work with Angélica M. Osorno)

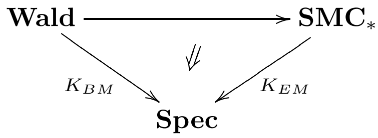
A very homotopical perspective on algebraic K-theory is that K-theory is a tool for building spectra from categorical data. For example, in influential work of the 70s and 80s, Segal [8] and Waldhausen [9] each construct a version of K-theory that produces spectra from certain types of categories. These constructions agree in the sense that appropriately equivalent categories yield weakly equivalent spectra. In fact, Waldhausen provides a comparison map from Segal's K-theory to his K-theory in his initial paper on the subject.

We should think of the equivalence between these K-theory constructions, as well as equivalences coming out of results like the May–Thomason theorem [7], as *additive* equivalences. These are comparisons of spectra—in other words, of spaces equipped with a highly coherent homotopy commutative operation—but they don't take into account any possible further structure, such as the multiplicative structure needed to build ring spectra.

In the 2000s, work of Elmendorf–Mandell [5] and Blumberg–Mandell [3] produced more structured versions of Segal and Waldhausen K-theory, respectively. These versions are "multiplicative," in the sense that appropriate notions of pairings of categories yield multiplication structure on their resulting spectra. Both K-theory constructions encode this multiplicativity in the language of multifunctors and multicategories.

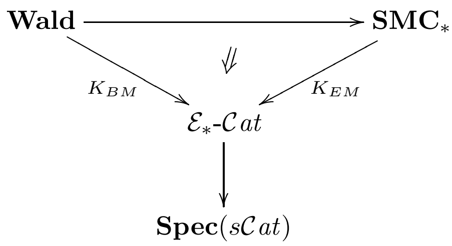
In our work, Osorno and I show that that the Elmendorf–Mandell and Blumberg–Mandell constructions agree as multiplicative versions of K-theory. Consequently, we get comparisons of ring spectra built from these two constructions. Furthermore, the same result also allows for comparisons of related constructions of spectrally-enriched categories. There is related work by Barwick [1], Blumberg–Gepner–Tabuada [2], and Gepner–Groth–Nikolaus [6] providing ∞-category level comparisons of several types of K-theory constructions. Our work differs in that it provides a structured "on-the-nose" comparison between these specific versions of K-theory. The main theorem can be stated as follows.

Theorem 1 ([4])**.** *There is a multinatural transformation of symmetric multifunctors fitting into the diagram*



The horizontal multifunctor from Waldhausen categories to strictly unital symmetric monoidal categories is given by endowing a Waldhausen category with a choice of wedge products.

In fact, to get the full type of equivalence necessary, this theorem must be refined in two ways. First, we wish to have an enriched comparison. Since the natural enrichment on the source category is in categories, we pass to categorical enrichment throughout by working with symmetric spectra in simplicial pointed categories. We further introduce an intermediate categorically-enriched symmetric multicategory to serve as the target of our natural transformation, so that the main construction is in fact a categorically-enriched multinatural transformation fitting into the following diagram of symmetric multifunctors:



The second refinement is to take care of the weak equivalences, which are integral to the definition of Waldhausen's K-theory functor. We show that with this set up, taking weak equivalences in the Waldhausen category "commutes" with the construction of the multinatural transformation and of the Elmendorf–Mandell version of K-theory. This allows us to prove the following result.

Theorem 2 ([4])**.** *For any Waldhausen category $\mathcal{C}$ for which Waldhausen's initial comparison map $K_{Wald}(\mathcal{C}) \to K_{Seg}(\mathcal{C})$ is an equivalence, the component at $\mathcal{C}$ of the multinatural transformation of Theorem 1 is an equivalence. In particular, this applies when $\mathcal{C}$ has suitably split cofibrations.*

The combination of these results implies that (commutative) ring spectra, E_∞-ring spectra and other operadically defined algebraic structures built out of Segal's and Waldhausen's K-theory constructions are equivalent as structured objects. Furthermore, spectrally-enriched categories (in the classical sense of Kelly) built from these constructions are also equivalent. Our main motivation for pursuing this work is in showing that Waldhausen- and Segal-type constructions of the spectral Burnside category are equivalent.

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Generalised Lie algebras in Algebra and Topology

Lukas Brantner

We discuss different generalisations of rational differential graded Lie algebras, and outline some recent applications to unstable homotopy theory [5][15], formal deformation theory [10], and the (generalised) homology of configuration spaces [7][8] away from characteristic zero.

1. Three applications of rational differential graded Lie algebras

Given a field k of characteristic zero, we recall the following notion:

Definition 1. *A (shifted) differential graded Lie algebra* $\mathfrak{g}$ *over* k *consists of a chain complex* $\ldots \to \mathfrak{g}_1 \xrightarrow{d} \mathfrak{g}_0 \xrightarrow{d} \mathfrak{g}_{-1} \to \ldots$ *of* k*-vector spaces together with bilinear maps* $[-,-] : \mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j-1}$ *such that for all* $x \in \mathfrak{g}_a$, $y \in \mathfrak{g}_b$, $z \in \mathfrak{g}_c$, *we have:*

(1) *Antisymmetry:* $[x,y] = (-1)^{ab}[y,x]$;
(2) *Jacobi identity:* $(-1)^{ac}[[x,y],z] + (-1)^{cb}[[z,x],y] + (-1)^{ba}[[y,z],x] = 0$;
(3) *Leibniz rule:* $d([x,y]) = -[dx,y] - (-1)^a[x,dy]$.

Remark 2. *The shifted grading convention arises naturally from Koszul duality; all Lie algebras appearing in this document are assumed to be shifted.*

Rational differential graded Lie algebras have several classical applications:

Rational Lie models. Quillen [27] established an equivalence $(\mathcal{S}_*)_{\mathbb{Q},\geq 2} \simeq \mathrm{Lie}_{\mathbb{Q},\geq 2}$ between rational simply connected pointed spaces and differential graded Lie algebras $\mathfrak{g}$ over $\mathbb{Q}$ with $\pi_i(\mathfrak{g}) = 0$ for $i < 2$. Under this correspondence, the rational n-sphere $S^n_{\mathbb{Q}}$ corresponds to the free Lie algebra on a class in degree n.

Rational homology of configuration spaces. Given a framed n-manifold M and an integer m, there is a (weight-preserving) isomorphism

$$(1) \qquad \bigoplus_k H_* \left(\mathrm{Conf}_k(M) \otimes_{\Sigma_k} S^m; \mathbb{Q}\right) \cong H_*^{\mathrm{Lie}} \left(H_c^{-*}(M; \mathbb{Q}) \otimes \mathrm{Free}_{\mathrm{Lie}^{\mathbb{Q}}}(x_{n+m})\right).$$

Here $H_*^{\mathrm{Lie}}(-)$ denotes Lie algebra homology, and $H_c^*(-;\mathbb{Q})$ is compactly supported cohomology. The isomorphism (1) is due to Knudsen [18], and generalises work of Bödigheimer–Cohen–Taylor [4], Félix–Thomas [14], Totaro [34], and others. It is very useful in practice; for example, we can read off that $H_*(\Omega^2 S^3; \mathbb{Q}) \cong \mathbb{Q}[1]$.

Rational deformation theory. Deformations of algebro-geometric objects over $\mathbb{Q}$ are controlled by rational differential graded Lie algebras. This general paradigm was first observed by Deligne [12], Drinfel'd [13], and Feigin, explored further by Hinich [16], Kontsevich–Soibelman [20], and Manetti [25], and finally formulated as an equivalence of ∞-categories by Lurie [22] and Pridham [26].

The Lurie-Pridham theorem identifies formal moduli problems over $\mathbb{Q}$, which encode deformation functors of algebro-geometric objects, with rational differential graded Lie algebras.

Remark 3. *These applications extend to general fields of characteristic zero.*

2. Settings away from characteristic zero

The three classical applications presented above use rational chain complexes; these model the ∞-category $\mathrm{Mod}_{\mathbb{Q}}$ of module spectra over $\mathbb{Q}$, i.e. $\mathbb{Q}$-local spectra.

It is possible to extend some of these results to other settings:

Modular settings. We could also work in chain complexes over a field k of characteristic p (e.g. $\mathbb{F}_p$), or over a complete local Noetherian base (such as $\mathbb{Z}_p$).

Chromatic settings. For every prime p, chromatic homotopy theory constructs infinitely many ring spectra $K(0)=\mathbb{Q}, K(1), K(2), \ldots$ known as Morava K-theories. For $h > 0$, these satisfy $K(h)_* \cong \mathbb{F}_p[v_h^{\pm 1}]$ with $|v_h| = 2(p^h - 1)$; they may be thought of as "generalised fields" sitting in between $\mathbb{Q}$ and $\mathbb{F}_p$. Accordingly, the ∞-category $\mathrm{Sp}_{K(h)}$ of $K(h)$-local spectra interpolates between rational and p-local spectra. As is customary, we suppress p from our notation for Morava K-theories.

3. Generalised Lie algebras and their applications

Away from characteristic 0, differential graded Lie algebras are not homotopically well-behaved; for example, their "free functor" fails to preserve quasi-isomorphisms. In recent years, more adequate substitutes were introduced for different applications:

	Lie models	Configuration spaces	Deformation theory
rational		Differential graded Lie algebras	
chromatic	Spectral Lie algebras $\big\downarrow E_*^{\wedge}(-)$ Hecke Lie algebras		
modular			Partition Lie algebras (derived algebraic geometry) Spectral partition Lie algebras (spectral algebraic geometry)

We give a brisk outline of the definitions and applications of these generalisations.

Spectral Lie algebras. Let $\mathcal{O}_{\mathrm{Comm}}$ be the commutative operad in spectra. Salvatore [31] and Ching [11] have defined the spectral Lie operad as the dualised bar construction $\mathbb{D}(\mathrm{Bar}(\mathcal{O}_{\mathrm{Comm}}))$; its algebras are called *spectral Lie algebras*. Over $\mathbb{Q}$, these are equivalent to the rational differential graded Lie algebras in Definition 1.

The free spectral Lie algebra on a spectrum X is given by

$$\mathrm{Lie}^s_k(X) = \bigoplus_n \mathbb{D}(\Sigma\Pi_n^{\diamond}) \otimes_{h\Sigma_n} X^{\otimes n},$$

where $\Sigma\Pi_n^{\diamond}$ is the unreduced-reduced suspension of the n^{th} partition poset. This makes spectral Lie algebras susceptible to methods from combinatorial topology [1].

Unstable chromatic homotopy theory. Spectral Lie algebras were first linked to unstable chromatic homotopy theory by Behrens and Rezk [5] who, for each pointed space X, constructed a comparison map $c_X : \Phi(X) \to \mathrm{TAQ}_{S_{K(h)}}(S^X_{K(h)})$ from the Bousfield–Kuhn functor on X to the topological André–Quillen cohomology of the $\mathbb{E}_\infty$-ring $S^X_{K(h)}$ – the latter is always a spectral Lie algebra.

The map c_X is an equivalence for X a sphere [3], and also for special unitary and symplectic groups [6]. In [9], we proved (with Heuts) that c_X fails to be an equivalence on wedges of spheres and Moore spaces.

Heuts [15] later equipped the Bousfield–Kuhn functor $\Phi(X)$ with the structure of a spectral Lie algebra, and used this to establish an equivalence between $K(h)$-local spectral Lie algebras and a certain ∞-category $\mathcal{M}_{K(h)}$ of periodic spaces.

Hecke Lie algebras. Lie algebras in $\mathrm{Sp}_{K(h)}$ are not amenable to explicit computations, as their homotopy groups involve the $K(h)$-local homotopy groups of spheres. However, work of Hopkins–Ravenel shows that $\mathrm{Sp}_{K(h)}$ is equivalent to $K(h)$-local modules over a height h Morava E-theory E with action by the stabiliser group $\mathbb{G}$.

In [7], we introduced *Hecke Lie algebras* to describe the operations acting on the homotopy groups of spectral Lie algebras in $\mathrm{Mod}_E^\wedge$, the ∞-category of $K(h)$-local E- modules. In particular, $E_*^\wedge(\Phi(X))$ is a Hecke Lie algebra for any pointed space X.

Very roughly, Hecke Lie algebras are Lie algebras in E_*-modules, equipped with an additional additive action by the cohomology of Rezk's ring Γ [29], which is closely related to the Hecke algebra of $\mathrm{GL}_n(\mathbb{Z}_p)$ [30]. There is an additional congruence at $p = 2$, and special care must be taken when composing operations.

These concrete algebraic structures facilitated recent computational advances:

Chromatic homotopy theory of configuration spaces. Computing the Morava K- or E-theory of the unordered configuration spaces of a manifold M is a hard problem.

For $M = \mathbb{R}^n$, it is of particular interest as the relevant groups parametrise Dyer–Lashof operations on $\mathbb{E}_n$-algebras. At *chromatic height* $h = 1$, the problem was essentially solved by Langsetmo [21]. In *dimensions* $n = 2, 3, 4$, Yamaguchi [35] and Tamaki [32] [33] computed the Morava K-theory groups with increasingly laborious methods. For general h and n, Ravenel stated a conjecture in [28].

Together with Hahn and Knudsen [8], we apply the theory of Hecke Lie algebras to a spectral generalisation of (1), which was originally established in [19]. This allows us to compute the Morava K- and E-homology groups (at a prime p) of the configuration space of p points in $\mathbb{R}^n$, for all heights h and all dimensions n.

We carry out similar computations for configuration spaces of punctured surfaces. Letting h tend to infinity, we can read off their previously unknown $\mathbb{F}_p$-homology.

One might hope to perform this computation without reference to E-theory by using spectral Lie algebras over $\mathbb{F}_p$. Their operations have been computed at $p = 2$ by Antolín-Camarena [2]. For p odd, partial progress has been made by Kjaer [17], but the Adem relations remain unknown. Our method from [7] does not immediately apply, as it uses $K(h)$-local Tate vanishing to identify orbits with fixed points.

However, algebraic geometry leads to two other notions of Lie algebras over $\mathbb{F}_p$:

Formal moduli. Infinitesimal deformations of a given algebro-geometric object over a field k are described by a corresponding *formal moduli problem*, which is a "functor of points" defined on suitable Artin k-algebras and satisfying a gluing axiom.

Away from characteristic 0, there are in fact two variants of formal moduli problems, as algebraic geometry can be based on simplicial commutative rings ("*derived algebraic geometry*") or on connective $\mathbb{E}_\infty$-rings ("*spectral algebraic geometry*").

Partition Lie algebras & spectral partition Lie algebras. Together with Mathew [10], we introduce two new generalisations of Lie algebras, called *partition Lie algebras* and *spectral partition Lie algebras*, over any base field k.

In characteristic 0, both recover the differential graded Lie algebras in Definition 1. In characteristic p, they are distinct from previously considered generalisations (such as spectral Lie algebras or simplicial/cosimplicial restricted Lie algebras).

We prove that our Lie algebras control formal moduli problems in derived and spectral algebraic geometry, respectively. This generalises the Lurie–Pridham theorem from characteristic 0 to base fields of arbitrary characteristic (such as $\mathbb{F}_p$); we also offer a version over mixed characteristic bases (like $\mathbb{Z}_p$).

Even when k is a field, our new Lie algebras are not governed by operads. Instead, they are algebras over monads $\mathrm{Lie}^{\pi}_{k,\Delta}$ and $\mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty}$ acting on the ∞-category of k-module spectra. These monads preserve filtered colimits, geometric realisations, and are given on *coconnective* objects $X \in \mathrm{Mod}_{k,\leq 0}$ by

$$\mathrm{Lie}^{\pi}_{k,\Delta}(X) = \bigoplus_n \widetilde{C}^*(\Sigma\Pi^{\diamond}_n, k) \otimes^{\Sigma_n} X^{\otimes n} \quad ; \quad \mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty}(X) = \bigoplus_n \widetilde{C}^*(\Sigma\Pi^{\diamond}_n, k) \otimes^{h\Sigma_n} X^{\otimes n}.$$

Here, $\widetilde{C}^*(-,k)$ denotes the reduced k-valued singular cochains of a space, whereas $(-)^{\Sigma_n}$ and $(-)^{h\Sigma_n}$ denote strict invariants and homotopy invariants, respectively. The precise definition of strict fixed points uses the genuine equivariant topology of partition complexes, and requires some care.

Outlook. As partition Lie algebras involve fixed points rather than orbits, one can adapt the arguments in [7] to compute their operations and relations. Following the strategy in [18] and [8] then leads to a new approach to the $\mathbb{F}_p$-homology of configuration spaces – a subject where many computations are yet to be done.

Two other tasks for the future are to provide a Lie algebraic description of deformations in chromatic contexts, and to construct Lie models for the modular homotopy type of spaces (Koszul dual to Mandell's commutative models in [24]).

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Formality with torsion coefficients

Joana Cirici
(joint work with Geoffroy Horel)

The notion of formality makes sense for almost any algebraic structure on a (co)chain complex, such as commutative dg-algebras, dg-Lie algebras, operads, or any algebraic object encoded by a coloured operad: such an algebraic object is *formal* if it is connected with its (co)homology by a string of morphisms preserving the algebraic structures and inducing isomorphisms in (co)homology.

Let R be a commutative ring. A topological space X is said to be *formal over* R if its singular cochain complex $C^*(X, R)$ with coefficients in R is formal as a dg-algebra. Similarly, we say that an operad $\mathcal{O}$ in topological spaces is *formal over* R if the operad $C_*(\mathcal{O}, R)$ is formal as a dg-operad in chain complexes.

There is also an abstract notion of functorial formality: let $\mathcal{C}$ be a symmetric monoidal category and $\mathcal{A}$ an abelian symmetric monodidal category. Consider a (lax) symmetric monoidal functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Ch}_*(\mathcal{A})$ with values in the category of chain complexes of $\mathcal{A}$. Then $\mathcal{F}$ is said to be *formal* if and only if there is a string of monoidal natural transformations from $\mathcal{F}$ to the composition $H_* \circ \mathcal{F}$ with the homology functor H_*, inducing quasi-isomorphisms on every object. Note that if $\mathcal{F}$ is a formal functor, then it sends algebraic structures (algebras, operads, monoids...) in $\mathcal{C}$ to formal algebraic structures in $\mathrm{Ch}_*(\mathcal{A})$. This notion was first introduced by Guillén-Navarro-Pascual-Roig in [6], who proved that the functor of singular chains $C_*(-, \mathbb{Q})$ with rational coefficients is formal when restricted to the symmetric monoidal category of compact Kähler manifolds. In [2] we extended this result to the symmetric monoidal category of all complex algebraic varieties whose weight filtration in cohomology satisfies certain purity properties, providing various applications to formality over $\mathbb{Q}$ of algebras and operads arising from algebraic geometry. The results of [2] strongly depend on mixed Hodge theory and as such, they are restricted to rational coefficients. However, via the étale counterpart of Deligne's theory of weights, we can obtain partial results of formality with torsion coefficients, as we next explain.

Let $K \hookrightarrow \mathbb{C}$ be a p-adic field with residue field $\mathbb{F}_q$. Given an algebraic variety X defined over K, we may consider its base change $X_{\overline{K}} := X \times_K \overline{K}$. By construction, $X_{\overline{K}}$ has an action of the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$. Then, the étale cohomology groups $H^*_{et}(X_{\overline{K}}, \mathbb{F}_\ell)$ carry a Frobenius automorphism φ^* for any prime $\ell \neq p$. In fact, this automorphism exists at the cochain level: there is a dg-algebra $C^*_{et}(X_{\overline{K}}, \mathbb{F}_\ell)$ together with a multiplicative endomorphism φ such that $H^*(\varphi) = \varphi^*$. Now, the base change $X_{\mathbb{C}} := X \times_K \mathbb{C}$ has an underlying analytic space X_{an} and its complex of singular cochains $C^*(X_{an}, \mathbb{F}_\ell)$ is naturally quasi-isomorphic to $C^*_{et}(X_{\overline{K}}, \mathbb{F}_\ell)$. As a consequence, to prove formality of the topological space X_{an} over $\mathbb{F}_\ell$ it suffices to prove formality for $C^*_{et}(X_{\overline{K}}, \mathbb{F}_\ell)$. The advantage of using étale chains is that they are endowed with an endomorphism which, under sufficiently nice conditions, produces useful decompositions leading to partial formality.

By sufficiently nice conditions we mean the following: Let $\alpha \in \mathbb{Q}$ be a positive rational number. We say that an algebraic variety $X \in \mathrm{Var}_K$ is *α-pure* if for all $n \geq 0$, the only eigenvalue of $H^n_{et}(X_{\overline{K}}, \mathbb{F}_\ell)$ is $q^{\alpha n}$.

We will need the following notion of partial formality. Let $N \geq 0$ be an integer. We say that a dg-algebra A is *N-formal* if there is a string of morphisms of dg-algebras from A to its homology $H_*(A)$ in such a way that the induced maps in degree i-homology are isomorphisms for all $i \leq N$.

Theorem 1.1 ([3]). *Let $X \in \mathrm{Var}_K$ be an α-pure algebraic variety defined over K, with $H^1_{et}(X_{\overline{K}}, \mathbb{F}_\ell) = 0$. Then the topological space X_{an} is N-formal over $\mathbb{F}_\ell$, where $N = \lfloor \frac{h-1}{\alpha} \rfloor$ and h is the order of q in $\mathbb{F}_\ell^\times$.*

Let us explain some direct applications of this result:

(1) The étale cohomology of projective space $\mathbb{P}^m_K$ is concentrated in even degrees and the Frobenious authomorphism acts by multiplication by q^i on $H^{2i}_{et}(\mathbb{P}^m_K, \mathbb{F}_\ell)$ so we have $\alpha = 1/2$. Since $\mathbb{P}^m_K$ is defined over $\mathbb{Z}$ we can choose $K = \mathbb{Q}_p$ such that q generates $\mathbb{F}_\ell^\times$, so that $h = \ell - 1$. Then the theorem says that $\mathbb{P}^m_K$ is $2(\ell-2)$-formal over $\mathbb{F}_\ell$. In particular, if $\ell > m+1$ then $\mathbb{P}^m_K$ is formal over $\mathbb{F}_\ell$.

(2) A similar argument applies to the moduli spaces $\overline{\mathcal{M}}_{0,n+1}$, giving formality of these spaces over $\mathbb{F}_\ell$ for any $n \leq \ell$.

(3) The theorem also applies to configuration spaces $\mathrm{Conf}_m(\mathbb{C}^d)$ of m points in $\mathbb{C}^d$, with $h = \ell - 1$ and $\alpha = d/(2d-1)$. Therefore these spaces are formal over $\mathbb{F}_\ell$ whenever $(m-1)d \leq \ell - 1$.

A recent approach to formality using homotopy transfer techniques has been developed by Drummond-Cole and Horel in [4]. Their theory allows to produce formality results with coefficients in the p-adic integers.

We also have a chain-operadic version of the above result, which can be stated abstractly as follows: consider the symmetric monoidal category T_α whose objects are given by pairs (C, φ) where C is a non-negatively graded chain complex of vector spaces over $\mathbb{F}_\ell$ with finite type homology and φ is an endomorphism of C such that the only eigenvalue of $H_n(\varphi)$ on $H_n(C)$ is $q^{\alpha n}$. Then, we have:

Theorem 1.2 ([3]). *Reasonable operads in T_α are N-formal, where $N = \lfloor \frac{h-1}{\alpha} \rfloor$ and h is the order of q in $\mathbb{F}_\ell^\times$.*

The proof of the above theorem follows after showing that the forgetful functor $T_\alpha \longrightarrow \mathrm{Ch}_*(\mathbb{F}_\ell)$ given by $(C, \varphi) \mapsto C$ is N-formal as an ∞-functor. This directly implies formality for reasonable (more precisely: for homotopically sound) operads. For instance, we may apply the above result to the operad of little disks. The idea is the following: there is a model of $C_*(E_2, \mathbb{F}_\ell)$ endowed with an action of the Grothendieck-Teichmüller group GT. Also, there is a surjective map $\mathcal{X}_\ell : \mathrm{GT} \to \mathbb{Z}^\times$. One may then choose p generating $\mathbb{F}_\ell^\times$ and $\varphi \in \mathrm{GT}$ such that $\mathcal{X}_\ell(\varphi) = p$. Then the pair $(C_*(E_2, \mathbb{F}_\ell), \varphi)$ is an object in T_α, with $\alpha = 1$. It follows that the operad of little discs E_2 is $(\ell-1)$-formal over $\mathbb{F}_\ell$.

In [1], Boavida de Brito and Horel construct a GT-action on the higher little discs E_n, thus obtaining partial formality over $\mathbb{F}_\ell$ also in this case.

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Atiyah duality for p-adic Lie groups

Dustin Clausen

Before doing any motiviation, set-up, or anything else, I want to describe the key construction which goes into the proof of the main result. This is a new kind of *p-adic cospecialization map*. Consider the ∞-category of $\mathbb{Q}_p^\times$-equivariant sheaves of spectra on $\mathbb{Q}_p$, or equivalently sheaves on the *quotient stack* $\mathbb{Q}_p/\mathbb{Q}_p^\times$. Here we are viewing $\mathbb{Q}_p$ as a topological space with an action of the topological group $\mathbb{Q}_p^\times$, and $\mathrm{Sh}(\mathbb{Q}_p/\mathbb{Q}_p^\times)$ is defined as the inverse limit of the ∞-cateogry of sheaves of spectra $\mathrm{Sh}(X)$ as X runs over the usual simplicial topological space presenting the quotient (the nerve of the action groupoid), with respect to the pullback functoriality.

The main construction is this. Suppose given any $\mathcal{F} \in \mathrm{Sh}(\mathbb{Q}_p/\mathbb{Q}_p^\times)$ such that the spectrum $\mathcal{F}_0$, the stalk of $\mathcal{F}$ at $0 \in \mathbb{Q}_p$, satisfies:

(1) The abelian group $\pi_d\mathcal{F}_0$ is torsion for all $d \in \mathbb{Z}$;
(2) $\pi_d(\mathcal{F}_0/n)$ finite for all $n > 0$ and $d \in \mathbb{Z}$.

Then there is a natural map

$$\mathcal{F}_0 \to \mathcal{F}_1$$

from the stalk at 0 to the stalk at 1, such that if $\mathcal{F}$ is a constant sheaf than this map is the natural identification.

If we were to replace $\mathbb{Q}_p$ by $\mathbb{R}$ in the above, one could construct such a map $\mathcal{F}_0 \to \mathcal{F}_1$ in a standard way by using the fact that the topology of $\mathbb{R}$ is determined by open intervals, which are contractible. But the p-adic case, where the topology is totally disconnected, the existence of such a cospecialization map is somewhat surprising and a more subtle construction is required.

Now, the main result is a generalization of a classic duality theorem of Lazard and Serre in the context of p-adic Lie groups. Let us recall the definition: a *p-adic Lie group* is a topological group G which admits an atlas modelled on open subsets of some $\mathbb{Q}_p^d$, such that the transition maps, and the group law, are *locally analytic*: in a neighborhood of any point, they are given by a convergent power series expansion. Standard examples are usual matrix groups such as $GL_n(\mathbb{Q}_p)$. But there is also the compact open subgroup $GL_n(\mathbb{Z}_p)$, and more generally the congruence subgroups of these,

$$G_m := \ker(GL_n(\mathbb{Z}_p) \to GL_n(\mathbb{Z}/p^m\mathbb{Z})).$$

These G_m in fact form a compact open neighborhood basis of the identity in $G = GL_n(\mathbb{Q}_p)$ consisting of p-adic Lie groups. The existence of such a system of neighborhood bases is completely general, and it distinguishes p-adic Lie groups from their real cousins, which have the opposite "no small subgroups" property. Another difference is that while the algebraic topology of a real Lie group is interesting even when you forget the group structure and just think of the topological space, the underlying topological space of a p-adic Lie group is devoid of interest; not even the dimension of G can be recovered from it. Only in the presence of the group structure does the theory become interesting from an algebraic topology point of view.

The duality result of Lazard concerns the *continuous group cohomology* of such a G with discrete, p-power torsion coefficients M. This cohomology, denoted

$$H^*(BG; M),$$

is defined as the cohomology of the subcomplex of the standard complex computing group cohomology, where you require the cochains $G^n \to M$ to be *continuous* functions. Also, M is to be equipped with a continuous G-action to make this well-defined.

The Lazard-Serre theorem, which we state just with constant coefficients for simplicity, is as follows: suppose that G is a compact and p-torsionfree p-adic Lie group, of dimension d. Then:

(1) There is a canonical free $\mathbb{Z}_p$-module of rank one δ_{BG} with continuous G-action and a canonical identification $H^d(BG; \delta_{BG}/p^n) = \mathbb{Z}/p^n\mathbb{Z}$ for all $n \geq 0$, such that for all $i \in \mathbb{Z}$ the cup product

$$H^i(BG; \mathbb{Z}/p^n\mathbb{Z}) \otimes_{\mathbb{Z}/p^n\mathbb{Z}} H^{d-i}(BG; \delta_{BG}) \to H^d(BG, \delta_{BG}/p^n) = \mathbb{Z}/p^n\mathbb{Z}$$

is a perfect pairing of finite $\mathbb{Z}/p^n\mathbb{Z}$-modules. (Note that $\mathbb{Z}/p^n\mathbb{Z}$ is injective as a module over itself.)

(2) This δ_{BG} can be explicitly described in terms of the adjoint representation of G on its Lie algebra: its defining character $G \to \mathbb{Z}_p^\times \subset \mathbb{Q}_p^\times$ is given by the determinant of the adjoint representation.

Thus, the *group cohomology* of a suitable p-adic Lie group satisfies Poincare duality, similarly to the *usual cohomology* of a compact real manifold. And similarly to how the orientation local system on a real manifold is controlled by the tangent bundle, so is δ_{BG} controlled by the adjoint representation.

Lazard proved this if you replace the p-torsionfree hypothesis with the condition that G have finite p-cohomological dimension, and also showed that this condition always holds if you shrink G sufficiently; it was Serre's contribution to show that, under the compactness hypothesis, p-torsionfreeness is equivalent to finite p-cohomological dimension.

Our main result is the following generalization from abelian group cohomology to spectrum cohomology. For G as in the Lazard-Serre theorem, define $\mathrm{Sh}(BG)$ to be the ∞-category of sheaves of spectra on BG, or equivalently G-equivariant sheaves of spectra on the point, defined as the limit of the cosimplicial diagram given by the sheaves on the simplicial topological space describing BG, similarly to the above discussion of $\mathbb{Q}_p/\mathbb{Q}_p^\times$. This $\mathrm{Sh}(BG)$ is one possible model for the

∞-category of "spectra with continuous G-action"; on homotopy groups it gives a *discrete* abelian group with continuous G-action. Also consider the full subcategory $\mathrm{Sh}_p(BG)$ of p-power torsion objects. This is actually a *unital* symmetric monoidal ∞-category under the usual smash product, though the unit looks a little funny (it is the derived p^∞-torsion in the constant sheaf on the sphere spectrum). Then:

(1) There is a canonical invertible object $\mathbb{D}_{BG} \in \mathrm{Sh}_p(BG)$ such that the left adjoint and right adjoint to the pullback $p^* : \mathrm{Sh}_p(*) \to \mathrm{Sh}_p(BG)$ differ by tensoring with $\mathbb{D}_{BG}$.
(2) $\mathbb{D}_{BG}$ can be explicitly described in terms of the adjoint representation of G on its Lie algebra $\mathfrak{g}$. (The proof says exactly how.)

In fact, our (1) is essentially a formal consequence of the Lazard-Serre (1). So the main point is (2), which is by no means a formal consequence of the Lazard-Serre (2), nor can Lazard's proof be adapted to this more general setting.

Let us explain the outline of the proof. The crucial thing to show is that the fiber of $\mathbb{D}_{BG}$ at the basepoint $* \to BG$ canonically identifies with the fiber of $\mathbb{D}_{B\mathfrak{g}}$ at the basepoint $* \to B\mathfrak{g}$; here we are simply viewing the Lie algebra $\mathfrak{g}$ as an abelian p-adic Lie group under addition. An "in-families" version of the same argument, applied to the adjoint action of G on itself, will provide the description posited in (2).

To give this identification we use a geometric construction borrowed from algebraic geometry, the *deformation to the normal bundle.* In our setting this provides a smooth family of p-adic Lie groups, parametrized by the topological space $\mathbb{Q}_p$, whose fiber at 1 is G and whose fiber at 0 is $\mathfrak{g}$. Moreover there is a $\mathbb{Q}_p^\times$-equivariant structure on this deformation. One produces a fiberwise version of the dualizing object $\mathbb{D}_{BG}$ for this family of p-adic Lie groups over $\mathbb{Q}_p/\mathbb{Q}_p^\times$, and then the p-adic cospecialization map indicated at the beginning of this lecture provides a comparison map

$$\mathbb{D}_{B\mathfrak{g}} \to \mathbb{D}_{BG}$$

which one can check to be an equivalence on homology.

To produce the fiberwise dualizing object, and run the above argument "in families", it is necessary to have a clean and canonical formalism for relative duality. We accomplish this by producing a *six functor formalism* in the sense of Grothendieck. Such an approach also has other advantages; for example it produces an analog of *compactly supported cohomology* which allows to remove the compact p-torsionfree hypotheses from our main theorem (at the cost of replacing ordinary cohomology with its compactly supported variant, just as when one generalizes Poincare duality for real manifolds away from the compact case.)

We note that this spectrum-level duality for p-adic Lie groups has also been recently investigated in work of Beaudry-Goerss-Hopkins-Stojanoska. They prove the identification of dualizing objects after restricting to certain finite subgroups of G, and give interesting applications to chromatic homotopy theory and duality in TMF.

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On the homotopy theory of stratified spaces

Peter Haine

Trying to understand invariants of stratified topological spaces, such as intersection cohomology, naturally leads to the question of what the correct homotopy theory of stratified topological spaces is. Just as in the classical setting, we would like a 'homotopy hypothesis' for stratified spaces

$$\begin{pmatrix}\text{A homotopy theory of}\\ \text{stratified topological spaces}\end{pmatrix} \simeq \begin{pmatrix}\text{Purely homotopical}\\ \text{objects}\end{pmatrix},$$

where

(1) The 'purely homotopical' side is simple to define and has excellent formal properties (e.g., is a presentable ∞-category).
(2) The 'topological' side is also simple to define and captures all examples of differential-topological interest (e.g., *topologically stratified spaces* in the sense of Goresky–MacPherson [5, §1.1]).
(3) The equivalence is given by MacPherson's *exit-path* construction.

Though many have attempted to construct such a homotopy theory, notably Henriques [6, 7], Ayala and Francis with Rozenblyum [1] and Tanaka [2], and Nand-Lal [11], a homotopy theory of stratified topological spaces satisfying (1)–(3) does not yet exist. We report on our recent preprint [8] where we define a new homotopy theory of stratified topological spaces satisfying these criteria, and show that all existing homotopy theories of stratified topological spaces embed into ours.

Stratified topological spaces & exit-paths

Definition. The *Alexandroff topology* on a poset P is the topology on the underlying set of P in which a subset $U \subset P$ is open if and only if $x \in U$ and $y \geq x$ implies that $y \in U$. We simply write $P \in \mathbf{Top}$ for the set P equipped with the Alexandroff topology.

The category of *P-stratified topological spaces* is the overcategory $\mathbf{Top}_{/P}$. If $s\colon T \to P$ is a P-stratified topological space, for each $p \in P$ we write $T_p := s^{-1}(p)$ for the p^{th} *stratum* of T.

MacPherson had the idea that the 'stratified homotopy type' of a P-stratified topological space T should be determined by its 'exit-path ∞-category' $\mathrm{Exit}_P(T)$ with:

(0) Objects: points of T.
(1) 1-morphisms: *exit-paths*, that is, paths in T that flow from lower to higher strata, and once they exit a stratum are not allowed to return.
(2) 2-morphisms: homotopies between exit-paths respecting stratifications.

$$\vdots$$

It is difficult to make a construction of $\mathrm{Exit}_P(T)$ that is both precise and useful, but the takeaway is that $\mathrm{Exit}_P(T)$ should be an ∞-category with a functor to the poset P with strata ∞-groupoids. Another way of saying this is that the functor $\mathrm{Exit}_P(T) \to P$ is conservative. This idea informs what the 'purely homotopical' side of a stratified homotopy hypothesis should be:

Definition. The ∞-category of *abstract P-stratified homotopy types* is the ∞-category

$$\mathbf{Str}_P := \mathbf{Cat}^{\mathrm{cons}}_{\infty,/P} \subset \mathbf{Cat}_{\infty,/P}$$

of ∞-categories C over P with conservative structure morphism $C \to P$.

The following 'exit-path simplicial set' construction due to Henriques [7] and Lurie [9, §A.6] is an attempt to make MacPherson's idea precise.

Construction. Let P be a poset. There is a natural stratification

$$\pi_P \colon |N(P)| \to P$$

of the geometric realization of the nerve $N(P)$ of P by the Alexandroff space P extended from the natural $[n]$-stratification $|\Delta^n| \to [n]$ of the standard topological n-simplex defined by the assignment

$$(t_0, \dots, t_n) \mapsto \max \{ i \in [n] \mid t_i \neq 0 \} \ .$$

If X is a simplicial set over $N(P)$, then we can stratify the geometric realization $|X|$ by composing the structure morphism $|X| \to |N(P)|$ with π_P. This defines a left adjoint functor $|-|_P : s\mathbf{Set}_{/N(P)} \to \mathbf{Top}_{/P}$ with right adjoint $\mathrm{Sing}_P \colon \mathbf{Top}_{/P} \to s\mathbf{Set}_{/N(P)}$ computed by the pullback of simplicial sets

$$\mathrm{Sing}_P(T) := N(P) \times_{\mathrm{Sing}(P)} \mathrm{Sing}(T) \ ,$$

where the morphism $N(P) \to \mathrm{Sing}(P)$ is adjoint to π_P.

Here the stratified story diverges from the classical story: the simplicial set $\mathrm{Sing}_P(T)$ generally is *not* a quasicategory. This creates a lot of technical problems if one attempts to prove a stratified homotopy hypothesis by proving a Quillen equivalence between a model structure on $s\mathbf{Set}_{/N(P)}$ presenting $\mathbf{Str}_P$ and a model structure on $\mathbf{Top}_{/P}$.

Nevertheless, when $\mathrm{Sing}_P(T)$ *is* a quasicategory, it has the properties we want out of an exit-path ∞-category. Write $\mathbf{Top}^{\mathrm{ex}}_{/P} \subset \mathbf{Top}_{/P}$ for the full subcategory spanned by those P-stratified topological spaces T for which the exit-path simplicial set $\mathrm{Sing}_P(T)$ is a quasicategory. Let W denote the class of morphisms in $\mathbf{Top}^{\mathrm{ex}}_{/P}$ that are sent to weak equivalences in the Joyal model structure under Sing_P (i.e., equivalences of ∞-categories). The following is our 'stratified homotopy hypothesis', which we regard as a precise form of [1, Conjecture 0.0.4]:

Theorem (H.)**.** *For any poset P, the induced functor*

$$\mathrm{Sing}_P\colon \mathbf{Top}^{\mathrm{ex}}_{/P}[W^{-1}] \to \mathbf{Str}_P$$

is an equivalence of ∞-categories.

Our proof is somewhat indirect. Using the main result of Chapter 7 of Douteau's thesis [4], which realizes pioneering ideas of Henriques [7], we show that a 'nerve' functor provides an equivalence between an ∞-category obtained from $\mathbf{Top}_{/P}$ by inverting a class of weak equivalences and and a Segal space model for $\mathbf{Str}_P$ introduced in work with Barwick and Glasman [3, §4.2]. This immediately implies that $\mathrm{Sing}_P\colon \mathbf{Top}^{\mathrm{ex}}_{/P}[W^{-1}] \to \mathbf{Str}_P$ is fully faithful, and a bit more careful analysis shows that it is also essentially surjective.

Comparisons to conically smooth stratified spaces

In work with Tanaka [2, §3], Ayala and Francis introduced *conically smooth structures* on stratified topological spaces, which they further studied in work with Rozenblyum [1]. Their homotopy theory of P-stratified spaces is the ∞-category obtained from the category $\mathbf{Con}_P$ of conically smooth P-stratified spaces by inverting the class H of stratified homotopy equivalences. The functor Sing_P sends stratified homotopy equivalences to equivalences of ∞-categories, hence descends to a functor $\mathbf{Con}_P[H^{-1}] \to \mathbf{Str}_P$. The Ayala–Francis–Rozenblyum 'stratified homotopy hypothesis' states that this functor is fully faithful. Hence we have a commutative triangle of fully faithful functors of ∞-categories

$$\begin{array}{ccc} \mathbf{Con}_P[H^{-1}] & \xhookrightarrow{\mathrm{Sing}_P} & \mathbf{Str}_P \\ \big\downarrow & \overset{\sim}{\nearrow}{\scriptstyle \mathrm{Sing}_P} & \\ \mathbf{Top}^{\mathrm{ex}}_{/P}[W^{-1}] & & \end{array},$$

where the vertical functor is induced by the functor $\mathbf{Con}_P \to \mathbf{Top}^{\mathrm{ex}}_{/P}$ forgetting conically smooth structures.

One of the major benefits of the ∞-category $\mathbf{Top}^{\mathrm{ex}}_{/P}[W^{-1}]$ over $\mathbf{Con}_P[H^{-1}]$ is that all *conically stratified topological spaces* fit into this framework [9, Theorem A.6.4], in particular topologically stratified spaces in the sense of Goresky–MacPherson, and even more particularly Whitney stratified spaces [10, 12], are conically stratified. Thus the ∞-category $\mathbf{Top}^{\mathrm{ex}}_{/P}[W^{-1}]$ captures most, if not all, examples of differential-topological interest. On the other hand, it is still unknown whether or not every Whitney stratified space admits a conically smooth structure [1, Conjecture 0.0.7].

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Chromatic Smith theory and thick tensor-ideals of finite G-spectra

Markus Hausmann

(joint work with T. Barthel, J. Greenlees, N. Naumann, T. Nikolaus, J. Noel, N. Stapleton)

1. Smith theory

Let G be a compact Lie group. Smith theory is a classical subject in algebraic topology, originating in a series of papers of P. A. Smith [Smi38, Smi39], which studies the relationship between the homology of a finite G-CW complex X and the homology of its fixed points X^G. One of the basic theorems of this theory is the following:

Theorem (Smith)**.** *Let P be a finite p-group and X a finite P-CW complex. Then, if (the underlying space of) X has trivial reduced mod p homology, so does the fixed point space X^P.*

This talk dealt with the question of whether there are similar statements when reduced mod p homology is replaced by generalized homology theories. Specifically, given two generalized reduced homology theories h_* and h'_* and a compact Lie group G, we study the following question:

Question 1. Does there exist a finite based G-CW complex X with $h_*(X) = 0$ but $h'_*(X^G) \neq 0$?

Every reduced homology theory extends uniquely to a homology theory on finite spectra (which we denote by the same name). The answer to Question 1 then only depends on the kernels $\ker(h_*)$ and $\ker(h'_*)$, i.e., the full subcategory of those finite spectra on which h_* respectively h'_* vanish. This perspective is useful, because such kernels always form a thick subcategory of finite spectra (they are closed under taking cofibers, desuspensions and retracts) and these were classified by Mike Hopkins and Jeff Smith [HS98]. For simplicity we from now on localize at a fixed prime p, where their result says the following: The thick subcategories of finite p-local spectra form a descending chain

$$C_0 \supset C_1 \supset \ldots \supset C_n \supset \ldots \supset C_\infty,$$

where C_n can be defined as the kernel of the $(n-1)$-st Morava K-theory $K(n-1)$ (or, equivalently, the kernel of $(n-1)$-st Lubin-Tate theory E_{n-1}), C_0 is the category of all p-local finite spectra and C_∞ is the kernel of mod p homology.

Hence, given any finite p-local spectrum X, the set of all $n \in \mathbb{N} \cup \{\infty\}$ such that $X \in C_n$ has a maximal element, and this number is called the *type* of n. For a finite CW-complex, the type is defined to be that of (the p-localization of) its suspension spectrum. Question 1 can therefore be rephrased as: Given a finite based G-CW complex X whose underlying space has type n, what are the possible values of the type of the fixed points X^G?

2. Type functions and thick tensor-ideals of finite G-spectra

More generally, given a finite based G-CW complex X, one can consider the following function, where $\mathrm{Sub}(G)$ denotes the set of closed subgroups of G:

$$\begin{aligned} \mathrm{type}_X\colon \mathrm{Sub}(G) &\to \mathbb{N} \cup \{\infty\} \\ H &\mapsto \mathrm{type}(X^H) \end{aligned}$$

We call *type*$_X$ the *type function of* X. These functions make sense more generally for finite genuine G-spectra using geometric fixed points $\Phi^H(X)$ in place of the point set level fixed points. This way the type function of a finite based G-CW complex agrees with the type function of its suspension spectrum. We obtain the following generalized version of Question 1:

Question 2. Which functions $\mathrm{Sub}(G) \to \mathbb{N} \cup \{\infty\}$ can be realized as the type function of a finite p-local G-spectrum?

This question also turns out to be interesting from the perspective of tensor-triangular geometry. We recall that given a triangulated category $\mathcal{C}$ with a compatible symmetric monoidal structure $\otimes$ (in our case the homotopy category of finite p-local G-spectra equipped with the smash product), a thick tensor-ideal is a thick subcategory $\mathcal{I}$ with the additional property that if $X \in \mathcal{I}$ and $Y \in \mathcal{C}$, then also $X \otimes Y \in \mathcal{I}$. Then we have the following, proved in [BS17] for finite groups and in [BGH] for compact Lie groups.

Proposition. Let X, Y be two finite p-local G-spectra. Then X and Y generate the same thick tensor-ideal if and only if their type functions agree.

Hence, Question 2 is equivalent to the classification of (finitely-generated) thick tensor-ideals of the category of finite p-local G-spectra.

3. The abelian case

The main result presented in this talk is the answer to Question 2 in the case where the compact Lie group is abelian. It was proved for $\mathbb{Z}/p$ in [BS17], for finite abelian groups in [BHN$^+$19], and for higher dimensional abelian compact Lie groups in [BGH].

Theorem 1. *Let A be an abelian compact Lie group, and $f : \mathrm{Sub}(A) \to \mathbb{N} \cup \{\infty\}$ a function. Then the following are equivalent:*

1. *There exists a finite p-local A-spectrum X such that $f = type_X$.*
2. *The following two conditions are satisfied:*
   - (a) *For every inclusion of closed subgroups $K \subset H$ of G such that $\pi_0(H/K)$ is a p-group, the inequality*
   $$f(H) \geq f(K) - \mathrm{rank}_p(\pi_0(H/K))$$
   is satisfied.
   - (b) *The function f is locally constant for the Hausdorff topology on* $\mathrm{Sub}(A)$.

Here, $\mathrm{rank}_p(-)$ denotes the p-rank of an abelian p-group, i.e., its minimal number of generators. For the definition of the Hausdorff topology on the set of closed subgroups, which is discrete for finite groups, see [tD79, Sec. 5.6]. The following examples illustrate special cases of the theorem:

Example 1 (Tori)**.** If X is a finite p-local $\mathbb{T}^r$-spectrum, then the type of the geometric fixed points $\Phi^{\mathbb{T}^r}(X)$ is at least the type of the underlying spectrum of X. In other words, if $K(n)_*(X)$ is trivial, then so is $K(n)_*(\Phi^{\mathbb{T}^r}(X))$.

Example 2 (Cyclic p-groups)**.** If X is a finite p-local C_{p^k}-spectrum, then the type of the geometric fixed points $\Phi^{C_{p^k}}(X)$ is at least the type of the underlying spectrum of X minus 1. In other words, if $K(n)_*(X)$ is trivial, then so is $K(n-1)_*(\Phi^{C_{p^k}}(X))$.

Example 3 (Elementary abelian groups)**.** For every $n \in \mathbb{N}$ there exists a finite p-local $(\mathbb{Z}/p)^{\times n}$-spectrum X such that
$$\mathrm{type}(\Phi^H(X)) = n - k$$
for every subgroup H of rank k. In particular, the type of the underlying spectrum is n and the type of the $(\mathbb{Z}/p)^{\times n}$-geometric fixed points is 0.

4. Chromatic blueshift of Tate constructions

The tool for showing the implication (1) $\implies$ (2) in Theorem 1 or the statements of Examples 1 and 2 is to determine the blueshift for generalized Tate constructions. To define this, let E be a non-equivariant spectrum. We can turn E into a genuine G-spectrum $\underline{E}$ by giving it the trivial action and forming the associated Borel-spectrum. We then define a new non-equivariant spectrum $\varphi^G(E)$ as the geometric fixed points of this genuine G-spectrum, i.e.,
$$\varphi^G(E) = \Phi^G(\underline{E}).$$

When $G = C_p$ is a cyclic group of prime order, $\varphi^{C_p}(E)$ is the usual Tate spectrum E^{tC_p} of G, and in general there is always a map $E^{tG} \to \varphi^G(E)$. When E is complex orientable one can also obtain $\varphi^G(E)$ by inverting all Euler classes of non-trivial irreducible G-representations in the function spectrum $F(BG_+, E)$.

The following was first observed by Strickland in unpublished work and later refined in [BS17, Sec. 9]. Here, $\ker(-)$ denotes the kernel of a homology theory on finite p-local spectra:

Proposition. Assume there exists a spectrum E such that $\ker(E_*) = C_n$ and $\ker(\varphi^G(E)_*) = C_m$. Then every finite p-local G-spectrum X whose underlying spectrum is of type at least n has geometric fixed points $\Phi^G(X)$ of type at least m.

The proof follows from formal properties of the geometric fixed point functor.

Making use of this proposition, one main ingredient in the proof of Theorem 1 is the computation of $\ker(\varphi^G(E)_*)$ in the case where E is a Lubin-Tate spectrum E_{n-1}.

Theorem 2 ([BHN$^+$19])**.** *Let A be a finite abelian p-group. Then*

$$\ker(\varphi^A(E_{n-1})) = C_{\max(n-\operatorname{rank}_p(A),0)}.$$

The phenomenon that Tate constructions lower chromatic height is known as 'blueshift' and has been studied in various forms, for example in [GS96, HS96, AMS98].

5. Examples for elementary abelian groups

In order to obtain finite $(\mathbb{Z}/p)^n$-spectra (in fact finite based $(\mathbb{Z}/p)^n$-CW complexes) with the properties of Example 3, one can make use of constructions involving partition complexes that have been studied previously in connection to the Goodwillie tower of the identity. Given $k \in \mathbb{N}$, let $|\pi_k|^\diamond$ denote the unreduced suspension of the geometric realization of the poset of non-trivial proper partitions of the set $\{1, \ldots, k\}$, equipped with its natural Σ_k-action. Furthermore, let $S^{\overline{\rho}_k}$ denote the sphere associated to the reduced standard real Σ_k-representation. Then we define

$$X(n) = U(p^n-1)_+ \wedge_{\Sigma_{p^n}} (|\pi_{p^n}|^\diamond \wedge S^{\overline{\rho}_{p^n}}),$$

where we view Σ_{p^n} as a subgroup of $U(p^n-1)$ via the reduced standard complex representation. By definition, $X(n)$ carries a $U(p^n-1)$-action by multiplication from the left, which we pull back to a $(\mathbb{Z}/p)^n$-action via the reduced regular complex representation. The following is then a combination of results of [AD01, AL17, AM99, Aro98, Mit85]:

Theorem 3 (Arone, Dwyer, Lesh, Mahowald, Mitchell)**.** *The finite $(\mathbb{Z}/p)^n$-CW complexes $X(n)$ satisfy the properties of Example 3, i.e., for every subgroup H of rank k the type of the fixed points $X(n)^H$ equals $n-k$.*

On-going work of Kuhn and Lloyd gives a different construction of finite p-local $(\mathbb{Z}/p)^n$-spectra with these properties, using machinery first described by Jeff Smith in his proof that finite spectra allow v_n-self maps. Their method also produces new equivariant spectra which provide an answer to Question 2 for the dihedral group with eight elements, but as of today the answer for general compact Lie groups remains open.

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Stratification for spaces with Noetherian mod p cohomology

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(joint work with Tobias Barthel, Natalia Castellana, and Gabriel Valenzuela)

This is a report on the two papers [1, 2], joint with Tobias Barthel, Natalia Castellana, and Gabriel Vanezuela. In these, we study the following question:

Question. Given a commutative Noetherian ring spectrum R (that is, π_*R graded Noetherian), can we give a classification of the thick subcategories of compact objects in Mod_R, or even a classification of the localizing subcategories of Mod_R?

A very general approach for answering this question has been developed in a series of papers by Benson, Iyengar, and Krause [4], who have used it to classify the localizing subcategories of the stable module category of a finite group.

Specialized to our case, their approach relies on the construction of certain local cohomology functors $\Gamma_{\mathfrak{p}}\colon \mathrm{Mod}_R \to \mathrm{Mod}_R$ for each homogeneous prime ideal $\mathfrak{p} \in \mathrm{Spec}^h(\pi_* R)$. The essential image $\Gamma_{\mathfrak{p}} \mathrm{Mod}_R$ forms a non-zero localizing subcategory of Mod_R, and we have the following definition.

Definition 1. We say that Mod_R is stratified by $\pi_* R$ if each $\Gamma_{\mathfrak{p}} \mathrm{Mod}_R$ is a minimal localizing subcategory of Mod_R.

The support of an R-module M is defined to be

$$\mathrm{supp}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}^h(\pi_* R) \mid \Gamma_{\mathfrak{p}} M \neq 0\},$$

and we have the following result.

Theorem 1 ([4]). *If Mod_R is stratified by $\pi_* R$, then there is a one-to-one correspondence between localizing subcategories of Mod_R and subsets of $\mathrm{Spec}^h(\pi_* R)$. The map giving this correspondence sends a localizing subcategory $\mathcal{U} \subseteq \mathrm{Mod}_R$ to the set $\cup_{M \in \mathcal{U}} \mathrm{supp}_R(M)$, with inverse that sends a subset $\mathcal{S} \subseteq \mathrm{Spec}^h(\pi_* R)$ to $\{M \in \mathrm{Mod}_R \mid \mathrm{supp}_R(M) \subseteq \mathcal{S}\}$.*

If $R = C^*(BG; \mathbb{F}_p)$ is the commutative ring spectrum of mod p cochains on a connected compact Lie group, then Benson and Greenlees [3] showed that $\mathrm{Mod}_{C^*(BG;\mathbb{F}_p)}$ is stratified by $H^*(BG; \mathbb{F}_p)$. The proof follows the general strategy of Benson, Iyengar, and Krause: one knows that the theorem is true when G is an elementary abelian group, and then one tries to use descent to deduce the result for an arbitrary connected compact Lie group. The crucial properties that make the descent work are the following:

(1) The map

$$\phi_G\colon C^*(BG; \mathbb{F}_p) \to \prod_{E<G} C^*(BE; \mathbb{F}_p)$$

induced by the inclusion of elementary abelian p-subgroups $E < G$ is biconservative; induction and coinduction along it detect trivial modules.

(2) The map ϕ_G is finite, that is $\prod_{E<G} C^*(BE; \mathbb{F}_p)$ is a compact $C^*(BG; \mathbb{F}_p)$-module.

(3) Quillen's stratification theorem, which tells us how $\mathrm{Spec}^h(H^*(BG; \mathbb{F}_p)$ can be decomposed in terms of the variety associated to its elementary abelian p-subgroups.

It is clear how to generalize condition (1) above to an arbitrary morphism $f\colon R \to S$ of commutative ring spectra. We are interested in not-necessarily finite morphisms of ring spectra, and so we want to remove condition (2) above. In order to state the generalization of condition (3), we note that dual to the theory of support, there is a theory of cosupport.

Definition 2 ([2]). Let $f\colon R \to S$ be a morphism of commutative Noetherian ring spectra, and let $\mathrm{res}_f\colon \mathrm{Spec}^h(\pi_* S) \to \mathrm{Spec}^h(\pi_* R)$. We say that f satisfies simple

Quillen lifting if for any $M \in \mathrm{Mod}_R$:

$$\mathrm{res}_f^{-1}\,\mathrm{res}_f\,\mathrm{supp}_S(\mathrm{Ind}_R^S M) = \mathrm{supp}_S(\mathrm{Ind}_R^S M) \quad \text{and}$$
$$\mathrm{res}_f^{-1}\,\mathrm{res}_f\,\mathrm{cosupp}_S(\mathrm{Coind}_R^S M) = \mathrm{cosupp}_R(\mathrm{Coind}_R^S M).$$

We then have the following.

Theorem 2 ([1, 2])**.** *Suppose that $f\colon R \to S$ is a biconservative morphism of commutative Noetherian ring spectra satisfying simple Quillen lifting. If* Mod_S *is stratified by π_*S, then* Mod_R *is stratified by π_*R.*

For example, Quillen's stratification implies that ϕ_G satisfies simple Quillen lifting, and so we can immediately extend the result of Benson and Greenlees to all compact Lie groups.

In [7], Broto, Levi, and Oliver introduced the concept of p-local compact groups as a common generalization of the notions of compact Lie groups, p-compact group [8] as well as fusion systems $\mathcal{F}$ on a finite group [6]. The data of a p-local compact group can be specified as a pair $(S, \mathcal{F})$ consisting of a discrete p-toral group S, and a saturated fusion system $\mathcal{F}$ on S. To such a pair, we can associate a classifying space $B\mathcal{F}$, and then we can consider the ring spectrum $R = C^*(B\mathcal{F}; \mathbb{F}_p)$. In order to apply our theory, we need the following.

Theorem 3 ([1])**.** *Let $(S, \mathcal{F})$ be a p-local compact group. Then:*

1. *The mod p cohomology ring $H^*(B\mathcal{F}; \mathbb{F}_p)$ is Noetherian.*
2. *A form of Quillen's stratification theorem holds for* $\mathrm{Spec}^h(H^*(B\mathcal{F}; \mathbb{F}_p))$.

Part (2) is the key input into the following.

Theorem 4 ([1])**.** *Let $(\mathcal{F}, S)$ be a p-local compact group, then the canonical morphism*

$$f\colon C^*(B\mathcal{F}; \mathbb{F}_p) \to \prod_{E \in \mathcal{E}(\mathcal{F})} C^*(BE; \mathbb{F}_p)$$

satisfies Quillen lifting. Here $\mathcal{E}(\mathcal{F})$ denotes a set of representatives of $\mathcal{F}$-isomorphism classes of elementary abelian subgroups of S.

We do not know that f is biconservative in general, however we have shown it when $(S, \mathcal{F})$ models a connected p-compact group, or when S is a finite p-group. In these cases, we deduce that $\mathrm{Mod}_{C^*(B\mathcal{F};\mathbb{F}_p)}$ is stratified by $H^*(B\mathcal{F}; \mathbb{F}_p)$.

In [2] we considered the more general situation when $R = C^*(X; \mathbb{F}_p)$ for a space X such that $H^*(X; \mathbb{F}_p)$ is Noetherian. One is immediately confronted with the problem of finding the elementary abelian 'subgroups' of X. Fortunately, work of Rector applies here [10]. Namely, one considers the category $\mathcal{R}(X)$ whose objects are pairs (E, f) where E is an elementary abelian group, and $f\colon H^*(X; \mathbb{F}_p) \to H^*(BE; \mathbb{F}_p)$ is a morphism of unstable algebras over the Steenrod algebra, such that $H^*(BE; \mathbb{F}_p)$ is a finitely-generated $H^*(X; \mathbb{F}_p)$-module. These assemble into a morphism

$$H^*(X; \mathbb{F}_p) \to \prod_{\mathcal{R}(X)} H^*(BE; \mathbb{F}_p).$$

If X is connected and p-good, then Lannes' theory applies [9], and shows that this can be realized as a morphism of commutative Noetherian ring spectra:

$$\phi_X : C^*(X; \mathbb{F}_p) \to \prod_{\mathcal{R}(X)} C^*(BE; \mathbb{F}_p).$$

Using Rector's work, we prove that ϕ_X always satisfies simple Quillen lifting, and deduce the following.

Theorem 5 ([2])**.** *Let X be a connected p-good space with Noetherian mod p cohomology, then $\mathrm{Mod}_{C^*(C;\mathbb{F}_p)}$ is stratified by $H^*(X;\mathbb{F}_p)$ if and only if ϕ_X is bi-conservative.*

Using a structure theorem for connected H-spaces due to Broto, Crespo, and Saumell [5], we deduce that if X is a connected H-space with Noetherian mod p cohomology ring, then $\mathrm{Mod}_{C^*(X;\mathbb{F}_p)}$ is stratified by $H^*(X;\mathbb{F}_p)$.

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Bökstedt periodicity and quotients of DVRs

Achim Krause

(joint work with Thomas Nikolaus)

This talk discusses an approach to compute $\mathrm{THH}(R)$ for R a quotient of a discrete valuation ring, following [6]. While the result in this generality is new, we also recover Brun's description [3] of $\mathrm{THH}(\mathbb{Z}/p^n)$ quite elegantly. On the way, we also review Bökstedt's classical result [2] on $\mathrm{THH}(\mathbb{F}_p)$ as well as Lindenstrauss-Madsens result [8] on $\mathrm{THH}(A)$ for A a complete discrete valuation ring.

Throughout these notes, we frequently consider an ordinary ring R as a $\mathbb{E}_\infty$ ring spectrum (i.e., the corresponding Eilenberg-MacLane spectrum) without distinguishing between them notationally. This is justified because the full subcategory of $\mathbb{E}_\infty$ ring spectra on those objects with homotopy groups concentrated in degree 0 is equivalent to the category of rings. We will thus freely write things such as $\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p$ (which more classically would be written $H\mathbb{F}_p \wedge H\mathbb{F}_p$). If the base is an ordinary ring, we will however write $- \otimes^{\mathbb{L}}_R -$ to avoid confusion with the underived tensor product. Since the Dold-Kan correspondence yields a symmetric-monoidal equivalence $\mathcal{D}(R) \to \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp})$, there is no essential difference between forming $- \otimes^{\mathbb{L}}_R -$ as the tensor product of the associated spectra, or classically in chain complexes.

1. Bökstedt periodicity

Recall that, for a ring spectrum R, its topological Hochschild homology is defined as

$$\mathrm{THH}(R) = R \otimes_{R \otimes_{\mathbb{S}} R^{\mathrm{op}}} R \simeq \mathrm{Bar}_R(R \otimes_{\mathbb{S}} R^{\mathrm{op}}).$$

We write $\mathrm{THH}(R; \mathbb{Z}_p)$ for the p-completion of the spectrum $\mathrm{THH}(R)$.

Classical (derived) Hochschild homology of an ordinary ring R is defined similarly to THH, as the derived tensor product

$$\mathrm{HH}(R) = R \otimes^{\mathbb{L}}_{R \otimes^{\mathbb{L}}_{\mathbb{Z}} R^{\mathrm{op}}} R.$$

For $R = \mathbb{F}_p$, one can directly compute $\mathbb{F}_p \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{F}_p$ by resolving $\mathbb{F}_p$, and one obtains an exterior algebra $\Lambda_{\mathbb{F}_p}(\varepsilon)$ over $\mathbb{F}_p$ on a generator of degree 1. One thus sees

$$\mathrm{HH}(\mathbb{F}_p) = \mathbb{F}_p \otimes^{\mathbb{L}}_{\mathbb{F}_p \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{F}_p} \mathbb{F}_p \cong \mathrm{Bar}_{\mathbb{F}_p}(\Lambda_{\mathbb{F}_p}(\varepsilon))$$

to be a divided power algebra on a generator in degree 2. Explicitly, as rings,

$$\mathrm{HH}_*(\mathbb{F}_p) \cong \mathbb{F}_p[x, x^{[p]}, x^{[p^2]}, \ldots]/(x^p, (x^{[p]})^p, \ldots).$$

Since this is already quite complicated, one would expect that $\mathrm{THH}(\mathbb{F}_p)$, where we replace the base ring $\mathbb{Z}$ by the sphere spectrum $\mathbb{S}$, becomes quite unwiedly. In fact, while $\mathbb{F}_p \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{F}_p$ is an exterior algebra, $\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p$ has homotopy given by the entire dual Steenrod algebra. It therefore comes as a surprise that $\mathrm{THH}(\mathbb{F}_p)$ is simpler than $\mathrm{HH}(\mathbb{F}_p)$:

Theorem 1 (Bökstedt periodicity)**.**

$$\mathrm{THH}_*(\mathbb{F}_p) \cong \mathbb{F}_p[x]$$

with x of degree 2.

We give a somewhat streamlined proof here. The essential observation is the following:

Lemma 2. *As an $\mathbb{E}_2$-algebra over $\mathbb{F}_p$, $\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p$ is free on a generator in degree* 1*, i.e.*

$$\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p \cong \mathbb{F}_p[\Omega^2 S^3].$$

Proof. We have $\pi_1(\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p) \cong \mathbb{F}_p$. Choosing a generator, we obtain an $\mathbb{E}_2$-map

$$\mathbb{F}_p[\Omega^2 S^3] \to \mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p$$

which is an isomorphism in degree 1. One then checks that under the action of $\mathbb{E}_2$ power operations, the homotopy of both sides is generated by an element in degree 1, and both sides have the same structure. For the left hand side, this is due to Araki-Kudo [7] and Dyer-Lashof [5], for the right-hand side this is due to Milnor [9] and Steinberger [4]. It follows that the map is an equivalence. □

Proof of Bökstedt's theorem. Using Lemma 2, we see

$$\mathrm{THH}(\mathbb{F}_p) \simeq \mathrm{Bar}_{\mathbb{F}_p}(\mathbb{F}_p[\Omega^2 S^3]) \simeq \mathbb{F}_p[\mathrm{Bar}_{\mathcal{S}}(\Omega^2 S^3)] \simeq \mathbb{F}_p[\Omega S^3],$$

from which the claim follows. □

One can actually reverse the logic and check that Theorem 1 implies 2 as well. Thus, any proof of Bökstedt's theorem also determines the structure of the dual Steenrod algebra (as an algebra together with $\mathbb{E}_2$-power operations).

Bökstedt's theorem holds more generally for k a perfect field of characteristic p, in the sense that

$$\mathrm{THH}_*(k) \cong k[x],$$

with x in degree 2, and can be deduced from the $\mathbb{F}_p$ version above by a basechange argument.

2. Bökstedt periodicity for DVRs

We let A be a complete mixed-characteristic discrete valuation ring with perfect residue field k. Let $\pi \in A$ denote a uniformizer. For example, we could consider $A = \mathbb{Z}_p$, $\pi = p$, or $\mathbb{Z}_p[\zeta_p]$, $\pi = \zeta_p - 1$.

$\mathrm{THH}(A)$ is complicated. However, we understand THH of $A/\pi \cong k$ by Bökstedt periodicity. We thus want to consider THH "relative" π.

Definition 3. *Define the $\mathbb{E}_\infty$ ring spectrum*

$$\mathbb{S}[z] := \mathbb{S}[\mathbb{N}] = \Sigma^\infty_+ \mathbb{N}.$$

For A, π as above, we obtain a map $\mathbb{S}[z] \to A$ sending $z \mapsto \pi$, and set

$$\mathrm{THH}(A/\mathbb{S}[z]) := A \otimes_{A \otimes_{\mathbb{S}[z]} A} A.$$

Theorem 4.

$$\mathrm{THH}_*(A/\mathbb{S}[z]; \mathbb{Z}_p) \cong A[x],$$

with x of degree 2.

Proof. As a p-complete A-module, $\mathrm{THH}_*(A/\mathbb{S}[z]; \mathbb{Z}_p)$ is also π-complete, since as ideals, $(\pi)^e = (p)$, where e is the ramification index of A. We have that

$$\mathrm{THH}(A/\mathbb{S}[z]) \otimes_A A/\pi \simeq \mathrm{THH}(A/\mathbb{S}[z]) \otimes_{\mathbb{S}[z]} \mathbb{S} \simeq \mathrm{THH}(k),$$

and so the cofiber of $\mathrm{THH}(A/\mathbb{S}[z]) \xrightarrow{\pi} \mathrm{THH}(A/\mathbb{S}[z])$ has homotopy groups given by k in each even degree. Completeness now implies the claim. □

This idea of working relative to the uniformizer $\mathbb{S}[z]$ to obtain a mixed characteristic version of Bökstedt periodicity goes back to Lurie and Scholze. The above version of Theorem 4 appeared first in [1].

For $A' = A/\pi^k$ a quotient of a complete DVR, we have a similarly nice description:

Proposition 5.

$$\mathrm{THH}_*(A'/\mathbb{S}[z]) \simeq A'[x]\langle y\rangle,$$

with x and y generators in degree 2, and the notation $R\langle y\rangle$ denoting the free divided power algebra over R on one generator.

Proof. We can write

$$A' = A \otimes_{\mathbb{S}[z]} (\mathbb{S}[z]/z^k),$$

and thus

$$\begin{aligned}\mathrm{THH}(A'/\mathbb{S}[z]) &\simeq \mathrm{THH}(A/\mathbb{S}[z]) \otimes_{\mathbb{S}[z]} \mathrm{THH}((\mathbb{S}[z]/z^k)/\mathbb{S}[z])\\ &\simeq \mathrm{THH}(A/\mathbb{S}[z]) \otimes_A \mathrm{HH}((A/\pi^k)/A)\\ &\simeq \mathrm{THH}(A/\mathbb{S}[z];\mathbb{Z}_p) \otimes_A \mathrm{HH}((A/\pi^k)/A).\end{aligned}$$

By Theorem 4, the homotopy groups of the left factor are given by A in each even degree. In particular, they are flat over A, by a Künneth argument $\mathrm{THH}_*(A'/\mathbb{S}[z])$ is just the tensor product of $A[x]$ and $\mathrm{HH}_*((A/\pi^k)/A)$. The latter is easily seen to be a divided power algebra over $A/\pi^k = A'$, since $A' \otimes^{\mathbb{L}}_A A'$ is an exterior algebra $\Lambda_{A'}(\varepsilon)$ on a generator in degree 1. □

3. Getting back to absolute THH

For an algebra R over $\mathbb{S}[z]$, given $\mathrm{THH}(R)$, one can recover $\mathrm{THH}(R/\mathbb{S}[z])$ by the formula

$$\mathrm{THH}(R/\mathbb{S}[z]) \simeq \mathrm{THH}(R) \otimes_{\mathrm{THH}(\mathbb{S}[z])} \mathbb{S}[z].$$

If R is in addition a $\mathbb{Z}$-algebra (e.g. an ordinary ring), one can write this further as

$$\mathrm{THH}(R/\mathbb{S}[z]) \simeq \mathrm{THH}(R) \otimes_{\mathbb{Z}\otimes_{\mathbb{S}}\mathrm{THH}(\mathbb{S}[z])} (\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{S}[z]). \tag{1}$$

We want to reverse this:

Proposition 6. *For a $\mathbb{Z}[z]$-algebra R, there is a multiplicative spectral sequence*

$$\mathrm{THH}_*(R/\mathbb{S}[z]) \otimes \Lambda(dz) \Rightarrow \mathrm{THH}_*(R),$$

with dz of degree 1. Similarly, we have a p-completed version of this spectral sequence.

Proof. This is obtained from observing that

$$\mathbb{Z} \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{S}[z]) \simeq \mathrm{HH}(\mathbb{Z}[z])$$

has homotopy groups $\mathbb{Z}[z] \otimes \Lambda(dz)$, i.e. $\pi_0 = \pi_1 = \mathbb{Z}[z]$. If we apply the Postnikov filtration to the right factor in

$$\mathrm{THH}(R) \simeq \mathrm{THH}(R) \otimes_{\mathbb{Z}\otimes_{\mathbb{S}}\mathrm{THH}(\mathbb{S}[z])} (\mathbb{Z} \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{S}[z])),$$

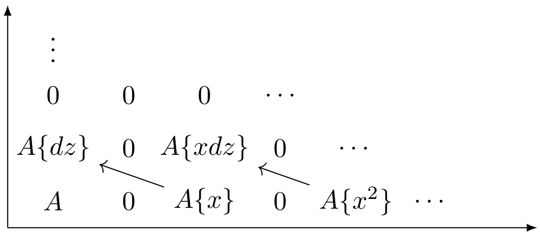
we thus obtain two copies of THH($R/\mathbb{S}[z]$) by equation (1), suitably shifted. The associated spectral sequence has the desired form. ☐

The resulting spectral sequence has two lines only, both of the form $\mathrm{THH}_*(R/\mathbb{S}[z])$. One could equivalently work with the fiber sequence

$$\Sigma\,\mathrm{THH}(R/\mathbb{S}[z]) \to \mathrm{THH}(R) \to \mathrm{THH}(R/\mathbb{S}[z]).$$

Example 7 (Lindenstrauss-Madsen [8])**.** For A a complete mixed-characteristic DVR with perfect residue field k and uniformizer π, we obtain a spectral sequence of the form

$$A[x] \otimes \Lambda(dz) \Rightarrow \mathrm{THH}_*(A; \mathbb{Z}_p).$$

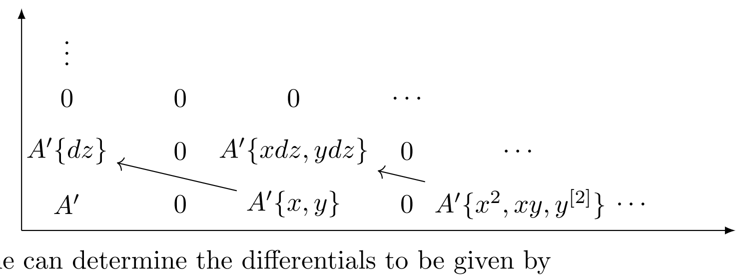


We have $\mathrm{THH}_1(A;\mathbb{Z}_p) \cong \mathrm{HH}_1(A;\mathbb{Z}_p)$, and one can further compute $\mathrm{HH}_1(A;\mathbb{Z}_p)$ to be $\Omega^1_{A/W(k)}$, where $W(k) \subseteq A$ denotes the p-typical Witt vectors of the residue field $k = A/\pi$. One can choose a minimal polynomial E for π over $W(k)$, such that $A = W(k)[z]/E(z)$, and then $\Omega^1_{A/W(k)} = A\{dz\}/E'(\pi)dz$. We thus get that $d_2(x) = E'(\pi)dz$ up to units, and we can choose our generator x such that this holds on the nose. By multiplicativity we get $d_2(x^n) = nE'(\pi)dz \cdot x^{n-1}$, and from the spectral sequence we recover

$$\mathrm{THH}_*(A;\mathbb{Z}_p) \cong \begin{cases} A \text{ for } * = 0, \\ A/nE'(\pi) \text{ for } * = 2n-1, \\ 0 \text{ otherwise.} \end{cases}$$

Example 8. For $A' = A/\pi^k$ a quotient of a DVR, the spectral sequence takes the form

$$A'[x]\langle y\rangle \otimes \Lambda(dz) \Rightarrow \mathrm{THH}_*(A/\pi^k).$$



Here, one can determine the differentials to be given by

$$d_2(x) = E'(\pi)dz$$
$$d_2(y) = k\pi^{k-1}dz,$$

together with multiplicativity and the fact that the differentials are compatible with the divided power structure, i.e. $d_2(y^{[i]}) = d_2(y) \cdot y^{[i-1]}$.

Again, since there is one nonzero entry only per total degree, the E_∞ page completely determines $\mathrm{THH}_*(A')$ multiplicatively. Thus, we obtain a description of $\mathrm{THH}_*(A')$ as the homology of an explicit dga. This does not seem to have a general closed-form expression, but there are interesting special cases. If k is small enough compared to the valuation of $E'(\pi)$, one can split off a polynomial generator. For $A = \mathbb{Z}_p$, this happens exactly if $k = 1$, in the Bökstedt case. If k is large enough, one can split off a divided power generator, recovering a generalisation of Brun's result on $\mathrm{THH}(\mathbb{Z}/p^k)$. For highly ramified A, there are "in between" cases which have a more complicated structure.

Theorem 9. *We have*

$$\mathrm{THH}_*(A') = H_*(A'[x]\langle y\rangle \otimes \Lambda(dz); \partial),$$

with $\partial x = E'(z)dz$, $\partial y^{[i]} = k\pi^{k-1}y^{[i-1]}dz$.

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New results about the equivariant stable homotopy Balmer spectrum

Nicholas J. Kuhn

(joint work with Chris Lloyd)

1. The $K(n)$ Smith Theory Problem

In [1], Balmer and Sanders study tensor triangulated ideals in the homotopy category of finite G–spectra for a finite group G. The general classification problem is reduced to a problem about finite p–groups as we now describe.

Let all spectra be localized at a prime p. By finite G–spectra we mean retracts of (p-local) finite G–CW spectra. If $H < G$ is a subgroup, we let $X^{\phi H}$ denote the geometric fixed point spectrum of a G–spectrum X. Let $K(n)$ denote the nth

Morava K–theory ring spectrum (at p), with $K(0) = H\mathbb{Q}$ and $K(\infty) = H\mathbb{F}_p$. Recall that X is of *type* n if $K(n)_*(X) \neq 0$, while $K(n-1)_*(X) = 0$.

K(n) Smith Theory Problem Let H be a subgroup of a finite p–group G. For what pairs (m, n) is it true that $K(m)_*(X^{\phi H}) = 0 \Rightarrow K(n)_*(X^{\phi G}) = 0$?

Example 1.1. This is true for all $H < G$ when $(m, n) = (\infty, \infty)$: this is the theorem that $\tilde{H}_*(Z^H; \mathbb{F}_p) = 0 \Rightarrow \tilde{H}_*(Z^G; \mathbb{F}_p) = 0$, when Z is a retract of a finite G–CW complex. This was proved by P.A.Smith in the 1940s.

2. The Realization Problem

The $K(n)$ Smith Theory Problem can be recast as follows.

Realization Problem Given $H < G$, for what pairs (m, n) does there exist a finite G–spectrum X with $X^{\phi G}$ of type n and $X^{\phi H}$ of type m?

Lemma 2.1. *Given $H < G$, if (m, n) can be realized, then one can also realize (m', n) for all $m' < m$, and (m, n') for all $n' > n$.*

Definition 2.2. Given n and $H < G$, let $r_n(H, G)$ denote the maximal r such that there exists a finite G–spectrum with $X^{\phi G}$ of type n and $X^{\phi H}$ of type $n + r$.

Realization Problem Restated Given $H < G$, compute $r_n(H, G)$ for all n.

3. Some reductions and a general upper bound

The next lemma includes some ways of reducing this problem to easier cases.

Lemma 3.1. *Let H be a subgroup of a finite p–group G.*
(a) *If $\alpha : G \xrightarrow{\sim} G$ is an automorphism, then $r_n(\alpha(H), G) = r_n(H, G)$.*
(b) *Given $N \lhd G$, $r_n(H, G) \geq r_n(HN, G) = r_n(HN/N, G/N)$.*
(c) *Given $H < L < G$, if, for all n, $r_n(H, L) \leq s$ and $r_n(L, G) \leq t$, then $r_n(H, G) \leq s + t$.*

To get a general upper bound on $r_n(H, G)$, the authors of [2] prove the following 'Blue Shift Theorem'. (The $k = 1$ case was proved first in [1].)

Theorem 3.2. $r_n(e, C_{p^k}) \leq 1$ *for all n.*

When combined with the previous lemma, this has a nice corollary.

Definition 3.3. Given a finite p–group G and $H < G$, let $r(H, G)$ be the minimal r such that there exists a chain of subgroups $H = L_0 \lhd L_1 \lhd \cdots \lhd L_r = G$ with L_i/L_{i-1} cyclic for all i.

Corollary 3.4. $r_n(H, G) \leq r(H, G)$ *for all n.*

The following question is unresolved in general.

Question Does $r_n(H, G) = r(H, G)$ for all $H < G$ and n?

4. Previously known lower bounds

One way to get lower bounds on $r_n(H,G)$ is to find examples. In [2] it is observed that examples in the literature constructed by Arone (channeling Mitchell), and analyzed by Arone and Lesh, have the following properties.

Example 4.1. There exist a finite $(C_p)^r$–spectrum X of type r with $X^{\phi(C_p)^r}$ of type 0. Thus $r \leq r_0(e,(C_p)^r)$.

Combined with previous results, one can deduce the next theorem from this.

Theorem 4.2. [2] *$r_n(H,G) = r(H,G)$ for all n for $H \lhd G$ and G/H abelian.*

5. A new result

From [2], one can also already deduce that $r_n(H,G) = r(H,G)$ for some other cases too. The first pair (H,G) *not* determined by previous work is (C, D_8) where C is any noncentral subgroup of order 2 in D_8, the dihedral group of order 8.

Theorem 5.1. *For all n, there exists a finite D_8–spectrum Y_n such that $Y_n^{\phi D_8}$ has type n and $Y_n^{\phi C}$ has type $n+2$. Thus $r_n(C,D_8) = 2 = r(C,D_8)$.*

To show this, and to resolve the general question for more groups, we need a source of new examples!

6. How to construct a type n complex

For simplicity, we will assume that $p = 2$ from now on.

Let $k_n(X)$ be the dimension of $K(n)_*(X)$ as a $K(n)_*$–vector space. Ravenel showed that $k_0(X) \leq k_1(X) \leq \cdots \leq k_\infty(X)$.

Here is a sneaky idea used by Jeff Smith in the 1980's.

Suppose given N, and an idempotent $e \in \mathbb{Z}_{(2)}[\Sigma_N]$ (so $e = e^2$). Given a a vector space V over a (possibly graded) field of characteristic 2, the vector space $eV^{\otimes N}$ will be a direct summand of the tensor product $V^{\otimes N}$.

Similarly, if X is a finite spectrum, $eX^{\wedge N}$ will be a wedge summand of the smash product $X^{\wedge N}$, and $K(n)_*(eX^{\wedge N}) = eK(n)_*(X)^{\otimes N}$.

Proposition 6.1. [3, Appendix C] *For each d, there exists N_d and an idempotent $e_d \in \mathbb{Z}_{(2)}[\Sigma_{N_d}]$ such that $e_d V^{\otimes N_d} \neq 0 \Leftrightarrow \dim V \geq d$.*

Corollary 6.2. *If $k_{n-1}(X) < d \leq k_n(X)$, and $Y = e_d X^{\wedge N_d}$, then Y has type n.*

Smash products play well with geometric fixed points, so we get an equivariant version of this corollary.

Theorem 6.3. *Given $H < G$, suppose there exists a finite G–spectrum X and d such that*

$$\clubsuit \qquad \max\{k_{m-1}(X^{\phi H}), k_{n-1}(X^{\phi G})\} < d \leq \min\{k_m(X^{\phi H}), k_n(X^{\phi G})\}.$$

If we let $Y = e_d X^{\wedge N_d}$, Y will be a finite G–spectrum such that $Y^{\phi G}$ has type n and $Y^{\phi H}$ has type m.

7. Examples from representation theory

Let V be a real representation of G. Then $V = \bigoplus_\lambda V_\lambda$, where λ runs through the irreducible representations, and V_λ is the corresponding isotypical summand of V. Let $\mathbb{RP}(V)$ be the G–space of lines in V.

Lemma 7.1. *$\mathbb{RP}(V)^G_+ = \bigvee_\lambda \mathbb{RP}(V_\lambda)_+$, where the wedge is over the one dimensional irreducibles.*

Example 7.2. The group D_8 has 5 irreducible real representations: four of dimension 1 – call them $\sigma_1, \dots, \sigma_4$ – and a two dimensional irreducible τ. Let $V_0 = 2\sigma_1 \oplus 2\sigma_2 \oplus 2\sigma_3 \oplus 2\sigma_4 \oplus \tau$. By the lemma, $\mathbb{RP}(V_0)^{D_8}_+ = \bigvee_4 \mathbb{RP}^1_+$, so $k_0(\mathbb{RP}(V_0)^{D_8}_+) = 8$. Now let $C < D_8$ be a noncentral subgroup of order 2. Restricted to C, $V_0 \simeq 5(\text{triv}) \oplus 5(\text{sign})$, so that $\mathbb{RP}(V_0)^C_+ = \bigvee_2 \mathbb{RP}^4_+$. Easy calculations show that $k_2(\mathbb{RP}(V_0)^C_+) = 8$ and $k_1(\mathbb{RP}(V_0)^C_+) = 4$. Thus we have

$$\diamondsuit \qquad k_1(\mathbb{RP}(V_0)^C_+)\} < 5 \leq \min\{k_2(\mathbb{RP}(V_0)^C_+), k_0(\mathbb{RP}(V_0)^{D_8}_+)\}.$$

$N_5 = 15$, and we conclude that if $Y_0 = e_5 \mathbb{RP}(V_0)^{\wedge 15}$, then Y_0 is a finite D_8–spectrum with $Y_0^{\phi D_8}$ of type 0 and $Y_0^{\phi C}$ of type 2.

The spectra Y_n as in Theorem 5.1 are similarly constructed for other n, using the representations $V_n = 2^{n+1}\sigma_1 \oplus 2^{n+1}\sigma_2 \oplus 2^{n+1}\sigma_3 \oplus 2^{n+1}\sigma_4 \oplus \tau$.

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Weight decompositions and automorphism groups of manifolds

Alexander Kupers

(joint work with Manuel Krannich)

We are interested in the classifying spaces $B\mathrm{Diff}_\partial(W_{g,1})$ of the topological group of diffeomorphisms of the manifolds $W_{g,1} = D^{2n} \# (S^n \times S^n)^{\# g}$ fixing a neighborhood of $\partial W_{g,1}$ pointwise. These classify smooth manifold bundles with fiber $W_{g,1}$ and trivialised boundary bundle, so their cohomology ring is the ring of characteristic classes of such bundles.

These characteristic classes were determined in a range depending on g by Galatius and Randal-Williams in terms of stable homotopy theory [GRW14, GRW18]. Rationally, their work shows that for $* \leq \frac{g-3}{2}$ the ring $H^*(B\mathrm{Diff}_\partial(W_{g,1}); \mathbb{Q})$ is a free polynomial algebra on classes κ_c of degree $|c| - 2n$, where c runs over a homogeneous basis of $H^{*>2n}(BO(2n)\langle n \rangle; \mathbb{Q})$. In low degrees with respect to g, however, very little is known.

Question. What is $H^*(B\mathrm{Diff}_\partial(W_{g,1});\mathbb{Q})$ for $2n \geq 6$ but g small, for instance $g = 1$?

We provide a complete answer to this question in degrees $* < 2n-5$ in terms of the cohomology of arithmetic groups (see Theorem 1.1). In degrees $* \geq 2n-5$, we still construct nontrivial classes, but unlike in the range $* < 2n-5$, these probably do not exhaust all of $H^*(B\mathrm{Diff}_\partial(W_{g,1};\mathbb{Q})$. Our results admit several generalisations, for instance to manifolds other than $W_{g,1}$.

1. The strategy

The diffeomorphism group of $W_{g,1}$ acts on the $\mathbb{Q}$-vector space $H^n := H^n(W_{g,1};\mathbb{Q})$ through its image G'_g in the automorphisms of $H^n(W_{g,1};\mathbb{Q})$ preserving the intersection product,

$$G'_g = \begin{cases} \mathrm{O}_{g,g}(\mathbb{Z}) & \text{if } n \text{ is even} \\ \mathrm{Sp}_{2g}(\mathbb{Z}) & \text{if } n = 1,3,7, \\ \mathrm{Sp}^q_{2g}(\mathbb{Z}) & \text{otherwise,} \end{cases}$$

where $\mathrm{Sp}^q_{2g}(\mathbb{Z})$ is the subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$ preserving the standard quadratic refinement with vanishing Arf invariant.

Let us fix some notation. We consider H^n as a graded $\mathbb{Q}$-vector space concentrated in degree 0. For a general graded $\mathbb{Q}$-vector space V we denote the k-fold grading shift by $V[k]$; for example, $H^n[-n]$ is concentrated in degree $-n$. Finally $V_{>0}$ denotes the truncation to strictly positive degrees, and $S^*(V)$ the free graded-commutative algebra on V.

Theorem 1.1 (Krannich–K.)**.** *For $2n \geq 6$ and $* < 2n-5$, there is a canonical isomorphism*

$$H^*(B\mathrm{Diff}_\partial(W_{g,1});\mathbb{Q}) \cong H^*\left(G'_g; W^* \otimes S^*(H^n[-n] \otimes (\pi_*(BO)\otimes\mathbb{Q}))_{>0}\right),$$

where $W^* = \mathbb{Q}[0] \oplus S^3(H^n[n])[-2n]$. The total degree of the right hand side is given as the sum of the cohomological degree and the degree of the coefficients.

As indicated above, we also prove that some of the classes of the right hand side still contribute nontrivially to $H^*(B\mathrm{Diff}_\partial(W_{g,1});\mathbb{Q})$ in the larger range $* \geq 2n-5$.

Example 1.2. Let us take $g = 1$ and $n = 7$. In this case $\mathrm{Sp}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$ and $S^3(H^n[n]) = 0$. The free graded commutative algebra

$$S^*(H^n[-n] \otimes (\pi_*(BO)\otimes\mathbb{Q}))_{>0}$$

decomposes into a direct sum of symmetric powers $\mathrm{Sym}^k H$ with $H \cong H^n$ the defining representation. The multiplicities of these representations can be determined by standard representation theory, so the computation of the right hand side of Theorem 1.1 reduces to the calculation of the cohomology $H^*(\mathrm{SL}_2(\mathbb{Z});\mathrm{Sym}^k H)$. These groups vanish in degrees $* \neq 1$, which one can see by using $\mathrm{vcd}(\mathrm{SL}_2(\mathbb{Z})) = 1$

and some invariant theory. Moreover, by the Eichler–Shimura isomorphism, we have that

$$\dim H^1(\mathrm{SL}_2(\mathbb{Z}), \mathrm{Sym}^k H) = \begin{cases} \dim \mathcal{M}_{k/2} + \dim \mathcal{S}_{k/2} & \text{if } k \text{ is even,} \\ 0 & \text{otherwise,} \end{cases}$$

with $\mathcal{M}_{k/2}$ the $\mathbb{C}$-vector space of level 1 modular forms of weight $k/2$, and $\mathcal{S}_{k/2}$ the subspace of cusp forms. The dimensions of both of these spaces are known [Shi94, Chapter 8].

Remark 1.3. Theorem 1.1 implies that $H^*(B\mathrm{Diff}_\partial(W_{g,1});\mathbb{Q})$ has many nontrivial contributions from automorphic forms for G'_g. For instance, combined with a result of Faltings [Fal83, Theorem 10] on Siegel modular forms, it shows that for n odd and $g(g+1)/2 < 2n-5$, the group $H^{g(g+1)/2}(B\mathrm{Diff}_\partial(W_{g,1});\mathbb{Q})$ is often non-zero.

1.1. **The strategy.** Every relative manifold bundle with fibre $(W_{g,1}, \partial W_{g,1})$ is in particular a relative fibration, together with a stable vector bundle trivialised over a section, with monodromy given by homotopy classes realisable by diffeomorphisms. Denoting by $B\mathrm{hAut}^*_\partial(T^s W_{g,1})$ the space classifying this data, our characteristic classes are pulled back along the natural map

$$B\mathrm{Diff}_\partial(W_{g,1}) \longrightarrow B\mathrm{hAut}^{*,\mathrm{Diff}}_\partial(T^s W_{g,1}) \simeq \mathrm{hAut}^{\mathrm{Diff}}_\partial(W_{g,1}) \mathbin{/\!\!/} \mathrm{Map}_*(W_{g,1}, BO)_0.$$

Thus, we need to solve two problems:

(1) compute the rational cohomology of $B\mathrm{hAut}^{*,\mathrm{Diff}}_\partial(T^s W_{g,1})$ and
(2) guarantee that some cohomology classes remain non-zero in $B\mathrm{Diff}_\partial(W_{g,1})$.

1.2. **Step (1): a weight decomposition.** Our strategy for Step (1) relies on the common principle that *spaces admit more endomorphisms after rationalising.*

In order to compute the cohomology of $\mathrm{hAut}^{\mathrm{Diff}}_\partial(W_{g,1}) \mathbin{/\!\!/} \mathrm{Map}_*(W_{g,1}, BO)_0$, we may rationalise BO to get a space

$$\mathrm{hAut}_\partial(W_{g,1}) \mathbin{/\!\!/} \mathrm{Map}_*(W_{g,1}, BO_\mathbb{Q})_0$$

with the same rational cohomology. The space $\mathrm{Map}_*(W_{g,1}, BO_\mathbb{Q})$ has an action of the space of endomorphisms $\mathrm{Map}_*(BO_\mathbb{Q}, BO_\mathbb{Q})$ which commutes with the action of $\mathrm{hAut}^{\mathrm{Diff}}_\partial(W_{g,1})$. As $BO_\mathbb{Q}$ splits as a product of Eilenberg–Mac Lane spaces $\prod K(\mathbb{Q}\{p_i\}; 4i)$, there are pointed self-maps of $BO_\mathbb{Q}$ which on homotopy groups have the effect of sending p_i to $a_i \cdot p_i$ for $a_i \in \mathbb{Q}$. These make Serre spectral sequence

$$E^2_{p,q} = H^p(B\mathrm{hAut}^{\mathrm{Diff}}_\partial(W_{g,1}); H^q(\mathrm{Map}_*(W_{g,1}, BO_\mathbb{Q})_0;\mathbb{Q}) \Rightarrow H^{p+q}(B\mathrm{hAut}^{*,\mathrm{Diff}}_\partial(T^s W_{g,1});\mathbb{Q})$$

into a spectral sequence of $\mathbb{Q}[\mathrm{End}(\pi_*(BO_\mathbb{Q}))]$-algebras.

There is an isomorphism of $\mathbb{Q}[\mathrm{End}(\pi_*(BO_\mathbb{Q}))]$-algebras

$$H^*(\mathrm{Map}_*(W_{g,1}, BO_\mathbb{Q})_0) \cong S^*(H^n[-n] \otimes \pi_*(BO_\mathbb{Q}))_{>0},$$

where $\mathrm{End}(\pi_*(BO_\mathbb{Q}))$ acts via $\pi_*(BO_\mathbb{Q})$ on the right hand side. This decomposes as a direct sum of *weight spaces* indexed by sequences $\mu = (\mu_1, \mu_2, \ldots)$ of integers which are eventually zero: on the μ-weight space the endomorphism determined by $p_i \mapsto a_i \cdot p_i$ acts by scaling with $\prod_i a_i^{\mu_i}$. Since the reduced cohomology of $W_{g,1}$ is concentrated in a single degree, for each weight $\mu = (\mu_1, \mu_2, \ldots)$ a non-zero μ-weight space only occurs in one degree. This has the consequence that there are no nontrivial differentials in the above spectral sequence and moreover solves all extension problems, resulting in a canonical isomorphism of the form

$$H^*(B\mathrm{hAut}_\partial^{*,\mathrm{Diff}}(T^s W_{g,1}); \mathbb{Q})$$

$$\cong H^*\left(B\mathrm{hAut}_\partial^{\mathrm{Diff}}(W_{g,1}); S^*(H^n[-n] \otimes \pi_*(BO_\mathbb{Q}))_{>0}; \mathbb{Q}\right).$$

The right hand side can be studied using the Serre spectral sequence for

$$B\mathrm{hAut}_\partial^{\mathrm{Diff}}(W_{g,1}) \to BG'_g,$$

one weight at a time, combined with work of Berglund–Madsen [BM14] who showed that the classifying space of the identity component $B\mathrm{hAut}_\partial^{\mathrm{id}}(W_{g,1})$ is coformal with homotopy Lie algebra $\pi_{*+1}(B\mathrm{hAut}_\partial^{\mathrm{id}}(W_{g,1})) \otimes \mathbb{Q}$ the graded Lie algebra of symplectic derivations of the free Lie algebra on $H^n[n-1]$. In degrees $* < 2n-2$, this Lie algebra of derivations is non-zero only in degree $n-1$, where it is given by $S^3(H^n[n])[-2n]$.

1.3. **Step (2): Morlet disjunction and embedding calculus.** We have two approaches for Step (2). In degrees $* < 2n-5$, the map

$$B\mathrm{Diff}_\partial(W_{g,1}) \to B\mathrm{hAut}_\partial^{*,\mathrm{Diff}}(T^s W_{g,1})$$

can be seen to be a rational cohomology isomorphism by factoring it through the classifying space for block bundles,

$$B\mathrm{Diff}_\partial(W_{g,1}) \longrightarrow B\widetilde{\mathrm{Diff}}_\partial(W_{g,1}) \longrightarrow B\mathrm{hAut}_\partial^{*,\mathrm{Diff}}(T^s W_{g,1})$$

and combining the following three results, all under the assumption $2n \geq 6$.

Theorem 1.4 (Morlet, [BLR75]). $\frac{\widetilde{\mathrm{Diff}}_\partial(D^{2n})}{\mathrm{Diff}_\partial(D^{2n})} \to \frac{\widetilde{\mathrm{Diff}}_\partial(W_{g,1})}{\mathrm{Diff}_\partial(W_{g,1})}$ *is* $(2n-5)$*-connected.*

Theorem 1.5 (Randal-Williams, [RW17]). $\frac{\widetilde{\mathrm{Diff}}_\partial(D^{2n})}{\mathrm{Diff}_\partial(D^{2n})}$ *is rationally* $(2n-5)$ *-connected.*

Theorem 1.6 (Berglund–Madsen, [BM14]). $B\widetilde{\mathrm{Diff}}_\partial(W_{g,1}) \to B\mathrm{hAut}_\partial^{*,\mathrm{Diff}}(T^s W_{g,1})$ *induces an isomorphism on rational cohomology.*

The second, harder approach for Step (2), which we do not discuss in this report, is based on approximating diffeomorphisms by self-embeddings and applying embedding calculus to show that some cohomology classes remain non-zero even in the larger range $* \geq 2n-5$.

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Chromatically localized algebraic K-theory

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(joint work with L. Meier and G. Tamme)

For a ring R, we will consider its (non-connective) algebraic K-theory spectrum $K(R)$. As a "global object" it is hard to compute, but nevertheless it satisfies some nice properties. The main goal of this talk is to extend some of these known nice properties to the chromatic context. One particular such property, which is the main motivation for our results, is the following proposition due to Waldhausen [4].

Proposition 1. *Let $f\colon A \to B$ be an n-connective map between connective ring spectra, with $n \geq 1$. Then the map $K(f)\colon K(A) \to K(B)$ is $(n+1)$-connective. If furthermore f is an equivalence after tensoring with $H\mathbb{Q}$, $\mathbb{S}[\frac{1}{\ell}]$, or $\mathbb{S}/\ell$, then the same is true for the map $K(f)$.*

We recall from [2, Definition 3.1] that a localizing invariant $E\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Sp}$ is called *truncating*, if for all connective $\mathbb{E}_1$-ring spectra A, the canonical map $E(A) \to E(\pi_0(A))$ is an equivalence. Here, we write $E(\mathrm{Perf}(A))$ simply as $E(A)$. Likewise, we say that E is truncating on a class $\mathcal{T}$ of connective ring spectra if the previous property holds for all $\mathbb{E}_1$-rings contained in this class.

From Waldhausen's result, one deduces for instance that rational K-theory is truncating on rationally discrete ring spectra, and likewise that ℓ-adic K-theory is truncating on $\mathbb{S}/\ell$ acyclic ring spectra. This says that arithmetically localized K-theory, by which we mean ℓ-adic K-theory $K(-)_\ell$ for ℓ a prime, or rational K-theory $K(-)_\mathbb{Q}$, is more accessible than the full K-theory spectrum.

We recall from [2, Corollary 3.5] that any truncating invariant in nilinvariant on discrete rings. We can then consider the following example

Example 1. *For all primes $\ell \neq p$ (including 0), the arithmetic localizations of $K(\mathbb{Z}/p^n)$ are explicitly known.*

Proof. We observe that $\mathbb{Z}/p^n \to \mathbb{F}_p$ is a surjection with nilpotent kernel. Thus the above observation implies that $K(\mathbb{Z}/p^n)_{\mathbb{Q}} \simeq K(\mathbb{F}_p)_{\mathbb{Q}}$ and $K(\mathbb{Z}/p^n)_\ell \simeq K(\mathbb{F}_p)_\ell$. Quillen computed the K-theory of all finite fields explicitly, so we can conclude the claim. □

However, the p-adic K-theory of $\mathbb{Z}/p^n$ is not fully understood. At this point, we make use of the new family of primes lying over the prime p, which appear in stable homotopy theory, i.e. in algebra over the sphere spectrum $\mathbb{S}$: The family of Morava K-theories $K(m)$ for $0 \leq m \leq \infty$. Here, we interpret $K(0) = H\mathbb{Q}$ and $K(\infty) = H\mathbb{F}_p$. We recall here that $\pi_*(K(m)) = \mathbb{F}_p[v_m^{\pm 1}]$ with $|v_m| = 2p^m - 2$. We will need to following companions to $K(m)$, namely the telescopes $T(m) = V(m)[v_m^{-1}]$ where $V(m)$ is a finite p-local type m spectrum equipped with a v_m-self map v_m. The main motivation for us was to find a new proof of the following theorem of Bhatt–Clausen–Mathew.

Theorem 1. *For every $n \geq 1$, $L_{K(1)}K(\mathbb{Z}/p^n)$ vanishes.*

Bhatt–Clausen–Mathew prove this using descent results in $K(1)$-local K-theory to pass to a perfectoid situation, and then perform a prismatic cohomology calculation. One might wonder whether one can give a proof of this result by showing that $K(1)$-local K-theory is truncating on a suitable class of $\mathbb{E}_1$-rings, and it is this approach that we will take here.

Our first main result in this direction is the following chromatic analog of Waldhausen's result above. Its proof relies on unstable chromatic homotopy theory, we briefly sketch it here.

Theorem 2. *Let $f\colon A \to B$ be an n-connective map between connective ring spectra which is a $T(i)$-local equivalence for all $0 \leq i \leq n$. Then the map $K(f)\colon K(A) \to K(B)$ is a again a $T(i)$-local equivalence.*

Proof. The assumptions imply that the fibre F of the map of *spaces* $\mathrm{BGL}(A) \to \mathrm{BGL}(B)$ is n-connected and has vanishing v_i-periodic homotopy groups. A result of Bousfield together with calculations of Ravenel–Wilson can be used to show that F is in fact $T(i)$-acyclic for $0 \leq i \leq n$. One then finds that the map $\Sigma^\infty\Omega^\infty K(A)_{\geq 1} \to \Sigma^\infty\Omega^\infty K(B)_{\geq 1}$ is a $T(i)$-local equivalence for $0 \leq i \leq n$. Making use of the Bousfield–Kuhn functor and an adjunction triangle identity, we find that the $K(A)_{\geq 1} \to K(B)_{\geq 1}$ is $T(i)$-locally a retract of the above map, and hence also an equivalence. Combining Waldhausen's result (for the case $i = 0$) with the fact that bounded above spectra are $T(i)$-acyclic for $i \geq 1$, the theorem follows. □

The rest of the talk was about collecting consequences of this result.

Corollary 1. *Let $0 < n < m$ be natural numbers. Then $L_{T(n)}K(K(m)) = 0$.*

Proof. There is a fibre sequence

$$K(\mathbb{F}_p) \to K(k(m)) \to K(K(m))$$

where $k(m)$ is the connective cover of $K(m)$. The map $k(m) \to \mathbb{F}_p$ is highly connected and a $T(i)$-equivalence for $i < m$, so we may apply the main theorem. □

Using a little trick which was explained in the talk, one can deduce the following lemma from Theorem 2 and [2, Main Theorem].

Lemma 1. *Let $f\colon A \to B$ be an $(n+1)$-connective map which is a $T(i)$-local equivalence for all $1 \leq i \leq n$. Then the map $K(f)\colon K(A) \to K(B)$ is also a $T(i)$-local equivalence for $1 \leq i \leq n$.*

From this and [2] we find the following results:

Theorem 3. *$K(1)$-local K-theory is truncating on $K(1)$-acyclic ring spectra. In particular, $K(1)$-local K-theory satisfies excision, nilinvariance and cdh-descent on ordinary discrete rings.*

Corollary 2. *The canonical map $L_{K(1)}K(\mathbb{Z}/p^n) \to L_{K(1)}K(\mathbb{F}_p)$ is an equivalence.*

It follows immediately from Quillen's calculations that $L_{K(1)}K(\mathbb{F}_p)$ vanishes, so we reproduce the above mentioned theorem of Bhatt–Clausen–Mathew. Similarly, we obtain the following result.

Corollary 3. *The canonical map $L_{K(1)}K(ku/\beta^n) \to L_{K(1)}K(\mathbb{Z})$ is an equivalence.*

We recall that p is the implicit prime present in the chromatic localizations.

Theorem 4. *Let $4p-4 \geq n \geq 2$. Then for any $m \geq 1$ we have that $L_{T(n)}K(\tau_{\leq m}\mathbb{S})$ vanishes. In particular, the same vanishing holds true for any ring spectrum A which is an algebra over $\tau_{\leq m}\mathbb{S}$, such as ku/β^n.*

This allows to prove the following generalization of Theorem 3 in higher chromatic heights. The corresponding results for $n = 0$ and $n = \infty$ hold without any assumption on the implicit prime.

Corollary 4. *For $4p-4 \geq n$, we find that $T(n)$-local K-theory is truncating on $T(1) \oplus \cdots \oplus T(n)$-acyclic ring spectra.*

We remark that $T(n)$-local K-theory is *not* truncating on $T(n)$-acylic ring spectra: ku is $T(2)$-acyclic, and $L_{T(2)}K(ku) \neq 0$ by calculations of Ausoni–Rognes [1], whereas $L_{T(2)}K(\mathbb{Z}) = 0$ by a result of Mitchell's [3]. We also remark that the conditions in Corollary 4 are not necessary in the following sense. By Mitchell's results one can rephrase Corollary 4 for $n \geq 2$ as to say that $T(n)$-local K-theory vanishes on $T(1) \oplus \cdots \oplus T(n)$-acyclic rings. From the work of Ausoni–Rognes and unpublished results of Clausen–Mathew–Naumann–Noel, it is known that $L_{T(3)}K(ku) = 0$, but ku is not $T(1)$-acyclic. This might lead to the following question:

Question. Is $T(3)$-local K-theory truncating on $T(2) \oplus T(3)$-acyclic rings?

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Decomposing C_2-equivariant spectra

Clover May

(joint work with Daniel Dugger, Christy Hazel)

Computations in $RO(G)$-graded Bredon cohomology can be challenging and are not well understood, even for $G = C_2$, the cyclic group of order two. A recent structure theorem for $RO(C_2)$-graded cohomology with coefficients in the constant Mackey functor $\underline{\mathbb{F}_2}$ substantially simplifies computations. The structure theorem says the cohomology of any finite C_2-CW complex decomposes as a direct sum of two basic pieces: cohomologies of representation spheres and cohomologies of spheres with the antipodal action. This decomposition lifts to a splitting at the spectrum level. In joint work with Dan Dugger and Christy Hazel we extend this result to a classification of compact modules over the Eilenberg-MacLane spectrum $H\underline{\mathbb{F}_2}$.

This talk had two parts: $RO(C_2)$-graded cohomology and the classification of $H\underline{\mathbb{F}_2}$-modules. I began with a crash course in $RO(C_2)$-graded cohomology, mainly setting some notation. This a bigraded theory and throughout the talk the coefficients were the constant Mackey functor $\underline{\mathbb{F}_2}$. The theory is represented by the genuine equivariant Eilenberg–MacLane spectrum $H\underline{\mathbb{F}_2}$.

The first important computation is the cohomology of a point $\mathbb{M}_2 = H^{*,*}(pt; \underline{\mathbb{F}_2})$. It was presented in the talk as described in [6] and later in [4] and [3], though the computation precedes these. Building on unpublished work of Stong, the computation has been reproduced several times and appears in [2] and [5]. It turns out that $\mathbb{M}_2$ is made up of a bigraded polynomial algebra in elements τ and ρ and has an element θ that is infinitely divisible by τ and ρ. This makes $\mathbb{M}_2$ a non-Noetherian ring.

The bigraded cohomology of any C_2-space is a module over $\mathbb{M}_2$ and algebraically there are many $\mathbb{M}_2$-modules. The cohomology of any representation sphere is just a shift $\tilde{H}^{*,*}(S^{p,q}; \underline{\mathbb{F}_2}) \cong \Sigma^{p,q}\mathbb{M}_2$ due to the suspension isomorphism. The cohomology of the n-dimensional sphere with the antipodal action S^n_a is not just a shift and can be written as

$$\mathbb{A}_n = H^{*,*}(S^n_a; \underline{\mathbb{F}_2}) \cong \mathbb{F}_2[\tau, \tau^{-1}, \rho]/(\rho^{n+1}).$$

Surprisingly, the structure theorem for $RO(C_2)$-graded cohomology from [3] says these are actually the only two types of modules needed to describe the cohomology of finite C_2-spaces. Thus the cohomology of a finite C_2-space depends only on representation spheres and antipodal spheres.

Theorem 1 (M. 2018 [3])**.** *If X is a finite C_2-CW complex then as an $\mathbb{M}_2$-module $H^{*,*}(X;\underline{\mathbb{F}_2})$ decomposes as*

$$H^{*,*}(X;\underline{\mathbb{F}_2}) \cong (\oplus_i \Sigma^{p_i,q_i}\mathbb{M}_2) \oplus (\oplus_j \Sigma^{r_j,0}\mathbb{A}_{n_j}).$$

The proof uses some interesting new facts. The first is that $\mathbb{M}_2$ is self-injective. The second is a Toda bracket, specifically $\langle \tau, \theta, \rho \rangle = 1$ with zero indeterminacy.

I recently showed this decomposition lifts to the spectrum level.

Theorem 2 (M. 2019 [3])**.** *If X is a finite C_2-CW spectrum there is a decomposition of $X \wedge H\underline{\mathbb{F}_2}$ into a wedge as follows*

$$X \wedge H\underline{\mathbb{F}_2} \simeq \left(\bigvee_i S^{p_i,q_i} \wedge H\underline{\mathbb{F}_2} \right) \vee \left(\bigvee_j S^{r_j,0} \wedge S^{n_j}_{a}{}_+ \wedge H\underline{\mathbb{F}_2} \right).$$

Now a question: does this decomposition describe more general $H\underline{\mathbb{F}_2}$-modules? Not quite. We can describe all finite $H\underline{\mathbb{F}_2}$-modules, but we need one more piece. The last piece is the cofiber of τ.

Theorem 3 (Dugger–Hazel–M. in progress)**.** *If Y is a finite $H\underline{\mathbb{F}_2}$-module then*

$$Y \simeq \left(\bigvee_i S^{p_i,q_i} \wedge H\underline{\mathbb{F}_2} \right) \vee \left(\bigvee_j S^{r_j,0} \wedge S^{n_j}_{a}{}_+ \wedge H\underline{\mathbb{F}_2} \right) \wedge \left(\bigvee_k S^{a_k,b_k} \wedge \mathrm{cof}(\tau^{n_k}) \right).$$

The proof uses very different techniques. It relies on the following Quillen equivalence.

Theorem 4 (Schwede–Shipley 2003 [7])**.** *There is a Quillen equivalence*

$$H\underline{\mathbb{F}_2} - \mathrm{Mod} \simeq Ch(\underline{\mathbb{F}_2}).$$

We prove the splitting of $H\underline{\mathbb{F}_2}$-modules by splitting all finite chain complexes of projective $\underline{\mathbb{F}_2}$-modules.

We are also able to describe the Balmer spectrum as defined in [1] for the compact objects in the derived category of $Ch(\underline{\mathbb{F}_2})$. There are three prime ideals, two of which are closed points. The first closed point contains direct sums of the chain complexes that correspond to $\mathbb{A}_n$ for all $n \geq 0$. The second contains direct sums of the chain complexes that correspond to $\mathrm{cof}(\tau^n)$ for all $n \geq 0$. The third prime ideal contains both types and its closure is the whole space.

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Calculations for equivariant topological modular forms

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(joint work with David Gepner)

Every even periodic cohomology theory h gives rise to a formal group, which can be defined as the formal spectrum $\operatorname{Spf} h^0(\mathbb{CP}^\infty)$. In case of K-theory, this formal group is isomorphic to the formal completion $\widehat{\mathbb{G}}_m$ of the multiplicative group scheme $\mathbb{G}_m = \operatorname{Spec}[t^{\pm 1}]$. In classical language, this corresponds to the formal group law $x + y + xy$.

In general, every commutative group scheme that is smooth of relative dimension 1 gives rise to a formal group by completion at the unit. Besides $\mathbb{G}_m$ essentially the only other examples of such group schemes are the additive group (corresponding to ordinary homology) and elliptic curves. An *elliptic cohomology theory* consists of an even-periodic cohomology theory h, an elliptic curve C over $h^0(\mathrm{pt})$ and an isomorphism of formal groups between $\operatorname{Spf} h^0(\mathbb{CP}^\infty)$ and $\widehat{C}$.

A natural demand is to extend elliptic cohomology theories to equivariant theories. Going one step back to K-theory, we observe that S^1-equivariant K-theory of a point is isomorphic to the representation ring $R(S^1) \cong \mathbb{Z}[t^{\pm 1}]$, where t corresponds to the tautological representation of $S^1 = U(1)$. Thus $K^0_{S^1}(X)$ becomes for every S^1-space a module over $\mathbb{Z}[t^{\pm 1}]$. As this is the coordinate ring of $\mathbb{G}_m$, the module $K^0_{S^1}(X)$ defines a quasi-coherent sheaf on $\mathbb{G}_m$.

Thus it becomes natural to expect that S^1-equivariant elliptic cohomology takes values in sheaves on the corresponding elliptic curve, an idea already present in the original work of Grojnowski [2] over the complex numbers and Greenlees over the rationals [1]. An idea of Lurie [3] [4] was to work fully derived and only pass to homotopy groups at the end. This relies heavily on spectral algebraic geometry; in particular Lurie had to define elliptic curves and formal groups over E_∞-rings. We will assume these in the following to represent functors valued in commutative topological groups instead of just E_∞-spaces. The following definition is a derived analogue of the notion of an elliptic cohomology theory:

Definition. An *oriented elliptic curve* consists of an E_∞ ring spectrum R, an elliptic curve C over R and an equivalence over R of formal groups between $\operatorname{Spf} E^{\mathbb{CP}^\infty}$ and $\widehat{C}$.

Given now an oriented elliptic curve (R, C), we want following Lurie to define a contravariant functor

$$R^{\mathrm{shv}}_{S^1} \colon (\mathcal{S}^{\mathrm{fin}}_G)^{op} \to \mathrm{QCoh}(C, \mathcal{O}_C)$$

from the ∞-category $\mathcal{S}^{\mathrm{fin}}_G$ of finite S^1-CW complexes to the ∞-category of quasi-coherent $\mathcal{O}_C$-modules. By Elmendorf's theorem, $\mathcal{S}^{\mathrm{fin}}_G$ embeds into space-valued presheaves on the orbit-category Orb_{S^1} as the sub-∞-category generated by finite colimits from the orbits S^1/H for closed subgroups $H \subset S^1$. Thus it suffices to specify $R^{\mathrm{shv}}_{S^1}$ on these orbits (with appropriate functoriality) if we demand that $R^{\mathrm{shv}}_{S^1}$ sends finite colimits in $\mathcal{S}^{\mathrm{fin}}_G$ to finite limits. We set $R^{\mathrm{shv}}_{S^1}(S^1/S^1)$ to be $\mathcal{O}_C$ and $R^{\mathrm{shv}}_{S^1/C_n} = (i_n)_*\mathcal{O}_{C[n]}$, where $i_n \colon C[n] \hookrightarrow C$ is the inclusion of the n-torsion. This is again in line with K-theory as $K_{S^1}(S^1/C_n) \cong R(C_n)$ and $\mathrm{Spec}\, R(C_n) = \mathbb{G}_m[n]$.

There are two methods to obtain more classical invariants from $R^{\mathrm{shv}}_{S^1}$. The first is to apply (sheafified) homotopy groups to obtain sheaves of abelian groups on the underlying classical elliptic curve of C, resulting in a Grojnowski style version of elliptic cohomology. The other is to take global sections, resulting in a functor

$$R_{S^1} \colon (\text{finite } S^1\text{-spaces})^{op} \to \text{ Spectra}$$

Taking homotopy groups results in an S^1-equivariant cohomology theory

$$R^*_{S^1} \colon (\text{finite } S^1\text{-spaces})^{op} \to \text{ graded abelian groups}$$

Actually, this is represented by an S^1-spectrum R with $R^{S^1} = R_{S^1}(\mathrm{pt})$.

This abstract theory leaves the question open how to calculate these objects, which we answer in the simplest case.

Theorem (Gepner–M.)**.** *There is an equivalence* $R^{S^1} = R_{S^1}(\mathrm{pt}) \simeq R \oplus \Sigma R$. *The map* $R \to R^{S^1}$ *is given by restriction along* $S^1 \to \{e\}$ *and* $\Sigma R \to R^{S^1}$ *by a transfer.*

While there is just one multiplicative groups, there are a lot of elliptic curves, resulting in many elliptic cohomology theories. There is one universal theory, called *topological modular forms* TMF associated with the moduli stack of all elliptic curves. While itself not an elliptic cohomology theory, it maps to all elliptic cohomology theories (associated with an oriented elliptic curve) and supports an equivariant theory as well. The naturality in the previous theorem implies:

Corollary. *We have an equivalence* $TMF^{S^1} \simeq TMF \oplus \Sigma TMF$.

Note that the homotopy groups of TMF are completely known and thus we obtain a complete calculation of $TMF^*_{S^1}(\mathrm{pt})$. Note moreover that (in contrast to K^{S^1}) the TMF-module TMF^{S^1} is dualizable, with dual $\Sigma^{-1}TMF^{S^1}$.

Actually, equivariant TMF can be defined for all compact Lie groups G, in particular resulting in fixed points TMF^G. Our results together with work of Lurie suggest the following conjecture.

Conjecture. *Let G be a compact Lie group and L its adjoint representation. Then* TMF^G *is a dualizable* TMF*-module with dual* $(\Sigma^{-L}TMF)^G$.

The case of G finite is a consequence of tempered ambidexterity, one of the main results of [4]. The case $G = S^1$ (and actually $G = (S^1)^r$) follows from the corollary above.

In particular, Lurie's result implies that TMF^{C_n} is a self-dual TMF-module.

Question. *Can one explicitly calculate TMF^{C_n} or at least its homotopy groups?*

Much of the difficulty lies in understanding explicitly the n-torsion points in the universal elliptic curve. This simplifies significantly if we invert n in the basis or even p-complete away from n. The following is one of the main results from [5].

Theorem (M.)**.** *For p not dividing n, the TMF-module TMF^{C_n} splits after p-completion into shifted copies of TMF, $TMF_1(2)$ (if $p = 3$) and $TMF_1(3)$ (if $p = 2$).*

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Descent property of (co)sheaves on manifolds via Thurston's fragmentation

Sam Nariman

Let $F : (\mathsf{Mfld}_n^{\partial})^{op} \to \mathsf{S}$ be a presheaf from the category of smooth n-manifolds (possibly with nonempty boundary) with smooth embeddings as morphisms to a convenient category of spaces S. For our purpose, it is enough to consider the category of simplicial sets or compactly generated Hausdorff spaces. Let F^h be the homotopy sheafification of F with respect to 1-good covers meaning contractible open sets whose nontrivial intersections are also contractible. One can describe the value of $F^h(M)$ as the space of sections of the bundle $\mathrm{Fr}(M) \times_{\mathrm{GL}_n(\mathbb{R})} F(\mathbb{R}^n) \to M$, where $\mathrm{Fr}(M)$ is the frame bundle of M. We say F satisfies an h-principle if the natural map from the functor to its homotopy sheafification

$$j : F(M) \to F^h(M),$$

induces a weak equivalence and we say it satisfies c-principle if the above map is a homology isomorphism. Some important examples of such presheaf in the manifold topology are the space of generalized Morse functions [Igu84], space of framed functions [Igu87], space of smooth functions on M^n that avoid singularities of codimension $n + 2$ (this is in general a c-principle, see [Vas92]), space of configuration of points with labels in a connected space [McD75], etc.

Given a fixed element of $s_0 \in F(M)$, one could also consider the compactly supported versions (precosheaf) $F_c(M, s_0)$ of these examples and still the natural map between F_c and F_c^h satisfies h-principle or c-principle. Proving that geometrically defined functorx of interest have nice homotopical properties (being homotopy (co)sheaf) is usually hard and it is the main step in proving h-principle theorems. Different techniques were developed [Gro86, EM02] to prove homotopical properties for certain geometric functors. But in the above examples, the known proofs are not "local to global" argument. In particular, they do not approach it by proving that F_c and F_c^h have descent property with respect to certain covers.

One common feature of the above examples is that $F(\mathbb{R}^n)$ is at least $(n-1)$-connected. So the fiber of the bundle whose compactly supported section space recovers $F_c^h(M)$ is at least $(n-1)$-connected. For such section spaces, there is a descent property known as *non-abelian Poincare duality* [Lur16, Theorem 5.5.6.6]. So it is expected that if $F(\mathbb{R}^n)$ is at least $(n-1)$-connected, proving h(c)-principle is equivalent to a descent property for $F_c(M)$. Inspired by Thurston's work in foliation theory, we introduce the notion of fragmentation for F_c as a way to prove a descent property for geometrically defined cosheaves. We talk about how fragmentation implies the known version of the non-abelian Poincare duality for space of sections and how it can be generalized when the connectivity of the hypothesis is relaxed.

1.1. **Non-abelian Poincare duality via fragmentation.** To state fragmentation property for the space of sections, let $\pi : E \to M$ be a Serre fibration over the manifold M. Let s_0 be a base section. By the support of a section s, we mean the closure of the points on which s differs from the base section s_0. Let $\mathrm{Sect}_c(\pi)$ be the space of compactly supported sections of the fiber bundle $\pi : E \to M$ equipped with the compact-open topology. Let $\mathrm{Sect}_\epsilon(\pi)$ denote the subspace of sections s such that the support of s can be covered by k geodesically convex balls of radius $2^{-k}\epsilon$ for some positive integer k.

Theorem 1.1 (Fragmentation property)**.** *If the fiber of π is at least $(n-1)$-connected, the inclusion*

$$\mathrm{Sect}_\epsilon(\pi) \hookrightarrow \mathrm{Sect}_c(\pi),$$

is a weak homotopy equivalence.

Remark 1.2. Thurston proved this property with the hypothesis that the fiber of π is at least n-connected.

One can improve on the same ideas to relax the connectivity hypothesis even more. For example, if the fiber of π is at least $(n-2)$-connected, one can show that

$$\mathrm{Sect}_\epsilon^{\mathrm{graph}}(\pi) \hookrightarrow \mathrm{Sect}_c(\pi),$$

is a weak homotopy equivalence where $\mathrm{Sect}_\epsilon^{\mathrm{graph}}(\pi)$ is the subspace of sections whose support is in a $2^{-k}\epsilon$-neighborhood of a graph with k vertices. Using Thurston's ideas in the foliation theory, one could prove the following c-principle theorem.

Definition 1.3. We say F is *good*, if it satisfies

- The subspace of elements with empty support in $F(M)$ is contractible.
- Let U and V be open disks. All embeddings $U \hookrightarrow V$ induces a homology isomorphism between $F_c(U)$ and $F_c(V)$.
- For an open subset U of a manifold M, the inclusion $F_c(U) \to F_c(M)$ is an open embedding.
- Let ∂_1 be the northern-hemisphere boundary of D^n. Let $F(D^n, \partial_1)$ be the subspace of $F(D^n)$ that restricts to the base element in a germ of ∂_1 inside D^n. We assume $F(D^n, \partial_1)$ is contractible.

Theorem 1.4 (N). *Let F be a good presheaf on manifolds such that*

- *$F(\mathbb{R}^n)$ is at least $(n-1)$-connected.*
- *It has the fragmentation property.*

Then F satisfies the c-principle.

Proving fragmentation property instead of descent property with respect to good covers for geometrically defined functors F_c is approachable using Thurston's ideas in foliation theory. For example one could prove Vassiliev c-principle theorem [Vas92] for space of smooth functions not having certain singularity via fragmentation technique.

1.2. **Relating two c-principle theorems in foliation theory.** Let $\mathrm{Vect}(M)$ denote the Lie algebra of smooth vector fields on a manifold M with its C^∞-topology and let $C^*_{GF}(\mathrm{Vect}(M))$ denote the Gelfand-Fuks cochains (continuous Chevalley-Eilenberg cochains). Bott and Segal showed that $C^*_{GF}(\mathrm{Vect}(-))$ has a descent property and used a local to global argument to find a zig-zag of quasi-isomorphism between $C^*_{GF}(\mathrm{Vect}(M))$ and real cochains of the space of sections of $\mathrm{Fr}(M) \times_{\mathrm{GL}_n(\mathbb{R})} F \to M$ where F is a $2n$-connected $\mathrm{GL}_n(\mathbb{R})$-space whose real cohomology $H^*(F;\mathbb{R})$ is isomorphic to the cohomology of $C^*_{GF}(\mathrm{Vect}(\mathbb{R}^n))$.

On the other hand, Thurston studied space of foliated trivial M-bundles [Thu74] and proved a c-principle for such a functor. More formally, one can represent this functor using the Lie algebra of vector fields as follows. Let

$$\mathsf{MC}_\bullet(\mathrm{Vect}(M)) := \mathsf{MC}(\Omega_{\mathrm{dR}}(\Delta^\bullet) \otimes \mathrm{Vect}(M)),$$

be the simplicial set given by smooth Maurer-Cartan elements of dgla $\Omega_{\mathrm{dR}}(\Delta^\bullet) \otimes \mathrm{Vect}(M)$.

He showed that $|\mathsf{MC}_\bullet(\mathrm{Vect}(M))|$ has the fragmentation property and proved it is homology isomorphic to a section space. For simplicity, suppose that M is parallelizable. (this assumption is to express the section space as a mapping space). Then the Thurston theorem states that there is a map

$$|\mathsf{MC}_\bullet(\mathrm{Vect}(M))| \to \mathrm{Map}(M, |\mathsf{MC}_\bullet(\mathrm{Vect}_0(\mathbb{R}^n))|), \tag{1}$$

where $\mathrm{Vect}_0(\mathbb{R}^n)$ is the formal vector fields on $\mathbb{R}^n$ (i.e. germs of vector fields at the origin). Thurston's theorem implies that the above map is a homology isomorphism. Inspired by rational homotopy theory, the mapping space

$\mathrm{Map}(W, |\mathsf{MC}_\bullet(\mathrm{Vect}(\mathbb{R}^n))|$ can be modeled by the Maurer-Cartan element of the dgla $\Omega_{\mathrm{dR}}(M) \otimes \mathrm{Vect}_0(\mathbb{R}^n)$.

Our goal is to enhance Thurston's theorem to a statement about the comparison between $\mathsf{MC}_\bullet(\Omega_{\mathrm{dR}}(W) \otimes \mathrm{Vect}_0(\mathbb{R}^n))$ and $\mathsf{MC}_\bullet(\mathrm{Vect}(W))$ that implies homology isomorphism after realization. This is inspired by the work of Haefliger on differential cohomology [Hae10] to relate these two theorems locally.

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Beilinson's fibre sequence in *K*-Theory via topological cyclic homology

Thomas Nikolaus

We will review several approximations to algebraic K-Theory: Hochschild homology, negative cyclic homology and topological cyclic homology. We review the results of Goodwillie, Dundas–McCarthy and Clausen–Mathew–Morrow which basically say that for nilpotent or henselian ideals, the difference between K-Theory and these approximations vanishes.

The main new result of this talk is a p-adic version of Goodwillie's result: for a commutative ring R, henselian along p, there is a fibre sequence

$$\Sigma HC(R; \mathbb{Q}_p) \to K(R; \mathbb{Q}_p) \to K(R/p; \mathbb{Q}_p)$$

This result is originally due to Beilinson under some additional assumptions on R. If time permits, we will discuss the proof using TC and applications.

The η-periodic motivic sphere spectrum and the connective Witt-theoretic J-spectrum

Kyle Ormsby
(joint work with Oliver Röndigs)

Infamously, the motivic Hopf map η is non-nilpotent in the motivic stable homotopy groups of the sphere spectrum. This is proved over any base field by Morel [4], but is easily seen over $\mathbb{R}$ since the real points of the unstable map $\eta : \mathbb{A}^2 \smallsetminus 0 \to \mathbb{P}^1$ are homotopy equivalent to the degree -2 map $S^1 \to S^1$. This is a first example of exotic behavior in motivic stable stems, and hints at the additional complexity necessary in any successful description of motivic nilpotence.

We may detect motivic homotopy classes on which η acts non-nilpotently by forming the *η-periodic sphere spectrum*

$$\eta^{-1}\mathrm{S} := \operatorname{hocolim}\left(\mathrm{S} \xrightarrow{\eta} \mathrm{S}^{-\alpha} \xrightarrow{\eta} \mathrm{S}^{-2\alpha} \xrightarrow{\eta} \cdots\right).$$

(Here we are using the "equivariant grading" in which $\mathrm{S}^{m+n\alpha} \simeq (S^1)^{\wedge m} \wedge (\mathbb{A}^1 \smallsetminus 0)^{\wedge n}$.) Following work over specific base fields by Andrews–Miller [1], Guillou–Isaksen [2, 3], and Wilson [8], we undertake the computation of the homotopy groups of $\eta^{-1}\mathrm{S}$ over an arbitrary base field via the slice spectral sequence. Our efforts are successful for a broad class of fields, as exhibited by the following theorem. Let $k^M_*(\mathsf{k})$ denote the mod 2 Milnor K-theory of k, let $\rho = [-1] \in k^M_1(\mathsf{k})$, let $W(\mathsf{k})$ denote the Witt ring of regular quadratic forms over k modulo the hyperbolic plane, and let sc denote slice completion as in [7].

Theorem 1 ([6, Theorem 4.8])**.** *Suppose that* k *is not of characteristic* 2 *and that* -1 *is a sum of four squares in* k*. Then, as a ring,*

$$\pi_\star \operatorname{sc}(\eta^{-1}\mathrm{S}) \cong W(\mathsf{k})[\eta^{\pm 1}, \sigma, \mu]/(\sigma^2)$$

where $|\eta| = \alpha$, $|\sigma| = 3 + 4\alpha$, *and* $|\mu| = 4 + 5\alpha$. *If additionally* $\operatorname{cd}\mathsf{k} < \infty$, *then* $\operatorname{sc}(\eta^{-1}\mathrm{S}) \simeq \eta^{-1}\mathrm{S}$ *and this is a computation of the η-periodic homotopy groups of the motivic sphere spectrum.*

The proof begins with the known computation of the E_1-page of the η-periodic slice spectral sequence [7]:

$$E_1^{*,\star} \cong H\mathbb{F}_2[\eta^{\pm 1}, \alpha_3, \alpha_4]/(\alpha_4^2).$$

Here H denotes the motivic Eilenberg-MacLane functor, $|\eta| = (1, \alpha)$, $|\alpha_3| = (3, 2 + 3\alpha)$, and $|\alpha_4| = (4, 4 + 5\alpha)$.

We then determine the d_1 differentials as an elaborate pattern of motivic Steenrod operations depicted in Figure 1. Some of these are detected by the unit map to connective Witt K-theory, kw, while others depend on a map $\sigma_\infty : \Sigma^3\mathrm{kw} \to \eta^{-1}\mathrm{S}$ that we construct over $\mathbb{C}$ via a cell presentation of kw.

This leads to an E_2-page of the form

$$E_2^{*,\star} \cong k^M_*(\mathsf{k})[\eta^{\pm 1}, \alpha_4, \alpha_5]/(\alpha_4^2)$$

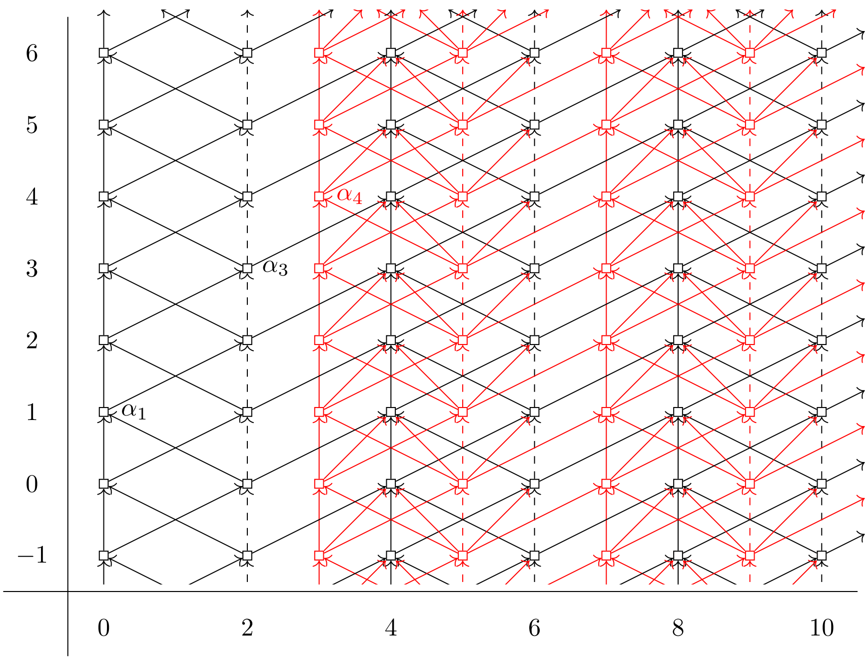

FIGURE 1. The first page of the η-periodic slice spectral sequence with its differentials. A $\square$ in position (m, n) represents a copy of $\pi_\star\Sigma^{m+n\alpha}H\mathbb{F}_2 \cong \Sigma^{m+n\alpha}k_*^M(\mathsf{k})[\tau]$ in slice degree n. The black portions of the diagram are detected by the unit map $\eta^{-1}\mathrm{S} \to \mathrm{kw}$, and the red portions are induced by $\sigma_\infty : \Sigma^3\mathrm{kw} \to \eta^{-1}\mathrm{S}$ over $\mathbb{C}$. Arrows with slope $-1/2$ represent τ, arrows with slope -1 represent ρ, solid vertical arrows represent Sq^2, dashed vertical arrows represent $\mathrm{Sq}^2 + \rho\mathrm{Sq}^1$, arrows with slope 1 represent $\mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^3$, and arrows with slope $1/2$ represent $\mathrm{Sq}^3\mathrm{Sq}^1$.

in which $|k_n^M(\mathsf{k})| = (0, -n\alpha)$ and $|\alpha_5| = (5, 4 + 5\alpha)$. In particular, this E_2-page is concentrated in nonnegative simplicial degrees congruent to 0 or 3 modulo 4. Moreover, we clearly have that the spectral sequence collapses at E_2 when $k_{>1}^M(\mathsf{k}) = 0$. In particular, we get collapse for $\mathsf{k} = \mathbb{F}_p$. A base change trick then proves the odd characteristic case of Theorem 1.

We then turn to the general characteristic 0 case, where we first show that $d_2 = 0$ over $\mathsf{k} = \mathbb{Q}$ via a Hilbert reciprocity argument. This in turn allows us to prove that there is a nondecreasing sequence of extended integers $r_k \in \mathbb{Z}_{\geq 3} \cup \{\infty\}$, $k \geq 2$, such that differentials $d_{r_k}\alpha_{2^k+1} = \rho^{r_k}\alpha_{2^k}\alpha_1^{r_k-1}$ and the Leibniz rule determine the spectral sequence. Over a general field k, we call such a sequence (if it exists) the *profile* of the η-periodic slice spectral sequence over k.

Theorem 2 ([6, Theorem 4.5])**.** *Let* k *denote a characteristic* 0 *field. The* η*-periodic slice spectral sequence over* k *is determined by the same profile as the* η*-periodic slice spectral sequence over* $\mathbb{Q}$*.*

The proof again relies on base change, but the details are more subtle. In particular, we invoke a deep theorem of Orlov–Vishik–Voevodsky [5, Theorem 3.3] to guarantee that the ρ^{r_k}-torsion in $k_*^M(\mathsf{k})[\eta^{\pm 1}]\{\alpha_{2^k+1}\}$ supports no differentials.

This essentially completes the proof of Theorem 1, but leaves open the question of determining the profile of a characteristic 0 field in which ρ is non-nilpotent. Inspired by the computations of [3], we make the following conjecture.

Conjecture 1 ([6, Conjecture 4.10])**.** *The profile of the* η*-periodic slice spectral sequence over* $\mathbb{R}$ *(and hence over every characteristic* 0 *field) is* $r_k = k + 1$*.*

We are not able to directly apply the results of [3] to prove this conjecture because the target of the η-periodic motivic Adams spectral sequence is not (necessarily) the same as $\pi_\star\,\mathrm{sc}(\eta^{-1}\mathrm{S})$.

If Conjecture 1 is true, then the homotopy groups of $\mathrm{sc}(\eta^{-1}\mathrm{S})$ exhibit a "Witt-theoretic image of J pattern." Indeed, we would get

$$\pi_m\,\mathrm{sc}(\eta^{-1}\mathrm{S}) \cong \begin{cases} W(\mathsf{k}) & \text{if } m = 0, \\ W(\mathsf{k})/2^{\nu_2(4\ell)+1} & \text{if } m = 4\ell - 1 > 0, \\ {}_{2^{\nu_2(4\ell)+1}}W(\mathsf{k}) & \text{if } m = 4\ell > 0, \\ 0 & \text{otherwise.} \end{cases}$$

This leads to the speculation that there is a short resolution of $\eta^{-1}\mathrm{S}_2^\wedge$ via connective Witt K-theory.

Let kw denote the connective cover of $\mathrm{KW} = \eta^{-1}\mathrm{KQ}$, where KQ is the motivic Hermitian K-theory spectrum. One can show that $\mathrm{kw}_2^\wedge$ admits an action of $\mathbb{Z}_2^\times/\{\pm 1\}$ by Adams operations when the virtual cohomological dimension of k is finite. Furthermore, for any $g \in \mathbb{Z}_2^\times/\{\pm 1\}$, $\psi^g - 1$ factors through $\Sigma^4\mathrm{kw}_2^\wedge$. Define the *connective Witt-theoretic J-spectrum* jw via the fiber sequence

$$\mathrm{jw} \longrightarrow \mathrm{kw}_2^\wedge \xrightarrow{\psi^3 - 1} \Sigma^4\mathrm{kw}_2^\wedge.$$

If true, the following conjecture easily implies Conjecture 1.

Conjecture 2 ([6, Conjecture 4.10])**.** *The Adams operations act on* $\pi_\star\mathrm{kw} \cong W(\mathsf{k})[\beta]$*,* $|\beta| = 4 + 4\alpha$*, via* $\psi^g(\beta^k) = g^{2k}\beta^k$*, and if* k *has finite virtual cohomological dimension, then*

$$\eta^{-1}\mathrm{S}_2^\wedge \simeq \mathrm{jw}.$$

Tom Bachmann has sketched a proof of Conjecture 2 via the effect of Adams operations on $HW_*\mathrm{kw}$. Work in progress of Bachmann and Mike Hopkins addresses precisely this question.

In one sense, a positive resolution of Conjecture 2 (that does not depend on slice techniques) will supersede the results explained above. Nonetheless, the η-periodic slice differentials which we have discovered should prove useful in more detailed analyses of the slice spectral sequence for the motivic sphere building on [7].

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Homotopy Theory of 2-categories

Viktoriya Ozornova
(joint work with Martina Rovelli)

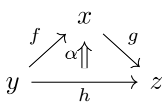
The long-term goal of this project joint with Julie Bergner and Martina Rovelli is understanding stratified simplicial sets as a model for (∞, n)-categories and comparing it with already existing models. We would like to report on first steps in this direction.

The theory of $(\infty, 1)$-categories, in particular in form of quasi-categories by Joyal and Lurie, has become ubiquitous in homotopy theory. Many further models of $(\infty, 1)$-categories are available, all of them are known to be equivalent and the equivalences between them are well-understood. However, the situation changes when we turn to (∞, n)-categories. The need for studying $(\infty, 2)$-categories, for example, arises when trying to incorporate analogs for non-invertible natural transformations in the context and in particular when looking at the totality of $(\infty, 1)$-categories.

The motivation for our desired model for (∞, n)-categories arises from the following classical consideration. If we consider ordinary categories, the nerve functor N exhibits a fully faithful embedding from small categories Cat into simplicial sets sSet. The essential image can be characterized in two different ways. A maybe more familiar way is using Segal condition: A simplicial set X is isomorphic to a nerve of a category if and only if the map $X_n \to X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$ induced by maps $(i, i+1)\colon [1] \to [n]$ is an isomorphism for any $n \geq 2$. Less well-known is maybe the description via inner horns: A simplicial set X is isomorphic to a nerve of a category if and only if it has unique right lifting property with respect to all inner horn inclusions $\Lambda^k[n] \hookrightarrow \Delta[n]$, for $0 < k < n$ and $n \geq 2$. The weakening of the former condition leads to the notion of complete Segal spaces, and the weakening of the latter leads to the notion of quasi-categories.

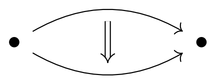
If one wants to mimick this approach for $(\infty, 2)$-categories, the first question to answer is what the analog of the nerve functor is. The variant we want to use is a version of Street nerve [Str87], due to Roberts–Street in the general case and also studied by Duskin and Gurski for 2-categories.

Similar to the usual nerve, the Duskin nerve of a strict 2-category is a simplicial set which is 3-coskeletal. The 0-simplices are precisely the objects, the 1-simplices are 1-morphisms. For the 2-morphisms, the situation becomes somewhat more complicated: Note that a 2-morphism in a 2-category has one 1-morphism as a source and one 1-morphism as a target. In contrast, a 2-simplex has to have three 1-simplices as its boundary. This is achieved by demanding one 2-simplex for every 2-morphism $h \overset{\alpha}{\Rightarrow} h'$ and every decomposition $h' = gf$:



The 3-simplices incorporate to a certain extent the composition of 2-morphisms.

The first surprise about Duskin nerve is that it is not fully faithful if we consider only strict 2-morphisms between strict 2-categories, and we will address this point in more detail in a moment. On a different note, Duskin nerve turns out to be complicated even for simple-looking 2-categories. For example, the 2-cell



has no non-trivial compositions. However, we were able to show the following, as a part of a larger class of examples:

Theorem 1 (O.–Rovelli)**.** *The Duskin nerve of the* 2*-cell has exactly* 2 *non-degenerate simplices in every dimension.*

As a remedy for the lack of fullness, Roberts–Street have introduced an additional structure on simplicial sets to make the nerve fully faithful. This leads to a notion of a *stratified simplicial set*, which is a simplicial set X together with subsets $tX_k \subset X_k$ of *thin* simplices containing all degenerate simplices. The idea behind the thin simplices is to remember the simplices inhabited by an identity. This notion turns out to be well-suited for categorical purposes:

Theorem 2 (Roberts–Street, Verity [Ver08a], Gurski [Gur09])**.** *The nerve* $N^{RS}\colon \mathrm{2Cat} \to \mathrm{Strat}$ *taking values in stratified simplicial sets and having thin simplices coming from the identities is fully faithful. Moreover, the essential image can be characterized in terms of strict horn lifting properties.*

Weakening the horn lifting condition leads to a proposed definition of a notion of $(\infty, 2)$-category (which also generalizes for $n > 2$) based on stratified simplicial sets by Riehl–Verity. However, Roberts–Street-stratification is too rigid for the purposes of homotopy theory, and that here, taking equivalences as thin simplices instead turns out to be the homotopically meaningful. As a first step showing the good properties of this definition, we were able to show:

Theorem 3 ([OR19])**.** *The nerve with the equivalence stratification defines a homotopically fully faithful embedding* $N^\natural\colon 2\mathrm{Cat} \to \mathrm{Strat}_{(\infty,2)}$.

Here, $\mathrm{Strat}_{(\infty,2)}$ denotes the category of stratified simplicial sets equipped with the homotopy theory for $(\infty,2)$-categories.

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Chromatic homotopy theory is algebraic when $p > n^2 + n + 1$

Piotr Pstragowski

A common theme in homotopy theory is to prove that certain classes of topological objects can be described by purely algebraic data. A famous example of this result is Quillen's rational homotopy theory, which describes simply connected rational spaces in terms of *dg*-Lie algebras [5].

Non-stable results are hard to come by, but in the stable context there is an extensive family of examples of homotopy theories which admit an *algebraic model* (that is, an equivalent triangulated category of algebraic nature), as well as examples of homotopy theories which are known to have no models of this type (such as the category of spectra, as proven by Schwede [6]).

In chromatic homotopy theory, one stratifies p-local homotopy theory by height n, and a common theme in this context is that answers turn out to be algebraic when the prime is much larger then the height. An example of this phenomena is a theorem of Bousfield, which shows that the $E(1)$-local homotopy theory admits an algebraic model for all $p > 2$ [2].

In this talk, we generalize Bousfield's result to all heights by showing that if $p > n^2 + n + 1$, then the homotopy category of $E(n)$-local spectra is equivalent to the derived category of $E(n)_*E(n)$. This gives a precise sense in which chromatic homotopy is algebraic at large primes, and implies that the corresponding $K(n)$-local Picard groups are algebraic [1].

Our methods are different from the ones traditionally used to prove algebraicity results, as developed by Franke [3], and are instead based on Goerss-Hopkins theory [4]. The general nature of the latter thus suggests that algebraicity results should be now also possible in other contexts, for example for modules over certain ring spectra.

A question left open by this work is what happens when $p \leq n^2 + n + 1$. At $p = 2$, it is known by the work of Roitzheim that the $E(1)$-local homotopy category does not admit algebraic models [7], and one expects that the same is true at all heights when the prime is sufficiently small. A result of this type would give some measure of the "fundamental non-algebraicity" of homotopy theory predicted by Mahowald's principle.

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Duality for topological modular forms

John Rognes

(joint work with Robert Bruner)

In joint work with Robert Bruner we have recovered a calculation of $\pi_*(tmf)$, originally due to Hopkins and Mahowald [1], with enough precision to define a map

$$a\colon \Sigma^{20} tmf_2^\wedge \longrightarrow I(tmf/(2^\infty, B^\infty, M^\infty))$$

of tmf-modules, where $B \in \pi_8(tmf)$ is a Bott element, $M \in \pi_{192}(tmf)$ is a Mahowald element, $tmf/(2^\infty, B^\infty, M^\infty)$ is the iterated homotopy cofiber of a cube of 2-, B- and M-localizations of tmf, and $I(-)$ denotes the Brown–Comenetz dual. Using descent along a map $tmf \to tmf_1(3) \simeq BP\langle 2\rangle$ we deduce that a is an equivalence. At the homotopy group level this implies a Pontryagin duality

$$\Theta N_i \cong \operatorname{Hom}(\Theta N_{170-i}, \mathbb{Q}/\mathbb{Z})$$

where N_* is the $\mathbb{Z}[B]$-submodule of $\pi_*(tmf)$ generated by the classes in degrees $0 \le * < 192$, $\pi_*(tmf) \cong N_* \otimes \mathbb{Z}[M]$, and $\Theta N_* \subset \Gamma_B N_*$ is the part of the Bott torsion in N_* that is not in degrees $* \equiv 3 \mod 24$.

As a consequence we obtain Stojanoska's theorem [2] that $\Sigma^{21} Tmf \simeq I_{\mathbb{Z}}(Tmf)$, where $I_{\mathbb{Z}}(-)$ denotes Anderson duality, also at the prime 2. Furthermore, we show that the image of the Hurewicz homomorphism $\pi_*(S) \to \pi_*(tmf)$ (when restricted to the cokernel of J) lies in the part $\Theta\pi_*(tmf)$ of $\Gamma_B \pi_*(tmf)$ that is not in degrees $* \equiv 3 \mod 24$. A conjecture of Mahowald asserts that this is precisely the Hurewicz image.

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Ambidexterity in Chromatic Homotopy Theory

Lior Yanovski

Background and Main Results. Given an abelian group A with an action of a finite group G, one has the classical norm map from the co-invariants to the invariants

$$\mathrm{Nm}_G : A_G \to A^G,$$

which is given by summation along the orbit $\mathrm{Nm}_G([a]) = \sum_{g\in G} ga$. Similarly, one can define the norm map for an action of a finite group on any object of an abelian (or even semiadditive) category, but the behavior of the norm map strongly depends on the category in question. Two fundamental examples to keep in mind:

- For $\mathcal{C} = \mathrm{Vec}_{\mathbb{Q}}$, the norm map is always an isomorphism (essentially because $|G|$ is invertible).
- For $\mathcal{C} = \mathrm{Vec}_{\mathbb{F}_p}$ the norm is usually not an isomorphism. E.g. for the trivial action of C_p on any vector space V, the norm map is identically zero.

Thus, the property that the norm map is an isomorphism seems to be a characteristic zero phenomena. Moving to homotopy theory and replacing abelian groups with spectra, we have more "prime fields". fixing a prime p, Morava K-theories provide in a sense an interpolation between characteristic 0 and characteristic p:

$$H\mathbb{Q} = K(0),\ K(1),\ \dots, K(n),\ ..., K(\infty) = H\mathbb{F}_p.$$

For any spectrum E, we denote by $\mathrm{Sp}_E \subseteq \mathrm{Sp}$ the ∞-category of E-local spectra. Even though p is never invertible on an object of $\mathrm{Sp}_{K(n)}$ for $n > 0$, we have the following remarkable result:

Theorem 1.1. *(Greenlees-Hovey-Sadofsky 1996) For $n < \infty$ and any $E \in \mathrm{Sp}_{K(n)}$ and an action of a finite group G on E, the norm map*

$$\mathrm{Nm} : E_{hG} \to E^{hG}$$

is an equivalence in $\mathrm{Sp}_{K(n)}$.

Another formulation is that the Tate construction X^{tG}, given as the cofiber of Nm in Sp, is $K(n)$-acyclic. Hence, the result is also known as "Tate vanishing". An action of G on E is a functor $F : BG \to \mathrm{Sp}_{K(n)}$, and the norm is a map:

$$\mathrm{Nm} : \operatorname{colim} F \to \lim F.$$

In this form, Hopkins and Lurie gave a strengthening of 1.1. Recall that a space A is called m-finite if all sets $\pi_n(A, a_0)$ are finite and A is m-truncated. A π-finite space is a space that is m-finite for some m.

Theorem 1.2. *(Hopkins-Lurie 2013) Let A be any π-finite space, $n < \infty$ and $F : A \to \mathrm{Sp}_{K(n)}$. There is a canonical (natural) equivalence*

$$\mathrm{Nm}_A : \operatorname{colim} F \xrightarrow{\sim} \lim F$$

Namely, they construct higher norms maps generalizing the usual ones and prove they are equivalences.

Remark 1.3. *A 1-finite space is a finite disjoint union of BG-s for finite G-s and the norm map is the sum of the classical norm maps. Thus, 1.2 generalizes 1.1.*

The $K(n)$-local categories have a close relative. Denote $T(n) = v_n^{-1} X(n)$, where $X(n)$ is some finite type n spectrum. There is a sequence of inclusions

$$\mathrm{Sp}_{K(n)} \subseteq \mathrm{Sp}_{T(n)} \subseteq \mathrm{Sp}$$

and a factorization of the localization functors $L_{K(n)} = L_{K(n)} L_{T(n)}$. Thus, the following is a strengthening of 1.1

Theorem 1.4. *(Kuhn 2003) 1.1 is true when we replace $\mathrm{Sp}_{K(n)}$ with $\mathrm{Sp}_{T(n)}$.*

Our first main result is:

Theorem 1.5. *(Carmeli, Schlank, Y) 1.2 is true when we replace $\mathrm{Sp}_{K(n)}$ with $\mathrm{Sp}_{T(n)}$.*

The proof of this theorem is inevitably different from the proof given by Hopkins and Lurie for the $K(n)$-local version. In particular, we start with a more elaborate analysis of 1-semiadditivity, and show that quite generally 1-semiadditivity can be bootstrapped to ∞-semiadditivity under relatively mild assumptions. In fact, we show more generally:

Theorem 1.6. *(Carmeli, Schlank, Y) Let R be a non-zero homotopy ring spectrum. The following are equivalent:*

1. Sp_R *is 1-semiadditive.*
2. Sp_R *is ∞-semiadditive.*
3. *There exists $0 \leq n < \infty$, such that $\mathrm{Sp}_{K(n)} \subseteq \mathrm{Sp}_R \subseteq \mathrm{Sp}_{T(n)}$.*

Higher Semiadditivity. The higher norms are constructed inductively on the truncatendness m of the space A. The main point is that in order to define the norm for an m-finite space A, one needs not only the existence of norm maps for $(m-1)$-truncated spaces, but the fact that they are *equivalences*. In particular, to even define the norm for 2-finite spaces, one needs the Tate vanishing result for group actions.

The best way to get a feeling for how the norm map is generalized *upwards*, is to actually unwind the definition of the usual map *downwards*. Recall that the usual norm map

$$\mathrm{Nm}_{BG} : E_{hG} \to E^{hG}$$

is defined roughly by *summing* over the orbit. The fact that we can sum maps is because of the canonical isomorphism from a co-product to the product:

$$E_1 \sqcup E_2 \sqcup \cdots \sqcup E_n \xrightarrow{\sim} E_1 \times E_2 \times \cdots \times E_n.$$

This map itself is a norm map with respect to a finite discrete space with n points (0-truncated) and is constructed by identities on the diagonal and zeros of the diagonal. The zero map $E_i \to E_j$ for $i \neq j$, exists because the unique map $0 \to 1$ from the initial object to the terminal object is an isomorphism. This map in turn is a norm for the empty space ((-1)-truncated).

Hopkins and Lurie organize this into a general categorical pattern, such that given a norm map for m-finite spaces, if it is an isomorphism we can use its inverse to construct a norm map for $(m+1)$-finite spaces.

Definition 1.7. *An ∞-category $\mathcal{C}$ is m-semiadditive if it is $(m-1)$-semiadditive and for all the norm maps for all m-finite spaces are equivalences. Every $\mathcal{C}$ is (-2)-semiadditive.*

Being 0-semiadditive is just being semiadditive and being (-1)-semiadditive is being pointed (having a zero object). With this terminology one can say:

- Greenlees-Hovey-Sadofsky: $\mathrm{Sp}_{K(n)}$ is 1-semiadditive.
- Hopkins-Lurie: $\mathrm{Sp}_{K(n)}$ is ∞-semiadditive.
- Kuhn: $\mathrm{Sp}_{T(n)}$ is 1-semiadditive.
- Carmeli-Schlank-Y: $\mathrm{Sp}_{T(n)}$ is ∞-semiadditive.

Just like the property of 0-semiadditivity (i.e. ordinary semiadditivity) of $\mathcal{C}$ induces a canonical operation of summation of finite sets of morphisms between a pair of objects of $\mathcal{C}$, so does higher semiadditivity induces an operation of summation (or "integration") of families of morphisms indexed by more general π-finite spaces. More explicitly, given object X and Y in an m-semiadditive ∞-category $\mathcal{C}$ and a map $\varphi : A \to \mathrm{Map}\,(X,Y)$ where A is an m-finite space, one can define a new morphism $\int_A \varphi \in \mathrm{Map}\,(X,Y)$ as the composition

$$X \xrightarrow{\Delta} \lim_A X \xrightarrow{\varphi} \lim_A Y \xrightarrow[\sim]{\mathrm{Nm}_A^{-1}} \operatorname{colim} Y \xrightarrow{\nabla} Y.$$

An important special case is where φ is constant on Id_X, in which case we denote $\int_A \mathrm{Id}_X$ by $[A]_X$ (those are actually the components of a natural endomorphism of the identity functor of $\mathcal{C}$).

Proof Sketch. As said above, the proof proceeds by induction on the level of semiadditivity m starting with $m = 1$. The inductive step is based on the following easy argument:

Lemma 1.8 (Transfer Principal)**.** *Given a fiber sequence*

$$A \to E \to B$$

with B connected, such that A and E are m-finite and $[A]$ is an isomorphism, then Nm_B is an isomorphism.

By the formal part of HL it is enough to show that the norm of $K\,(C_p, m+1)$ is an isomorphism.

Definition 1.9. *We call a space A* good *if it is a connected m-finite p-space with $\pi_m\,(A) \neq 0$.*

For every good space A, there is a fiber sequence

$$A \to E \to K(C_p, m+1).$$

Thus, if we can find any good space A with $[A] \in \pi_0(\mathbb{S}_{T(n)})$ invertible, we are done. The next observation is that we can check whether $[A]$ is invertible using Morava E-theory. Namely, the Hurewicz map

$$f : \pi_0 \mathbb{S}_{T(n)} \to \pi_0(E_n) \simeq \mathbb{Z}_p[[u_1, \ldots, u_{n-1}]]$$

detects invertibility. This follows from the following

Lemma 1.10 (Conservativity Principal)**.** *The localization functor* $\mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{K(n)}$ *is conservative on dualizable objects.*

In fact, the image of f is contained in the constants $\mathbb{Z}_p$, so the question becomes how to find a good space A, such that $f([A])$ has zero p-adic valuation? The rough idea is that by the results of Ravenel and Wilson, $f([K(C_p, m)])$ is non zero, so it has *finite* p-adic valuation and using 1-semiadditivity we can construct an operation which reduces p-adic valuation.

1-Semiadditivity. Let $\mathcal{C}$ be a 1-semiadditive stable p-local symmetric monoidal ∞-category and R an $\mathbb{E}_\infty$-ring object in $\mathcal{C}$. An element $x \in \pi_0 R$ is by definition a map $x : 1 \to R$ up to homotopy. Using the $\mathbb{E}_\infty$-structure, we get a new element $x^p : BC_p \to \mathrm{Map}(1, R)$ and we can define

$$\delta(x) = \int_{BC_p} (x - x^p) \in \pi_0 R.$$

The resulting function $\delta : \pi_0 R \to \pi_0 R$ is an additive p-derivation. That is, it satisfies:

(1) $\delta(0) = \delta(1) = 0$.
(2) For all $x, y \in R$:

$$\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p}.$$

(This implies that $x \mapsto x^p + p\delta(x)$ is additive).

In other words, it behaves very much like the operation $\delta_0(x) = \frac{x - x^p}{p}$ on $\mathbb{Z}_p$. Recall the map

$$f : \pi_0 \mathbb{S}_{T(n)} \to \mathbb{Z}_p \subseteq \pi_0(E_n).$$

The key properties of δ that are needed to complete the proof sketch above are:

- $f(\delta(x)) = \delta_0(f(x))$
- For a good space A, the element $\delta([A])$ is of the form $[A'] - [A'']$ for good spaces A' and A''.

Thus, starting with $A = K(C_p, m)$, if $[A]$ is not invertible, we can replace A with a good space A' or A'' from above, with a smaller p-adic valuation. This procedure stops after a finite number of steps producing a good space A such that $[A]$ is invertible and thus completing the proof of the $(m+1)$-semiadditivity of $\mathrm{Sp}_{T(n)}$.

Real cobordism, Morava *E*-theories and the dual Steenrod algebra

Mingcong Zeng
(joint work with Lennart Meier)

If one wish to understand stable homotopy groups of sphere $\pi_*(S)$ from the chromatic perspective, Morava E-theories and their homotopy fixed points cannot be avoided. Fix a height n, its crucial to understand the $K(n)$-local sphere $L_{K(n)}S$. By [2], $L_{K(n)}S \simeq E_n^{h\mathbb{G}_n}$, where the right hand side is the homotopy fixed point spectrum of Morava E_n by the big Morava stabilizer group $\mathbb{G}_n$. Theoretically one can approach the computation using the homotopy fixed point spectral sequence, but it is very difficult in practice. The situation becomes more reasonable if one replace $\mathbb{G}$ by one of its finite subgroups F and consider E_n^{hF}. When the height $n = 1, 2$, one can recover $E_n^{h\mathbb{G}_n}$ from various E_n^{hF} using finite resolutions and E_n^{hF} are much more computable.

Now assume that $p = 2$, we know that there is a finite subgroup $C_{2^k} \subset \mathbb{G}_n$ if and only if $2^{k-1}|n$. For $n = 1, 2$, $E_1^{hC_2}$ and $E_2^{hC_4}$ are well-understood. In general, when either the height and the order of subgroup grows, the computation becomes more difficult. A new approach to this computation through equivariant stable homotopy theory is given in [3]. Let $MU_{\mathbb{R}}$ be the C_2-spectrum obtained by the complex cobordism MU with complex conjugate, then it is shown that all E_n treated as C_2-ring spectra receive orientations from $MU_{\mathbb{R}}$. Furthermore, let $N_2^{2^k} : Sp^{C_2} \to Sp^{C_{2^k}}$ be the Hill-Hopkins-Ravenel norm functor in [4], then by the norm-forget adjunction in commutative rings, we have

Theorem 1.1 ([3])**.** *There is a C_{2^k}-equivariant homotopy ring map*

$$N_2^{2^k} MU_{\mathbb{R}} \to E_n.$$

If E_n supports a C_{2^k}-action.

Localizing at 2, $MU_{\mathbb{R}}$ splits into wedge of suspensions of $BP_{\mathbb{R}}$, the real Brown-Peterson spectrum. Let $BP^{((C_{2^k}))} := N_2^{2^k} BP_{\mathbb{R}}$, one can build versions of Morava E-theories from them by truncation, inverting a suitable element and then $K(n)$-localization. Therefore one can apply computations of $BP^{((C_{2^k})}$ to Morava E-theories. In general, a thorough understanding of the C_{2^k}-spectrum $BP^{((C_{2^k}))}$ can tell us a lot about $E_n^{hC_{2^k}}$ for all heights n that supports a C_{2^k}-action.

The main tool of computation around $BP^{((C_{2^k}))}$ is the equivariant slice filtration [4], which is an analog of the motivic very effective slice filtration in the category of C_{2^k}-spectra. Let $H \subset G = C_{2^k}$, ρ_H be the regular representation $\mathbb{R}[H]$ and S^{ρ_H} be its one point compactification. A slice cell is a G-space of the form $G/H_+ \wedge S^{i\rho_H}$ for $i \in \mathbb{Z}$. $S_{>n}$, the slice n-positive category is the subcategory of G-spcetra generated by slice cells of underlying dimension larger than n under weak equivalence, wedge and cofibre. Let $P^n(-)$ be the Bousfield localization with respect to $S_{>n}$, we have

Definition 1.2. *The slice tower of a G-spectrum X is the tower $P^n(X) \to P^{n-1}(X)$. The fibre $P_n^n(X) \to P^n(X) \to P^{n-1}(X)$ is called the n-th slice of X.*

The corresponding $RO(G)$-graded spectral sequence of Mackey functors $E_2 = \underline{\pi}_\bigstar P_n^n(X) \Rightarrow \underline{\pi}_\bigstar(X)$ is the slice spectral sequence(SSS) of X.

One of the main theorem of Hill-Hopkins-Ravenel determines the slices of $BP^{((G))}$.

Theorem 1.3 ([4]).

$$P_i^i(BP^{((G))}) = \begin{cases} * & i<0 \text{ or } i \text{ is odd} \\ H\underline{\mathbb{Z}} & i=0 \\ W_i \wedge H\underline{\mathbb{Z}} & i>0 \text{ and } i \text{ is even} \end{cases}$$

Where W_i is a wedge of slice cells of dimension i.

Therefore, the E_2-page of the slice spectral sequence can be read off from $\underline{\pi}_\bigstar H\underline{\mathbb{Z}}$.

One very important trick of computing differentials in the slice spectral sequence is localizing the whole spectral sequence at the Euler class of some representation spheres. Let σ be the sign representation of G and λ_i be the G-representation on $\mathbb{R}^2$ by rotating $e^{\frac{2^i}{2^n}2\pi i}$ and $a_V : S^0 \to S^V$ be the embedding of north and south pole. We have

Proposition 1.4. $a_{\lambda_i}^{-1}SSS(BP^{((G))})$ *computes* $\underline{\pi}_\bigstar(a_{\lambda_i}^{-1}BP^{((G))})$.

The first example is inverting a_σ. $(a_\sigma^{-1}BP^{((G))})^G$ is its G-geometric fixed point $\Phi^G(BP^{((G))}) \simeq H\mathbb{F}_2$. One the other hand, $a_\sigma^{-1}SSS(BP^{((G))})$ is nontrivial and in fact, there is only one pattern of differential that can give the correct answer. When we invert a_{λ_i}, the smaller i is, the closer $a_{\lambda_i}^{-1}SSS(BP^{((G))})$ looks like to $SSS(BP^{((G))})$. In fact,

Proposition 1.5. *In the integral page, the map*

$$SSS(BP^{((G))}) \to a_{\lambda_0}^{-1}SSS(BP^{((G))})$$

is an isomorphism in positive filtration and in filtration 0 *it is a surjection and its kernel consists of permanent cycles.*

This means that in non-negative filtration $SSS(BP^{((G))})$ and its a_{λ_0}-localization contains exactly the same information. But what does $a_{\lambda_0}^{-1}SSS(BP^{((G))})$ compute?

Theorem 1.6.

$$(a_{\lambda_0}^{-1}BP^{((G))})^G \simeq (N_1^{2^{k-1}}H\mathbb{F}_2)^{C_{2^{k-1}}} \simeq (N_1^{2^{k-1}}H\mathbb{F}_2)^{hC_{2^{k-1}}}.$$

The second equivalence is the main theorem of [1].

Specialize to $G = C_4$, we see that $a_{\lambda_0}^{-1}SSS(BP^{((C_4))})$ computes $(N_1^2 H\mathbb{F}_2)^{hC_2}$. What is special in this case is that if we consider the Tate fixed point instead of homotopy fixed point, we have

Theorem 1.7 ([5]).

$$(N_1^2 H\mathbb{F}_2)^{tC_2} \simeq H\mathbb{F}_2.$$

The homotopy fixed point and Tate spectral sequence for $N_1^2 H\mathbb{F}_2$ has input the dual Steenrod algebra A_* with conjugate action. The above theorem means that in both spectral sequence, there are a lot of nontrivial differentials. Using $a_{\lambda_0}^{-1} SSS(BP^{((C_4))})$, we can understand these differentials and $\pi_*(N_1^2 H\mathbb{F}_2)^{hC_2}$ in a range.

Theorem 1.8. *The first* 6 *homotopy groups of* $(N_1^2 H\mathbb{F}_2)^{hC_2}$ *are*

i	0	1	2	3	4	5	6
$\pi_i(N_1^2 H\mathbb{F}_2)^{hC_2}$	$\mathbb{Z}/4$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/4 \oplus \mathbb{Z}/2$

Where the black summands are from transfers of permutation submodules in A_* *and the red summands are the interesting parts.*

By comparing the spectral sequences, we can also translate the family of slice differentials in [4] into the Tate spectral sequence.

Theorem 1.9. *Let* $x \in \tilde{H}^{-1}(C_2; \mathbb{F}_2)$ *be the generator of Tate cohomology and* $k > 0$. *Then in the Tate spectral sequence of* $N_1^2 H\mathbb{F}_2$, $d_i(x^{2^k}) = 0$ *when* $i < 2^{k+1}-1$ *and*

$$d_{2^{k+1}-1}(x^{2^k}) = \xi_k \overline{\xi}_k x^{-2^k+1}.$$

In the current statues, we can understand $N_1^2 H\mathbb{F}_2$ by understanding $a_{\lambda_0}^{-1} SSS(BP^{((C_4))})$. However, if one has a different way of computing $N_1^2 H\mathbb{F}_2$, then it should tell us a lot of information about $a_{\lambda_0}^{-1} SSS(BP^{((C_4))})$, thus $BP^{((C_4))}$ itself, and $E_{2n}^{hC_4}$ for all n.

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